Governing equations of single- and multi-phase flow in one-dimensional pipelines

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Executive Summary

In this report the governing equations for single- and multi-phase flow in one-dimensional pipelines are derived starting from the three-dimensional compressible Navier-Stokes equations. For single-phase flow the governing partial differential equations are directly integrated over a pipe segment, leading to one-dimensional equations for cross-sectional averaged quantities. For multi-phase flow an averaging procedure is first employed to obtain the 'macroscopic' governing partial differential equations. These equations are then integrated over a pipe segment in a way similar to the single-phase case. A distinction is made between stratified flow and dispersed flow. Although both require a different derivation, the resulting equations are very similar for both types of flow. For other flow patterns a similar derivation is valid. The resulting equations provide a solid basis for future improvements of Compas, Shell's in-house 1D multiphase flow simulator. Some terms that can be directly improved in Compas are: hydraulic gradients, area variations and the effect of injections or leaks on the momentum and energy equations.

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1. Introduction

An important aspect of the design of pipelines for the transport of oil and gas is the description of multiphase flow behaviour in pipes, such as temperature, pressure, velocity and hold-ups. The flow of fluids (either gas or liquids) in pipelines is governed by the Navier-Stokes equations, which describe three-dimensional, unsteady, compressible, turbulent flow. However, solving these equations numerically is prohibitively expensive, because the range of time and length scales in pipeline flows is too large. It is therefore necessary to simplify the physical model and reduce the full 3D Navier-Stokes equations to a more tractable model. For multiphase flows in long pipelines we proceed by averaging the governing equations over the pipe cross-sectional area, leading to a set of 'one-dimensional' equations that describe how fluid properties change along the pipeline axis. The purpose of this report is to derive these 'one-dimensional' equations from the Navier-Stokes equations for both single- and multi-phase flow in pipelines, and to explicitly list the required assumptions. With regard to the hydraulic guidelines [8], in which the one-dimensional multiphase equations are also stated, this report offers a more extensive and fundamental derivation of these equations. The recent review report of van Zwieten [18] can be seen as complementary; it provides a literature review which is directed mainly towards numerical models, while the focus of the current report is on physical modelling. The indispensable help of Patricio Rosen Esquivel (PTE/EPFA), Ruud Henkes (PTE/EPFA) and Joost van Zwieten (TU Delft) in writing this report is greatly appreciated.

The equations derived in this report are used primarily for the development of Compas, which is Shell's in-house 1D multiphase flow simulator. Important questions, such as 'should friction terms be present in the energy equation?', 'how to take into account hydraulic gradients?', 'how do area variations enter in the equations?', 'how do injections or leaks affect the momentum and energy equation?' are treated here.

This report is structured as follows: chapter 2 discusses single phase flow and chapter 3 multiphase flow. A number of appendices give more information on the governing equations in differential form, definitions for stratified flow, averaging theorems, and additional forms of the energy equation.

2. Single-phase flow

The governing equations for single phase fluid flow can be found in many textbooks; see for example [11, 12]. We restrict ourselves to laminar flow. In this chapter these equations are applied to a pipe segment, and several simplifications are made to arrive at one-dimensional equations. An important step in obtaining the governing equations is the choice of a coordinate system. We employ an orthogonal curvilinear coordinate system along a generic spatial curve, as introduced by Germano [5]. The interested reader is referred to appendix A.

2.1. Conservation of mass

The integral form of the equation describing conservation of mass over a domain Ω (fixed in space and time, so the cross-section is assumed to be independent of time) with boundary $\partial\Omega$ reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\rho\,\mathrm{d}\Omega+\int_{\partial\Omega}\rho\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S=0, \tag{2.1}$$

where $\rho(\boldsymbol{x},t)$ is the density, $\boldsymbol{u}(\boldsymbol{x},t)$ the velocity and $\boldsymbol{n}$ the outward unit normal vector to $\partial\Omega$. We prefer to start with the integral form of the equations, because it is the most physical form, that requires less smoothness of the variables than the differential form and allows the presence of discontinuities (which can appear, for example, at interfaces in multiphase flows). In case a mass source G (injection or leak) is present on part of the boundary, part of the boundary integral in (2.1) is given by

$$\int_{\partial\Omega_{\text{source}}}\rho\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S=\int_{\partial\Omega_{\text{source}}}G\,\mathrm{d}S, \tag{2.2}$$

where $\partial\Omega_{\text{source}}$ is the part of the boundary where a leak or injection occurs; G has units of $\mathrm{kg}/(\mathrm{m}^2\mathrm{s})$.

The integral form (2.1) can be applied to a (three-dimensional) segment of a pipe occupying a domain Ω with boundary $\partial\Omega$, see figure 2.1. The boundary consists of three parts: inflow ①, outflow ② and pipe surface ③. On the pipe surface the no-slip condition applies and on the in- and outflow planes the outward normal is aligned with $\boldsymbol{e}_s$.

This results into:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\{\rho\}\,(t)V\right)+\langle\rho u_s\rangle\,(s_2,t)A(s_2)-\langle\rho u_s\rangle\,(s_1,t)A(s_1)=0, \tag{2.3}$$

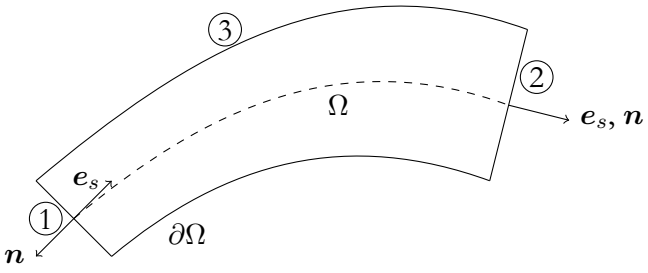

Figure 2.1.: Pipe segment.

or in short notation:

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\{\rho\}\, V\right) + \langle \rho u_s \rangle_2\, A_2 - \langle \rho u_s \rangle_1\, A_1 = 0,} \tag{2.4}$$

where $u_s = \boldsymbol{u} \cdot \boldsymbol{e}_s$ and the volume average $\{.\}$ and area average $\langle . \rangle$ are defined by equations (C.1) and (C.2), respectively. The volume V and area A also follow from these definitions:

$$V = \int_\Omega \mathrm{d}\Omega \qquad A_1 = \int_{S_1} \mathrm{d}S. \tag{2.5}$$

Equation (2.4) is exact and still in integral form.

The differential form can be obtained by taking a small finite volume, with sides $A(s)$ and $A(s+\delta s)$ and volume $V(s) \approx A(s)\delta s$ (neglecting curvature effects), yielding

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\{\rho\}\,(t) A(s)\delta s\right) + \langle \rho u_s \rangle\,(s+\delta s, t) A(s+\delta s) - \langle \rho u_s \rangle\,(s,t) A(s) = 0. \tag{2.6}$$

Dividing by δs and taking the limit $\delta s \to 0$ we get

$$\frac{\partial}{\partial t}\left(\langle \rho \rangle\,(s,t) A(s)\right) + \frac{\partial}{\partial s}\left(\langle \rho u_s \rangle\,(s,t) A(s)\right) = 0. \tag{2.7}$$

An alternative way of deriving this equation is by starting from the differential form (A.1) and integrating it over a cross-sectional area, which requires the use of Gauss' theorem for surface integrals.

If ρ is assumed uniform over the pipe cross section, $\langle \rho u_s \rangle = \rho \langle u_s \rangle$ and $\langle \rho \rangle = \rho$, and one obtains

$$\frac{\partial}{\partial t}(\rho(s,t) A(s)) + \frac{\partial}{\partial s}(\rho(s,t) \langle u_s \rangle\,(s,t) A(s)) = 0. \tag{2.8}$$

However, a more generally applicable assumption is to approximate the cross-sectional average of products in terms of the product of averages (see appendix C, equation (C.8)):

$$\langle fg \rangle \approx \langle f \rangle \langle g \rangle\,. \tag{2.9}$$

The error in this equation is on the order of D^2, with D the pipe diameter. This yields

$$\frac{\partial}{\partial t}(\rho(s,t)A(s)) + \frac{\partial}{\partial s}(\rho(s,t)u_s(s,t)A(s)) = 0, \tag{2.10}$$

or in short notation

$$\boxed{\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial s}(\rho u_s A) = 0,} \tag{2.11}$$

where we left out the indication of averaging, so *all variables should be interpreted as cross-sectional averages*. A similar equation is shown in [1, 8]. Note that, since A is assumed independent of time, it can be taken outside the derivative in the first term. Mass sources or sinks will appear on the right hand side as

$$\int_{C_{\text{source}}} G \, \mathrm{d}C. \tag{2.12}$$

2.2. Conservation of momentum

The integral form of the equation describing conservation of momentum reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho \boldsymbol{u} \, \mathrm{d}\Omega + \int_{\partial\Omega} \rho \boldsymbol{u}\boldsymbol{u} \cdot \boldsymbol{n} \, \mathrm{d}S = -\int_{\partial\Omega} p\boldsymbol{n} \, \mathrm{d}S + \int_{\partial\Omega} \boldsymbol{\tau} \cdot \boldsymbol{n} \, \mathrm{d}S + \int_\Omega \rho \boldsymbol{g} \, \mathrm{d}\Omega, \tag{2.13}$$

where p is the pressure, $\boldsymbol{\tau}$ is the viscous stress tensor, and $\boldsymbol{g} = -g\boldsymbol{e}_y$ denotes gravity. If mass sources G (injections or leaks) are present, see equation (2.2), they also contribute to the streamwise momentum equation, viz.

$$\int_{\partial\Omega_{\text{source}}} G u_s \, \mathrm{d}S, \tag{2.14}$$

but this depends on how the injection takes place, i.e., on the streamwise velocity of the injected stream. In [13] it is assumed that injections/leaks at the inlet or outlet are done with the local velocity and local pressure; in [1] it is assumed that injections are at right angles to the pipe wall, so no streamwise momentum is added ($u_s = 0$). The latter approach is currently taken in Compas.

We evaluate the momentum equation for the pipe segment shown in figure 2.1. The streamwise component of momentum is obtained by taking the dot product with the unit vector $\boldsymbol{e}_s$. In order to get $\boldsymbol{e}_s$ inside the volume integral term, we assume that the pipe centerline is straight ($\kappa = 0$), i.e. we neglect curvature, as for the mass equation. This leads to the following expressions.

The unsteady term:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho \boldsymbol{u} \, \mathrm{d}\Omega \cdot \boldsymbol{e}_s = \frac{\mathrm{d}}{\mathrm{d}t}\left(\{\rho u_s\} V\right). \tag{2.15}$$

The convective term:

$$\int_{\partial\Omega} \rho \boldsymbol{u}\boldsymbol{u} \cdot \boldsymbol{n}\, \mathrm{d}S \cdot \boldsymbol{e}_s = \left\langle \rho u_s^2 \right\rangle_2 A_2 - \left\langle \rho u_s^2 \right\rangle_1 A_1. \tag{2.16}$$

The pressure term:

$$\int_{\partial\Omega} p\boldsymbol{n}\, \mathrm{d}S \cdot \boldsymbol{e}_s = \langle p \rangle_2 A_2 - \langle p \rangle_1 A_1 + \langle p\boldsymbol{e}_s \cdot \boldsymbol{n} \rangle_3 A_3, \tag{2.17}$$

where A_3 is the area of the outer pipe surface and $\boldsymbol{n}$ is outward unit normal vector. The last term is nonzero if the pipe diameter is changing as function of s (expanding/contracting pipes).
The gravity term:

$$\int_{\Omega} \rho \boldsymbol{g}\, \mathrm{d}\Omega \cdot \boldsymbol{e}_s = -\int_{\Omega} \rho g \boldsymbol{e}_y \cdot \boldsymbol{e}_s\, \mathrm{d}\Omega = -g \left\{\rho \sin\varphi\right\} V. \tag{2.18}$$

The angle $\varphi(s)$ is the angle of the pipe centerline with respect to the horizontal ($\boldsymbol{e}_s \cdot \boldsymbol{e}_x = \cos\varphi$), see equation (A.21).
The viscous term:

$$\int_{\partial\Omega} \boldsymbol{\tau} \cdot \boldsymbol{n}\, \mathrm{d}S \cdot \boldsymbol{e}_s = \int_{\partial\Omega} \tau_{ss}(\boldsymbol{e}_s \cdot \boldsymbol{n}) + \tau_{sr}(\boldsymbol{e}_r \cdot \boldsymbol{n}) + \tau_{s\theta}(\boldsymbol{e}_\theta \cdot \boldsymbol{n})\, \mathrm{d}S. \tag{2.19}$$

If the cross section is circular, then $\boldsymbol{e}_r \cdot \boldsymbol{n} = \cos\zeta$, and $\boldsymbol{e}_\theta \cdot \boldsymbol{n} = 0$ on $\partial\Omega_3$. The components of the stress tensor in curvilinear coordinates, including curvature effects, are given in appendix A.3.

This gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\{\rho u_s\} V) + \left\langle \rho u_s^2 \right\rangle_2 A_2 - \left\langle \rho u_s^2 \right\rangle_1 A_1 = -(\langle p \rangle_2 A_2 - \langle p \rangle_1 A_1) - \langle p\boldsymbol{e}_s \cdot \boldsymbol{n} \rangle_3 A_3 \\ - g \sin\varphi \left\{\rho\right\} V + \langle \tau_{ss} \rangle_2 A_2 - \langle \tau_{ss} \rangle_1 A_1 + \langle \tau_{\mathrm{tot}} \rangle_3 A_3, \end{aligned} \tag{2.20}$$

where τ_{tot} is given by:

$$\tau_{\mathrm{tot}} = (\boldsymbol{\tau} \cdot \boldsymbol{n}) \cdot \boldsymbol{e}_s = \tau_{ss}(\boldsymbol{e}_s \cdot \boldsymbol{n}) + \tau_{sr}(\boldsymbol{e}_r \cdot \boldsymbol{n}) + \tau_{s\theta}(\boldsymbol{e}_\theta \cdot \boldsymbol{n}). \tag{2.21}$$

Like equation (2.4), this equation is still exact and in integral form.

To further simplify the model we make the assumption that at the entrance and exit of the pipe segment the viscous stress τ_{ss}, is negligible compared to the pressure stress.

