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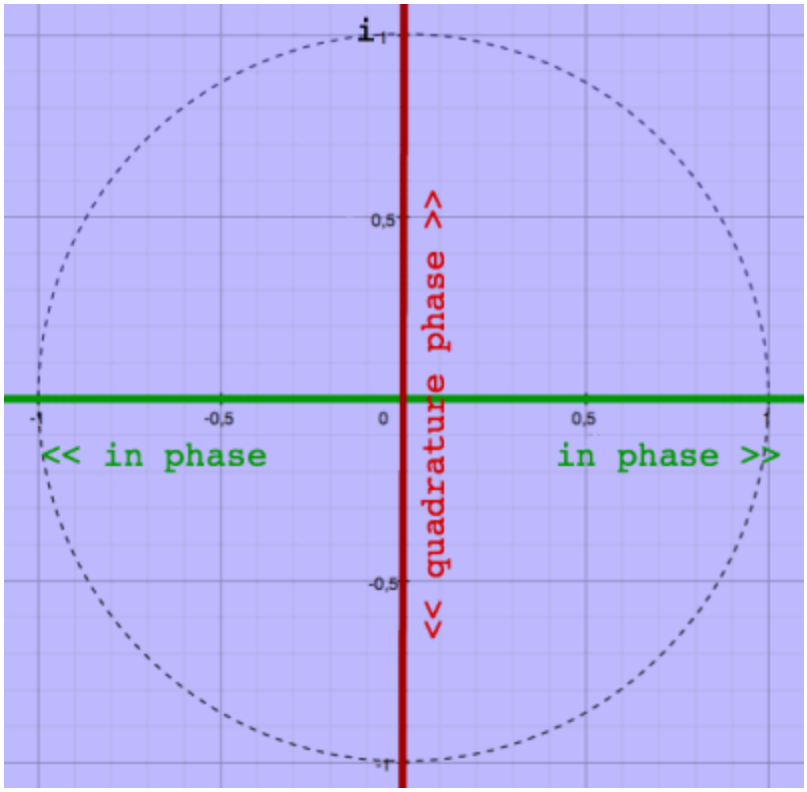
complexify a real signal

(page under construction)

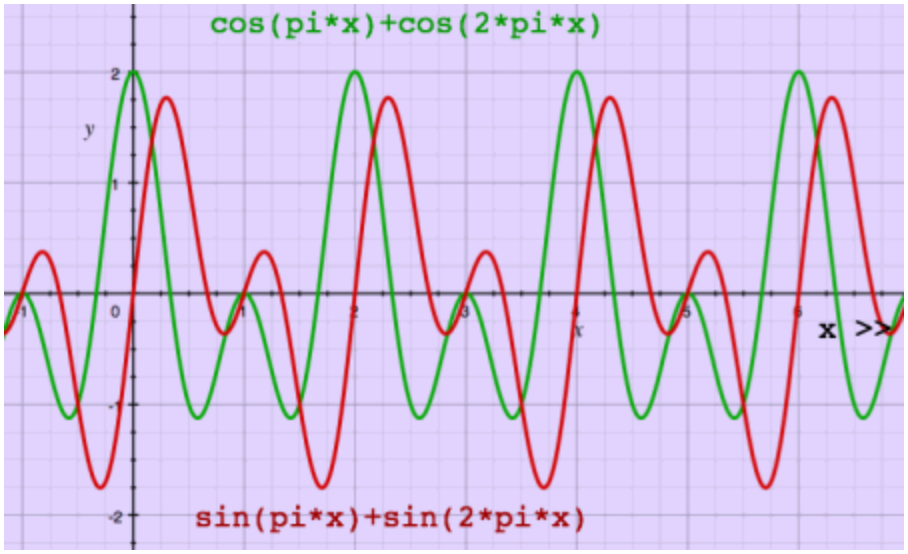
on Hilbert Transform and all-pass filtering

Complex signals are valuable, because they offer an opportunity to calculate instantaneous energy, amplitude and frequency. That is why complex signals are dubbed 'analytic signals'. The signal can be analysed sample-wise instead of frame-wise, and sometimes such fast access to analysis is welcome.

To derive an analytic signal from a real signal, a -pi/2 radian phase shifted version of the real signal must be made, an imaginary phase. In engineering terms, that extra signal is called quadrature phase, because pi/2 radians make a square angle on the complex plane. The mathematics of the transform were developed by the German mathematician David Hilbert, hence the title Hilbert Transform.

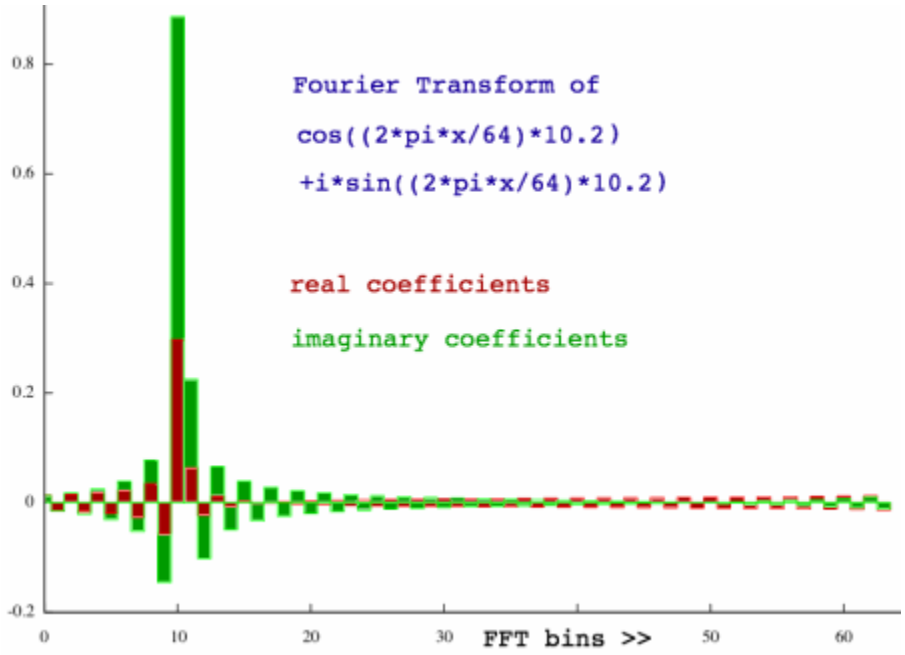


For each frequency, a -pi/2 radian phase shift translates to a known time-shift: a quarter of a cycle in the positive direction on the timeline, meaning [frequency/4] seconds delay. Because the time-shift is different for each frequency, it is not trivial to shift a compound signal by a fixed arc length. The example below with only two harmonic components serves to illustrate that:

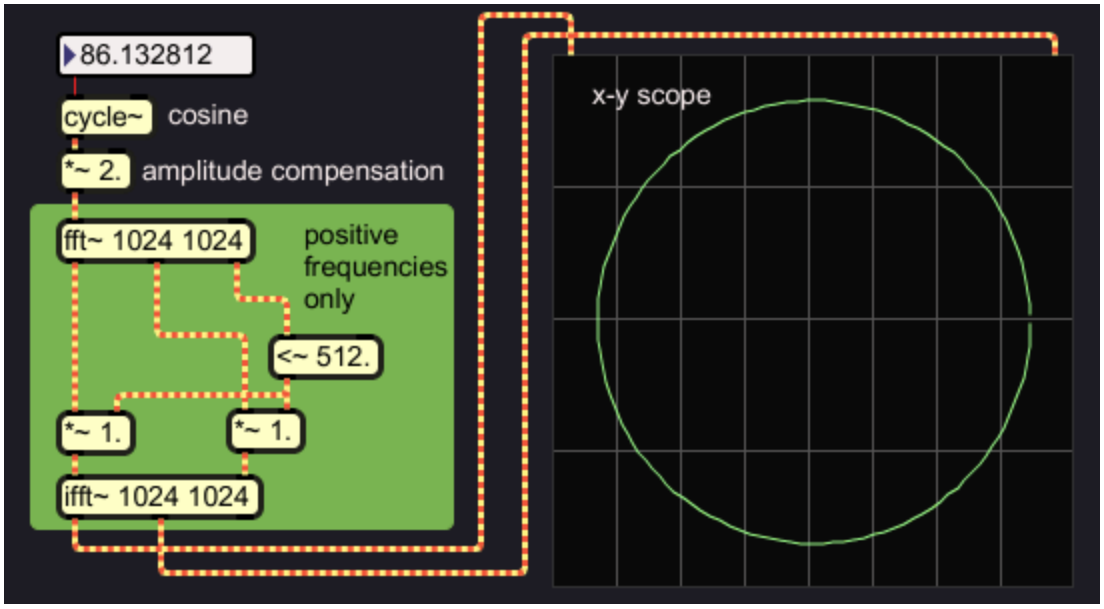


It is useful to first have a look at real and complex signals in frequency domain. Here is the spectrum of a real signal, with positive frequencies in the lower half of the spectrum and negative frequencies in the upper half. The spectrum is mirror-symmetric:

