Concentric Maclaurin Spheroids: theory and practice

Summary of the theory and practice of modeling rotating fluid planets by Hubbard's Concentric Maclaurin Spheroids technique. These notes provide the mathematical basis for using gravity measurements to learn about planetary interiors. Mathematical statements are checked to my satisfaction unless otherwise noted (important exception is the addition theorem). Primary references are (Zharkov and Trubitsyn, 1978) for the mathematical foundation, and (Hubbard, 2012, 2013) and Bill's personal notes for the CMS theory.

Contents

I. THEORY

A. Definitions and notation

Given a density distribution $\rho(\mathbf{r}')$ inside the planet, the *total potential* is defined as

$$U(\mathbf{r}) = V(\mathbf{r}) + Q(\mathbf{r}) \tag{1}$$

where

$$V(\mathbf{r}) = G\int \rho(\mathbf{r}')/\left|\mathbf{r}-\mathbf{r}'\right|\, d\tau' \tag{2}$$

is the gravitational potential and

$$Q(\mathbf{r}) = \frac{1}{2}\omega^2 r^2 \sin^2\theta \tag{3}$$

is the centrifugal potential; we will use $d\tau$ for a volume element, ω is the angular rotation velocity (assumed constant and usually about a principal axis), $r = |\mathbf{r}|$, and θ is the angle from the rotation axis. *Note that Hubbard and Zharkov and Trubitsyn use an unusual positive potential presumably for algebraic convenience. This also means accelerations are given by the positive gradient of potential.*

When the planet is in hydrostatic equilibrium the level surfaces of potential are also level surfaces of pressure and of density. Including the free surface where the pressure $p = 0$. This can be shown rigorously (e.g. Batchelor, 1967) and is also fairly intuitive. Finding the shape of the planet is therefore reduced to the problem of finding the level surfaces $U(\mathbf{r}) = \text{constant}$. This problem is easy to state but hard to solve, essentially because the volume of space over which the integral above is taken is unknown and must be found as part of the self-consistent solution.

Equilibrium figures which differ only slightly from spheres are called *spheroids*. A dimensionless parameter describing the importance of rotation is

$$q = \frac{\omega^2 a^3}{GM} \tag{4}$$

where M is the planet's mass and a is the *equatorial radius.*

B. Decomposition of potential into spherical harmonics

The expression for the gravitational potential, eq. (2) can be written as a sum of powers of r using the decomposition of $1/|\mathbf{r}-\mathbf{r}'|$ in Legendre polynomials (see LEGENDRE.PDF):

$$\begin{aligned}\frac{1}{|\mathbf{r}-\mathbf{r}'|} &= \frac{1}{r\sqrt{1-2t(r'/r)+(r'/r)^2}} = \\ &= \begin{cases}\frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(t), & r>r',\\ \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^{-n-1}P_n(t), & r<r',\end{cases}\end{aligned} \tag{5}$$

where γ is the angle between the radius vectors $\mathbf{r}$ and $\mathbf{r}'$ and $t=\cos\gamma$. If $r>r'$ for all points where $\rho(\mathbf{r}')>0$ (i.e. inside the planet) then the potential is called *external.* The Legendre polynomials are given by Rodrigues's formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{6}$$

(although this is not the easiest way to obtain them explicitly) for $x\in[-1,1]$ and luckily there is no confusion about normalization. It will be useful to remember that

$$P_0 = 1, \qquad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}, \qquad \int_{-1}^{1} P_{n>0}(t')\,dt' = 0.$$

and that $P_n(1)=1$. The gravitational potential in terms of Legendre polynomials is

$$V(\mathbf{r}) = \frac{G}{r}\sum_{n=0}^{\infty}\int \rho(\mathbf{r}')P_n(t)(r'/r)^{\alpha}\,d\tau' \tag{7}$$

where

$$\alpha = \begin{cases} n, & r>r',\\ -(n+1), & r<r',\end{cases}$$

and the integration is over the (as yet unknown) volume of the planet.

The expansion in Legendre polynomials is compact and neat but it is of little utility because it does not separate terms arising from the mass distribution from those arising from the location where the potential is to be evaluated. In spherical polar coordinates the variable

$$t = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

mixes the colatitude θ and longitude φ of the integration variable and the point of measurement. The salvation comes from the *addition theorem for spherical harmonics* (**which I can't derive**):

$$P_n(\cos\psi) = P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos\big[m(\varphi-\varphi')\big], \tag{8}$$

with the associated Legendre functions

$$P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x). \tag{9}$$

Unfortunately there is not a universal consensus on the exact form of the P_n^m functions. If a different normalization is used then the following expressions will all be somewhat different. With the aid of the addition theorem we can decompose the potential as[1]

$$V(\mathbf{r}) = \frac{G}{r}\Bigg(\sum_{n=0}^{\infty} P_n(\cos\theta)\int_\tau \rho(\mathbf{r}')P_n(\cos\theta')(r'/r)^\alpha\,d\tau' + \sum_{n=1}^{\infty}\sum_{m=1}^{n} P_n^m(\cos\theta)\cos(m\varphi)\int_\tau \frac{2(n-m)!}{(n+m)!}\rho(\mathbf{r}')P_n^m(\cos\theta')\cos(m\varphi')\Big(\frac{r'}{r}\Big)^\alpha\,d\tau' + \sum_{n=1}^{\infty}\sum_{m=1}^{n} P_n^m(\cos\theta)\sin(m\varphi)\int_\tau \frac{2(n-m)!}{(n+m)!}\rho(\mathbf{r}')P_n^m(\cos\theta')\sin(m\varphi')\Big(\frac{r'}{r}\Big)^\alpha\,d\tau'\Bigg) \tag{10}$$

with α as before.