We will apply the same limiting process as in section 2.1. To do so, the pressure and viscous stresses are rewritten for infinitesimal pipeline segments. The integral over the pipe surface is written as

$$\begin{aligned} \langle p\boldsymbol{e}_s \cdot \boldsymbol{n} \rangle_3 A_3 &= \int_{A_3} p\boldsymbol{e}_s \cdot \boldsymbol{n}\, \mathrm{d}S \\ &= \int_{A_3} p\boldsymbol{e}_s \cdot \boldsymbol{n}\, \mathrm{d}C \frac{\mathrm{d}s}{\boldsymbol{e}_r \cdot \boldsymbol{n}} = -\int_{A_3} p \frac{\sin\zeta}{\cos\zeta} \mathrm{d}C\mathrm{d}s = -\int_{A_3} p \frac{\partial R}{\partial s} \mathrm{d}C\mathrm{d}s, \end{aligned} \tag{2.22}$$

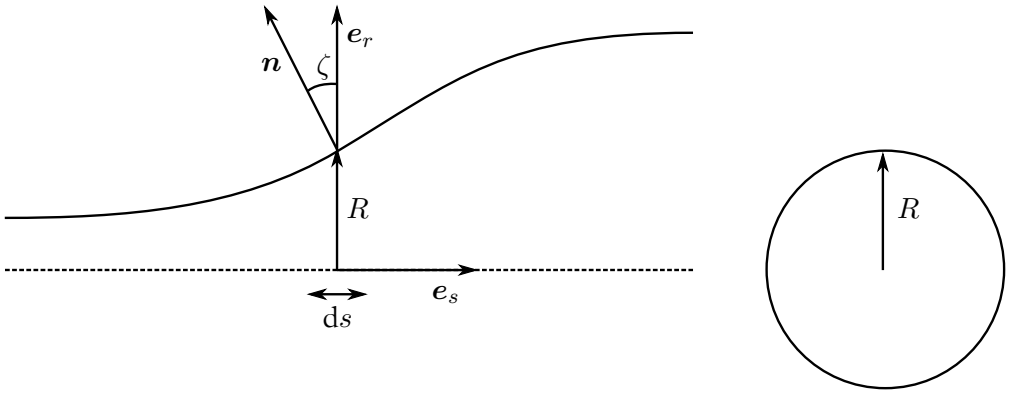

Figure 2.2.: Expanding pipe.

where $R(s,\theta)$ is the distance from the pipe centerline to the pipe surface and $\tan\zeta = \partial R/\partial s$. The viscous stresses are written as

$$\langle \tau_{\mathrm{tot}} \rangle_3 = \frac{1}{A_3}\int_S \tau_{\mathrm{tot}}\,\mathrm{d}S = \frac{1}{A_3}\int_s\int_C \frac{\tau_{\mathrm{tot}}}{\cos\zeta}\,\mathrm{d}C\mathrm{d}s, \tag{2.23}$$

so it is convenient to define an average stress $\bar{\tau}$ along the perimeter as follows:

$$\bar{\tau}_{\mathrm{tot}} = \frac{1}{C}\int_C \frac{\tau_{\mathrm{tot}}}{\cos\zeta}\,\mathrm{d}C. \tag{2.24}$$

Taking the limit $\delta s \to 0$ leads to the following differential equation:

$$\frac{\partial}{\partial t}(\langle\rho\rangle\,\langle u_s\rangle\,A) + \frac{\partial}{\partial s}(\langle\rho\rangle\,\langle u_s\rangle^2\,A) = -\frac{\partial}{\partial s}(\langle p\rangle\,A) + \int_C p\frac{\partial R}{\partial s}\mathrm{d}C - \langle\rho\rangle\,gA\sin\varphi + \bar{\tau}_{\mathrm{tot}}C, \tag{2.25}$$

The first integral on the right side can be written with Leibniz' rule as

$$\frac{\partial}{\partial s}(\langle p\rangle\,A) = \frac{\partial}{\partial s}\int_{A(s)} p\,\mathrm{d}S = \int_{A(s)}\frac{\partial p}{\partial s}\mathrm{d}S + \int_C p\frac{\partial R}{\partial s}\mathrm{d}C. \tag{2.26}$$

The boundary integral is exactly equal to the boundary integral in (2.25), and therefore we obtain

$$\frac{\partial}{\partial t}(\langle\rho\rangle\,\langle u_s\rangle\,A) + \frac{\partial}{\partial s}(\langle\rho\rangle\,\langle u_s\rangle^2\,A) = -\left\langle\frac{\partial p}{\partial s}\right\rangle A - \langle\rho\rangle\,gA\sin\varphi + \bar{\tau}_{\mathrm{tot}}C, \tag{2.27}$$

where again the cross-sectional average of products are approximated in terms of the product of averages. Leaving out the indication of averages leads to

$$\boxed{\frac{\partial}{\partial t}(\rho u_s A) + \frac{\partial}{\partial s}(\rho u_s^2 A) = -\frac{\partial p}{\partial s}A - \rho g A\sin\varphi + \bar{\tau}_{\mathrm{tot}}C.} \tag{2.28}$$

Note that taking $\left\langle \frac{\partial p}{\partial s} \right\rangle = \frac{\partial \langle p \rangle}{\partial s} = \frac{\partial p}{\partial s}$ is consistent with setting $\langle p \rangle = p$. We stress again that ρ, p and u_s are cross-sectional averaged quantities that depend on s and t; the geometric quantities A, C and φ are assumed to only depend on s. Equation (2.27) corresponds to the momentum equation in the hydraulic guidelines [8]. If A depends on s, the first term on the right hand side cannot be brought in conservative form - this is a consequence of the 'quasi-1D' approach.

In general, the viscous normal stress at the wall is neglected compared to the viscous shear stress, and for a circular pipe one obtains (cf. equation (2.23))

$$\tau_{\text{tot}} = \tau_{sr} \cos\zeta \quad \Rightarrow \quad \bar{\tau}_{\text{tot}} = \frac{1}{C} \int_C \tau_{sr} \, \mathrm{d}C, \tag{2.29}$$

where

$$\tau_{sr} = \mu \left(\frac{\partial u_r}{\partial s} + \frac{\partial u_s}{\partial r} \right) = \mu \frac{\partial u_s}{\partial r}, \tag{2.30}$$

and the last equality holds due to the one-dimensional flow assumption: u_r, $u_\theta \ll u_s$, so $\boldsymbol{u} \approx u_s \boldsymbol{e}_s$.

Conservation of momentum in the other coordinate directions can be obtained by taking the inner product of (2.13) with $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$. These conservation equations do not yield very useful information, because the component of gravity in these directions, averaged over the cross-section, vanishes. Instead, we take the inner product in $\boldsymbol{N}$-direction. For simplicity, take a pipe with a constant circular cross-section. With the one-dimensional flow assumption the unsteady, convective and viscous terms are negligible. Conservation of momentum in $\boldsymbol{N}$-direction reduces therefore to a balance of gravity and pressure forces:

$$\begin{aligned} \int_0^{2\pi} p \cos(\theta) \, R \, \mathrm{d}\theta &= - \int_0^{2\pi} \int_0^R \rho g \cos(\varphi) \, r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= - \langle \rho \rangle \, g \cos\varphi A. \end{aligned} \tag{2.31}$$

In differential form the above balance reduces simply to

$$\frac{\partial p}{\partial h} = -\rho g \cos\varphi, \tag{2.32}$$

where the derivative with respect to h indicates the $\boldsymbol{N}$-direction. In other words:

$$\boxed{p(s,h) = p(s,h_{\text{ref}}) - \rho g (h - h_{\text{ref}}) \cos\varphi,} \tag{2.33}$$

assuming that the density is uniform over the cross-section. By taking the reference pressure in the center of the pipe, the momentum equation in s-direction is unchanged (integration of an odd function over the symmetric contour is zero). However, in multiphase flow the variation of pressure with height leads to so-called hydraulic gradients; this will be discussed in section 3.2.2.

2.3. Conservation of energy

The integral form of the equation describing conservation of (specific total) energy E reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho E \,\mathrm{d}\Omega + \int_{\partial\Omega} \rho E \boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = \\ -\int_{\partial\Omega} p\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_{\partial\Omega}(\boldsymbol{\tau}\cdot\boldsymbol{u})\cdot\boldsymbol{n}\,\mathrm{d}S - \int_{\partial\Omega}\boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_\Omega \rho\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}\Omega + \int_\Omega \dot{q}\,\mathrm{d}\Omega. \quad (2.34)$$

The total energy changes due to work done by pressure forces, friction forces, heat conduction ($\boldsymbol{q}$), gravity forces, and heat sources $\dot{q}$. Heat conduction is typically modelled with Fourier's law: $\boldsymbol{q} = -k\nabla T$.
The specific total energy E is the sum of specific internal energy e and kinetic energy:

$$E = e + \frac{1}{2}|\boldsymbol{u}|^2. \quad (2.35)$$

Note that the total energy E does not change due to viscous effects (except at inflow or outflow boundaries) - see also appendix D. Mass sources G also change the total energy; part of the boundary integrals in (2.34) will be given by

$$\int_{\partial\Omega_{\text{source}}} G(E + p/\rho)\,\mathrm{d}S, \quad (2.36)$$

where we ignored viscous stresses. A similar form is suggested by [13].
The *mechanical* energy equation, which can be found by taking the inner product of the momentum equation with the velocity vector $\boldsymbol{u}$, features the term $\boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau})$. Subtracting this equation from the total energy equation gives the thermal energy equation (for e), which features the dissipation term $(\boldsymbol{\tau} : \nabla\boldsymbol{u})$, since $\nabla\cdot(\boldsymbol{\tau}\cdot\boldsymbol{u}) = \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau}) + \boldsymbol{\tau} : \nabla\boldsymbol{u}$. The thermal energy e increases due to viscous dissipation. In appendix D the kinetic and internal energy equations are discussed in more detail.

Again, evaluating the integral expression for the pipe segment in figure 2.1 gives

$$\frac{\mathrm{d}}{\mathrm{d}t}(\{\rho E\}\,V) + \langle \rho E u_s\rangle_2\,A_2 - \langle \rho E u_s\rangle_1\,A_1 = -(\langle p u_s\rangle_2\,A_2 - \langle p u_s\rangle_1\,A_1) + \\ \langle \tau_{ss}u_s + \tau_{sr}u_r + \tau_{s\theta}u_\theta\rangle_2\,A_2 - \langle \tau_{ss}u_s + \tau_{sr}u_r + \tau_{s\theta}u_\theta\rangle_1\,A_1 + \\ -(\langle q_s\rangle_2\,A_2 - \langle q_s\rangle_1\,A_1 + \langle \boldsymbol{q}\cdot\boldsymbol{n}\rangle_3\,A_3) + \{\rho\boldsymbol{g}\cdot\boldsymbol{u}\}\,V + \{\dot{q}\}\,V. \quad (2.37)$$

The effect of a changing pipe diameter does not show up in the pressure term due to the fact that $u = 0$ on the wall. Now employing the same assumptions as for the momentum equation, and additionally:

- one-dimensional flow: $u_\theta, u_r \ll u_s$, so $\boldsymbol{g}\cdot\boldsymbol{u} \approx -g u_s \sin\varphi$,
- neglecting the work done by viscous stresses compared to the work done by pressure stress at the inlet and outlet (consistent with neglecting normal viscous stress compared to pressure stress, as done in section 2.2),

- neglect streamwise heat conduction $\frac{\partial}{\partial s}(\langle q_s \rangle A)$ compared to transversal heat conduction.

leads to

$$\frac{\partial}{\partial t}(\langle \rho \rangle \langle E \rangle A) + \frac{\partial}{\partial s}(\langle \rho \rangle \langle E \rangle \langle u_s \rangle A) = -\frac{\partial}{\partial s}(\langle p \rangle \langle u_s \rangle A) - \bar{Q}C - \langle \rho \rangle g \langle u_s \rangle A \sin\varphi + \dot{q}A. \tag{2.38}$$

Here $\bar{Q}$ is the average heat conduction along the perimeter:

$$\bar{Q} = \frac{1}{C}\int_C \frac{\boldsymbol{q}\cdot\boldsymbol{n}}{\cos\zeta}\mathrm{d}C. \tag{2.39}$$

If the streamwise heat conduction $\frac{\partial}{\partial s}(q_s A)$ is not neglected, it can be combined with $\bar{Q}$ to yield a non-conservative term (like the pressure):

$$\frac{\partial}{\partial s}(q_s A) + \frac{1}{C}\int_C \frac{\boldsymbol{q}\cdot\boldsymbol{n}}{\cos\zeta}\mathrm{d}C = \frac{\partial q_s}{\partial s}A + \frac{1}{C}\int_C q_r \,\mathrm{d}C. \tag{2.40}$$

Upon introducing the (specific) total enthalpy H as:

$$H := E + \frac{p}{\rho}, \tag{2.41}$$

and leaving out the indication of averages, the following energy equation is obtained:

$$\frac{\partial}{\partial t}(\rho E A) + \frac{\partial}{\partial s}(\rho H u_s A) = -\bar{Q}C - \rho g u_s A \sin\varphi + \dot{q}A. \tag{2.42}$$

This is equivalent to the one in [8]. A possible way to model the heat transfer term Q is in terms of an overall heat transfer coefficient (also known as U-value); for more information and other approaches see [8]. The term $\dot{q}$ represents the total heat source added to the system.

2.4. Summary

We summarize the mass, momentum and energy equations in differential form for 1D single-phase flow:

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial s}(\rho u_s A) = 0, \tag{2.43}$$

$$\frac{\partial}{\partial t}(\rho u_s A) + \frac{\partial}{\partial s}(\rho u_s^2 A) = -\frac{\partial p}{\partial s}A - \rho g A \sin\varphi + \bar{\tau}_{\mathrm{tot}}C, \tag{2.44}$$

$$\frac{\partial}{\partial t}(\rho E A) + \frac{\partial}{\partial s}(\rho H u_s A) = -\bar{Q}C - \rho g u_s A \sin\varphi + \dot{q}A. \tag{2.45}$$

In addition, there is the (decoupled) momentum equation in normal direction:

$$\frac{\partial p}{\partial h} = -\rho g \cos\varphi. \tag{2.46}$$

The system of equations becomes closed when the density ρ and internal energy e are expressed in terms of pressure and temperature via an equation of state. Furthermore, the wall friction and heat flux have to provided. The wall friction is typically determined based on the Reynolds number and the wall roughness via

The assumptions behind these equations are:

- the radial and circumferential velocities are much smaller than the streamwise velocity;
- the pipeline curvature is neglected;
- the density is uniform over the cross-section;
- the cross-sectional average of products of quantities is approximated by the product of the cross-sectional averages;
- the pressure is to be interpreted as the pressure in the center of the pipeline;
- viscous stresses (and the work they perform) in streamwise direction are neglected compared to pressure stresses in streamwise direction;
- streamwise heat conduction is neglected.

3. Multi-phase flow

The equations of motion of single phase flow also hold for multiphase flow when considering a local fluid volume occupied by a certain phase (i.e. in a *microscopic* sense) and when appropriate interface conditions between the phases are taken into account. Some general references are [2, 10, 11, 14, 19]. For each conservation equation we first derive the multiphase flow equations for *dispersed flow* by employing a volume-averaging technique, and integrate the resulting equations over a pipe segment (see figure 3.1). For *stratified flow* the governing equations are derived by integrating over the volume occupied by a certain phase: no volume-averaging is required.

