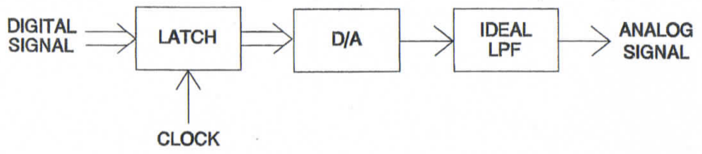
**3-3** An IF signal with a bandwidth of 50 KHz (±25 kHz) is to be sampled at a 150 kHz rate.

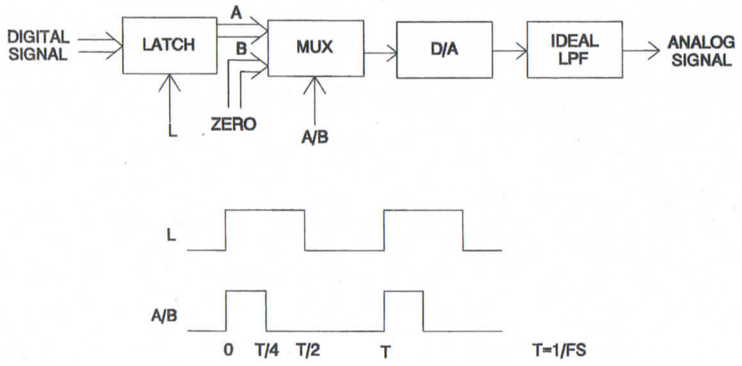
a) What are the permissible IF center frequency ranges between 10.0 and 10.25 MHz for which no aliasing occurs?
b) If the IF center frequency and the sample frequency are optimally chosen, what is the minimum allowable sample frequency? Using this sample frequency, what is the closest IF frequency to 10 MHz? Sketch the sampled spectrum.

**3-4** A D/A converter is used to convert a sampled data signal to an analog signal. The sample rate is $f_s$ = 50 kHz, and the passband of the digital signal is 30 Hz to 15 kHz.

a) If the circuit of the figure below is used, what is the frequency response at 15 kHz compared to the response at 1 kHz? Derive the formulas required.




b) If a shorter duty cycle is used for the sample-and-hold function as shown in the following figure, the frequency response can be improved in the upper portion of the spectrum. Recalculate the 15 kHz response relative to the 1 kHz for this case.




# 4

# Processing Complex Signals

Much of the power of digital signal processing in communications equipment is the product of its the ability to form and manipulate complex signals. These complex signals often possess a frequency spectrum that is not realizable using real signals. A complex signal may have nonsymmetrical positive and negative frequency components, and this is often used to advantage. The concepts of complex signals are considerably different from those of real signals as we normally understand them. Therefore, it is necessary for students and engineers to study these concepts in detail if they are to take full advantage of the power of digital signal processing. This chapter discusses many of these concepts, and the reader is urged to take the time to obtain a thorough understanding of these basic ideas prior to proceeding with the rest of the material in the book.

## POSITIVE AND NEGATIVE FREQUENCIES

We will first consider an ordinary signal of the form

$$V = \cos(2\pi ft) \tag{4.1}$$

It is possible to count the number of zero crossings of this signal in a given second. This is often called the *frequency*. From the zero crossings, it is impossible to distinguish if the frequency is positive or negative. A signal of the form

$$V = \cos(-2\pi ft) \tag{4.2}$$

has exactly the same number of zero crossings. The number of zero crossings evidently does not tell all there is to know about the frequency of a signal. Now let us consider the example of a rotating shaft with a disk attached. The disk has a handle protruding as shown in Fig. 4.1.

The disk is rotating in the positive (counterclockwise) direction. The graph shows the vertical displacement of the handle with time. The graph represents the view an observer would have if he were looking at the edge of the disk from the right. The observer sees the handle moving up and down following the function $\sin(2\pi ft)$, as shown in the graph. The observer at the right edge of the paper cannot determine if the disk is rotating clockwise or counterclockwise. Now consider a second observer looking at the edge of the disk from the bottom of the page, as shown in Fig. 4.2. This displacement as a function of time is plotted in the figure. The observer sees the handle moving according to the function $\cos(2\pi ft)$. However, like the first observer, he cannot tell if the disk is moving in the clockwise or counterclockwise direction.

If the observer looks at both projections simultaneously, it is possible to determine the direction of rotation. This is shown in Fig. 4.3. We now have two representations of the signal of the form:

Real axis: $\cos(2\pi ft)$
Imaginary axis: $\sin(2\pi ft)$

This gives rise to a representation of the signal of the form

$$Z = \cos(2\pi ft) + j\sin(2\pi ft) \tag{4.3}$$

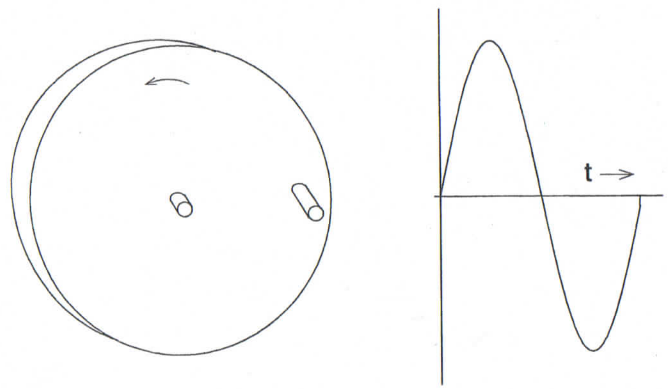



**FIGURE 4.1** Rotating disk with handle



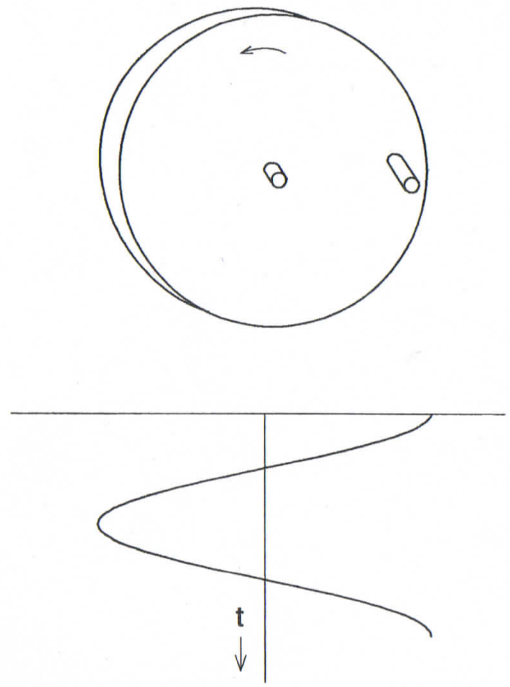



**FIGURE 4.2** Rotating disk with handle, bottom view

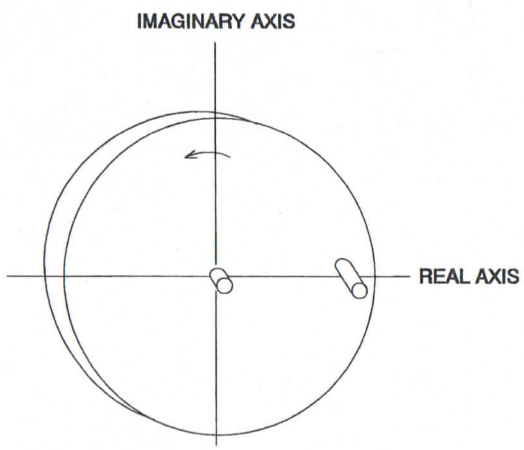



**FIGURE 4.3** Complex representation of rotation disk

For a positive rotation (counterclockwise) the cosine decreases as the sine increases (goes positive). If the disk were rotating in the clockwise (negative) direction, the sine would go negative as the cosine decreased. Thus, it is possible to distinguish the direction of rotation, and we see that a complex representation is required. Since the cos(2πft) or sin(2πft) functions taken individually are real only, they must have both positive and negative frequency components. The vector representation of the cos function is shown in Fig. 4.4. Since the two vectors are rotating in opposite directions, the sum of the imaginary component resulting from the addition of the two vectors always equals zero.

Thus, the frequency spectrum of cos(2πft) has the form

$$F(f) = \frac{1}{2}\delta(f-f_0) + \frac{1}{2}\delta(f+f_0) \tag{4.4}$$

as derived in Eq. (2.39). The frequency spectrum is plotted in Fig. 4.5.

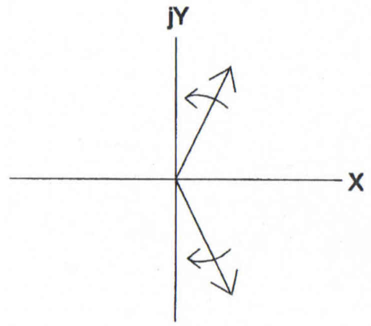



**FIGURE 4.4** Vector representation of x = cos (2π ft)

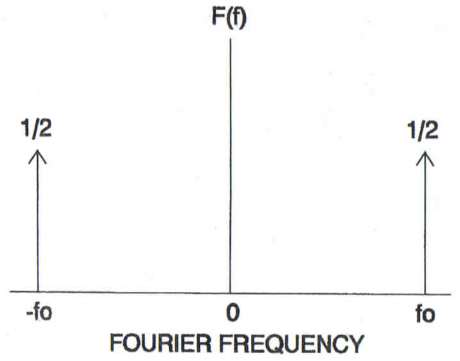



**FIGURE 4.5** Frequency spectrum of cos(2π ft)



Now consider a single rotating vector as shown in Fig. 4.6. This vector can be represented as

$$Z = e^{j2\pi f_0 t} = \cos(2\pi f_0 t) + j\sin(2\pi f_0 t) \tag{4.5}$$

A complex exponential in the form of Eq. (4.5) may represent a single positive or negative frequency. The Fourier transform of this signal, as derived in Eq. (2.35) and shown in Fig. 2.12, is

$$F(f) = \delta(f-f_0) \tag{4.6}$$

The Fourier transform of a real signal has several properties which are useful in signal processing. These properties follow from the discussion above. Suppose f(t) is a real signal with a corresponding Fourier transform F(f). Since f(t) is real, it must have a frequency spectrum in which the corresponding positive and negative frequencies are of equal magnitude. A typical function of this type is shown in Fig. 4.7.

