7. Homogeneous Equilibrium Model

7.1. Governing equations

The Homogeneous Equilibrium Model (HEM) for the 1D isothermal two-fluid equations read:

$$\frac{\partial}{\partial t}(\rho_g A_g) + \frac{\partial}{\partial x}(\rho_g u_m A_g) = 0 \tag{7.1}$$

$$\frac{\partial}{\partial t}(\rho_l A_l) + \frac{\partial}{\partial x}(\rho_l u_m A_l) = 0 \tag{7.2}$$

$$\frac{\partial}{\partial t}(\rho_m u_m A) + \frac{\partial}{\partial x}(\rho_m u_m^2 A + pA) + \rho_m g_s A + T_w = 0 \tag{7.3}$$

The liquid holdup A_l and gas holdup A_g have to fill up the total area of the pipe A, which allows a direct relation between A_g and A_l: $A_l + A_g = A$. The last term of Equation 7.3 represents the effect of wall friction which is expressed as:

$$T_w = \frac{A}{2D} f_w \rho_m u_m \left|u_m\right| \tag{7.4}$$

Here f_w is a friction factor calculated with the Churchill correlation:

$$f_w = 8\left(\left(\frac{8}{Re}\right)^{12} + (a+b)^{-1.5}\right)^{1/12} \tag{7.5}$$

where

$$a = \left(-2.457\log\left(\left(\frac{7}{Re}\right)^{0.9} + 0.27\frac{\epsilon}{D}\right)\right)^{16} \tag{7.6}$$

$$b = \left(\frac{37530}{Re}\right)^{16} \tag{7.7}$$

Here ϵ is the roughness of the pipe and D the pipe diameter. Re is the Reynolds number which is defined as

$$Re = \frac{\rho_m u_m D}{\mu_m} \tag{7.8}$$

Here ρ_m and μ_m are the mixture density and mixture dynamic viscosity respectively, defined as

$$\rho_m = (\rho_g A_g + \rho_l A_l)/A \tag{7.9}$$

$$\mu_m = (\mu_g A_g + \mu_l A_l)/A \tag{7.10}$$

The model is closed by defining an equation of state for the gas density $\rho_g = \rho_g(p)$ (ρ_l is taken constant, $\rho_l = \rho_{l0}$):

$$\rho_g(p) = \frac{\rho_{norm} T_{norm}}{p_{norm}} \frac{p}{T} \tag{7.11}$$

7.1.1. IFP case

We now specify the relevant constants we use for simulating the IFP test case described in Omgba 2004. The test case describes a ramp up of the gas mass flow rate at the inlet of a 10 km pipe filled with liquid and gas. The liquid mass flow rate is constant during the ramp up with a value of 20 kg/s. The gas mass flow rate is initially $0.2\,kg/s$ and is then doubled to $0.4\,kg/s$ in a time span of 10 seconds in a linear fashion. As a result, a transient will move through the pipe and a new steady state is developed. The pressure at the outlet is fixed at $p_{outlet} = 10^6\,Pa$.

The constants for this case study are defined in Table 7.1. As can be noted from the table, we consider an actual fluid temperature of $T = 278K$.

ρ_{norm}	$1\,kg/m^3$
T_{norm}	$300\,K$
p_{norm}	$10^5\,Pa$
p_{outlet}	$10^6\,Pa$
T	$278\,K$
D	$0.146\,m$
L	$10\,km$
ϵ	$10^{-8}\,m$
ρ_{l0}	$1000\,kg/m^3$
μ_l	$8.9 * 10^{-4}\,Pas$
μ_g	$1.8 * 10^{-5}\,Pas$

Table 7.1: Values of relevant constants for IFP test case.

7.1.2. Eigenvalue analysis

The HEM described in Equation 7.1, 7.2, and 7.3 can be written in vector notation as:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \mathbf{S} = \mathbf{0}. \tag{7.12}$$

Here $\mathbf{U}$ is the conservative variable vector, $\mathbf{U} = [\rho_g A_g, \rho_l A_l, \rho_m u_m A]^T$, $\mathbf{F}$ is the flux vector $\mathbf{F} = [\rho_g u_g A_g, \rho_l u_l A_l, \rho_m u_m^2 A + pA]^T$, and $\mathbf{S}$ is the vector containing the source terms, $\mathbf{S} = [0, 0, \rho_m g_s A + T_w]^T$

