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Rank-sum test – U-test

- The U-test is used for two samples. The other names are;
  - (i) Wilcoxon test
  - (ii) Man-Whitney test
- We are going to test the hypothesis that we are sampling identical continuous populations against the alternative that the two populations have unequal means.
- To illustrate the procedure, suppose we want to compare the kinds of emergency flares on the basis of the following burning times.

Brand A	14.9	11.3	13.2	16.6	17.0	14.1	15.4	13.0	16.9
Brand B	15.2	19.8	14.7	18.3	16.2	21.2	18.9	12.2	15.3	19.4

Steps

1. Arrange the values in ascending order
2. Rank them jointly

Brand A

11.3	13.0	13.2	14.1	14.9	15.4	16.6	16.9	17.0
1	3	4	5	7	10	12	13	14

Brand B

12.2	14.7	15.2	15.3	16.2	18.3	18.9	19.4	19.8	21.2
2	6	8	9	11	15	16	17	18	19

→ The values of brand A occupy ranks 1, 3, 4, 5, 7, 10, 12, 13, and B occupy 2, 6, 8, 9, 11, 15, 16, 17, 18 and 19.

→ For ties we assign the mean of the ranks which they jointly occupy

- If there is an appropriate difference between the means of the two observations, most of the lower ranks are likely to go to the values of one sample while most of the higher ranks will go to the values of the other sample
- The test is based on the values of W_1 which is the sum of the ranks of the first sample or W_2 which is the sum of the ranks of the second sample.
- It doesn't matter whether we use W_1 or W_2 for if there are n_1 values in the first sample and n_2 values in the second sample, $W_1 + W_2$ is the sum of the first $n_1 + n_2$ integers i.e. $W_1 + W_2 =$

$$\frac{(n_1 + n_2)(n_1 + n_2 + 1)}{2}$$

- For any pair of values of W_1 and W_2. Therefore, the test based on W_1 and W_2 are equivalent. In actual practice we seldom use the test based on the statistics W_1 and W_2 instead we use the related statistics

$$U_1 = W_1 - \frac{n_1(n_1+1)}{2}$$

$$U_2 = W_2 - \frac{n_2(n_2+1)}{2}$$

or $U = \min(U_1, U_2)$

- The resulting tests are all equivalent to the one based on W_1 and W_2 but have the advantage that they label themselves more readily to the tables of critical values i.e

$$u_1 + u_2 = w_1 - \frac{n_1(n_1+1)}{2} + w_2 - \frac{n_2(n_2+1)}{2}$$

$$= \frac{(n_1+n_2)(n_1+n_2+1)}{2} - \frac{n_1(n_1+1)}{2} -$$

$$\frac{n_2(n_2+1)}{2}$$

$$= n_1^2 + n_1n_2 + n_1 + n_2n_1 + n_2 + n_2^2 - n_1^2 - n_1 - n_2^2 - n_2$$

$$= 2(u_1+u_2) = 2n_1n_2$$

$\Rightarrow$ $u_1 + u_2 = n_1n_2$ i.e the sum of $u_1 + u_2$ is always n_1n_2 and both of these random variables take on the same range of the values from 0 to n_1n_2.

- We use the following table for testing $H_0 : \mu_1 = \mu_2$ Vs the various alternatives.

Alternative hypothesis	Reject H_0 if
$\mu_1 \mp \mu_2$	$u' \le u_\alpha$
$\mu_1 > \mu_2$	$u_2 < u_{2\alpha}$
$\mu_1 < \mu_2$	$u_1 \le u_{2\alpha}$

where the level of significance is α for each test

- The critical values of u are such that u_α is the largest value for which $u \le u_\alpha$ does not exceed α.

Example

1) Two brands of flares have the following burning times

Brand A	14.9	11.3	13.2	16.6	17.0	14.1
	15.4	13.0	16.9			
Brand B	15.2	19.8	14.7	18.3	16.2	21.2
	18.9	12.2	15.3	19.4		

Test at $\alpha = 0.05$ level of significance whether the two samples come from identical continuous populations ~~of~~ or whether the mean burning times of B flares ie outshines A.

Solution

We want to test $H_0: \mu_1 = \mu_2$ vs $H_1: \mu_1 < \mu_2$

Arrange the values in ascending order (rank them jointly)

Brand A	11.3	13.0	13.2	14.1	14.9	15.4	16.6
	1	3	4	5	7	10	12
	16.9	17.0					
	13	14					
Brand B	12.2	14.7	15.2	15.3	16.2	18.3	18.9
	2	6	8	9	11	15	16
	19.4	19.8	21.2				
	17	18	19				

We reject H_0 if $u_1 \le u_{2\alpha}$

thus $w_1 = 1+3+4+5+7+10+12+13+14$
$= 69$

$\Rightarrow u_1 = w_1 = \frac{n_1(n_1+1)}{2} = 69 - \frac{9(10)}{2}$
$= 24$

for $n_1 = 9,\ n_2 = 10$

$u_{2\alpha} = u_{0.10} = 24$

Thus we conclude since $24 \leq 24$ we reject H_0 and conclude that on average brand A flakes as a mean value time that is less than that of brand B.