The above expansion is general, not requiring hydrostatic equilibrium or principal axis rotation. If the planet is fluid then at equilibrium the rotation will always be about a principal axis. If we take the polar axis (call it the z axis) to coincide with the rotation axis then symmetry requires that $\rho(\mathbf{r})$ and therefor $V(\mathbf{r})$ cannot depend on longitude φ and must include only even powers of $\cos\theta$ (for symmetry about the equator). In this case we write a simpler expansion involving only ordinary Legendre polynomials of only even degree:

$$V(r,\theta) = \frac{G}{r}\sum_{n=0}^{\infty}\left(r^{-2n}D_{2n} + r^{2n+1}D'_{2n}\right)P_{2n}(\cos\theta) \tag{11a}$$

with

$$D_n = \int_{r'<r} \rho(\mathbf{r}')(r')^n P_n(\cos\theta')\,d\tau', \tag{11b}$$

$$D'_n = \int_{r'>r} \rho(\mathbf{r}')(r')^{-n-1} P_n(\cos\theta')\,d\tau'. \tag{11c}$$

The coefficients D_n are usually replaced with the non-dimensional coefficients $J_n = D_n/(Ma^n)$.

C. The external potential

If the potential is to be evaluated at a point away from the surface of the planet then $r > r'$ for all differential volume elements in the integral expressions above. The general form eq. (10) can be rearranged slightly and rewritten in a form more convenient for comparison with measured quantities:

$$V_e = \frac{GM}{r}\Bigg(1 - \sum_{n=1}^{\infty}(a/r)^n J_n P_n(\cos\theta) + {} + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(a/r)^n P_n(\cos\theta)\Big[C_{nm}\cos(m\varphi) + S_{nm}\sin(m\varphi)\Big]\Bigg), \tag{12a}$$

with the coefficients

$$Ma^n J_n = -\int \rho(\mathbf{r}')(r')^n P_n(\cos\theta')\,d\tau' = -D_n, \tag{12b}$$

$$Ma^n C_{nm} = \frac{2(n-m)!}{(n+m)!}\int \rho(\mathbf{r}')(r')^n P_n^m(\cos\theta')\cos(m\varphi')\,d\tau', \tag{12c}$$

$$Ma^n S_{nm} = \frac{2(n-m)!}{(n+m)!}\int \rho(\mathbf{r}')(r')^n P_n^m(\cos\theta')\sin(m\varphi')\,d\tau'. \tag{12d}$$

(Remember $M = \int \rho\,d\tau'$ is the planet's mass and a is the equatorial radius.) Different normalizations of the associated Legendre functions are sometimes used leading to expansions with the same form but

[1] Note typo in Z&T.

different meaning of the J_n, C_{nm}, and S_{nm} coefficients. There is no easy way to guard against errors or guess which normalization was used if it is not explicitly given. The decomposition can also be carried out with the complex form of the Legendre functions leading to even more confusion.

A natural choice of reference frame can eliminate many of the coefficients in the expansion (12). First, if the origin of coordinates is chosen at the center of mass of the planet, $\mathbf{R} = (X, Y, Z)$ then

$$-MaJ_1 = \int z'\,dm' = MZ = 0,$$
$$MaC_{11} = \int x'\,dm' = MX = 0, \text{ and}$$
$$MaS_{11} = \int y'\,dm' = MY = 0$$

so that J_1, C_{11}, and S_{11} vanish. Next, we can relate the expansion coefficients to the moments of inertia. We designate those:

$$B = \int_\tau \rho(\mathbf{r}')(x'^2 + z'^2)\,d\tau', \tag{13a}$$
$$A = \int_\tau \rho(\mathbf{r}')(y'^2 + z'^2)\,d\tau', \tag{13b}$$
$$C = \int_\tau \rho(\mathbf{r}')(x'^2 + y'^2)\,d\tau', \tag{13c}$$

for the principal moments and

$$D = \int_\tau \rho(\mathbf{r}')\,y'z'\,d\tau', \tag{13d}$$
$$E = \int_\tau \rho(\mathbf{r}')\,x'z'\,d\tau', \tag{13e}$$
$$F = \int_\tau \rho(\mathbf{r}')\,x'y'\,d\tau', \tag{13f}$$

for the diagonal, so-called *products of inertia*, also called *centrifugal moments*. The relations with the expansion coefficients in eq. (12) become clear by writing out the degree 2 Legendre functions in Cartesian coordinates. It is easy to show by direct comparison that

$$-a^2MJ_2 = \frac{A+B}{2} - C, \quad \text{and} \quad a^2MC_{22} = \frac{B-A}{4}, \tag{14a}$$

and also that

$$D = a^2MS_{21}, \quad E = a^2MC_{21}, \quad \text{and} \quad F = 2a^2MS_{22}. \tag{14b}$$

