STATISTICS AND ACTURIAL SCIENCE

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DESIGN AND ANALYSIS OF EXPERIMENTS

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1. INTRODUCTION

In design of experiments we study the effects of treatments or assignable causes and the effects of Random errors or natural causes on the yields are obtained from plots or experimental units of a design. As an illustrations consider a study where the effects of fertilizers ,effects of soil fertility are to be studied on the effects of a crop yield like maize. The effects of assignable causes since these are assigned by the experimenter.Random errors or chances are those which cannot be manipulated.

Suppose we have v treatments and B blocks.In Randomized block design each block will have v plots.The v treatments will be assigned to the plots in a randomized way such that each plot will have the same probability of getting any treatments

Also each block is said to be homogeneous in fertility.

Lets denote byY_{ij} the yields of the plot which received treatment i and is in the plot j. Then we can model the yield is

$$y_{ij} = u + t_i + b_j + e_{ij}$$

$$1 = 1, 2, ...v$$

$$j - 1.......b$$

u = Constant common to all plots
t_i = Effect of treatment i
b_j = Effect of block j
e_{ij}= Error term where
$E(e_{ij}) = 0$ and $V(e_{ij}) = {s_e}^2$
The total variation of the yield is defined as

$$s_t^2 = \sum_{i=1}^{v}\sum_{s=1}^{b}(y_{ij} - \bar{y})^2$$

and called (Total sum of squares)

This total variation can be partitioned into that which is from assignable causes and that from random causes. We use ANOVA technique to separate and test for the causes which have got effects.

We first get the least square estimate of the parameters u, t_i and b_j. The least square estimates are one of the parameters that are obtained by minimizing the sum of the squares of the error term

$$s^2 = \sum_{i}^{v}\sum_{j}^{b} e_{ij}^2 = \sum_{i}^{v}\sum_{j}^{b}(y_{ij} - u - t_i - b_j)^2$$

For u, with respect to u minimize s^2 subject to $\sum t_i = \sum b_j = 0$ given

$$\frac{ds^2}{du} = -2\sum_i^v\sum_j^b (y_{ij} - u - t_i - b_j) = 0$$

$$= \sum_i^v\sum_j^b (y_{ij} - u - t_i - b_j) = 0$$

$$= \sum_i^v\sum_j^b u = \sum\sum y_{ij} - \sum\sum t_i - \sum\sum b_j$$

$$V_b\hat{u} = \sum\sum y_{ij}$$

$$\hat{u} = \frac{\sum\sum y_{ij}}{Vb} y_{ij} = \bar{y}$$

For t_i

$$\frac{ds^2}{dt_i} = -2\sum_{j=1}^b (y_{ij} - u - t_i - b_j) = 0$$

$$= \sum_{j=1}^s t_i = \sum y_{ij} - b\hat{u} - \sum b_j$$

or

$$\hat{t_i} = \frac{\sum y_{ij}}{b} - \hat{u} = \bar{y_i} - \bar{y}$$

For b_j

$$\frac{ds^2}{db_j} = -2\sum (y_{ij} - u - t_i - b_j) = 0$$

$$= \sum b_j = \sum y_{ij} - \sum u_i$$

$$\hat{b_j^w} = \frac{\sum y_{ij}}{V} - \hat{u} = \bar{y_j} - \hat{u}$$

The total sum of squares

$$S_t^2 = \sum_i^v\sum_j^b (y_{ij} - \bar{y})^2$$

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$$= \sum_i \sum_j (\bar{y}_i - \bar{y} + \bar{y}_j - \bar{y} + y_{ij} - \bar{y}_i - \bar{y}_j + \bar{y})^2$$

$$= \sum_i \sum_j (\bar{y}_i - \bar{y})^2 + \sum_i \sum_j (\bar{y}_j - \bar{y})^2 + \sum \sum (y_{ij} - \bar{y}_i - \bar{y}_j + y)^2$$

$$= b \sum_{i=} (\bar{y}_i - \bar{y})^2 + V \sum_{j=} (\bar{y}_j - \bar{y})^2 + \sum \sum (y_{ij} - \bar{y}_i - \bar{y}_j + \bar{y})^2$$

$$= s_t^2 + s_b^2 + s_e^2$$

where

$$s_t^2 = b \sum (\bar{y}_i - \bar{y})^2$$

- Sum of squares due to treatments with $v-1$ df

$$s_b^2 = v \sum_i^v (\bar{y}_j - \bar{y})^2$$

-ss due to blocks with $b-1$ df

$$s_e^2 = \sum \sum (y_{ij} - \bar{y}_i - \bar{y}_j + \bar{y})^2$$

-ss due to error with $(vb-1) - (v-1) - (b-1) = (v-1)(b-1)$ df

The mean ss due to a source is the ss divided by the degrees of freedom.

$$MS_t^2 = \frac{S_t^2}{v-1}$$

$$MS_b^2 = \frac{S_b^2}{b-1}$$

$$MS_e^2 = \frac{S_e^2}{(v-1)(b-1)}$$

1.1. Expectations of sum of squares

Fixed effects design with i treatment and j blocks : t_i is a constant , b_j is a constant

From;

$$y_{ij} = M + b_j + t_i + e_{ij}$$

We get any yield due to treatment i

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$$\bar{y_i} = \frac{\sum_j^b y_{ij}}{b} = m + t_i + \bar{e_i}$$

$$\bar{y_j} = \frac{\sum_i^v y_{ij}}{v} = m + b_j + \bar{e_j}$$

$$\bar{y} = \frac{\sum\sum y_{ij}}{vb} = m + \bar{e}$$

$$E(S_t^2) = E[b\sum(\bar{y_i} - \bar{y})^2] = Eb(\sum_i^v [t_i + \bar{e_i} - \bar{e}]^2)$$

$$= b\sum_i^v E(t_i + \bar{e_i} - \bar{e})^2 E(t_i + \bar{e_i} - \bar{e})^2 = E(t_i^2 + \bar{e_i}^2 + \bar{e} + 2t_i\bar{e_i} - 2t_i\bar{e} - 2\bar{e}\cdot\bar{e})$$

$$\Rightarrow E(t_i^2) = t_i^2, E(t_i\bar{e_i}) = 0, E(t_i\bar{e}) = 0$$

$$E(\bar{e_i}^2) = E(\frac{\sum_j^b e_{ij}}{b})^2 = E[\sum_{j=1}^b \frac{e_{ij}^2}{b^2} + \sum_{j\neq j}\sum_1 \frac{e_{ij}e_{ij}^{'}}{b^2}]$$

$$= \frac{\sum_j^b \sigma e^2}{b^2} = \frac{b\sigma e^2}{b^2} = \frac{\sigma e^2}{b}$$

Also

$$E(\bar{e^2}) = E(\frac{\sum\sum e_{ij}}{vb})^2 = E[\frac{\sum\sum e_{ij}^2}{(vb)^2} + \frac{\sum_i\sum e_{ij}e_i^{'}j^{'}}{vb}]$$

$i \neq i^{'}$ or $j \neq j^{'}$

$$= \frac{\sum_i^v\sum_j^b \sigma e}{(vb)^2} = \frac{\sigma e^2}{vb}$$

Also

$$E(\bar{e_i}\cdot\bar{e}) = E[\sum_j^b \frac{e_{ij}}{b}\cdot\frac{\sum\sum e_{ij}}{vb}]$$

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$$= E[\frac{\sum_{j=1}^{b} e_{ij}^2}{vb^2} + \sum_i \sum_{j \neq j'} \frac{e_{ij}e_i^{'}j^{'}}{vb}]$$

$$= \frac{b\sigma e^2}{vb^2} = \frac{\sigma e^2}{vb}$$

$$\Rightarrow E(s_t) = b\sum_i^v (t_i^2 + \frac{\sigma e^2}{b} + \frac{\sigma e^2}{vb} - \frac{2\sigma e^2}{vb})$$

$$= b\sum_{i=1}^v (t_i^2 + \frac{\sigma e^2}{b} - \frac{\sigma e^2}{vb})$$

$$= b\sum t_i^2 + \frac{vb\sigma e^2}{b} - \frac{vb\sigma e^2}{vb}$$

$$= v\sigma e^2 - \sigma e^2 + b\sum_{i=1}^t t_i^2$$

$$= (v-1)\sigma e^2 + b\sum_{i=1}^v t_i^2$$

This implies :-

$$E(ms_t^2) = E(\frac{s_t^2}{v-1}) = \sigma e^2 + \frac{b\sum t_i^2}{v-i}$$

Similarly;

$$E(s_b^2) = (b-1)\sigma e^2 + v\sum_{j=1}^b b_j^2$$

and

$$E(ms_b^2) = \sigma e^2 + \frac{v\sum_{j=1}^b b_j^2}{b-1}$$

$$E(s_e^2) = [\sum\sum(y_{ij} - \bar{y_i} - \bar{y_j} + \bar{y})^2]$$

$$= E[\sum_i^v \sum_j^b (e_{ij} - \bar{e}_i - \bar{e}_j + \bar{e})^2]$$

$$= \sum_{i}^{v}\sum_{j}^{b}(\sigma e^2 - \frac{\sigma e^2}{b} - \frac{\sigma e^2}{v} + \frac{\sigma e^2}{vb})$$

$$= vb(\sigma e^2 - \frac{\sigma e^2}{b} - \frac{\sigma e^2}{v} + \frac{\sigma e^2}{vb})$$

$$= (vb - v - b + 1)\sigma e^2 = (v-1)(b-1)\sigma e^2$$

$$E(ms_e^2) = E(\frac{s_e^2}{(v-1)(b-1)}) = \sigma e^2$$

1.2. Testing hypothesis

$$H_0 : t_1 = t_2 = ... = t_v = 0,$$

If H_0 is true , then

$$E(ms_t^2) = E(ms_e^2) = \sigma e^2$$

This implies that if H_0 is true , then the statistic:-

$$F_1 = \frac{ms_t^2}{ms_e^2} \approx 1$$

Large values of F_1 would imply that some $t_i's$ due not zero leading to resection of H_0.

F_1 has a fisher F distribution with

$$[v-1, (v-1)(b-1)]df$$

It's resected if

$$F_1 > F_{1-\alpha}[v-1, (v-1)(b-1)]$$

Consider the hypothesis:-

$$H_0 : b_1 = b_2 = ... = b_b = 0$$

If H_0 is true ,then

$$E(ms_b^2) = E(ms_e^2) = \sigma e^2$$

and

$$F_2 = \frac{ms_b^2}{ms_e^2} \approx 1$$

H_0 is resected if

$$F_2 > F_{1-\alpha}[b-1, (v-1)(b-1)]$$

1.3. ANOVA

Source	d.f	ss	mss	F
Treats	$v-1$	s_t^2	T	$F_1 = \frac{T}{E}$
Blocks	$b-1$	s_b^2	B	$F_2 = \frac{B}{E}$
Error	$(v-1)(b-1)$	s_e^2	E	
Total	$vb-1$	s_T^2		

Note that:

$$s_t^2 = b\sum_i^v \bar{y}_i^2 - vb\bar{y}^2$$

$$= b\sum_i^v\left(\frac{\sum_j^b y_{ij}}{b}\right)^2 - vb\left(\frac{\sum\sum y_{ij}}{vb}\right)^2$$

$$= \sum_i^v \frac{(\sum_j^b y_{ij})}{b} - \frac{G^2}{vb}$$

$$= \sum_{i=1}^v \frac{T_i^2}{b} - \frac{G^2}{vb}$$

where:-
$T_i$is total observations with treat i
G is the grand total
Similarly;

$$s_b^2 = \sum_{j=1}^b \frac{B_j^2}{v} - \frac{G^2}{vb}$$

where
B_j is the total observation in block j
and
$s_e^2 = s_T^2 - s_t^2 - s_b^2$

Exercise 1. Construct the ANOVA for the experiment with the layout below

	t_1	t_2	t_3	t_4	
B_1	5	10	15	12	42
B_2	3	6	19	14	42
B_3	2	14	16	18	50
	10	30	50	44	134

