Concentric Maclaurin Spheroids: theory and practice

Summary of the theory and practice of modeling rotating fluid planets by Hubbard's Concentric Maclaurin Spheroids technique. These notes provide the mathematical basis for using gravity measurements to learn about planetary interiors. Mathematical statements are checked to my satisfaction unless otherwise noted (important exception is the addition theorem). Primary references are (Zharkov and Trubitsyn, 1978) for the mathematical foundation, and (Hubbard, 2012, 2013) and Bill's personal notes for the CMS theory.

Contents

I. THEORY

A. Definitions and notation

Given a density distribution $\rho(\mathbf{r}')$ inside the planet, the *total potential* is defined as

$$U(\mathbf{r}) = V(\mathbf{r}) + Q(\mathbf{r}) \tag{1}$$

where

$$V(\mathbf{r}) = G\int \rho(\mathbf{r}')/\left|\mathbf{r}-\mathbf{r}'\right|\, d\tau' \tag{2}$$

is the gravitational potential and

$$Q(\mathbf{r}) = \frac{1}{2}\omega^2 r^2 \sin^2\theta \tag{3}$$

is the centrifugal potential; we will use $d\tau$ for a volume element, ω is the angular rotation velocity (assumed constant and usually about a principal axis), $r = |\mathbf{r}|$, and θ is the angle from the rotation axis. *Note that Hubbard and Zharkov and Trubitsyn use an unusual positive potential presumably for algebraic convenience. This also means accelerations are given by the positive gradient of potential.*

When the planet is in hydrostatic equilibrium the level surfaces of potential are also level surfaces of pressure and of density. Including the free surface where the pressure $p = 0$. This can be shown rigorously (e.g. Batchelor, 1967) and is also fairly intuitive. Finding the shape of the planet is therefore reduced to the problem of finding the level surfaces $U(\mathbf{r}) =$ constant. This problem is easy to state but hard to solve, essentially because the volume of space over which the integral above is taken is unknown and must be found as part of the self-consistent solution.

Equilibrium figures which differ only slightly from spheres are called *spheroids*. A dimensionless parameter describing the importance of rotation is

$$q = \frac{\omega^2 a^3}{GM} \tag{4}$$

where M is the planet's mass and a is the *equatorial radius.*

B. Decomposition of potential into spherical harmonics

The expression for the gravitational potential, eq. (2) can be written as a sum of powers of r using the decomposition of $1/|\mathbf{r}-\mathbf{r}'|$ in Legendre polynomials (see LEGENDRE.PDF):

$$\begin{aligned} \frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r\sqrt{1-2t(r'/r)+(r'/r)^2}} = \\ &= \begin{cases} \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(t), & r > r', \\ \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^{-n-1} P_n(t), & r < r', \end{cases} \end{aligned} \tag{5}$$

where γ is the angle between the radius vectors $\mathbf{r}$ and $\mathbf{r}'$ and $t = \cos\gamma$. If $r > r'$ for all points where $\rho(\mathbf{r}') > 0$ (i.e. inside the planet) then the potential is called *external*. The Legendre polynomials are given by Rodrigues's formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{6}$$

(although this is not the easiest way to obtain them explicitly) for $x \in [-1,1]$ and luckily there is no confusion about normalization. It will be useful to remember that

$$P_0 = 1, \qquad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}, \qquad \int_{-1}^{1} P_{n>0}(t')\,dt' = 0.$$

and that $P_n(1) = 1$. The gravitational potential in terms of Legendre polynomials is

$$V(\mathbf{r}) = \frac{G}{r}\sum_{n=0}^{\infty}\int \rho(\mathbf{r}')P_n(t)(r'/r)^{\alpha}\,d\tau' \tag{7}$$

where

$$\alpha = \begin{cases} n, & r > r', \\ -(n+1), & r < r', \end{cases}$$

and the integration is over the (as yet unknown) volume of the planet.