We now adopt as primitive variable vector $\mathbf{W} = [Al, p, u_m]^T$ and define Jacobian matrix $\mathbf{F_W} = \frac{\partial \mathbf{F}}{\partial \mathbf{W}}$ and $\mathbf{U_W} = \frac{\partial \mathbf{U}}{\partial \mathbf{W}}$:

$$\mathbf{F_W} = \begin{pmatrix} -u_m\,\rho_g(p) & A_g\,u_m\,\frac{\partial}{\partial p}\rho_g(p) & A_g\,\rho_g(p) \\ u_m\,\rho_l(p) & A_l\,u_m\,\frac{\partial}{\partial p}\rho_l(p) & A_l\,\rho_l(p) \\ {u_m}^2\,\frac{\partial}{\partial A_l}\rho_m(p, A_l)\;A & \left({u_m}^2\,\frac{\partial}{\partial p}\rho_m(p, A_l)+1\right)\,A & 2\,u_m\,\rho_m(p, A_l)\;A \end{pmatrix} \tag{7.13}$$

$$\mathbf{U_W} = \begin{pmatrix} -\rho_g(p) & A_g\,\frac{\partial}{\partial p}\rho_g(p) & 0 \\ \rho_l(p) & A_l\,\frac{\partial}{\partial p}\rho_l(p) & 0 \\ u_m\,\frac{\partial}{\partial A_l}\rho_m(p, A_l)\,A & u_m\,\frac{\partial}{\partial p}\rho_m(p, A_l)\,A & \rho_m(p, A_l)\,A \end{pmatrix} \tag{7.14}$$

Making use of the definition of the Jacobians, we multiply Equation 7.12 by ${\mathbf{U_W}}^{-1}$ and obtain the HEM in quasi linear form:

$$\boxed{\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}(\mathbf{W})\frac{\partial \mathbf{W}}{\partial x} + {\mathbf{U_W}}^{-1}\mathbf{S} = \mathbf{0}.} \tag{7.15}$$

where $\mathbf{A}(\mathbf{W}) = {\mathbf{U_W}}^{-1}\mathbf{F_W}$. We can calculate $\mathbf{A}(\mathbf{W})$ analytically, as an analytic expression for ${\mathbf{U_W}}^{-1}$ exists. We will now substitute the relevant equations of state described in Section 7.1, and obtain for $\mathbf{A}(\mathbf{W})$:

$$\mathbf{A}(\mathbf{W}) = \begin{pmatrix} u_m & 0 & A_l \\ 0 & u_m & \frac{p\,A}{A_g} \\ 0 & \frac{{c_g}^2\,A}{A_l\,\rho_{10}\,{c_g}^2+A_g\,p} & u_m \end{pmatrix} \tag{7.16}$$

The eigenvalues and corresponding eigenvectors of $\mathbf{A}$, that is $\mathbf{A} = \mathbf{R}\boldsymbol{\Lambda}\mathbf{R}^{-1}$, which are used for characteristic boundary treatment, can now readily be calculated. Here $\boldsymbol{\Lambda}$ contains the eigenvalues $[\lambda_1, \lambda_2, \lambda_3]^T$ on the diagonal and $\mathbf{R}$ contains the right eigenvectors of $\mathbf{A}$.

$$\boldsymbol{\Lambda} = \begin{pmatrix} \frac{{A_g}^{\frac{3}{2}}\,p\,u_m - A\,c_g\,\sqrt{p}\,\sqrt{A_l\,\rho_{10}\,{c_g}^2+A_g\,p}+\sqrt{A_g}\,A_l\,{c_g}^2\,\rho_{10}\,u_m}{{A_g}^{\frac{3}{2}}\,p+\sqrt{A_g}\,A_l\,{c_g}^2\,\rho_{10}} & 0 & 0 \\ 0 & \frac{{A_g}^{\frac{3}{2}}\,p\,u_m + A\,c_g\,\sqrt{p}\,\sqrt{A_l\,\rho_{10}\,{c_g}^2+A_g\,p}+\sqrt{A_g}\,A_l\,{c_g}^2\,\rho_{10}\,u_m}{{A_g}^{\frac{3}{2}}\,p+\sqrt{A_g}\,A_l\,{c_g}^2\,\rho_{10}} & 0 \\ 0 & 0 & u_m \end{pmatrix} \tag{7.17}$$

$$\mathbf{R} = \begin{pmatrix} -\frac{\sqrt{A_g}\,A_l\,\sqrt{A_l\,\rho_{10}\,{c_g}^2+A_g\,p}}{A\,c_g\,\sqrt{p}} & \frac{\sqrt{A_g}\,A_l\,\sqrt{A_l\,\rho_{10}\,{c_g}^2+A_g\,p}}{A\,c_g\,\sqrt{p}} & 1 \\ -\frac{\sqrt{p}\,\sqrt{A_l\,\rho_{10}\,{c_g}^2+A_g\,p}}{\sqrt{A_g}\,c_g} & \frac{\sqrt{p}\,\sqrt{A_l\,\rho_{10}\,{c_g}^2+A_g\,p}}{\sqrt{A_g}\,c_g} & 0 \\ 1 & 1 & 0 \end{pmatrix} \tag{7.18}$$

$$\mathbf{R}^{-1} = \begin{pmatrix} 0 & -\frac{\sqrt{A_g}\,c_g}{2\,\sqrt{p}\,\sqrt{A_l\,\rho_{10}\,{c_g}^2+A_g\,p}} & \frac{1}{2} \\ 0 & \frac{\sqrt{A_g}\,c_g}{2\,\sqrt{p}\,\sqrt{A_l\,\rho_{10}\,{c_g}^2+A_g\,p}} & \frac{1}{2} \\ 1 & -\frac{A_g\,A_l}{A\,p} & 0 \end{pmatrix} \tag{7.19}$$