The frequency spectrum of an even real signal is of this form. If the real signal has odd symmetry (i.e., f(t) = –f(–t)), the frequency spectrum will be imaginary and will have odd symmetry. This is shown in Fig. 4.8. If f(t) is real but has neither even or odd symmetry, its frequency spectrum will be complex and the real part will be even:

$$\mathrm{Re}\,F(f) = \mathrm{Re}\,F(-f) \tag{4.7}$$

while the imaginary part will be odd:

$$\mathrm{Im}\,F(f) = -\mathrm{Im}\,F(-f) \tag{4.8}$$

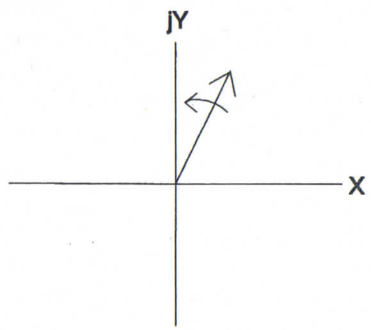



**FIGURE 4.6** Single rotating vector

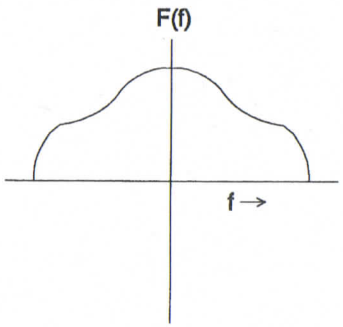



**FIGURE 4.7** Fourier transform of even, real signal

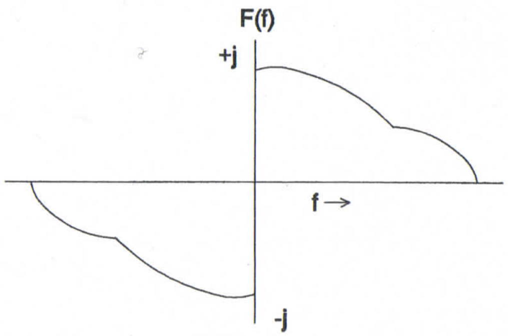



**FIGURE 4.8** Fourier transform of odd, real signal

## COMPLEX SIGNALS

The frequency spectrum of a complex signal does not have the symmetrical properties discussed in the previous section on real signals. We will refer to a signal as complex if neither its real part nor its imaginary part is zero. In some instances, the sideband structure is shifted by frequency translation. We will refer to these as complex translated signals. A subset of complex signals are the analytic signals which have special properties of interest. To provide a formal definition: *an analytic signal is a complex function having only positive or only negative frequencies.*

We will proceed to developing an example of an analytic signal. To accomplish this, let F(f) be the frequency spectrum of a signal whose time function, f(t), is real. Let $F_p(f)$ be the positive half of the spectrum so that

$$F_p(f) = F(f) \text{ for } f \geq 0 \tag{4.9}$$



$$F_p(f) = 0 \text{ for } f \leq 0$$

Let $F_n(f)$ be the negative half of the frequency spectrum so that

$$F_n(f) = F(f) \text{ for } f < 0 \tag{4.10}$$

$$F_n(f) = 0 \quad \text{for } f \geq 0$$

Then

$$F(f) = F_n(f) + F_p(f) \tag{4.11}$$

Now, let f(t) pass through a Hilbert transformer as shown in Fig. 4.9.

A Hilbert transformer delays the signal inversely with frequency so that all positive frequencies are delayed by 90 degrees, and all negative frequencies are advanced by 90 degrees. The Hilbert transform of a signal is represented by placing a caret (^) above the function.

The frequency spectrum of the signal $\hat{f}(t)$ is now given by

$$\hat{F}(f) = -jF_p + jF_n(f) \tag{4.12}$$

Now, let us make a signal z(t), which is given by

$$z(t) = f(t) + j\hat{f}(t) \tag{4.13}$$

It consists of two real signals, f(t), which are called the *in-phase* component, I(t) = f(t), and Q(t), which is called the *quadrature* component, Q(t) = $\hat{f}(t)$. The Q signal, if we think of it as existing on a wire, is treated as though it were multiplied by "j" in all operations. The frequency spectrum of the analytic signal z(t) is given by

$$Z(f) = F(f) + j\hat{F}(f) \tag{4.14}$$


**FIGURE 4.9** Signal passing through Hilbert transformer

Substituting for F(f) and $\hat{F}(f)$ from Eqs. (4.11) and (4.12) gives

$$Z(f) = F_p(f) + F_n(f) + j[-jF_p(f) + jF_n(f)] \tag{4.15}$$

which simplifies to

$$Z(f) = 2F_p(f) \tag{4.16}$$

Thus, we have created a signal that consists only of the positive frequency components of the original real signal f(t). This is illustrated in Fig. 4.10. If we had defined $z(t) = f(t) - j\hat{f}(t)$, it would be found that Z(f) consists of only the negative components of f(t). This is illustrated in Fig. 4.11. It is a general rule, then, that the positive frequency spectrum of a signal, f(t), may be obtained by passing the signal through a Hilbert transformer and creating a new signal $z(t) = f(t) + j\hat{f}(t)$. The negative frequency spectrum may be obtained by forming the signal $z(t) = f(t) - j\hat{f}(t)$. The reader should note that this rule is applicable whether or not f(t) is a real signal. In the explanation, we considered f(t) to be a real signal for clarity; however, nothing used in the deviation is restrictive to real signals. The proof of

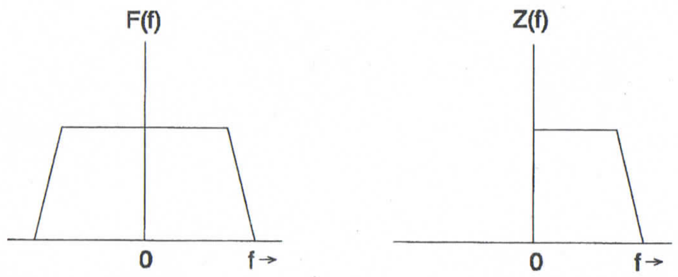



**FIGURE 4.10** Frequency spectrum of f(t) and z(t)

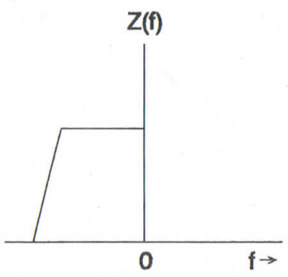



**FIGURE 4.11** Frequency spectrum of signal $z(t) = f(t) - j\,\hat{f}(t)$



this conclusion for a complex signal is left as an exercise for the reader (see Problem 4-1).

In order to make the concept of an analytic signal more plausible, let us consider the example of a cosine wave:

$$v(t) = A\cos(2\pi f_0 t) \tag{4.17}$$

Let

$$z(t) = v(t) + j\hat{v}(t) \tag{4.18}$$

Now, the frequency spectrum of a cosine wave was shown to be

$$v(t) = \frac{1}{2}A\delta(f - f_0) + \frac{1}{2}A\delta(f + f_0) \tag{4.19}$$

The Hilbert transform of the cosine wave is a sine wave, so that

$$\hat{v}(t) = \sin(2\pi f_0 t) \tag{4.20}$$

The frequency spectrum of the sine wave is

$$v(f) = -j\frac{1}{2}A\delta(f - f_0) + j\frac{1}{2}A\delta(f + f_0) \tag{4.21}$$

Figure 4.12 shows these frequency spectra in graphic form. The spectrum points in the upper graph correspond to A $\cos(2\pi f_0 t)$, while the second graph is the spectrum of V $\sin(2\pi f_0 t)$. Note that the ordinate in this graph is the imaginary axis. The third graph is the frequency spectrum of the second graph multiplied by "j," since the analytic signal formed is $z(t) = v(t) + j\,\hat{v}(t)$. Note that the ordinate is again real. Finally, the lower graph, which is the sum of the first and third graphs, represents the frequency spectrum of the analytic signal. This result could also have been obtained be using the trigonometric identity

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{4.22}$$

as follows:

$$z(t) = A\cos(2\pi f_0 t) + \sin(2\pi f_0 t) = Ae^{j2\pi f_0 t} \tag{4.23}$$

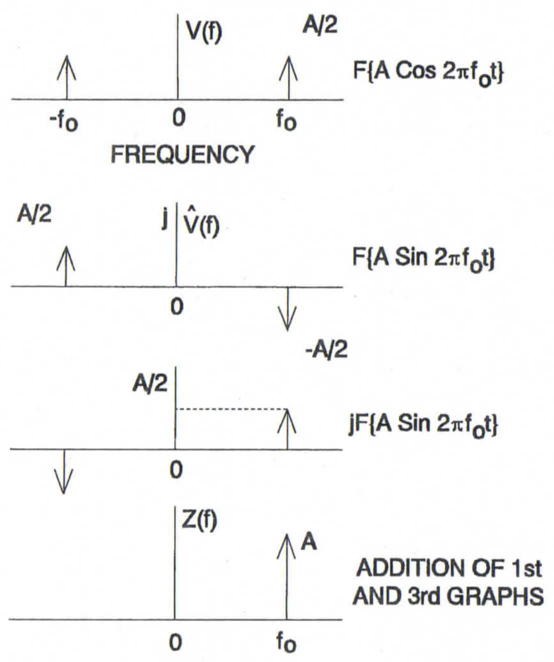



FIGURE 4.12 Frequency spectra of analytic signal z(t) = v(t) + j $\hat{v}$ (t) and its components

The Fourier transform of an exponential is given by

$$F\{Ae^{j2\pi f_0 t}\} = A\delta(f - f_0) \tag{4.24}$$

as we conclude in the lower graph in Fig. 4.12.

We now turn our attention to the frequency spectra of the I(f) and Q(f) components of a complex signal. It will be found that if we examine only the real or only the imaginary part, all the isolated positive components of the signal are duplicated on the negative side, and all the isolated negative components of the signal are duplicated on the positive side. Figure 4.13 illustrates this property.

Note that each component appears both on the positive and negative side in the right-hand plot and that the components are reduced to ½ their original amplitude. It should be noted that any symmetrical or antisymmetrical components in a complex signal need not follow this rule and may be present only in the I or Q component. Thus, Fig. 4.13 represents only the case when the positive and negative components of the complex signal are distinctly separate. It will frequently occur in signal processing work that a signal is analytic and has only positive or only negative frequencies. We may then use the I(t) component to



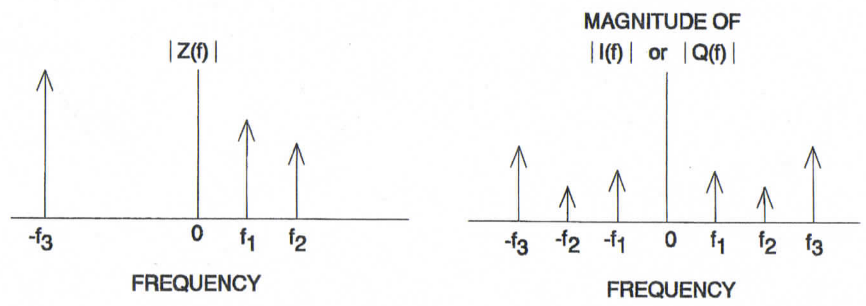



FIGURE 4.13 Frequency spectrum of complex signal and its I or Q component only

form a real signal that will, of course, have a symmetrical sideband structure. I this is not the case, it is necessary to use a Hilbert transformer to first form a analytic signal which has only positive or only negative components. The I(t component can then be used to form a real output signal. This is commonly don to separate the upper and lower sidebands in a single sideband receiver.