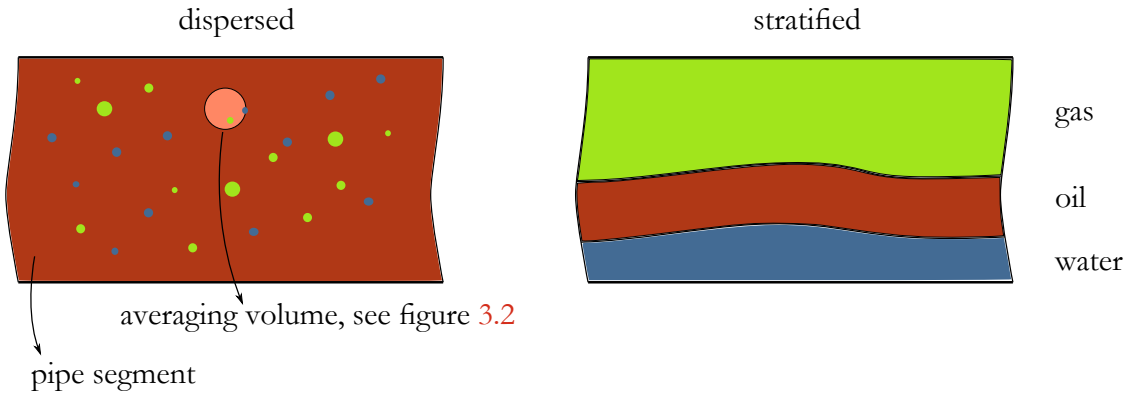

Figure 3.1.: Dispersed and stratified flow.

3.1. Conservation of mass

3.1.1. Dispersed flow

In contrast to section 2.1, we consider a *moving* control volume $\Omega_\beta(\boldsymbol{x}, t)$, occupied by a phase β. This is necessary because we consider phases that are moving with respect to each other. The control volume has a part of the boundary which is not moving (indicated by S_β) and parts that are moving with a local velocity $\boldsymbol{w}(x, t)$ (indicated by $S_{\beta\gamma}$). This distinction will be useful when employing the averaging approach. Conservation of mass for this phase β is then given by [14]:

$$\frac{\partial}{\partial t}\int_{\Omega_\beta} \rho_\beta \,\mathrm{d}\Omega + \int_{S_\beta} \rho_\beta \boldsymbol{u}_\beta \cdot \boldsymbol{n}_\beta \,\mathrm{d}S + \sum_{\gamma,\gamma\neq\beta} \int_{S_{\beta\gamma}} \rho_\beta (\boldsymbol{u}_\beta - \boldsymbol{w}) \cdot \boldsymbol{n}_\beta \,\mathrm{d}S = 0. \tag{3.1}$$

Compared to the single phase case, equation (2.1), an additional term has appeared, which is due to the fact that a moving instead of a stationary volume is considered. Between two phases β and γ the following interface condition holds:

$$\rho_\beta(\boldsymbol{u}_\beta - \boldsymbol{w}) \cdot \boldsymbol{n}_\beta + \rho_\gamma(\boldsymbol{u}_\gamma - \boldsymbol{w}) \cdot \boldsymbol{n}_\gamma = 0. \tag{3.2}$$

If no mass transfer due to phase transition takes place, the velocity field is continuous through the interface, and the interface moves with the local fluid velocity:

$$\boldsymbol{u}_\beta = \boldsymbol{u}_\gamma = \boldsymbol{w}. \tag{3.3}$$

It is generally not feasible to solve for each phase and interface position separately. One has to resort to averaging techniques to obtain the conservation equations in a *macroscopic* sense. We will employ *volume averaging*, similar to [7, 10, 14]; for other averaging techniques (e.g. time-averaging, ensemble averaging) see [3, 11]. The basic assumption in the volume-averaging process is that an averaging volume exists with a length scale much smaller than that of the system under consideration, but much larger than the characteristic size of the interfacial structures: scale separation.

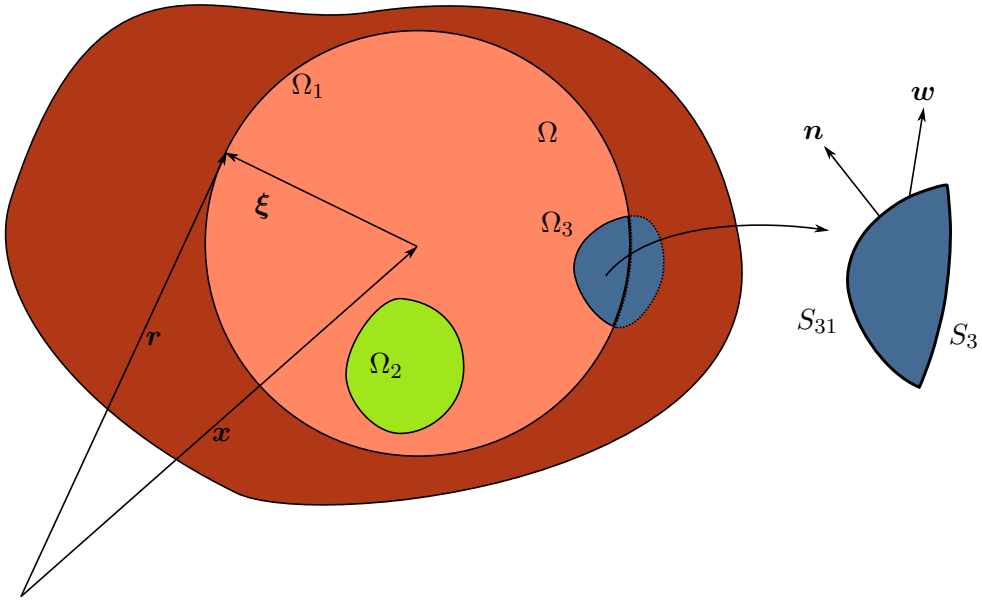

Figure 3.2.: Detail of figure 3.1: averaging volume $\Omega(\boldsymbol{x})$ containing part of phase 3 (water, blue), phase 2 (gas, green), and part of phase 1 (oil, red).

We consider a phase (phase 1) in which two other phases (phases 2 and 3) are dispersed, see figure 3.2. To each point $\boldsymbol{x}$ in space we attach such a volume $\Omega(\boldsymbol{x})$ (a 'representative elementary volume' [7]), fixed in time and occupied by different moving, non-overlapping, immiscible phases β, such that $\Omega(\boldsymbol{x}) = \Omega_1(\boldsymbol{x}, t) \cup \Omega_2(\boldsymbol{x}, t) \cup \ldots$. Within $\Omega(\boldsymbol{x})$ the (global) position vector of a point with respect to the inertial reference frame is given by

$\boldsymbol{r}$, and the (local) position vector with respect to the center of the volume ($\boldsymbol{x}$) by $\boldsymbol{\xi}$:

$$\boldsymbol{r} = \boldsymbol{x} + \boldsymbol{\xi}. \tag{3.4}$$

The volume of $\Omega(\boldsymbol{x})$ is the same for each averaging volume and is given by

$$V = \int_{\Omega(\boldsymbol{x})} \mathrm{d}\Omega. \tag{3.5}$$

By defining the phase distribution function $\gamma_\beta(\boldsymbol{r}, t)$ as

$$\gamma_\beta(\boldsymbol{r}, t) := \begin{cases} 1 & \boldsymbol{r} \in \Omega_\beta(\boldsymbol{x}, t), \\ 0 & \text{else}, \end{cases} \tag{3.6}$$

the volume occupied by phase β can be expressed as

$$V_\beta(\boldsymbol{x}, t) = \int_{\Omega(\boldsymbol{x})} \gamma_\beta(\boldsymbol{r}, t)\, \mathrm{d}\Omega = \int_{\Omega_\beta(\boldsymbol{x},t)} \mathrm{d}\Omega, \tag{3.7}$$

such that

$$V = \sum_\beta V_\beta(\boldsymbol{x}, t). \tag{3.8}$$

Define the volume fractions

$$\alpha_\beta(\boldsymbol{x}, t) := \frac{V_\beta(\boldsymbol{x}, t)}{V}, \tag{3.9}$$

and consequently $\sum_\beta \alpha_\beta(\boldsymbol{x}, t) = 1$. Furthermore we define a ('moving') volume average operator as

$$\{f_\beta\}(\boldsymbol{x}, t) := \frac{1}{V_\beta(\boldsymbol{x}, t)} \int_{\Omega_\beta(\boldsymbol{x},t)} f_\beta(\boldsymbol{r}, t)\, \mathrm{d}\Omega. \tag{3.10}$$

This is known as the *intrinsic* or *phasic* average (note the difference with the volume average (C.1)). The *partial* average is found by division by the total volume V instead of the phase volume V_β, see e.g. [10]. To each point in space and time an average value of f_β is associated - as such it should be regarded as a 'sliding' or 'moving' average. In the macroscopic field the average volume represents a physical *point* [7]. This is possible due to the assumption of scale separation. Note that in the single phase case volume- (or area-) averaged quantities were also introduced, but they did not depend on position explicitly.

The boundary of $\Omega_\beta(\boldsymbol{x}, t)$, namely $\partial\Omega_\beta(\boldsymbol{x}, t)$ consists of parts of the outer boundary of the averaging volume, S_β, (where a phase is 'cut' by the boundary) and parts interfacing the other phases, $S_{\beta\gamma}$ (see figure 3.2):

$$\partial\Omega_\beta(\boldsymbol{x}, t) = S_\beta(\boldsymbol{x}) \bigcup_{\gamma \neq \beta} S_{\beta\gamma}(\boldsymbol{x}, t). \tag{3.11}$$

With the definitions above the continuity equation (3.1) can be volume-averaged. Note that the integrands depend on the coordinate $\boldsymbol{r}$ (or $\boldsymbol{\xi}$) and t, and the domain and boundary depend on $\boldsymbol{x}$ and t. The second term in (3.1) can be written with the following identity (see e.g. [14])

$$\int_{S_\beta(\boldsymbol{x},t)} \boldsymbol{A}_\beta(\boldsymbol{r},t)\cdot\boldsymbol{n}_\beta \,\mathrm{d}S = \nabla_{\boldsymbol{x}}\cdot\int_{\Omega_\beta(\boldsymbol{x},t)} \boldsymbol{A}_\beta(\boldsymbol{r},t)\,\mathrm{d}\Omega. \tag{3.12}$$

This is not very intuitive, since S_β does not form the complete boundary of Ω_β (S_β is not closed). See appendix C for a derivation. In terms of volume averages, conservation of mass can then be expressed as

$$\boxed{\frac{\partial}{\partial t}(\alpha_\beta\{\rho_\beta\}) + \nabla\cdot(\alpha_\beta\{\rho_\beta\boldsymbol{u}_\beta\}) = \Psi_\beta,} \tag{3.13}$$

where the interfacial mass transfer terms are given by

$$\Psi_\beta = -\frac{1}{V}\sum_{\gamma,\gamma\neq\beta}\int_{S_{\beta\gamma}} \rho_\beta(\boldsymbol{u}_\beta-\boldsymbol{w})\cdot\boldsymbol{n}_\beta\,\mathrm{d}S. \tag{3.14}$$

An alternative derivation of (3.13) that starts from the differential form of the governing equations (and used in many literature sources) is presented in appendix C.3. In contrast to (2.4), the time derivative is still a partial derivative, because the volume averaged quantities are a function of space and time.

Equation (3.13) is still in integral form, but the region of integration is the averaging volume, which is a point compared to the size of the domain. In the macroscopic field, equation (3.13) can be interpreted as being in differential form, and we write it as

$$\frac{\partial}{\partial t}(\alpha_\beta\rho_\beta) + \nabla\cdot(\alpha_\beta\rho_\beta\boldsymbol{u}_\beta) = \Psi_\beta. \tag{3.15}$$

The quantities in this equation are locally volume-averaged values; the average of products is approximated as product of averages. Furthermore, Ψ_β is not computed from (3.14), because the details of the interfaces between the phases are not resolved. Equation (3.15) now represents the governing differential equation for three-dimensional multiphase flow, to be solved on a one-dimensional pipe segment.

Equation (3.15) is integrated over a pipe segment, and the limiting process $\delta s \to 0$ is applied. Then, after applying the same 1D assumptions as in the single-phase case, section 2.1, the following differential equation results:

$$\boxed{\frac{\partial}{\partial t}(\alpha_\beta\rho_\beta A) + \frac{\partial}{\partial s}(\alpha_\beta\rho_\beta u_{s,\beta}A) = \Psi_\beta A.} \tag{3.16}$$

The volume fraction α_β has become an area fraction in the limit of an infinitesimally small pipe segment. Formally, the variables in this equation represent the area average

of volume averaged quantities - the governing equations have been integrated twice. In the 1D models applied in Compas the mass transfer term Ψ_β is computed as a function of local pressure and temperature [16]. The mass transfer terms satisfy the property

$$\sum_\beta \Psi_\beta = 0. \tag{3.17}$$

In case an injection or leak is present, a term $G_\beta A$ will be added to equation (3.16). An equation for the bulk can be obtained by summing (3.1) over the different phases, and using (3.17):

$$\boxed{\sum_\beta \left(\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta A) + \frac{\partial}{\partial s}(\alpha_\beta \rho_\beta u_{s,\beta} A) \right) = 0.} \tag{3.18}$$

3.1.2. Stratified flow

It is questionable whether equation (3.16) holds in the case of stratified flow, because in stratified flow the averaging volume is certainly not much larger than the characteristic size of the interfacial structures. In the case of stratified flow equation (3.1) can also be applied directly in a macroscopic sense (for simplicity without interphasial mass transfer):

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\{\rho_\beta\}(t) V_\beta(t)\right) + \langle \rho_\beta u_{s,\beta} \rangle (s_2,t) A_\beta(s_2,t) - \langle \rho_\beta u_{s,\beta} \rangle (s_1,t) A_\beta(s_1,t) = 0. \tag{3.19}$$

We stress that the integration volume V_β is completely different here than in the derivation of the dispersed flow equations. The differential form becomes, after applying the same 1D assumptions as in the single-phase case,

$$\boxed{\frac{\partial}{\partial t}(\rho_\beta A_\beta) + \frac{\partial}{\partial s}(\rho_\beta u_{s,\beta} A_\beta) = 0.} \tag{3.20}$$

This corresponds to the equations in [8, 9]. For an infinitesimally small element, the volume fraction equals $\alpha_\beta = A_\beta / A$, so this equation can be written in a form similar to (3.16) (omitting the mass transfer terms):

$$\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta A) + \frac{\partial}{\partial s}(\alpha_\beta \rho_\beta u_{s,\beta} A) = 0. \tag{3.21}$$

3.2. Conservation of momentum

3.2.1. Dispersed flow

Conservation of momentum for each phase is given by [14]:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\beta} \rho_\beta \boldsymbol{u}_\beta \,\mathrm{d}\Omega + \int_{S_\beta} \rho_\beta \boldsymbol{u}_\beta \boldsymbol{u}_\beta \cdot \boldsymbol{n}_\beta \,\mathrm{d}S + \sum_{\gamma,\gamma\neq\beta} \int_{S_{\beta\gamma}} \rho_\beta \boldsymbol{u}_\beta (\boldsymbol{u}_\beta - \boldsymbol{w}) \cdot \boldsymbol{n}_\beta \,\mathrm{d}S \\ = -\int_{\partial\Omega_\beta} p_\beta \boldsymbol{n}_\beta \,\mathrm{d}S + \int_{\partial\Omega_\beta} \boldsymbol{\tau}_\beta \cdot \boldsymbol{n}_\beta \,\mathrm{d}S + \int_{\Omega_\beta} \rho_\beta \boldsymbol{g} \,\mathrm{d}\Omega. \end{aligned} \tag{3.22}$$

The interface condition reads (see e.g. [10]):

$$\rho_\beta \boldsymbol{u}_\beta(\boldsymbol{u}_\beta - \boldsymbol{w})\cdot\boldsymbol{n}_\beta + \rho_\gamma \boldsymbol{u}_\gamma(\boldsymbol{u}_\gamma - \boldsymbol{w})\cdot\boldsymbol{n}_\gamma = (-p_\beta \boldsymbol{I} + \boldsymbol{\tau}_\beta)\cdot\boldsymbol{n}_\beta + (-p_\gamma \boldsymbol{I} + \boldsymbol{\tau}_\gamma)\cdot\boldsymbol{n}_\gamma + \boldsymbol{f}. \quad (3.23)$$

$\boldsymbol{f}$ represents the surface tension force acting on the interface between the phases. Surface tension can be modelled by taking

$$\boldsymbol{f} = \sigma_{\beta\gamma}\kappa\boldsymbol{n} - \nabla\sigma_{\beta\gamma}, \quad (3.24)$$

where κ is the curvature of the interface and σ the surface tension coefficient. For sign convention see e.g. [6].

