Concentric Maclaurin Spheroids: theory and practice

Summary of the theory and practice of modeling rotating fluid planets by Hubbard's Concentric Maclaurin Spheroids technique. These notes provide the mathematical basis for using gravity measurements to learn about planetary interiors. Mathematical statements are checked to my satisfaction unless otherwise noted (important exception is the addition theorem). Primary references are (Zharkov and Trubitsyn, 1978) for the mathematical foundation, and (Hubbard, 2013) and Bill's personal notes for the CMS theory.

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I. DEFINITIONS AND NOTATION

Given a density distribution $\rho(\mathbf{r}')$ inside the planet, the *total potential* is defined as

$$U(\mathbf{r}) = V(\mathbf{r}) + Q(\mathbf{r}) \tag{1}$$

where

$$V(\mathbf{r}) = G\int \rho(\mathbf{r}')/\left|\mathbf{r}-\mathbf{r}'\right|\, d\tau' \tag{2}$$

is the gravitational potential and

$$Q(\mathbf{r}) = \frac{1}{2}\omega^2 r^2 \sin^2\theta \tag{3}$$

is the centrifugal potential; we will use $d\tau$ for a volume element, ω is the angular rotation velocity (assumed constant and usually about a principal axis), $r = |\mathbf{r}|$, and θ is the angle from the rotation axis. *Note that Hubbard and Zharkov and Trubitsyn use an unusual positive potential presumably for algebraic convenience. This also means accelerations are given by the positive gradient of potential.*

When the planet is in hydrostatic equilibrium the level surfaces of potential are also level surfaces of pressure and of density. Including the free surface where the pressure $p = 0$. This can be shown rigorously (e.g. Batchelor, 1967) and is also fairly intuitive. Finding the shape of the planet is therefore reduced to the problem of finding the level surfaces $U(\mathbf{r}) =$ constant. This problem is easy to state but hard to solve, essentially because the volume of space over which the integral above is taken is unknown and must be found as part of the self-consistent solution.

Equilibrium figures which differ only slightly from spheres are called *spheroids*. A dimensionless parameter describing the importance of rotation is

$$q = \frac{\omega^2 a^3}{GM} \tag{4}$$

where M is the planet's mass and a is the *equatorial radius*.

II. DECOMPOSITION OF POTENTIAL INTO SPHERICAL HARMONICS

The expression for the gravitational potential, eq. (2) can be written as a sum of powers of r using the decomposition of $1/|\mathbf{r}-\mathbf{r}'|$ in Legendre polynomials (see LEGENDRE.PDF):

$$\begin{aligned}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r\sqrt{1-2t(r'/r)+(r'/r)^2}} = \\ &= \begin{cases}\frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(t), & r>r', \\ \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^{-n-1} P_n(t), & r<r',\end{cases}\end{aligned} \tag{5}$$

where γ is the angle between the radius vectors $\mathbf{r}$ and $\mathbf{r}'$ and $t=\cos\gamma$. If $r>r'$ for all points where $\rho(\mathbf{r}')>0$ (i.e. inside the planet) then the potential is called *external*. The Legendre polynomials are given by Rodrigues's formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{6}$$

(although this is not the easiest way to obtain them explicitly) for $x\in[-1,1]$ and luckily there is no confusion about normalization. It will be useful to remember that

$$P_0 = 1, \qquad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}$$

and that $P_n(1)=1$. The gravitational potential in terms of Legendre polynomials is

$$V(\mathbf{r}) = \frac{G}{r}\sum_{n=0}^{\infty}\int \rho(\mathbf{r}')P_n(t)(r'/r)^k\, d\tau' \tag{7}$$

where

$$k = \begin{cases} n, & r>r', \\ -(n+1), & r<r',\end{cases}$$

and the integration is over the (as yet unknown) volume of the planet.

The expansion in Legendre polynomials is compact and neat but it is of little utility because it does not separate terms arising from the mass distribution from those arising from the location where the potential is to be evaluated. In spherical polar coordinates the variable

$$t = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

mixes the colatitude θ and longitude φ of the integration variable and the point of measurement. The salvation comes from the *addition theorem for spherical harmonics* (**which I can't derive**):

$$P_n(\cos\psi) = P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos\big[m(\varphi-\varphi')\big], \tag{8}$$

with the associated Legendre functions

$$P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x). \tag{9}$$

Unfortunately there is not a universal consensus on the exact form of the P_n^m functions. If a different normalization is used then the following expressions will all be somewhat different. With the aid of the addition theorem we can decompose the potential as[1]

$$
\begin{aligned}
V(\mathbf{r}) = \frac{G}{r}\Bigg(&\sum_{n=0}^{\infty} P_n(\cos\theta)\int_\tau \rho(\mathbf{r}')P_n(\cos\theta')(r'/r)^k\,d\tau' \\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{n} P_n^m(\cos\theta)\cos(m\varphi)\int_\tau \frac{2(n-m)!}{(n+m)!}\rho(\mathbf{r}')P_n^m(\cos\theta')\cos(m\varphi')\left(\frac{r'}{r}\right)^k d\tau' \\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{n} P_n^m(\cos\theta)\sin(m\varphi)\int_\tau \frac{2(n-m)!}{(n+m)!}\rho(\mathbf{r}')P_n^m(\cos\theta')\sin(m\varphi')\left(\frac{r'}{r}\right)^k d\tau'\Bigg)
\end{aligned} \tag{10}
$$

with k as before.

