5. Characteristic boundary treatment

We start with the 1D two fluid equations in a channel to derive characteristic boundary treatment . . .

$$\frac{\partial}{\partial t}(\rho_g A_g) + \frac{\partial}{\partial x}(\rho_g u_g A_g) = 0 \tag{5.1}$$

$$\frac{\partial}{\partial t}(\rho_l A_l) + \frac{\partial}{\partial x}(\rho_l u_l A_l) = 0 \tag{5.2}$$

$$\frac{\partial}{\partial t}(\rho_g u_g A_g) + \frac{\partial}{\partial x}(\rho_g u_g^2 A_g + pA_g) - p\frac{\partial A_g}{\partial x} - \rho_g g_n A_g \frac{\partial A_g}{\partial x} + \rho_g g_s A_g + friction = 0 \tag{5.3}$$

$$\frac{\partial}{\partial t}(\rho_l u_l A_l) + \frac{\partial}{\partial x}(\rho_l u_l^2 A_l + pA_l) - p\frac{\partial A_l}{\partial x} + \rho_l g_n A_l \frac{\partial A_l}{\partial x} + \rho_l g_s A_l + friction = 0 \tag{5.4}$$

The liquid holdup A_l and gas holdup A_g have to fill up the total area of the pipe A_{tot} (which is taken constant in this analysis) , which allows a direct relation between A_g and A_l: $A_l + A_g = A_{tot}$.

The last two terms of Eq. 5.3 and Eq. 5.4 represent gravity and an expression for friction. They do not include derivatives of unknown quantities and are identified as source terms. Grouping these two terms, as they will be treated in the same manner in the characteristic boundary treatment, and grouping terms with terms $\partial A_g/\partial x$, $\partial A_l/\partial x$ we find:

$$\frac{\partial}{\partial t}(\rho_g A_g) + \frac{\partial}{\partial x}(\rho_g u_g A_g) = 0 \tag{5.5}$$

$$\frac{\partial}{\partial t}(\rho_l A_l) + \frac{\partial}{\partial x}(\rho_l u_l A_l) = 0 \tag{5.6}$$

$$\frac{\partial}{\partial t}(\rho_g u_g A_g) + \frac{\partial}{\partial x}(\rho_g u_g^2 A_g + pA_g) - (p + \rho_g g_n A_g)\frac{\partial A_g}{\partial x} + S_1 = 0 \tag{5.7}$$

$$\frac{\partial}{\partial t}(\rho_l u_l A_l) + \frac{\partial}{\partial x}(\rho_l u_l^2 A_l + pA_l) - (p - \rho_l g_n A_l)\frac{\partial A_l}{\partial x} + S_2 = 0 \tag{5.8}$$

We now adopt as primitive variable vector $\mathbf{W} = [Ag, p, u_g, u_l]^T$ and define Jacobian matrix $\mathbf{Q} = \frac{\partial \mathbf{T}}{\partial \mathbf{W}}$, where $\mathbf{T} = [\rho_g u_g A_g, \rho_l u_l A_l, \rho_g u_g^2 A_g + pA_g, \rho_l u_l^2 A_l + pA_l]^T$:

$$\begin{bmatrix} \frac{\partial \rho_g A_g}{\partial t} \\ \frac{\partial \rho_l A_l}{\partial t} \\ \frac{\partial \rho_g u_g A_g}{\partial t} \\ \frac{\partial \rho_l u_l A_l}{\partial t} \end{bmatrix} + \mathbf{Q} \begin{bmatrix} \frac{\partial A_g}{\partial x} \\ \frac{\partial p}{\partial x} \\ \frac{\partial u_g}{\partial x} \\ \frac{\partial u_l}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -(p + \rho_g g_n A_g)\frac{\partial A_g}{\partial x} \\ -(p + \rho_g g_n A_g)\frac{\partial A_g}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ S_1 \\ S_2 \end{bmatrix} = \mathbf{0}. \tag{5.9}$$