We apply again volume-averaging to the governing equations. Splitting the surface integrals on the right-hand side of (3.22) by using (3.11) and subsequently applying (3.12), gives conservation of momentum in terms of volume averages as

$$\frac{\partial}{\partial t}(\alpha_\beta\{\rho_\beta\boldsymbol{u}_\beta\}) + \nabla\cdot(\alpha_\beta\{\rho_\beta\boldsymbol{u}_\beta\otimes\boldsymbol{u}_\beta\}) = -\nabla(\alpha_\beta\{p_\beta\}) + \nabla\cdot(\alpha_\beta\{\boldsymbol{\tau}_\beta\}) + \alpha_\beta\{\rho_\beta\boldsymbol{g}\} + \boldsymbol{M}_\beta, \quad (3.25)$$

where

$$\boldsymbol{M}_\beta = \frac{1}{V}\sum_{\gamma,\gamma\neq\beta}\int_{S_{\beta\gamma}}(-p_\beta\boldsymbol{I} + \boldsymbol{\tau}_\beta - \rho_\beta\boldsymbol{u}_\beta(\boldsymbol{u}_\beta - \boldsymbol{w}))\cdot\boldsymbol{n}_\beta\,\mathrm{d}S \quad (3.26)$$

is the interphasial momentum transfer. The last term can be interpreted as the transfer of momentum associated with mass transfer. Summation over all phases gives

$$\sum_\beta \boldsymbol{M}_\beta = -\frac{1}{V}\sum_{\substack{\beta,\gamma\\ \gamma\neq\beta}}\int_{S_{\beta\gamma}}\boldsymbol{f}\,\mathrm{d}S. \quad (3.27)$$

If surface tension can be neglected we have

$$\sum_\beta \boldsymbol{M}_\beta = 0. \quad (3.28)$$

Equation (3.25) should be interpreted as the partial differential equation governing the macroscopic field. Taking the streamwise component of the momentum equation, integrating over a pipe segment, taking the limit $\delta s \to 0$, and making the same assumptions as in the single phase case (section 2.2), leads to

$$\frac{\partial}{\partial t}(\alpha_\beta\rho_\beta u_{s,\beta}A) + \frac{\partial}{\partial s}(\alpha_\beta\rho_\beta u_{s,\beta}^2 A) = -A\frac{\partial\alpha_\beta p_\beta}{\partial s} + \alpha_\beta\bar{\tau}_{\mathrm{tot},\beta}C - \alpha_\beta\rho_\beta g A\sin\varphi + M_{s,\beta}A, \quad (3.29)$$

where C is the outer perimeter. Like in the single-phase case, the model is non-conservative, since A appears outside the partial derivative of the pressure term. Note that the pressure term should be understood as

$$A\left\langle\frac{\partial\alpha_\beta p_\beta}{\partial s}\right\rangle = \int_{A(s)}\frac{\partial}{\partial s}(\alpha_\beta p_\beta)\,\mathrm{d}S. \quad (3.30)$$

Compared to the single-phase case the momentum equation is effectively the same, except for the presence of the interphasial momentum transfer. For the effect of mass sources G_β on the momentum equation, see section 2.2.
The momentum equation in the transverse direction provides again insight into the behaviour of the pressure across the pipe cross-section. Neglecting all terms in the transverse momentum equation per phase, except for pressure and gravity, leads to:

$$\frac{\partial p_\beta}{\partial h} = -\rho_\beta g \cos\varphi. \tag{3.31}$$

Solving this equation leads to a form similar to (2.33). Assuming that all phases share the same reference pressure, i.e., $p_\beta(s, h_{\text{ref}}) = p(s, h_{\text{ref}})$, the pressure variation with height is given by

$$p_\beta(s, h) = p(s, h_{\text{ref}}) - \rho_\beta g(h - h_{\text{ref}}) \cos\varphi. \tag{3.32}$$

By choosing h_{ref} in the center of the pipe, the momentum equation in s-direction remains unchanged, like in the single phase case:

$$\begin{aligned} \int_{A(s)} \frac{\partial}{\partial s}(\alpha_\beta p_\beta)\,\mathrm{d}S &= \int_{A(s)} \frac{\partial}{\partial s}\left(\alpha_\beta p(s, h_{\text{ref}}) - \alpha_\beta \rho_\beta g(h - h_{\text{ref}}) \cos\varphi\right)\,\mathrm{d}S \\ &= \int_{A(s)} \frac{\partial}{\partial s}(\alpha_\beta p(s, h_{\text{ref}}))\,\mathrm{d}S. \end{aligned} \tag{3.33}$$

The momentum equation per phase therefore reads

$$\boxed{\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta u_{s,\beta} A) + \frac{\partial}{\partial s}(\alpha_\beta \rho_\beta u_{s,\beta}^2 A) = -A\frac{\partial \alpha_\beta p}{\partial s} + \alpha_\beta \bar{\tau}_{\text{tot},\beta} C - \alpha_\beta \rho_\beta g A \sin\varphi + M_{s,\beta} A,} \tag{3.34}$$

and p is the pressure in the center of the pipe. In this dispersed flow model no hydraulic gradients are present. The use of a single pressure leads to the problem that the model becomes conditionally hyperbolic. Fortunately, in practice equation (3.34) is not always solved, but a 'total', or 'bulk', momentum equation is solved together with a slip relation, not suffering from loss of hyperbolicity [4]. This total momentum equation can be obtained by summing equation (3.34) over all phases, leading to

$$\boxed{\sum_\beta \left(\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta u_{s,\beta} A) + \frac{\partial}{\partial s}(\alpha_\beta \rho_\beta u_{s,\beta}^2 A)\right) = -A\frac{\partial p}{\partial s} + \sum_\beta \alpha_\beta \bar{\tau}_{\text{tot},\beta} C - \sum_\beta \alpha_\beta \rho_\beta g A \sin\varphi.} \tag{3.35}$$

Note that it is possible to use a more complete transverse momentum equation than equation (3.31) (e.g. the Ransom & Hicks model [15], see also [18]), but this is not common in the 1D multiphase flow models employed in the oil- and gas industry.

3.2.2. Stratified flow

Similar to section 3.1.2 we derive the momentum equation in case of stratified flow (no interphasial momentum transfer due interphasial mass transfer). The momentum equa-

tion in streamwise direction becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(\{\rho_\beta u_{s,\beta}\}\, V_\beta) + \left\langle \rho_\beta u_{s,\beta}^2 \right\rangle_2 (A_\beta)_2 - \left\langle \rho u_{s,\beta}^2 \right\rangle_1 (A_\beta)_1 = -\left(\langle p_\beta \rangle_2 (A_\beta)_2 - \langle p_\beta \rangle_1 (A_\beta)_1 \right) - \sum_\gamma \langle p \boldsymbol{e}_s \cdot \boldsymbol{n}_{\beta\gamma} \rangle A_{\beta\gamma} + \sum_\gamma \langle \tau_{\mathrm{tot},\beta\gamma} \rangle A_{\beta\gamma} - g \sin\varphi \{\rho_\beta\} V_\beta. \quad (3.36)$$

The viscous stress can be expressed in terms of cylindrical coordinates (like equation (2.21)), but when considering an interface in stratified flow it is more conveniently expressed in terms of

$$\tau_{\mathrm{tot},\beta\gamma} = \tau_{ss,\beta} \boldsymbol{n}_{\beta\gamma} \cdot \boldsymbol{e}_s + \tau_{sB,\beta} \boldsymbol{n}_{\beta\gamma} \cdot \boldsymbol{e}_B + \tau_{sN,\beta} \boldsymbol{n}_{\beta\gamma} \cdot \boldsymbol{e}_N. \quad (3.37)$$

The summation over γ includes $\gamma = \beta$, which constitutes the pipeline surface. We define the angle $\zeta_{\beta\gamma}$ as the angle between the interface and the streamwise direction. The differential form reads

$$\frac{\partial}{\partial t}(\rho_\beta u_{s,\beta} A_\beta) + \frac{\partial}{\partial s}(\rho_\beta u_{s,\beta}^2 A_\beta) = -\frac{\partial}{\partial s}\left(\int_{A_\beta} p_\beta \mathrm{d}S \right) - \int_{C_\beta} p_\beta \frac{\sin\zeta_{\beta\gamma}}{\cos\zeta_{\beta\gamma}} \mathrm{d}C - \rho_\beta g A_\beta \sin\varphi + \sum_\gamma \bar{\tau}_{\mathrm{tot},\beta\gamma} C_{\beta\gamma}. \quad (3.38)$$

Averages of products have been approximated by products of averages: the density and velocity should be interpreted as cross-sectional averages. The pressure is left in integral form in order to demonstrate the effect of pressure variation in transverse direction (hydraulic gradients). Similar to the single-phase case, the second term on the right-hand sides indicates that the pressure along the interface or along the pipe perimeter has a contribution in streamwise direction when the pipe diameter or the interface position is changing as a function of s. Upon expanding the first integral on the right-hand side with Leibniz' rule, one finds again that a non-conservative term remains, which reads

$$-\int_{A_\beta(s)} \frac{\partial p_\beta}{\partial s} \mathrm{d}A. \quad (3.39)$$

The contribution of the viscous forces on the interface with another phase γ is given by

$$\bar{\tau}_{\mathrm{tot},\beta\gamma} = \frac{1}{C_{\beta\gamma}} \int_{C_{\beta\gamma}} \frac{\tau_{\mathrm{tot},\beta\gamma}}{\cos\zeta_{\beta\gamma}} \mathrm{d}C. \quad (3.40)$$

The viscous stresses are generally dominated by shear stress, so

$$\tau_{\mathrm{tot},\beta\gamma} = \begin{cases} \tau_{sN} \boldsymbol{n}_{\beta\gamma} \cdot \boldsymbol{e}_N = \tau_{sN} \cos\zeta_{\beta\gamma}, \quad \tau_{sN} \approx \mu \frac{\partial u_s}{\partial h} & \text{interface, } \beta \neq \gamma \\ \tau_{sr} \boldsymbol{n}_\beta \cdot \boldsymbol{e}_r = \tau_{sr} \cos\zeta_\beta, \quad \tau_{sr} \approx \mu \frac{\partial u_s}{\partial r} & \text{pipe surface (if circular), } \beta = \gamma. \end{cases} \quad (3.41)$$

Here h is positive in the direction of decreasing density. Since $\cos\zeta_{\beta\gamma}$ appears both in (3.40) and (3.41), there is no dependence on the angle of the interface in the resulting

equations. In case of three phases (water, oil, gas), and taking into account the direction of the outward unit normals, the viscous contributions read, for each phase,

$$W: \quad C_W\bar{\tau}_W + C_{WO}\bar{\tau}_{WO}, \tag{3.42}$$
$$O: \quad C_O\bar{\tau}_O - C_{WO}\bar{\tau}_{WO} + C_{OG}\bar{\tau}_{OG}, \tag{3.43}$$
$$G: \quad C_G\bar{\tau}_G - C_{OG}\bar{\tau}_{OG}. \tag{3.44}$$

where

$$\bar{\tau}_W = \frac{1}{C_W}\int_{C_W} \mu\frac{\partial u_s}{\partial r}\mathrm{d}C, \tag{3.45}$$

$$\bar{\tau}_{WO} = \frac{1}{C_{WO}}\int_{C_{WO}} \mu\frac{\partial u_s}{\partial n}\mathrm{d}C, \tag{3.46}$$

and similarly for the other quantities. Note that for attached flow $\frac{\partial u_s}{\partial r} < 0$ so the pipe-surface contributions $\bar{\tau}_W$, $\bar{\tau}_O$, $\bar{\tau}_G$ are generally *negative* (shear stresses cause a pressure drop). The negative signs in (3.43) and (3.44) are a result of the direction of the outward unit normal on the interfaces. Often, $\bar{\tau}_{WO}$ and $\bar{\tau}_{OG}$ are positive, since the lower density fluids often move with higher velocity and drag the higher density fluids. The momentum equation per phase now reads

$$\frac{\partial}{\partial t}\left(\rho_\beta u_{s,\beta} A_\beta\right) + \frac{\partial}{\partial s}\left(\rho_\beta u_{s,\beta}^2 A_\beta\right) = -\frac{\partial p_\beta}{\partial s}A_\beta - \rho_\beta g A_\beta \sin\varphi + \sum_\gamma \bar{\tau}_{\mathrm{tot},\beta\gamma} C_{\beta\gamma}. \tag{3.47}$$

Within each phase the pressure varies linearly with height. Interface condition (3.23) shows that at the interface the pressure can jump due to surface tension. This is neglected here. We take again three phases: water, oil, and gas. The pressure at the interface between oil and gas is chosen as the reference pressure: $p(s, h_{\mathrm{ref}}) = p_{OG}(s)$ [8]. This gives the following piecewise linear pressure profile:

$$p_G = p_{OG} - \rho_G g\cos\varphi(h - h_{OG}), \tag{3.48}$$
$$p_O = p_{OG} - \rho_O g\cos\varphi(h - h_{OG}), \tag{3.49}$$
$$p_W = p_{OG} - \rho_O g\cos\varphi(h_{WO} - h_{OG}) - \rho_W g\cos\varphi(h - h_{WO}). \tag{3.50}$$

Inserting these profiles into (3.39) gives the hydraulic gradients HG_β, see appendix B.2. The hydraulic gradients are responsible for surface waves (Kelvin-Helmholtz instabilities). Without the variation of pressure with height (i.e. all phases share the same pressure), the multi-phase flow equations are not able to reproduce surface waves (physical issue) and are not unconditionally hyperbolic (mathematical issue).