The expansion in Legendre polynomials is compact and neat but it is of little utility because it does not separate terms arising from the mass distribution from those arising from the location where the potential is to be evaluated. In spherical polar coordinates the variable

$$t = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

mixes the colatitude θ and longitude φ of the integration variable and the point of measurement. The salvation comes from the *addition theorem for spherical harmonics* (**which I can't derive**):

$$P_n(\cos\psi) = P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos\big[m(\varphi-\varphi')\big], \tag{8}$$

with the associated Legendre functions

$$P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x). \tag{9}$$

Unfortunately there is not a universal consensus on the exact form of the P_n^m functions. If a different normalization is used then the following expressions will all be somewhat different. With the aid of the addition theorem we can decompose the potential as[1]

$$
\begin{aligned}
V(\mathbf{r}) = \frac{G}{r}\Bigg(& \sum_{n=0}^{\infty} P_n(\cos\theta) \int_\tau \rho(\mathbf{r}') P_n(\cos\theta')(r'/r)^\alpha \, d\tau' \\
& + \sum_{n=1}^{\infty}\sum_{m=1}^{n} P_n^m(\cos\theta)\cos(m\varphi) \int_\tau \frac{2(n-m)!}{(n+m)!} \rho(\mathbf{r}') P_n^m(\cos\theta')\cos(m\varphi')\left(\frac{r'}{r}\right)^\alpha d\tau' \\
& + \sum_{n=1}^{\infty}\sum_{m=1}^{n} P_n^m(\cos\theta)\sin(m\varphi) \int_\tau \frac{2(n-m)!}{(n+m)!} \rho(\mathbf{r}') P_n^m(\cos\theta')\sin(m\varphi')\left(\frac{r'}{r}\right)^\alpha d\tau' \Bigg)
\end{aligned} \tag{10}
$$

with α as before.

The above expansion is general, not requiring hydrostatic equilibrium or principal axis rotation. If the planet is fluid then at equilibrium the rotation will always be about a principal axis. If we take the polar axis (call it the z axis) to coincide with the rotation axis then symmetry requires that $\rho(\mathbf{r})$ and therefor $V(\mathbf{r})$ cannot depend on longitude φ and must include only even powers of $\cos\theta$ (for symmetry about the equator). In this case we write a simpler expansion involving only ordinary Legendre polynomials of only even degree:

$$
V(r,\theta) = \frac{G}{r}\sum_{n=0}^{\infty}\left(r^{-2n}D_{2n} + r^{2n+1}D'_{2n}\right) P_{2n}(\cos\theta) \tag{11a}
$$

with

$$
D_n = \int_{r'<r} \rho(\mathbf{r}')(r')^n P_n(\cos\theta')\, d\tau', \tag{11b}
$$

$$
D'_n = \int_{r'>r} \rho(\mathbf{r}')(r')^{-n-1} P_n(\cos\theta')\, d\tau'. \tag{11c}
$$

The coefficients D_n are usually replaced with the non-dimensional coefficients $J_n = D_n/(Ma^n)$.

C. The external potential

If the potential is to be evaluated at a point away from the surface of the planet then $r > r'$ for all differential volume elements in the integral expressions above. The general form eq. (10) can be rearranged slightly and rewritten in a form more convenient for comparison with measured quantities:

$$
\begin{aligned}
V_e = \frac{GM}{r}\Bigg(1 & - \sum_{n=1}^{\infty}(a/r)^n J_n P_n(\cos\theta) + \\
& + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(a/r)^n P_n(\cos\theta)\Big[C_{nm}\cos(m\varphi) + S_{nm}\sin(m\varphi)\Big]\Bigg),
\end{aligned} \tag{12a}
$$

with the coefficients

$$
Ma^n J_n = -\int \rho(\mathbf{r}')(r')^n P_n(\cos\theta')\, d\tau' = -D_n, \tag{12b}
$$

$$
Ma^n C_{nm} = \frac{2(n-m)!}{(n+m)!}\int \rho(\mathbf{r}')(r')^n P_n^m(\cos\theta')\cos(m\varphi')\, d\tau', \tag{12c}
$$

$$
Ma^n S_{nm} = \frac{2(n-m)!}{(n+m)!}\int \rho(\mathbf{r}')(r')^n P_n^m(\cos\theta')\sin(m\varphi')\, d\tau'. \tag{12d}
$$

