MATH 215

Proof #4

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April 17, 2018

Theorem 0.1. *For every integer $n \in N$, it follows that* $1^3 + 2^3 + 3^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$ *(1)*

Proof. Case n = 1:

Then (1) becomes: $1^3 = \frac{1^2(1+1)^2}{4} = \frac{4}{4} = 1$, which is true

Case n = k: Now assume the statement is true for some integer n = k $\geq$ 1, that is assume $1^3 + 2^3 + 3^3 + ... + k^3 = \frac{k^2(k+1)^2}{4}$ is true.

Case n = k+1:

We have (1) becomes: $1^3 + 2^3 + 3^3 + ... + k^3 + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$ (2)

Since case n = k is true, we have (2) is equivalent to:

$$\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$$

$$k^2(k^2 + 2k + 1) + 4(k^3 + 3k^2 + 3k + 1) = (k^2 + 2k + 1)(k^2 + 4k + 4)$$

$$k^4 + 2k^3 + k^2 + 4k^3 + 12k^2 + 12k + 4 = k^4 + 4k^3 + 4k^2 + 2k^3 + 8k^2 + 8k + k^2 + 4k + 4$$

$$k^4 + 6k^3 + 13k^2 + 12k + 4 = k^4 + 6k^3 + 13k^2 + 12k + 4$$

Therefore, $\frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{(k+1)^2(k+2)^2}{4}$, which means the statement is true for n = k+1. $\square$

Theorem 0.2. *Prove that* $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{n^2} \leq 2 - \frac{1}{n}$

Proof. Case n = 1:

Then (1) becomes: $1 \leq 2 - \frac{1}{1} = 1$, which is true

Case n = k: Now assume the statement is true for some integer n = k $\geq$ 1, that is assume $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{k^2} \leq 2 - \frac{1}{k}$ (2) is true.

Case n = k+1:

We have (1) becomes: $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$

Since case n = k is true, we add $\frac{1}{(k+1)^2}$ to both side of (1) then have:

$\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}(2)$

Right side of the statement above is equivalent to: $2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{k^2+k+1}{k(k+1)} = 2 - \frac{k^2+1}{k(k+1)} - \frac{1}{k+1}$

Consequently, (2) is equivalent to : $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + ... + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} = 2 - \frac{k^2+1}{k(k+1)} - \frac{1}{k+1} \leq 2 - \frac{1}{k+1}$.

Therefore the statement is true with n = k+1. $\square$

Theorem 0.3. *If* $n \in \mathbb{N}$, *then* $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^n - 1} + \frac{1}{2^n} \geq 1 + \frac{n}{2}$.

Proof. Let $S(n) = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{2^n-1} + \frac{1}{2^n}$

Case 1: $n = 1$ then $S(1) = 1 + \frac{1}{2}$

Case 2: n = k

Assume that statement is true for $n = k \geq 1$, which is $S(k) \geq 1 + \frac{k}{2}$

Case 3: n = k+1 We want to show: $S(k+1) \geq 1 + \frac{k+1}{2}$

We have that:

$S(k+1) = S(k) + \frac{1}{2^k+1} + \frac{1}{2^k+2} + ... + \frac{1}{2^{k+1}-1} + \frac{1}{2^{k+1}} = S(k) + \sum_{i=1}^{2^k} \frac{1}{2^k+i}$

So $S(k+1) \geq S(k) + \sum_{i=1}^{2^k} \frac{1}{2^{k+1}} = S(k) + \frac{2^k}{2^{k+1}} \geq 1 + \frac{k}{2} + \frac{1}{2} = 1 + \frac{k+1}{2}$ as desired.

Therefore, by the Principle of Mathematical Induction the statement is true $\square$

Theorem 0.4. *Prove that* $3^1 + 3^2 + 3^3 + ... + 3^n = \frac{3^{n+1}-3}{2}$ *for every* $n \in \mathbb{N}$

Proof. Let $S(n) = 3^1 + 3^2 + 3^3 + ... + 3^n$

Case 1: $n = 1$, then $S(1) = 3 = \frac{3^{1+1}-3}{2}$

Case 2: n = k

Assume that statement is true for $n = k \geq 1$, which is $S(k) = \frac{3^{k+1}-3}{2}$

Case 3: n = k+1

We want to show: $S(k+1) = \frac{3^{k+2}-3}{2}$

We have that:

$S(k+1) = S(k) + 3^{k+1} = \frac{3^{k+1}-3}{2} + 3^{k+1} = \frac{3(3^{n+1})-3}{2} = \frac{3^{k+2}-3}{2}$ as desired

Therefore, by the Principle of Mathematical Induction, $3^1 + 3^2 + 3^3 + ... + 3^n = \frac{3^{n+1}-3}{2}$ for every $n \in \mathbb{N}$ $\square$