Suppose we have a complex signal, z(t), which has both positive and negativ components. To obtain only the positive components (upper sideband), the signa must be passed through a Hilbert transformer. A new complex signal is formed b taking

$$w(t) = z(t) + j\hat{z}(t) \tag{4.25}$$

This is shown in Fig. 4.14. Note that the component $\hat{I}$ is not itself imaginary. Sinc we wish to have $j\hat{I}$, this is equivalent to adding it to the imaginary wire $Q_2$. Sinc the component $\hat{Q}$ is already on the imaginary wire, taking jQ is equivalent to mu

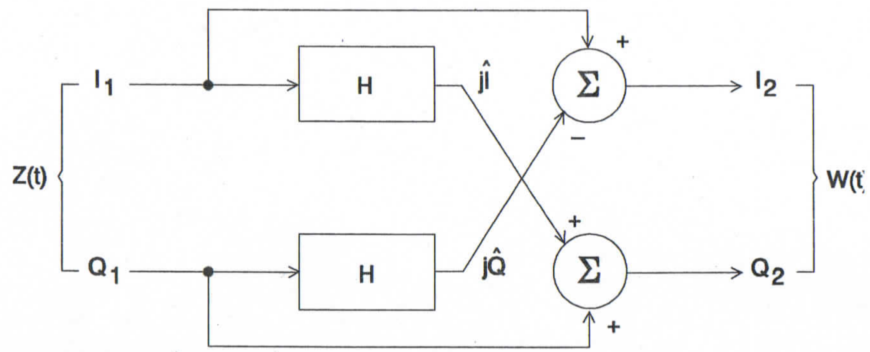



FIGURE 4.14 Hilbert transform of a complex signal to select upper sideband

tiplying it by $j^2 = -1$. This component is therefore subtracted from the real wire $I_2$. Now the analytic signal w(t) contains only the upper sidebands of z(t). Therefore, a real signal that contains no sidebands corresponding to the lower sidebands of z(t) can be formed from $I_2$. There is no need to provide hardware to generate both $I_2$ and $Q_2$, and an upper sideband separator can be constructed as shown in Fig. 4.15. The lower sideband could have been selected by forming the signal

$$w(t) = z(t) - j\hat{z}(t) \tag{4.26}$$

which simply reverses the sign of the lower signal entering the summing point in Fig. 4.15. If we had used the Q(t) component, all the frequency components would have been shifted by 90 degrees.

## FREQUENCY TRANSLATION

Frequency translation is one of the most powerful techniques in digital signal processing. It can be performed on either real or complex signals. Conversely, in the analog domain, frequency translation (also called mixing) is usually limited to operations on real signals because of the difficulty of precisely matching the I and Q channels. The frequency translation theorem has already been developed in Chapter 2 [see Eq. (2.35)] and is the basis for the following discussion.

One of the advantages of mixing, particularly in the analog domain, is that fixed-frequency filters and amplifiers can be used in a superheterodyne receiver. The frequencies can also be selected to minimize filter cost and spurious responses. In digital systems, frequencies are often translated to baseband or to near base-

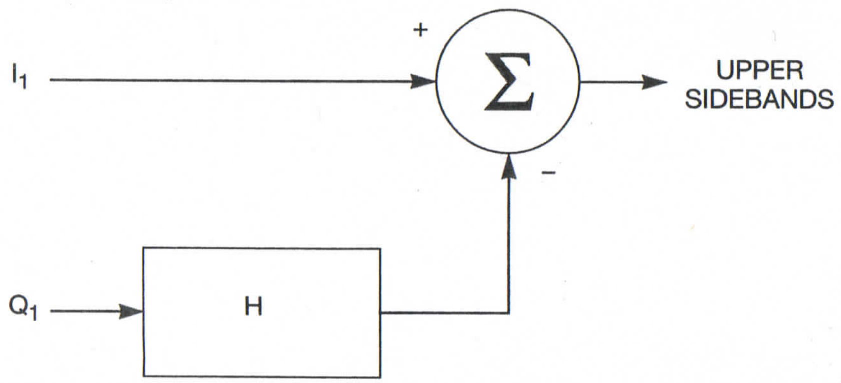



**FIGURE 4.15** Upper sideband selector using the $I_2(t)$ component



band, so that a low sample rate can be used. Indeed, as we shall see later, the signals are sometimes translated so the center of the desired channel is at zero frequency, with half the signal consisting of positive frequencies and half the signal consisting of negative frequencies. Obviously, both I and Q signals are required to keep the components separate.

As indicated above, analog mixers are frequently used in superheterodyne receivers. At the present time, a receiver using digital signal processing is also likely to use an analog front end and one or more analog IF frequencies. The analog portion of a typical digital receiver is shown in Fig. 4.16. It is advisable to provide a bandpass filter at the input of the receiver. This filter is often very broad, and it removes any very large signals which are out of the tuning range of the receiver. For example, in a communications receiver, this filter might remove TV signals or radar frequency signals. The signal after broadband filtering is then frequency translated to the first IF frequency. In some cases a small amount of gain may be provided to overcome the loss in a passive mixer if a low noise figure is required in the receiver.

The function of the analog mixer can be derived from the Fourier frequency translation theorem as follows. From Eqs. 2.30 and 2.32, we see that if a signal f(t) has a frequency spectrum F(f), and the signal is multiplied by $e^{j2\pi f_0 t}$, the resulting frequency spectrum is $F(f - f_0)$. This is the original spectrum translated up in frequency by $f_0$. The result is shown graphically in Fig. 4.17. For the analog mixer, the injection is real and may be of the form

$$y(t) = A\cos(2\pi f_{LO} t) = A\left[\left(e^{j2\pi f_{LO} t} + e^{-j2\pi f_{LO} t}\right)/2\right] \tag{4.27}$$

where

$f_{LO}$ = the local oscillator frequency

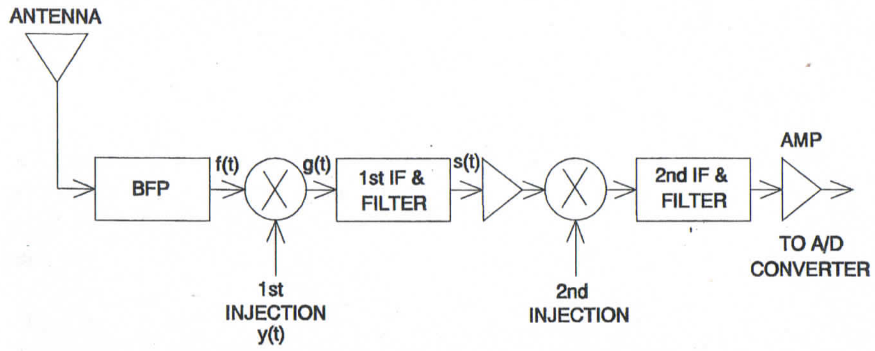



**FIGURE 4.16** Analog portion of receiver with digital signal processing

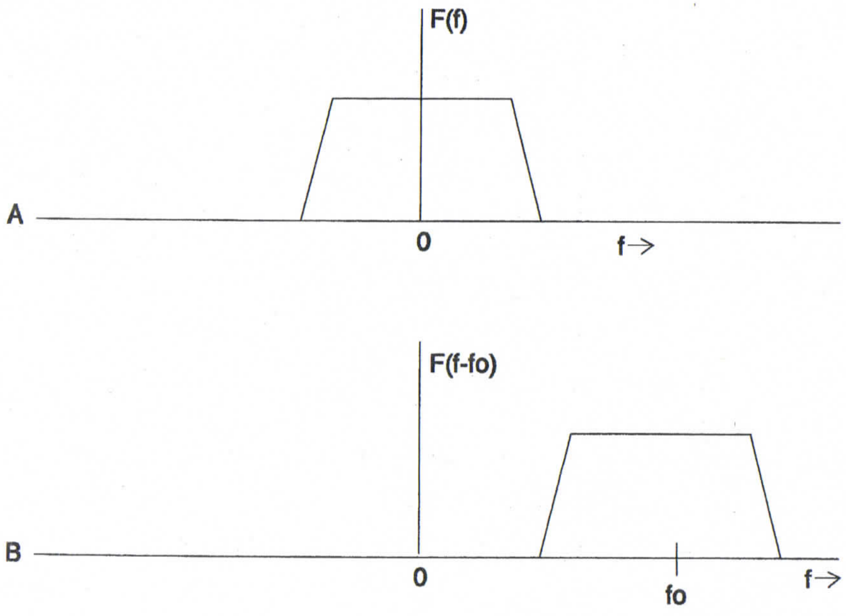



**FIGURE 4.17** Frequency spectrum of (a) basic signal, f(t), and (b) translated version, $f(t)\, e^{j2\pi f_0 t}$

Because we have the transform pair

$$f(t) \leftrightarrow F(f) \tag{4.28}$$

for the signal, it follows from the frequency translation theorem that

$$A \cos(2\pi f_{LO} t)\, f(t) \leftrightarrow \frac{1}{2} A\ F(f - f_{LO}) + \frac{1}{2} A\ F(f + f_{LO}) \tag{4.29}$$

This is shown graphically in Fig. 4.18, where $f_c$ is the carrier frequency of the incoming signal. Since real signals are used, both sum and difference products are generated. These are represented by the signal centered at $f_c + f_{LO}$ and $f_c - f_{LO}$. Of course, since the mixer output is real, both of these spectra are also present on the negative frequency axis. A fairly sharp bandpass filter must be used following the mixer to remove the undesired mixer product. Assume, in this case, that the difference signal is required. Then the filter removes the signal at $f_c + f_{LO}$. The filter output is shown in Fig. 4.19.



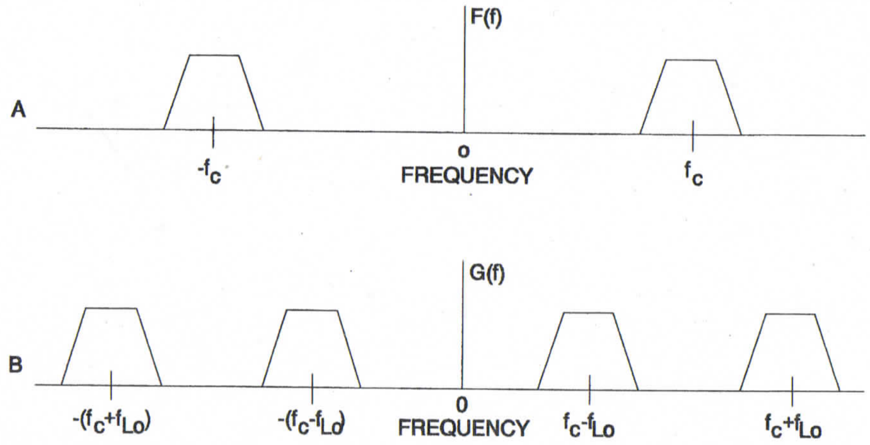



**FIGURE 4.18** Frequency spectrum of (a) input signal and (b) mixer output

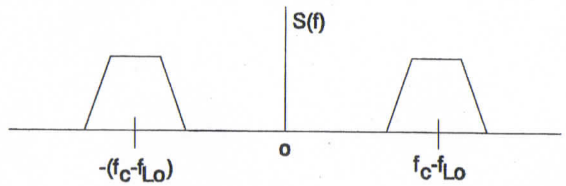



**FIGURE 4.19** Frequency spectrum of filter output

For the receiver in Fig. 4.16, if the frequency coverage were from 2 to 30 MHz and the IF frequency were 100 MHz, the injection frequency would be made variable between 102 to 130 MHz so that

$$f_{LO} - f_{Sig} = 100\ \text{MHz} \tag{4.30}$$

Since a real analog mixer is used, an incoming signal whose frequency satisfies the relationship $f_{Sig} - f_{LO} = 100$ MHz would also pass through the mixer. This is referred to as the *image frequency*. In this case, the image frequency range would be from 202 to 230 MHz. These frequencies can easily be eliminated by a low pass filter between the antenna and the mixer.