(Remember $M = \int \rho\, d\tau'$ is the planet's mass and a is the equatorial radius.) Different normalizations of the associated Legendre functions are sometimes used leading to expansions with the same form but

[1] Note typo in Z&T.

different meaning of the J_n, C_{nm}, and S_{nm} coefficients. There is no easy way to guard against errors or guess which normalization was used if it is not explicitly given. The decomposition can also be carried out with the complex form of the Legendre functions leading to even more confusion.

A natural choice of reference frame can eliminate many of the coefficients in the expansion (12). First, if the origin of coordinates is chosen at the center of mass of the planet, $\mathbf{R} = (X, Y, Z)$ then

$$\begin{aligned} -MaJ_1 &= \int z'\, dm' = MZ = 0, \\ MaC_{11} &= \int x'\, dm' = MX = 0, \text{ and} \\ MaS_{11} &= \int y'\, dm' = MY = 0 \end{aligned}$$

so that J_1, C_{11}, and S_{11} vanish. Next, we can relate the expansion coefficients to the moments of inertia. We designate those:

$$B = \int_\tau \rho(\mathbf{r}')(x'^2 + z'^2)\, d\tau', \tag{13a}$$

$$A = \int_\tau \rho(\mathbf{r}')(y'^2 + z'^2)\, d\tau', \tag{13b}$$

$$C = \int_\tau \rho(\mathbf{r}')(x'^2 + y'^2)\, d\tau', \tag{13c}$$

for the principal moments and

$$D = \int_\tau \rho(\mathbf{r}')\, y'z'\, d\tau', \tag{13d}$$

$$E = \int_\tau \rho(\mathbf{r}')\, x'z'\, d\tau', \tag{13e}$$

$$F = \int_\tau \rho(\mathbf{r}')\, x'y'\, d\tau', \tag{13f}$$

for the diagonal, so-called *products of inertia*, also called *centrifugal moments*. The relations with the expansion coefficients in eq. (12) become clear by writing out the degree 2 Legendre functions in Cartesian coordinates. It is easy to show by direct comparison that

$$-a^2MJ_2 = \frac{A+B}{2} - C, \quad \text{and} \quad a^2MC_{22} = \frac{B-A}{4}, \tag{14a}$$

and also that

$$D = a^2MS_{21}, \quad E = a^2MC_{21}, \quad \text{and} \quad F = 2a^2MS_{22}. \tag{14b}$$

Usually $A = B < C$ so that $a^2MJ_2 = (C - A)$. Also, if we align the coordinate axes with the planet's principal axes of inertia then the centrifugal moments vanish and

$$S_{21} = C_{21} = S_{22} = 0. \tag{15}$$

And of course it is still true that for a fluid planet at equilibrium the density must be independent of longitude and symmetrical about the equator and therefore

$$D_{2n+1} = J_{2n+1} = 0, \quad \forall n. \tag{16}$$

Finally, expressing the centrifugal potential (3) in terms of P_2:

$$Q(\mathbf{r}) = \frac{1}{3}\omega^2 r^2 \big[1 - P_2(\cos\theta)\big] \tag{17}$$

and using the small parameter q (4) the total external potential is

$$V_e(\mathbf{r}) = \frac{GM}{r}\left[1 - (a/r)^2 J_2P_2 - (a/r)^4 J_4P_4 - (a/r)^6 J_6P_6 - \cdots + \frac{1}{3}(r/a)^3\big(1 - P_2(\cos\theta)\big)\, q\right]. \tag{18}$$

D. A constant density spheroid

In the special case where $\rho = \text{const.}$ the integrals in the definitions of the gravity coefficients (11) are greatly simplified. In fact for this case there is a closed analytic solution showing that the equilibrium surface is an ellipsoid with ellipticity related to the dimensionless rotation parameter

$$m = \frac{3\omega^2}{4\pi G\rho}. \tag{19}$$

We are not interested so much in the closed form solution itself but rather in the simplified form of the gravity coefficients.