Gathering all terms leads to the following momentum equation per phase for a stratified flow:

$$\boxed{\frac{\partial}{\partial t}\left(\rho_\beta u_{s,\beta} A_\beta\right) + \frac{\partial}{\partial s}\left(\rho_\beta u_{s,\beta}^2 A_\beta\right) = -\frac{\partial p}{\partial s}A_\beta - \mathrm{HG}_\beta - \rho_\beta g A_\beta \sin\varphi + \sum_\gamma \bar{\tau}_{\mathrm{tot},\beta\gamma} C_{\beta\gamma},} \tag{3.51}$$

where $p = p_{OG}$. The total momentum equation, obtained by summing over all phases, reads

$$\boxed{\sum_\beta \left(\frac{\partial}{\partial t} \left(\rho_\beta u_{s,\beta} A_\beta \right) + \frac{\partial}{\partial s} (\rho_\beta u_{s,\beta}^2 A_\beta) \right) = -\frac{\partial p}{\partial s} A - \sum_\beta \mathrm{HG}_\beta - \sum_\beta \rho_\beta g A_\beta \sin\varphi + \sum_\beta \bar{\tau}_{\mathrm{tot},\beta} C_\beta.} \tag{3.52}$$

The viscous stress summation only contains the wall contributions, since the interphasial terms cancel each other. The hydraulic gradients do not cancel from this equation.

3.3. Conservation of energy

3.3.1. Dispersed flow

Proceeding as in the previous sections, we write conservation of total energy for each phase, $E_\beta = e_\beta + \frac{1}{2}|\boldsymbol{u}_\beta|^2$,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Omega_\beta} \rho_\beta E_\beta \,\mathrm{d}\Omega + \int_{S_\beta} \rho_\beta E_\beta \boldsymbol{u}_\beta \cdot \boldsymbol{n}_\beta \,\mathrm{d}S + \sum_{\gamma,\gamma\neq\beta} \int_{S_{\beta\gamma}} \rho_\beta E_\beta (\boldsymbol{u}_\beta - \boldsymbol{w}) \cdot \boldsymbol{n}_\beta \,\mathrm{d}S = \\ &- \int_{\partial\Omega_\beta} p_\beta \boldsymbol{u}_\beta \cdot \boldsymbol{n}_\beta \,\mathrm{d}S + \int_{\partial\Omega_\beta} (\boldsymbol{\tau}_\beta \cdot \boldsymbol{u}_\beta) \cdot \boldsymbol{n}_\beta \,\mathrm{d}S - \int_{\partial\Omega_\beta} \boldsymbol{q}_\beta \cdot \boldsymbol{n}_\beta \,\mathrm{d}S + \int_{\Omega_\beta} \rho_\beta \boldsymbol{g} \cdot \boldsymbol{u}_\beta \,\mathrm{d}\Omega + \int_{\Omega_\beta} \dot{q}_\beta \mathrm{d}\Omega. \end{aligned} \tag{3.53}$$

and the interface conditions

$$\begin{aligned} &\rho_\beta E_\beta (\boldsymbol{u}_\beta - \boldsymbol{w}) \cdot \boldsymbol{n}_\beta + \rho_\gamma E_\gamma (\boldsymbol{u}_\gamma - \boldsymbol{w}) \cdot \boldsymbol{n}_\gamma = \\ &\quad ((-p_\beta \boldsymbol{I} + \boldsymbol{\tau}_\beta) \cdot \boldsymbol{u}_\beta) \cdot \boldsymbol{n}_\beta + ((-p_\gamma \boldsymbol{I} + \boldsymbol{\tau}_\gamma) \cdot \boldsymbol{u}_\gamma) \cdot \boldsymbol{n}_\gamma - \boldsymbol{q}_\beta \cdot \boldsymbol{n}_\beta - \boldsymbol{q}_\gamma \cdot \boldsymbol{n}_\gamma + \boldsymbol{f} \cdot \boldsymbol{w}. \end{aligned} \tag{3.54}$$

The last term represents work done by surface tension, see e.g. [19].

Applying local volume averaging as before yields the following equation:

$$\begin{aligned} \frac{\partial}{\partial t} (\alpha_\beta \{\rho_\beta E_\beta\}) + \nabla \cdot (\alpha_\beta \{\rho_\beta E_\beta \boldsymbol{u}_\beta\}) &= -\nabla \cdot (\alpha_\beta \{p_\beta \boldsymbol{u}_\beta\}) \\ &+ \nabla \cdot (\alpha_\beta \{\boldsymbol{\tau}_\beta \cdot \boldsymbol{u}_\beta\}) - \nabla \cdot (\alpha_\beta \{\boldsymbol{q}_\beta\}) + \alpha_\beta \{\rho_\beta \boldsymbol{g} \cdot \boldsymbol{u}_\beta\} + \alpha_\beta \{\dot{q}_\beta\} + \zeta_\beta, \end{aligned} \tag{3.55}$$

where the interfacial energy transfer terms are given by

$$\zeta_\beta = \frac{1}{V} \sum_{\gamma,\gamma\neq\beta} \int_{S_{\beta\gamma}} (-p_\beta \boldsymbol{u}_\beta + \boldsymbol{\tau}_\beta \cdot \boldsymbol{u}_\beta - \boldsymbol{q}_\beta - \rho_\beta E_\beta (\boldsymbol{u}_\beta - \boldsymbol{w})) \cdot \boldsymbol{n}_\beta \,\mathrm{d}S, \tag{3.56}$$

which have the property

$$\sum_\beta \zeta_\beta = -\sum_{\substack{\beta,\gamma \\ \gamma\neq\beta}} \int_{S_{\beta\gamma}} \boldsymbol{f} \cdot \boldsymbol{w} \,\mathrm{d}S. \tag{3.57}$$

The right-hand side is zero if surface tension can be neglected. For the effect of mass sources G_β on the energy equation, see section 2.3.

The differential equation for the energy of each phase reads (for assumptions see section 2.3)

$$\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta E_\beta A) + \frac{\partial}{\partial s}\left(\alpha_\beta \rho_\beta E_\beta u_{s,\beta} A\right) = -\frac{\partial}{\partial s}(\alpha_\beta p_\beta u_{s,\beta} A) - \alpha_\beta \bar{Q}_\beta C - \alpha_\beta \rho_\beta g u_{s,\beta} A \sin\varphi + \alpha_\beta \dot{q}_\beta A + \zeta_\beta A. \quad (3.58)$$

The pressure term can be added to the second (convective) term to yield a formulation in terms of the total enthalpy, $H_\beta = h_\beta + \frac{1}{2}|\boldsymbol{u}_\beta|^2 = E_\beta + p/\rho_\beta$:

$$\boxed{\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta E_\beta A) + \frac{\partial}{\partial s}\left(\alpha_\beta \rho_\beta H_\beta u_{s,\beta} A\right) = -\alpha_\beta \bar{Q}_\beta C - \alpha_\beta \rho_\beta g u_{s,\beta} A \sin\varphi + \alpha_\beta \dot{q}_\beta A + \zeta_\beta A.} \quad (3.59)$$

In Compas it is assumed that the temperature of all phases is the same, and the energy equation of all phases are added, giving

$$\boxed{\sum_\beta \left(\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta E_\beta A) + \frac{\partial}{\partial s}\left(\alpha_\beta \rho_\beta H_\beta u_{s,\beta} A\right)\right) = -\sum_\beta \alpha_\beta \bar{Q}_\beta C - \sum_\beta \alpha_\beta \rho_\beta g u_{s,\beta} A \sin\varphi + \dot{q} A.} \quad (3.60)$$

Ledaflow is an example of a 1D multiphase simulator that employs (3.59) instead of (3.60), allowing for a different temperature for each phase.

3.3.2. Stratified flow

To derive the energy equation for stratified flow, equation (3.53) is directly applied in macroscopic sense. Like for single phase flow, the viscous stresses do not perform work on the boundary of the pipe, and we neglect the work done by streamwise viscous stresses and the effect of streamwise heat conduction. However, work can be done by viscous stresses at the interface of the phases, and we need to evaluate the viscous stress tensor multiplied by the velocity at the interface:

$$(\boldsymbol{\tau}\cdot\boldsymbol{u})\cdot\boldsymbol{n} = \tau_{ss} u_s \boldsymbol{e}_s\cdot\boldsymbol{n} + \tau_{sr} u_s \boldsymbol{e}_r\cdot\boldsymbol{n} + \tau_{s\theta} u_s \boldsymbol{e}_\theta\cdot\boldsymbol{n} = \tau_{\text{tot}} u_s. \quad (3.61)$$

The result is the following integral form of the energy equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\{\rho_\beta E_\beta\} V_\beta\right) + \langle \rho_\beta E_\beta u_{s,\beta}\rangle_2 (A_\beta)_2 - \langle \rho_\beta E_\beta u_{s,\beta}\rangle_1 (A_\beta)_1 = -\left(\langle p_\beta u_{s,\beta}\rangle_2 (A_\beta)_2 - \langle p_\beta u_{s,\beta}\rangle_1 (A_\beta)_1\right) + \sum_{\gamma,\gamma\neq\beta} \langle \tau_{\text{tot},\beta\gamma} u_{s,\beta}\rangle A_{\beta\gamma} + \sum_\gamma \langle Q_{\beta\gamma}\rangle A_{\beta\gamma} + \{\rho \boldsymbol{g}\cdot\boldsymbol{u}\} V_\beta + \{\dot{q}\} V_\beta. \quad (3.62)$$

The work done by pressure forces at the interface is neglected, since the normal velocity of the interface is assumed to be small. However, the work done on the inflow and outflow is not zero and, like the hydraulic gradients in the momentum equation, contain a hydrostatic contribution:

$$\langle p_\beta u_{s,\beta} \rangle A_\beta = \int_{A_\beta} p_\beta u_{s,\beta} \, \mathrm{d}S =$$
$$\begin{cases} \int_{A_G} p_{\mathrm{OG}} u_G \, \mathrm{d}S - \rho_G u_G g \cos\varphi \left[(R - h_{\mathrm{OG}}) A_G + \frac{1}{12} C_{OG}^3 \right] \\ \int_{A_O} p_{\mathrm{OG}} u_O \, \mathrm{d}S - \rho_O u_O g \cos\varphi \left[(R - h_{\mathrm{OG}}) A_O - \frac{1}{12} (C_{OG}^3 - C_{WO}^3) \right] \\ \int_{A_W} p_{\mathrm{OG}} u_W \, \mathrm{d}S - \rho_O u_W g \cos\varphi (h_{\mathrm{WO}} - h_{\mathrm{OG}}) A_W - \rho_W u_W g \cos\varphi \left[(R - h_{\mathrm{WO}}) A_W - \frac{1}{12} C_{WO}^3 \right] \end{cases} \tag{3.63}$$

In the limit of $\delta s \to 0$, and after some rewriting, one finds the same hydraulic gradients as in the momentum equation, now multiplied with the phase velocity:

$$-\frac{\partial}{\partial s} \left(p_\beta u_{s,\beta} A_\beta \right) = -\frac{\partial}{\partial s} \left(p_{\mathrm{OG}} u_{s,\beta} A_\beta \right) - u_{s,\beta} \mathrm{HG}_\beta, \tag{3.64}$$

where we neglected $\frac{\partial}{\partial s} \left(\rho_\beta u_{s,\beta} g \cos\varphi \right)$, see appendix B.2. Proceeding as before then leads to the following differential equation:

$$\boxed{\begin{aligned} \frac{\partial}{\partial t} (\rho_\beta E_\beta A_\beta) + \frac{\partial}{\partial s} \left(\rho_\beta H_\beta u_{s,\beta} A_\beta \right) = & -u_{s,\beta} \mathrm{HG}_\beta \\ & + \sum_{\gamma, \gamma \neq \beta} \bar{\tau}_{\mathrm{tot},\beta\gamma} u_{s,\beta} C_{\beta\gamma} - \sum_\gamma \bar{Q}_{\beta\gamma} C_{\beta\gamma} - \rho_\beta g u_{s,\beta} A_\beta \sin\varphi + \dot{q}_\beta A_\beta. \end{aligned}} \tag{3.65}$$

Summation over all phases leads to the total energy equation, in which the interphasial terms cancel:

$$\boxed{\begin{aligned} \sum_\beta \left(\frac{\partial}{\partial t} (\rho_\beta E_\beta A_\beta) + \frac{\partial}{\partial s} \left(\rho_\beta H_\beta u_{s,\beta} A_\beta \right) \right) = & - \sum_\beta u_{s,\beta} \mathrm{HG}_\beta - \sum_\beta \bar{Q}_\beta C_\beta \\ & - \sum_\beta \rho_\beta g u_{s,\beta} A_\beta \sin\varphi + \sum_\beta \dot{q}_\beta A_\beta. \end{aligned}} \tag{3.66}$$

Compared to the total energy equation for dispersed flow, (3.60), the only difference is the presence of hydraulic gradients and the expression for the heat transfer with the wall.

3.4. Summary

We summarize the mass, momentum and energy equations in differential form for 1D multi-phase flow. For dispersed flow, the approach taken is to apply volume averaging to the governing equations on a scale much smaller than the characteristic scale of the system (such as pipe diameter), leading to differential equations for locally volume averaged

quantities. Subsequently, these differential equations are again integrated over a pipe segment, and then reduced to differential form by considering an infinitesimally small pipe segment. For stratified flow the local volume averaging is not needed, and the governing equations are directly obtained by integrating over the area a phase occupies in a pipe segment.

The main assumptions are:

- the radial and circumferential velocities are much smaller than the streamwise velocity;
- the pipeline curvature is neglected;
- the density is uniform over the cross-section;
- the cross-sectional average of products of quantities is approximated by the product of the cross-sectional averages;
- viscous stresses (and the work they perform) in streamwise direction are neglected compared to pressure stresses in streamwise direction;
- streamwise heat conduction is neglected;
- the different phases are immiscible;
- an averaging procedure based on the assumption of scale separation can be carried out: the quantities in the equations represent the area-average of a local volume-averaged quantity (the equations have been integrated twice);
- for stratified flow the interphasial terms associated with mass transfer have been ignored, as well as the work done by pressure forces on the interface.