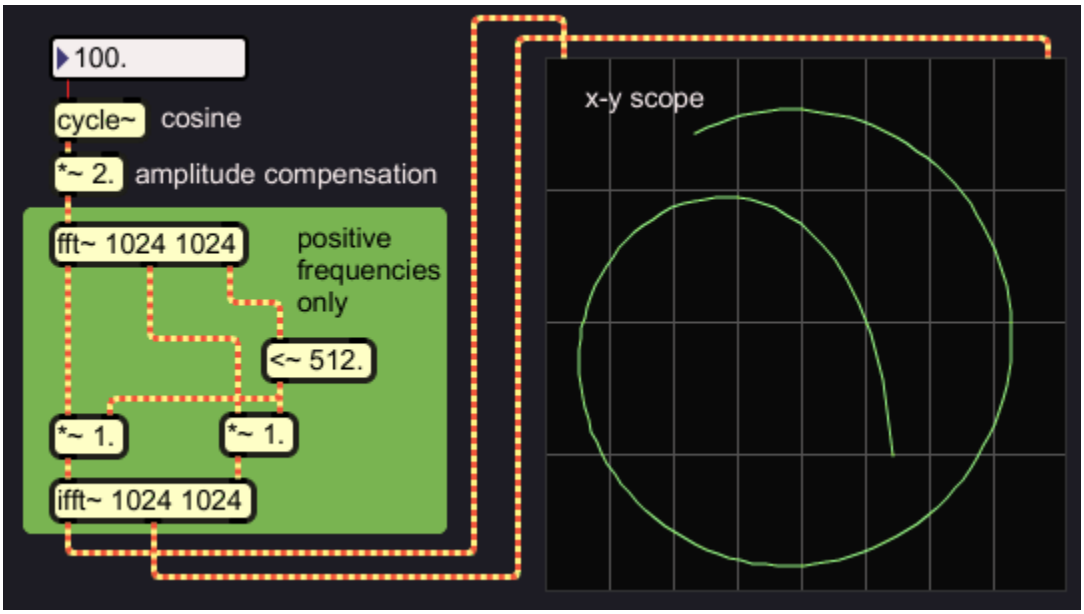
The spectrum of a complex signal with the same frequency is single-sided. It has only positive frequencies, filling the first half of the spectrum up to Nyquist. Analysed with a DFT, there is always spectral leakage and therefore the negative frequency coefficients still have small non-zero values:



Inversely, it should be possible to get a complex signal by simply erasing the negative frequencies from a real signal's spectrum. That is what I am going to check with complex FFT in a Max/MSP patch. To avoid the leakage problem in this experiment, I tune the input test signal to an FFT harmonic. The real and imaginary inverse FFT outputs are connected to an x-y scope so the orthogonality (perpendicularity) of the phases can be checked:

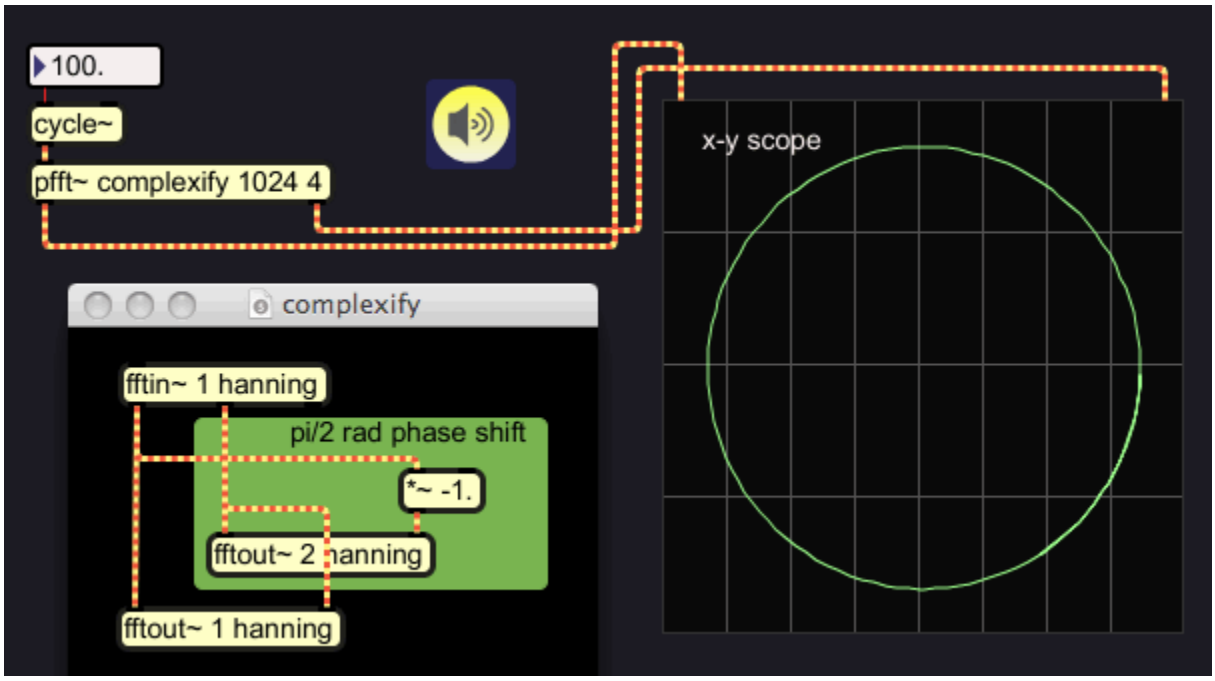


That works as expected. The scope trace would be elliptical or a straight line if the phases were not orthogonal. It could be done for arbitrary input signals, when proper windowing and overlap is applied to the FFT. Otherwise you get this:



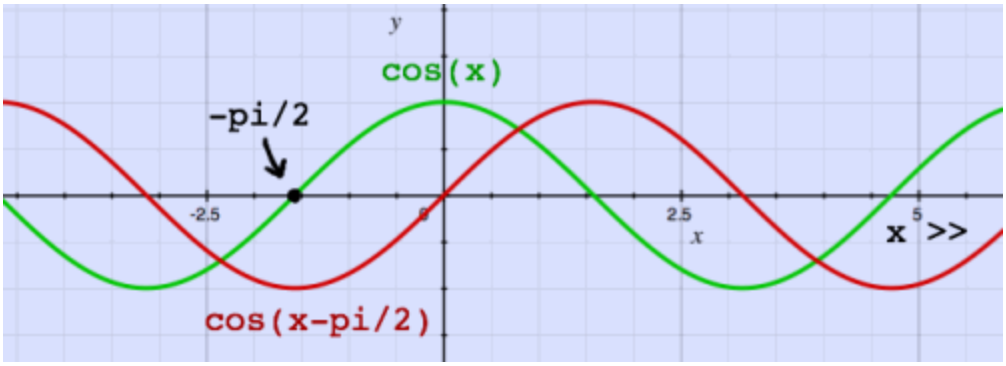
Below is another method to get complex phases via frequency domain.

It is done with one real FFT and two real IFFT's. The first IFFT does the in-phase or real phase output, and the second IFFT does the quadrature phase or imaginary phase output. The spectrum is rotated over -pi/2 radian by feeding the FFT imaginary coefficients in the second IFFT real input, and the real coefficients sign-flipped in the second IFFT imaginary input. The patch does four times overlap and Hann windows before and after transform. The complex output is accurate for frequencies from double the FFT fundamental up till near Nyquist.



