Rank sum test : The H-test (Kruskal-Wallis test)

- The H-test / Kruskal-Wallis test is a generalization of the rank-sum test to the case where we test the null hypothesis that K-samples come from identical continuous distributions.
- As in the U-test the data are ranked jointly from low to high as though they constitute one sample.
- Let R_i be the sum of the ranks of the values of the i^{th} sample.
- We base the test on the statistic

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} n_i \left(R_i/n_i - \frac{n+1}{2} \right)^2 \quad - (1)$$

Therefore the H-statistic is proportional to the weighted mean of the squared difference $\left(R_i/n_i - \frac{n+1}{2} \right)^2$ where R_i/n_i is the mean rank for sample i and $(n+1)/2$ is the mean rank of all the data. It follows that the null hypothesis must be rejected for large values of H.

- Expression 1 can be simplified as

$$\left(R_i/n_i - \frac{n+1}{2} \right)^2 = \frac{R_i^2}{n_i^2} - 2\left(R_i/n_i \right)\left(\frac{n+1}{2} \right) + \left(\frac{n+1}{2} \right)^2$$

$$= \frac{R_i^2}{n_i^2} - \frac{R_i}{n_i}(n+1) + \frac{(n+1)^2}{4}$$

$$= n_i \left(R_i/n_i - \frac{n+1}{2} \right)^2 = \frac{R_i^2}{n_i} - R_i(n+1) + \frac{n_i(n+1)^2}{4}$$

$$= \sum_{i=1}^{k} n_i \left(\frac{R_i}{n_i} - \frac{n+1}{2} \right)^2$$

$$= \sum_{i=1}^{k} R_i^2/n_i - (n+1) \sum_{i=1}^{k} R_i + \frac{(n+1)^2}{4} \sum_{i=1}^{k} n_i$$

But

$$\sum_{i=1}^{k} n_i = n$$

$$= \sum_{i=1}^{k} \frac{R_i^2}{n_i} - (n+1) \cdot \frac{n(n+1)}{2} + \frac{(n+1)^2}{4} n \quad \text{--- (*)}$$

$$= \sum_{i=1}^{k} R_i^2/n_i - \frac{n(n+1)^2}{4}$$

Substituting in (1) we find that

$$H = \frac{12}{n(n+1)} \left[\sum_{i=1}^{k} R_i^2/n_i - n\frac{(n+1)^2}{4} \right]$$

$$H = \frac{12}{n(n+1)} \sum_{i=1}^{k} R_i^2/n_i - 3(n+1) \quad \text{--- (2)}$$

- For every small values of k and n_i the test of the null hypothesis may be based on special tables but since the sampling distribution of H depends on the values of ~~R_i~~ n_i, it is impossible to tabulate it in compact form, hence the test is usually based on the large sample theory that the distribution of H can be approximately closely with a χ distribution with $k-1$ degrees of freedom.

Example

1) A sampling of the acidity of rain on 10 randomly selected rainfall was recorded at 3 different locations i.e, 1, 2, 3.

1	2	3
4.45	4.60	4.55
4.02	4.27	4.31
4.13	4.31	4.84
3.51	3.88	4.67
4.42	4.49	4.28
3.89	4.22	4.95
4.18	4.54	4.72
3.95	4.76	4.63
4.07	4.36	4.36
4.29	4.21	4.47

Use the Kruskal-Walis test at $\alpha = 0.05$ to test whether or not there is a difference in the average acidity of rain in the 3 locations.

Solution

$k = 3$

$n = n_1 + n_2 + n_3 = 10 + 10 + 10$

$\Rightarrow n = 30$

$\sum_{i=1}^{k} n_i = n$

For each location arrange the values in ascending order.