Analog mixers have several additional shortcomings of which the designer should be aware. These exist because of nonlinearities in the mixer. An ideal mixer would multiply the signal by the injection and produce only the sum and difference frequencies. Because of small nonlinearities; however, the sums and differences of harmonies are also created at very low levels. For example,

$f_{LO} - 2f_{Sig}$, $2f_{LO} - 2f_{Sig}$, $3f_{LO} - 2f_{Sig}$, etc. are also created. The higher-order distortion products are usually designated by the sum of harmonic numbers that resulted in the product. In the example above, the spurious products would be designated the 3rd, 4th, and 5th order products, respectively. The level of the products varies greatly with the mixer being used. In some cases, tables are available from the manufacturer to assist engineers in determining whether a particular product is small enough to be ignored. Even if a perfect mixer could be obtained, it would still be necessary to remove the undesired image product. Complex mixers could eliminate the problem; however, they are very difficult to construct with analog components due to the exacting amplitude and phase matching requirements. Consequently, they are seldom used. Complex digital mixers are very practical, however, and will be discussed following the treatment of analog mixers.

Because real analog mixers generate both sum and difference frequencies, precautions must be taken to eliminate input frequencies from the mixer if the image falls into the desired IF passband. For example, if a 100 MHz signal is being received and is mixed with 110 MHz to produce a 10 MHz IF, an incoming signal frequency of 120 MHz will also produce a 10 MHz IF. Thus, the 120 MHz signal must be removed by filters preceding the mixer. The image frequency is always located twice the IF frequency from the desired signal. For this reason, a high first IF frequency is often chosen. For example, for the receiver shown in Fig. 4.16, a 100 MHz first IF frequency was chosen so the image frequency is 200 MHz away and can easily be prevented from reaching the mixer by a lowpass filter. Another distortion that must be considered is the in-band intermodulation. This occurs when two closely spaced off-frequency signals enter the mixer to produce a low-level in-band signal. The most troublesome of these signals is the third-order product. Suppose the receiver is tuned to receive a signal at frequency $f_0$. Two equal level signals are then applied to the receiver, one at $f_1 = (f_0 + \Delta f)$ and the second at $f_2 = (f_0 + 2\Delta f)$. The value of $\Delta f$ is normally small, in the 1 to 100 kHz range. The third-order product is then at

$$2f_1 - f_2 = 2(f_0 + \Delta f) - (f_0 + 2\Delta f) = f_0 \qquad (4.31)$$

It can be shown that for a cubic nonlinearity, the magnitude of the intermodulation products are proportional to $V_1V_2^2$ or $V_1^2V_2$, where $V_1$ and $V_2$ are the magnitude of the input signals. This can be shown by forming the product $[(V_1 \cos \omega_1 t)(V_2 \cos \omega_2 t)]^3$ and applying the appropriate trigonometric identity. The proof is left as an exercise for the student (see Problem 4-2). The mixer performance, insofar as third-order products are concerned, can be represented by a nonlinearity preceding an ideal mixer. The equivalent circuit is shown in Fig. 4.20. If equal level signals of magnitude V are applied to the mixer input, the third-order products are proportional to $V^3$. This is shown in Fig. 4.21, along with a plot of the output for an on channel input.



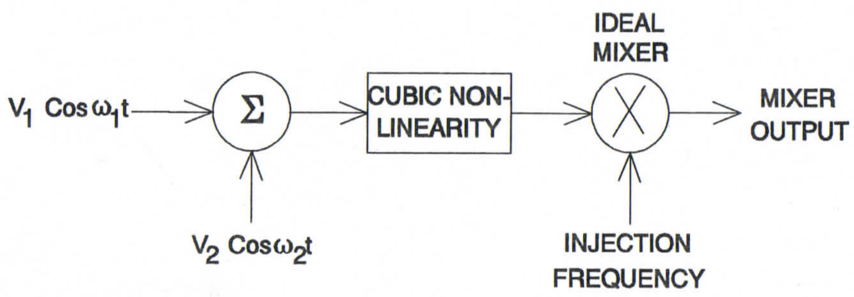



**FIGURE 4.20** Third-order equivalent circuit of mixer

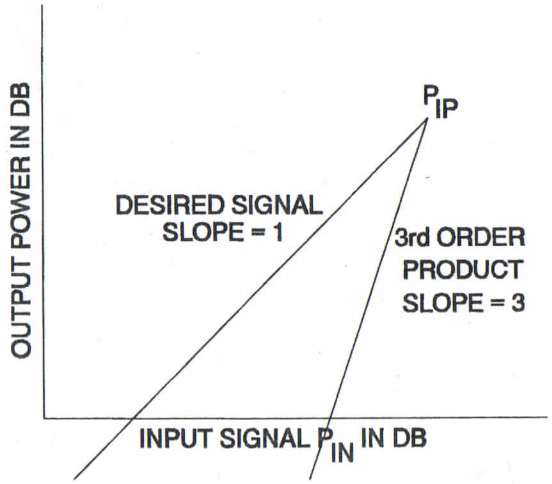



**FIGURE 4.21** Slope of desired and intermodulation products

Since the slope of the intermodulation product is three times that of the desired signal, the lines would eventually intersect if the signals could be made large enough without changing the mixer characteristics. In practice, the lines are generated using low-level signals and then extrapolated until they intersect. The point of intersection is called the third-order intercept point, $P_{IP}$. The power level of the intermodulation product can then be calculated for any input power $P_{in}$ using the equation

$$P_{IM} = \frac{P_{in}^3}{P_{IP}^2} \qquad (4.32)$$

Taking 10 log of each side and referencing the signals to 1 milliwatt gives

$$(P_{IM})_{dBm} = 3(P_{in})_{dBm} - 2(P_{IP})_{dBm} \tag{4.33}$$

The third-order intercept point is often specified for a mixer, and it may be expected to be in the range of a few dBm to perhaps 40 to 42 dBm for a very high-level mixer. The third-order intercept point is normally specified in terms of input signal levels rather than in terms of output signals. The converse is true for amplifiers, which are often modeled as an ideal amplifier followed by the nonlinearity. As a rule of thumb for an amplifier, the third-order intercept point may be in the vicinity of 10 dB above the 1 dB compression point, which is often specified; this may vary from design to design, however. The intermodulation produced for each amplifier should be checked, and the most stringent requirement may be for the stage immediately preceding the A/D, where the signal is the largest. Other stages preceding the analog IF filters may also contribute because higher levels of performance are often specified for signals further off channel, and the first stages in a receiver may receive little protection from the filters. For amplifiers, the relationship of Eq. (4.33) is sometimes written in terms of the ratio, R, of the output power to the power of the generated intermodulation product. Thus if

$$R = 10 \log \frac{P_{OUT}}{P_{IM}} \text{ in dB} \tag{4.34}$$

then

$$(P_{IP})_{dBm} = \frac{1}{2}R + (P_{OUT})_{dBm} \tag{4.35}$$

where $P_{IP}$ here is the third-order intercept point referred to the amplifier output. It can also be shown that for higher order intermodulation products of order n, Eq. (4.35) takes on the form (see *ANSI X3.4.* [81])

$$(P_{IPn})_{dBm} = \frac{1}{n-1}R + (P_{OUT})_{dBm} \tag{4.36}$$

where

$P_{IPn}$ = the nth order output intercept

R = ratio of signal power to power level of nth order intermodulation component



Digital mixers overcome most of the problems associated with analog mixers and, in many cases, can be considered ideal. Complex digital mixers that produce only the desired output product are also used frequently. These are discussed in detail later in this section. Prior to examining digital mixers in detail, we shall address a more concise way to represent the spectra of digital signals. Basically, the result is that the spectrum need only be considered over the frequency interval from $-f_s/2$ to $f_s/2$ and, if properly treated, will give correct results. Consider the example of a complex signal

$$x(nT) = I(nT) + jQ(nT) \tag{4.37}$$

as shown in Fig. 4.22.

Let this signal be multiplied by a second signal, $y(nT) = e^{j2\pi f_0 nT}$, to produce an output signal

$$w(nT) = x(nT)\,y(nT) \tag{4.38}$$

According to the frequency translation theorem, the spectrum of w(nT) is the spectrum x(nT) translated up by an amount $f_0$. The spectrum of w(nT) is shown in Fig. 4.23. Now consider the same example, but represent the spectrum only in the region from $-f_s/2$ to $f_s/2$, where $f_s$ is the sample frequency. The truncated signal

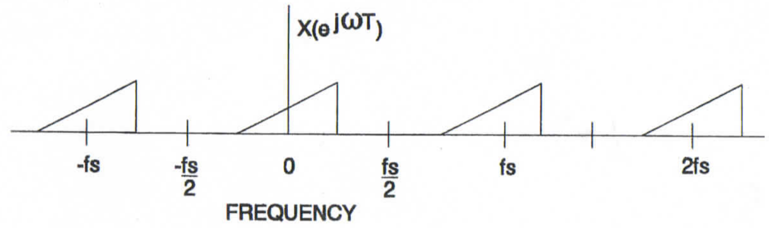



**FIGURE 4.22** Portion of spectrum of x(nT)

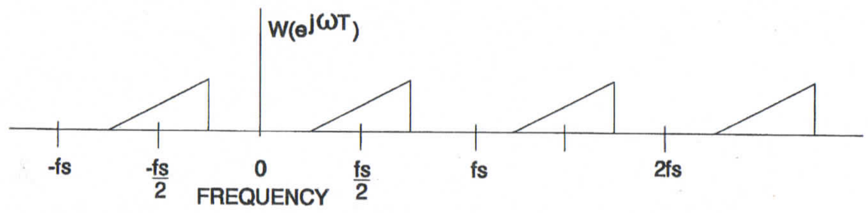



**FIGURE 4.23** Portion of spectrum w(nT)

spectrum of x(nT) is shown in Fig. 4.24 and is simply that part of the spectrum in Fig. 4.22 that falls between $-f_s/2$ and $f_s/2$. We now translate the signal up resulting from multiplication by y(nT). This is shown in Fig. 4.25. Note that as the spectrum is moved to the right, that part that disappears beyond $f_s/2$ must be added starting on the left at $-f_s/2$. This is that portion of the spectrum in Fig. 4.23 that falls in the range from $-f_s/2$ to $f_s/2$. With this consideration, then, any signal that is cut off above $f_s/2$ will be reinstated on the left from $-f_s/2$. The two representations give identical results.