For the potential on the surface of a constant-density spheroid we can write eqs. (11) in polar coordinates:

$$D_n = \rho \int_\tau (r')^n P_n(\cos\theta')\, d\tau' = 2\pi\rho \int_0^\pi d\theta' \int_0^{r(\theta')} (r')^n P_n(\cos\theta')(r')^2 \sin\theta'\, dr' \tag{20}$$

and moving to the variable $\mu = \cos\theta$ we have

$$D_n = \frac{2\pi\rho}{n+3} \int_{-1}^1 d\mu'\, P_n(\mu') r(\mu')^{n+3}. \tag{21}$$

Note on convergence.

Moving now to the non-dimensional radius $\xi(\mu) = r(\mu)/a$ and remembering that only even degree coefficients contribute we can write

$$Ma^n J_n = -D_n = -\frac{4\pi\rho a^{n+3}}{n+3} \int_0^1 d\mu'\, P_n(\mu')\xi(\mu')^{n+3}. \tag{22}$$

And finally, remembering that

$$M = \frac{4\pi\rho a^3}{3} \int_0^1 d\mu'\, \xi(\mu')^3, \tag{23}$$

we have the general expression for J_n:

$$J_n = -\frac{3}{n+3} \frac{\int_0^1 d\mu'\, P_n(\mu')\xi(\mu')^{n+3}}{\int_0^1 d\mu'\, \xi(\mu')^3}. \tag{24}$$

We can now write an implicit equation for $r(\mu)$ by requiring it to be a level surface, i.e., to be an curve of constant *total* potential, including the centrifugal term. The total potential at a point on the surface is (eq. (18))

$$U(r,\mu) = \frac{GM}{r}\Big[1 - \sum_{k=1}^{\infty} \left(\frac{a}{r}\right)^{2k} J_{2k} P_{2k}(\mu)\Big] + \frac{1}{3} r^2 \omega^2 [1 - P_2(\mu)]. \tag{25}$$

We require that the potential at any point on the surface equal the potential on the equator:

$$U(a,0) = \frac{GM}{a}\Big[1 - \sum_{k=1}^{\infty} J_{2k} P_{2k}(0)\Big] + \frac{1}{2} a^2 \omega^2. \tag{26}$$

In non-dimensional form the implicit equation for the equilibrium surface is

$$\frac{1}{\xi}\Big[1 - \sum_{k=1}^{\infty} \xi^{-2k} J_{2k} P_{2k}(\mu)\Big] + \frac{q}{3}\xi^2 \big[1 - P_2(\mu)\big] - \frac{q}{2} - 1 + \sum_{k=1}^{\infty} J_{2k} P_{2k}(0) = 0. \tag{27}$$

We have two sets of coupled equations. Equation (27) for the shape of the surface given the gravity coefficients J_n, and equations (24) for the gravity coefficients given the level surface $\xi(\mu)$. We proceed with an iterative solution. Given a value of q and initial guesses for J_n and/or ξ we numerically integrate

