# 9. Steady state equations for the two-fluid model for stratified pipeflow

## 9.1. Governing equations

The two-fluid model for 1D isothermal two-phase stratified pipeflow reads:

$$\frac{\partial}{\partial t}(\rho_g A_g) + \frac{\partial}{\partial x}(\rho_g u_g A_g) = 0 \tag{9.1}$$

$$\frac{\partial}{\partial t}(\rho_l A_l) + \frac{\partial}{\partial x}(\rho_l u_l A_l) = 0 \tag{9.2}$$

$$\frac{\partial}{\partial t}(\rho_g u_g A_g) + \frac{\partial}{\partial x}(\rho_g u_g^2 A_g) + A_g \frac{\partial p}{\partial x} - \frac{\partial H_g}{\partial x} + S_1 = 0 \tag{9.3}$$

$$\frac{\partial}{\partial t}(\rho_l u_l A_l) + \frac{\partial}{\partial x}(\rho_l u_l^2 A_l) + A_l \frac{\partial p}{\partial x} - \frac{\partial H_l}{\partial x} + S_2 = 0 \tag{9.4}$$

The liquid holdup $A_l$ and gas holdup $A_g$ have to fill up the total area of the pipe $A_{tot}$, which allows a direct relation between $A_g$ and $A_l$: $A_l + A_g = A_{tot}$. $\frac{\partial H_l}{\partial x}$ and $\frac{\partial H_g}{\partial x}$ represent the hydraulic gradients in conservative form where $H_g$ and $H_l$ are defined as:

$$H_g = ((R - h_{lg})A_g + (1/12)(P_{lg}^3))\rho_g g \tag{9.5}$$

$$H_l = ((R - h_{lg})A_l - (1/12)(P_{lg}^3))\rho_l g \tag{9.6}$$

Here $R$ is the pipe radius, $h_{lg}$ is the height of the gas-liquid interface and $P_{lg}$ is the size of the gas-liquid interface. $S_1$ and $S_2$ contain source terms. For the moment we only consider horizontal pipe flow with no other source terms than frictional source terms. In that case $S_1$ and $S_2$ read:

$$S_1 = F_g + F_{lg} \tag{9.7}$$

$$S_2 = F_l - F_{lg} \tag{9.8}$$

Here $F_g$ is the friction of the gas with the wall, $F_l$ is the friction of the liquid with the wall, and $F_{lg}$ is the interfacial friction. The frictional terms are further discussed in section 9.1.2.

The model is closed by defining an equation of state for the gas density $\rho_g = \rho_g(p)$ ( $\rho_l$ is taken constant, $\rho_l = \rho_{l0}$ ):

$$\rho_g(p) = \frac{\rho_{norm} T_{norm}}{p_{norm}} \frac{p}{T} = \frac{p}{c_g^2} \tag{9.9}$$

Here $c_g$ is the speed of sound in gas. We now will develop the steady state two-fluid model equations.

### *9.1.1. Steady state equations*

As a first in reducing the governing equations 9.1, 9.2, 9.3, and 9.4 to their steady state form we drop the time dependent terms, and integrate equations 9.1, 9.2:

$$\rho_g u_g A_g = M_g \tag{9.10}$$

$$\rho_l u_l A_l = M_l \tag{9.11}$$

$$\frac{\partial}{\partial x}(\rho_g u_g^2 A_g) + A_g \frac{\partial p}{\partial x} - \frac{\partial H_g}{\partial x} + S_1 = 0 \tag{9.12}$$

$$\frac{\partial}{\partial x}(\rho_l u_l^2 A_l) + A_l \frac{\partial p}{\partial x} - \frac{\partial H_l}{\partial x} + S_2 = 0 \tag{9.13}$$

Here $M_g$ and $M_l$ are the mass flow of the gas and liquid respectively. In steady state they are constant everywhere in the pipe, and they are considered as a known quantity. We use these definitions of the mass flow to simplify the first terms of equation 9.12 and 9.13:

$$M_g \frac{\partial u_g}{\partial x} + A_g \frac{\partial p}{\partial x} - \frac{\partial H_g}{\partial x} + S_1 = 0 \tag{9.14}$$

$$M_l \frac{\partial u_l}{\partial x} + A_l \frac{\partial p}{\partial x} - \frac{\partial H_l}{\partial x} + S_2 = 0 \tag{9.15}$$

In addition we use equation 9.10 and 9.11 to express $u_g$ and $u_l$ as function of $A_l$ and $p$, that is:

$$u_g(A_l, p) = M_g/(\rho_g(A_{tot} - A_l)) \tag{9.16}$$

$$u_l(A_l, p) = M_l/(\rho_l A_l) \tag{9.17}$$

We now substitute these expressions for $u_g$ and $u_l$ and the equation of state equation 9.9 into equation 9.14 and 9.15 to obtain two equations with two unknowns $A_l$ and $p$. By collecting terms we can write equation 9.14 and 9.15 in the following form :

$$c_1 \frac{\partial p}{\partial x} + c_2 \frac{\partial A_l}{\partial x} + S_1 = 0 \tag{9.18}$$

$$c_3 \frac{\partial p}{\partial x} + c_4 \frac{\partial A_l}{\partial x} + S_2 = 0 \tag{9.19}$$