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2. Extended model with interractive effects

sometimes we want to study whether there is interraction effects between the treatments and block files.Assuming there are v treatments and b blocks , we make n observations from each plot then kthobservations in each plot i of block j can be denoted by

$$y_{ijk} = \mu + t_i + b_j + \theta_{ij} + e_{ijk}$$

where μ-costant common to all terms

θ_{ijk}-measre effects of interraction of treatment i and block j

b_j-effect of block j

t$_i$-effect of treatment i

$$\theta_{ijk} \sim N(0, \sigma_e^2)$$

i=1,2...v and j=1,2,3,4...b and k=1,2,3,4....n

The estimates of the parameters μ,t_i, b_j,$\theta_{ijk}$are obtained from minimising

$$s^2 = \sum_i^v \sum_j^b \sum_k^n (y_{ijk} - \mu - t_i - b_j - \theta_{ij})^2$$

subject to:

$$\hat{\mu} = \bar{y} = \frac{\sum_i^v \sum_j^b \sum_k^n}{vbn}$$

$$\hat{t} = \bar{y}_i - \bar{y} = \frac{\sum_j^b \sum_k^n}{bn} - \bar{y}$$

$$\hat{\mu} = \bar{y}_j - \bar{y} = \frac{\sum_i^v \sum_k^n}{vn} - \bar{y}$$

$$\theta_{ij} = \bar{y}_{ij} - y_i - y_j + \bar{y}$$

sum of squares are :

$$S_T^2 = \sum \sum \sum (y_{ijk} - \bar{y})^2$$

$$= \sum \sum \sum (\bar{y}_i - \bar{y} + \bar{y}_j - \bar{y} + \bar{y}_{ij} - \bar{y}_i - \bar{y}_j + \bar{y} + \bar{y}_{ijk} - \bar{y}_{ij})^2$$

$$= \sum \sum \sum (\bar{y}_i - \bar{y})^2 + \sum \sum \sum (\bar{y}_j - \bar{y})^2$$

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$$+\sum\sum\sum(\bar{y}_{ij}-\bar{y}_i-\bar{y}_j+\bar{y})^2+\sum\sum\sum(\bar{y}_{ijk}-\bar{y}_{ij})^2$$

$$=bn\sum_i^v(\bar{y}_i-\bar{y})^2+vn\sum_j^b(\bar{y}_j-\bar{y})^2+n\sum_i^n\sum_j^b(\bar{y}_{ij}-\bar{y}_i-\bar{y}_j+\bar{y})^2+\sum\sum\sum(y_{ijk}-\bar{y}_{ij})^2$$

$$=S_t^2+S_b^2++S_\theta^2+S_e^2$$

where S_θ^2 is sum of squares due to interaction

2.1. Expectation of sum of squares

From

$$y_{ijk}=\mu+t_i+b_j+\theta_{ij}+e_{ijk}$$

we get:-

$$\bar{y}_{ij}=\frac{\sum_k Y_{ijk}}{n}=\mu+t_i+b_j+\theta_{ij}+\bar{e}_{ij}$$

$$\bar{y}_i=\frac{\sum_j^b\sum_k^n y_{ijk}}{bn}=\mu+t_i+\bar{e}_i$$

$$\bar{y}_j=\frac{\sum_i^v\sum_k^n y_{ijk}}{vn}=\mu+b_j+\bar{e}_j$$

$$\bar{y}=\mu+\bar{e}$$

$$E(S_t^2)=E(bn\sum(\bar{y}_i-\bar{y})^2)=bn\sum_i^v E(t_i+\bar{e}_i-\bar{e})^2=bn\sum_i^v(t_i^2+\frac{\sigma e^2}{bn}-\frac{\sigma e^2}{vbn})$$

$$vbn(\frac{\sigma e^2}{bn}-\frac{\sigma e^2}{vbn})+bn\sum_i t_i^2=(v-1)\sigma e^2+bn\sum t_i^2$$

$$E(MS_t^2)=\sigma e^2+\frac{bn\sum_i^v t_i^2}{v-1}$$

Similarly

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$$E(S_b^2) = (b-1)\sigma e^2 + vn\sum_j^b b_j^2$$

$$E(MS_b^2) = \sigma e^2 + \frac{vn\sum_{bj}^2}{b-1}$$

$$E(S_\theta^2) = E[n\sum_i^v\sum_j^b(\bar{y}_{ij} - \bar{y}_i - \bar{y}_j + \bar{y})^2]$$

$$= n\sum_i^v\sum_j^b E(\theta_{ij} - \bar{e}_{ij} - \bar{e}_i - \bar{e}_j + \bar{e})^2$$

$$n\sum_i^v\sum_j^b(\theta_{ij}^2 + \frac{\sigma_e^2}{n} - \frac{\sigma_e^2}{bn} - \frac{\sigma_e^2}{vn} + \frac{\sigma_e^2}{vbn})$$

$$(vb - v - b + 1)\sigma_e^2 + n\sum_i^v\sum_j^b\theta_{ij}^2$$

$$E(mS_\theta^2) = \sigma_e^2 + \frac{n\sum\sum\theta_{ij}^2}{(v-1)(b-1)}$$

$$E(S_e^2) = E[\sum\sum\sum(y_{ijk} - \bar{y_{ij}})^2]$$

$$= \sum\sum\sum E(e_{ij} - \bar{e}_{ij})^2$$

$$= \sum\sum\sum(\sigma_e^2 - \frac{\sigma_e^2}{n})$$

$$= vbn\sigma_e^2 - vb\sigma_e^2$$

$$= vb(n-1)\sigma_e^2$$

$$E(ms_e^2) = \sigma_e^2$$

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2.2. Testing hypothesis

1.
$$H_0 : t_1 = t_2 = ... = t_v = 0$$

2.
$$H_0 : b_1 = b_2 = ... = b_j = 0$$

3.
$$H_0 : \theta_{ij} = 0,$$

$$\forall i, j$$

4.

source	df	ss	mss	F
Treat	v-1	S_t^2	T	$F_1 = \frac{T}{E}$
block	b-1	S_b^2	B	$F_2 = \frac{B}{E}$
interraction	(v-1)(b-1)	S_θ^2	θ	$F_3 = \frac{\Theta}{E}$
Error	vb(n-1)	S_e^2	E	
Total	vbn-1	S_T^2		

3. ESTIMABILITY AND BEST ESTIMATES OF LINEAR FUNCTIONS OF UNKNOWN PARAMETERS

Assume we can observe independent variables$y_1, y_2, y_3.....................y_n, \quad var(y_i) = \sigma^2 \quad \forall_i$ and are independent of each other hence their covariance is zero.

$$E(y_1) = a_{11}P_1 + a_{21}P_2 + ... + a_{m1}P_m$$

$$E(y_2) = a_{12}P_1 + a_{22}P_2 + ... + a_{m2}P_m$$

$$E(y_n) = a_{1n}P_n + a_{2n}P_n + ... + a_{mn}P_m$$

where $P_1, P_2.......P_m$ are unkown parameters $a_{ij's}$ are known constants. In matrix notation we get:

$$E(\mathbf{y}) = A^{'}\mathbf{P}$$

$$Var(\mathbf{y}) = I_n\sigma^2$$

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And therefore we can write the model as:

$$\mathbf{Y} = A^{'}\mathbf{P} + \mathbf{e}$$

or $y_i = a_{1i} + P_1 + a_{2i}P_2 + ... + a_{m1}P_m + e_i$ where $E(e_i) = 0 \quad var(e_i) = \sigma^2 \quad cov(e_ie_j) = 0$ also assume

$$Rank(A) = n_0 \leq m < n$$

consider the linear function:

$$\mathbf{L}^{'}\mathbf{P} = L_1P_1 + L_2P_2 + ... + L_mP_m$$

we are interested in establishing the condition underwhich $L^{'}P$ is estimable . If $L^{'}P$ is estimable then there must exist a linear function of observed values y_i say:

$$\mathbf{C'y} = c_1y_1 + c_2y_2 + ... + c_ny_n$$

such that $E(\mathbf{Cy}) = \mathbf{l}\,\mathbf{P} \qquad \forall \;\; P$,We say that $C^{'}Y$ is unbiased estimator of $L^{'}P$. Therefore

$E(\mathbf{Cy}) = \mathbf{l}\,\mathbf{P}$ implies that $c^{'}E(\mathbf{y}) = \mathbf{L}^{'}\mathbf{P}$ or $\mathbf{C}A^{'}\mathbf{P} = \mathbf{L}^{'}\mathbf{P}$,But $\mathbf{L} = AC$ implies that $\mathbf{L}$ is a linear combination of the rows of A.Hence we see that

$$Rank(A, L) = Rank(A)$$

- **Theorem**

The necessary and sufficient condition for the linear function **LP** of the unknown parameters p to be linearly estimable is that $Rank(A, L) = Rank(A)$

3.1. Estimation and error spaces

consider the model:

$$\mathbf{Y} = A^{'}\mathbf{P} + \epsilon$$

where

$$E(\mathbf{Y}) = A^{'}\mathbf{P}$$

$A_{m\times n}$ is a $m \times n$ matrix of rank n_o. This implies that there are n_o linearly independent vectors in A which generate the other vectors in A. The vector space generated by these n_o vectors is called the estimation space. It is denoted by $V(A)$. Since we had n observations we had expected a square matrix of n observation $n \times n$.

Consider $n - n_o = n_e$ other vectors which are all orthogonal to other n_o vectors.The matrix formed by these n_e vectors is denoted by E and the vector space generated by the $V(E)$.This vector space is called the error space .A linear function say $g^{'}(y)$ is said to belong to the error space if $E(\mathbf{g}\,\mathbf{y}) = 0$. This implies that $g^{'}A^{'}\mathbf{P} = 0$ or $A\mathbf{g} = 0$, $EA^{'}$, $AE^{'} = 0$.

- **Theorem**

If $\mathbf{l'P}$ is estimable then there exist a unique linear function $\mathbf{C}\mathbf{y}$such that

1. $E(\mathbf{CY}) = \underset{\sim}{L}'\underset{\sim}{Y} \qquad \forall \underset{\sim}{P}$
2. $V(\mathbf{CY})$ is at least almost unbiased estimator of $\mathbf{L'P}$

Proof

1. Let $\mathbf{L'P}$ be estimable and let $\mathbf{d'Y}$be unbiased estimator of $\mathbf{L'P}$. Let $\mathbf{d} = \mathbf{c} + \mathbf{e}$where $\mathbf{c}\ \epsilon V(A)$ and $\mathbf{e}\ \epsilon V(E)$ .Then

$$\mathbf{d'Y} = \mathbf{C'Y} + \mathbf{e'Y}$$

$$E(\mathbf{d'Y}) = E(\mathbf{C'Y}) + 0 = \mathbf{L'Y}$$

This implies that if $\mathbf{L'Y}$ is estimable then there musst exist a linear function$\mathbf{cY}$ belonging to $V(A)$ which is unbiased estimator of $\mathbf{L'Y}$.Suppose it is not unique,let c_1 be another vector in $V(A)$ such that $E(\mathbf{c_1\ Y}) = \mathbf{L'P}$

Let

$$c_0 = c - c_1$$

then

$$E(c_0^{'}\mathbf{Y}) = E(c^{'}\mathbf{Y}) - E(c_1^{'}\mathbf{Y})$$

$$= \mathbf{L'P} - \mathbf{L'P} = 0 \qquad \forall\ P$$

1. $$\Rightarrow E(c_0^{'}\mathbf{Y}) = 0 \qquad \forall P$$

   But this implies that $c_0^{'}\epsilon V(E)$ and also $c_0^{'}\epsilon\ V(A)$ .The only vector belonging to both $V(A)$ and $V(E)$ is$\mathbf{0}$. Hence $\mathbf{c} - \mathbf{c_0} = 0 \Rightarrow \mathbf{c} = \mathbf{c_1}$and $\mathbf{c}$ is unique.