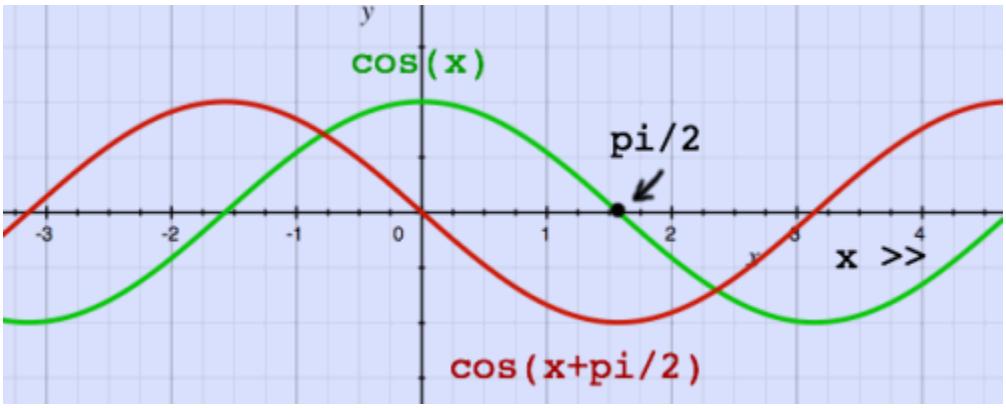
It is only for the purpose of illustration that I use an FFT to make a real signal complex. Because in the end, I do not want to wait for that whole FFT to happen before I have my complex signal.

To perform Hilbert Transform in time domain, must be done by convolution. That means filtering. I need to know the filter impulse response. A filter that makes a sine out of each cosine by phase-shifting that cosine with -pi/2 radians. Even this simple statement confuses me so often. Why a negative shift? Let me just plot this, to be shure why sin(x) = cos(x-pi/2):

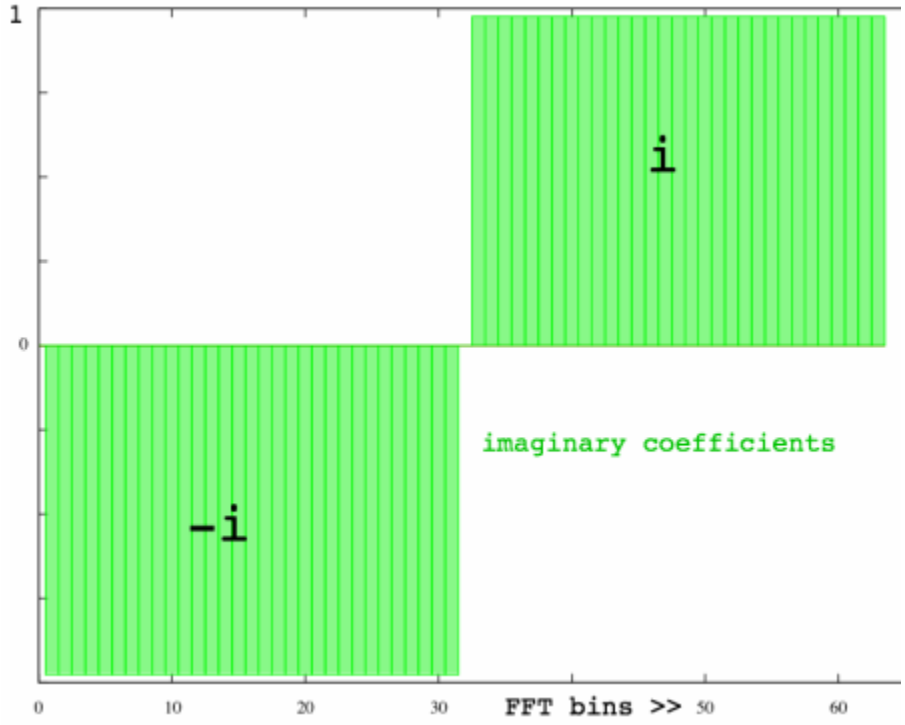


For Hilbert Transform, the phase shift must be done for all frequencies, except for DC and the Nyquist frequency, which can not be phase shifted by pi/2 rad because they are pure real. For the negative frequencies, cos(x) must be transformed to -sin(x). This goes:

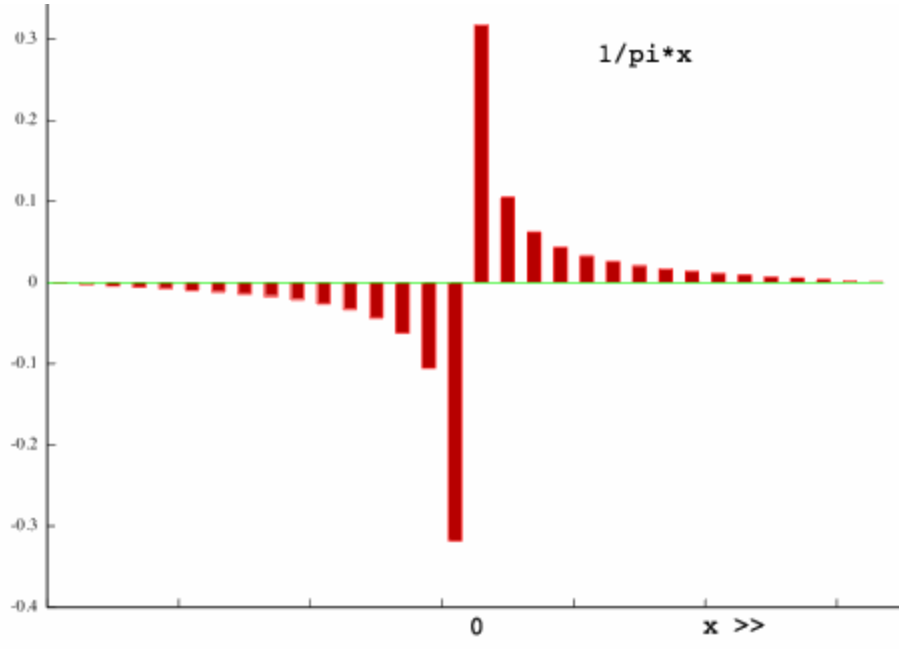
-sin(x) = cos(x+pi/2)



The phase shifts should be effectuated by convolution, but how? Again, frequency domain can help to clarify. Convolution in time domain is equivalent to multiplication in frequency domain. See the pages 'Convolution' and 'Fast Convolution' for illustrations on this topic. All -pi/2 and pi/2 phase shifts for Hilbert Transform could be performed in frequency domain as multiplications with -i for positive frequencies (in the first part of the spectrum) and i for negative frequencies (in the second part). This is what I actually did in the last Max/MSP patch above. -i and i are imaginary coefficients of magnitude -1 and 1:



To find the impulse response for Hilbert Transform in time domain, this spectrum must be inverse-FFT'd. It is first normalised, because there would be too much energy in the impulse response otherwise. The normalisation factor is [0.5 / N], where 0.5 accounts for the fact that the correlation weight is distributed over positive and negative frequencies. After IFFT, this pure real signal is the result:



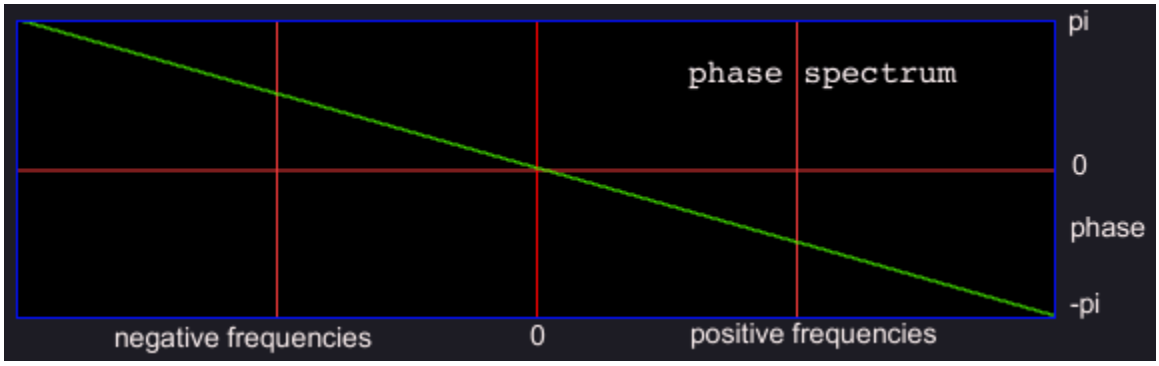
The impulse response for a Hilbert Transform filter is 1/pi*x, but only for uneven x. It represents a sum of sinewaves. The impulse response is not by nature restricted to a defined length.