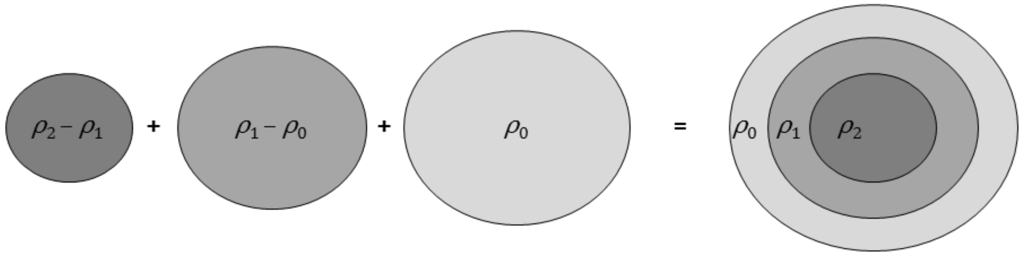

FIG. 1 Superposition of constant-density spheroids has the same gravitational potential as a layered spheroid.

eqs. (24). When the integration routine requires the value of ξ at some point μ' we obtain it by numerically solving eq. (27), using the current values J_n. The result is an updated value of J_n. We repeat this procedure until all J_n values converge to the chosen tolerance, usually to machine precision, at which point we have our self consistent solution: a constant-density spheroid in hydrostatic equilibrium.

A side note on practical implementation. The numerical integration of (24) can be done in any way we prefer but in practice it is best to use a scheme that evaluates the integrand at a constant number of fixed points. Gauss-Legendre integration works well because $\xi(\mu)$ is expected to be a low-order polynomial, and a relatively manageable number of abscissas, μ_α, $\alpha = 1, 2, \ldots, L$ are needed to yield precise results.

E. Concentric Maclaurin spheroids

The constant-density spheroid is not of great interest by itself but it can be used in the calculation of the potential of a fluid planet with any density profile. The fundamental idea is that, by the principle of superposition, the potential of a layered spheroid with N layers of density $\rho_0, \rho_1, \ldots, \rho_{N-1}$ is equal to the algebraic sum of potentials of N constant-density spheroids with densities $\rho_0, (\rho_1 - \rho_0), \ldots, (\rho_{N-1} - \rho_{N-2})$. Figure 1 illustrates this idea. We look for the total potential of the system of concentric, constant-density (Maclaurin) spheroids. As before the surface of each spheroid must be a level surface of (total) potential and this will provide us with implicit equations for the shape functions. The point of this is, of course, that we can use as many layers/concentric spheroids as we like and thus approximate an arbitrary density profile to any desired accuracy.

1. Definitions

The spheroids are labeled $i = 0, \ldots, N-1$ with the largest, corresponding to the outermost layer in the layered planet, labeled $i = 0$. The equatorial radii are labeled a_i, such that $a_0 > a_1 > \cdots > a_{N-1}$. We define $\delta\rho_i = \rho_i - \rho_{i-1}$ for $i > 0$ and $\delta\rho_0 = \rho_0$. The surface of spheroid i is given by the function $r_i(\mu = \cos\theta)$ and we still assume a fluid planet at equilibrium with the associated symmetry. We will often use dimensionless shape functions $\xi_i = r_i/a_0$.

2. The starting point

The starting point is equations (11) reproduced here for convenience:

$$V(r,\mu) = \frac{G}{r}\sum_{k=0}^{\infty}\left(r^{-2k}D_{2k} + r^{2k+1}D'_{2k}\right)P_{2k}(\mu) \tag{28a}$$

with

$$D_k = \int_{r'<r} \rho(\mathbf{r}')(r')^k P_k(\mu')\,d\tau', \tag{28b}$$

$$D'_k = \int_{r'>r} \rho(\mathbf{r}')(r')^{-k-1} P_k(\mu')\,d\tau'. \tag{28c}$$

3. The measurable J_n coefficients are the sum of $J_{i,n}$

By the principal of superposition the total gravitational potential at a point (r, μ) **external** to the planet is

$$V_{\rm ext} = \frac{G}{r}\left(\sum_{k=0}^{\infty} D_{0,2k} r^{-2k} P_{2k}(\mu) + \sum_{k=0}^{\infty} D_{1,2k} r^{-2k} P_{2k}(\mu) + \cdots + \sum_{k=0}^{\infty} D_{N-1,2k} r^{-2k} P_{2k}(\mu)\right) \tag{29}$$

with

$$D_{i,2k} = \frac{2\pi\delta\rho_i}{2k+3}\int_{-1}^{1} d\mu'\, P_{2k}(\mu') r_i(\mu')^{2k+3} \tag{30}$$

