B

The signed-Rank test (Wilcoxon signed-rank test)

- As we have seen the sign test is very easy to perform but since we utilize only the signs of the differences between the observations M_0 (mean) in one sample case, it tends to be wasteful of information.
- An alternative non-parametric test i.e the Wilcoxon signed rank test is less wasteful in that it takes into account the magnitude of the difference.
- In this test we rank the difference without regard to the signs assigning rank one to the smallest differences in magnitude in absolute value, rank two to the second smallest differences in magnitude absolute value and rank n to the largest difference in absolute value.
- Zero differences are ignored. If the absolute value of two or more differences are the same we assign each one the mean of the ranks, which they jointly occupy
- The signed rank test is based on T^+ the sum of the ranks assigned to the +ve difference, T^- the sum of ranks assigned to the −ve difference
- Therefore $T = \min(T^+, T^-)$
- We note that T^+ and T^- takes the values in the interval 0 to $\frac{n(n+1)}{2}$.
- T^- and T^+ are symmetric about $n(n+1)/4$.
Assignment: Find out how.
- The critical ~~values~~ regions for various tests is given below;

Alternative hypothesis	critical region - rejection
$m \neq m_0$	$T \leq T_\alpha$
$m > m_0$	$T^- \leq T_{2\alpha}$
$m < m_0$	$T^+ \leq T_{2\alpha}$

Example

1) The following are 15 measurements of the octane rating of a certain kind of gasoline

97.5, 95.2, 97.3, 96.0, 96.8, 100.3, 97.4, 97.4, 95.3 93.2, 99.1, 96.1, 97.6, 98.2, 98.5, 94.9

Find the signed-rank test at $\alpha = 0.05$ level of significance to determine whether or not the mean ~~of~~ octane rating of the given kind of gasoline is 98.5.

Solution

We wish to test $H_0: \mu = 98.5$ Vs $H_1: \mu \neq 98.5$. Therefore we reject H_0 if $T \leq T_{0.05}$ for the appropriate value of n. We subract 98.5 from each value and rank the absolute difference i.e

(In the next page)

$T^+ = 2 + 8 = 10$ (sum of ranks assigned to the +ve differences)

$T^- = 1 + 3 + 4 + 5 + 6 + 9 + 10 + 11 + 12 + 13 + 14 = 95$

$T = \min(T^+, T^-) = \min(10, 95)$

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Measurements	$x_i - \mu$	rank	
97.5	−1	4	
95.2	−3.3	12	
97.3	−1.2	6	
96.0	−2.5	10	
96.8	−1.7	7	
100.3	+1.8	8	+ difference
97.4	−1.1	5	
95.3	−3.2	11	
93.2	−5.3	14	
99.1	+0.6	2	+ difference
96.1	−2.4	9	
97.6	−0.9	3	
98.2	−0.3	1	
98.5	0.0	—	disregard - zero
94.9	−3.6	13	

Note: – Rank absolute value starting from the smallest difference. &
– Disregard zero difference

$n = 14$, $T_{0.05} = 21$ (from critical regions for Wilcoxon - sign rank test)

Since $10 < 21$, we reject H_0.

Note: $n = 14$, disgard one value (zero diff)
Rejection criterion is determined by the alternative hypothesis.

Sighed - Rank test for paired data

- We have the observations $(x_1, y_1), (x_2, y_2) \dots (x_n, y_n)$
- Let μ_1 be the mean of x's and μ_2 be the mean of y's.
- We want to test $H_0: \mu_1 = \mu_2$ vs $H_1: \mu_1 \neq \mu_2$ or $H_1: \mu_1 > \mu_2$ or $H_1: \mu_1 < \mu_2$. We use the table below.

Alternative hypothesis	Reject H_0 if
$\mu_1 \neq \mu_2$	$T \leq T\alpha$
$\mu_1 > \mu_2$	$T^- \leq T2\alpha$
$\mu_1 < \mu_2$	$T^+ \leq T2\alpha$

For $n \geq 15$, it is considered reasonable to assume the distribution of T^+ is approximately normal. To perform the sign-rank test based on this assumption we need the following results

Theorem: The mean and variance of T^+ are

$$E[T^+] = n(n+1)/4$$

$$Var(T^+) = \frac{n(n+1)(2n+1)}{24}$$

Proof: If the sample is of size n, the ranks to be assigned are $1, 2, 3, \dots, n$.