Usually $A = B < C$ so that $a^2MJ_2 = (C - A)$. Also, if we align the coordinate axes with the planet's principal axes of inertia then the centrifugal moments vanish and

$$S_{21} = C_{21} = S_{22} = 0. \tag{15}$$

And of course it is still true that for a fluid planet at equilibrium the density must be independent of longitude and symmetrical about the equator and therefore

$$D_{2n+1} = J_{2n+1} = 0, \quad \forall n. \tag{16}$$

Finally, expressing the centrifugal potential (3) in terms of P_2:

$$Q(\mathbf{r}) = \frac{1}{3}\omega^2 r^2 \left[1 - P_2(\cos\theta)\right] \tag{17}$$

and using the small parameter q (4) the total external potential is

$$V_e(\mathbf{r}) = \frac{GM}{r}\left[1 - (a/r)^2 J_2 P_2 - (a/r)^4 J_4 P_4 - (a/r)^6 J_6 P_6 - \cdots + \frac{1}{3}(r/a)^3 \big(1 - P_2(\cos\theta)\big)\, q\right]. \tag{18}$$

D. A constant density spheroid

In the special case where $\rho = \text{const.}$ the integrals in the definitions of the gravity coefficients (11) are greatly simplified. In fact for this case there is a closed analytic solution showing that the equilibrium surface is an ellipsoid with ellipticity related to the dimensionless rotation parameter

$$m = \frac{3\omega^2}{4\pi G\rho}. \tag{19}$$

We are not interested so much in the closed form solution itself but rather in the simplified form of the gravity coefficients.

For the potential on the surface of a constant-density spheroid we can write eqs. (11) in polar coordinates:

$$D_n = \rho \int_\tau (r')^n P_n(\cos\theta')\, d\tau' = 2\pi\rho \int_0^\pi d\theta' \int_0^{r(\theta')} (r')^n P_n(\cos\theta')(r')^2 \sin\theta'\, dr' \tag{20}$$

and moving to the variable $\mu = \cos\theta$ we have

$$D_n = \frac{2\pi\rho}{n+3} \int_{-1}^{1} d\mu'\, P_n(\mu') r(\mu')^{n+3}. \tag{21}$$

(Important note on convergence. Using the D_n coefficients when the integration volume is the whole interior of the planet has been the subject of much debate. The problem is that our starting poing is an expansion in powers of (r'/r) and we use this expansion in regions where the series diverges. The claim is that after integration the series converge unconditionally due to perfect cancellation of the diverging terms. But I was not able to follow this argument or even determine whether the proof is general or applies only to ellipsoids. Zharkov and Trubitsyn (1978) claim that this expansion is entirely justified, and Hubbard et al. (2014) claims this is true but only if q is sufficiently small. Hubbard also provides some numerical tests that show this expansion gives the same result as the more complicated expansion using both D_n and D'_n (e.g. Kong et al., 2013). I will leave this wrinkle aside for now and hopefully come back to it some day.)

Moving now to the non-dimensional radius $\xi(\mu) = r(\mu)/a$ and remembering that only even degree coefficients contribute we can write

$$Ma^n J_n = -D_n = -\frac{4\pi\rho a^{n+3}}{n+3} \int_0^1 d\mu'\, P_n(\mu')\xi(\mu')^{n+3}. \tag{22}$$

And finally, remembering that

$$M = \frac{4\pi\rho a^3}{3} \int_0^1 d\mu'\, \xi(\mu')^3, \tag{23}$$

we have the general expression for J_n:

$$J_n = -\frac{3}{n+3} \frac{\int_0^1 d\mu'\, P_n(\mu')\xi(\mu')^{n+3}}{\int_0^1 d\mu'\, \xi(\mu')^3}. \tag{24}$$

We can now write an implicit equation for $r(\mu)$ by requiring it to be a level surface, i.e., to be an curve of constant *total* potential, including the centrifugal term. The total potential at a point on the surface is (eq. (18))

$$U(r,\mu) = \frac{GM}{r}\left[1 - \sum_{k=1}^{\infty} \left(\frac{a}{r}\right)^{2k} J_{2k} P_{2k}(\mu)\right] + \frac{1}{3} r^2\omega^2 [1 - P_2(\mu)]. \tag{25}$$

We require that the potential at any point on the surface equal the potential on the equator:

$$U(a,0) = \frac{GM}{a}\left[1 - \sum_{k=1}^{\infty} J_{2k} P_{2k}(0)\right] + \frac{1}{2} a^2\omega^2. \tag{26}$$

