Theory of figures

Summary of the theory of figures, providing the mathematical basis for using gravity measurements to learn about planetary interiors. Mathematical statements are checked to my satisfaction unless otherwise noted (important exception is the addition theorem). The primary reference is (Zharkov and Trubitsyn, 1978).

Contents

I. DEFINITIONS AND NOTATION

Given a density distribution $\rho(\mathbf{r}')$ inside the planet, the *total potential* is defined as

$$U(\mathbf{r}) = V(\mathbf{r}) + Q(\mathbf{r}) \tag{1}$$

where

$$V(\mathbf{r}) = G \int \rho(\mathbf{r}')/\left|\mathbf{r} - \mathbf{r}'\right| \, d\tau' \tag{2}$$

is the gravitational potential and

$$Q(\mathbf{r}) = \frac{1}{2}\omega^2 r^2 \sin^2\theta \tag{3}$$

is the centrifugal potential; we will use $d\tau$ for a volume element, ω is the angular rotation velocity (assumed constant and usually about a principal axis), $r = |\mathbf{r}|$, and θ is the angle from the rotation axis. *Note that Zharkov and Trubitsyn use an unusual positive potential, presumably for algebraic convenience. This also means accelerations are given by the positive gradient of potential.*

When the planet is in hydrostatic equilibrium the level surfaces of potential are also level surfaces of pressure and of density. Including the free surface where the pressure $p = 0$. This can be shown rigorously (e.g. Batchelor, 1967) and is also fairly intuitive. Finding the shape of the planet is therefore reduced to the problem of finding the level surfaces $U(\mathbf{r}) =$ constant. This problem is easy to state but hard to solve, essentially because the volume of space over which the integral above is taken is unknown and must be found as part of the self-consistent solution.

Equilibrium figures which differ only slightly from spheres are called *spheroids.* A dimensionless parameter describing the importance of rotation is

$$q = \frac{\omega^2 a^3}{GM} \tag{4}$$

where M is the planet's mass and a is the *equatorial radius.*

II. DECOMPOSITION OF POTENTIAL INTO SPHERICAL HARMONICS

The expression for the gravitational potential, eq. (2), can be written as a sum of powers of r using the decomposition of $1/|\mathbf{r}-\mathbf{r}'|$ in Legendre polynomials (see LEGENDRE.PDF):

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r\sqrt{1-2t(r'/r)+(r'/r)^2}} = \\ = \begin{cases} \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^n P_n(t), & r > r', \\ \frac{1}{r}\sum_{n=0}^{\infty}(\frac{r'}{r})^{-n-1} P_n(t), & r < r', \end{cases} \tag{5}$$

where γ is the angle between the radius vectors $\mathbf{r}$ and $\mathbf{r}'$ and $t = \cos\gamma$. If $r > r'$ for all points where $\rho(\mathbf{r}') > 0$ (i.e. inside the planet) then the potential is called *external*. The Legendre polynomials are given by Rodrigues's formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \tag{6}$$

(although this is not the easiest way to obtain them explicitly) for $x \in [-1,1]$ and luckily there is no confusion about normalization. It will be useful to remember that

$$P_0 = 1, \qquad P_2(t) = \frac{3}{2}t^2 - \frac{1}{2}, \qquad \int_{-1}^{1} P_{n>0}(t')\,dt' = 0.$$

and that $P_n(1) = 1$. The gravitational potential in terms of Legendre polynomials is

$$V(\mathbf{r}) = \frac{G}{r}\sum_{n=0}^{\infty}\int \rho(\mathbf{r}')P_n(t)(r'/r)^{\alpha}\,d\tau' \tag{7}$$

where

$$\alpha = \begin{cases} n, & r > r', \\ -(n+1), & r < r', \end{cases}$$

and the integration is over the (as yet unknown) volume of the planet.