- For each rank, the probability it will be assigned a +ve or -ve difference are both $1/2$ when H_0 is true

$$Z_i = \begin{cases} 1 & \text{if } x_i > y_i \\ 0 & \text{if } x_i < y_i \end{cases} \quad \text{for } i = 1, 2 \dots n$$

Thus $T^+ = 1\cdot Z_1 + 2\cdot Z_2 + 3\cdot Z_3 + \dots + n\cdot Z_n$

$Z_1, Z_2 \dots Z_n$ are independent random variables having a Bernoulli distribution with $p = 1/2$ since $E[Z_i] = 1/2$ and $Var(Z_i) = 1/4$ for $i = 1, 2 \dots n$

Thus $E[T^+] = E[Z_i]$

$$= E[1\cdot Z_1 + 2\cdot Z_2 + \dots + n\cdot Z_n)$$

$$= 1\,E[Z_1] + 2\,E[Z_2) + \dots + n\,E[Z_n]$$

$$= 1\cdot 1/2 + 2\cdot 1/2 + \dots + n\cdot 1/2$$

$$= \frac{1 + 2 + \dots + n}{2} = \frac{n(n+1)}{4}$$

Normally $1 + 2 + \dots + n = \dfrac{n(n+1)}{2}$

$$\Rightarrow \frac{n(n+1)}{2}\cdot 1/2 = \frac{n(n+1)}{4}$$

Likewise

$$Var(T^+) = Var(1\cdot Z_1 + 2\cdot Z_2 + \dots + n\cdot Z_n)$$

$$= 1^2 Var(Z_1) + 2^2 Var(Z_2) + n^2 Var(Z_n)$$

$$= 1^2\cdot 1/4 + 2^2\cdot 1/4 + \dots + n^2\cdot 1/4$$

$$= \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{4}$$

$$1^2 + 2^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow \frac{n(n+1)(2n+1)}{6}\cdot 1/4$$

$$= \frac{n(n+1)(2n+1)}{24}$$

Example

The following are the weights in pounds before and after 16 persons who stayed on a certain weight ~~de~~ reducing diet for 4 weeks

Before	After	Before	After
147.0	137.9	166.8	158.5
183.5	178.2	131.9	130.4
232.1	219.0	150.3	149.3
161.6	163.8	197.2	189.1
197.5	193.5	159.8	159.1
206.3	207.4	171.7	173.2
177.0	180.6		
215.4	203.2		
147.7	149.0		
208.1	195.4		

Use the sign-rank test to test at $\alpha = 0.05$ level of significance whether the weight reducing baland diet is effective.

Solution

We are testing $H_0: \mu_1 = \mu_2$ Vs. $H_1: \mu_1 > \mu_2$

We reject H_0 if $Z_{calc} \geq Z_{0.05} = 1.645$

where $$Z = \frac{T^+ - E[T^+]}{\sqrt{Var(T^+)}}$$

The difference between the respective pairs are given on the next page

pair	diff	rank	
1	9.1	13	+
2	7.3	10	+
3	13.1	16	+
4	-2.2	5	
5	4.0	8	+
6	4.9	9	
7	-3.6	7	
8	12.2	14	+
9	-1.3	3	
10	12.7	15	+
11	8.3	12	+
12	-2.5	6	
13	1.0	2	+
14	8.1	11	+
15	0.7	1	+
16	-1.5	4	

NB: Ranks in absolute values

$n = 16$, $T^+ = 13 + 10 + 16 + 8 + 9 + 14 + 15 + 12 + 12 + 11 + 1)$

$= 111$

$$E(T^+) = \frac{n(n+1)}{4} = \frac{16(17)}{4} = 68$$

$$Var(T^+) = \frac{n(n+1)(2n+1)}{24} = \frac{16(17)(33)}{24}$$

$= 374$

$$Z = \frac{111 - 68}{\sqrt{374}} = 2.22$$

$Z_\alpha = Z_{0.05} = 1.645$

$2.22 > 1.645$, thus we reject H_0

Alternatively: Reject H_0 if $T^- \le T_{2\alpha}$

$T^- = 25 \ (5+7+3+6+4)$

$T_{0.10} = 35$

$T^- = 25 < T_{0.10} = 35$, thus reject H_0.