Even when the time frame will be confined to a practical length of choice, the left side of the impulse response poses a problem. That part of the filter is non-causal, it has points in negative time, meaning that we have to collect a corresponding number of samples in a delay line before they can be processed. Implementing this as a Finite Impulse Response filter, would just be the slow convolution version of the spectrum manipulation showed earlier. Of course, this would not bring any advantage.

complexification using all-pass filters

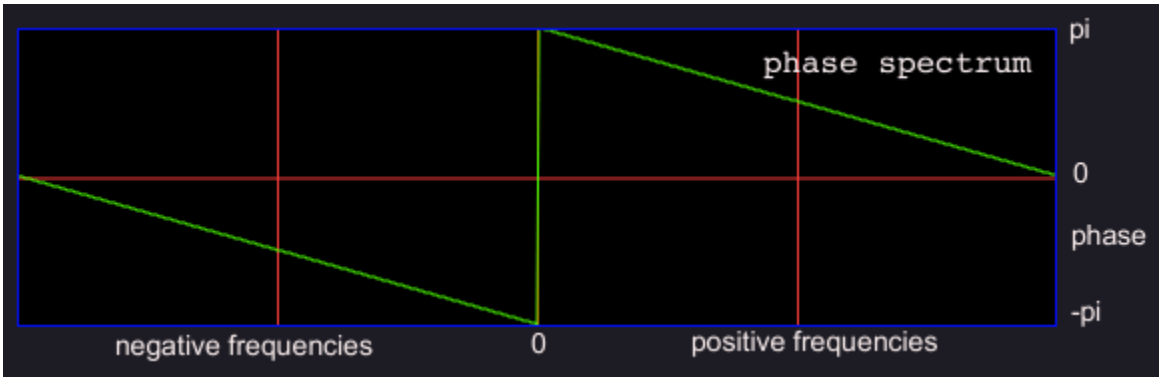
Probably I should be prepared for another compromise. If I can live with a real and imaginary phase which are *both* phase-shifted respective to the input signal, the complexification can be done with help of phase-shifting all-pass filters. This is not Hilbert Transform in the proper sense, since it does not produce a phase orthogonal to the input. Anyhow, if the output phases show a 90 degree phase difference, it will do fine for my analysis purposes.

This is actually a completely different route to get complex signals, so I will start from scratch. Digital time domain filters add delayed input- and/or output values to the signal, eventually with a scaling factor and/or sign-flip inbetween. The phase-shift effect of delaying a stream of sample values varies per frequency. DC can not be phase-shifted by delay. With a one sample delay, positive frequencies are shifted by an angle inbetween zero and -pi radians:



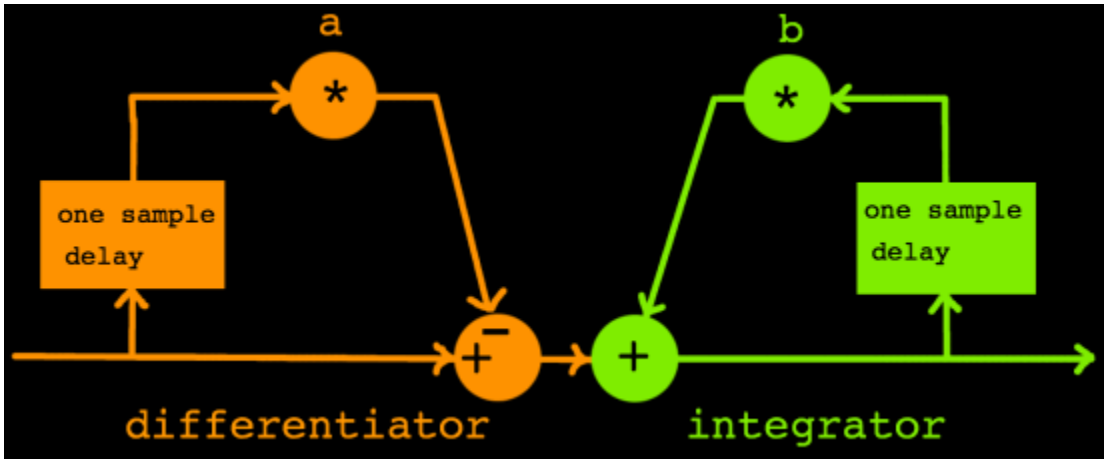
phase shifts of frequencies when one sample delayed

A sign-flip does a rotation over pi radian for all frequencies. The combination of one sample delay and sign-flip has this spectral effect:



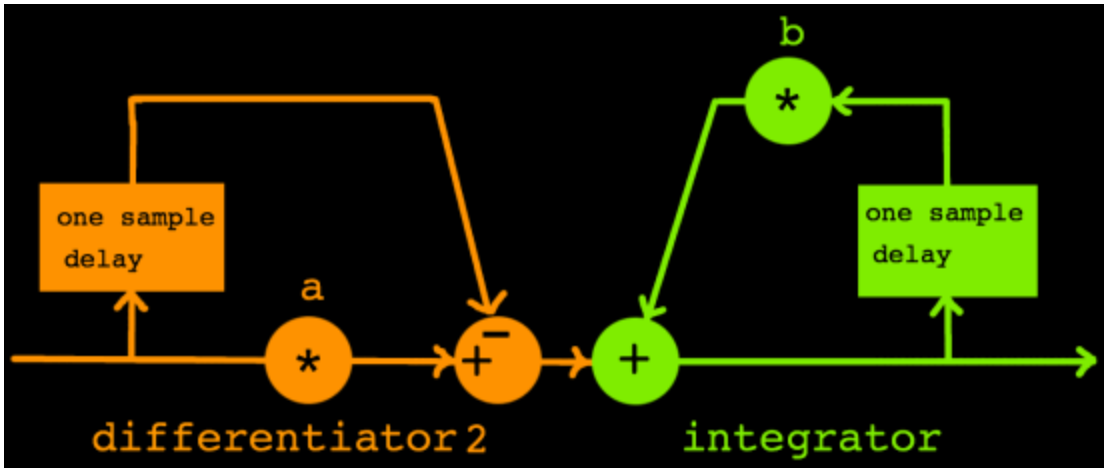
phase shifts of frequencies one sample delayed and sign-flipped

One sample delay and sign-flip is applied in differentiators. And differentiators, combined with integrators, can make all-pass filters under certain conditions. A differentiator is a feed-forward filter. It looks a bit like the integrator figuring on the previous pages, but there is no recirculation, and the delayed/scaled sample is subtracted from the input, instead of summed to it. Here is a differentiator/integrator flow graph:



y[n] = x[n] - a*x[n-1] + b*y[n-1]

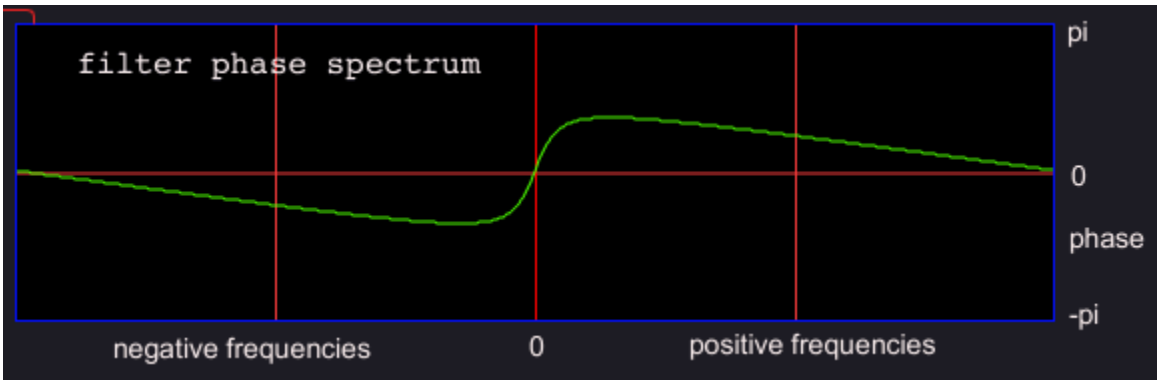
Since integration is the inverse process of differentiation, the filtering effect of the differentiator could be undone by the integrator when their coefficients a and b were set to the same value. For all frequencies, DC included, the amplitude and phase response of the filter combination would be flat at unity amplitude and phase zero. Of course, such a filter without any effect would be of no use, and in practical applications the coefficients would have different values to render desired filter effects. I just mentioned the do-nothing filter as a comparison for what comes next: the alternative differentiator type. It's filter coefficient is not in the feed-forward line, but directly on the input:



Implemented this way, the differentiator has the same amplitude response as the earlier sketched one, but the phase response is altered. So we have two differentiator types. When x[n] is the differentiator input and y[n] is it's output, the following filter expressions apply to them:

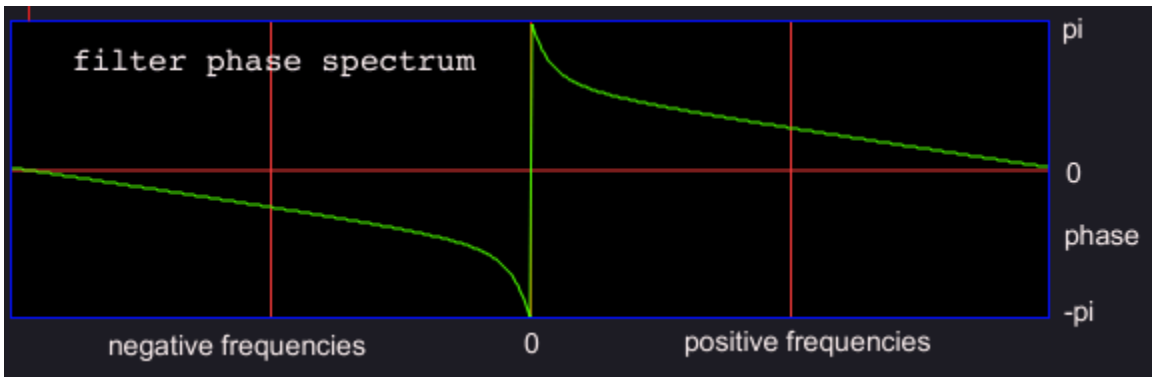
```
y[n] = x[n] - a*x[n-1]   differentiator type 1
y[n] = a*x[n] - x[n-1]   differentiator type 2
```

I will plot example phase spectra for the two types to illustrate the spectacular difference between them. For the moment, the integrator is not taken into consideration. I use Max/MSP's filtergraph~ object to compute and plot the spectra. The filter coefficient 'a' is 0.9 in both cases. Here is the phase response of differentiator type 1:



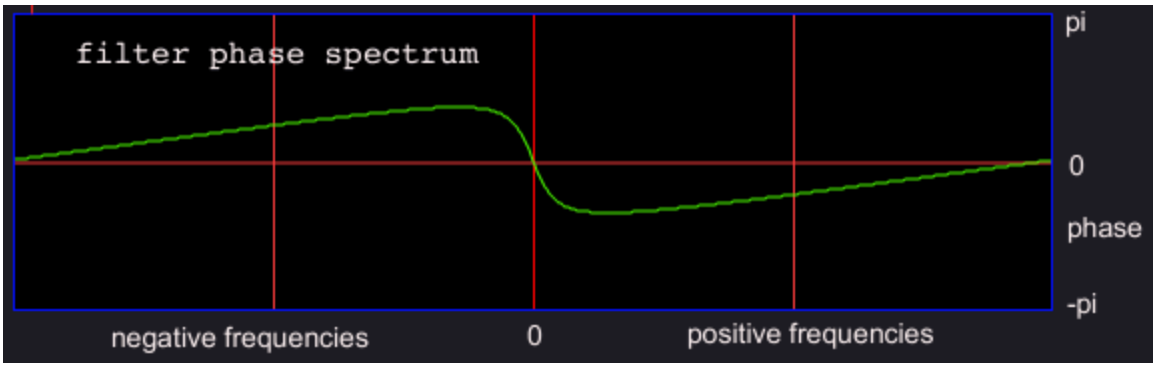
differentiator type 1 with: y[n] = x[n] - 0.9*x[n-1]

And here is the response of differentiator type 2:



differentiator type 2 with: y[n] = 0.9*x[n] - x[n-1]

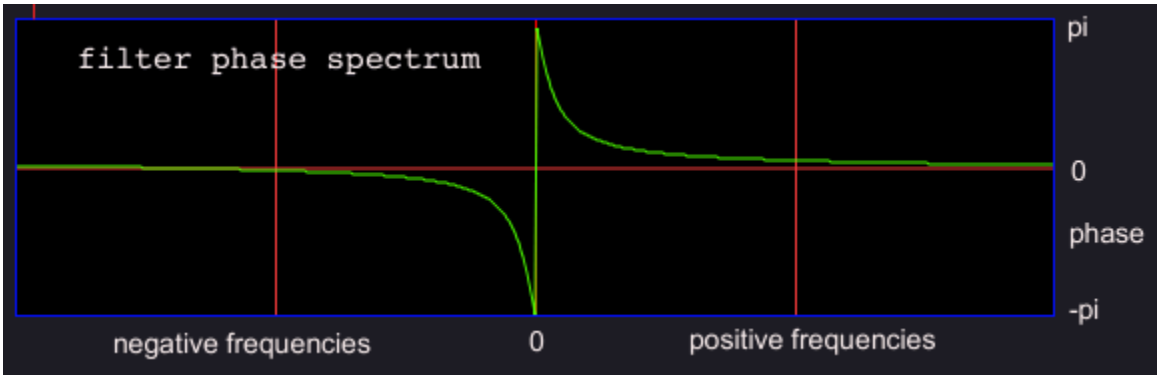
To demonstrate what the differentiator types do in combination with an integrator, I first plot the integrator phase response separately. Here it is, for filter coefficient b=0.9:



integrator with: y[n] = x[n] + 0.9*y[n-1]

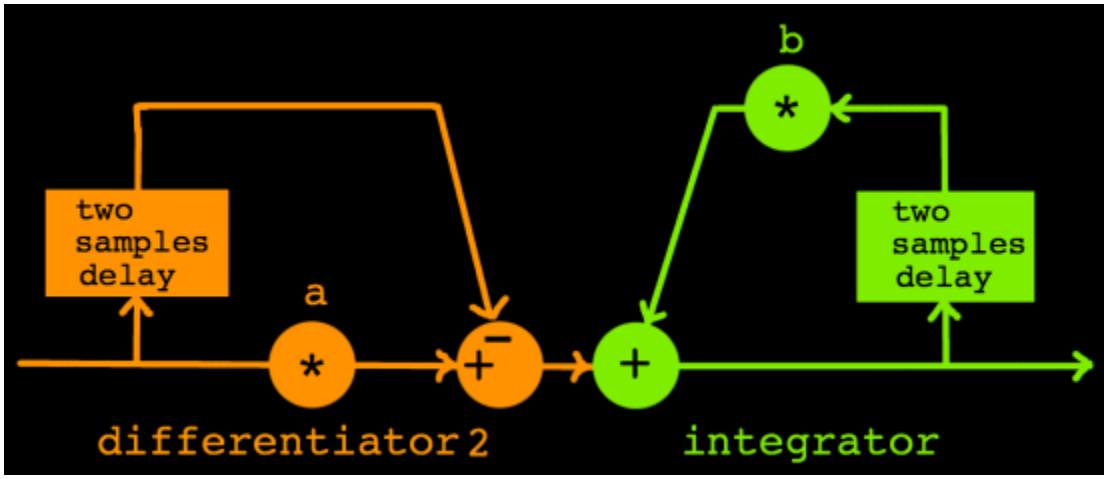
The curve is exactly the inverse of the differentiator type 1 curve. These phase shifts will cancel each other in a combined filter. I am not going to plot the flat phase response resulting from that, a dull line at phase zero.