(see eq.(20)). Notice also that

$$\sum_{i=0}^{N-1} D_{i,0} = \sum_{i=0}^{N-1} \frac{2\pi\delta\rho_i}{3}\int_{-1}^{1} d\mu'\, r_i(\mu')^3 = M. \tag{31}$$

The dimensionless coefficients are all normalized by the same radius

$$M a_0^{2k} J_{i,2k} = -D_{i,2k} \tag{32}$$

so that the external potential can be written as

$$V_{\rm ext}(r,\mu) = \frac{GM}{r}\left(1 - \sum_{i=0}^{N-1}\sum_{k=1}^{\infty} J_{i,2k}(r/a_0)^{-2k} P_{2k}(\mu)\right). \tag{33}$$

Assuming for now that the infinite series on k converges we can change the order of summation and write

$$V_{\rm ext}(r,\mu) = \frac{GM}{r}\left(1 - \sum_{k=1}^{\infty}\left(\sum_{i=0}^{N-1} J_{i,2k}\right)(r/a_0)^{-2k} P_{2k}(\mu)\right), \tag{34}$$

which shows that the coefficients J_n measurable by spacecraft are the simple algebraic sum of the individual $J_{i,2k}$, given by

$$J_{i,2k} = -\left(\frac{3}{2k+3}\right)\frac{\delta\rho_i \int_{-1}^{1} d\mu'\, P_{2k}(\mu')\xi_i(\mu')^{2k+3}}{\sum_{j=0}^{N-1}\delta\rho_j \int_{-1}^{1} d\mu'\, \xi_j(\mu')^3}. \tag{35}$$

This is the equivalent to eq. (24) for a single Maclaurin spheroid.

As before, we will use an iterative process. We will numerically evaluate the integrals in eq.(35). The shape functions of the surfaces of the spheroids, $\xi_i(\mu')$ in the integrands, will be numerically solved using current values of $J_{i,2k}$, and the process will continue until the coefficients $J_{i,2k}$ for all layers remain unchanged through successive iterations. However, unlike in the case of a single spheroid, the $J_{i,2k}$ are not sufficient to calculate the shape functions of interior spheroids. We need an expression for the total potential at an interior point on the boundary between spheroid i and spheroid $i-1$.

4. The potential at an interior point

Consider a point $B = (r_j(\mu), \mu)$ on the level surface of spheroid j. The contribution to the potential from any spheroid $i \geq j$ is the external potential:

$$V_{i\geq j}(r_j(\mu),\mu) = \frac{G}{r_j}\sum_{k=0}^{\infty} D_{i,2k} r_j(\mu)^{-2k} P_{2k}(\mu). \tag{36}$$

The contribution from a spheroid $i < j$ has two parts. The external part, from the sphere of radius r_j inside spheroid i:

$$V_{i<j,\rm ext}(r_j,\mu) = \frac{G}{r_j}\frac{4\pi}{3}\delta\rho_i r_j(\mu)^3. \tag{37}$$

And the internal part, from the oblate region in spheroid i where $r' > r_j$:

$$V_{i<j,\mathrm{int}}(r_j,\mu) = \frac{G}{r_j(\mu)} \sum_{k=0}^{\infty} r_j(\mu)^{2k+1} P_{2k}(\mu)\, 2\pi\, \delta\rho_i \int_{-1}^{1} d\mu'\, P_{2k}(\mu') \int_{r_j(\mu)}^{r_i(\mu')} dr'\, (r')^{-2k-1} (r')^2. \tag{38}$$

(Note typo in eq. (15) of (Hubbard, 2013).)