The above expansion is general, not requiring hydrostatic equilibrium or principal axis rotation. If the planet is fluid then at equilibrium the rotation will always be about a principal axis. If we take the polar axis (call it the z axis) to coincide with the rotation axis then symmetry requires that $\rho(\mathbf{r})$ and therefor $V(\mathbf{r})$ cannot depend on longitude φ and must include only even powers of $\cos\theta$ (for symmetry about the equator). In this case we write a simpler expansion involving only ordinary Legendre polynomials of only even degree:

$$
V(r,\theta) = \frac{G}{r}\sum_{n=0}^{\infty}\left(r^{-2n}D_{2n} + r^{2n+1}D'_{2n}\right)P_{2n}(\cos\theta) \tag{11a}
$$

with

$$
D_n = \int_{r'<r} \rho(\mathbf{r}')(r')^n P_n(\cos\theta')\,d\tau', \tag{11b}
$$

$$
D'_n = \int_{r'>r} \rho(\mathbf{r}')(r')^{-n-1} P_n(\cos\theta')\,d\tau'. \tag{11c}
$$

The coefficients D_n are usually replaced with the non-dimensional coefficients $J_n = D_n/(Ma^n)$.

III. THE EXTERNAL POTENTIAL

If the potential is to be evaluated at a point exterior to the surface of the planet then $r > r'$ for all differential volume elements in the integral expressions above. The general form eq. (10) can be rearranged slightly and rewritten in a form more convenient for comparison with measured quantities:

$$
\begin{aligned}
V_e = \frac{GM}{r}\Bigg(1 - &\sum_{n=1}^{\infty}(a/r)^n J_n P_n(\cos\theta) + \\
&+\sum_{n=1}^{\infty}\sum_{m=1}^{n}(a/r)^n P_n(\cos\theta)\Big[C_{nm}\cos(m\varphi) + S_{nm}\sin(m\varphi)\Big]\Bigg),
\end{aligned} \tag{12a}
$$

with the coefficients

$$
Ma^n J_n = -\int \rho(\mathbf{r}')(r')^n P_n(\cos\theta')\,d\tau' = -D_n, \tag{12b}
$$

$$
Ma^n C_{nm} = \frac{2(n-m)!}{(n+m)!}\int \rho(\mathbf{r}')(r')^n P_n^m(\cos\theta')\cos(m\varphi')\,d\tau', \tag{12c}
$$

$$
Ma^n S_{nm} = \frac{2(n-m)!}{(n+m)!}\int \rho(\mathbf{r}')(r')^n P_n^m(\cos\theta')\sin(m\varphi')\,d\tau'. \tag{12d}
$$

(Remember $M = \int \rho\,d\tau'$ is the planet's mass and a is the equatorial radius.) Different normalizations of the associated Legendre functions are sometimes used leading to expansions with the same form but

[1] Note typo in Z&T.

different meaning of the J_n, C_{nm}, and S_{nm} coefficients. There is no easy way to guard against errors or guess which normalization was used if it is not explicitly given. The decomposition can also be carried out with the complex form of the Legendre functions leading to even more confusion.

A natural choice of reference frame can eliminate many of the coefficients in the expansion (12). First, if the origin of coordinates is chosen at the center of mass of the planet, $\mathbf{R} = (X, Y, Z)$ then

$$-MaJ_1 = \int z'\,dm' = MZ = 0,$$
$$MaC_{11} = \int x'\,dm' = MX = 0, \text{ and}$$
$$MaS_{11} = \int y'\,dm' = MY = 0$$

so that J_1, C_{11}, and S_{11} vanish. Next, we can relate the expansion coefficients to the moments of inertia. We designate those:

$$B = \int_\tau \rho(\mathbf{r}')(x'^2 + z'^2)\,d\tau', \tag{13a}$$
$$A = \int_\tau \rho(\mathbf{r}')(y'^2 + z'^2)\,d\tau', \tag{13b}$$
$$C = \int_\tau \rho(\mathbf{r}')(x'^2 + y'^2)\,d\tau', \tag{13c}$$

for the principal moments and

$$D = \int_\tau \rho(\mathbf{r}')\,y'z'\,d\tau', \tag{13d}$$
$$E = \int_\tau \rho(\mathbf{r}')\,x'z'\,d\tau', \tag{13e}$$
$$F = \int_\tau \rho(\mathbf{r}')\,x'y'\,d\tau', \tag{13f}$$

for the diagonal, so-called *products of inertia*, also called *centrifugal moments*. The relations with the expansion coefficients in eq. (12) become clear by writing out the degree 2 Legendre functions in Cartesian coordinates. It is easy to show by direct comparison that

$$-a^2MJ_2 = \frac{A+B}{2} - C, \quad \text{and} \quad a^2MC_{22} = \frac{B-A}{4}, \tag{14a}$$

and also that

$$D = a^2MS_{21}, \quad E = a^2MC_{21}, \quad \text{and} \quad F = 2a^2MS_{22}. \tag{14b}$$

