**Concentric Maclaurin Spheroid theory: N-layer model**

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Because the gravitational potential is linear in the density , we may use a principle of superposition, such that the total potential at any point in space is the sum of the partial potentials produced by concentric constant-density spheroids. Figure 1 illustrates this concept for a three-layer model.

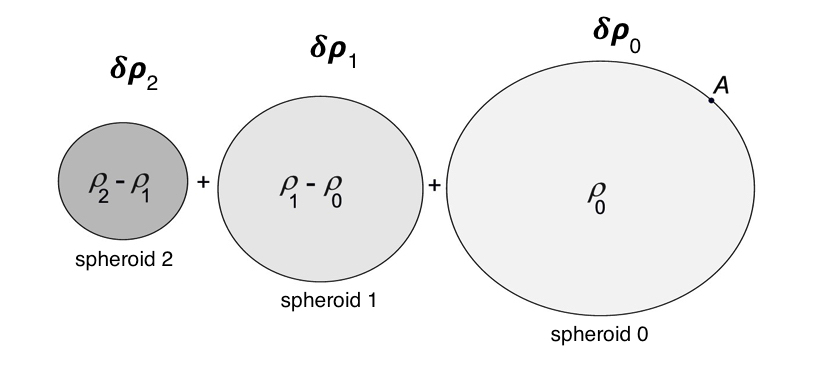


Fig. 1 – Method of superposition of Maclaurin spheroids, for the case . The innermost spheroid has a density , the middle spheroid has a density , and the outer spheroid has a density . We define for , and .

When the spheroids are stacked (i.e., made concentric), the total densities become The point “A” is a typical point on the outermost level surface.

Let the equatorial radius of the outermost spheroid be , and let the equatorial radii of the nested spheroids be . The total gravitational potential at some point “A” on the outermost level surface is

where

etc., where the relation is the surface equipotential of the *i*-th layer. The zero-degree values are given by

and so we have

Dimensionless forms:

Let

and

The total gravitational potential at surface point “A” can thus be rewritten

or, rearranging the order of summation,

where

The external potential measured at an arbitrary point on the planet’s surface can be simply written as

where

The total value of a given (measurable in principle by a spacecraft) is just a linear superposition of the contribution from each of the spheroids. We can use expression (3) to exhibit the relative contributions (weight functions) for the zonal harmonics.

The CMS method proceeds by iteration. We determine the shapes of the equipotential surfaces of the spheroids, , and then we use those shapes to calculate the harmonic coefficients for each layer. The resulting equations for the surface shapes are then fed into the next iteration. However, the are not sufficient to calculate the equipotential shapes of interior spheroids. If there are no interior spheroids because we have only a single Maclaurin spheroid, they would be sufficient. To see what else we need, let us compute the total gravitational potential at an arbitrary point “B” on the interface (level surface) between spheroid and spheroid