The expansion in Legendre polynomials is compact and neat but it is of little utility because it does not separate terms arising from the mass distribution from those arising from the location where the potential is to be evaluated. In spherical polar coordinates the variable

$$t = \cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi-\varphi')$$

mixes the colatitude θ and longitude φ of the integration variable and the point of measurement. The salvation comes from the *addition theorem for spherical harmonics* (**which I can't derive**):

$$P_n(\cos\gamma) = P_n(\cos\theta)P_n(\cos\theta') + 2\sum_{m=1}^{n}\frac{(n-m)!}{(n+m)!}P_n^m(\cos\theta)P_n^m(\cos\theta')\cos\big[m(\varphi-\varphi')\big], \tag{8}$$

with the associated Legendre functions

$$P_n^m(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x). \tag{9}$$

(Unfortunately there is not a universal consensus on the exact form of the P_n^m functions. If a different normalization is used then the following expressions will all be somewhat different.) With the aid of the addition theorem we can decompose the potential as[1]

$$V(\mathbf{r}) = \frac{G}{r}\Bigg(\sum_{n=0}^{\infty} P_n(\cos\theta)\int_\tau \rho(\mathbf{r}')P_n(\cos\theta')(r'/r)^\alpha\,d\tau'$$
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{n} P_n^m(\cos\theta)\cos(m\varphi)\int_\tau \frac{2(n-m)!}{(n+m)!}\rho(\mathbf{r}')P_n^m(\cos\theta')\cos(m\varphi')\Big(\frac{r'}{r}\Big)^\alpha\,d\tau' \tag{10}$$
$$+\sum_{n=1}^{\infty}\sum_{m=1}^{n} P_n^m(\cos\theta)\sin(m\varphi)\int_\tau \frac{2(n-m)!}{(n+m)!}\rho(\mathbf{r}')P_n^m(\cos\theta')\sin(m\varphi')\Big(\frac{r'}{r}\Big)^\alpha\,d\tau'\Bigg)$$

with α as before.

The above expansion is general, not requiring hydrostatic equilibrium or principal axis rotation. If the planet is fluid then at equilibrium the rotation will always be about a principal axis. If we take the polar axis (call it the z axis) to coincide with the rotation axis then symmetry requires that $\rho(\mathbf{r})$ and therefore $V(\mathbf{r})$ cannot depend on the longitude φ and that $\rho(r,\theta)$ must include only even powers of $\cos\theta$ (for symmetry about the equator). In this case we can write a simpler expansion involving only ordinary Legendre polynomials of only even degree:

$$V(r,\theta) = \frac{G}{r}\sum_{n=0}^{\infty}\left(r^{-2n}D_{2n} + r^{2n+1}D'_{2n}\right)P_{2n}(\cos\theta) \tag{11a}$$

with

$$D_n = \int_{r'<r}\rho(\mathbf{r}')(r')^n P_n(\cos\theta')\,d\tau', \tag{11b}$$

$$D'_n = \int_{r'>r}\rho(\mathbf{r}')(r')^{-n-1} P_n(\cos\theta')\,d\tau'. \tag{11c}$$

The coefficients D_n are usually replaced with the non-dimensional coefficients $J_n = D_n/(Ma^n)$.

III. THE EXTERNAL POTENTIAL

If the potential is to be evaluated at a point away from the surface of the planet then $r > r'$ for all differential volume elements in the integral expressions above. The general form eq. (10) can be rearranged slightly and rewritten in a form more convenient for comparison with measured quantities:

$$V_e = \frac{GM}{r}\Bigg(1 - \sum_{n=1}^{\infty}(a/r)^n J_n P_n(\cos\theta) + \\ + \sum_{n=1}^{\infty}\sum_{m=1}^{n}(a/r)^n P_n(\cos\theta)\Big[C_{nm}\cos(m\varphi) + S_{nm}\sin(m\varphi)\Big]\Bigg), \tag{12a}$$