Usually $A = B < C$ so that $a^2MJ_2 = (C - A)$. Also, if we align the coordinate axes with the planet's principal axes of inertia then the centrifugal moments vanish and

$$S_{21} = C_{21} = S_{22} = 0. \tag{15}$$

And of course it is still true that for a fluid planet at equilibrium the density must be independent of longitude and symmetrical about the equator and therefore

$$D_{2n+1} = J_{2n+1} = 0, \quad \forall n. \tag{16}$$

Finally, expressing the centrifugal potential (3) in terms of P_2:

$$Q(\mathbf{r}) = \frac{1}{3}\omega^2 r^2 \left[1 - P_2(\cos\theta)\right] \tag{17}$$

and using the small parameter q (4) the total external potential is

$$V_e(\mathbf{r}) = \frac{GM}{r}\left[1 - (a/r)^2 J_2 P_2 - (a/r)^4 J_4 P_4 - (a/r)^6 J_6 P_6 - \cdots + \frac{1}{3}(r/a)^3\big(1 - P_2(\cos\theta)\big)\,q\right]. \tag{18}$$

IV. A CONSTANT DENSITY SPHEROID

In the special case where $\rho = \text{const.}$ the integrals in the definitions of the gravity coefficients (11) are greatly simplified. In fact this case there is a closed analytic solution showing that the equilibrium surface is an ellipsoid with ellipticity related to the dimensionless rotation parameter

$$m = \frac{3\omega^2}{4\pi G\rho}. \tag{19}$$

We are not interested so much in the closed form solution itself but rather in the simplified form of the gravity coefficients.

For the external potential of a rotating fluid in equilibrium we can write eqs. (12) in polar coordinates:

$$D_n = \rho \int_\tau (r')^n P_n(\cos\theta')\, d\tau' = 2\pi\rho \int_0^\pi d\theta' \int_0^{r(\theta')} (r')^n P_n(\cos\theta')(r')^2 \sin\theta'\, dr' \tag{20}$$

and moving to the variable $\mu = \cos\theta$ we have

$$D_n = \frac{2\pi\rho}{n+3} \int_{-1}^{1} d\mu'\, P_n(\mu') r(\mu')^{n+3}. \tag{21}$$

Moving now to the non-dimensional radius $\xi(\mu) = r(\mu)/a$ and remembering that only even degree coefficients contribute we can write

$$M a^n J_n = -D_n = -\frac{4\pi\rho a^{n+3}}{n+3} \int_0^1 d\mu'\, P_n(\mu') \xi(\mu')^{n+3}. \tag{22}$$

And finally, remembering that

$$M = \frac{4\pi\rho a^3}{3} \int_0^1 d\mu'\, \xi(\mu')^3, \tag{23}$$

we have the general expression for J_n:

$$J_n = -\frac{3}{n+3} \frac{\int_0^1 d\mu'\, P_n(\mu')\xi(\mu')^{n+3}}{\int_0^1 d\mu'\, \xi(\mu')^3}. \tag{24}$$

We can now write an implicit equation for $r(\mu)$ by requiring it to be a level surface, i.e. to be an curve of constant potential. That is, constant *total* potential, including the centrifugal term. The total potential at a point on the surface is (eq. (18))

$$U(r,\mu) = \frac{GM}{r}\Big[1 - \sum_{k=1}^{\infty} \Big(\frac{a}{r}\Big)^{2k} J_{2k} P_{2k}(\mu)\Big] + \frac{1}{3} r^2 \omega^2 [1 - P_2(\mu)]. \tag{25}$$

We require that the potential at any point on the surface equal the potential on the equator:

$$U(a,0) = \frac{GM}{a}\Big[1 - \sum_{k=1}^{\infty} J_{2k} P_{2k}(0)\Big] + \frac{1}{2} a^2 \omega^2. \tag{26}$$

In non-dimensional form the implicit equation for the equilibrium surface is

$$\frac{1}{\xi}\Big[1 - \sum_{k=1}^{\infty} \xi^{-2k} J_{2k} P_{2k}(\mu)\Big] + \frac{q}{3}\xi^2 \big[1 - P_2(\mu)\big] - \frac{q}{2} - 1 + \sum_{k=1}^{\infty} J_{2k} P_{2k}(0) = 0. \tag{27}$$

We have two sets of coupled equations. Equation (27) for the shape of the surface given the gravity coefficients J_n, and equations (24) for the gravity coefficients given the level surface $\xi(\mu)$. We proceed with an iterative solution. Given a value of q and initial guesses for J_n and/or ξ we numerically integrate eqs. (24). When the integration routine requires the value of ξ at some point μ' we obtain it by numerically solving eq. (27), using the current values J_n. This results in updated values of J_n. We iterate this procedure until all J_n values converge to the chosen tolerance – usually to machine precision!

References

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