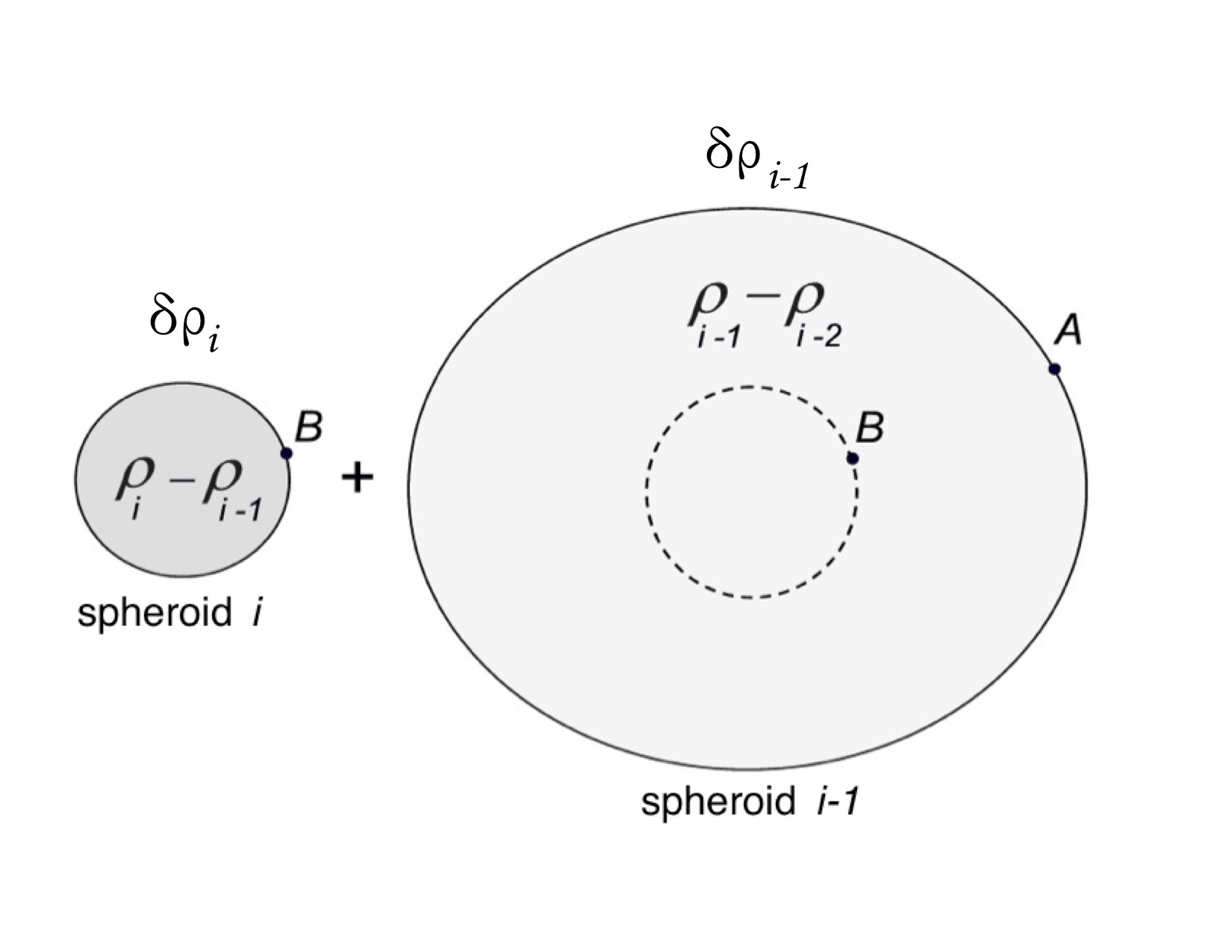


Fig. 2 – Schematic diagram illustrating the computation of three contributions to the gravitational potential at point “B” located at on an interior interface. B lies on the oblate surface of the left-hand spheroid of density . B also lies on a spherical surface of radius in the right-hand spheroid of density .

The three contributions to the potential at “B” are:

1. , the external potential due to the mass (with constant density ) within the spherical region (dashed circle in Fig. 2);
2. , the internal potential due to the mass (with constant density ) external to the spherical region (dashed circle in Fig. 2);
3. , the external potential at point “B” due to the mass distribution with constant density (left-hand shaded distribution in Fig. 2):

We easily calculate contribution (1). We give this potential the subscript because it is one of the three contributions to the potential at the surface of spheroid :

Contribution (2) is given by:

Note: comparing this with Eq. (15) of ApJ 768:43 (2013), the above expression (4) includes a factor which is missing from Eq. (15). Check:

dimensions for Eq. (15) of the *ApJ* paper:

which is wrong.

Dimensions for Eq. (4):

which is correct. So there is a typo in Eq. (15) of the *ApJ* paper. The correct form is given by expression (4).

Separating out the term in Eq. (4), we rewrite contribution (2) as

Contribution (3) at point “B” due to the mass distribution with constant density (left-hand shaded distribution in Fig. 2) is given by:

where

When we add the three contributions together, we find that both and contain terms that vary as with no dependence. We lump these terms together and give the coefficient a double-prime superscript for convenience, as follows.

Sum of the three contributions:

or

or

or

where

as before,

[converting to dimensionless variables:

so

to be evaluated on the level surface ]

for , and

for

for

and

Dimensionless forms:

Let

So

for , and

for

for

and

so

To get the total potential on the th level surface, we must sum the additional contributions from spheroids external to this surface (indices less than ) and interior to this surface (indices greater than ). We get

in agreement with Eq. (29) of the *ApJ* paper.