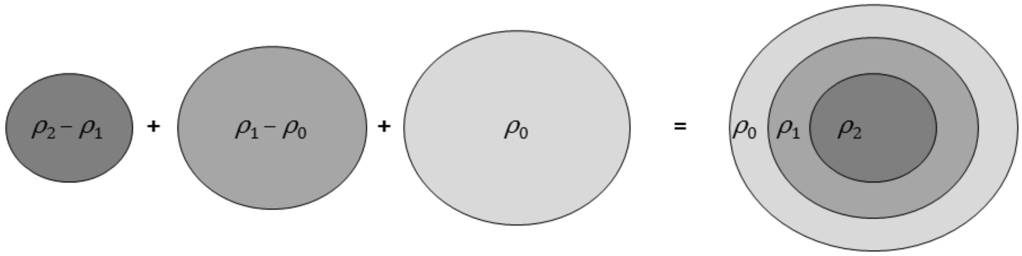

FIG. 1 Superposition of constant-density spheroids has the same gravitational potential as a layered spheroid.

In non-dimensional form the implicit equation for the equilibrium surface is

$$\frac{1}{\xi}\Big[1-\sum_{k=1}^{\infty}\xi^{-2k}J_{2k}P_{2k}(\mu)\Big]+\frac{q}{3}\xi^2\big[1-P_2(\mu)\big]-\frac{q}{2}-1+\sum_{k=1}^{\infty}J_{2k}P_{2k}(0)=0. \tag{27}$$

We have two sets of coupled equations. Equation (27) for the shape of the surface given the gravity coefficients J_n, and equations (24) for the gravity coefficients given the level surface $\xi(\mu)$. We proceed with an iterative solution. Given a value of q and initial guesses for J_n and/or ξ we numerically integrate eqs. (24). When the integration routine requires the value of ξ at some point μ' we obtain it by numerically solving eq. (27), using the current values J_n. The result is an updated value of J_n. We repeat this procedure until all J_n values converge to the chosen tolerance, usually to machine precision, at which point we have our self consistent solution: a constant-density spheroid in hydrostatic equilibrium.

A side note on practical implementation. The numerical integration of (24) can be done in any way we prefer but in practice it is best to use a scheme that evaluates the integrand at a constant number of fixed points. Gauss-Legendre integration works well because $\xi(\mu)$ is expected to be a low-order polynomial, and a relatively manageable number of abscissas, μ_α, $\alpha = 1, 2, \ldots, L$ are needed to yield precise results.

E. Concentric Maclaurin spheroids

The constant-density spheroid is not of great interest by itself but it can be used in the calculation of the potential of a fluid planet with any density profile. The fundamental idea is that, by the principle of superposition, the potential of a layered spheroid with N layers of density $\rho_0, \rho_1, \ldots, \rho_{N-1}$ is equal to the algebraic sum of potentials of N constant-density spheroids with densities $\rho_0, (\rho_1{-}\rho_0), \ldots, (\rho_{N-1}{-}\rho_{N-2})$. Figure 1 illustrates this idea. We look for the total potential of the system of concentric, constant-density (Maclaurin) spheroids. As before the surface of each spheroid must be a level surface of (total) potential and this will provide us with implicit equations for the shape functions. The point of this is, of course, that we can use as many layers/concentric spheroids as we like and thus approximate an arbitrary density profile to any desired accuracy.

1. Definitions

The spheroids are labeled $i = 0, \ldots, N-1$ with the largest, corresponding to the outermost layer in the layered planet, labeled $i = 0$. The equatorial radii are labeled a_i, such that $a_0 > a_1 > \cdots > a_{N-1}$. We define $\delta\rho_i = \rho_i - \rho_{i-1}$ for $i > 0$ and $\delta\rho_0 = \rho_0$. The surface of spheroid i is given by the function $r_i(\mu = \cos\theta)$ and we still assume a fluid planet at equilibrium with the associated symmetry. We will often use dimensionless shape functions $\xi_i = r_i/a_0$.

2. The starting point

The starting point is equations (11) reproduced here for convenience:

$$V(r,\mu) = \frac{G}{r}\sum_{k=0}^{\infty}\left(r^{-2k}D_{2k} + r^{2k+1}D'_{2k}\right)P_{2k}(\mu) \tag{28a}$$

with

$$D_k = \int_{r'<r} \rho(\mathbf{r}')(r')^k P_k(\mu')\,d\tau', \tag{28b}$$

$$D'_k = \int_{r'>r} \rho(\mathbf{r}')(r')^{-k-1} P_k(\mu')\,d\tau'. \tag{28c}$$

3. The measurable J_n coefficients are the sum of $J_{i,n}$

By the principal of superposition the total gravitational potential at a point (r,μ) away from the planet (so $r' < r$) is

$$V_{\text{ext}} = \frac{G}{r}\left(\sum_{k=0}^{\infty} D_{0,2k}r^{-2k}P_{2k}(\mu) + \sum_{k=0}^{\infty} D_{1,2k}r^{-2k}P_{2k}(\mu) + \cdots + \sum_{k=0}^{\infty} D_{N-1,2k}r^{-2k}P_{2k}(\mu)\right) \tag{29}$$

with

$$D_{i,2k} = \frac{2\pi\delta\rho_i}{2k+3}\int_{-1}^{1} d\mu'\,P_{2k}(\mu')r_i(\mu')^{2k+3} \tag{30}$$