Conservation equations dispersed flow, per phase

$$\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta A) + \frac{\partial}{\partial s}(\alpha_\beta \rho_\beta u_{s,\beta} A) = \Psi_\beta A, \tag{3.67}$$

$$\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta u_{s,\beta} A) + \frac{\partial}{\partial s}(\alpha_\beta \rho_\beta u_{s,\beta}^2 A) = -A\frac{\partial \alpha_\beta p}{\partial s} + \alpha_\beta \bar{\tau}_{\mathrm{tot},\beta} C - \alpha_\beta \rho_\beta g A \sin\varphi + M_{s,\beta} A, \tag{3.68}$$

$$\frac{\partial}{\partial t}(\alpha_\beta \rho_\beta E_\beta A) + \frac{\partial}{\partial s}\left(\alpha_\beta \rho_\beta H_\beta u_{s,\beta} A\right) = -\alpha_\beta \bar{Q}_\beta C - \alpha_\beta \rho_\beta g u_{s,\beta} A \sin\varphi + \alpha_\beta \dot{q}_\beta A + \zeta_\beta A. \tag{3.69}$$

Conservation equations dispersed flow, total

$$\sum_{\beta}\left(\frac{\partial}{\partial t}(\alpha_{\beta}\rho_{\beta}A)+\frac{\partial}{\partial s}(\alpha_{\beta}\rho_{\beta}u_{s,\beta}A)\right)=0, \tag{3.70}$$

$$\sum_{\beta}\left(\frac{\partial}{\partial t}(\alpha_{\beta}\rho_{\beta}u_{s,\beta}A)+\frac{\partial}{\partial s}(\alpha_{\beta}\rho_{\beta}u_{s,\beta}^{2}A)\right)=-A\frac{\partial p}{\partial s}+\sum_{\beta}\alpha_{\beta}\bar{\tau}_{\mathrm{tot},\beta}C-\sum_{\beta}\alpha_{\beta}\rho_{\beta}gA\sin\varphi, \tag{3.71}$$

$$\sum_{\beta}\left(\frac{\partial}{\partial t}(\alpha_{\beta}\rho_{\beta}E_{\beta}A)+\frac{\partial}{\partial s}\left(\alpha_{\beta}\rho_{\beta}H_{\beta}u_{s,\beta}A\right)\right)=-\sum_{\beta}\alpha_{\beta}\bar{Q}_{\beta}C-\sum_{\beta}\alpha_{\beta}\rho_{\beta}gu_{s,\beta}A\sin\varphi+\dot{q}A. \tag{3.72}$$

Conservation equations stratified flow, per phase

$$\frac{\partial}{\partial t}\left(\rho_{\beta}A_{\beta}\right)+\frac{\partial}{\partial s}\left(\rho_{\beta}u_{s,\beta}A_{\beta}\right)=0, \tag{3.73}$$

$$\begin{aligned}\frac{\partial}{\partial t}\left(\rho_{\beta}u_{s,\beta}A_{\beta}\right)+\frac{\partial}{\partial s}(\rho_{\beta}u_{s,\beta}^{2}A_{\beta})=&-\frac{\partial p}{\partial s}A_{\beta}-\mathrm{HG}_{\beta}\\&+\sum_{\gamma}\bar{\tau}_{\mathrm{tot},\beta\gamma}C_{\beta\gamma}-\rho_{\beta}gA_{\beta}\sin\varphi,\end{aligned} \tag{3.74}$$

$$\begin{aligned}\frac{\partial}{\partial t}(\rho_{\beta}E_{\beta}A_{\beta})+\frac{\partial}{\partial s}\left(\rho_{\beta}H_{\beta}u_{s,\beta}A_{\beta}\right)=&-u_{s,\beta}\mathrm{HG}_{\beta}+\sum_{\gamma,\gamma\neq\beta}\bar{\tau}_{\mathrm{tot},\beta\gamma}u_{s,\beta}C_{\beta\gamma}\\&-\sum_{\gamma}\bar{Q}_{\beta\gamma}C_{\beta\gamma}-\rho_{\beta}gu_{s,\beta}A_{\beta}\sin\varphi+\dot{q}_{\beta}A_{\beta}.\end{aligned} \tag{3.75}$$

Conservation equations stratified flow, total

$$\sum_{\beta}\left(\frac{\partial}{\partial t}\left(\alpha_{\beta}\rho_{\beta}A\right)+\frac{\partial}{\partial s}\left(\alpha_{\beta}\rho_{\beta}u_{s,\beta}A\right)\right)=0, \tag{3.76}$$

$$\begin{aligned}\sum_{\beta}\left(\frac{\partial}{\partial t}\left(\rho_{\beta}u_{s,\beta}A_{\beta}\right)+\frac{\partial}{\partial s}(\rho_{\beta}u_{s,\beta}^{2}A_{\beta})\right)=&-\frac{\partial p}{\partial s}A-\sum_{\beta}\mathrm{HG}_{\beta}\\&+\sum_{\beta}\bar{\tau}_{\mathrm{tot},\beta}C_{\beta}-\sum_{\beta}\rho_{\beta}gA_{\beta}\sin\varphi,\end{aligned} \tag{3.77}$$

$$\begin{aligned}\sum_{\beta}\left(\frac{\partial}{\partial t}(\rho_{\beta}E_{\beta}A_{\beta})+\frac{\partial}{\partial s}\left(\rho_{\beta}H_{\beta}u_{s,\beta}A_{\beta}\right)\right)=&-\sum_{\beta}u_{s,\beta}\mathrm{HG}_{\beta}-\sum_{\beta}\bar{Q}_{\beta}C_{\beta}\\&-\sum_{\beta}\rho_{\beta}gu_{s,\beta}A_{\beta}\sin\varphi+\sum_{\beta}\dot{q}_{\beta}A_{\beta}.\end{aligned} \tag{3.78}$$

4. Conclusions

In this report we have presented the governing equations for single- and multi-phase flow in pipelines, as summarized on pages 23-24. All assumptions that were made in the derivation have been listed consistently, giving an overview of which physical effects can, and which cannot be captured by the one-dimensional equations. The discussion has focused on single-phase and multi-phase flow, where for the latter the distinction between stratified (segregated) and dispersed flow was made. Other flow patterns used in Compas, being slug, annular and homogeneous flow can be grouped under these types: slug flow can be handled as dispersed flow, annular flow as stratified flow, homogeneous flow as single-phase flow.

The resulting equations future a number of terms such as wall friction, interphasial friction and interphasial mass transfer, that have not been detailed in this report. They are in general obtained from the Shell Flow Correlations (SFC). Furthermore, the dependence of density and internal energy on pressure and temperature is determined by a thermodynamic package like Shell's Physical Properties and Thermodynamics Software (SPPTS).

The differential equations in this report have been derived starting from the governing equations in volume formulation. This makes this report a useful starting point for the derivation of discretization methods for both finite difference and finite volume methods. The equations as presented here include a number of physical terms that should be improved in the current Compas code, such as hydraulic gradients, varying pipe diameter, and the effect of mass sources on momentum and energy equation.

Appendix A.

Governing equations in curvilinear coordinate system

A.1. Equations in vector form (coordinate-free)

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0. \tag{A.1}$$

With the definition of the substantial derivative, $\frac{D}{Dt} = \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla$, this can be equivalently written as:

$$\frac{D\rho}{Dt} = -\rho \nabla \cdot \boldsymbol{u}. \tag{A.2}$$

Incompressible flow:

$$\frac{D\rho}{Dt} = 0 \quad \longleftrightarrow \quad \nabla \cdot \boldsymbol{u} = 0. \tag{A.3}$$

Conservation of momentum:

$$\frac{\partial \rho \boldsymbol{u}}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} + \rho \boldsymbol{g}, \tag{A.4}$$

where $\otimes$ indicates a tensor (dyadic) product. The left side can also be written as $\rho \frac{D\boldsymbol{u}}{Dt}$ by employing the mass conservation equation. For Newtonian fluids the viscous stress tensor is expressed as

$$\boldsymbol{\tau} = \mu(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T) + \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I}. \tag{A.5}$$

Conservation of total energy $E = e + k$ (sum of specific internal and specific kinetic energy):

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho E \boldsymbol{u}) = -\nabla \cdot (p\boldsymbol{u}) + \nabla \cdot (\boldsymbol{\tau} \cdot \boldsymbol{u}) - \nabla \cdot \boldsymbol{q} + \rho \boldsymbol{g} \cdot \boldsymbol{u}. \tag{A.6}$$

The left side can also be written as $\rho \frac{DE}{Dt}$ by employing the mass conservation equation. Conservation of kinetic energy $k = \frac{1}{2}|\boldsymbol{u}|^2$ (inner product of (A.4) with $\boldsymbol{u}$):

$$\frac{\partial \rho k}{\partial t} + \nabla \cdot (\rho k \boldsymbol{u}) = -\boldsymbol{u} \cdot \nabla p + \boldsymbol{u} \cdot (\nabla \cdot \boldsymbol{\tau}) + \rho \boldsymbol{g} \cdot \boldsymbol{u}. \tag{A.7}$$

Conservation of mechanical energy e:

$$\frac{\partial \rho e}{\partial t} + \nabla \cdot (\rho e \boldsymbol{u}) = -p \nabla \cdot \boldsymbol{u} + \boldsymbol{\tau} : \nabla \boldsymbol{u} - \nabla \cdot \boldsymbol{q}. \tag{A.8}$$

A.2. Coordinate system

An important step in obtaining the governing equations of fluid flow in pipes is the choice of a coordinate system. We follow the approach of Germano [5], who introduced an orthogonal curvilinear coordinate system along a generic spatial curve, being the centerline of the pipe. Such a coordinate system is suitable for general pipe flows, including curved, non-circular and expanding or contracting pipes. It allows us to systematically list the assumptions that lead to the derivation of the one-dimensional equations. The spatial curve is described by the vector $\boldsymbol{R}(s)$, parameterized by the arc length s. Such a curve can be completely described in terms of the curvature $\kappa(s)$ and the torsion $\tau(s)$. The (unit) tangent, normal and binormal vectors to the curve are given by

$$\boldsymbol{T}(s) = \frac{\mathrm{d}\boldsymbol{R}(s)}{\mathrm{d}s}, \qquad \boldsymbol{N}(s) = \frac{1}{\kappa}\frac{\mathrm{d}\boldsymbol{T}(s)}{\mathrm{d}s}, \qquad \boldsymbol{B}(s) = \boldsymbol{T}(s) \times \boldsymbol{N}(s). \tag{A.9}$$

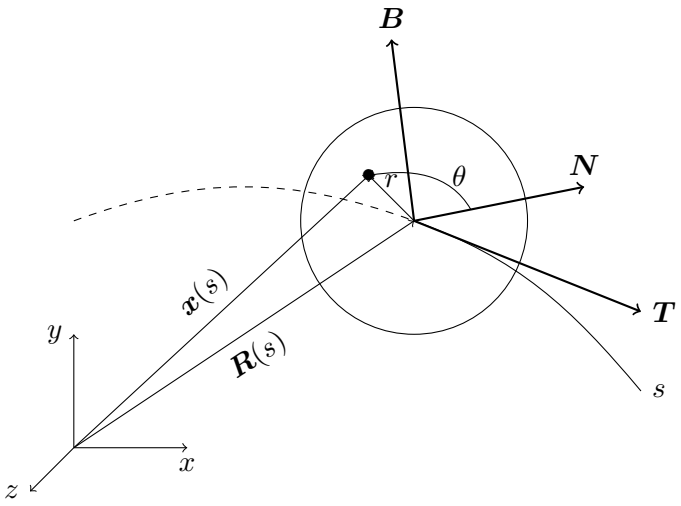

Figure A.1.: Coordinate system.

The coordinate system has one unit vector aligned with the tangential vector $\boldsymbol{T}$, and the other two unit vectors lie in the plane spanned by $\boldsymbol{N}$ and $\boldsymbol{B}$. In this plane we define the polar coordinates r and θ. This gives the coordinate system (s, r, θ). A Cartesian vector $\boldsymbol{x}$ expressed in terms of (s, r, θ) reads

$$\boldsymbol{x}(s, r, \theta) = \boldsymbol{R}(s) + r\cos(\theta + \phi(s) + \phi_0)\boldsymbol{N}(s) + r\sin(\theta + \phi(s) + \phi_0)\boldsymbol{B}(s), \tag{A.10}$$

where

$$\phi(s) = -\int_{s_0}^{s} \tau(\tilde{s}) d\tilde{s}, \tag{A.11}$$

$$\phi_0 = \pi/2. \tag{A.12}$$

The choice (A.11) make the coordinate system orthogonal. The choice (A.12) is to obtain for the case $\tau = 0$ the system ordinarily used for the flow in a curved pipe. The unit vectors are determined by the derivatives of the transformation. For example,

$$\begin{aligned}\frac{\partial \boldsymbol{x}}{\partial s} =& \boldsymbol{R}'(s) + r\cos(\theta+\phi(s)+\phi_0)\boldsymbol{N}'(s) - r\sin(\theta+\phi(s)+\phi_0)\phi'\boldsymbol{N}(s) + \\ & r\sin(\theta+\phi(s)+\phi_0)\boldsymbol{B}'(s) + r\cos(\theta+\phi(s)+\phi_0)\phi'\boldsymbol{B}(s). \end{aligned} \tag{A.13}$$

With the help of $\phi' = -\tau$ and the Frenet-Serret formulae, this simplifies to

$$\frac{\partial \boldsymbol{x}}{\partial s} = \boldsymbol{T}(s)(1+\kappa r\sin(\theta+\phi(s))), \tag{A.14}$$

and the resulting unit vector is given by

$$\boldsymbol{e}_s(s) = \frac{1}{\left|\frac{\partial \boldsymbol{x}}{\partial s}\right|}\frac{\partial \boldsymbol{x}}{\partial s} = \boldsymbol{T}(s). \tag{A.15}$$

Similar derivations yield the other unit vectors:

$$\boldsymbol{e}_r(s,\theta) = \frac{1}{\left|\frac{\partial \boldsymbol{x}}{\partial r}\right|}\frac{\partial \boldsymbol{x}}{\partial r} = \boldsymbol{B}(s)\cos(\theta+\phi) - \boldsymbol{N}(s)\sin(\theta+\phi), \tag{A.16}$$

$$\boldsymbol{e}_\theta(s,\theta) = \frac{1}{\left|\frac{\partial \boldsymbol{x}}{\partial \theta}\right|}\frac{\partial \boldsymbol{x}}{\partial \theta} = -\boldsymbol{B}(s)\sin(\theta+\phi) - \boldsymbol{N}(s)\cos(\theta+\phi). \tag{A.17}$$

The transformation factors h_i follow from

$$h_s = \left|\frac{\partial \boldsymbol{x}}{\partial s}\right| = 1+\kappa r\sin(\theta+\phi), \tag{A.18}$$

$$h_r = \left|\frac{\partial \boldsymbol{x}}{\partial r}\right| = 1, \tag{A.19}$$

$$h_\theta = \left|\frac{\partial \boldsymbol{x}}{\partial \theta}\right| = r. \tag{A.20}$$

Orthogonality is expressed by $g_{sr} = g_{s\theta} = g_{r\theta} = 0$, where $g_{sr} = \frac{\partial \boldsymbol{x}}{\partial s}\cdot\frac{\partial \boldsymbol{x}}{\partial r}$. With these transformation metrics the governing equations in curvilinear coordinates can be written down. Note that for negligible curvature ($\kappa \approx 0$) the classical cylindrical coordinate system is retrieved. In that case the curve is essentially planar, and the binormal vector $\boldsymbol{B}$ points 'out of the page'.