The potential at the center of the spheroid array is given by

Note: comparing this with Eq. (49) of ApJ 768:43 (2013), the above expression (8) differs. Only terms with with survive in the limit .

So there is a typo in Eq. (49) of the *ApJ* paper. The correct form is given by expression (8), or by setting in Eq. (49) of the *ApJ* paper.

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Dimensionless forms:

Let

So

for , and

for

for

and

so

Thus,

*Potential at center:*

Parameterizations:

Define the mean density (can compute initial approximation from input parameters by setting ):

Final value comes from converged model.

Now we set up parameters for the nested spheroids.

Let

The could be equally spaced between 0 and 1, or made more dense where needed.

Next, we specify the density step at each layer:

The units are unimportant since everything gets renormalized in the end. After we have a converged model, we require:

For the single Maclaurin spheroid , there is only one interface, the outer surface . The iterative approach given by Hubbard 2012, ApJL, 756, L15 works like this. Guess initial values for . Use these values to solve for the function on the Gaussian quadrature points. Then use these numbers to recompute the . Iterate until converged.

For the case , there are multiple interfaces. The outermost interface is a special case since there are no spheroids external to it.

Equation for the shape of each interface:

*Interface 0:*

or

*Interface j:*

Calculation of moments from the shape of each interface:

for , and

for

for

and

We can start off the iterative process with any array of values for the . During a given iteration, we must cycle through all the interfaces. After we decide that the iterations have converged because all the interfaces and gravitational moments remain unchanging within a specified tolerance, we multiply all the in Eq. (9) by a single scale factor to obtain the prescribed total mass.

*Calculation of equation of state:*

Equation of HE

Since density is constant across a layer, we have

from top to bottom of layer. Units for calculation:

where

So

or

Table of e.o.s. points:

|  |  |  |
| --- | --- | --- |
| (top of outermost spheroid) |  |  |
| (bottom of outermost spheroid) |  |  |
| (top) |  |  |
| (bottom) |  |  |
| (top) |  |  |
| (bottom) |  |  |
| … | … | … |
| (top) |  |  |
| (center of planet) |  |  |

Before developing the N-layer theory, I worked out a 2-layer theory because there were some published examples in the literature for comparison. Here is the 2-layer theory, which is subsumed by the N-layer theory:

**2-layer theory (precedes the foregoing N-layer theory)**

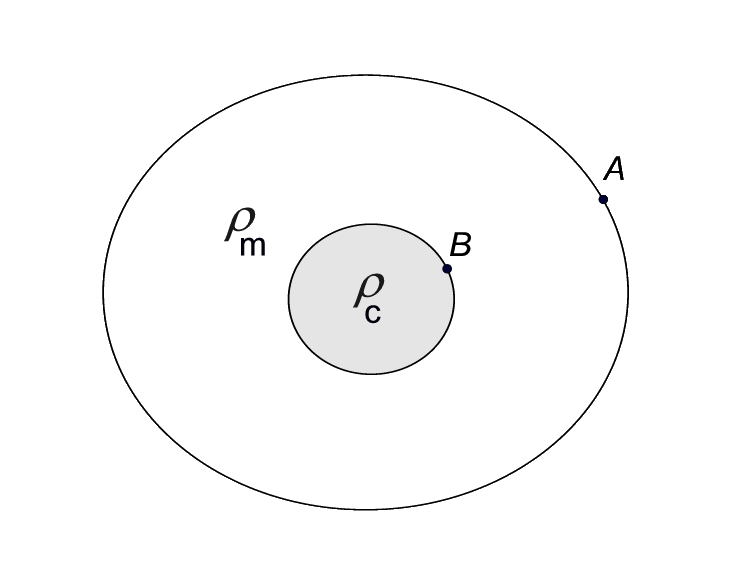
We assume a liquid core with constant density and a liquid mantle with constant density . The core surface is an equipotential expressed as and the mantle surface (overall surface of the planet) is an equipotential expressed as .

Fig. 1 – Two-layer Maclaurin spheroid.