(see eq.(20)). Notice also that

$$\sum_{i=0}^{N-1} D_{i,0} = \sum_{i=0}^{N-1}\frac{2\pi\delta\rho_i}{3}\int_{-1}^{1} d\mu'\,r_i(\mu')^3 = M. \tag{31}$$

The dimensionless coefficients are all normalized by the same radius

$$Ma_0^{2k}J_{i,2k} = -D_{i,2k} \tag{32}$$

so that the external potential can be written as

$$V_{\text{ext}}(r,\mu) = \frac{GM}{r}\left(1 - \sum_{i=0}^{N-1}\sum_{k=1}^{\infty} J_{i,2k}(r/a_0)^{-2k}P_{2k}(\mu)\right). \tag{33}$$

Since the infinite series on k converges[2] we can change the order of summation and write

$$V_{\text{ext}}(r,\mu) = \frac{GM}{r}\left(1 - \sum_{k=1}^{\infty}\left(\sum_{i=0}^{N-1} J_{i,2k}\right)(r/a_0)^{-2k}P_{2k}(\mu)\right), \tag{34}$$

which shows that the coefficients J_n measurable by spacecraft are the simple algebraic sum of the individual $J_{i,2k}$, given by

$$J_{i,2k} = -\left(\frac{3}{2k+3}\right)\frac{\delta\rho_i\int_{-1}^{1} d\mu'\,P_{2k}(\mu')\xi_i(\mu')^{2k+3}}{\sum_{j=0}^{N-1}\delta\rho_j\int_{-1}^{1} d\mu'\,\xi_j(\mu')^3}. \tag{35}$$

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[2] Away from the planet the convergence of the Legendre expansion in r'/r is not in question, so there is no problem with the claim that the measured J_n are the sum of layer $J_{i,n}$. But for the actual calculation of $J_{i,n}$ by eq. (35) we need the stronger claim that the Legendre expansion in D is valid also on the surface of the spheroid.

This is the equivalent to eq. (24) for a single Maclaurin spheroid.

As before, we will use an iterative process. We will numerically evaluate the integrals in eq.(35). The shape functions of the surfaces of the spheroids, $\xi_i(\mu')$ in the integrands, will be numerically solved using current values of $J_{i,2k}$, and the process will continue until the coefficients $J_{i,2k}$ for all layers remain unchanged through successive iterations. However, unlike in the case of a single spheroid, the $J_{i,2k}$ are not sufficient to calculate the shape functions of interior spheroids. We need an expression for the total potential at an interior point on the boundary between spheroid i and spheroid $i-1$.

4. The potential at an interior point

Consider a point $B = (r_j(\mu), \mu)$ on the level surface of spheroid j. The contribution to the potential from any spheroid $i \geq j$ is the external potential:

$$V_{i \geq j}(r_j(\mu), \mu) = \frac{G}{r_j} \sum_{k=0}^{\infty} D_{i,2k} r_j(\mu)^{-2k} P_{2k}(\mu). \tag{36}$$

The contribution from a spheroid $i < j$ has two parts. The external part, from the sphere of radius r_j inside spheroid i:

$$V_{i<j,\mathrm{ext}}(r_j, \mu) = \frac{G}{r_j} \frac{4\pi}{3} \delta\rho_i r_j(\mu)^3. \tag{37}$$

And the internal part, from the oblate region in spheroid i where $r' > r_j$:

$$V_{i<j,\mathrm{int}}(r_j, \mu) = \frac{G}{r_j(\mu)} \sum_{k=0}^{\infty} r_j(\mu)^{2k+1} P_{2k}(\mu)\, 2\pi\, \delta\rho_i \int_{-1}^{1} d\mu'\, P_{2k}(\mu') \int_{r_j(\mu)}^{r_i(\mu')} dr'\, (r')^{-2k-1} (r')^2. \tag{38}$$

(Note typo in eq. (15) of (Hubbard, 2013).)

The rest is just algebra but here are a few points to keep in mind while we work. First, since we now have r' to a negative power in the inner integrand we have to do the $k=1$ term, the integral of $(r')^{-1}$, separately. Second, the lower bound of the inner integral is independent or μ'. This means that the outer integral will have a term that is simply a constant times $P_{2k}(\mu')$. Recall that the orthogonality of Legendre polynomials requires that $\int_{-1}^{1} P_{n>0}(x)\, dx = 0$ so that this term is identically zero except when $k=0$. So we will also need to do the $k=0$ term separately. Third, we will lump an r_j^2 term together with the contribution from $V_{i<j,\mathrm{ext}}$. Also we will use the oblate symmetry to take the integrals over μ from 0 to 1. Here we go.