In contrast, the differentiator-type2/integrator combination add their phase-shifts. Instead of a do-nothing filter, you have a leave-the-amplitudes-but-shift-the-phases filter, concisely called all-pass filter. The net phase response with coefficient 0.9 looks like this:



all-pass filter with: y[n] = 0.9*x[n] - x[n-1] + 0.9*y[n-1]

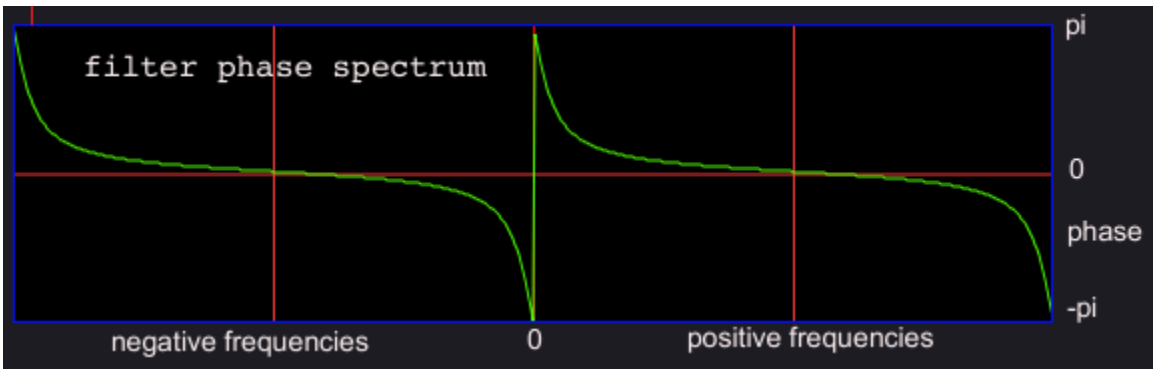
The same all-pass trick can be done with two samples delay time (or more, like is done in reverb simulation). The filter flow graph remains the same, only the delay time is increased:



y[n] = a*x[n] - x[n-2] + b*y[n-2]

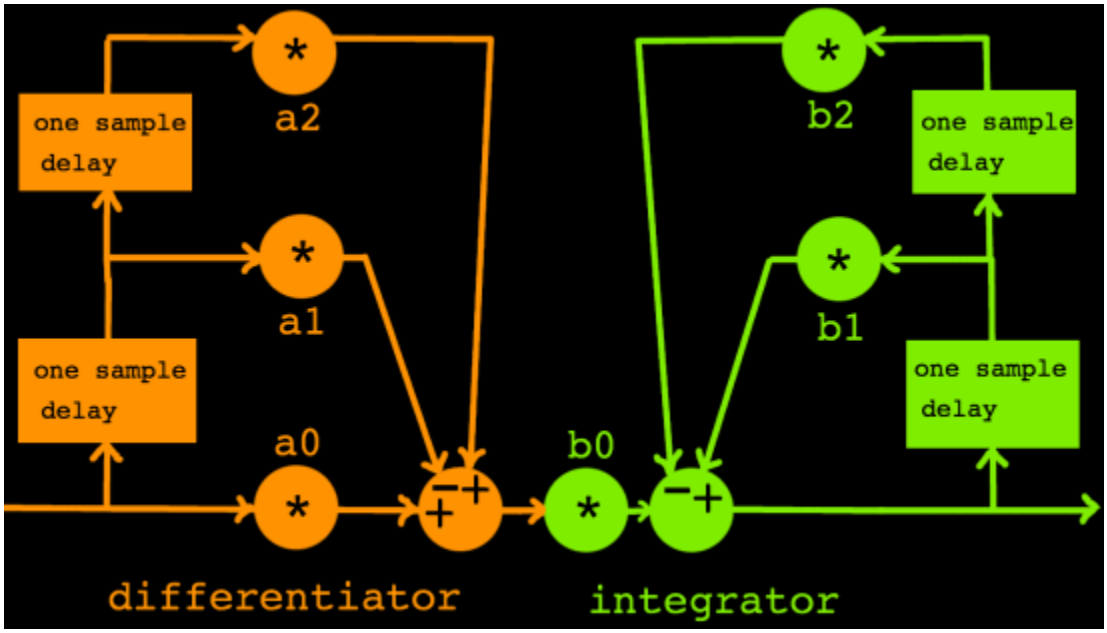
Using two samples delay instead of one sample, is equivalent to downsampling the filter's time resolution with a factor 2. The total

phase shift over the spectrum is doubled, as compared to the all-pass filter with one sample delay:



all-pass filter with: y[n] = 0.9*x[n] - x[n-2] + 0.9*y[n-2]

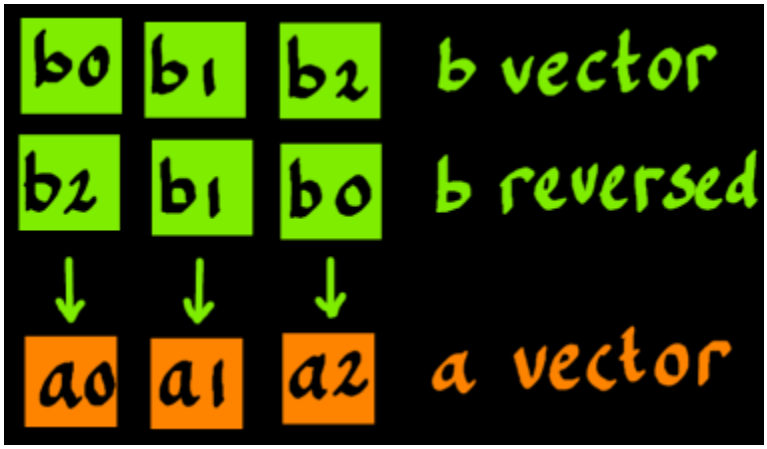
There is also the option to use one and two samples delay combined in an all-pass biquad filter, which has this signal flow graph:



Each section has two delays and three coefficients. The b0

coefficient is often factored out for efficiency reasons.

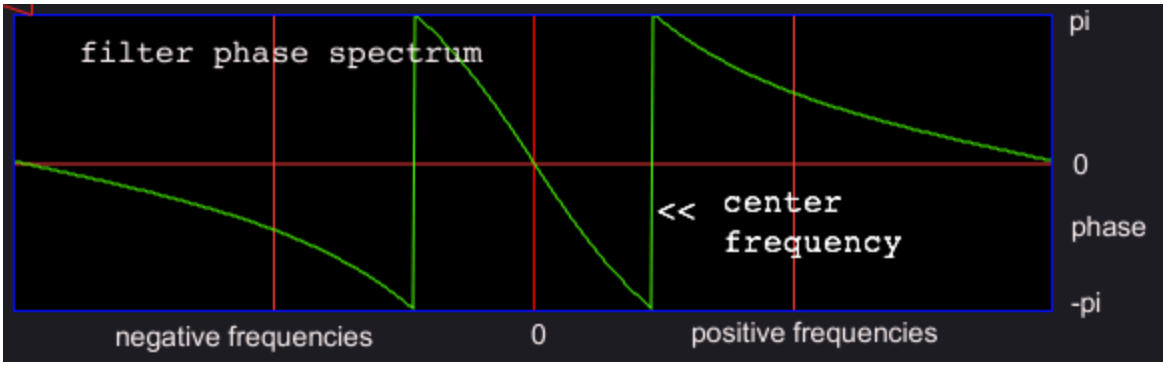
It is not so obvious how the coefficients should be set equal for both sections, to make an all-pass filter. This deserves special attention. The a and b coefficients should be treated as vectors, arrays. The b vector must be computed first, and then time-reversed to find the a vector. The b coefficients translate to the a coefficients in the following way:



A two-pole two-zero all-pass filter is thus characterized by these equalities:

a0 = b2
a1 = b1
a2 = b0

Below is an example of what the biquad all-pass can do to the phase spectrum. The phase-wrap that was at DC with the simpler filter is now duplicated. It indicates the filter center frequency, since half of the phase-shift is between DC and the phase-wrap, while the other half is between the wrap and Nyquist. With a biquad all-pass, the center frequency can be positioned at any location of choice within the spectrum.

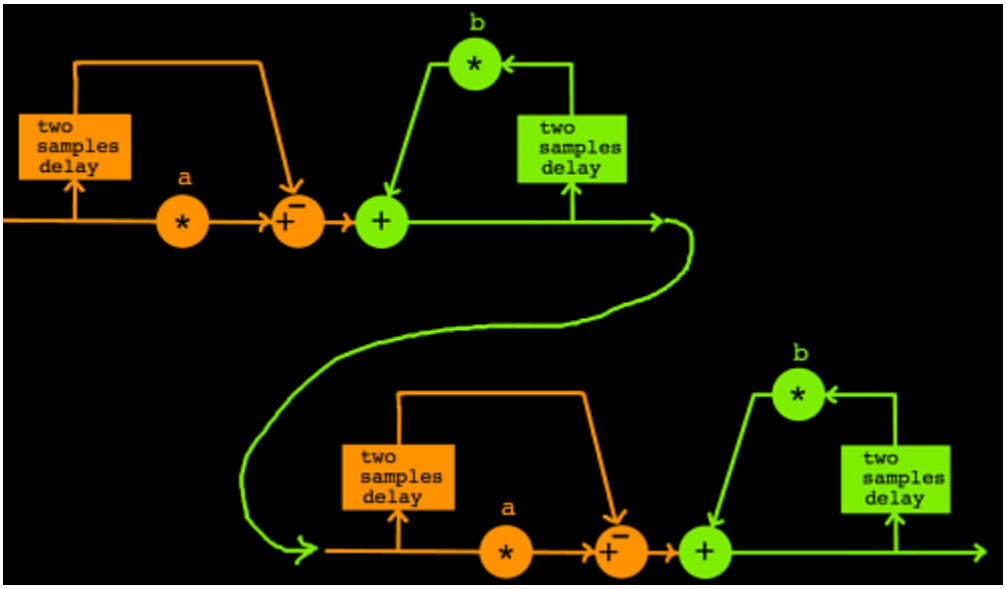


all-pass filter: y[n] = 0.2*[x] - 0.9*x[n-1] + x[n-2] + 0.9*y[n-1] - 0.2*y[n-2]

Whatever all-pass filter type and coefficients we use, the shift at DC and Nyquist frequency will always be zero, pi radian, or a multiple of pi. The phase response is tied like cymbalom strings to the pi radian grid. Flexible strings they are, but a pure -pi/2 radian phase shift for all frequencies is certainly out of the question.