- When both n_1 and n_2 are greater than 8, it is considered reasonable to assume that the distribution of ~~μ_1 and μ_2~~ u_1 and u_2 can be approximated closely by the normal distribution. To perform the given rank sum test on the basis of this assumption, we need the following results

Theorem: Under the null hypothesis, the mean and variance of ~~μ_1 and μ_2~~ u_1 and u_2 are

1) $E[u_1] = E[u_2] = \frac{n_1 n^2}{2}$ and

2) $var(u_1) = var(u_2) = \frac{n_1 n_2 (n_1 + n_2 + 1)}{12}$

Example

The following are the number of minutes it took random samples of 15 men and 12 women to complete a written test given for their renewal of their driving licences

Men : 9.9, 7.4, 8.9, 9.1, 7.7, 9.7, 11.8, 9.2, 10.0, 10.2, 9.5, 10.8, 8.0, 11.0, 7.5

Women: 8.6, 10.9, 9.8, 10.7, 9.4, 10.3, 7.3, 11.5, 7.6, 9.3, 8.8, 9.6

Use the U-test based on the normal approximation to test $H_0 : \mu_1 = \mu_2$ Vs $H_1 : \mu_1 \neq \mu_2$ where μ_1 and μ_2 are the average amount of time it takes men and women to complete the test respectively

Solution

Re-arrange the data in ascending order.

7.3,	7.4,	7.5,	7.6,	7.7,	8.0,	8.6,	8.8,	8.9
1	2	3	4	5	6	7	8	9
9.1,	9.2,	9.3,	9.4,	9.5,	9.6,	9.7,	9.8,	9.9,
10	11	12	13	14	15	16	17	18
10.0	10.2	10.3,	10.7,	10.8,	10.9,	11.0,	11.5	11.8
19	20	21	22	23	24	25	26	27

The underlined values are for men

$n_1 = 15,$

$n_2 = 12,$

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Ranks for men $\Rightarrow w_1 = 2+3+5+6+9+10+11+14+16+18+19+20+23+25+27 = 208$

Ranks for Women $\Rightarrow w_2 = 1+4+7+8+12+13+15+17+21+22+24+26$

$= 170$

$$u_1 = w_1 - \frac{n_1(n_1+1)}{2}$$

$$= 208 - \frac{15(16)}{2} = 88$$

$$u_2 = w_2 - \frac{n_2(n_2+1)}{2}$$

$$= 170 - \frac{12(13)}{2} = 92$$

$$\Rightarrow u = \min(u_1, u_2) = \min(88, 92)$$

$$= 88$$

Thus $E[u] = \frac{n_1 n_2}{2} = \frac{15 * 12}{2} = 90$

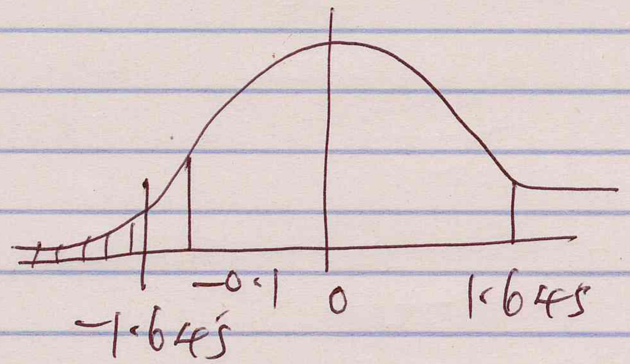
$$Var(u) = \frac{n_1 n_2 (n_1 + n_2 + 1)}{12}$$

$$= \frac{15 * 12 (15 + 12 + 1)}{12}$$

$$= 420$$

$$Z = \frac{U - E[U]}{\sqrt{Var(U)}} = \frac{88-90}{\sqrt{420}} = -0.098 \simeq -0.1$$

$$Z_\alpha = Z_{0.05} = 1.645, \quad H_0 : \mu = \mu_2$$
$$H_1 : \mu_1 \neq \mu_2 \Rightarrow \text{2 sided}$$



since $-1.645 < -0.1$ we do no reject the null hypothesis (-0.1 is not in the rejection region)

Exercises

1) The following are the amount of time in mins which took a random sample of 20 technicians to peform a task

18.1, 20.3, 18.3, 15.6, 22.5, 16.8, 17.6, 16.9, 18.2, 17.0, 19.3, 16.5, 19.5, 18.6, 20.0, 18.8, 19.1, 17.6, 18.5, 18.0

Use the sign test at $\alpha = 0.05$ to test the null hypothesis that the measurement constitute a random sample from a continous pdf with mean $\mu = 19.4$ min against the 2 sided alternative $\mu \neq 19.4$ mins. Use the Binomial test.

2) Rework problem 1 using the normal approximation to the Binomial. Use ~~[illegible]~~ sign-rank test also.