2. Since $\mathbf{dy}$is an unbiased estimator of $\mathbf{L'P}$ then

$$Var(\mathbf{dy}) = \mathbf{d}^{'}Var(\mathbf{y})\mathbf{d} = \mathbf{d}^{'}\mathbf{d}\sigma$$

$$= (\mathbf{c}^{'} + \mathbf{e}^{'})(\mathbf{c} + \mathbf{e})\sigma^2$$

$$= \mathbf{cc}^{'}\sigma^2 + (\mathbf{c}^{'}\mathbf{e})\sigma^2$$

$$= Var(\mathbf{c}^{'}\mathbf{y}) + Var(\mathbf{e}^{'}\mathbf{y}) \geq Var(\mathbf{c}^{'}\mathbf{y})$$

equality holds only if $\mathbf{e} = 0$

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3.2. Least Squares Estimates of P

From the model

$$\mathbf{Y} = A^{'}P + e$$

the least squares estimates of **P** is obtained by minimizing

$$S^2 = \mathbf{ee}^{'} = \sum_{i=1}^{n} e_i^2 = (Y - A^{'}P)^{'}(\mathbf{Y} - A^{'}\mathbf{P})$$

with respect to **P** .This gives

$$\frac{\delta S^2}{\delta \mathbf{P}} = \frac{\delta}{\delta P}(Y^{'}Y - 2\mathbf{P}^{'}AY + \mathbf{P}^{'}AA^{'}\mathbf{P}$$

$$= 2AY + 2AA^{'}\mathbf{P}$$

This implies the normal equations for **P** are $AA^{'}P = A\mathbf{Y}$Usually $AA^{'}P$ would not be of full rank and hence the solution for $\underset{\sim}{P}$ would not be unique.The solution depends on the additional equation broughtinto the model.

3.3. Solutions of Systems of equations

Consider the systems of linear equations,

$$A\mathbf{x} = \mathbf{Y}$$

The systems is consistent if it has solutions otherwise it is inconsistent.If Ahas an inverse, then

$$\mathbf{x} = A^{-1}\mathbf{Y}$$

sometimes Amay not have an inverse but the system is consistent.

3.4. Conditional Inverse of a Matrix

Let $A_{m\times n}$ be an $m \times n$ matrix, then the conditional or generalised inverse of A is the matrix is $A^*_{m\times n}$such that:

$$A^*AA^* = A^*$$

and

$$AA^*A = A$$

If A is non singular then

$$A^* = A^{-1}$$

$$AA^*A = A$$

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then it follows that

$$A^{-1}AA^*AA^{-1} = A^{-1}AA^{-1}$$

then

$$A^* = A^{-1}$$

3.5. Theorem

If $A^*_{n\times m}$ the conditional inverse of $A_{m\times n}$ and if the equations $A\mathbf{x} = \mathbf{y}$ are consistent then:

1. $\mathbf{x_1} = A^*\mathbf{y}$ is a solution of $A\mathbf{x} = \mathbf{y}$
2. If $\mathbf{x_1} = A^*\mathbf{Y}$is a solution of $A\mathbf{x} = \mathbf{Y}$, then $A^*$is a conditional inverse of A

Proof

1. Since $A\mathbf{x} = \mathbf{y}$ is a consistent systems of equations, there exists asolution say $\mathbf{k}$such that $A\mathbf{k} = \mathbf{y}$.

But

$$A\mathbf{x} = AA^*\mathbf{y} = AA^*Ak$$

this implies that

$$A\mathbf{x_1} = AA^*A\mathbf{k} = A\mathbf{k}$$

1. $$\Rightarrow \mathbf{x_1} = \mathbf{k}$$

   hence $\mathbf{x_1}$is a solution of $A\mathbf{x} = \mathbf{y}$

2. Since $\mathbf{x_1} = A^*\mathbf{Y}$is a solution of $A\mathbf{x} = \mathbf{y}$ then

   $$A\mathbf{x_1} = \mathbf{y}$$

   $$A\mathbf{x_1} = AA^*\mathbf{y} = AA^*A\mathbf{x_1}$$

   $$\Rightarrow A = AA^*A$$

   This implies that $A^*$is a condition inverse of A

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3.6. Getting a condition inverse of A matrix

Reduce the augmented matrix (A/I) to canonical form to get (I/A^*)

Example

consider the matrix

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 1 & 4 \end{pmatrix}$$

reducing to the canonical form :

$$(A/I) = \begin{array}{c} a \\ b \\ c \end{array} \left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$= \begin{array}{c} a \\ b \\ c \end{array} \left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & -1 & -1 & -3 & 0 & 2 \end{array}\right)$$

$$b + c$$

$$\left(\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & -4 & 2 & 2 \end{array}\right)$$

$$a - b \quad , b + c$$

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 2 & 2 & -2 & 0 \\ 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -4 & 2 & 2 \end{array}\right)$$

$$\frac{a}{2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & -4 & 2 & 2 \end{array}\right)$$

$$\Rightarrow A^* = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now

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$$AA^*A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ - & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 1 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 3 & 1 & 4 \end{pmatrix} = A$$

from

$$AA^{'}\underset{\sim}{P} = A\mathbf{y}$$

$$\mathbf{P} = (AA^{'})^* A\mathbf{Y}$$

$$var(\mathbf{l}'\mathbf{P}) = \mathbf{l}var(\hat{p})\mathbf{l}'$$

$$= \mathbf{l}'var((AA^{'})^* A\mathbf{y})$$

$$= \mathbf{l}'(AA^{'})^* Avar(\mathbf{Y})A^{'}(AA^{'})^* l$$

$$= \mathbf{l}'(AA^{'})^* AA^{'}(AA^{'})^* l\sigma^2$$

$$= \mathbf{l}'(AA^{'})l\sigma^2$$

3.7. sum of squares due to a linear function $\mathbf{c}^{'}\mathbf{Y}$

Let $\mathbf{c}'\mathbf{Y} = c_1Y_1 + c_2Y_2 + \cdots + c_nY_n$ be a linear function. The sum of squares due to this linear function is:

$$S^2(\mathbf{c}'\mathbf{Y}) = \frac{(\mathbf{c}'\mathbf{Y})^2}{\mathbf{cc}}$$

The expectation of $S^2(\mathbf{c}'\mathbf{Y})$ is

$$E[S^2(\mathbf{c}'\mathbf{Y})] =$$

$$\frac{E(\mathbf{c}'\mathbf{c})^2}{\mathbf{c}'\mathbf{c}} = \frac{v(\mathbf{c}'\mathbf{Y})}{\mathbf{c}'\mathbf{c}} + \frac{[E(\mathbf{c}'\mathbf{Y})]^2}{\mathbf{c}'\mathbf{c}}$$

$$= \frac{\mathbf{c}'\mathbf{c}\sigma^2}{\mathbf{c}'\mathbf{c}} + \frac{E(\mathbf{c}'\mathbf{Y})}{\mathbf{c}'\mathbf{c}}$$

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$$= \sigma^2 + \frac{E(\mathbf{c}'\mathbf{Y})}{\mathbf{c}'\mathbf{c}}$$

This implies that if $\mathbf{c}\epsilon v(E)$, $\Rightarrow E(\mathbf{c}'\mathbf{Y}) = 0$ then $S^2(\mathbf{c}'\mathbf{Y})$ will be unbiased estimator for σ^2

4. Treatment contrast

Consider v treatment t_1 , t_1 $\cdots$, t_v .Let $\mathbf{l}'\mathbf{t}$be a linear function;

$$\mathbf{l}'\,\mathbf{t} = l_1t_1 + l_2t_2 + l_3t_3 + \ \ ...... + l_vt_v$$

then $\mathbf{l}'\mathbf{t}$ is said to be a treatment contrast if

$$\sum_{i=1}^{v} l_i = 0$$

Two treatment contrast say

$$k_1 = l_{11}t_1 + l_{12}t_2 + ... + l_{1v}t_v$$

$$k_2 = l_{21}t_1 + l_{22}t_2 + ... + l_{2v}t_v$$

are said to be orthogonal if

$$\sum_{i=1}^{v} l_{1i}l_{2i} = 0$$

we shall see later that only only treatment contrast are estimable.

5. Analysis of generalised block design

Consider a design with v treatments and b blocks. Block j has kj plots. Treat i occurs $r_i$times ;

$$\Rightarrow \sum_{i=1}^{v} r_i = \sum_{i=1}^{v} r_i = n$$

where n is the total number of plots . The observation of plot u , $u = 1, 2, 3, 4....., n$ can be modelled as ;

$$y_u = g + a_{1u}t_1 + a_{2u}t_2 + \cdots\cdots + a_{vu}t_v + l_{1u}b_1 + l_{2u}b_2 + \cdots\cdots + l_{bu}b_b + e_u$$

where

$$a_{iu} = \begin{cases} 1 & if \quad treatment\ i\ is\ in\ plot\ u \\ 0 & otherwise \end{cases}$$

$$l_{ju} = \begin{cases} 1 & if \quad plot\ u\ is\ in\ block\ j \\ 0 & otherwise \end{cases}$$

g is a constant common to all plots
t_i is effect of treatment i
b_j is effect of block j
$e_u \sim\ N(0, \sigma_e^2)$,
the n observations can be written as :

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_u \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ \vdots \\ 1 \end{pmatrix} g + \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{v1} \\ a_{12} & a_{22} & \cdots & a_{v2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1u} & a_{2u} & \cdots & a_{vu} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{vn} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ \vdots \\ \vdots \\ \vdots \\ t_v \end{pmatrix}$$

$$+ \begin{pmatrix} l_{11} & l_{21} & \cdots & l_{b1} \\ l_{12} & l_{22} & \cdots & l_{b2} \\ \vdots & \vdots & \vdots & \vdots \\ l_{1u} & l_{2u} & \cdots & l_{bu} \\ \vdots & \vdots & \vdots & \vdots \\ l_{1n} & l_{2n} & \cdots & l_{bn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ \vdots \\ b_b \end{pmatrix} + \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ \vdots \\ \vdots \\ e_n \end{pmatrix}$$

$$= \mathbf{y} = \mathbf{j}\, g + A^{'}\mathbf{t} + \mathbf{l}'\mathbf{b} + \mathbf{e}$$

$$= \begin{pmatrix} J & A^{'} & L^{'} \end{pmatrix} \begin{pmatrix} g \\ \mathbf{t} \\ \mathbf{b} \end{pmatrix} + \mathbf{e}$$

$$= \mathbf{A}^{'}\mathbf{P} + \mathbf{e}$$

where $\mathbf{J} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$, $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{v1} & a_{v2} & \cdots & a_{vn} \end{pmatrix}$, $L = \begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ l_{b1} & l_{b2} & \cdots & l_{bn} \end{pmatrix}$, $\mathbf{e} = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{pmatrix}$, $\mathbf{t}^{'} = \begin{pmatrix} t_1 & t_2 & \cdots & t_v \end{pmatrix}$, $\mathbf{b}' = \begin{pmatrix} b_1 & b_2 & \cdots & b_b \end{pmatrix}$, $A^* = \begin{pmatrix} J^{'} \\ A \\ L \end{pmatrix}_{(1+v+b)\times n}$, $\mathbf{P} = \begin{pmatrix} g \\ \underset{\sim}{t} \\ \underset{\sim}{b} \end{pmatrix}$,

$$\Rightarrow E(\mathbf{y}) = A^{'}\mathbf{P} \forall\ \mathbf{P}$$

$$v(\mathbf{y}) = I_n\sigma^2$$

5.1. Normal Equations for estimating P

$$AA' = A\mathbf{y}$$

$$AA' = \begin{pmatrix} J' \\ A \\ L \end{pmatrix} \begin{pmatrix} J & A' & L' \end{pmatrix}$$

$$= \begin{pmatrix} JJ' & J'A' & J'L' \\ AJ & AA' & AA' \\ LJ & LA' & LL' \end{pmatrix}$$