$$\begin{aligned}
V_{i<j,\mathrm{int}}(r_j, \mu) = 2\pi G \delta\rho_i \int_0^1 d\mu' \,[r_i(\mu')^2 - r_j(\mu)^2] + \\
4\pi G \delta\rho_i\, r_j(\mu)^2 P_2(\mu) \int_0^1 d\mu'\, P_2(\mu')[\ln r_i(\mu') - \ln r_j(\mu) + \\
2\pi G \delta\rho_i \sum_{k=2}^{\infty} r_j(\mu)^{2k} P_{2k}(\mu) \frac{2}{2-2k} \int_0^1 d\mu'\, P_{2k}(\mu')[r_i(\mu')^{2-2k} - r_j(\mu)^{2-2k}] \;= \\
2\pi G \delta\rho_i \int_0^1 d\mu'\, r_i(\mu')^2 - 2\pi G \delta\rho_i r_j(\mu)^2 + \\
4\pi G \delta\rho_i\, r_j(\mu)^2 P_2(\mu) \int_0^1 d\mu'\, P_2(\mu') \ln r_i(\mu') + \\
2\pi G \delta\rho_i \sum_{k=2}^{\infty} r_j(\mu) P_{2k}(\mu) \frac{2}{2-2k} \int_0^1 d\mu'\, P_{2k}(\mu') r_i(\mu')^{2-2k}.
\end{aligned} \tag{39}$$

And combined with the contribution $V_{i<j,\mathrm{ext}}$ we have

$$\begin{aligned} V_{i<j} = -\frac{2\pi G\delta\rho_i}{3} r_j(\mu)^2 + 2\pi G\delta\rho_i \int_0^1 d\mu'\, r_i(\mu')^2 + \\ 4\pi G\delta\rho_i\, r_j(\mu)^2 P_{2\mu} \int_0^1 d\mu'\, P_2(\mu') \ln r_i(\mu') + \\ 2\pi G\delta\rho_i \sum_{k=2}^{\infty} r_j(\mu)^{2k} P_{2k}(\mu)\, \frac{2}{2-2k} \int_0^1 d\mu'\, P_{2k}(\mu') r_i(\mu')^{2-2k}. \end{aligned} \tag{40}$$

We can write the potential as a sum of powers of r_j with D-like coefficients so that it looks more like eq. (36):

$$V_{i<j}(r_j(\mu), \mu) = G \sum_{k=0}^{\infty} D'_{i,2k} r_j(\mu)^{2k} P_{2k}(\mu) + G\, D''_{i,0}\, r_j(\mu)^2, \tag{41}$$

with

$$D'_{i,2k} = \frac{4\pi}{2-2k} \delta\rho_i \int_0^1 d\mu'\, P_{2k}(\mu') r_i(\mu')^{2-2k}, \qquad k \neq 1, \tag{42a}$$

$$D'_{i,2} = 4\pi\delta\rho_i \int_0^1 d\mu'\, P_2(\mu') \ln r_i(\mu'), \qquad k = 1, \tag{42b}$$

and

$$D''_{i,0} = -\frac{2\pi}{3} \delta\rho_i. \tag{42c}$$

For the non-dimensional coefficients let

$$M a_0^{2k} J_{i,2k} = -D_{i,2k}, \tag{43a}$$

$$M a_0^{-(2k+1)} J'_{i,2k} = -D'_{i,2k}, \tag{43b}$$

and

$$M a_0^{-3} J''_{i,0} = -D''_{i,0}, \tag{43c}$$

and eqs. (36) and (41) become

$$V_{i\geq j} = -\frac{GM}{r_j} \sum_{k=0}^{\infty} J_{i,2k} \left(\frac{r_j}{a_0}\right)^{-2k} P_{2k}(\mu) \tag{44}$$

and

$$V_{i<j} = -\frac{GM}{r_j} \left[\sum_{k=0}^{\infty} J'_{i,2k} \left(\frac{r_j}{a_0}\right)^{2k+1} P_{2k}(\mu) + J''_{i,0} \left(\frac{r_j}{a_0}\right)^3 \right]. \tag{45}$$

And finally, we put it all together. Using non-dimensional coefficients and the normalized radius $\xi = r/a_0$, the total gravitational potential at point $B = (\xi_j(\mu), \mu)$ on the equilibrium surface of spheroid j summing contributions from all spheroids is

$$\begin{aligned} V(\xi_j, \mu) = -\frac{GM}{a_0} \frac{1}{\xi_j(\mu)} \Bigg[\sum_{i=j}^{N-1} \sum_{k=0}^{\infty} J_{i,2k} \xi_j(\mu)^{-2k} P_{2k}(\mu) + \\ \sum_{i=0}^{j-1} \sum_{k=0}^{\infty} J'_{i,2k} \xi_j(\mu)^{2k+1} P_{2k}(\mu) + \sum_{i=0}^{j-1} J''_{i,0} \xi_j(\mu)^3 \Bigg]. \end{aligned} \tag{46}$$

