Analytic Signals and Hilbert Transform Filters

A signal which has no negative-frequency components is called an *analytic signal*.[4.12] Therefore, in continuous time, every analytic signal $z(t)$ can be represented as

$$z(t) = \frac{1}{2\pi}\int_0^{\infty} Z(\omega)e^{j\omega t}d\omega$$

where $Z(\omega)$ is the complex coefficient (setting the amplitude and phase) of the positive-frequency complex sinusoid $\exp(j\omega t)$ at frequency ω .

Any real sinusoid $A\cos(\omega t + \phi)$ may be converted to a positive-frequency complex sinusoid $A\exp[j(\omega t + \phi)]$ by simply generating a phase-quadrature component $A\sin(\omega t + \phi)$ to serve as the ``imaginary part'':

$$Ae^{j(\omega t+\phi)} = A\cos(\omega t + \phi) + jA\sin(\omega t + \phi)$$

The phase-quadrature component can be generated from the in-phase component by a simple quarter-cycle time shift.[4.13]

For more complicated signals which are expressible as a sum of many sinusoids, a *filter* can be constructed which shifts each sinusoidal component by a quarter cycle. This is called a *Hilbert transform filter*. Let $\mathcal{H}_t\{x\}$ denote the output at time t of the Hilbert-transform filter applied to the signal x . Ideally, this filter has magnitude 1 at all frequencies and introduces a phase shift of $-\pi/2$ at each positive frequency and $+\pi/2$ at each negative frequency. When a real signal $x(t)$ and its Hilbert transform $y(t) = \mathcal{H}_t\{x\}$ are used to form a new complex signal $z(t) = x(t) + jy(t)$, the signal $z(t)$ is the (complex) *analytic signal* corresponding to the real signal $x(t)$. In other words, for any real signal $x(t)$, the corresponding analytic signal $z(t) = x(t) + j\mathcal{H}_t\{x\}$ has the property that all ``negative frequencies'' of $x(t)$ have been ``filtered out.''

To see how this works, recall that these phase shifts can be impressed on a complex sinusoid by multiplying it by $\exp(\pm j\pi/2) = \pm j$. Consider the positive and negative frequency components at the particular frequency ω_0 :

$$\begin{aligned} x_+(t) &\isdef e^{j\omega_0 t} \\ x_-(t) &\isdef e^{-j\omega_0 t} \end{aligned}$$

Now let's apply a -90 degrees phase shift to the positive-frequency component, and a $+90$ degrees phase shift to the negative-frequency component:

$$\begin{aligned} y_+(t) &= e^{-j\pi/2} e^{j\omega_0 t} = -j e^{j\omega_0 t} \\ y_-(t) &= e^{j\pi/2} e^{-j\omega_0 t} = j e^{-j\omega_0 t} \end{aligned}$$

Adding them together gives

$$\begin{aligned} z_+(t) &\isdef x_+(t) + jy_+(t) = e^{j\omega_0 t} - j^2 e^{j\omega_0 t} = 2e^{j\omega_0 t} \\ z_-(t) &\isdef x_-(t) + jy_-(t) = e^{-j\omega_0 t} + j^2 e^{-j\omega_0 t} = 0 \end{aligned}$$

and sure enough, the negative frequency component is filtered out. (There is also a gain of 2 at positive frequencies.)

For a concrete example, let's start with the real sinusoid

$$x(t) = 2\cos(\omega_0 t) = e^{j\omega_0 t} + e^{-j\omega_0 t}.$$

Applying the ideal phase shifts, the Hilbert transform is

$$\begin{aligned} y(t) &= e^{j\omega_0 t - j\pi/2} + e^{-j\omega_0 t + j\pi/2} \\ &= -je^{j\omega_0 t} + je^{-j\omega_0 t} = 2\sin(\omega_0 t). \end{aligned}$$

The analytic signal is then

$$z(t) = x(t) + jy(t) = 2\cos(\omega_0 t) + j2\sin(\omega_0 t) = 2e^{j\omega_0 t},$$

by Euler's identity. Thus, in the sum $x(t) + jy(t)$, the negative-frequency components of $x(t)$ and $jy(t)$ cancel out, leaving only the positive-frequency component. This happens for any real signal $x(t)$, not just for sinusoids as in our example.