The only thing we can do is to deform two 'strings', phase responses, in such a way that they are approximately parallel over a maximized portion of their length.

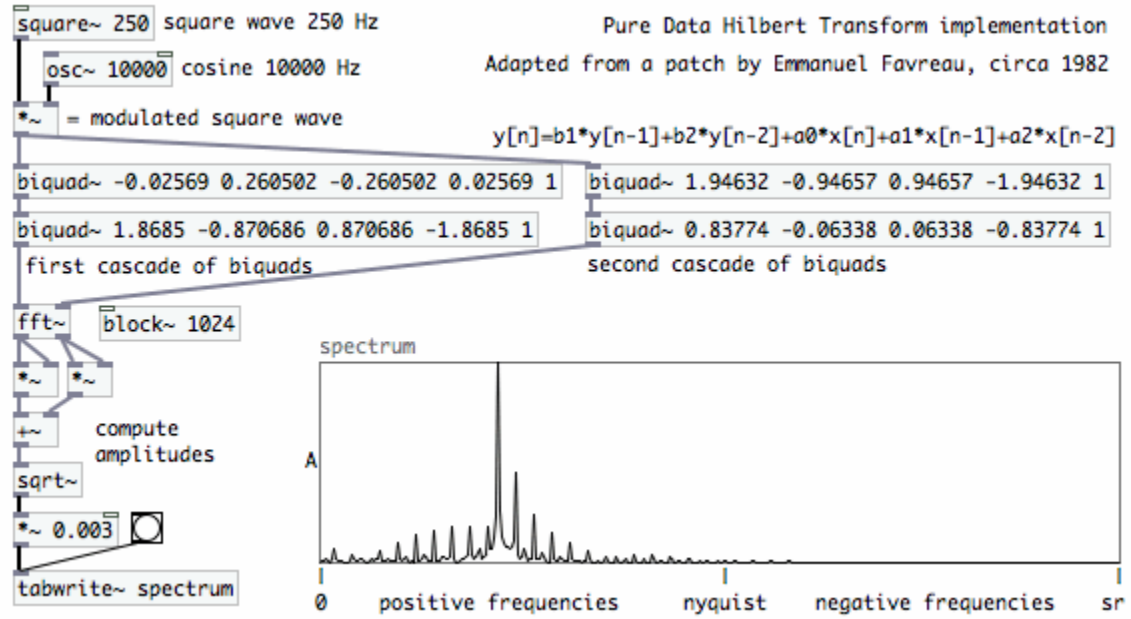
To accomplish this, one all-pass filter per output phase is not enough. For each phase, there must be a cascade of them. In itself, a cascade is not a complicated concept. It just means that there are two ore more filters in series, like here:



The phase shift of the cascade is just the sum of all the filter's shifts. Nonetheless, the math involved in cascade coefficient calculation is complicated, and generally left to specialised filter design software. For a start, I will review a couple of methods which were published with coefficients included, and compare these.

biquads

With only two biquads in cascade per phase, it is already possible to complexify real signals over a wide range of frequencies. Pure Data has this method as a patch, so we can see how it is implemented and learn from it. In the pd patch below, the cascades are positioned in such a way that they give positive frequencies. A modulated square wave serves as test input signal. The full complex spectrum shows that the negative frequencies are gone indeed:



The output of this constellation is approximately orthogonal starting from around 80 Hz (at sampling rate 44K1) till half the Nyquist frequency. After Nyquist/2 there is a gradual decrease in the phase interval, and near Nyquist the outputs show zero phase difference. Max/Msp's hilbert~ object may have a procedure comparable to pd's

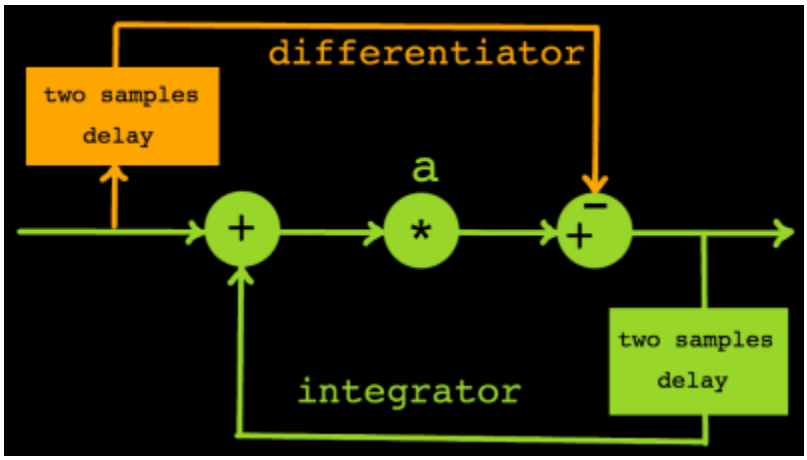
hilbert~. It's accuracy in the low spectrum half is better. Possibly it has more filters per cascade, but the object source is not public so it can not be compared. When implemented in C, each all-pass biquad can be efficiently computed as:

```
y[n] = a0*(x[n]+y[n-2]) + a1*(y[n-1]-x[n-1]) + x[n-2]
```

With two multiplications per biquad, and a minimum of 2x2 biquads, a complex signal can be computed at a minimum price of 8 multiplications plus a number of additions / subtractions.

polyphase IIR

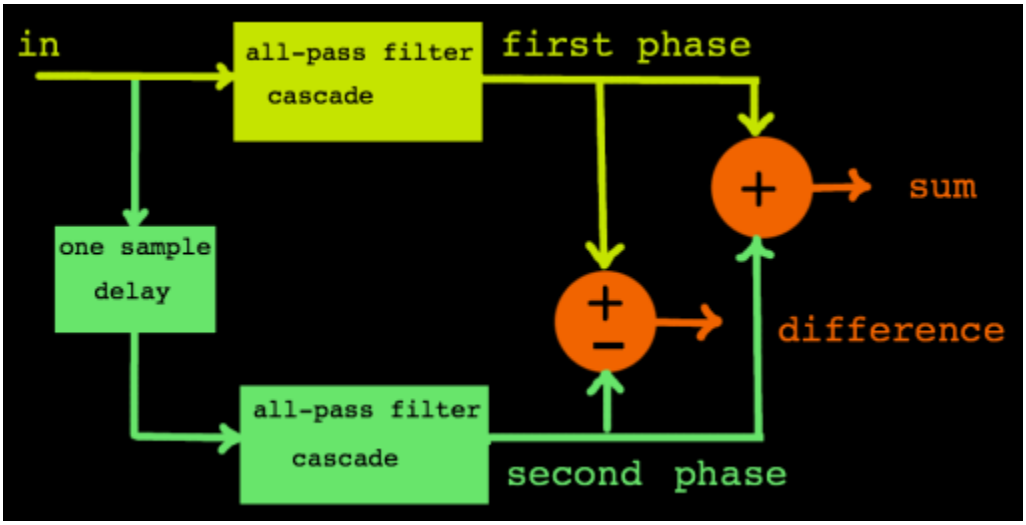
It seems that the elementary two-sample delay all-pass filters, sketched earlier on this page, are popular building blocks in the multirate filtering scene. Let me redraw the differentiator / integrator combination as an all-pass yin and yang unit to illustrate it's elegance and efficiency:




```
y[n] = a*(x[n] + y[n-2]) - x[n-2]
```