With the coefficients

$$J_{i,2k} = -\left(\frac{3}{2k+3}\right)\frac{\delta\rho_i \int_0^1 d\mu'\, P_{2k}(\mu')\xi_i(\mu')^{2k+3}}{\sum_{m=0}^{N-1}\delta\rho_m \int_0^1 d\mu'\,\xi_m(\mu')^3}, \tag{47a}$$

$$J'_{i,2k} = -\left(\frac{3}{2-2k}\right)\frac{\delta\rho_i \int_0^1 d\mu'\, P_{2k}(\mu')\xi_i(\mu')^{2-2k}}{\sum_{m=0}^{N-1}\delta\rho_m \int_0^1 d\mu'\,\xi_m(\mu')^3}, \qquad k \neq 1, \tag{47b}$$

$$J'_{i,2} = -3\,\frac{\delta\rho_i \int_0^1 d\mu'\, P_2(\mu')\ln\xi_i(\mu')}{\sum_{m=0}^{N-1}\delta\rho_m \int_0^1 d\mu'\,\xi_m(\mu')^3}, \qquad k = 1, \tag{47c}$$

$$J''_{i,0} = \frac{1}{2}\,\frac{\delta\rho_i}{\sum_{m=0}^{N-1}\delta\rho_m \int_0^1 d\mu'\,\xi_m(\mu')^3}. \tag{47d}$$

(For the surface, $j = 0$ layer the second double sum in (46) is dropped.)

5. A self-consistent interior model

With eq. (46) at hand we can construct a self-consistent shape for an array of N layers with densities ρ_i and equatorial radii a_i, given a rotation parameter q. We use an iterative method similar to the one in sec. I.D. Start with an initial guess for the values of all J-like coefficients in eqs. (47). The simplest guess is to use values corresponding to a perfectly spherical planet: $J_{i,0} = -(\rho_i - \rho_{i-1})/\rho_{N-1}$, $J'_{i,0} = (3/2)J_{i,0}$, $J''_{i,0} = -(1/2)J_{i,0}$, and all other coefficients equal to zero. We then reevaluate all the coefficients by numerically integrating eqs. (47). To evaluate the integrands we need the shape functions, $\xi_i(\mu')$. We obtain those using the *current* values of the J-coefficients. The implicit equations defining the shape functions are obtained by equating the *total* potential (including a centrifugal term) at $(\xi_i(\mu'), \mu')$ to the value computed at $(a_i/a_0, 0)$. An implicit equation is solved, numerically, for every point where an integrand is evaluated in each of eqs. (47) so this will be a computationally expensive operation. The result of an iteration is a new set of values for J coefficients. The next iteration will use these new values when computing the shape functions, and will therefore return slightly different updated values for Js. Iterations continue until successive updates no longer modify any J above the specified tolerance.

The resulting converged model is a self-consistent mass distribution. In other words, the shape of the layered spheroid is such that gravitational accelerations balance centrifugal accelerations everywhere. However the mass distribution is probably not consistent with a realistic equation-of-state. Indeed up to now the composition of the planet was not even mentioned. To obtain a useful interior model of a fluid planet we employ a second level of iterations. Computing the pressure inside layers of constant density (by integrating, numerically, from the surface down) we compare the value to the pressure predicted by a chosen barotrope meant to represent the planet's composition, and adjust the layer densities accordingly. We then run the first level iterations again to converge again to a self-consistent mass distribution, and repeat the process until successive iterations no longer modify the layer densities above a specified tolerance.

In principle this completes the description of the concentric Maclaurin spheroid technique for modeling a rotating fluid. However there are many details that are useful to consider when implementing the method in practice, and some that are necessary. These are the subject of part II.

II. PRACTICE

As always, when we begin to implement a theoretical model as an actual computer program we discover many small details that are not particularly important or interesting for a general understanding of the theory but are nonetheless necessary to consider for a practical implementation. I describe those here using the notation of Hubbard (2013) which I am hoping will become standard notation for CMS models. I also mention some details that are particular to my MATLAB implementation, set apart like this.

1. Truncating infinite series

Look at eq. (46). Obviously we must truncate the series at some $k = k_{\max}$. But how to choose $k_{\max}$ for the desired accuracy? It's easier to experiment than it is to analyze, and

it turns out that truncating at $n = 2k_{\mathbf{max}} = 30$ leaves a remainder of $\sim 10^{-12}$.

2. Rescaling

Suppose we want to construct a modest CMS model with $N = 100$ layers and $k_{\max} = 15$. Look at eq. (46), for the $j = 100$ layer. The $k = 10$ term in the first double sum contains a product of $J_{100,20}$ and the huge number $\xi_{100}(\mu)^{-20} \sim (10^{-2})^{-20}$. But look now at eqs. (47). The integrand in the expression for $J_{100,20}$ is on the order $\sim(10^{-2})^{23}$, more than enough to subdue the huge value of ξ^{-2k} and make the contribution to the potential from the degree 20 moment of the smallest layer negligible, as expected. Similarly, when we come to calculate, say, $J'_{100,20}$ the integrand is huge, of order $\sim(10^{-2})^{-18}$, and the corresponding term in eq. (46) is a product of this huge number with a factor of order $\sim(10^{-2})^{21}$, so that the contribution to the potential is again small, as expected. This is fine in principle, but in practice in order to preserve numerical accuracy it is important to avoid multiplying and dividing by huge numbers. We can do this by rescaling the dimensionless multipole moments and associated equations.