Now

$$J'J = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = n$$

$$J'A' = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{v1} \\ a_{12} & a_{22} & \cdots & a_{v2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{vn} \end{pmatrix} = \begin{pmatrix} r_1 & r_2 & \cdots & r_v \end{pmatrix} = \mathbf{r}'$$

$$J'L' = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & \cdots & l_{b1} \\ l_{12} & l_{22} & \cdots & l_{b2} \\ \vdots & \vdots & \ddots & \vdots \\ l_{1n} & l_{2n} & \cdots & l_{bn} \end{pmatrix} = \begin{pmatrix} k_1 & k_2 & \cdots & k_b \end{pmatrix} = \mathbf{k}'$$

$$AA' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{v1} & a_{v2} & \cdots & a_{vn} \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{v1} \\ a_{12} & a_{22} & \cdots & a_{v2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{vn} \end{pmatrix} = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_v \end{pmatrix} = D_r$$

$$AL' = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{v1} & a_{v2} & \cdots & a_{vn} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & \cdots & l_{b1} \\ l_{12} & l_{22} & \cdots & l_{b2} \\ \vdots & \vdots & \ddots & \vdots \\ l_{1n} & l_{2n} & \cdots & l_{bn} \end{pmatrix}$$

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$$= \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ n_{v1} & n_{v2} & \cdots & n_{vb} \end{pmatrix} = N, \quad incidence \;\; matrix$$

where $n_{ij}$is the number of times treatment i occur in block j.

$$LL^{'} = \begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{b1} & l_{b2} & \cdots & l_{bn} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & \cdots & l_{b1} \\ l_{12} & l_{22} & \cdots & l_{b2} \\ \vdots & \vdots & \ddots & \vdots \\ l_{1n} & l_{2n} & \cdots & l_{bn} \end{pmatrix} = \begin{pmatrix} k_1 & 0 & \cdots & 0 \\ 0 & k_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_b \end{pmatrix} = D_k$$

$$A\mathbf{y} = \begin{pmatrix} J^{'} \\ A \\ L \end{pmatrix} \mathbf{y} = \begin{pmatrix} J^{'}\mathbf{y} \\ A\mathbf{y} \\ L\mathbf{y} \end{pmatrix}$$

$$J^{'}\mathbf{y} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{v} y_i = G$$

thye grand total .

$$A\mathbf{y} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{v1} & a_{v2} & \cdots & a_{vn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_v \end{pmatrix} = \mathbf{T}$$

where $T_i$is the total observation of treatment i

$$L\mathbf{y} = \begin{pmatrix} l_{11} & l_{12} & \cdots & l_{1n} \\ l_{21} & l_{22} & \cdots & l_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ l_{b1} & l_{b2} & \cdots & l_{bn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} B_1 \\ B_2 \\ \vdots \\ B_b \end{pmatrix} = \mathbf{B}$$

where $B_j$is the total observations in block j .

The normal equations:

$$AA^{'}\mathbf{P} = A\mathbf{y} = \begin{pmatrix} n & \mathbf{r}' & \mathbf{k}' \\ \mathbf{r} & D_r & N \\ \mathbf{k} & N^{'} & D_k \end{pmatrix} \begin{pmatrix} g \\ \mathbf{t} \\ \mathbf{b} \end{pmatrix} = \begin{pmatrix} G \\ \mathbf{T} \\ \mathbf{B} \end{pmatrix}$$

$$ng + \mathbf{r}'\mathbf{t} + \mathbf{k}'\mathbf{b} = G \qquad \ldots\ldots\ldots\mathbf{??}$$

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$$\mathbf{r}\,g + D_r\mathbf{t} + N\mathbf{b} = \mathbf{T} \qquad \ldots\ldots\ldots\mathbf{??}$$

$$\mathbf{k}g + N^{'}\mathbf{t} + D_k\mathbf{b} = \mathbf{B} \qquad \ldots\ldots\mathbf{??}$$

These are $1 + v + b$ equations in $1 + v + b$ unknowns .
But

$$\sum_{i=1}^{v} T_i = \sum_{j=1}^{b} B_j = G$$

These implies that there are $(1 + v + b) - 2 = v + b - 1$ independent equations . Here solutions which are not unique are obtained by having additional equations,

$$\sum_{i=1}^{v} t_i = \sum_{j=1}^{b} b_j = 0$$

To solve for $\underset{\sim}{t}$ premultiply 3 with $ND_k^{-1}$and subtract from *equation* 2 to get

$$(\mathbf{r} - ND_k^{-1}\mathbf{k})g + (D_r - ND_kN^{'})\mathbf{t} = \mathrm{T} - ND_k^{-1}\mathbf{B}$$

But

$$ND_K^{-1}\mathbf{k} = N\begin{pmatrix} \frac{1}{k_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{k_b} \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_b \end{pmatrix} = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ n_{v1} & n_{v2} & \cdots & n_{vb} \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_v \end{pmatrix} = \mathbf{r}$$

$$\Rightarrow (\mathbf{r} - ND_k^{-1}\mathbf{k})g = (r - \mathbf{r})g = 0$$

$$\Rightarrow (D_r - ND_k^{-1}N^{'})\mathbf{t} = \mathbf{T} - ND_k^{-1}\mathbf{B}$$

$$c\mathbf{t} = \mathbf{Q}\cdots\mathbf{??}$$

equation $*$ gives the normal equations for $\mathbf{t}$ where

$$c = D_r - ND_k^{-1}N^{'}$$

$$Q_i = T_i - \sum_{j=1}^{v} \frac{n_{ij}B_j}{kj}$$

is the adjusted treatment total for treatment i

$$c = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1v} \\ c_{21} & c_{22} & \cdots & c_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ c_{v1} & c_{v2} & \cdots & c_{vv} \end{pmatrix} = \begin{bmatrix} r_1 - \sum \frac{n_{ij}^2}{kj} & -\sum \frac{n_{ij}n_{ij}}{kj} & \cdots & -\sum \frac{n_{ij}n_{vj}}{kj} \\ -\sum \frac{n_{ij}n_{2j}}{kj} & r_2 - \sum \frac{n_{2j}^2}{kj} & \cdots & -\sum \frac{n_{2j}n_{vj}}{kj} \\ \vdots & \vdots & \ddots & \vdots \\ -\sum \frac{n_{ij}n_{vj}}{kj} & -\sum \frac{n_{vj}n_{2j}}{kj} & \cdots & r_v - \sum \frac{n_{vj}^2}{kj} \end{bmatrix}$$

is a symetric matrix .

We note that $\mathbf{J}c = C(D_r - ND_kN^{'})$ where

$$\underset{\mathbf{J}_v}{'} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times v} = \mathbf{J}_v D_r - \mathbf{J}ND_kN^{'}$$

Now we see that

$$\underset{v}{\mathbf{J}'}D_r = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_v \end{pmatrix} = \begin{pmatrix} r_1 & r_2 & \cdots & r_v \end{pmatrix} = \mathbf{r}$$

$$\mathbf{J}ND_kN^{'} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ n_{v1} & n_{v2} & \cdots & n_{vb} \end{pmatrix} D_k^{'}N^{'}$$

$$= \begin{pmatrix} k_1 & k_2 & \cdots & k_b \end{pmatrix} \begin{pmatrix} \frac{1}{k_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{k_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{k_b} \end{pmatrix} N^{'} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \vdots & \vdots & \ddots & \vdots \\ n_{v1} & n_{v2} & \cdots & n_{vb} \end{pmatrix} =$$

$$\begin{pmatrix} r_1 & r_2 & \cdots & r_v \end{pmatrix} = \mathbf{r} \Rightarrow \mathbf{J}c_v = \mathbf{r} - \mathbf{r} = \mathbf{0}$$

This implies that one of the rows (columns) of c is a linear combination of the rest. Here $R(c) \leq v-1$,Hence only $v-1$ treatment effects are estimable.

From

$$\underset{v}{\mathbf{J}^{'}}c = \underset{v}{\mathbf{J}} \begin{pmatrix} \underset{1}{\mathbf{c}} & \underset{2}{\mathbf{c}} & \cdots & \underset{v}{\mathbf{c}} \end{pmatrix} = \mathbf{0} \Rightarrow \underset{v}{\mathbf{J}^{'}}\underset{v}{\mathbf{C}} = \sum_{i^{'}-1}^{v} c_{i^{'}i} = 0$$

Hence in

$$c\mathbf{t} = \begin{pmatrix} \underset{1}{\mathbf{c}^{'}} \\ \underset{2}{\mathbf{c}^{'}} \\ \vdots \\ \underset{v}{\mathbf{c}^{'}} \end{pmatrix} \begin{pmatrix} \underset{1}{\mathbf{t}} \\ \underset{2}{\mathbf{t}} \\ \vdots \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} \mathbf{c}'\mathbf{t} \\ \mathbf{c}'\mathbf{t} \\ \vdots \\ \underset{v}{\mathbf{c}^{'}}\mathbf{t} \end{pmatrix}$$

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where $\mathbf{c}'\mathbf{t}$is a treatment contrast. This implies that only $v-1$ contrasts are estimable. But if we have v treatments , only $v-1$ of the treatment contrasts are independent . Here if $R(c) \leq v-1$, then all treatment contrasts are estimable.

- **To estimate b**

premultiply equation 2 with ND_r^{-1} and subtract from 3

$$\Rightarrow (\mathbf{k} - N^{'}D_r^{-1}\mathbf{r})g + (D_k - N^{'}D_rN)b = \mathbf{B} - N^{'}D_r^{-1}\mathbf{T}$$

$$= 0 + (D_k - N^{'}D_rN)b = \mathbf{B} - N^{'}D_r^{-1}\mathbf{T} = F\mathbf{b} = \mathbf{W}$$

where

$$F = (D_k - N^{'}D_rN)b \text{ , } \mathbf{W} = \mathbf{B} - N^{'}D_r^{-1}\mathbf{T} \text{ , } W_i = B_j - \sum_{i=1}^{v}\frac{n_{ij}T_i}{r_i}$$

W_i is the block total, Also $\mathbf{J}F = 0$, Hence only b-1 block effects are estimable.

Exercise

show that only treatment contrasts are estimable.

6. Connected Designs

Connectedness of a design is an important aspect of designs, because in designs which are connected all treatment contrasts are estimable.

- **Definitions:**

1. A traetment and a block design are said to be connected if the treatment occur in that block.
2. Two treatment or two blocks are said to be connected if it is possible to move from one to the other one by means of a chain consisting alternatingly of treatments and blocks such that only two consecutive members of the chain are associated.Thus if $i_0$and $i_k$are connected then i_0j_1 , i_1j_2 ......$i_{k-1}J_ki_k$
3. A design is said to be connected if evfery treatment and every block are connected . In aconnected design , every treatment contrasts are etimable .

7. Sum of squares due to k linear functions say dy..........$\mathbf{d_K y}$

We say that sum of squares due to $\mathbf{d}'\mathbf{y}$ is

$$S^2(\mathbf{d'y}) = \frac{(\mathbf{d'y})^2}{\mathbf{dd'}}$$

If we have $\mathbf{d_1 y}$..........$\mathbf{d_k y}$ we can write in matrix form a combined linear function as ;

$$D\mathbf{y} = \begin{pmatrix} \mathbf{d}'_{\mathbf{1}}\mathbf{y} \\ \mathbf{d}_{\mathbf{2}}\mathbf{y} \\ \vdots \\ \mathbf{d}_{\mathbf{k}}\mathbf{y} \end{pmatrix}$$

and the sum of squares

$$S^2(\mathbf{d}'_{\mathbf{1}}\mathbf{y}\text{..........}\mathbf{d}'_{\mathbf{k}}\mathbf{y}) = S^2(D\mathbf{y}) = \mathbf{y}'D'(DD')^*D\mathbf{y}$$

Note that if $(DD')^*$ has an inverse then

$$S^2(Dy) = \mathbf{y}'D'(DD')^{-1}D\mathbf{y}$$

Suppose (DD') is of rank say r . Then we can get r orthogonal vectors say $\mathbf{b}_{1}$............$\mathbf{b}_{-r}$ which generate the row vectors of D.