To calculate the potential within the body, we break the calculation into two problems which can be linearly superposed:

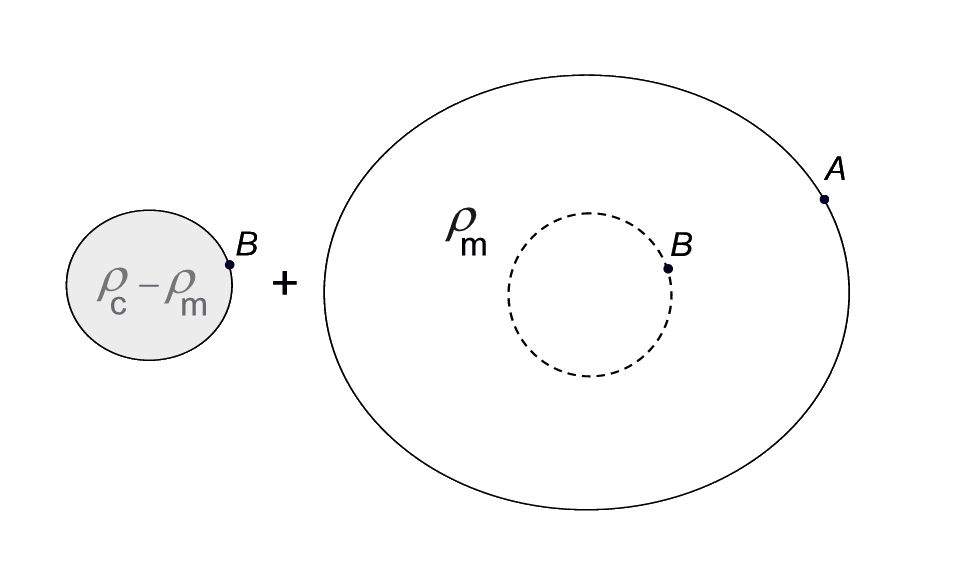


Fig. 2 – Superposition of uniform-density core and Maclaurin spheroid.

The convergent expansion for the potential at point A at coordinates external to both mass distributions is given by

where

and

The zero-degree values are given by

= core volume (core density – mantle density) = core mass – excluded mass. Approximately,

so

Now

= total planet volume mantle density = planet mass – core mass + excluded mass. Thus .

To obtain the potential at point B at coordinates internal to and external to we use the expansion

Now

So

or

or

Consider the limiting expression for a spherical planet (terms only) with no core:

Correct.

Rewrite as

where

as before,

for , and

for

for

and

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Digression for debugging purposes:

Let’s write out to lowest (zeroth) order only:

\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_

Dimensionless forms:

Let

That is as customary, but note that has dimensions of , so define

for .

Now let and , so

or

Rewrite external potential at A:

or

where

or (general two-layer Maclaurin result)

Rewrite potential at B, external to core and interior to mantle:

or

where

and

Recall for the one-layer case we have

*Formulation for the two-layer case:*

1. Adopt and and thus value for
2. Adopt the ratio
3. Adopt the ratio
4. Adopt any initial guess for the , ,
5. Solve for referencing the equatorial potential:

The total potential in the corotating frame is

We solve for the outermost by equating to its equatorial value :

or in dimensionless form

1. Solve for referencing the equatorial potential:

The total potential in the corotating frame is

We solve for the shape of the core surface by equating to its equatorial value :

or in dimensionless form

1. Calculate

and

with

Final dimensionless expression for the shape of the core surface :

Cycle through 5-7 until converged.

**First guesses:**

Set all coefficients with equal to zero.

Set

To test, set and verify that the equipotential at the core surface is an ellipsoid given by the Maclaurin theory.

Result of test: passed with flying colors.

**Calculation of total potential at surface, first interface, and center:**

*Surface potential:*

where

We may also write in terms of , its value at the equator:

*Potential on first interface:*

*Potential at center:*

Now

So

or

where

*Calculation of density in layers:*

So

*Calculation of equation of state:*

Equation of HE

Since density is constant across a layer, we have

from top to bottom of layer. Units for calculation:

So

or

So for our two-layer model, we have three values for pressure and density:

*Surface*

*First interface*

and

so

*Center*

for debugging:

pressure profile in a uniform-density sphere (grabbed this on the web)

Convert to pup:

So