The following geometric quantities will be useful in the next sections: $\varphi(s)$ is the angle of the centerline with respect to the horizontal:

$$\boldsymbol{e}_x\cdot\boldsymbol{e}_s(s) = \boldsymbol{e}_y\cdot\boldsymbol{e}_N(s) = \cos\varphi(s), \qquad \boldsymbol{e}_y\cdot\boldsymbol{e}_s(s) = \sin\varphi(s). \tag{A.21}$$

$\zeta(s,r,\theta)$ is the angle which the normal to the pipe surface, $\boldsymbol{n}(s,r,\theta)$, makes with respect to the tangent to the pipe centerline:

$$\boldsymbol{e}_s(s)\cdot\boldsymbol{n}(s,r,\theta) = -\sin\zeta(s,r,\theta), \tag{A.22}$$

$$\boldsymbol{e}_r(s,\theta)\cdot\boldsymbol{n}(s,r,\theta) = \cos\zeta(s,r,\theta), \tag{A.23}$$

$$\boldsymbol{e}_\theta(s,\theta)\cdot\boldsymbol{n}(s,r,\theta) = 0. \tag{A.24}$$

Note that the last two equalities only hold for pipes with circular cross-section. ζ is positive if the pipe is expanding, and negative if it is contracting.

A.3. Equations in curvilinear coordinates

With a curvilinear, orthogonal coordinate system established, vectors and tensors can be expressed in terms of a basis. For example, the velocity vector $\boldsymbol{u}$ can be written as

$$\boldsymbol{u} = u_s\boldsymbol{e}_s + u_r\boldsymbol{e}_r + u_\theta\boldsymbol{e}_\theta, \tag{A.25}$$

or, in more general notation,

$$\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3. \tag{A.26}$$

Since the coordinate system is orthogonal, a component of a vector follows as

$$u_i = \boldsymbol{u}\cdot\boldsymbol{e}_i. \tag{A.27}$$

Conservation of mass reads:

$$\frac{\partial\rho}{\partial t} + \frac{1}{h_1h_2h_3}\left[\frac{\partial\left(h_2h_3\rho u_1\right)}{\partial x_1} + \frac{\partial\left(h_1h_3\rho u_2\right)}{\partial x_2} + \frac{\partial\left(h_1h_2\rho u_3\right)}{\partial x_3}\right] = 0. \tag{A.28}$$

Conservation of momentum quickly becomes complicated, see e.g. [12]. For future reference, we give the components of the viscous stress tensor along direction 1:

$$\tau_{11} = 2\mu\left(\frac{1}{h_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_1h_2}\frac{\partial h_1}{\partial x_2} + \frac{u_3}{h_3h_1}\frac{\partial h_1}{\partial x_3}\right) + \lambda\nabla\cdot\boldsymbol{u}, \tag{A.29}$$

$$\tau_{12} = \tau_{21} = \mu\left(\frac{h_2}{h_1}\frac{\partial}{\partial x_1}\left(\frac{u_2}{h_2}\right) + \frac{h_1}{h_2}\frac{\partial}{\partial x_2}\left(\frac{u_1}{h_1}\right)\right), \tag{A.30}$$

$$\tau_{13} = \tau_{31} = \mu\left(\frac{h_3}{h_1}\frac{\partial}{\partial x_1}\left(\frac{u_3}{h_3}\right) + \frac{h_1}{h_3}\frac{\partial}{\partial x_3}\left(\frac{u_1}{h_1}\right)\right). \tag{A.31}$$

Evaluating these expressions by employing (A.18)-(A.20) gives

$$\tau_{ss} = 2\mu K\left(\frac{\partial u_s}{\partial s} + u_r\kappa\sin(\theta+\phi) + u_\theta\kappa\cos(\theta+\phi)\right) + \lambda\nabla\cdot\boldsymbol{u}, \tag{A.32}$$

$$\tau_{sr} = \tau_{rs} = \mu\left(K\frac{\partial u_r}{\partial s} + \frac{1}{K}\frac{\partial}{\partial r}\left(Ku_s\right)\right), \tag{A.33}$$

$$\tau_{s\theta} = \tau_{\theta s} = \mu\left(rK\frac{\partial}{\partial s}\left(\frac{u_\theta}{r}\right) + \frac{1}{rK}\frac{\partial}{\partial\theta}\left(Ku_s\right)\right), \tag{A.34}$$

where $K = 1/h_s = 1/(1+\kappa r\sin(\theta+\phi))$. Note that Stokes' hypothesis sets $\lambda = -\frac{2}{3}\mu$.

Appendix B.

Stratified flow

In this appendix we compute a number of geometrical quantities that are required to compute the friction terms in the momentum equation(s) in case of stratified flow.

B.1. Definitions

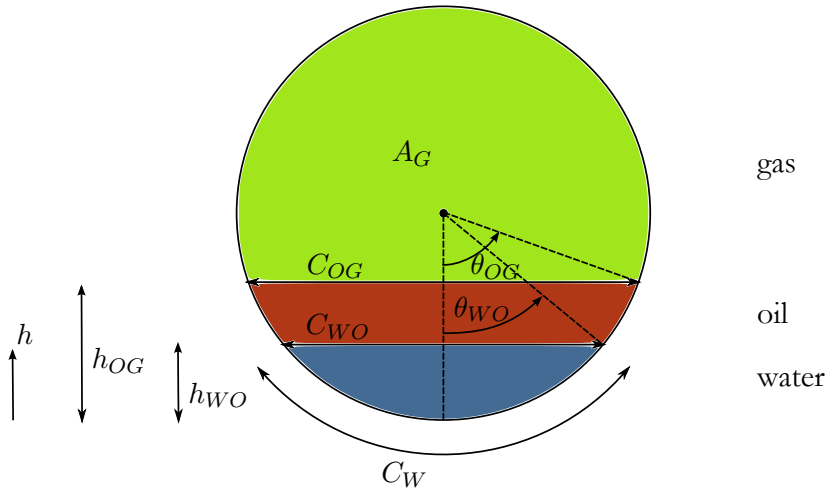

Figure B.1.: Stratified flow layout and definitions.

Length of interfacial area:

$$C_{WO} = 2R\sin\theta_{WO}, \tag{B.1}$$
$$C_{OG} = 2R\sin\theta_{OG}. \tag{B.2}$$

'Wetted' perimeter:

$$C_W = 2R\theta_{WO}, \tag{B.3}$$
$$C_O = 2R(\theta_{OG} - \theta_{WO}), \tag{B.4}$$
$$C_G = 2R(\pi - \theta_{OG}). \tag{B.5}$$

Interface height:

$$h_{WO} = R - R\cos\theta_{WO} \tag{B.6}$$
$$h_{OG} = R - R\cos\theta_{OG}. \tag{B.7}$$

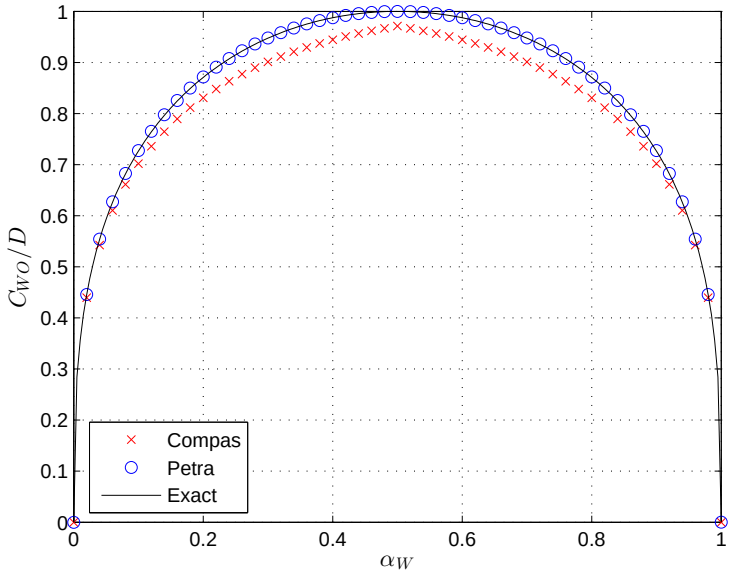

Figure B.2.: Interfacial perimeter as function of hold-up.

The area occupied by a phase is computed by the following integral (one can also do this geometrically, but the integral approach can be easily extended to determine the hydraulic gradients, done in the next section):

$$\begin{aligned} A_\beta &= \int_{\Omega_\beta} \mathrm{d}A = \int_{h_1}^{h_2} 2R\sin\theta\,\mathrm{d}h \\ &= 2\int_{\theta_1}^{\theta_2} R^2\sin^2\theta\,\mathrm{d}\theta = R^2\left((\theta_2-\theta_1)-\frac{1}{2}(\sin 2\theta_2-\sin 2\theta_1)\right). \end{aligned} \tag{B.8}$$

The hold-up, defined as $\alpha_\beta := A_\beta/A$, then follows simply as:

$$\alpha_W = \frac{1}{\pi}(\theta_{WO} - \frac{1}{2}\sin 2\theta_{WO}), \tag{B.9}$$

$$\alpha_O = \frac{1}{\pi}(\theta_{OG} - \theta_{WO} - \frac{1}{2}(\sin 2\theta_{OG} - \sin 2\theta_{WO})), \tag{B.10}$$

$$\alpha_G = \frac{1}{\pi}(\pi - \theta_{OG} + \frac{1}{2}\sin 2\theta_{OG}). \tag{B.11}$$

Expressing the interfacial perimeter in terms of hold-up is necessary when computing the friction terms given the current hold-ups. For example, for the water fraction, first express θ_{WO} in terms of α_W:

$$\pi\alpha_W \approx \theta_{WO} - \frac{1}{2}(2\theta_{WO} - \frac{4}{3}\theta_{WO}^3 + \mathcal{O}(\theta^5)) \Rightarrow \theta_{WO} \approx \left(\frac{3}{2}\alpha_W\pi\right)^{1/3}, \tag{B.12}$$

and then C_{WO} follows from (B.1). This expression is currently used in Compas and shown in figure B.2. The maximum difference with the exact result is almost 0.05. It is relatively easy to improve this expression, using for example the expression in the Petra manual [13]:

$$\theta_{WO} = \pi\alpha_W + \left(\frac{3}{2}\pi\right)^{1/3}\left(1 - 2\alpha_W + \alpha_W^{1/3} - (1-\alpha_W)^{1/3}\right). \tag{B.13}$$

This expression has a maximum error of 0.001. See figure B.2 for a comparison of the approximations.

B.2. Hydraulic gradients

In case of two-phase flow, hydraulic gradients lead to integrals of the form

$$\int_{A_O(s)} \frac{\partial p_O}{\partial s}\mathrm{d}A = \int_{A_O(s)} \frac{\partial}{\partial s}\left(p_{OG} - \rho_O g\cos\varphi(h - h_{OG})\right)\mathrm{d}A. \tag{B.14}$$

We focus on the second part of the integrand:

$$\begin{aligned}
&\int_{A_O(s)} \frac{\partial}{\partial s}\left(\rho_O g\cos\varphi(h - h_{OG})\right)\mathrm{d}A \\
&\quad= \int_{A_O(s)} \frac{\partial}{\partial s}\left(\rho_O g\cos\varphi\right)(h - h_{OG})\mathrm{d}A + \int_{A_O(s)} \left(\rho_O g\cos\varphi\right)\frac{\partial}{\partial s}(h - h_{OG})\mathrm{d}A \\
&\quad= \frac{\partial}{\partial s}\left(\rho_O g\cos\varphi\right)\int_{A_O(s)} (h - h_{OG})\mathrm{d}A - \rho_O g\cos\varphi A_O\frac{\partial h_{OG}}{\partial s}.
\end{aligned} \tag{B.15}$$

The integral can be evaluated with a similar parametrization as (B.8):

$$\begin{aligned}
\int_{A_O} h\,\mathrm{d}A &= 2\int_\theta R(1-\cos\theta)R^2\sin^2\theta\,\mathrm{d}\theta \\
&= \left[\theta - \sin 2\theta - \frac{2}{3}\sin^3\theta\right]_0^{\theta_{OG}} = RA_O - \frac{1}{12}C_{OG}^3.
\end{aligned} \tag{B.16}$$

Gathering all terms then leads to

$$\begin{aligned}
\int_{A_O(s)} \frac{\partial p_O}{\partial s}\mathrm{d}A = \int_{A_O(s)} \frac{\partial p_{OG}}{\partial s}\mathrm{d}A + \rho_O g\cos\varphi A_O\frac{\partial h_{OG}}{\partial s} \\
- \frac{\partial}{\partial s}\left(\rho_O g\cos\varphi\right)\left[(R - h_{OG})A_O - \frac{1}{12}C_{OG}^3\right].
\end{aligned} \tag{B.17}$$

The last term is not present in [9]. The term in square brackets is generally not small; e.g. if the contact is in the middle of the pipe we have $\left[(R - h_{OG})A_O - \frac{1}{12}C_{OG}^3\right] = -\frac{2}{3}R^3$. However, if density and inclination variations are small, then the last term can be neglected.

An alternative derivation of the above equation can be done by using Leibniz' rule, expressing the integral of the derivative in terms of the derivative of the integral and a boundary term. This leads to the same equation.

For three-phase flow a similar derivation can be performed. The result is:

$$\int_{A_W(s)} \frac{\partial p_W}{\partial s} \mathrm{d}A = \left\langle \frac{\partial p_{OG}}{\partial s} \right\rangle A_W + \underbrace{(\rho_W - \rho_O) g \cos\varphi A_W \frac{\partial h_{WO}}{\partial s} + \rho_O g \cos\varphi A_W \frac{\partial h_{OG}}{\partial s}}_{\mathrm{HG}_W} - \frac{\partial}{\partial s}\left(\rho_O g \cos\varphi\right)(h_{WO} - h_{OG}) A_W - \frac{\partial}{\partial s}\left(\rho_W g \cos\varphi\right)\left[(R - h_{WO}) A_W - \frac{1}{12} C_{WO}^3\right], \tag{B.18}$$

$$\int_{A_O(s)} \frac{\partial p_O}{\partial s} \mathrm{d}A = \int_{A_O(s)} \frac{\partial p_{OG}}{\partial s} \mathrm{d}A + \underbrace{\rho_O g \cos\varphi A_O \frac{\partial h_{OG}}{\partial s}}_{\mathrm{HG}_O} - \frac{\partial}{\partial s}\left(\rho_O g \cos\varphi\right)\left[(R - h_{OG}) A_O - \frac{1}{12} C_{OG}^3\right], \tag{B.19}$$

$$\int_{A_G(s)} \frac{\partial p_G}{\partial s} \mathrm{d}A = \int_{A_G(s)} \frac{\partial p_{OG}}{\partial s} \mathrm{d}A + \underbrace{\rho_G g \cos\varphi A_G \frac{\partial h_{OG}}{\partial s}}_{\mathrm{HG}_G} - \frac{\partial}{\partial s}\left(\rho_G g \cos\varphi\right)\left[(R - h_{OG}) A_G + \frac{1}{12} C_{OG}^3\right]. \tag{B.20}$$

We have defined the terms with underbraces as the hydraulic gradients HG_β; in principle the other terms involving $\frac{\partial}{\partial s}(\rho g \cos\varphi)$ should also be involved, but they are probably small for low-speed (almost incompressible) flows.