Let $\mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{pmatrix}$ then

$$S^2(D\mathbf{y}) = S^2(B\mathbf{y}) = \mathbf{y}'B'(B'\mathbf{B})^*B\mathbf{y}$$

but

$$BB' = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_r \end{pmatrix} \begin{pmatrix} b_! & b_2 & \cdots & b_r \end{pmatrix} = \begin{pmatrix} \mathbf{b}'_1\mathbf{b}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{b}'_2\mathbf{b}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{b}'_r\mathbf{b}_r \end{pmatrix}$$

$$(BB')^{-1} = \begin{pmatrix} \frac{1}{\mathbf{b}'_1 b} & 0 & \cdots & 0 \\ 0 & \frac{1}{\mathbf{b}'_2 b} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{\mathbf{b}'_r b_r} \end{pmatrix}$$

Hence $S^2(D\mathbf{y}) = S^2(B\mathbf{y}) = \mathbf{y}B^{'}(B\mathbf{B'})^{-1}B\mathbf{y} = \sum limits$

Expection:

$$s^2\left(D\mathbf{y}\right)i$$

$$E\left(s^2\left(Dy\right)\right) = E\left(s^2\left(B\mathbf{y}\right)\right) = \sum_{i}^{r} \frac{F\left(\mathbf{b}_{2i}\mathbf{y}_2\right)^2}{b_{2i}b_{2i}}$$

$$= \sum \left[\frac{v\left(b_1y_1\right)}{\mathbf{b}_i\mathbf{b}_i} = \frac{\left[E\left(b_1y_1\right)\right]^2}{\mathbf{b}_i\mathbf{b}_i}\right]$$

$$= r\delta e^2 + \sum_{i=1}^{r} \frac{E\left(\mathbf{b}_1\mathbf{y}_1\right)^2}{\mathbf{b}_i\mathbf{b}_i}$$

if all b_i belong to error space then $\frac{s^2(B\mathbf{y})}{r}$ is an unbiased estimate of δe^2

Sum square due to adjusted treatments

we saw that the normal equation fot t are

$$\mathbf{Ct} = Q$$

$$\hat{\mathbf{t}} = \mathbf{C}Q,\ \hat{t} = \mathbf{Q}^{'}\mathbf{C}$$

The functions

$$\mathbf{CQ} = \begin{pmatrix} \mathbf{C}_1 & Q \\ \mathbf{C}_2 & Q \\ \vdots & \vdots \\ \mathbf{C}_\vee & Q \end{pmatrix} \text{ are linear functions}$$

if $\mathbf{y}$ and hence the sum of squares due to them is

$$S^2\left(\mathbf{C}Q\right) = Q^{'}\mathbf{C}\left(\mathbf{CC}\right)\mathbf{C}Q$$

But the matrix

$$\mathbf{C}^{'}\left(\mathbf{CC}^{'}\right)\mathbf{C} = \mathbf{C}\left(\mathbf{CC}\right)\mathbf{C}$$

$$\mathbf{CC} = \mathbf{C}$$

$$\left(D_r - ND_k^{-1}N^{'}\right)\left(D_r - ND_k^{-1}N^{'}\right) =$$

$$\left(D_r - ND_k^{-1}N^{'}\right)$$

$$\Rightarrow \mathbf{C}^1\left(\mathbf{CC}\right)\mathbf{C} = \mathbf{C}\left(\mathbf{CC}\right)\mathbf{C} = \mathbf{CCC} = \mathbf{C}$$

Similarly

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$$\mathbf{C}(\mathbf{C})\,\mathbf{C} = \mathbf{C}\,(\mathbf{CC})\,\mathbf{C} = \mathbf{CCC} = \mathbf{C}$$

Hence

$$s_t^2\,(adjusted) = s^2\,(\mathbf{C}Q) = Q^1\mathbf{C}Q$$

$$= \hat{t}Q$$

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Unadjusted treatment sum of aquares is

$$s_t^2\,(unadj) = \sum_{\imath=1}^{\wedge} \frac{.T_1^2}{r_1} - \frac{G^2}{n}$$

unadjusted block sum of squares is

$$s_b^2\,(unadj) = \sum_{\imath=i}^{b} \frac{Bj^2}{k_j} - \frac{G^2}{n}$$

Adjusted treatment sum of square is

$$s_t^2\,(adj) = \hat{\mathbf{t}}_\imath^1\mathbf{Q} = Q^1\mathbf{CQ}$$

Testing hypothesis

We saw that

$$E\left[s_t^2\,(unadj)\right] = (v-1)\,\delta e^2 + b\sum_{i}^{\wedge} t_i^2$$

or

$$E\left[ms_t^2\,(unadj)\right] = \delta e^2 + \frac{b\sum t_i^2}{v-1} = \delta e^2 + (Non\ zero)$$

This implies

$$E\left(Ms_t^2 unadj\right) = \delta e^2$$

only if

$$t_1 = t_2 = \ldots = t_v = 0$$

on the other hand

$$E\left(s_t^2\,(adj)\right) = E\left(Q^{'}\mathbf{C}Q\right)$$

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$$= E\left(trace\, Q'\mathbf{C}Q\right)$$

$$= E\, trace\left(\mathbf{C}QQ^1\right)$$

$$= trace\mathbf{C}E\left(QQ^1\right)$$

$$= trace\mathbf{C}\left[v\left(Q\right) + E\left(Q\right)E\left(Q^1\right)\right]$$

$$= trace\mathbf{CC}\delta e^2 + trace\mathbf{C}E\left(Q\right)E\left(Q^1\right)$$

$$= \left(v-1\right)\delta e^2 + trace\mathbf{CC}tt\mathbf{C}$$

But if

$$H0 : t_1 = t_2 = \ldots = t_v$$

is true then

$$= E\left(s_t^2\left(adj\right)\right) = \left(v-1\right)\delta e^2 + 0$$

because
$\mathbf{Cj}v = 0$
Hence the adjusted treatment sum of squares can be used to fist the hypothesis

$$H0 : t_1 = t_2 = \cdots = t_v$$

source	d.f	ss	mss	F
treatment	$v-1$	$s_t^2\left(adj\right) = \hat{t}QT$		
blocks	$b-1$	$s_b^2\left(adj\right) = \frac{B_j^2}{k_j} - \frac{G^2}{n}$		
Error	$n-v-b+1$	$s_e^2 = s_T^2 - s_t^2\left(adj\right) - s_b^2\left(unadj\right)$		
Total	n	$s_T^2 = \sum_{u=1}^n y_u^2 - \frac{G^2}{n}$		

For$H_0 : b_1 = b_2 = \cdots = b_b$

source	d.f	ss	mss	F
treatment	$v-1$	$s_t^2\left(unadj\right) = \frac{\sum Ti^2}{r_i-} - \frac{G^2}{n}$		
blocks	$b-1$	$s_b^2\left(adj\right) = \hat{b}_i 1w$		
Error	$n-v-b+1$	$s_e^2 = s_T^2 - s_t^2\left(unadj\right) - s_b^2\left(adj\right)$		
Total	$n-1$	$s_T^2 = \sum_{u=1}^n y_u^2 - \frac{G^2}{n}$		

source mss F
treats $-$

Blocks B $f_2 = \frac{B}{E}$
error E
Total

Theorem

-Adjusted ss due to treatment plus unadjusted ss due to blocks is equal to unadjusted ss due to treatment plus adjusted sum of squares due to blocks.

Proof

$$\mathbf{Y} = A^1\mathbf{p} + \underset{-}{\varepsilon}$$

normal equation

$$AA^1\mathbf{p} = -A\mathbf{y}$$

$$\hat{\mathbf{P}} = (AA)\, A\mathbf{Y}$$

ss due to estimation space is

$$\mathbf{y}^1 A\,(AA)\, A\mathbf{Y} = \hat{\mathbf{p}}^1 A\mathbf{y}$$

But

$$\hat{p} = \left(\hat{g}\hat{\mathbf{t}}^1\hat{\mathbf{p}}^1\right)$$

Hence ss due to estimation space is:

$$\hat{p}^{'} Ay = \left(\hat{g}\hat{\mathbf{t}}^{'}\hat{\mathbf{b}}^{'}\right)\begin{pmatrix} G \\ \mathbf{T} \\ \mathbf{B} \end{pmatrix}$$

$$\hat{p}A\mathbf{y} = yG + \hat{\mathbf{t}}^{'}\mathbf{T} + \hat{\mathbf{b}}^{'}\mathbf{B}$$

From the equations

$$ng + \mathbf{r}^{'}\mathbf{t} + \mathbf{k}^{'}\mathbf{b} = G \cdots\cdots (1)$$

$$\mathbf{r}g + D_r\mathbf{t} + N\mathbf{b} = \mathbf{T} \cdots\cdots (2)$$

$$\mathbf{k}g + N^{'}\mathbf{t} + D_k\mathbf{b} = B \cdots\cdots (3)$$

we get

$$\hat{\mathbf{b}} = D_k^{-'}\mathbf{B} - D_k^{-'}\mathbf{k}g - D_k^{-'}N^{'}\mathbf{t}$$

put

$$\hat{g} = \frac{G}{n}$$

Substitute $\hat{\mathbf{b}_1}$ and $\hat{g}$ in $\hat{\mathbf{p}}A\mathbf{y}$
to get

$$\hat{g}G + \hat{\mathbf{t}}' \left(\mathbf{T} - ND_k^{-'}\mathbf{B} \right) + B'D_k^{-'}B_\imath - \mathbf{K}'D_k^{-'}\mathbf{B}\hat{g}$$

$$= \frac{G^2}{n} + \hat{\mathbf{t}}'\mathbf{Q} + \sum_{j=1}^{b} \frac{B_j^2}{k_j} - \frac{G^2}{n}$$

$$= \frac{G^2}{n} + s_t^2\left(adj\right) + s_b^2\left(unadj\right)$$

from (2) we get

$$\hat{\mathbf{t}} = D_r^{-'}\mathbf{T} - D_r^{-'}\mathbf{r}\hat{g} - D_r^{-'}N\hat{\mathbf{b}}$$

substitutes in $\hat{p}A\mathbf{y}$ to get

$$\hat{p}A\mathbf{y} = \hat{g}G + \mathbf{T}'O_r^{-'}t - r'o_r^{-'}D_r^{-'}T\hat{p} + \hat{b}'\left(B - N'D_r^{-'}B\right)$$

$$= \frac{G^2}{n} + \sum_{i=1}^{\vee} \frac{Tr^2}{r^i} - \frac{G^2}{n} + \hat{b}'w$$

$$= \frac{G^2}{n} + s_t^2\left(unadj\right) + s_b^2\left(adj\right)$$

Theorem

$$var\left(Q\right) = c\delta^2, var\left(T\right) = D_r\delta,^2 var\left(B\right) = D_k\delta^2$$

$$var\begin{pmatrix} G \\ I \\ b \end{pmatrix} = var\left(Ay\right) = AA'\delta^2$$

$$= \begin{pmatrix} n & \mathbf{r}' & k' \\ \mathbf{r} & D_r & N \\ \mathbf{k} & N' & D_k \end{pmatrix}\delta^2$$

$$var\left(G\right) = n\delta^2, var\left(I\right) = D_r\delta^2, var\left(B\right) = D_k\delta^2$$

$$var\left(g\mathbf{I}\right) = \mathbf{r}\delta^2, var\left(G, \mathbf{B}\right) = k\delta^2$$

$$var\,(T,B) = N\delta^2$$

$$var\begin{pmatrix} Q \\ B \end{pmatrix} = var\begin{pmatrix} T- & ND_k^{-'} & B \\ . & . & . \\ . & B & . \end{pmatrix} =$$

$$var\begin{pmatrix} IT_v & -ND_k^{-'} \\ 0 & Ib \end{pmatrix}\begin{pmatrix} T \\ B \end{pmatrix}$$

$$= \begin{pmatrix} T_v & -ND_k^{-'} \\ 0 & Ib \end{pmatrix}\begin{pmatrix} D_r & N \\ N' & D_k \end{pmatrix}\begin{pmatrix} T_v & o \\ D_k^{-'}N' & T_b \end{pmatrix}$$

$$var\begin{pmatrix} Q \\ B \end{pmatrix} = \begin{pmatrix} D_{r-}ND_k^{-'}N' & 0 \\ 0 & D_k \end{pmatrix}\delta^2$$

$$= \begin{pmatrix} c\delta^2 & 0 \\ 0 & 0 \end{pmatrix} 0$$

Hence

$$var\,(\mathbf{Q}) = C\delta^2$$

$$var\,(\mathbf{Q}, \mathbf{B}) = 0$$

This implies **Q** and **B** are additive but δ **T** and **B** are not

7.1. Balanced incomplete block design

It is a block design with v treatment $b$blocks where each block has k plots such that :

1. A treatment occurs once or dues not occur in a block i.e $nij = o, 1$
2. Each treat occurs r time in the design
3. Any pair of treatment occurs together in λ blocks.