Thus we must consider the bars at $-f_s/2$ to be the same point. A positive frequency, $f_p$, which is pushed off the graph reappears as a negative frequency with a value $-(f_s - f_p)$. This graphical representation of the signals will be used in much of the remainder of the book.

Now, let us consider the implementation of a digital mixer. First, consider a real input signal, x(nT), multiplied by a complex exponential

$$y(nT) = e^{j\omega_0 nT} = \cos(\omega_0 nT) + j\sin(\omega_0 nT)$$

This is referred to as a *half-complex* mixer. The block diagram is shown in Fig. 4.26.

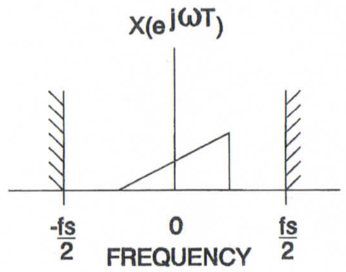



**FIGURE 4.24** Abbreviated spectrum of X(nT)

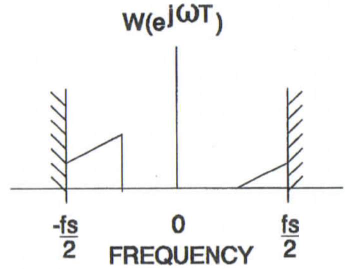



**FIGURE 4.25** Abbreviated spectrum of w(nT)



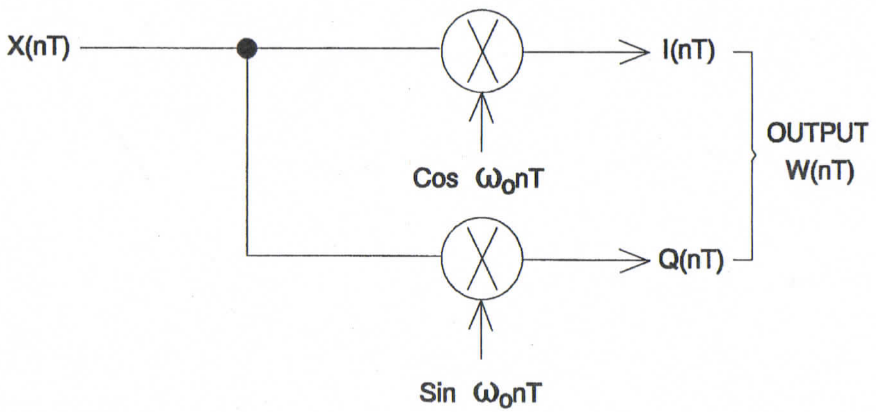



**FIGURE 4.26** Half-complex mixer

The half-complex mixer produces a complex signal from a real input. The output signal can be determined by convolution, as shown in Fig. 4.27. The upper graph shows the frequency spectrum of the input signal. The graph in the center is the negative of the injection. The convolution integral [see Eq. (2.42)], requires that one of the signals be reversed in time. The output at f = 0 is then the sum of the product of the corresponding points on the upper two graphs. The output at any other frequency is found by displacing the signal that was reversed and forming a similar sum of products. Since the injection in this case is a single impulse, the output spectrum is simply a displaced replica of the input spectrum. We can also have a half-complex mixer if a complex input signal is mixed with a real injection. This is shown in Fig. 4.28, with a typical input and output spectrum shown in Fig. 4.29. In this case, the output signal is also complex. Note that, since the injection signal is real, it has symmetrical positive and negative components. This is illustrated in the center graph of Fig. 4.29. The convolution integral requires that the spectrum of one of the signals be reversed; however, since the injection is symmetrical, the reversal results in an identical spectrum.

We now consider the operation of a full complex mixer. Let the input signal be

$$x(nT) = I(nT) + jQ(nT) \tag{4.39}$$

and the injection be

$$y(nT) = e^{j2\pi f_0 nT} = \cos(2\pi f_0 nT) + j\sin(2\pi f_0 nT) \tag{4.40}$$

The output is the product

$$w(nT) = x(nT)\,y(nT) \tag{4.41}$$

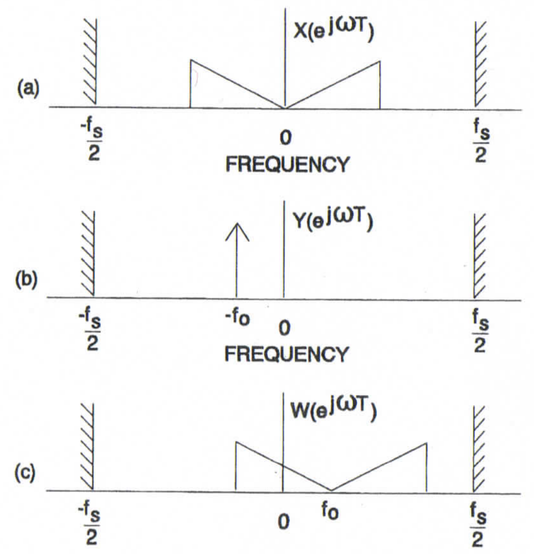



**FIGURE 4.27** Frequency spectrum of mixer input and output for exponential injection signal

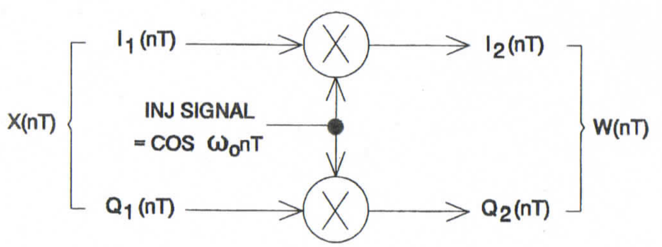



**FIGURE 4.28** Half-complex mixer with real injection signal

which, expanded, is

$$w(nT) = [I + jQ]\,[\cos(\omega_0 nT) + j\,\sin(\omega_0 nT)] \tag{4.42}$$

$$w(nT) = [I\cos(\omega_0 nT) - Q\sin(\omega_0 nT)] + j\,[Q\cos(\omega_0 nT) + I\sin(\omega_0 nT)] \tag{4.43}$$

This leads to the block diagram of a complex mixer shown in Fig. 4.30. In this case, all signals are complex, and four real multiplications are required, plus two additions. Note that in the case of the real output, a subtraction rather than an addition is required. In many cases, the additional complexity of a full-complex implementation is justified because of the ideal frequency translation that is obtained. The typical signal spectra are shown in Fig. 4.31 for the complex mixer.



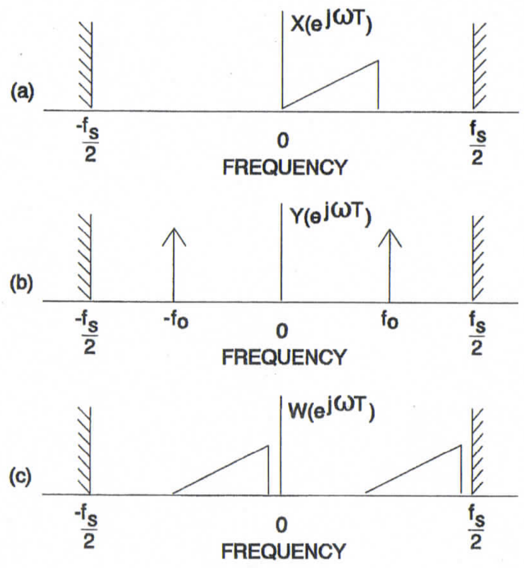



**FIGURE 4.29** Frequency spectrum of mixer with complex input and real injection

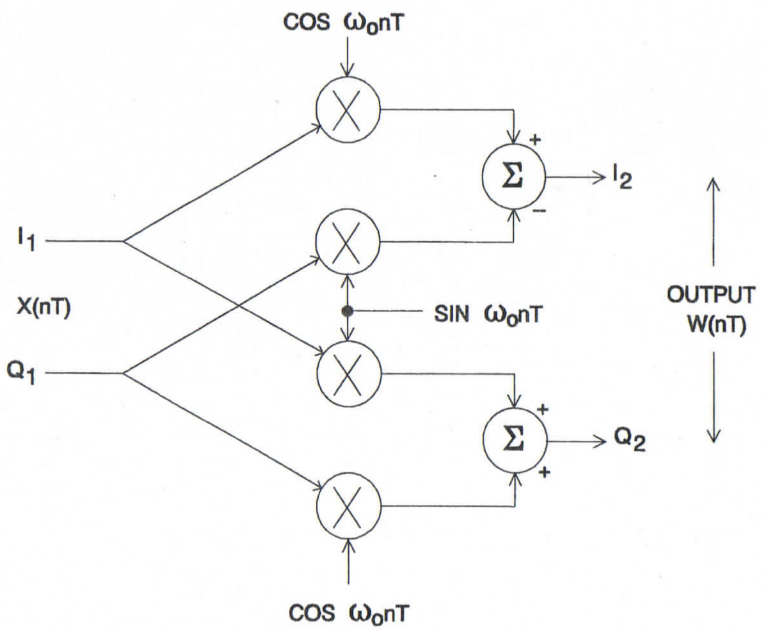



**FIGURE 4.30** Block diagram of complex mixer

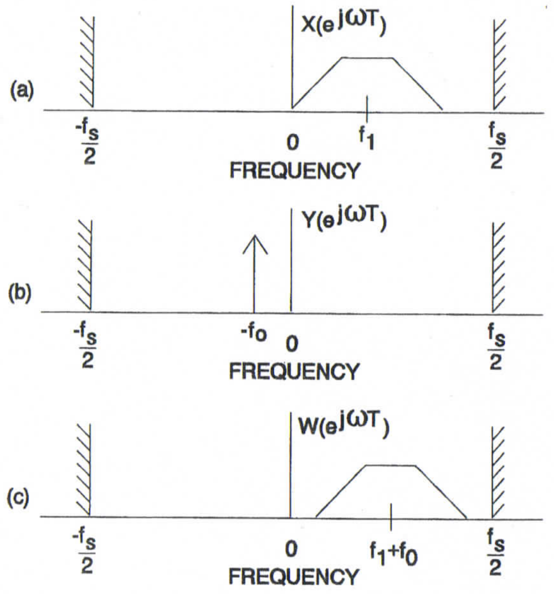



**FIGURE 4.31** Frequency spectrum of (a) input, (b) injection, and (c) output of complex mixer

The injection frequency $y(nT) = e^{j2\pi f_0 T}$ is shown reversed in the center graph of the figure. Note that only one output sideband is generated within the $-f_s/2$ to $f_s/2$ interval shown.

Of course, we also can have a real digital mixer that generates both sum and difference products as an analog mixer. Only a single multiplication is then required. A real digital mixer has the advantage of avoiding an analog mixer's nonlinearities. The digital mixer is an ideal mixer if infinite precision arithmetic is used. The use of truncated arithmetic causes a small amount of low-level intermodulation. These products may be in the order of the least significant bit of the digital word. Unlike an analog mixer, the distortion products in a digital mixer may not decrease if the signal level is reduced. Indeed, in some cases, the distortion level may actually increase at a lower signal level.