The hydraulic gradients contain terms of the form $A_O \frac{\partial h_{\mathrm{OG}}}{\partial s}$; both A_O and h_{OG} can be expressed in terms of hold-ups following section B.1.

In Petra [13] a similar approach is followed, but the reference pressure is taken at the top of the pipe instead of at the oil-gas interface.

Appendix C.
Averaging

C.1. Definitions

The volume average of a scalar quantity $f(\boldsymbol{x}, t)$ associated with a volume Ω (such as density):

$$\{f\}(t) := \frac{1}{V}\int_{\Omega} f(\boldsymbol{x}, t)\,\mathrm{d}\Omega, \tag{C.1}$$

where $V = \int_{\Omega}\mathrm{d}\Omega$, assumed to be independent of time. The area average of a scalar quantity $g(\boldsymbol{x}, t)$ associated with a surface S (such as pressure):

$$\langle g\rangle(t) := \frac{1}{A}\int_{S} g(\boldsymbol{x}, t)\,\mathrm{d}S, \tag{C.2}$$

where $A = \int_S \mathrm{d}S$, which is not necessarily the cross-sectional area of the pipe. One can also define the area average of a scalar quantity $f(\boldsymbol{x}, t)$ associated with a volume, by considering a pipe segment $V = A\Delta s$ and taking the limit of (C.1) for $\Delta s \to 0$:

$$\langle f\rangle(s, t) := \frac{1}{A}\int_{S} f(\boldsymbol{x}, t)\,\mathrm{d}S. \tag{C.3}$$

This cross-sectional area average still depends on s, since

$$\lim_{\delta s\to 0}\{f\}(t) = \langle f\rangle(s, t). \tag{C.4}$$

C.2. Average of products and product of averages

The commutation error between the operations of averaging and taking products can be evaluated as follows. Take for Ω a cuboid with sides of length Δ_k, $k = 1\ldots 3$, and let $\Delta := \max_k \Delta_k$. Assuming f is sufficiently smooth, its average can be expressed in terms of its value and its derivatives in the midpoint (summation over k is implied):

$$\{f\} = f + \frac{1}{24}\Delta_k^2\frac{\partial^2 f}{\partial k^2} + \mathcal{O}(\Delta^4). \tag{C.5}$$

The average of a product of functions is then given by

$$\{fg\} = (fg) + \frac{1}{24}\Delta_k^2\frac{\partial^2 (fg)}{\partial k^2} + \mathcal{O}(\Delta^4), \tag{C.6}$$

and the product of averages by:

$$\begin{aligned}\{f\}\{g\} &= \left(f + \frac{1}{24}\Delta_k^2\frac{\partial^2 f}{\partial k^2} + \mathcal{O}(\Delta^4)\right)\left(g + \frac{1}{24}\Delta_k^2\frac{\partial^2 g}{\partial k^2} + \mathcal{O}(\Delta^4)\right), \\ &= fg + \frac{1}{24}f\Delta_k^2\frac{\partial^2 g}{\partial k^2} + \frac{1}{24}g\Delta_k^2\frac{\partial^2 f}{\partial k^2} + \mathcal{O}(\Delta^4), \\ &= fg + \frac{1}{24}\Delta_k^2\frac{\partial^2 (fg)}{\partial k^2} + \frac{1}{12}\Delta_k^2\frac{\partial f}{\partial k}\frac{\partial g}{\partial k} + \mathcal{O}(\Delta^4).\end{aligned} \tag{C.7}$$

The difference between the two is therefore $\mathcal{O}(\Delta^2)$:

$$\{fg\} - \{f\}\{g\} = \mathcal{O}(\Delta^2). \tag{C.8}$$

For a circular cross-section instead of a cuboid the same result can be derived, if the cross-section is radially symmetric (such as a circle).

C.3. Proofs

Proof of (3.12), following [14]:

$$\int_{\Omega_\beta(\boldsymbol{x},t)} \boldsymbol{A}_\beta(\boldsymbol{r},t)\,\mathrm{d}\Omega = \int_{\Omega(\boldsymbol{x},t)} \boldsymbol{A}_\beta(\boldsymbol{r},t)\gamma_\beta(\boldsymbol{r},t)\,\mathrm{d}\Omega = \int_{\Omega(t)} \boldsymbol{A}_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\gamma_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\,\mathrm{d}\boldsymbol{\xi} \tag{C.9}$$

In local $\boldsymbol{\xi}$ coordinates the integral over the entire domain (last term) is not dependent on $\boldsymbol{x}$. Taking the divergence:

$$\begin{aligned}\nabla_{\boldsymbol{x}} \cdot \int_{\Omega(t)} \boldsymbol{A}_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\gamma_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\,\mathrm{d}\boldsymbol{\xi} &= \int_{\Omega(t)} \nabla_{\boldsymbol{x}} \cdot \left(\boldsymbol{A}_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\gamma_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\right)\,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{\Omega(t)} \nabla_{\boldsymbol{\xi}} \cdot \left(\boldsymbol{A}_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\gamma_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\right)\,\mathrm{d}\boldsymbol{\xi} \\ &= \int_{\partial\Omega(t)} \boldsymbol{A}_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\gamma_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\cdot\boldsymbol{n}\,\mathrm{d}S \\ &= \int_{S_\beta(\boldsymbol{x},t)} \boldsymbol{A}_\beta(\boldsymbol{x}+\boldsymbol{\xi},t)\cdot\boldsymbol{n}\,\mathrm{d}S.\end{aligned} \tag{C.10}$$

The third step in this derivation should actually be performed by splitting $\Omega(t)$ in domains for the different phases, and applying Gauss' divergence theorem to each subdomain.

An alternative proof of equation (3.13) starts with the differential form (A.1). Integrating (A.1) over the moving volume $\Omega_\beta(\boldsymbol{x},t)$ gives:

$$\int_{\Omega_\beta(\boldsymbol{x},t)} \frac{\partial\rho}{\partial t}\,\mathrm{d}\Omega + \int_{\Omega_\beta(\boldsymbol{x},t)} \nabla_{\boldsymbol{r}} \cdot (\rho\boldsymbol{u})\,\mathrm{d}\Omega = 0. \tag{C.11}$$

Note that the integrands are depending on $\boldsymbol{r}$ and t. The first integral, of the temporal derivative, can be changed into the temporal derivative of the integral by employing Reynolds' transport theorem:

$$\frac{\partial}{\partial t}\int_{\Omega_\beta(\boldsymbol{x},t)} f(\boldsymbol{r},t)\,\mathrm{d}\Omega = \int_{\Omega_\beta(\boldsymbol{x},t)} \frac{\partial f}{\partial t}(\boldsymbol{r},t)\,\mathrm{d}\Omega + \int_{\partial\Omega_\beta(\boldsymbol{x},t)} f(\boldsymbol{r},t)\boldsymbol{w}\cdot\boldsymbol{n}_\beta\,\mathrm{d}S. \tag{C.12}$$

where $\boldsymbol{w}$ is the interface velocity. The second integral, of the divergence, cannot be simply expressed in terms of a surface integral only (with the divergence theorem), due to the fact that the volume over which we integrate depends on $\boldsymbol{x}$. The generalized divergence theorem reads

$$\int_{\Omega_\beta(\boldsymbol{x},t)} \nabla_{\boldsymbol{r}}\cdot\boldsymbol{f}(\boldsymbol{r},t)\,\mathrm{d}\Omega = \nabla_{\boldsymbol{x}}\cdot\int_{\Omega_\beta(\boldsymbol{x},t)} \boldsymbol{f}(\boldsymbol{r},t)\,\mathrm{d}\Omega + \int_{\partial\Omega_\beta(\boldsymbol{x},t)} \boldsymbol{f}(\boldsymbol{r},t)\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{C.13}$$

See for example [7]. Application of (C.12) and (C.13) to (C.11) yields

$$\frac{\partial}{\partial t}\int_{\Omega_\beta(\boldsymbol{x},t)} \rho_\beta\,\mathrm{d}\Omega - \int_{\partial\Omega_\beta(\boldsymbol{x},t)} \rho_\beta\boldsymbol{w}\cdot\boldsymbol{n}_\beta\,\mathrm{d}S + \nabla_{\boldsymbol{x}}\cdot\int_{\Omega_\beta(\boldsymbol{x},t)} \rho_\beta\boldsymbol{u}_\beta\,\mathrm{d}\Omega + \int_{\partial\Omega_\beta(\boldsymbol{x},t)} \rho_\beta\boldsymbol{u}_\beta\cdot\boldsymbol{n}\,\mathrm{d}S = 0. \tag{C.14}$$

In case equation (3.11) holds, the last integral in (C.13) is over $S_{\beta\gamma}$, and $\boldsymbol{w} = 0$ on S_β. Upon introducing the volume average (3.10) the last equation reduces to (3.13).

Appendix D.

Alternative forms of the energy equation

The conservation equations for mass, momentum and energy, discussed in chapter 2, are three 'independent' equations corresponding to different physical laws. Other conservation equations can be derived. For example, conservation of mechanical (kinetic) energy can be obtained by taking the inner product of the momentum equations with $\boldsymbol{u}$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho\frac{1}{2}|\boldsymbol{u}|^2\,\mathrm{d}\Omega + \int_{\partial\Omega}\rho\frac{1}{2}|\boldsymbol{u}|^2\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = -\int_\Omega \boldsymbol{u}\cdot\nabla p\,\mathrm{d}\Omega + \int_\Omega \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau})\,\mathrm{d}\Omega + \int_\Omega \rho\boldsymbol{g}\cdot\boldsymbol{u}\,\mathrm{d}\Omega. \tag{D.1}$$

Subtracting this equation from the total energy equation (2.34) leads to conservation of internal (thermal) energy:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho e\,\mathrm{d}\Omega + \int_{\partial\Omega}\rho e\boldsymbol{u}\cdot\boldsymbol{n}\,\mathrm{d}S = -\int_\Omega p\nabla\cdot\boldsymbol{u}\,\mathrm{d}\Omega + \int_\Omega(\boldsymbol{\tau}:\nabla\boldsymbol{u})\,\mathrm{d}\Omega - \int_{\partial\Omega}\boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{D.2}$$

To obtain this equation the following identity has been used:

$$\nabla\cdot(\boldsymbol{\tau}\cdot\boldsymbol{u}) = \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau}) + \boldsymbol{\tau}:\nabla\boldsymbol{u}. \tag{D.3}$$

The integral version of this identity reads

$$\int_{\partial\Omega}(\boldsymbol{\tau}\cdot\boldsymbol{u})\cdot\boldsymbol{n}\,\mathrm{d}S = \int_\Omega \boldsymbol{u}\cdot(\nabla\cdot\boldsymbol{\tau})\,\mathrm{d}\Omega + \int_\Omega(\boldsymbol{\tau}:\nabla\boldsymbol{u})\,\mathrm{d}\Omega. \tag{D.4}$$

It is important to note that in the total energy equation the viscous stresses appear as surface integral, while in the internal energy equation they appear as a volume integral. On no-slip boundaries the total energy E *does not change* due to viscous effects, but the internal energy e always *increases* due to viscous dissipation in the interior of the fluid because $\boldsymbol{\tau}:\nabla\boldsymbol{u} > 0$: deformation of the fluid results in heat generation. This increase in internal energy by viscous dissipation is a loss term in the kinetic energy equation. The sum of the two is zero.

For 1D steady single-phase flow with constant cross-sectional area, the equations become:

$$\frac{\mathrm{d}}{\mathrm{d}s}(\rho u) = 0, \tag{D.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(p+\rho u^2) = \tau_w C/A - \rho g \sin\varphi, \tag{D.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\rho u E) = -\frac{\mathrm{d}}{\mathrm{d}s}(pu) - \rho g u \sin\varphi - Q_w C/A, \tag{D.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\rho u \frac{1}{2}u^2) = -u\frac{\mathrm{d}p}{\mathrm{d}s} - \rho g u \sin\varphi - \Phi, \tag{D.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}s}(\rho u e) = -p\frac{\mathrm{d}u}{\mathrm{d}s} + \Phi - Q_w C/A. \tag{D.9}$$

Note that τ_w is generally negative. The 1D kinetic energy equation has been obtained by starting from the momentum equation in differential form, multiplying it by u, integrating over a pipe segment, applying boundary conditions, and then taking the limit of an infinitesimally small segment. One can also take another approach, namely first integrating over a pipe segment, obtaining the 1D momentum equation, and then multiplying by u. The two approaches are shown in the commuting diagram below:

$$\begin{array}{ccc} \text{3D momentum pde, (A.4)} & \xrightarrow{\cdot\boldsymbol{u}} & \text{3D kin. energy pde, (A.7)} \\ \Big\downarrow \text{integrate, apply BC, average, limit } \delta s\to 0 & & \Big\downarrow \text{integrate, apply BC, average, limit } \delta s\to 0 \\ \text{1D momentum pde, (D.6)} & \xrightarrow{\times u} & \text{1D kin. energy pde, (D.8)} \end{array}$$

In order for the diagram to commute, we must have:

$$\Phi = -\tau_w u C/A. \tag{D.10}$$

It is somewhat puzzling that upon multiplying the 1D momentum equation with u, the wall shear stress seems to perform work, while this is not possible in the 3D equations. Eventually, the discussion on whether the wall shear stress actually performs work, and if this equals the dissipation, is not most relevant. The most important conclusion, which can also be derived without considering the kinetic and mechanical energy equations, is that (in steady, constant area subsonic pipe flow) *friction causes a pressure drop, a temperature decrease, a density decrease, and a velocity increase* [17]. This is already valid for a thermally perfect gas ($h = h(T)$). In the more general case that $h = h(p,T)$, there is an additional effect causing a temperature drop, known as the Joule-Thomson effect. It is found by rewriting the total energy equation as [8]

$$C_p \mathrm{d}T = -\underbrace{\left(\frac{\partial h}{\partial p}\right)_T \mathrm{d}p}_{(i)} - \underbrace{\mathrm{d}\left(\frac{1}{2}u^2\right)}_{(ii)} - \underbrace{g\sin\varphi\,\mathrm{d}s}_{(iii)} - \underbrace{\frac{Q_w C}{\dot{m}A}\mathrm{d}s}_{(iv)}, \tag{D.11}$$

where $\dot{m} = \rho u$. The temperature can decrease due to (i) the Joule-Thomson effect, (ii) friction (giving a velocity increase), (iii) gravity, (iv) heat conduction via the pipe wall.

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