The pressure integral

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Derivation of the pressure integral. From Clayton.

The microscopic source of pressure in a *perfect gas* is particle bombardment. The reflection of these particles from a real or imagined surface inside the gas results in transfer of momentum to that surface. By Newton's second law, that momentum transfer exerts a force on the surface. The average force per unit area is called the pressure.

In *thermal equilibrium* the angular distribution of particle momenta should be isotropic. Imagine a surface inside the gas with a normal $\hat{\mathbf{n}}$. If a particle of momentum $\mathbf{p}$, inclined at an angle θ to $\hat{\mathbf{n}}$, is *specularly reflected* from the surface, the momentum transferred is normal to the surface and equals $\Delta p_{\mathrm{n}} = 2p\cos\theta$. Let $F(\theta, p)\,d\theta\,dp$ be the number of particles with momentum magnitude p in the range dp striking the surface per unit area per unit time from all directions inclined at angle θ in the range $d\theta$ to the normal. The contribution to the pressure from these particles is

$$dP = 2p\cos\theta\,F(\theta, p)\,d\theta\,dp. \tag{1}$$

The total pressure then is

$$P = \int_{\theta=0}^{\pi/2}\int_{p=0}^{\infty} 2p\cos\theta\,F(\theta, p)\,d\theta\,dp. \tag{2}$$

In thermodynamic equilibrium, the angular distribution of momenta is isotropic. The distribution of momenta magnitudes is given by statistical mechanics. The flux $F(\theta, p)\,d\theta\,dp$ may be written as the product of the number density of particles with momentum in the given range times the volume of such particles capable of striking the unit surface in unit time. This volume is equal to $\cos\theta$ times the velocity associated with momentum of magnitude p, denoted v_p. That is,

$$F(\theta, p)\,d\theta\,dp = v_p\cos\theta\,n(\theta, p)\,d\theta\,dp, \tag{3}$$

where $n(\theta, p)\,d\theta\,dp$ is the number density of particles with these momenta. Because of isotropy the fraction of particles moving in the cone of directions is the fraction of solid angle subtended by this cone. So if $n(p)\,dp$ is the total number density of particles with momentum magnitude p in dp then

$$\frac{n(\theta, p)\,d\theta\,dp}{n(p)\,dp} = \frac{2\pi\sin\theta\,d\theta}{4\pi}. \tag{4}$$

The gas pressure is therefore

$$P = \int_0^{\pi/2}\int_0^{\infty} 2p\cos\theta\,v_p\cos\theta\,n(p)\,dp\,\frac{1}{2}\sin\theta\,d\theta. \tag{5}$$

The integration over angles yields

$$P = \frac{1}{3}\int_0^{\infty} p v_p n(p)\,dp. \tag{6}$$

Equation (6) is known as the *pressure integral.* It is valid for a perfect isotropic "gas". The relationship of v_p to p depends on relativistic considerations. The distribution $n(p)$ depends on the type of particles and can require quantum statistical mechanics.