# Math 215, Spring 2018 - Assignment # 5

Name: Hieu Tran

1. **Proposition:** If $x$ is an even integer, then $x^2$ is even.
   Proof:
   Assume $x$ is even.
   Then $x = 2a$ where $a$ is some integer.
   So $x^2 = (2a)^2 = 4a^2 = 2(2a^2)$
   So $x^2 = 2k$ where $k = 2a^2 \in \mathbb{Z}$
   Therefore $x^2$ is even.

2. **Proposition:** If $a$ is an odd integer, then $a^2 + 3a + 5$ is odd.
   Proof:
   Assume $a$ is odd.
   Then $a = 2k + 1$ where $k$ is some integer.
   So $a^2 + 3a + 5 = (2k+1)^2 + 3(2k+1) + 5 = 4k^2 + 10k + 9 = 2(2k^2 + 5k + 4) + 1$
   So $x^2 = 2h + 1$ where $h = 2k^2 + 5k + 4 \in \mathbb{Z}$
   Therefore $a^2 + 3a + 5$ is odd.

3. **Proposition:** Suppose $x, y \in \mathbb{Z}$. If $x$ and $y$ are odd, then $xy$ is odd.
   Proof:
   Assume $x, y$ are odd
   Then $x = 2k + 1, y = 2h + 1$ where $k, h$ are some integers.
   So $xy = (2k+1)(2h+1) = 4kh + 2k + 2h + 1 = 2(kh + k + h) + 1$
   So $xy = 2l + 1$ where $l = kh + h + k \in \mathbb{Z}$
   Therefore $xy$ is odd.

4. **Proposition:** Suppose $a, b, c \in \mathbb{Z}$. If $a|b$ and $a|c$, then $a|(b + c)$.
   Proof:
   Assume $a|b$ and $a|c$.
   Then $b = am, c = an$ where $m, n$ are some integers.
   So $b + c = am + an = a(m + n)$
   So $b + c = ak$ where $k = m + n \in \mathbb{Z}$
   Therefore $a|(b + c)$.

5. **Proposition:** Suppose $a$ is an integer. If $5|2a$, then $5|a$.
   Proof:
   Assume $5|2a$.
   Then $2a = 5k$ where $k$ are some integers.
   Since 5 does not devide 2, 5 is a devisor of $a$
   Therefore $5|a$.

6. **Proposition:** Suppose $a, b, c, d \in \mathbb{Z}$. If $a|b$ and $c|d$, then $ac|bd$.
Proof:
Assume $a|b$ and $c|d$.
Then $b = am, d = cn$ where $m, n$ are some integers.
So $bd = (am)(cn) = (ac)(mn)$
So $bd = (ac)k$ where $k = mn \in \mathbb{Z}$
Therefore $ac|bd$.

7. **Proposition:** If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$
Proof:
Assume $0 < x < 4$.
Then $x > 0$ and $4 - x > 0$.
So $x(4-x) \geq 0$ (1)
So $\frac{4}{x(4-x)} - 1 = \frac{4-4x+4x^2}{x(4-x)} = \frac{(1-2x)^2}{x(4-x)} \geq 0$
Therefore $\frac{4}{x(4-x)} \geq 1$

8. **Proposition:** Suppose $x, y \in \mathbb{R}$. If $x^2 + 5y = y^2 + 5x$, then $x = y$ or $x + y = 5$.
Proof:
Assume $x^2 + 5y = y^2 + 5x$.
Then $(x^2 - y^2) - (5x - 5y) = 0$
So $(x - y)(x + y) - 5(x - y) = 0$
So $(x - y)(x + y - 5) = 0$
Hence $x - y = 0$ or $x + y - 5 = 0$
Therefore $x = y$ or $x + y = 5$.

9. **Proposition:** Suppose $a, b$ and $c$ are integers. If $a^2|b$ and $b^3|c$, then $a^6|c$
Proof:
Assume $a^2|b$ and $b^3|c$.
Then $b = a^2k$ and $c = b^3l$ where $k, l$ are some integers.
So $c = b^3l = (a^2k)^3l = a^6k^3l$
So $c = a^6m$ where $m = k^3l \in \mathbb{Z}$
Therefore $a^6|c$.

10. **Proposition:** If $n \in \mathbb{N}$, then $n^2 = 2\binom{n}{2} + \binom{n}{1}$
Proof:
<u>Case 1:</u> Suppose n = 1
Then $n^2 = 1$ and $2\binom{1}{2} + \binom{1}{1} = 1$
Thus $n^2 = 2\binom{n}{2} + \binom{n}{1}$.
<u>Case 2:</u> Suppose $n \in \mathbb{N}$ and $n > 1$
Then $2\binom{n}{2} + \binom{n}{1} = 2\frac{n!}{2!(n-2)!} + \frac{n!}{1!(n-1)!} = 2\frac{(n-2)!(n-1)n}{2(n-2)!} + \frac{(n-1)!n}{(n-1)!} = (n-1)n + n = n^2$
Hence $n^2 = 2\binom{n}{2} + \binom{n}{1}$. These cases show that $n^2 = 2\binom{n}{2} + \binom{n}{1}$.

11. **Proposition:** If $n \in \mathbb{N}$ and $n \geq 2$, then the numbers $n!+2, n!+3, n!+4, n!+5, ..., n!+n$ are all composite.

Proof:
Suppose $n \geq 2$ and k is some integer such that $2 \leq k \leq n$.
Then $n! + k = (k-1)!k\frac{n!}{k!} + k = k((k-1)!\frac{n!}{k!} + 1)$
So $n! + k = ka$ where $a = (k-1)!\frac{n!}{k!} + 1 \in \mathbb{Z}$
So $n! + k$ is composite.
Therefore $n! + 2, n! + 3, n! + 4, n! + 5, ..., n! + n$ are all composite.

12. **Proposition:** If $a, b, c \in \mathbb{N}$ and $c \leq b \leq a$, then $\binom{a}{b}\binom{b}{c} = \binom{a}{b-c}\binom{a-b+c}{c}$
Proof:
Assume $a, b, c \in \mathbb{N}$ and $c \leq b \leq a$.
Then $b - c > 0$ and $a - b + c > 0$.
Then $\binom{a}{b-c}\binom{a-b+c}{c} = \frac{a!}{(b-c)!(a-b+c)!}\frac{(a-b+c)!}{c!(a-b)!} = \frac{a!}{(b-c)!c!(a-b)!} = \frac{a!}{(a-b)!b!}\frac{b!}{(b-c)!c!} = \binom{a}{b}\binom{b}{c}$
Therefore $\binom{a}{b}\binom{b}{c} = \binom{a}{b-c}\binom{a-b+c}{c}$