with the coefficients

$$Ma^n J_n = -\int \rho(\mathbf{r}')(r')^n P_n(\cos\theta')\,d\tau' = -D_n, \tag{12b}$$

$$Ma^n C_{nm} = \frac{2(n-m)!}{(n+m)!}\int \rho(\mathbf{r}')(r')^n P_n^m(\cos\theta')\cos(m\varphi')\,d\tau', \tag{12c}$$

$$Ma^n S_{nm} = \frac{2(n-m)!}{(n+m)!}\int \rho(\mathbf{r}')(r')^n P_n^m(\cos\theta')\sin(m\varphi')\,d\tau'. \tag{12d}$$

Remember, $M = \int \rho\, d\tau'$ is the planet's mass and a is the equatorial radius.

[1] Note typo in Z&T.

(Different normalizations of the associated Legendre functions are often used leading to expansions with the same form but different meaning of the J_n, C_{nm}, and S_{nm} coefficients. There is no easy way to guard against errors or guess which normalization was used if it is not explicitly given. The decomposition can also be carried out with the complex form of the Legendre functions leading to even more confusion.)

A natural choice of reference frame can eliminate some of the coefficients in the expansion (12). First, if the origin of coordinates is chosen at the center of mass of the planet, $\mathbf{R} = (X, Y, Z)$, then

$$\begin{aligned} -MaJ_1 &= \int z'\, dm' = MZ = 0, \\ MaC_{11} &= \int x'\, dm' = MX = 0, \text{ and} \\ MaS_{11} &= \int y'\, dm' = MY = 0 \end{aligned}$$

so that J_1, C_{11}, and S_{11} vanish. Next, we can relate the expansion coefficients to the moments of inertia. We designate those:

$$B = \int_\tau \rho(\mathbf{r}')(x'^2 + z'^2)\, d\tau', \tag{13a}$$

$$A = \int_\tau \rho(\mathbf{r}')(y'^2 + z'^2)\, d\tau', \tag{13b}$$

$$C = \int_\tau \rho(\mathbf{r}')(x'^2 + y'^2)\, d\tau', \tag{13c}$$

for the principal moments and

$$D = \int_\tau \rho(\mathbf{r}')\, y'z'\, d\tau', \tag{13d}$$

$$E = \int_\tau \rho(\mathbf{r}')\, x'z'\, d\tau', \tag{13e}$$

$$F = \int_\tau \rho(\mathbf{r}')\, x'y'\, d\tau', \tag{13f}$$

for the diagonal, so-called *products of inertia*, also called *centrifugal moments*. The relations with the expansion coefficients in eq. (12) become clear by writing out the degree 2 Legendre functions in Cartesian coordinates. It is easy to show by direct comparison that

$$-a^2 MJ_2 = \frac{A+B}{2} - C, \quad \text{and} \quad a^2 MC_{22} = \frac{B-A}{4}, \tag{14a}$$

and also that

$$D = a^2 MS_{21}, \quad E = a^2 MC_{21}, \quad \text{and} \quad F = 2a^2 MS_{22}. \tag{14b}$$

Usually $A = B < C$ so that $a^2 MJ_2 = (C - A)$. Also, if we align the coordinate axes with the planet's principal axes of inertia then the centrifugal moments vanish and

$$S_{21} = C_{21} = S_{22} = 0. \tag{15}$$

And of course it is still true that for a fluid planet at equilibrium the density must be independent of longitude and symmetric about the equator and therefore

$$D_{2n+1} = J_{2n+1} = 0, \quad \forall n. \tag{16}$$

Finally, expressing the centrifugal potential (3) in terms of P_2:

$$Q(\mathbf{r}) = \frac{1}{3}\omega^2 r^2 \big[1 - P_2(\cos\theta)\big] \tag{17}$$

and using the small parameter q (4) the total external potential is

$$V_e(\mathbf{r}) = \frac{GM}{r}\left[1 - (a/r)^2 J_2 P_2 - (a/r)^4 J_4 P_4 - (a/r)^6 J_6 P_6 - \cdots + \frac{1}{3}(r/a)^3 \big(1 - P_2(\cos\theta)\big)\, q\right]. \tag{18}$$

