Over-simplified single-size model

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Abstract

Notes to myself on the simple model. The purpose of this model is to estimate a characteristic grain size in a steady state sedimentation scenario. Basically, I try to find one grain size in each zone that equalizes the typical times of sedimentation and coagulation.

ATMOSPHERIC DATA

I read the gas density and temperature at 60 *locations* in the atmosphere from file, e.g., ATM.DAT. I convert this to 59 *zones*, or layers, with l_k using the values in Z_{k+1}, T_{k+1}, and ρ^{gas}_{k+1}. In other words, the gas density in the first zone is taken from the *second* line of ATM.DAT, and so on. (Should I use mean values instead? Probably not.)

GRAIN INDEPENDENT PROPERTIES

I denote by l the layer thickness, i.e., $l = |\text{diff}(Z)|$. The acceleration of gravity at each zone is

$$g = \frac{GM_{\text{core}}}{Z^2}, \tag{1}$$

where $M_{\text{core}} = 6.9 \times 10^{25}\,\text{kg}$ is the core mass of the protoplanet, and $G = 6.673 \times 10^{-11}\,\text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2}$ is the universal gravitational constant. The gas number density is

$$n_{\text{gas}} = \frac{\rho_{\text{gas}} N_{\text{A}}}{\mu}, \tag{2}$$

where $N_{\text{A}} = 6.02\text{e}23\,\text{mole}^{-1}$ is Avogadro's number and $\mu = 2.3\text{e-}3\,\text{kg/mole}$ is the mean molecular weight of the gas. The thermal velocity of the gas molecules is given by

$$v^{\text{gas}}_{\text{th}} = \sqrt{\frac{8K_{\text{B}} T N_{\text{A}}}{\pi\mu}}. \tag{3}$$

(Boltzmann's constant in mks is $K_{\text{B}} = 1.38e - 23\,\text{J}/^{\circ}\text{K}$.) Finally, the mean free path of a gas molecule is given by

$$\text{mfp} = \frac{1}{\sqrt{2} n_{\text{gas}} \sigma}, \tag{4}$$

with $\sigma = 1\text{e-}19\,\text{m}^2$ as the molecular collision cross section. Table I gives some typical values of these dynamic properties.

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TABLE I. Typical values of grain independent atmospheric and dynamic properties.

Z [m]	T [°K]	g [m/s^2]	ρ_{gas} [kg/m^3]	n_{gas} [1/m^3]	$v_{\text{th}}^{\text{gas}}$ [m/s]	mfp$_{\text{gas}}$ [m]
5.9e9	154	1.3e-4	7.4e-8	1.9e19	1.2e3	3.6e-1
2.1e9	168	1.0e-3	2.8e-6	7.3e20	1.2e3	9.7e-3
1.2e9	257	3.3e-3	7.0e-5	1.8e22	1.5e3	3.8e-4

SEDIMENTATION SPEED AND TIME

The Knudsen number is much greater than one in most of the zones and for most conceivable grain sizes. The only exception is deep in the atmosphere, and even then only for the largest grains. I therefore use Epstein drag in my calculation. The drag force then is given by

$$F_{\text{drag}} = \frac{4\pi a^2 \rho_{\text{gas}} v_{\text{th}}^{\text{gas}} v_{\text{sed}}}{3}, \tag{5}$$

and the sedimentation speed can be found by setting $F = mg$:

$$v_{\text{sed}} = \frac{\rho_{\text{grain}} g}{\rho_{\text{gas}} v_{\text{th}}^{\text{gas}}}\, a. \tag{6}$$

The time to sediment out of the layer is l/v_{sed}. The sedimentation time for a 10 micron grain ranges from a few years in the uppermost layer to hundreds of years deeper in the atmosphere.

COAGULATION TIME

In a steady state the total mass of dust coming into a layer per unit time is equal to the total mass going out of the layer at the same time. There is therefore a simple expression for the flux F going through each layer,

$$F = F_0 \frac{Z_0^2}{Z^2}, \tag{7}$$

where Z_0 is the "end" of the atmosphere, and in each layer,

$$F = \frac{4\pi}{3} a^3 \rho_{\text{grain}} n_{\text{grain}} v_{\text{sed}}. \tag{8}$$

(At this time there is no source term of dust representing breakup of planetesimals etc.) I take $F = 4\text{e-}12\,\mathrm{kg\,m^{-2}\,s^{-1}}$ based on 1e-8 Earth masses a year falling onto the planet from the solar nebula. I therefore have n_{grain}, the number density of grains in each zone:

$$n_{\mathrm{grain}} = \frac{3F}{4\pi a^3 \rho_{\mathrm{grain}} v_{\mathrm{sed}}}. \tag{9}$$

The mean distance between grains is

$$\mathrm{mfp}_{\mathrm{grain}} = \frac{1}{\sqrt{2} n_{\mathrm{grain}} \sigma_{\mathrm{grain}}}. \tag{10}$$

For σ, the grains collision cross section, Morris takes $4\pi^2$, because if the center of one grain gets closer than $2a$ to the center of another grain they will collide. So

$$\mathrm{mfp}_{\mathrm{grain}} = \frac{1}{\sqrt{2} 4\pi a^2 n_{\mathrm{grain}}}. \tag{11}$$

The random speed of the grain due to Brownian motion is

$$v_{\mathrm{th}}^{\mathrm{grain}} = \sqrt{\frac{8K_{\mathrm{B}}T}{\pi m_{\mathrm{grain}}}}, \tag{12}$$

and $t_{\mathrm{coag}} = \mathrm{mfp}_{\mathrm{grain}}/v_{\mathrm{th}}^{\mathrm{grain}}$.

CHARACTERISTIC GRAIN SIZE

Setting $t_{\mathrm{sed}} = t_{\mathrm{coag}}$ and solving for a yields

$$a = \left[\frac{3\sqrt{12K_{\mathrm{B}}T}}{\pi} \frac{Fl\rho_{\mathrm{gas}}^2 (v_{\mathrm{th}}^{\mathrm{gas}})^2}{\rho_{\mathrm{grain}}^{7/2} g^2} \right]^{2/9}. \tag{13}$$

Substituting in Eqs. (7) and (3) we get

$$a = \left[\frac{48\sqrt{3} N_{\mathrm{A}} (K_{\mathrm{B}}T)^{3/2} F_0 Z_0^2 \rho_{\mathrm{gas}}^2 l}{\pi^2 \mu \rho_{\mathrm{grain}}^{7/2} g^2 Z^2} \right]^{2/9}. \tag{14}$$