The rest is just algebra but here are a few points to keep in mind while we work. First, since we now have r' to a negative power in the inner integrand we have to do the $k=1$ term, the integral of $(r')^{-1}$, separately. Second, the lower bound of the inner integral is independent or μ'. This means that the outer integral will have a term that is simply a constant times $P_{2k}(\mu')$. Recall that the orthogonality of Legendre polynomials requires that $\int_{-1}^{1} P_{n>0}(x)\, dx = 0$ so that this term is identically zero except when $k=0$. So we will also need to do the $k=0$ term separately. Third, we will lump an r_j^2 term together with the contribution from $V_{i<j,\mathrm{ext}}$. Also we will use the oblate symmetry to take the integrals over μ from 0 to 1. Here we go.

$$\begin{aligned} V_{i<j,\mathrm{int}}(r_j,\mu) = 2\pi G\delta\rho_i \int_0^1 d\mu'\, [r_i(\mu')^2 - r_j(\mu)^2] + \\ 4\pi G\delta\rho_i\, r_j(\mu)^2 P_2(\mu) \int_0^1 d\mu'\, P_2(\mu')[\ln r_i(\mu') - \ln r_j(\mu) + \\ 2\pi G\delta\rho_i \sum_{k=2}^{\infty} r_j(\mu)^{2k} P_{2k}(\mu)\, \frac{2}{2-2k} \int_0^1 d\mu'\, P_{2k}(\mu')[r_i(\mu')^{2-2k} - r_j(\mu)^{2-2k}] = \\ 2\pi G\delta\rho_i \int_0^1 d\mu'\, r_i(\mu')^2 - 2\pi G\delta\rho_i r_j(\mu)^2 + \\ 4\pi G\delta\rho_i\, r_j(\mu)^2 P_2(\mu) \int_0^1 d\mu'\, P_2(\mu') \ln r_i(\mu') + \\ 2\pi G\delta\rho_i \sum_{k=2}^{\infty} r_j(\mu) P_{2k}(\mu)\, \frac{2}{2-2k} \int_0^1 d\mu'\, P_{2k}(\mu') r_i(\mu')^{2-2k}. \end{aligned} \tag{39}$$

And combined with the contribution $V_{i<j,\mathrm{ext}}$ we have

$$\begin{aligned} V_{i<j} = -\frac{2\pi G\delta\rho_i}{3} r_j(\mu)^2 + 2\pi G\delta\rho_i \int_0^1 d\mu'\, r_i(\mu')^2 + \\ 4\pi G\delta\rho_i\, r_j(\mu)^2 P_2\mu \int_0^1 d\mu'\, P_2(\mu') \ln r_i(\mu') + \\ 2\pi G\delta\rho_i \sum_{k=2}^{\infty} r_j(\mu)^{2k} P_{2k}(\mu)\, \frac{2}{2-2k} \int_0^1 d\mu'\, P_{2k}(\mu') r_i(\mu')^{2-2k}. \end{aligned} \tag{40}$$

We can write the potential as a sum of powers of r_j with D-like coefficients so that it looks more like eq. (36):

$$V_{i<j}(r_j(\mu),\mu) = G \sum_{k=0}^{\infty} D'_{i,2k} r_j(\mu)^{2k} P_{2k}(\mu) + G\, D''_{i,0}\, r_j(\mu)^2, \tag{41}$$

with

$$D'_{i,2k} = \frac{4\pi}{2-2k} \delta\rho_i \int_0^1 d\mu'\, P_{2k}(\mu') r_i(\mu')^{2-2k}, \qquad k \neq 1, \tag{42a}$$

$$D'_{i,2} = 4\pi\delta\rho_i \int_0^1 d\mu'\, P_2(\mu') \ln r_i(\mu'), \qquad k = 1, \tag{42b}$$