$A\ BIB$ design has 5 parameters v, b, r, k, λ satisfying.

- $vr = bk$
- $r\,(k-1) = \lambda\,(v-1)$

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$$\binom{V}{2}\lambda = \binom{\lambda}{2}b$$

$$\frac{v\left(v-1\right)}{2}\lambda = \frac{kck-1}{2}b$$

$$\lambda v\left(v-1\right) = bk\left(k-1\right)$$

$$\lambda v\left(v-1\right) = vr\left(k-1\right)$$

$$\lambda\left(v-1\right) = r\left(k-1\right)$$

Theorem

In a $B, B, D, b \geq v$ or $v \geq k$,Fisher's inequality

Proof

Let

$$N = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \cdots & \cdots & \cdots & \cdots \\ nv_1 & nv_2 & \cdots & nv_b \end{pmatrix}$$

The index matrix. then obviously

$$R\left(N\right) = R\left(NN^{'}\right)$$

$$NN^{'} = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \cdots & \cdots & \cdots & \cdots \\ nv_1 & nv_2 & \cdots & nv_b \end{pmatrix} \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \cdots & \cdots & \cdots & \cdots \\ nv_{1b} & nv_{2b} & \cdots & nv_b \end{pmatrix}$$

$$= \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \lambda & \lambda & \cdots & r \end{pmatrix}$$

def

$$NN^{'} = \begin{bmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \lambda & \lambda & \cdots & r \end{bmatrix} =$$

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$$\begin{bmatrix} r+\lambda(v-1) & r+\lambda(v-1) & \cdots & r+\lambda(v-1) \\ \lambda & \lambda & \lambda & \lambda \\ \lambda & \lambda & \lambda & \lambda \\ \cdots & \ldots & \ldots & \ldots \\ \lambda & \lambda & \ldots & \ldots \\ \lambda & \lambda & \ldots & \ldots \end{bmatrix}$$

Subtract	1st column	from the rest of columns:			
def NN'	$r+\lambda(v-1)$	0	0	$\cdots$	0
	λ	$r-\lambda$	0	$\cdots$	0
	λ	0	$r-\lambda$	$\cdots$	0
	$\cdots$	$\cdots$			
	$\cdots$	$\cdots$			
	λ	0	0		$r-\lambda$

$$= [r+\lambda(v-1)](r-\lambda)^{v-1}$$
$$= [r+r(k-1)](r-\lambda)^{v-1}$$

$$= rk(r-\lambda)^{v-1}$$

Hence $r \neq \lambda$
def $NN' \neq 0$
$R(NN') = R(N) = min(v,b)$ but NN' is of dimension $v \times v$
Hence
$v = min(v,b) \Rightarrow b \geq v$
From $vr = bk$, $r = k$

- **Symmetric BIB Design**

A BIB design is said to by symmetric if $v = b$, $r = k$

- **Complementary Design of a BIB Design**

Consider a BIB design with parameter r, v, b, k, λ.A complementary design of this design is obtained by removing all the treats in a block and replacing them with treats which did not clear in that block.

The new design will have the following parameters
$v' = v$ $b' = b$ $r' = b-r$, $k' = v-k$
to get

$$\lambda', r'\left(k'-1\right) = \lambda'\left(r'-1\right)$$

$$\lambda'(v-1)^{ok} = (b-r)(v-k-1)$$

$$\Rightarrow \lambda' = b-2r+\lambda$$

- **Analysis of BIB Design**

Adjustment treatment total

$Q = T - ND_k^{-1}B$

Normal equation

$$\subset t = Q \Rightarrow \hat{t} = \subset^{\star} Q$$

$$\subset = \begin{pmatrix} \subset_{11} & \subset_{12} & \dots & \subset_{1v} \\ \subset_{21} & \subset_{22} & \dots & \subset_{2v} \\ \dots & \dots & \dots & \dots \\ \subset 1v & \subset_{2v} & \dots & \subset_{vv} \end{pmatrix}$$

$$\mathbf{C}_{ii} = r_i - \sum_{j=i}^{n} \frac{n_{ij}}{k_j} = r - \frac{r}{k} = r\left(1 - \frac{1}{k}\right)$$

$$\subset'_{ii} = -\sum_{j}^{b} \frac{n_{ij}n_{ij}}{k_j} = -\frac{\lambda}{k}$$

From $\mathbf{C}t = \mathbf{Q}$

we get

$$\mathbf{C}_i \mathbf{t}_i + \mathbf{C}_{i2}t_2 + \cdots\cdots + \mathbf{C}_{ii}t_i + \cdots\cdots \mathbf{C}_{iv}t_v = Q_i$$

$$= -\frac{\lambda}{k}\hat{\sum_{i=1}} t_i + \frac{\lambda}{k}t_i + r\left(1 - \frac{1}{k}\right)t_i = Q_i$$

But

$\sum_{i=1}^{\vee} t_i = 0$

Hence

$$r\left(1 - \frac{1}{k}\right)ti + \frac{\lambda}{k}t_i + Q_i or \frac{r\left(k-1\right)}{k}t_i + \frac{\lambda}{k}t_i = \frac{\lambda\left(v-1\right)}{k}t_i + \frac{\lambda}{k}t_i = \frac{\lambda v}{k}t_i = Q_i$$

This implies

$\hat{t_i} = \frac{k}{\lambda r}Q_i$

sum of squares due for adjusted treatments is $\therefore$

$$s_t^2\left(adj\right) = \hat{t}^{1}Q = \frac{k}{\lambda v}\left(Q_1, Q_{2,\dots\dots}Q_{\vee}\right)\begin{pmatrix} Q_1, \\ Q_2 \\ \dots \\ Q_{\vee} \end{pmatrix} = \frac{k}{\lambda\vee}\hat{\sum_{i=1}} Q_i^2$$

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source	df	ss	mss	F
Treatments	$v-1$	$s_t^2\,(adj) = \hat{t}Q$		
Blocks	$b-1$	$s_b^2\,(adj) = \sum_j^b \frac{B_j^2}{k_j} - \frac{G^2}{vr}$		
Error	$vr-v-b+1$	$s_e^2 = s_T^2 - s_t^2\,(adj) - s_b^2\,(unadj)$		
Total	$vr-1$	$s_T^2 = \sum y_n^2 - \frac{G^2}{n}$		

$c+d$
Source mss F
Treat T $F_1 = \frac{T}{E}$
Blocks $-$
error E

7.2. Partially Balanced Incomplete Block Design($PBIBD$)

-Consider a block design with v treats and b blocks each block having k plots & each treatment occuring r times in the design.The design is said to be a $PBIB$ design if it satiafies the followingg conditions:

1. A treat does not occur more than once in the design
2. Write every treatment all the other treats fall into m groups with corresponding sizes $n_1, n_2, \cdots, nm$ s/t every treat in the c^{th} group occurs λ_i times in the given treatments.λ_i & $n_i$aree independent of whichever treat you start with.
3. For the given treatment a treat belonging to the i^{th} group is called an i^{th} associated of treat β then treat β is an i^{th} associate of treat α. Also the number p^i_{jk} which denotes the n of j^{th}associates of α which are k^{th} associate of B is independent of the treatment pair (α, β) for p^i_{jk} consider 1st associates say 1 and 2.

$p^1_{11} = 0$ $p^1_{12} = 2$ $p^1_{13} = 0$
$p^1_{21} = 2$ $p^1_{22} = 0$ $p^1_{23} = 1$
$p^1_{31} = 0$ $p^1_{32} = 1$ $p^1_{33} = 0$

$$p^1 = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{mat of 1st associates}$$

for
p^{i2}_{jk} consider 1 and 3
$p^2_{11} = 2$ $p^2_{12} = 0$ $p^2_{13} = 1$
$p^2_{21} = 0$ $p^2_{22} = 2$ $p^2_{23} = 0$
$p^2_{31} = 0$ $p^2_{32} = 1$ $p^2_{33} = 0$

$$p^2 = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix} \text{ mat of 2nd associates}$$

$$p^3 = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

- **Properties of PBIB Designs**

1. $vr = bk$
2. $v - 1 = n_1 + n_2 + \cdots = nm$
3. $r(k-1) = \lambda n_1 + \lambda n_2 + \cdots \lambda mnm$
4. $\sum_{k=1}^{m} p_{jk}^{i} = \left\{ \begin{array}{cc} n_j - 1, if & i = j \\ n_j if & i \neq j \end{array} \right\}$
5. $n_i p_{ik}^{i} = n_j p_{ik}^{j} = nkpij^k$

8. Lattice Designs

Two Dimensional

- Lattice designs belong to the class of $pBiB$ design.The construction of a $2-D$ lattice design is as follows,consider k^2 treatment which are placed in $k \times k$ each treatment in one if the k^2cells

1	t_1	t_2			t_k
2	t_k+1	$t_k + 2$			
	t_1	t_2		t_i	t_k
k	$t_k(k-1)+1$	$t_k(k-1)+2$			t_{k^2}

for all treats lying in the same row assign them to a block to get k blocks between all treats occuring in the same column assign them to a block to get another k blocks.

- The resulting design is a $PBIB$ design with the following parameters.

$v = k^2$, $b = 2k$, $r = 2$, $k = k$ each that assigned twice because you pick a row with it then a –

Group 1
$n_1 = 2(k-1)$
$\lambda_1 = 1$
Group 2
$n_2 = (k-1)^2$
$\lambda_2 = 0$

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Analysis

$$c\mathbf{t} = \mathbf{Q}$$

$$\mathbf{c}_i^1\mathbf{t} = c_{1i}t_i + c_{i2}t_2 + c_{i3}t_3 + \cdots + c_{ii}t_i + \cdots c_{ir}t_r = Q_i$$

$$c_{ii} = ri - \frac{ri}{k} = 2\left(1 - \frac{1}{k}\right)$$

$$= -\frac{\sum n_{ij}n_{ij}}{k_j} =$$

$\frac{-i}{k}$ if i and i^1 are in in the same block o_1 otherwise

Let$s_p\left(t_i\right)$ denote the sum of all treats occuring in the same row as t_i, $t_i$inclusive

$$s_R\left(t_i\right) = t_i + t_2 + \cdots + t_i + \cdots + t_k$$

Similarly let

$$s_c\left(t_i\right) = t_i + t_2 + \cdots + t_i + \cdots + t_k$$

Then

$$c_it = 2\left(1 - \frac{1}{k}\right)t_i - \frac{1}{k}s_R\left(t_i\right) - \frac{1}{c}s_c\left(t_i\right) + \frac{2}{k}t_i = Q_1$$

$$\Rightarrow 2t_i - \frac{1}{k}s_R\left(t_i\right) - \frac{1}{k}s_c\left(t_i\right) = Q_i \cdots (1)$$

Sum (1)over rows to get

$$2s_R\left(t_i\right) - \frac{1}{k}s_Rs_R\left(t_i\right) - \frac{1}{k}s_Rs_c\left(t_i\right) = s_R\left(Q_i\right)$$

Note

$s_Rs_R\left(t_i\right) = s_R\left(t_i + t_2 + \cdots + t_i + \cdots + t_k\right) = s_R\left(t_i\right) + s_R\left(t_2\right) + \cdots + s_R\left(t_i\right) + \cdots + s_R\left(t_k\right) = ks_R\left(t_i\right)$

$s_Rs_c\left(t_i\right) = s_R\left(t_i + \cdots + t_k\right) = s_R\left(t_i\right) + s_R\left(t_2\right) + \cdots + s_R\left(t_i\right) + \cdots + s_R\left(t_k\right) = \sum_{i=1}^{k^2} t_i = 0$

$s_R\left(Q_i\right) = Q_1 + Q_2 + \cdots + Q_k \Rightarrow 2s_R\left(t_i\right) - s_R\left(t_i\right) = s_R\left(Q_i\right)$

or

$$s_R\left(t_i\right) = s_R\left(Q_i\right)$$

and

$$s_c\left(t_i\right) = s_c\left(Q_i\right)$$

Substituting (2) in (1) gives

$$\mathbf{t}_i^{\hat{}} = \frac{1}{2}\left(Q_i + \frac{1}{k}s_R\left(Q_i\right) + \frac{1}{k}s_c\left(Q_i\right)\right)$$