The computation load for a digital mixer can sometimes be reduced by the judicious choice of the injection frequency. One case where a great savings can be realized is when the injection frequency is 1/4 the sample rate. In this case,

$$y(nT) = e^{j2\pi f_{INJ} nT} = \cos(2\pi f_{INJ} nT) + j\sin(2\pi f_{INJ} nT) \tag{4.44}$$

where

$T$ = the sample period given by $T = 1/f_s$



Then, the product $2\pi f_{INJ}/f_s$ becomes $2\pi/4 = \pi/2$. The injection sequence then becomes

$$\cos\left(\frac{n\pi}{2}\right) + j\sin\left(\frac{n\pi}{2}\right)$$

The cosine sequence is given by

$$\left\{\cos\left(\frac{n\pi}{2}\right)\right\} = 1, 0, -1, 0, \ldots \tag{4.45}$$

and the sine sequence is given by

$$\left\{\sin\frac{n\pi}{2}\right\} = 0, 1, 0, -1, \ldots \tag{4.46}$$

Physically, this mixer can be implemented in hardware by a circuit to complement the signal and two demultiplexing circuits. One realization of the mixer is shown in Fig. 4.32. The amount of hardware is quite minimal compared to the need for a high-speed multiplier. The circuit forms the negative of the signal by taking the ones compliment rather than the exact twos complement. It should be noted that this results in the addition of a small negative dc component to the signal. The average value of this error is –1/2 (least significant bit), which may be significant

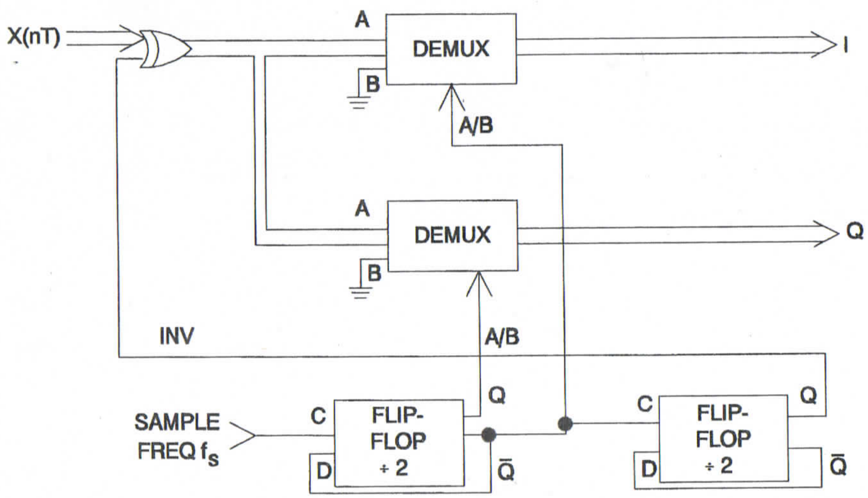



**FIGURE 4.32** Hardware implementation of digital mixer

if the signal is later mixed, and it could cause an audio tone. A discrete tone should be about 50 dB below the signal so it cannot be heard be the human ear.

The circuit in Fig. 4.32 works as follows: The sample frequency, $f_s$, is divided by two, and the outputs of the divide-by-two flip-flop cause the I and Q multiplexers to generate a zero output on alternate samples by selecting the B input to the demultiplexers. The A/B select signal is further divided by two to generate the complement command signal. This signal causes the input to be negated for two samples after the first two samples have passed. Assuming the flip-flops were originally reset, this generates the sequences

$$\{I\} = x(0), 0, -x(2), 0, \ldots \tag{4.47}$$

$$\{Q\} = 0, x(1), 0, -x(3), \ldots \tag{4.48}$$

Another way mixing can be accomplished is by sample rate decimation. This amounts to harmonic resampling of the signal using a slower sample rate, and can only be done if the signal bandwidth has been reduced such that aliasing will not occur. Figure 4.33 shows the nomenclature used for resampling and for decimation, where s(t) is a periodic impulse train with a repetition rate $f_{s2}$. The decimation ratio is normally an integer, so the procedure amounts to simply taking every Rth input sample and discarding the rest.

Figure 4.34 depicts the spectrum for a narrow band signal that is translated to baseband by decimation using a ratio R = 4. A considerable variety of effects can be achieved using resampling, and the designer should add this technique to his "bag of tricks."

## HILBERT TRANSFORMERS

Because Hilbert transformers are frequently used in digital signal processing, and because they play such an important role in forming analytic signals, it is appropriate to include a section that will enable us to understand how they can be built and how they function. From our previous discussion, we know that a Hilbert

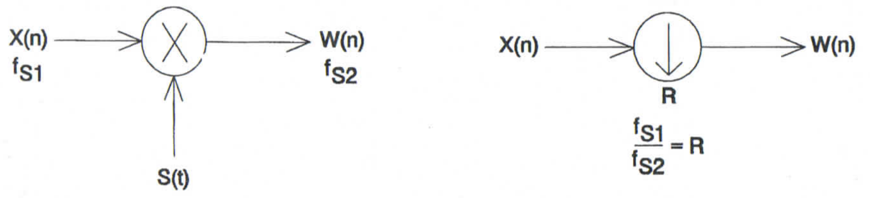



FIGURE 4.33 Resampling and decimation representation of mixing by decimation



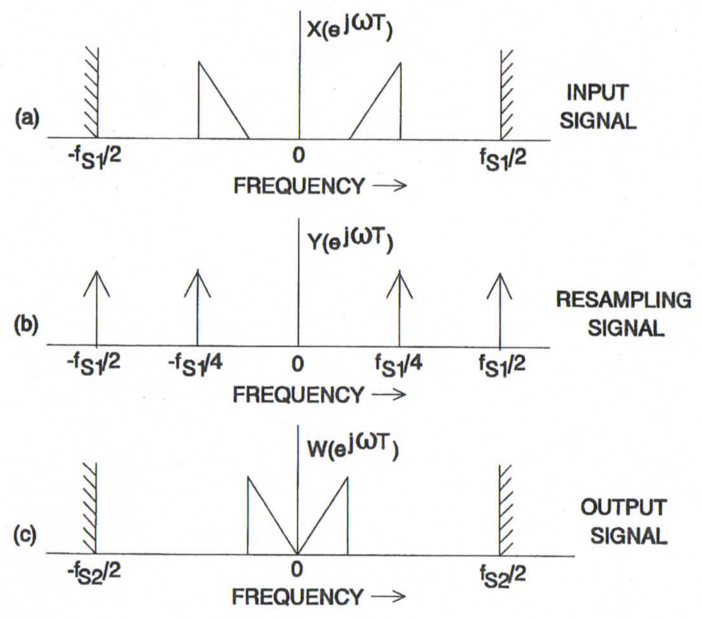



FIGURE 4.34 Spectrum of signal resampled at 1/4 the original rate

transformer shifts all positive frequencies by –90 degrees and all negative frequencies by +90 degrees. This amounts to providing a time delay that is inversely proportional to the absolute frequency. Formally we may write the frequency response as

$$\begin{aligned} H(f) &= -j \quad f \geq 0 \\ H(f) &= \ \ j \quad f < 0 \end{aligned} \tag{4.49}$$

We shall use this definition to derive the impulse response of an ideal Hilbert transformer. The derivation is accomplished by taking the inverse Fourier transformer of the frequency response. The derivation can be simplified by taking advantage of a trick. Fourier transforms have the property that if

$$h(t) \leftrightarrow H(f) \tag{4.50}$$

then

$$(-j2\pi t)\, h(t) \leftrightarrow dH(f)/df$$

The proof of this property is left as an exercise (see Problem 4-7). A plot of the frequency response from Eq. (4.49) is given in Fig. 4.35.

The derivative of the response is

$$dH(f)/df = -2j\delta(f) = G(f) \tag{4.51}$$

The impulse response is

$$g(t) = F^{-1}\{G(f)\} = \int_{-\infty}^{\infty} -2j\delta(f)\, e^{j2\pi ft} dt \tag{4.52}$$

$$g(t) = -2j \tag{4.53}$$

From Eq. (4.50) we also have:

$$(-j2\pi t)\, h(t) = -2j \tag{4.54}$$

and

$$h(t) = 1/\pi t \tag{4.55}$$

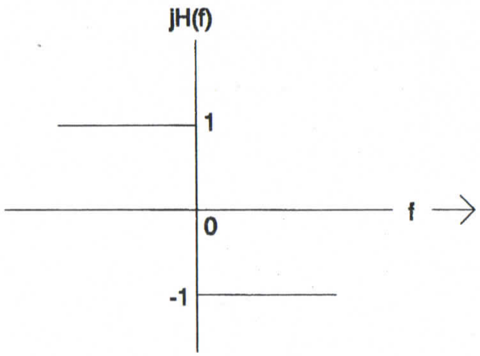



FIGURE 4.35 Frequency response of ideal Hilbert transformer



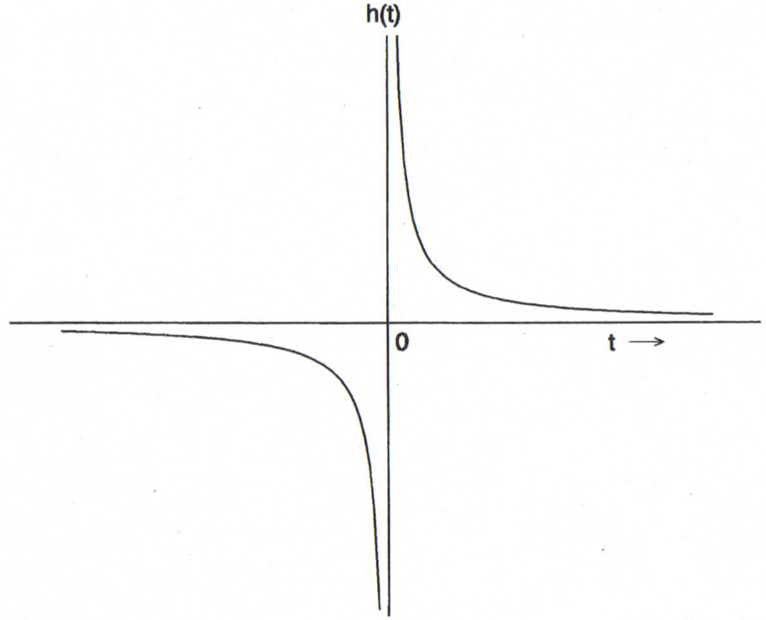



FIGURE 4.36 Impulse response of ideal Hilbert transformer

This is plotted in Fig. 4.36. With background on the ideal continuous Hilbert transformer we will now examine an ideal digital Hilbert transformer. The frequency response is required to be

$$H(e^{j\omega t}) = -j \quad \omega \geq 0 \tag{4.56}$$

$$H(e^{j\omega t}) = j \quad \omega < 0$$

The frequency response of any digital filter, as we will see in Chapter 5, is periodic with a period equal to the sample frequency. For this discussion, a normalized sample frequency of 1 Hz will be used. Therefore, the period of the response is $2\pi$. Figure 4.37 shows the periodic frequency response of the Hilbert transformer. Since the graph is periodic, we can use the Fourier series to determine the coefficients of an exponential series representing the plot.