and

$$D''_{i,0} = -\frac{2\pi}{3} \delta\rho_i. \tag{42c}$$

For the non-dimensional coefficients let

$$Ma_0^{2k} J_{i,2k} = -D_{i,2k}, \tag{43a}$$

$$Ma_0^{-(2k+1)} J'_{i,2k} = -D'_{i,2k}, \tag{43b}$$

and

$$Ma_0^{-3} J''_{i,0} = -D''_{i,0}, \tag{43c}$$

and eqs. (36) and (41) become

$$V_{i \geq j} = -\frac{GM}{r_j} \sum_{k=0}^{\infty} J_{i,2k} \left(\frac{r_j}{a_0}\right)^{-2k} P_{2k}(\mu) \tag{44}$$

and

$$V_{i<j} = -\frac{GM}{r_j} \left[\sum_{k=0}^{\infty} J'_{i,2k} \left(\frac{r_j}{a_0}\right)^{2k+1} P_{2k}(\mu) + J''_{i,0} \left(\frac{r_j}{a_0}\right)^3 \right]. \tag{45}$$

And finally, we put it all together. Using non-dimensional coefficients and the normalized radius $\xi = r/a_0$, the total gravitational potential at point $B = (\xi_j(\mu), \mu)$ on the equilibrium surface of spheroid j summing contributions from all spheroids is

$$\begin{aligned} V(\xi_j, \mu) = -\frac{GM}{a_0} \frac{1}{\xi_j(\mu)} \Bigg[& \sum_{i=j}^{N-1} \sum_{k=0}^{\infty} J_{i,2k} \xi_j(\mu)^{-2k} P_{2k}(\mu) + \\ & \sum_{i=0}^{j-1} \sum_{k=0}^{\infty} J'_{i,2k} \xi_j(\mu)^{2k+1} P_{2k}(\mu) + \sum_{i=0}^{j-1} J''_{i,0} \xi_j(\mu)^3 \Bigg]. \end{aligned} \tag{46}$$

With the coefficients

$$J_{i,2k} = -\left(\frac{3}{2k+3}\right) \frac{\delta\rho_i \int_0^1 d\mu' \, P_{2k}(\mu') \xi_i(\mu')^{2k+3}}{\sum_{m=0}^{N-1} \delta\rho_m \int_0^1 d\mu' \, \xi_m(\mu')^3}, \tag{47a}$$

$$J'_{i,2k} = -\left(\frac{3}{2-2k}\right) \frac{\delta\rho_i \int_0^1 d\mu' \, P_{2k}(\mu') \xi_i(\mu')^{2-2k}}{\sum_{m=0}^{N-1} \delta\rho_m \int_0^1 d\mu' \, \xi_m(\mu')^3}, \qquad k \neq 1, \tag{47b}$$

$$J'_{i,2} = -3 \, \frac{\delta\rho_i \int_0^1 d\mu' \, P_2(\mu') \ln \xi_i(\mu')}{\sum_{m=0}^{N-1} \delta\rho_m \int_0^1 d\mu' \, \xi_m(\mu')^3}, \qquad k = 1, \tag{47c}$$

$$J''_{i,0} = \frac{1}{2} \, \frac{\delta\rho_i}{\sum_{m=0}^{N-1} \delta\rho_m \int_0^1 d\mu' \, \xi_m(\mu')^3}. \tag{47d}$$

II. PRACTICE (TO BE CONTINUED)

References

G.K. Batchelor. *An introduction to fluid dynamics.* Cambridge Univ. Press, Cambridge, UK, 1967.

W. B. Hubbard. High-precision MacLaurin-Based models of rotating liquid planets. *The Astrophysical Journal Letters*, 756:L15, 2012. ISSN 0004-637X. doi: 10.1088/0004-637X/768/1/43.

W. B. Hubbard. Concentric Maclaurin Spheroid Models of Rotating Liquid Planets. *The Astrophysical Journal*, 768(1):43, 2013. ISSN 0004-637X. doi: 10.1088/0004-637X/768/1/43.

Vladimir N Zharkov and V P Trubitsyn. *Physics of Planetary Interiors.* Tucson, AZ, 1978.