It is possible to to get c now because $t^{\hat{}} = c\mathbf{Q}$ and $\mathbf{t}_i^{\hat{}} = c_{ii}Q_2 + c_{i2}Q_2 + \cdots + c_{i1}Q_1 + \cdots + c_{iv}o_v$

This gives the coefficients of Q_i $i = 1\ 2 \cdots v$ in $t_i^{\hat{}}$ as $c_i = \frac{1}{2}\left(1 + \frac{1}{k} + \frac{i}{k}\right) = \frac{1}{2} + \frac{1}{k}$

$c_{ii} = \frac{1}{2k}$ if treat i and i occur in the same row or column

Anova

source	df	ss	mss	f
Treatments	$k^2 - 1$	$s_t^2\,(adj) = \hat{t}Q$	T	$F_1 = \frac{T}{E}$
Blocks	$2k - 1$	$s_b^2\,(adj) = \frac{\sum B_j^2}{k} - \frac{G^2}{2k^2}$	$-$	
Error	$k - 1^2$	$s_e^2 = s_T^2 - s_t^2\,(adj) - s_b^2\,(unadj)$	E	
Totals	$2k - 1$	$s_T^2 = \sum y_n^2 - \frac{G^2}{2k^2}$		

9. Finite plane projective Geometries Axiums for existance of plane projection Geometry

- Consider a system with paints and lines s.t a point may or may not lie on a line. We say the point is incident on the line if it lies on the line or the line passes through it.
- The system is said to form a finite plane projective geometry if it satisfies the following axiums

1. Any two points are incident with a line.
2. Any two points are not incident with more than one line
3. Two lines are incident with the point
4. All points are not incident with the same line
5. There are atleast 3 different points on a line
6. The no of points incident with any line is finite say set

Theorem

There are $s^2 + s + 1$ points in the geometry

Proof

$\downarrow$

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proof

Consider point o through this point there are $s+1$ lines each line besides point o has s more points. Hence then total no of points would be

$s\,(s+1)+1=s^2+s+1$

Theorem

There are s^2+s+1 lines in the geometry if points are treatments & lines blocks we can construct a symmetric BIB design

$v=s^2+s+1$
$b=s^2+s+1$
$r=s+1$
$k=s+1$
$\lambda=1$

Finite fields

Let p be an integer let any integral number n be equivalent to the remainder when n is divided by p. then we have a set containing the possible remainders when n is divided by p. This set is reffered as module p.

If r is the remainder when n is divided by p then $r\in$ module p i.e $r\in$ module p and $n{=}r$ module p.

Example

let $p=7$.

Consider all integers
$\frac{n}{7}=k7+r$
mod $7=\{0,1,2,3,4,5,6\}$
$7=0$ mod 7, $8=1$ mod7, $9=2$ mod 7
$14=4$ mod 7 ,$12=5$ mod 7, $14=0$ mod 7

- if $p$is prime then all the arithmetiic operations of x and /are closed. if this is satisfied the ste is reffered to as Galois field denoted by$GF(P)$.

Example take 3 and 4

$3+4=7=0$ mod 7
$3-4=-1=-1+0=-1+7=6$ mod 7
$3\times 4=12=5$ mod 7
$\frac{3}{4}=\frac{3+0}{4}=\frac{3+3.7}{4}=6$ mod 7
$\frac{2}{5}=\frac{2+0}{5}=\frac{2+4.7}{5}=6$ mod 7

Points in finite plane projective geometry are of the form (X_0,X_1,X_2) where $X_iE\in GF\,(P)$ where not all X_1^{-1} are zero.

- Also two points (X_0,X_1,X_2) and (X_0^1,X_1^1,X_1^1) are equivalent. if

$X_i^1 = pX_i,$
where
$p \in GF\left(p\right)$
NOTE
Let $s = p$ then we talk of $GF\left(s\right)$. S can be a power of a prime p line $s = p^n$

- The total number of points in the geometry is $\therefore$ $p_2 = \frac{s^3 - i}{s-1}$

$s^3 - 1$ because $(0, 0, 0)$ is not a member and divide by $s - 1$ because p can take $s - 1$ non-zero values.

- A line in the geometry is of the form $a_0X_0 + a_1X_1 + a_2X_2 = 0$ where a are element of $GF\left(s,\right)$ $X_i \in GF\left(s\right)$ not all $a_i^1$are zero.

Hence the total number of lines

$$\frac{s^3 - 1}{s - 1} = s^2 + s + 1$$

Example
Construct a BIB design with 7 treatments and 7 blocks using $F.P.P.G.$

Solution
points are of the form

$$p_1\left(1, 0, 0\right), p_2\left(0, 1, 0\right), p_3\left(0, 0, 1\right), p_4\left(1, 1, 0\right), p_5\left(1, 0, 1\right), p_6\left(0, 1, 1\right), p_7\left(1, 1, 1\right)$$

lines GF(2) =mod 2 = $[0, 1]$

$$L_1 = X_0 = 0, (p_{2,}p_3, p_6)$$

$$L_2 = X_1 = 0, (p_{1,}p_3, p_5)$$

$$L_3 = X_2 = 0, (p_{1,}p_{2,}p_5)$$

$$L_4 = X_0 + X_1 = 0, (p_{3,}p_{4,}p_7)$$

$$L_5 = X_0 + X_2 = 0, (p_{2,}p_{5,}p_7)$$

$$L_6 = X_1 + X_2 = 0, (p_{1,}p_6, p_7)$$

$$L_7 = X_0 + X_1 + X_2 = 0, (p_{4,}p_5, p_6)$$

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$$v = 7, b = 7, r = 3, k = 3, X = 1$$

with

$$v = 13, b = 13$$

$$v = s^2 + s + 1$$

$$13 = s^2 + s + 1$$

$$s^2 + s - 12 = 0$$

$$pq = -12$$

$$p + q = 1$$

$$\left(s^2 + 4\right)(s - 3) = 0$$

$$s^2 + 4 = 0$$

$$s - 3 = 0$$

$$s = 3$$

p can take $s - 1 = 3 - 1 = 2$ non zero values
$GF\left(3\right) = \bmod 3 = \{0, 1, 2\}$

$$p_1\left(1,0,0\right), p_2\left(0,1,0\right), p_3\left(0,0,1\right), p_4\left(1,1,0\right), p_5\left(1,0,1\right),$$

$$p_6\left(0,1,1,\right), p_7\left(1,1,1\right), p_8\left(1,2,0\right), p_9\left(1,0,2\right), p_{10}\left(0,1,2\right), p_{11}\left(1,1,2\right)$$

$$p_{12}\left(1,2,1\right), p_{13}\left(2,1,1\right)$$

$$v = 13, b = 113, r = 4, k = 4, X = 1$$

$$a_0X_0 + a_1X_1 + a_2X_2 = 0$$

9.1. Finite plane Eudidean Geometry

Axioms of Eudidean Geometry

1. Two different points are incident with one line
2. Two different points are incident with more than one line
3. Through any point not incident with a given line, there passes one and only one line which has no common point with the given line, this line is said to be parallel to the given line.All other lines through the point has a common point with the given line.
4. All points are not incident with the same line
5. There are atleast two different point on a line
6. The no of points incidennt with a line is finite say s with $s$is prime usually points will have the form

$X_1, X_2, \ldots, X_1, X_2 \in GF(s)$
The total number of points $= s^2$ this time they can be zero

- The no of lines $v = s^2$ $b = s^2 + s$
- To construct a finite plane Eudidean Geometry, remove one line in finite plane projective geometry and all the points on that line from the Geometry.

Projective

$$v = s^2 + s + 1$$

$$b = s^2 + s + 1$$

$$r = s + 1$$

$$k = s + 1$$

$$X = 1$$

Euclidean

$$v = s^2$$

$$b = s^2 + s$$

$$r = s + 1$$

$$k = s$$

$$X = 1$$

$$s = 2$$

$$GF(2) = [0, 1]$$

$$p_1(0,0), p_2(0,1), p_3(1,0), p_4(1,1)$$

$$a_1X_1 + a_2X_2 = \alpha, \alpha t GF(s)$$

$$X_1 = 0\,(p_1, p_2)$$

$$X_2 = 0\,(p_1, p_3)$$

$$X_1 + X_2 = 0\,(p_1, p_4)$$

$$X_1 = 1\,(p_3, p_4)$$

$$X_2 = 0\,(p_2, p_4)$$

$$X_1 + X_2 = 1\,(p_2, p_3)$$

Projective

$p_1\,(1,0,0)\,p_2\,(0,1,0)\,p_3\,(0,0,1)\,p_4\,(1,1,0)\,p_5\,(1,0,1)\,p_6\,(0,1,1)\,p_7\,(1,1,1)$

$$L_1 = X_0 = 0, (p_{2,}p_3, p_6)$$

$$L_2 = X_1 = 0, (p_{1,}p_3, p_5)$$

$$L_3 = X_2 = 0, (p_{1,}p_{2,}p_5)$$

$$L_4 = X_0 + X_1 = 0, (p_{3,}p_{4,}p_7)$$

$$L_5 = X_0 + X_2 = 0, (p_{2,}p_{5,}p_7)$$

$$L_6 = X_1 + X_2 = 0, (p_{1,}p_6, p_7)$$

$$L_7 = X_0 + X_1 + X_2 = 0, (p_{4,}p_5, p_6)$$

Suppose you remove the last line and all its points

$$L_1 : X_0 = 0, (p_{2,}p_3)$$

$$L_2 : X_1 = 0, (p_{1,}p_3)$$

$$L_3 : X_2 = 0, (p_{1,}p_{2,})$$

$$L_4 : X_0 + X_1 = 0, (p_{3,}p_7)$$

$$L_5 : X_0 + X_2 = 0, (p_{2,}p_7)$$

$$L_6 : X_1 + X_2 = 0, (p_{1,}p_7)$$

Points that have remained :p_1, p_2, p_3, p_7
$\Rightarrow p_1\,(1,0,0)p_2\,(0,1,0)\ p\,(0,0,1)\ p_7\,(1,1,1)$
But we want to get XX from these points
Now suppose you removed the last line then you get (aX)

9.2. Finite Eudidean Geometry of Dimesions

We have seen that points in a finite plane Eudidean geometry are of the form (X_1, X_2)We ca have finite eudidean geometry of dimension n whose points are of the form

$(X_1, X_2 \ldots, \chi_N)$

where

$X_1 \in GF\left(s\right), s = p$ or $s = p^n$

where n is an integer.

A linear equation of the form $a_1X_1 + a_2X_2 + \ldots + a_nX_n = \alpha, \alpha \in GF\left(s\right)$ defines a line or a flat in that finite Eudidean geometry.These Eudidean geometries are important in construction of factorial designs

Projective equation to 0

Eudidean to $\alpha, \alpha t GF\left(s\right)$

$a_1X_1 + a_2X_2 + \ldots + a_NX_N = \alpha, \alpha \in GF\left(s\right)$

$GF(s) = \{\alpha_0, \alpha_1 \ldots \alpha s - 1\}$

The possible parrallel lines (or feats) are:

$a_1X_1 + a_2X_2 + \ldots + a_nX_n = \alpha 0$

$a_1X_1 + a_2X_2 + \ldots + a_nX_n = \alpha 1$

$a_1X_1 + a_2X_2 + \ldots + a_n\chi_n = \alpha s - 1$

We say all these flats belongs to pencil $p\left(a_1, a_2 \ldots, a_N\right)$ where not a^i are zeros

The geometry has s^Npoints and these points are distributed equally within the flats of this pencil

Each flat has $\frac{s^N}{s} = s^{N-1}$ points

Example

Let

$s = 2, N = 3$

the points are $(X_1, X_2, X3)$

$GF\left(s\right) = (0, 1)$

$p_1\left(0,0,0\right), p_2\left(1,0,0\right), p_3\left(0,1,0\right), p_4\left(0,0,01\right), p_5\left(1,1,0\right), p_6\left(1,0,1,\right), p_7\left(0,1,1\right), p_8\left(1,1,1\right)$

Consider pencil $p\left(1, 0, 1\right)$

The flats of this pencil are:

$$X_1 + X_3 = 0\left(p_1, p_3, p_6, p_8\right)$$

$$X_1 + X_3 = 1\left(p_2, p_4, p_5, p_7\right)$$

because $s = 2$ there are only two flats.