IV. LEVEL SURFACES – PRELIMINARY RESULTS

For a slowly rotating planet all level surfaces (surfaces of constant potential) differ only slightly from spheres. We would like to describe them in the form

$$r(\theta) = l\big[1 + l_{2n}(l)P_{2n}(t)\big]. \tag{19}$$

The coefficients l_{2n} are of order q^n (**WHY?**) and l is some characteristic length. Note that here I begin to use an implicit summation convention[2]. Also remember that $t = \cos\theta$.

We can get some preliminary relations simply by plugging in either the equatorial radius (a), the polar radius (b) or the mean radius (s), into eq. (19), with $\theta = 0$ or $\theta = \pi/2$. When we use the equatorial radius a we denote the coefficients $l_n = a_n$ and similarly for the other variables. For example, since $a = r(\pi/2)$ we know that

$$a_{2n}(a)P_{2n}(0) = 0 \tag{20}$$

(implicit summation!) and, since $b = r(0)$,

$$\sum_{n=0}^{\infty} b_{2n}(b) = 0. \tag{21}$$

Now, with $l = a$ and $\theta = 0$ we find that

$$b = a\big[1 + \sum_{n=0}^{\infty} a_{2n}\big] \tag{22}$$

and we can describe the *flattening* of a level surface:

$$e(a) = \frac{a-b}{a} = -\sum_{n=0}^{\infty} a_{2n}. \tag{23}$$

We will also often use the mean radius of a level surface, s, defined by

$$4\pi s^3/3 = \int\!\!\int\!\!\int r^2(\theta)\,dr\,d\theta\,d\phi = (2\pi/3)\int_{-1}^{1} r^3(\theta)\,dt. \tag{24}$$

Again, playing the substitution games, we can find several relations:

$$a/s = 1 + s_{2n}P_{2n}(0) = 1 + s_0 - \frac{1}{2}s_2 + \frac{3}{8}s_4 - \frac{5}{16}s_6 + \cdots, \tag{25a}$$

$$b/s = 1 + \sum_{n=0}^{\infty} s_{2n}, \tag{25b}$$

and

$$e(s) = \frac{\sum s_{2n}[P_{2n}(0) - 1]}{1 + \sum s_{2n}P_{2n}(0)}. \tag{25c}$$

The coefficients s_n themselves are coupled by substituting eq. (19) in the definition of s, to get

$$\int_{-1}^{1} \big[1 + s_{2n}P_{2n}(t)\big]^3\,dt = 2. \tag{26}$$

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[2] The summation is on $n \geq 0$. The constant, $n = 0$ coefficients are not of order one though because of the explicit $+1$ in eq. (19). An odd choice, I think, but whatever.

Finally, we will sometimes use a small parameter $m = \omega^2 s^3/GM$ instead of the parameter $q = \omega^2 a^3/GM$. The advantage of m is that

$$m = \frac{3\omega^2}{4\pi G\bar{\rho}} \tag{27}$$

where $\bar{\rho}$ is the average density of the planet. We find the relation between m and q by using eqs. (25):

$$q/m = (a_1/s_1)^3 = \left[1 + s_{2n}(s)P_{2n}(0)\right]^3 = 1 + 3s_0 - \frac{3}{2}s_2 + \frac{3}{4}s_2^2 + \frac{9}{8}s_4 + \cdots . \tag{28}$$

The index 1 on the coefficients refers to the outermost level surface, i.e., the surface of the planet.

References

G.K. Batchelor. *An introduction to fluid dynamics.* Cambridge Univ. Press, Cambridge, UK, 1967.
Vladimir N Zharkov and V P Trubitsyn. *Physics of Planetary Interiors.* Tucson, AZ, 1978.