Transition to Advanced Math - Assignment 7 - Chapter 7

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1. Suppose x ∈ Z. Then x is even if and only if 3x + 5 is odd.
   Solution:
   First, we prove that if x is even, then 3x+5 is odd.
   Assume that x is even, then x = 2k with k is some integers.
   Then 3x+5 = 3.2k+5 = 6k+5.
   Hence, if x is even, 3x + 5 is odd.

   Conversely, suppose 3x+5 odd, then 3x+5 = 2k+1.
   We then have 3x + 4 = 2k $\iff$ 3x = 2k - 4. Since 2k - 4 is an even number, then 3x is also an even number. We then observe that 2 and 3 are two prime numbers that have the greatest common factor is 1. Therefore, x must be an even number in order to have 3x is even.

   With two sides are proved above, we then have x is even if and only if 3x+5 is odd.

7. Suppose x, y ∈ R. Then $(x+y)^2 = x^2 + y^2$ if and only if x = 0 or y = 0.

   Solution:
   First, we prove that if $(x+y)^2 = x^2 + y^2$, then x = 0 or y = 0.
   $(x+y)^2 = x^2 + y^2 \iff$ 2xy = 0.
   The equation above is true if and only if x = 0, or y = 0, or x = y = 0.

   Conversely, suppose x = 0 or y = 0, then 2xy = 0. We then add both side of equation above with $x^2 + y^2$ (> 0) to have: $x^2 + 2xy + y^2 = x^2 + y^2$.
   This new equation equals to $(x+y)^2 = x^2 + y^2$
   With two sides are proved above, we then have $(x+y)^2 = x^2 + y^2$ if and only if x = 0 or y = 0.

8. Suppose a,b ∈ Z. Prove that a ≡ b (mod 10) if and only if a ≡ b (mod 2) and a ≡ b (mod 5).

   Solution:
   First we prove that if a ≡ b (mod 10), then a ≡ b (mod 2) and a ≡ b (mod 5).
   Suppose a ≡ b (mod 10). This means 10 | (ab), so there is an integer k for which a - b = 10k.
   From this we get ab = 2(5n), which implies 2 — (ab), so a ≡ b (mod 2). But we also get a - b = 5(2n), which implies 5 | (ab), so a ≡ b (mod 5).

   Conversely, suppose a ≡ b (mod 2) and a ≡ b (mod 5). Since a ≡ b (mod 2) we get 2 — (ab), so there is an integer k for which ab = 2k. Therefore, ab is even. Also, from a ≡ b (mod 3) we get 5 — (a b), so there is an integer l for which: a - b = 5l.

   But since we know a b is even, it follows that l must be even also, for if it were odd then a - b = 5l would be odd (because a - b would be the product of two odd integers). Hence l = 2m for some integer m. Thus a - b = 5l = 5.2m = 10m. This means 10 | (a - b), so a ≡ b (mod 10).

17. There is a prime number between 90 and 100.
Solution:
In the range from 90 to 100, there are only two possible prime numbers which are 91 and 97. However, 91 = 13.7. We then prove that 97 is a prime number.

We then test factor of 97 by divide 97 to all prime number which is smaller than $\sqrt{97}$. We then have 97 is not divisible by 2, 3, 5, 7. Therefore, 97 is a prime number

20. There exists an n ∈ N for which 11 | (2n-1)
Solution:
11 is a prime number which has only two factors are 1 and 11. Therefore, we have two cases:
Case 1: 2n-1 = 1 if and only if n = 0
Case 2: 2n-1 = 11 if and only if n = 6.
Therefore, there true exists an n ∈ N for which 11 | (2n-1). For example, n = 0.

28. Prove the division algorithm: If a,b ∈ N, there exist unique integers q, r for which a = bq + r, and 0 ≤ r < b. (A proof of existence is given in Section 1.9, but uniqueness needs to be established too.)
Solution:
Existence Take from section 1.9
Given integers a,b with a,b ∈ N, form the set A = $\{a - xb : x \in Z, 0 \leq a - xb\} \subseteq \{0, 1, 2, 3, ...\}$

In general, let r be the smallest element of the set A. Then r = a - qb for some x = q ∈ Z, so a = qb + r. Moreover, $0 \leq r < b$, as follows. The fact that r ∈ A ⊆ $\{0, 1, 2, 3...\}$ implies 0 ≤ r.
In addition, it cannot happen that r ≥ b: If this were the case, then the non-negative number r - b = (a - qb) - b = a - (q+1)b having form a-xb would be a smaller element of A than r, which is contradict with the fact that r was chosen as the smallest element of A. Since r ≱ b, it must be that r < b. Therefore $0 \leq r < b$.
We've now produced a q and an r for which a = qb + r and $0 \leq r < b$

Uniqueness: Suppose that we have to sets of integers q, r and q', r' satisfying
a = bq + r with $0 \leq r < b$
and a = bq'+ r' with $0 \leq r' < b$
We want to show that q' = q and r' = r. We then have b.(q' - q) + r' - r = 0.

Since $0 \leq r < b$ and $0 \leq r' < b$, we have $b < r - r' < b$. From this we obtain $-1 < q' - q = (r - r')/b < 1$
Since q' q ∈ Z, this implies that q' q = 0. This in turn implies that r' r = 0.