Cascades of them are used in arrays, where each cascade represents a separate phase. Theoretically there can be any number of cascades, and therefore any number of phases. That is why the concept is called 'Polyphase Infinite Impulse Response', abbreviated polyphase IIR. A dual-phase version of the polyphase IIR principle, with it's possible

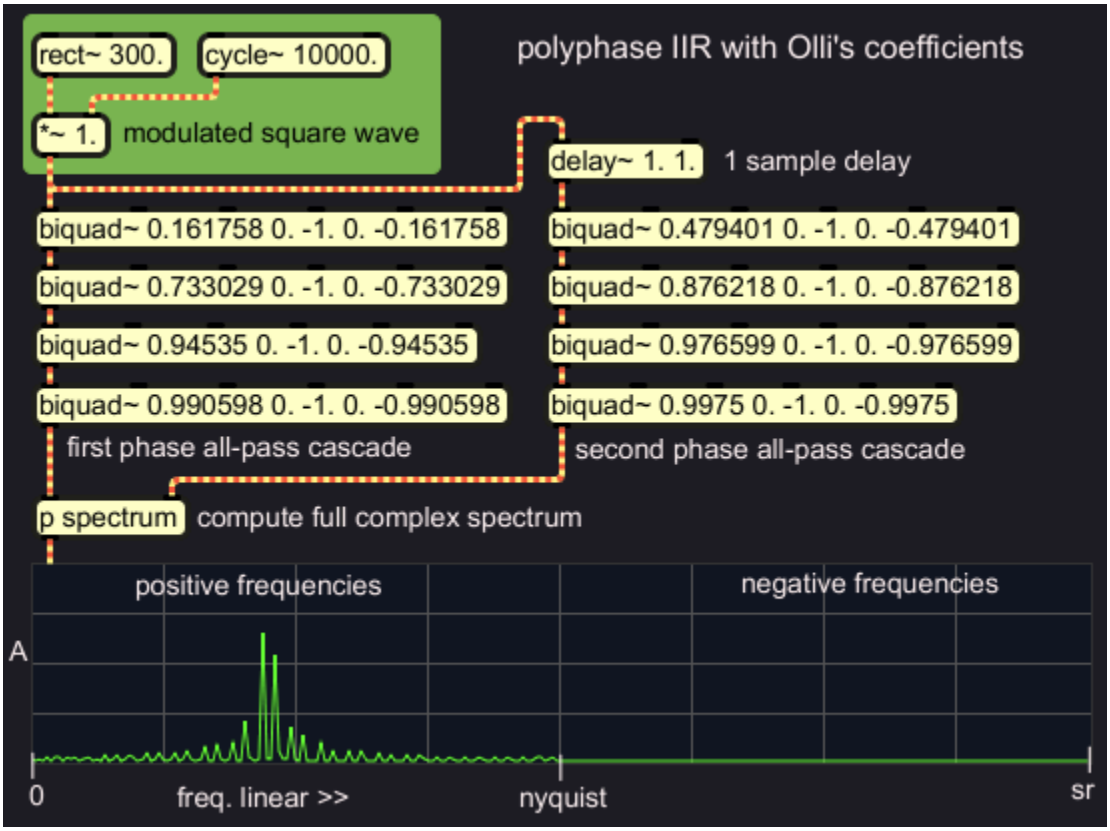
output signals, can be sketched like this:



The combined output of *different* all-pass filter cascades can produce an amplitude filtering effect as result of phase cancellations and summations. At first sight, this seems an elaborate way to implement a filter. But it is actually efficient because of the one-coefficient units inside. The polyphase set-up offers interesting options, like extracting sum and difference output with complementary spectra, and phase decomposition before the filters. A typical application is halfband filtering for a downsampling filterbank. If you want to search the internet on this topic, do not forget the IIR term, because there is also polyphase FIR, and both are used in multirate filtering. Many texts on polyphase IIR are by Artur Krukowski and Izzet Kale.

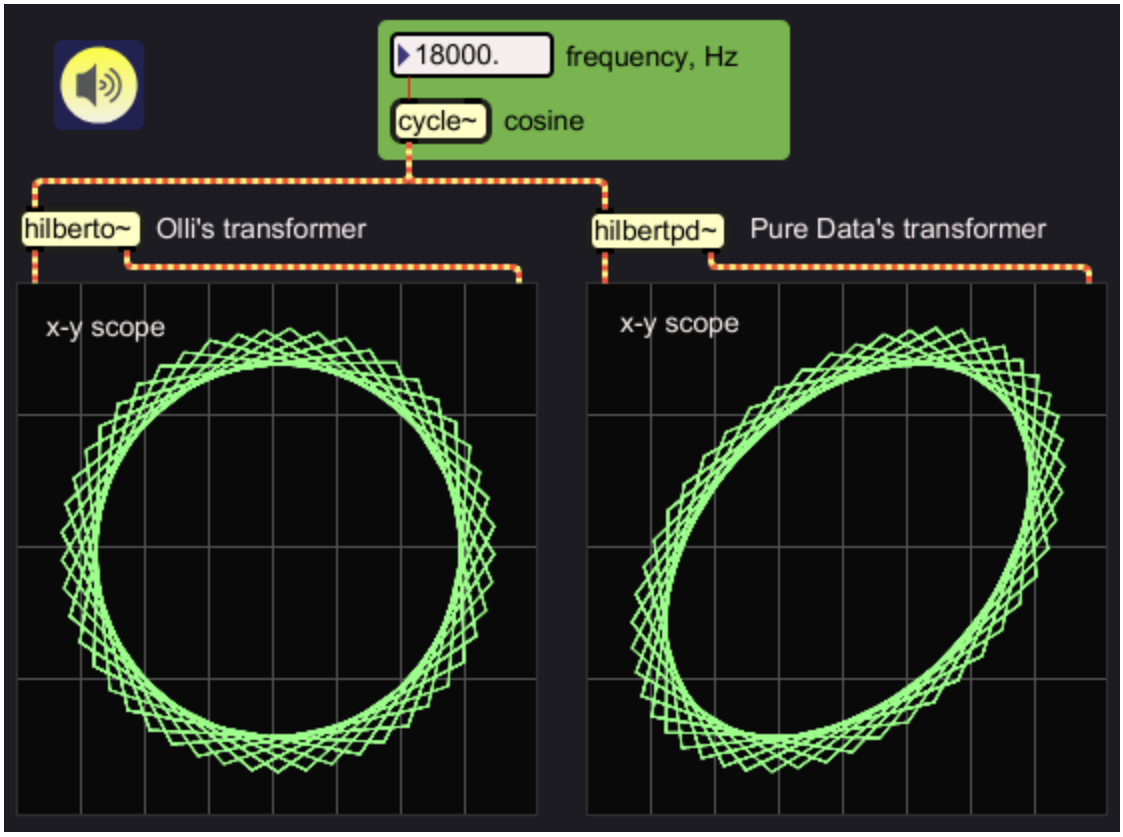
The technique can also be applied to produce output phases with pi/2 phase interval. That is why I mention it here, of course. Coefficients for 4-section cascades are calculated and published by Olli Niemitalo on http://yehar.com/blog/. It was also his presentation that put me on the track of polyphase IIR.

I tried Olli's filters in Max/MSP, using biquads. The one-sample delay coefficients are just set to zero. I swapped Olli's first and second cascades to get positive frequencies. Here is the Max patch with a modulated square wave as input and showing the full spectrum of the complex output:



Max/Msp's biquad~ has: y[n] = a0*x[n]+a1*x[n-1]+a2*x[n-2]-b1*x[n-1]-b2*x[n-2]

Notice that this routine could be done with a total of 8 multiplications as well. Compared to Pure Data's implementation, Olli's filters produce a better complex signal. Phase orthogonality starts around 40 Hz (at 44K1 sampling rate). The specifications show symmetry over Nyquist/2, therefore this transform is also quite accurate in the highest frequency range. The figure below shows a comparison at 18000 Hz. The lines in the scope traces are just interpolations between the actual datapoints.



Although regular audio has little energy in the highest frequency range, this can be different for certain (manipulated) signals.

All toghether, polyphase IIR seems more promising to me than the classic biquad method. That is why I want to find more details of this technique. How to design a polyphase IIR Hilbert Transformer of higher order? Laurent de Soras has published a C++ resampling library using polyphase IIR halfband filtering, with pi/2 phaser functions as a bonus. See http://ldesoras.free.fr/index.html. I am now examining this library too see if and how it can be applied for analysis in a real time system like Max/MSP or Pure Data. This page will be updated as soon as I have results.

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