This is discussed in greater detail in Chapter 5, on digital filters. For the present discussion, it is sufficient to note that the x axis represents angular frequency rather than time, as it normally does in the Fourier series. Also, the period of the Fourier series is $2\pi$. Making these substitutions into Eq. (2.7) and using $H(e^{j\omega})$ for s(t), the coefficients are given by

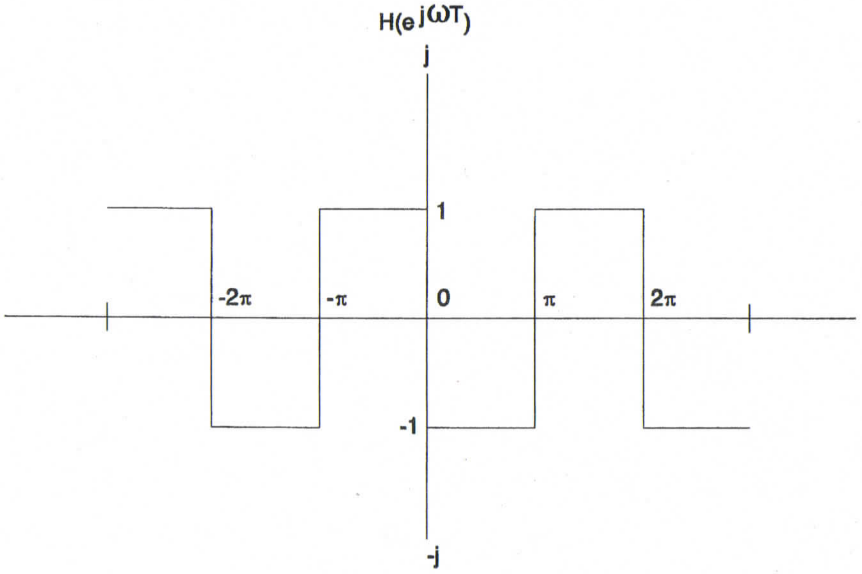



**FIGURE 4.37** Periodic frequency response of a digital Hilbert transformer

$$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{-jn\omega}\, d\omega \tag{4.57}$$

and

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} C_n e^{j\omega n} \tag{4.58}$$

Now, letting $z = e^{j\omega T}$ with T = 1 in Eq. (4.58)

$$H(z) = \sum_{k=-\infty}^{\infty} C_n z^n \tag{4.59}$$

From this equation, we see that the impulse response of the digital Hilbert transformer is given by

$$h(n) = C_{-n} \tag{4.60}$$



Solving for the values of $C_n$ using Eq. (4.57) gives

$$C_{-n} = \frac{1}{2\pi}\int_{-\pi}^{0} je^{j\omega n}\, d\omega - \frac{1}{2\pi}\int_{0}^{\pi} je^{j\omega n}\, d\omega \tag{4.61}$$

$$h(n) = \frac{2}{\pi}\,\frac{\sin^2\left(\frac{\pi n}{2}\right)}{n} \qquad n \text{ is not} = 0$$

$$h(n) = 0 \qquad n = 0$$

A plot showing the impulse response is given in Fig. 4.38.

In practice, the Hilbert transformer is approximated by a finite number of taps. It is also necessary to allow a signal delay of 1/2 the number of taps multiplied by the sample time. This can be accomplished by taking the untransformed output from the center of the transformer, as shown in Fig. 4.39.

Since the coefficients are truncated, some methods of optimization may lead to the presence of small, even-numbered coefficients, even though the ideal transformer has only odd coefficients. It is interesting to note that the error from truncating the filter does not cause phase errors but, rather, distorts the amplitude response. Most FIR digital filter design programs include an option to design Hil-

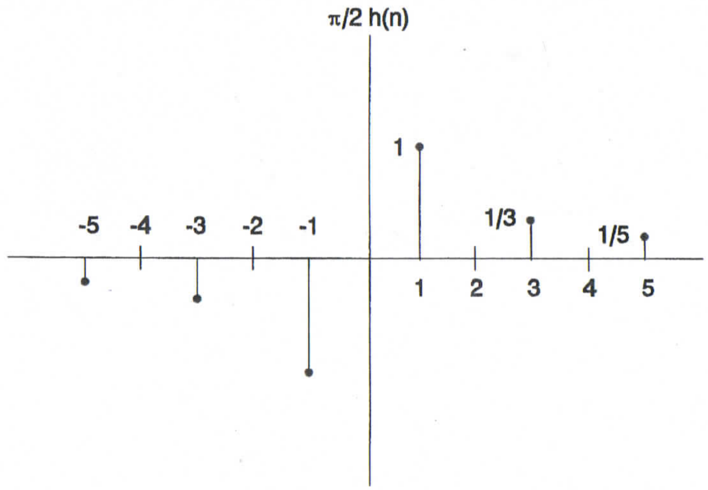



**FIGURE 4.38** Impulse response of digital Hilbert transformer

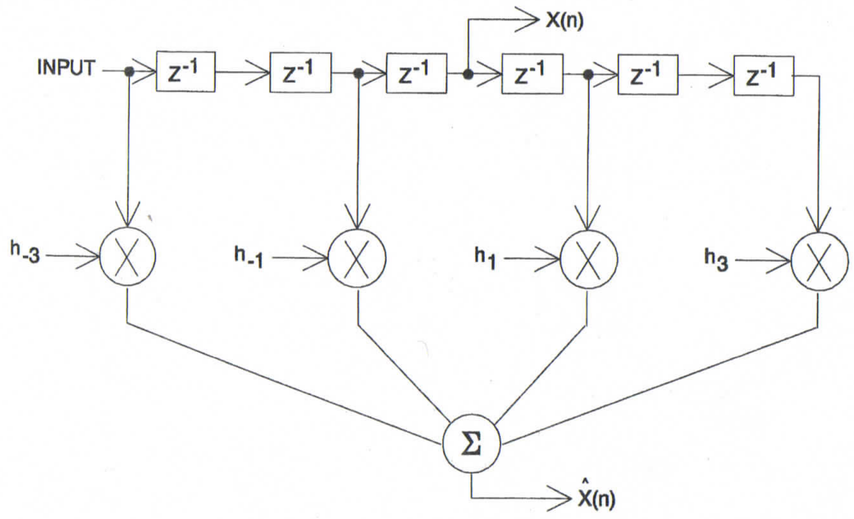



**FIGURE 4.39** Realization of digital Hilbert transformer

bert transformers, so the engineer can determine the coefficients and the required length from the ripple specifications. Filter design programs are discussed further in Chapter 5. An excellent treatment of the Hilbert transformer is also given in Rabiner and Schafer[10].

Another way to obtain a Hilbert transform pair is to incorporate the transformer in a filter. The procedure can also be viewed as designing a complex filter, and this approach is detailed in Chapter 5. We shall describe it here as designing a Hilbert transform pair. The general procedure is to design two bandpass filters with identical inputs. The filters are designed so that one filter has a 90 degree phase shift with respect to the other. Filters of this type can be designed by starting with a lowpass filter such as shown in Fig. 4.40.

Let the impulse response of the filter be h(t). This filter can be transformed to a real bandpass filter having a response as shown in Fig. 4.41. This can be accomplished by multiplying the impulse response of the lowpass filter by $2\cos(2\pi f_0 t)$ so the bandpass filter impulse response is

$$h_{BP}(t) = 2h(t)\cos(2\pi f_0 t) \tag{4.62}$$

The frequency response can be determined by using the frequency translation theorem and rewriting Eq. (4.62) in the form

$$h_{BP}(f) = h(t)e^{j2\pi f_0 t} + h(t)e^{-2\pi f_0 t} \tag{4.63}$$



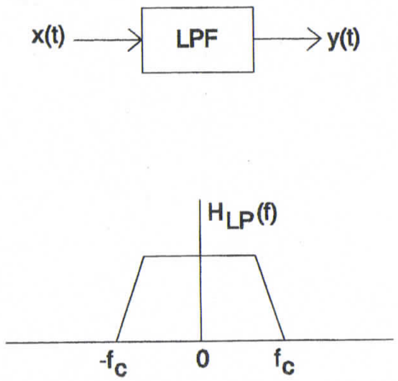



**FIGURE 4.40** Lowpass filter and its frequency response

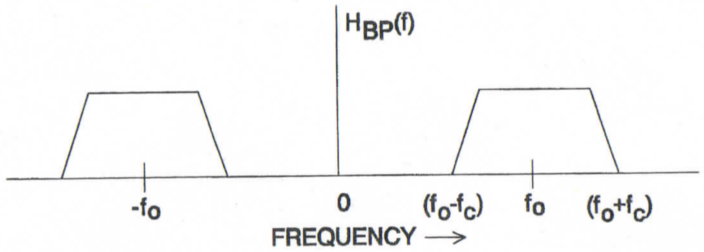



**FIGURE 4.41** Frequency response of bandpass filter

Then, the frequency response is

$$H_{BP}(f) = H_{LP}(f-f_0) + H_{LP}(f+f_0) \tag{4.64}$$

Since we are dealing with a real filter, it has identical responses to positive and negative frequencies. Because this is the case, it should be noted that the expected bandpass filter response is obtained only if the positive and negative response selections in Fig. 4.41 do not overlap. If the skirt selectively is too broad or if $f_0$ is less than $f_c$, some overlap will occur. The low frequency response will then be the sum of the overlapping section, as predicted by Eq. (4.64).

Now, suppose we wish to include a Hilbert transformer in a second filter with an identical frequency response magnitude. The desired response can be obtained by modifying Eq. (4.64) to obtain

$$\hat{H}_{BP}(f) = -jH_{LP}(f-f_0) + jH_{LP}(f+f_0) \tag{4.65}$$

By the frequency translation theorem, the impulse response is given by

$$\hat{h}_{BP}(t) = -je^{j2\pi f_0 t}h_{LP}(t) + je^{-j2\pi f_0 t}h_{LP}(t) \tag{4.66}$$

Factoring out the common term $h_{LP}(t)$ and rewriting gives

$$\hat{h}_{BP}(t) = h_{LP}(t)\left[\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{j}\right]$$

We recognize this as

$$\hat{h}_{BP}(t) = 2h_{LP}(t)\sin(s\pi f_0 t) \tag{4.67}$$

Equations (4.62) and (4.67), then, give us the method for designing a pair of filters whose outputs can be used as the signal and its Hilbert transform. As we shall see in Chapter 5, the impulse response of a digital FIR filter is simply composed of its coefficient values. Therefore, if we design a lowpass digital FIR filter and multiply the coefficients of the filter by $\cos(2\pi f_0 t)$ and $\sin(2\pi f_0 t)$, a Hilbert pair with center frequency $f_0$ is obtained. We shall also see in Chapter 5 that to maintain a linear phase response the impulse response must be symmetrical about the center of the filter. Therefore, the actual multipliers should include an addition delay of $T(N-1)/2$ in each filter. This maintains linear phase in the cosine filter. Here, N is the total number of taps, and T is the sample time. Therefore, the actual transformations use are:

$$h_{BP}(n) = 2h_{LP}(n)\cos\left(2\pi f_0\left[n - \frac{(N-1)}{2}\right]T\right) \tag{4.68}$$

and

$$\hat{h}_{BP}(n) = 2h_{LP}(n)\sin\left(2\pi f_0\left[n - \frac{(N-1)}{2}\right]T\right) \tag{4.69}$$

Figure 4.42 shows the resulting physical configuration.