Here $c_1$, $c_2$, $c_3$, and $c_4$ are coefficients which are a function of $A_l$ and $p$. When neglecting the hydraulic gradients in equation 9.14 and 9.15, we find for $c_1$, $c_2$, $c_3$, and $c_4$:

$$c_1 = (A_{tot} - A_l) - \frac{c_g^2 M_g^2}{p^2 (A_{tot} - A_l)} \tag{9.20}$$

$$c_2 = \frac{c_g^2 M_g^2}{p (A_{tot} - A_l)^2} \tag{9.21}$$

$$c_3 = A_l \tag{9.22}$$

$$c_4 = \frac{-M_l^2}{A_l^2 \rho_{l0}} \tag{9.23}$$

In principle the hydraulic gradients can be included, but this will lead to more complicated expressions for $c_1$, $c_2$, $c_3$, and $c_4$. As the hydraulic gradients have very little effect on a typical steady state pipeline calculation, we choose to neglect it here. We stress, however, that without loss of generality it is possible to include the hydraulic gradients.

We can manipulate equation 9.18 and 9.19 to write a system of two coupled ODE's for $A_l$ and $p$.

$$\frac{\partial p}{\partial x} = \frac{S_2 c_2 - S_1 c_4}{c_1 c_4 - c_2 c_3} \tag{9.24}$$

$$\frac{\partial A_l}{\partial x} = \frac{-(S_2 c_1 - S_1 c_3)}{c_1 c_4 - c_2 c_3} \tag{9.25}$$

This set of two ODE's can be integrated along the pipe coordinate $x$ with an appropriate initial condition for $p$ and $A_l$. As an appropriate initial condition the value at the outlet boundary of the pipe can be taken were often the outlet pressure is known. The liquid hold up $A_l$ for this outlet pressure can be estimated by neglecting velocity gradients as well as hydraulic gradients in equation 9.14 and 9.15, resulting in:

$$\frac{\partial p}{\partial x} - \frac{S_1}{(A_{tot} - A_l)} = 0 \tag{9.26}$$

$$\frac{\partial p}{\partial x} - \frac{S_2}{A_l} = 0 \tag{9.27}$$

Subtracting the two equations above leads to an equation which can be solved for $A_l$ at the outlet.

### *9.1.2. Fluid friction model*

The source terms $S_1$ and $S_2$ defined equation 9.7 and 9.8 in contain the fluid friction terms. They are expressed as:

$$F_g = P_g \frac{1}{2} f_g \rho_g u_g |u_g| \tag{9.28}$$

$$F_l = P_l \frac{1}{2} f_l \rho_l u_l |u_l| \tag{9.29}$$

$$F_{lg} = P_{lg}\frac{1}{2}f_{lg}\rho_g(u_g - u_l)\,|(u_g - u_l)| \tag{9.30}$$

Here $f_g$, $f_l$, and $f_{lg}$ are the corresponding friction factors which are calculated with the Churchill correlation, which gives an explicit expression for the (Fanning) friction factor $f$.

$$f = 2\left(\left(\frac{8}{Re}\right)^{12} + (a+b)^{-1.5}\right)^{1/12} \tag{9.31}$$

where

$$a = \left(-2.457\log\left(\left(\frac{7}{Re}\right)^{0.9} + 0.27\frac{\epsilon}{D}\right)\right)^{16} \tag{9.32}$$

$$b = \left(\frac{37530}{Re}\right)^{16} \tag{9.33}$$

Here $\epsilon$ is the roughness of the pipe and $D$ the pipe diameter. $Re$ is the Reynolds number. To obtain the friction factor for the liquid $f_l$ we insert for $Re$ the liquid Reynolds number $Re_l$ in the formula above, which is defined as:

$$Re_l = \frac{\rho_l u_l D_l}{\mu_l} \tag{9.34}$$

where $D_l$ is hydraulic diameter for the liquid phase defined as:

$$D_l = \frac{4A_l}{P_l} \tag{9.35}$$

Similar, in order to obtain $f_g$ we use the gas Reynolds number $Re_g$ defined as:

$$Re_g = \frac{\rho_g u_g D_g}{\mu_g} \tag{9.36}$$

with $D_g$ is hydraulic diameter for the liquid phase defined as:

$$D_g = \frac{4A_g}{P_g + P_{lg}} \tag{9.37}$$

The interfacial friction factor $f_{lg}$ is taken to be equal to $f_g$, that is $f_{lg} = f_g$, for $f_g \geq 0.014$. If $f_g < 0.014$, $f_{lg} = 0.014$.