1	2	3
3.51	3.88	4.28
3.89	4.21	4.31
3.95	4.22	4.36
4.02	4.27	4.47
4.07	4.31	4.55
4.13	4.36	4.63
4.18	4.49	4.67
4.29	4.54	4.72
4.42	4.60	4.84
4.45	4.76	4.95

– Rank the values jointly, we get

3.51,	3.88,	3.89,	3.95,	4.02,	4.07,	4.13,	4.18
1	2	3	4	5	6	7	8
4.21,	4.22,	4.27,	4.28,	4.29,	4.31,	4.31,	4.36,
9	10	11	12	13	14	15	16
4.36,	4.42,	4.45,	4.47,	4.49,	4.54,	4.55,	4.60
17	18	19	20	21	22	23	24
4.63,	4.67,	4.72,	4.76,	4.84,	4.95		
25	26	27	28	29	30		

Note: 4.31, 4.31 (14, 15) $14 + 15 = 29/2 = 14.5$

4.36, 4.36 (16, 17) $16 + 17 = 33/2 = 16.5$

Ranks for location 1: $1 + 3 + 4 + 5 + 6 + 7 + 8 + 13 + 18 + 19$

II : $24 + 11 + 14.5 + 2 + 21 + 10 + 22 + 28 + 16.5 + 9$

III : $13 + 14.5 + 29 + 26 + 12 + 30 + 27 + 25 + 16.5 + 20$

$R_1 = 84,$
$R_2 = 158$
$R_3 = 223$

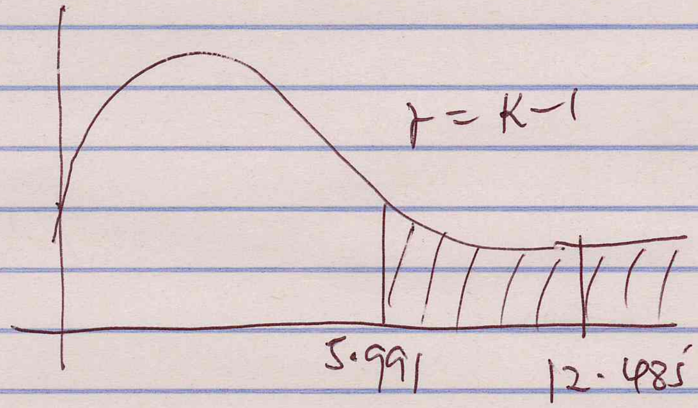
Therefore $H = \frac{12}{n(n+1)} \sum_{i=1}^{k} R_i^2 / n_i - 3(n+1)$

$$= \frac{12}{30(31)} \sum_{i=1}^{3} R^2/10 - 3(n+1)$$

$$= \frac{12}{30(31)} \left(\frac{84^2}{10} + \frac{158^2}{10} + \frac{223^2}{10} \right) - 3(31)$$

$$= \frac{12}{930} \left(\frac{84^2}{10} + \frac{158^2}{10} + \frac{223^2}{10} \right) - 93$$

$\Rightarrow H = 12.4835$



$n = 30, \; k = 3, \; r = 3-1 = 2$

$\Rightarrow \chi^2_{0.05}(r) = \chi^2_{0.05}(2) = 5.991$

$12.485 > 5.991$, we reject H_0.

Exercise

1) In an example to determine which of the 3 missles systems is preferable, the propellant burning rate is measured. The dat after coding are given in the table below. Use the Kruskal-Walis test at $\alpha = 0.05$ to test the hypothesis that the propellant burning rate are the same for the three missile systems.

Propellant burning rate

1	2	3
24.0	23.2	18.4
16.7	19.8	19.1
22.8	18.1	17.3
19.8	17.6	19.7
18.9	20.2	18.9
	17.8	18.8
		19.3

2) The following data represents the operating time in hours for 3 types of scientific pocket calculators before a recharge is required

Calculator

A	B	C
4.9	5.5	6.4
6.1	5.4	6.8
4.3	6.2	5.6
4.6	5.8	6.5
5.3	5.5	6.5
	5.2	6.3
	4.8	6.6

Use Kruskal-Wallis test at $\alpha = 0.01$ to test the hypothesis that the operating times for all 3 calculators are the same