The application of complex mixers and Hilbert transformers in combination can often be used to accomplish interesting results. For example, the circuit shown



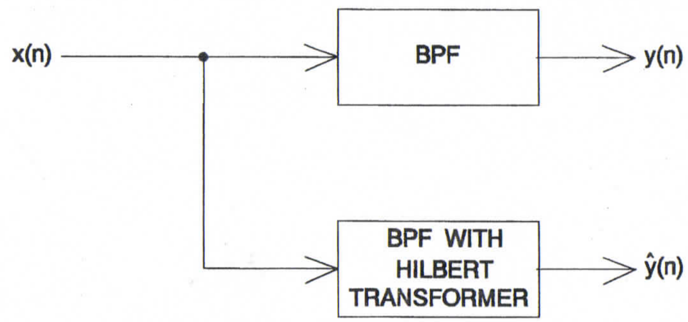



**FIGURE 4.42** Block diagram of Hilbert transformer pair

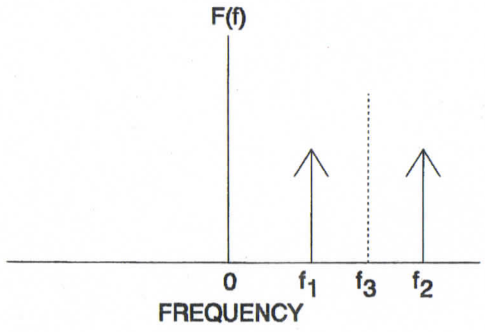



**FIGURE 4.43** Analytic signal consisting of two positive frequencies

in Fig. 4.44 can be used to separate the two signals shown in Fig. 4.43. The first portion of the frequency separator translates the composite signal down by an amount $f_3$, so that the original tone at $f_1$ becomes negative by an amount $f_1 - f_3$. The original signal at $f_2$ remains positive and has a new frequency of $f_2 - f_3$. The Q signal is then passed through a Hilbert transformer, and the difference $I_2(n) - I_3(n)$ selects the positive frequency $(f_2 - f_3)$ but rejects the negative frequency $(f_1 - f_3)$. The positive frequency selection circuit here is the same circuit derived earlier in Fig. 4.15a. Changing the sign of $I_3(n)$ in the summation gives $Z'(n)$, which responds only to the negative frequency $f_1 - f_3$.

The circuit in Fig. 4.44 is most useful to separate the upper and lower sidebands in an independent sideband receiver. It will be seen again later, in connection with these receivers.

The construction of a Hilbert transform filter pair and utilizing $I_2(n) + jI_3(n)$ can also be viewed as the construction of a complex filter having only a positive frequency response. Complex filters are treated in Chapter 5.

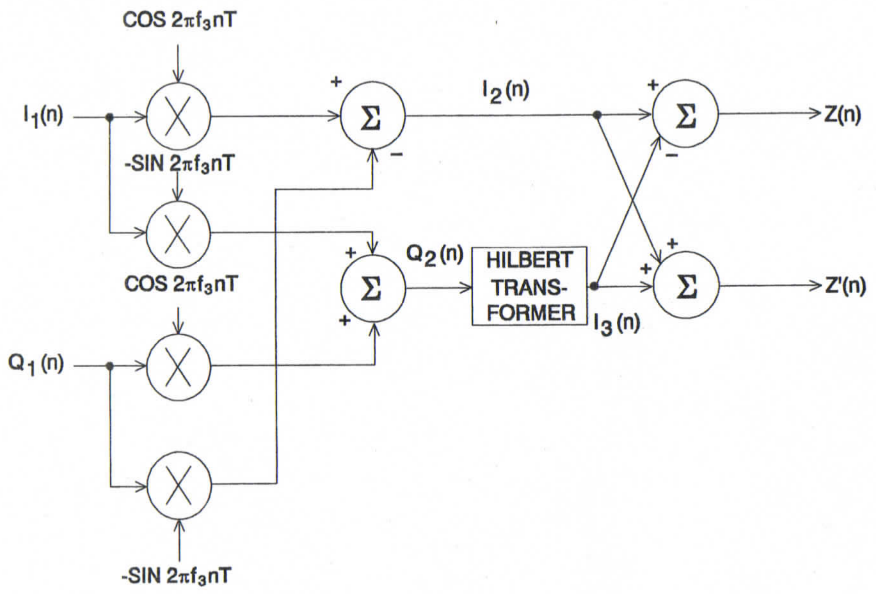



**FIGURE 4.44** Block diagram of frequency separator

## PROBLEMS

**4-1** An analytic signal is formed by passing a complex signal $C(t) = I(t) + jQ(t)$ through a Hilbert transformer as shown below:

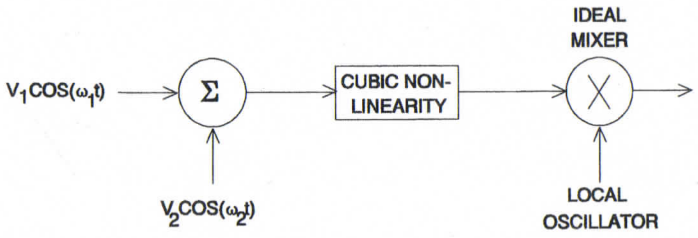

The frequency spectrum of I(t) is given by I(f), and the frequency spectrum of Q(t) is Q(f). Show that the frequency spectrum of the analytic signal

$$Z(t) = C(t) + j\hat{C}(t)$$

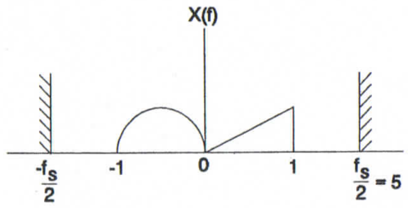
consists only of the positive frequencies of C(t).

**4-2** An analog mixer has two signal inputs: $V_1 \cos \omega_1 t$ and $V_2 \cos \omega_2 t$ as shown in the following illustration. Show that the third-order intermodulation products are proportional to $V_1 V_2^2$ or $V_1^2 V_2$.

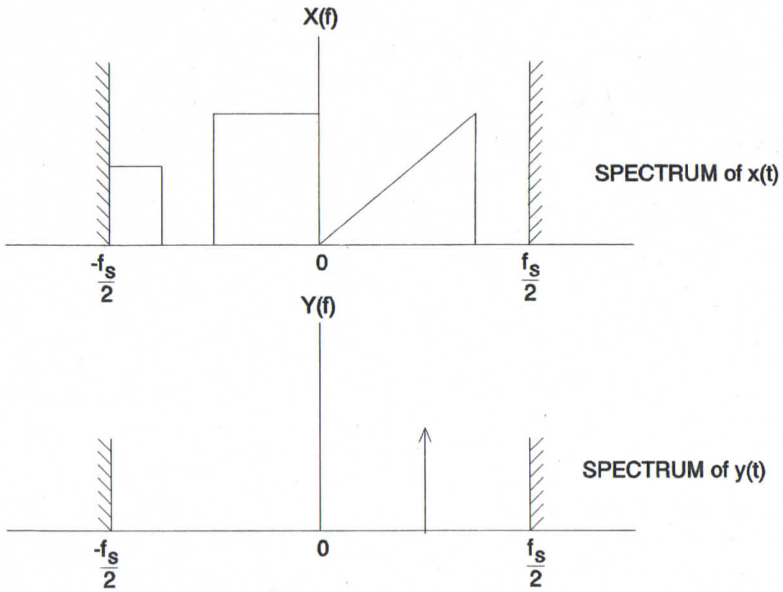





**4-3** A digital signal with a spectrum X(f), as shown below, is mixed by multiplication with an injection signal $\cos(4\pi nT)$, where $f_s = 1/T = 10$ Hz. Sketch the frequency spectrum of the injection signal and the spectrum of the output signal.




**4-4** A real digital signal x(t) is complex mixed by multiplication by y(t). The respective spectra are shown below. Sketch the output spectrum of W(t), where $W(t) = x(t)y(t)$.



**4-5** A real signal consists of two sine waves, so that

$$f(t) = 3 \cos(10t) + 5 \sin(4t)$$

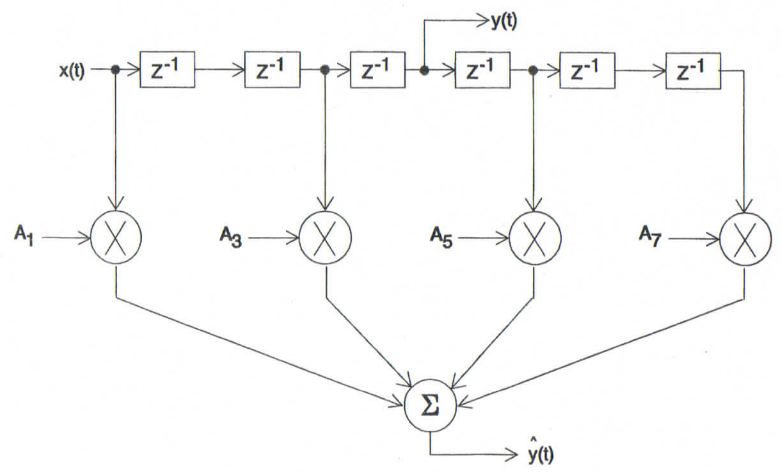
An analytic signal is constructed from f(t) using a Hilbert transformer so that

$$g(t) = f(t) + j\hat{f}(t)$$

a) Sketch the frequency response of g(t)
b) Using the frequency components of cos ωt and sin ωt, and the phase shift properties of a Hilbert transformer, find the frequency spectrum of g(t) by adding the components of f(t) and $j\hat{f}(t)$ to show that the result in part a) is correct.

**4-6** A digital Hilbert transformer of the form shown below has tap weights $A_1 = -1/3$, A3 = –1, A5 = 1, A7 = 1/3.

a) **If the sample frequency is 1 Hz, write an expression for the z-transform of the transfer function $\hat{Y}(z)/Y(z)$.**
b) Sketch the magnitude and phase of the frequency response for 0 < f < 1/2 Hz using the signal y(t) at the center of the transformer as the reference point.




**4-7** Given the Fourier transform pair $h(t) \leftrightarrow H(f)$, show that the transform pair below follows.

$$(-j\pi t)\, h(t) \leftrightarrow dH(f)/df$$



**4-8** A signal with the frequency spectrum shown below is sampled at a 50 kHz rate.

a) Sketch the frequency spectrum of the signal for all frequencies between –120 and +120 kHz.
b) Sketch the frequency spectrum of the sampled signal using $f_s/2$ limited nomenclature.