$\mathbf{Q}$ can be computed analytically and follows (using $\partial A_g = -\partial A_l$):

$$\mathbf{Q} = \begin{bmatrix} u_g\rho_g & A_g u_g \frac{\partial \rho_g}{\partial p} & A_g\rho_g & 0 \\ -u_l\rho_l & A_l u_l \frac{\partial \rho_l}{\partial p} & 0 & A_l\rho_l \\ u_g^2\rho_g + p & A_g(u_g^2 \frac{\partial \rho_g}{\partial p} + 1) & 2A_g u_g \rho_g & 0 \\ -u_l^2\rho_l - p & A_l(u_l^2 \frac{\partial \rho_l}{\partial p} + 1) & 0 & 2A_l u_l \rho_l \end{bmatrix} \tag{5.10}$$

We can now combine the second and third term in Eq. 5.9 to obtain:

$$\begin{bmatrix} \frac{\partial \rho_g A_g}{\partial t} \\ \frac{\partial \rho_l A_l}{\partial t} \\ \frac{\partial \rho_g u_g A_g}{\partial t} \\ \frac{\partial \rho_l u_l A_l}{\partial t} \end{bmatrix} + \mathbf{B} \begin{bmatrix} \frac{\partial A_g}{\partial x} \\ \frac{\partial p}{\partial x} \\ \frac{\partial u_g}{\partial x} \\ \frac{\partial u_l}{\partial x} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ S_1 \\ S_2 \end{bmatrix} = \mathbf{0}. \tag{5.11}$$

where $\mathbf{B}$ is

$$\mathbf{B} = \begin{bmatrix} u_g\rho_g & A_g u_g \frac{\partial \rho_g}{\partial p} & A_g\rho_g & 0 \\ -u_l\rho_l & A_l u_l \frac{\partial \rho_l}{\partial p} & 0 & A_l\rho_l \\ u_g^2\rho_g - \rho_g g_n A_g & A_g(u_g^2 \frac{\partial \rho_g}{\partial p} + 1) & 2A_g u_g \rho_g & 0 \\ -u_l^2\rho_l - \rho_l g_n A_l & A_l(u_l^2 \frac{\partial \rho_l}{\partial p} + 1) & 0 & 2A_l u_l \rho_l \end{bmatrix} \tag{5.12}$$

We can write Eq. 5.11 more compact using conservative variable vector $\mathbf{U} = [\rho_g A_g, \rho_l A_l, \rho_g u_g A_g, \rho_l u_l A_l]^T$ and source term vector $\mathbf{S} = [0, 0, S1, S2]^T$

$$\frac{\partial \mathbf{U}}{\partial t} + \mathbf{B} \cdot \frac{\partial \mathbf{W}}{\partial x} + \mathbf{S} = \mathbf{0}. \tag{5.13}$$

We now define the Jacobian $\mathbf{P} = \frac{\partial \mathbf{U}}{\partial \mathbf{W}}$ and multiply Eq. 5.13 by $\mathbf{P}^{-1}$ to obtain the two-fluid equations in quasi linear form:

$$\boxed{\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}(\mathbf{W}) \frac{\partial \mathbf{W}}{\partial x} + \mathbf{P}^{-1}\mathbf{S} = \mathbf{0}.} \tag{5.14}$$

where $\mathbf{A}(\mathbf{W}) = \mathbf{P}^{-1}\mathbf{B}$. Just like $\mathbf{B}$, also $\mathbf{P}$ and $\mathbf{P}^{-1}$ can be calculated be analytically, which allows an analytic expression for $\mathbf{A}(\mathbf{W})$:

$$\mathbf{A}(\mathbf{W}) =$$
$$\begin{pmatrix} \frac{A_g\, u_l\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, u_g\, \rho_g\, \frac{\partial}{\partial p}\rho_l}{A_g\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l} & \frac{A_g\, A_l\, (u_g - u_l)\, \frac{\partial}{\partial p}\rho_g\, \frac{\partial}{\partial p}\rho_l}{A_g\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l} & \frac{A_g\, A_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l}{A_g\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l} & -\frac{A_g\, A_l\, \rho_l\, \frac{\partial}{\partial p}\rho_g}{A_g\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l} \\ \frac{\rho_g\, \rho_l\, (u_g - u_l)}{A_g\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l} & \frac{A_g\, u_g\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, u_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l}{A_g\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l} & \frac{A_g\, \rho_g\, \rho_l}{A_g\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l} & \frac{A_l\, \rho_g\, \rho_l}{A_g\, \rho_l\, \frac{\partial}{\partial p}\rho_g + A_l\, \rho_g\, \frac{\partial}{\partial p}\rho_l} \\ -g & \frac{1}{\rho_g} & u_g & 0 \\ -g & \frac{1}{\rho_l} & 0 & u_l \end{pmatrix} \tag{5.15}$$

We stress that no assumptions have been made and an equation of state has yet to be specified. As one of the most simple examples we could take the equation of state of the liquid incompressible

$\rho_l = \rho_{l0}$ and the gas compressible $\rho_g = p/c_g^2$. Here c_g^2 is the speed of sound in the gas. Substituting these equations of state into Eq. 5.15 yields:

$$\mathbf{A}(\mathbf{W}) = \begin{pmatrix} u_l & 0 & 0 & -A_l \\ \frac{p\,(u_g - u_l)}{A_g} & u_g & p & \frac{p\,A_l}{A_g} \\ -g & \frac{{c_g}^2}{p} & u_g & 0 \\ -g & \frac{1}{\rho_{l0}} & 0 & u_l \end{pmatrix} \tag{5.16}$$

5.1. Characteristic equations

To derive characteristic equations, from which time dependent equations for the boundary points can be obtained, we determine the eigenvalues and corresponding eigenvectors of $\mathbf{A}$, that is $\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}$ where $\mathbf{\Lambda}$ contains the eigenvalues $[\lambda_1, \lambda_2, \lambda_3, \lambda_4]^T$ on the diagonal and $\mathbf{R}$ contains the right eigenvectors of $\mathbf{A}$. For even the most simple example of $\mathbf{A}$, Eq. 5.16, an analytical expression is hard to obtain and is pages long. In practice the eigenvectors and eigenvalues are therefor determined numerically instead. This also guarantees more flexibility for more complex equations of state. The current approach is thus to provide the model with $\mathbf{A}$ and determine $\mathbf{R}$, $\mathbf{\Lambda}$ and $\mathbf{R}^{-1}$ numerically.

Using $\mathbf{A} = \mathbf{R}\mathbf{\Lambda}\mathbf{R}^{-1}$ we now multiply Eq. 5.14 with $\mathbf{R}^{-1}$ and find

$$\mathbf{R}^{-1}\frac{\partial \mathbf{W}}{\partial t} + \mathbf{\Lambda}\mathbf{R}^{-1}\frac{\partial \mathbf{W}}{\partial x} + \mathbf{R}^{-1}\mathbf{P}^{-1}\mathbf{S} = \mathbf{0} \tag{5.17}$$

To gain more insight in the numerical implementation we write 5.17 in component form, denoting the elements of $\mathbf{R}^{-1}$ as $R^{-1}_{i,j}$.