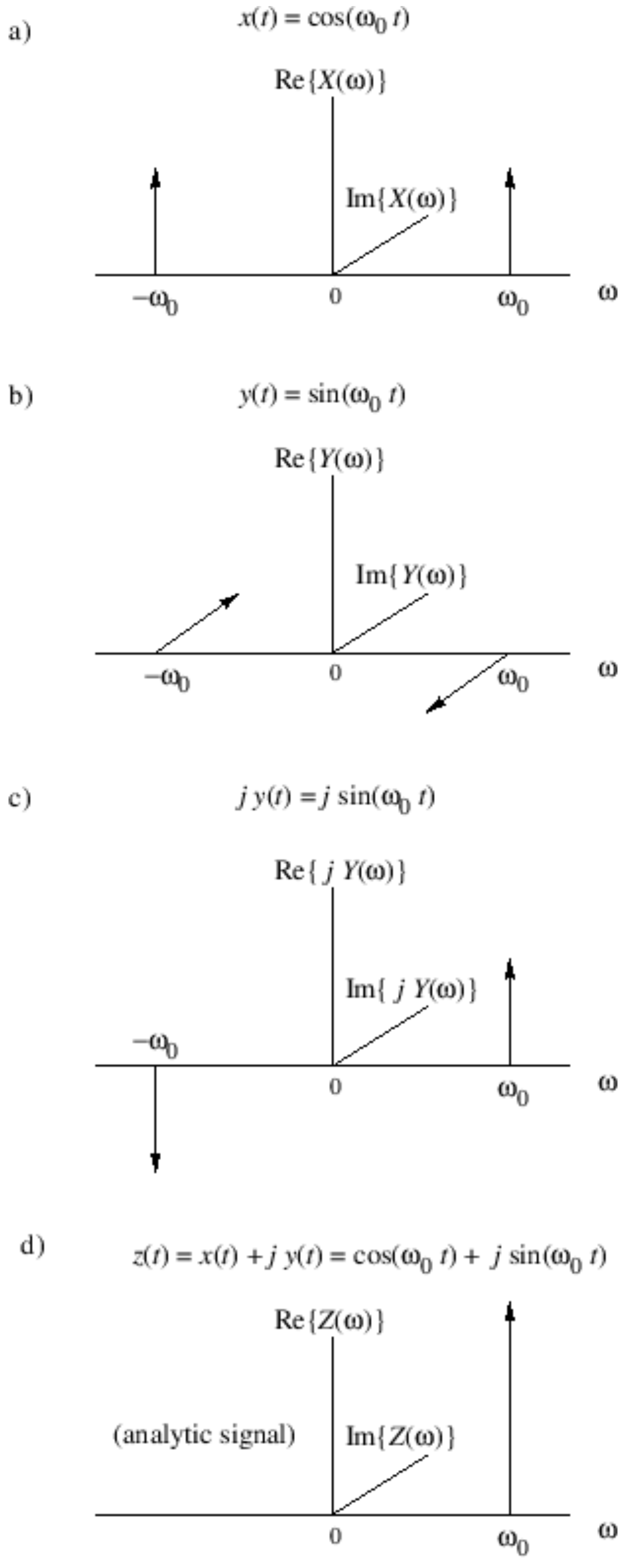

Figure 4.16: Creation of the analytic signal $z(t) = e^{j\omega_0 t}$ from the real sinusoid $x(t) = \cos(\omega_0 t)$ and the derived phase-quadrature sinusoid $y(t) = \sin(\omega_0 t)$, viewed in the frequency domain. a) Spectrum of x. b)

Figure 4.16 illustrates what is going on in the frequency domain. At the top is a graph of the spectrum of the sinusoid $\cos(\omega_0 t)$ consisting of impulses at frequencies $\omega = \pm\omega_0$ and zero at all other frequencies (since $\cos(\omega_0 t) = (1/2)\exp(j\omega_0 t) + (1/2)\exp(-j\omega_0 t)$). Each impulse amplitude is equal to $1/2$. (The amplitude of an impulse is its algebraic area.) Similarly, since $\sin(\omega_0 t) = (1/2j)\exp(j\omega_0 t) - (1/2j)\exp(-j\omega_0 t) = -(j/2)\exp(j\omega_0 t) + (j/2)\exp(-j\omega_0 t)$, the spectrum of $\sin(\omega_0 t)$ is an impulse of amplitude $-j/2$ at $\omega = \omega_0$ and amplitude $+j/2$ at $\omega = -\omega_0$. Multiplying $y(t)$ by j results in $j\sin(\omega_0 t) = (1/2)\exp(j\omega_0 t) - (1/2)\exp(-j\omega_0 t)$ which is shown in the third plot, Fig.4.16c. Finally, adding together the first and third plots, corresponding to $z(t) = x(t) + jy(t)$, we see that the two positive-frequency impulses *add in phase* to give a unit impulse (corresponding to $\exp(j\omega_0 t)$), and at frequency $-\omega_0$, the two impulses, having opposite sign, *cancel* in the sum, thus creating an analytic signal z , as shown in Fig.4.16d. This sequence of operations illustrates how the negative-frequency component $\exp(-j\omega_0 t)$ gets *filtered out* by summing $\cos(\omega_0 t)$ with $j\sin(\omega_0 t)$ to produce the analytic signal $\exp(j\omega_0 t)$ corresponding to the real signal $\cos(\omega_0 t)$.

As a final example (and application), let $x(t) = A(t)\cos(\omega t)$, where $A(t)$ is a slowly varying amplitude envelope (slow compared with ω). This is an example of *amplitude modulation* applied to a sinusoid at ``carrier frequency'' ω (which is where you tune your AM radio). The Hilbert transform is very close to $y(t) \approx A(t)\sin(\omega t)$ (if $A(t)$ were constant, this would be exact), and the analytic signal is $z(t) \approx A(t)e^{j\omega t}$. Note that AM *demodulation*[4.14]is now nothing more than the *absolute value*. *I.e.*, $A(t) = \vert z(t)\vert$. Due to this simplicity, Hilbert transforms are sometimes used in making *amplitude envelope followers* for narrowband signals (*i.e.*, signals with all energy centered about a single ``carrier'' frequency). AM demodulation is one application of a narrowband envelope follower.

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