Define $\lambda_i \equiv (a_i/a_0)$ and $\zeta_i \equiv \xi_i/\lambda_i = r_i/a_i$. I call $\zeta_i(\mu)$ the *renormalized shape functions*. Notice that $\zeta_i(\mu') \lesssim 1$, always. In the definition of the $J'_{100,20}$, for example, the large numerical value can be pulled out of the integral:

$$J'_{i,2k} = -\left(\frac{3}{2-2k}\right)\frac{\delta\rho_i\lambda_i^{2-2k}\int_0^1 d\mu'\,P_{2k}(\mu')\zeta_i(\mu')^{2-2k}}{\sum_{m=0}^{N-1}\delta\rho_m\lambda_m^3\int_0^1 d\mu'\,\zeta_m(\mu')^3}, \qquad k\neq 1,$$

the expected huge value now coming from λ_i^{2-2k}. We can prevent multiplication by this huge value by rescaling the mulipole moment itself. Defining $\widetilde{J}'_{i,2k} = J'_{i,2k}\lambda_i^{2k+1}$, the potentially huge number is deferred, we shall see, to the expression for the potential where, we shall see, it is subdued by an even larger number, avoiding the expected loss of accuracy. Similarly, the $J_{i,2k}$ moments are rescaled to $\widetilde{J}_{i,2k} = J_{i,2k}/\lambda^{2k}$ to match the change of variables from ξ to ζ. And although the rescaling is not necessary for numerical reason in the case of $J''_{i,0}$ we do not want to maintain two sets of variables so we perform the same change of variables and rescaling moments by $\widetilde{J}''_{i,0} = J''_{i,0}\lambda_i^3$. Eqs. (47) are then replaced by eqs. (48), involving no numerically huge numbers:

$$\widetilde{J}_{i,2k} = -\left(\frac{3}{2k+3}\right)\frac{\delta\rho_i\lambda_i^3\int_0^1 d\mu'\,P_{2k}(\mu')\zeta_i(\mu')^{2k+3}}{\sum_{m=0}^{N-1}\delta\rho_m\lambda_m^3\int_0^1 d\mu'\,\zeta_m(\mu')^3}, \tag{48a}$$

$$\widetilde{J}'_{i,2k} = -\left(\frac{3}{2-2k}\right)\frac{\delta\rho_i\lambda_i^3\int_0^1 d\mu'\,P_{2k}(\mu')\zeta_i(\mu')^{2-2k}}{\sum_{m=0}^{N-1}\delta\rho_m\lambda_m^3\int_0^1 d\mu'\,\zeta_m(\mu')^3}, \qquad k\neq 1, \tag{48b}$$

$$\widetilde{J}'_{i,2} = -3\,\frac{\delta\rho_i\lambda_i^3\int_0^1 d\mu'\,P_2(\mu')\ln\zeta_i(\mu')}{\sum_{m=0}^{N-1}\delta\rho_m\lambda_m^3\int_0^1 d\mu'\,\zeta_m(\mu')^3}, \qquad k=1, \tag{48c}$$

$$\widetilde{J}''_{i,0} = \frac{1}{2}\,\frac{\delta\rho_i\lambda_i^3}{\sum_{m=0}^{N-1}\delta\rho_m\lambda_m^3\int_0^1 d\mu'\,\zeta_m(\mu')^3}. \tag{48d}$$

(To understand the $k = 1$ term recall that $\int_0^1 P_2(\mu')\,d\mu' = 0$.)

Now, in terms of the renormalized shape functions and the rescaled multipole moments eq. (46) is replaced by:

$$\begin{aligned} V_{\mathrm{pu}}(\zeta_j,\mu) = \frac{a_0}{GM}V(\zeta_j,\mu) = &\\ -\frac{1}{\lambda_j\zeta_j(\mu)}\Bigg[\sum_{i=j}^{N-1}\sum_{k=0}^{k_{\max}}\left(\frac{\lambda_i}{\lambda_j}\right)^{2k}\widetilde{J}_{i,2k}\zeta_j(\mu)^{-2k}P_{2k}(\mu) + &\\ \sum_{i=0}^{j-1}\sum_{k=0}^{k_{\max}}\left(\frac{\lambda_j}{\lambda_i}\right)^{2k+1}\widetilde{J}'_{i,2k}\zeta_j(\mu)^{2k+1}P_{2k}(\mu) + \sum_{i=0}^{j-1}\left(\frac{\lambda_j}{\lambda_i}\right)^3\widetilde{J}''_{i,0}\zeta_j(\mu)^3\Bigg]. \end{aligned} \tag{49}$$

Notice how the ratios of λs in the double sums are always less than or equal to one, and the ζ shape functions also have values close to but less than one, as long as the oblateness is not extreme. In this form it is easy to see that the expansions in multipole moments, with expressions (48) for the coefficients, are expected to converge rapidly, allowing us to choose a a sensible k_{max}. We also introduced the non-dimensional potential V_{pu}. The subscript stands for planetary units.

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