$$\begin{bmatrix} R^{-1}_{11}\frac{\partial A_g}{\partial t} + R^{-1}_{12}\frac{\partial p}{\partial t} + R^{-1}_{13}\frac{\partial u_g}{\partial t} + R^{-1}_{14}\frac{\partial u_l}{\partial t} \\ R^{-1}_{21}\frac{\partial A_g}{\partial t} + R^{-1}_{22}\frac{\partial p}{\partial t} + R^{-1}_{23}\frac{\partial u_g}{\partial t} + R^{-1}_{24}\frac{\partial u_l}{\partial t} \\ R^{-1}_{31}\frac{\partial A_g}{\partial t} + R^{-1}_{32}\frac{\partial p}{\partial t} + R^{-1}_{33}\frac{\partial u_g}{\partial t} + R^{-1}_{34}\frac{\partial u_l}{\partial t} \\ R^{-1}_{41}\frac{\partial A_g}{\partial t} + R^{-1}_{42}\frac{\partial p}{\partial t} + R^{-1}_{43}\frac{\partial u_g}{\partial t} + R^{-1}_{44}\frac{\partial u_l}{\partial t} \end{bmatrix} + \begin{bmatrix} \lambda_1\left(R^{-1}_{11}\frac{\partial A_g}{\partial x} + R^{-1}_{12}\frac{\partial p}{\partial x} + R^{-1}_{13}\frac{\partial u_g}{\partial x} + R^{-1}_{14}\frac{\partial u_l}{\partial x}\right) \\ \lambda_2\left(R^{-1}_{21}\frac{\partial A_g}{\partial x} + R^{-1}_{22}\frac{\partial p}{\partial x} + R^{-1}_{23}\frac{\partial u_g}{\partial x} + R^{-1}_{24}\frac{\partial u_l}{\partial x}\right) \\ \lambda_3\left(R^{-1}_{31}\frac{\partial A_g}{\partial x} + R^{-1}_{32}\frac{\partial p}{\partial x} + R^{-1}_{33}\frac{\partial u_g}{\partial x} + R^{-1}_{34}\frac{\partial u_l}{\partial x}\right) \\ \lambda_4\left(R^{-1}_{41}\frac{\partial A_g}{\partial x} + R^{-1}_{42}\frac{\partial p}{\partial x} + R^{-1}_{43}\frac{\partial u_g}{\partial x} + R^{-1}_{44}\frac{\partial u_l}{\partial x}\right) \end{bmatrix} + \mathbf{K} = \mathbf{0} \tag{5.18}$$

Here $\mathbf{K} = \mathbf{R}^{-1}\mathbf{P}^{-1}\mathbf{S}$. If the flow remains subsonic, we expect two of the numerically determined λ to be positive, while the other two are expected to be negative. At the left boundary point we can then use the left running characteristic equations, which follows from the interior, to set up time dependent equations for the left boundary point.

In practice this is done as follows: $\mathbf{A}$ is evaluated at the left boundary point. The eigenvalues $\mathbf{\Lambda}$, right eigenvector matrix $\mathbf{R}$, and $\mathbf{R}^{-1}$ are determined numerically. As $\mathbf{S}$ is also evaluated at the left boundary point, $\mathbf{K}$ can now be calculated. Subsequently, two equations from Eq. 5.18 are selected which correspond to the two negative eigenvalues (at the right boundary point the positive eigenvalues are selected). The spatial derivatives in these two equations can be determined from the interior with finite difference. The only unknowns left are the time derivatives of the primitive variables for which we now can solve. As we have four time derivatives of primitive variables, ($\frac{\partial A_g}{\partial t}$, $\frac{\partial p}{\partial t}$, $\frac{\partial u_g}{\partial t}$, and $\frac{\partial u_l}{\partial t}$), and two equations, two time derivatives have to be specified and the other two will follow. For reflective boundary conditions, for example, we set $\frac{\partial u_g}{\partial t} = 0$ and $\frac{\partial u_l}{\partial t} = 0$, which allows us to calculate $\frac{\partial A_g}{\partial t}$ and $\frac{\partial p}{\partial t}$. MH: THIS MAY NOT ALWAYS BE CONVENIENT, FOR

EXAMPLE IF WE WANT TO SPECIFY MASS INLET WITH CERTAIN MASS FLUX, WE RATHER DO THAT IN TERMS OF CONSERVATIVE VARIABLES -> WE MAY DERIVE AND DO A TEST CASE FOR TIME DEP. BC IN TERMS OF CONSERVATIVE VARIABLES TOO