A flats is not necessary a straight line.

Two pencils $p\left(a_1, a_2 \ldots a_m\right)$ and $p\left(a_1, a_2 \ldots a_m\right)$ are equal if

$$a_i = pa, a_i, a_i \in GF\left(s\right)$$

Hence in an m-dimensional Eudidean geometry total number of pencils

$$\frac{s^m - 1}{s - 1}$$

Example

for

$$s = 2, m = 3$$

$$\frac{s^m - 1}{s - 1} = \frac{2^3 - 1}{2 - 1} = 7$$

The seven pencils are:
$p_1\,(1,0,0)\,, p_2\,(0,1,0)\,, p_3\,(0,0,1)\,, p_4\,(1,1,0)\,, p_5\,(1,0,1)\,, p_6\,(0,1,1,)\,, p_7\,(1,1,1)$

10. Latin squares

- A latin square is an arrangement of s objects in an $s \times s$ square such that each object appears once and only once in each row and in each column eg

A	B	C
B	C	A
C	A	B

10.1. Graeco latin squares

Consider two latin squares,suppose the two squares have objects if different characteristics. If when superimposed on each other part pair of the objects appears oncce & only once in the superimposed square.

The two squares are said to be othergonal and the squares resulting from superimposing is called a Graeco latin square

A	B	C
B	C	A
C	A	B

X	Y	δ
δ	χ	Y
Y	δ	χ

Superimposing

$A\chi$	BY	$C\delta$
$B\delta$	$C\chi$	AY
CY	$A\delta$	$B\chi$

A glaecon latin square

- **Construction of latin squares**
  - If we have $s \times s$ squares, the max n of pthorgonal latin squares that can be formed/constructed is $s-1$. Denote the squares by $L_1, L_2 \ldots L_j \ldots L_{s-1}$
  - $S = P$ wher p is prime for latin square $L_j$the n which is entitled in the X^{th} and Y^{th} column is given by

$(jX + y)$ mod p

where

$X = 0, 1 \ldots s-1, y = 0, 1, \ldots s-1$

Suppose $s = p = 5$

y/X	0	1	$\ldots$	$\ldots$	$s-1$
0	0	1	2	$\ldots$	$s-1$
1	j	$j+1$	$j+2$	$\ldots$	$j+s-1$
2	$2j$	$2j+1$	$2j+2$	$\ldots$	$2j+s-1$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$s-1$	$j(s-1)$	$j(s-1)$+1	$j(s-1)+2$	$\ldots$	$j(s-1)+(s-1)$

Example

Suppose $p = 5$ and L_3 is to be constructed then:

$L_3 \Rightarrow$	0	1	2	3	4
0	0	1	2	3	4
1	3	4	0	1	2
2	1	2	3	4	0
3	4	0	1	2	3
4	2	3	4	0	1

Li

$L_1 \Rightarrow$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

For L_3

$0 - A$

$1 - 13$

$2 = C$

$3 = D$

$4 = E$

For L_1

$0 - v$

$1 = w$

$2 - X$

$3 - Y$
$4 - \delta$

	Age 1	Age 2	Age 3	Age 4	Age 5
Bread 1	Av	Bw	CX	DY	$E\delta$
Bread 2	Dw	EX	AY	$B\delta$	Cv
Bread 3	BX	$A\delta$	Bv	Cw	Aw
Bread 4	EY	$A\delta$	Bv	Cw	DX
Bread 5	$C\delta$	Dw	Ew	AX	BY

- **Analysis of latin square design**

Consider a latin square with S columns and S treatments columns & treatments

The model is

$$y_{ijk} = M + r_i + c_j + t_k + e_{ijk}$$

M is a constant common to all plots
r_i measure effect of row $a, i, i = u, 1 \ldots s-1]$
c_j measure effect of column $j, j = u, 1 \ldots s-1$
t_k measure effect of treat $k, k = 0, 1, \ldots, s-1$

$$e_{ijk} \aleph\aleph \left(o, \delta e^2\right)$$

The parameters $m, r_i, \subset_j \&\ t_k$ are estimated by minimizing

$$s^2 = \sum_i^s \sum_j^s \sum_k^s \left(y_{ijk} - m - r_i + c_j - t_k\right)^2$$

Which gives

$$\hat{m} = \frac{\sum\sum\sum y_{ijk}}{s^2} = \bar{y} = \frac{G}{s^2}$$

$$\hat{C}_j = \frac{\sum_j^s \sum_k^s yijk}{s} - \frac{G}{s^2} = \bar{y_j} - \bar{y}$$

$$\hat{t}k = \frac{\sum_i \sum_j yijk}{s} - \frac{G}{s^2} = \bar{y_k} - \bar{y}$$

The total sum of squares is partitimed as follows:
$s_T^2 = \sum_i \sum_j \sum_k \left(y_{ijk} - \bar{y}\right)^2$
$=\sum_i \sum_j \sum_k \left(\bar{y_i} - \bar{y} + \bar{y} + \bar{y_k} + y_{ijk} - y_i - y_j - y_k + 2\bar{y}\right)^2$
$= s\sum_i \left(\hat{y_i} - \bar{y}\right)^2 + s\sum_j \left(\bar{y_j} - \bar{y}\right)^2 + s\sum_k \left(\bar{y_k} - \bar{y}\right)^2$
$\sum\sum\sum \left(y_{ijk} - y_i - y_j - y_k + 2ij\right)^2$
$= s_r^2 + s_c^2 + s_t^2 + s_e^2$

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More
$s_r^2 = s\sum_i (\bar{y}_i - \bar{y})^2 = \sum_{i=1}^{s} \frac{R_i^2}{s} - \frac{G^2}{s^2}$
sum of squares due to rows and R_i is total observations in row i

$$s_t^2 = s\sum_j (\bar{y}_j - \bar{y})^2 = \sum_j \frac{C_j^2}{s} - \frac{G^2}{s^2}$$

the sum of squares due columns and C_j is total observations of column $j.(s-1)\, d.f$

$$s_t^2 = s\sum_k (\bar{y}_k - \bar{y})^2 = \sum_k^s \sum \frac{T_k^2}{s} - \frac{G^2}{s^2}$$

is the ss due to treatments and $T_k$is total observation of plots with treatment $k.\,(s-1)\, d.f$

$$s_e^2 = \sum\sum\sum (y_{ijk} - \bar{y}_i - \bar{y}_j - \bar{y}_k + 2_{ij})^2$$

is the ss due to error with $(s-2)\,(s-1)\, d.f$

Anova

Source	$d.f$	ss	$m.sf$	f
Rows	$s-1$	s_r^2	R	$F_1 = \frac{R}{E}$
Columns	$s-1$	s_c^2	C	$F_1 = \frac{C}{E}$
Treatments	$s-1$	s_t^2	T	$F_1 = \frac{T}{E}$
Error	$(s-1)\,(s-2)$	s_e^2	E	
Total	s^2-1	s_T^2		

- **Analysis of graeco-latin square Design**

Consider a Graeco latin square with S rows and S columns & treatments
The model is

$$y_{ijkL} = m + r_i + \subset_j + t_k + \alpha_L + e_{ijkL}$$

m is a constant common to all plots
r_i measures effect of row $i, i = o, 1, 2, \ldots, s-1$
$\subset_j$ measures effect of column $j, j = 0, 1, 2, \ldots, s-1$
t_k measures effect of treat $k, k = 0, 1, \ldots, s-1$
α_L measures effect of L^{th} treat in 2nd square
$L = 0, 1, 2, \ldots, s-1$
$e_{ijkL=N(0,\delta e^2)}$
The parameters $m, r_i, \subset_j, t_k, \alpha_i$ are estimated by minimising

$$s^2 = \sum_i^s \sum_j^s \sum_k^s \sum_L^s (y_{ijkL} - N - r_i - l_j - t_k - \alpha_l)^2$$

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Which gives

$$\hat{m} = \frac{\sum_i \sum_j \sum_k \sum_L y_{ijkL}}{s^2} = \bar{y} = \frac{G}{s^2}$$

$$\hat{r_i} = \frac{\sum_j \sum_k \sum_L y_{ijkl}}{s} - \frac{G}{s^2} = \bar{y_i} - \bar{y}$$

$$\hat{C_j} = \frac{\sum_i \sum_k \sum_L y_{ijkL}}{s} - \frac{G}{s^2} = \bar{y_i} - \bar{y}$$

$$\hat{t_k} = \frac{\sum_i \sum_s \sum_L y_{ijkL}}{s} - \frac{G}{s^2} = \bar{y_k} - \bar{y}$$

$$\hat{\alpha_l} = \frac{\sum_i \sum_j \sum_k y_{ijkL}}{s} - \frac{G}{s^2}$$

The total sum squares is partitioned as follow

$$s_T^2 = \sum_i \sum_j \sum_k \sum_l y_{ijkL} \left(y_{ijkL} - m - r_i - C_j - t_k - \alpha_L\right)^2$$

$$= \sum_i \sum_j \sum_k \sum_L \left(\bar{y_i} - \bar{y} + \bar{y_j} + \bar{y_k} - \bar{y} + y_L - \bar{y} - y_{ijkL} - y_i - y_j - y_k - y_\imath + 3\bar{y}\right)^2$$

$$s \sum_i \left(\bar{y_i} - \bar{y}\right)^2 + s \sum_j \left(\bar{y_j} - \bar{y}\right)^2 + s \sum_k \left(\bar{y_k} - \bar{y}\right)^2 + s \sum_L \left(\bar{y_L} - \bar{y}\right)^2 + \sum \sum \sum \sum \left(y_{ijkl} - y_i - y_j -\right.$$

$$= s_r^2 + s_c^2 + s_{t1}^2 + s_{t2}^2 + s_e^2$$

Where

$$s_r^2 = s \sum_i \left(\bar{y_i} - \bar{y}\right)^2 = \sum_{i=1}^{n} \frac{Ri^2}{s} - \frac{G^2}{s^2}$$

is the sum of squares due to rows and R_i is total observations in row i

$$s_C^2 = s \sum_j \left(\bar{y_j} - \bar{y}\right)^2 = \sum_j \frac{C_j^2}{s} - \frac{G^2}{s^2}$$

The sum of squares due to columns and C_j is the total observation of column j $(s-1)\,d.f$

$$s_{t1}^2 = s \sum_k \left(\bar{y_k} - \bar{y}\right)^2 = \sum_k^{s} \frac{T_k^2}{s} - \frac{G^2}{s^2}\,(s-1)\,d.f$$

the ss due to treatment $.T_k$ is total observation of plots with treatment

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$k\,(s-1)\,d.f$

$$s^2_{t2} = s\sum(\bar{y}_i - \bar{y})^2 = \sum \frac{T_L^2}{s} - \frac{G^2}{s^2}$$

the ss due to treatment of the 2nd latin squares with $(s-1)\,d.f$ and T_L is total of all observations with i^{th} treatment in 2nd latin square

$$s^2_e = \sum\sum\sum\sum(y_{ijkL} - \bar{y_i} - \bar{y_j} - \bar{y_k} - y_L \bar{+} 3\bar{y})^2$$

is ss due to error with $(s-1)(s-3)\,d.f$

Anova

Source	$d.f$	ss	$m.sf$	f
Rows	$s-1$	s^2_T	R	$F_1 = \frac{R}{E}$
Columns	$s-1$	s^2_c	C	$F_2 = \frac{C}{E}$
Treatments sq 1	$s-1$	s^2_{t1}	T_1	$F_3 = \frac{T_1}{E}$
Treatments sq 2	$(s-1)$	s^2_{t2}	T_2	$F_4 = \frac{T_2}{E}$
Error	$(s-1)\,(s-3)$	s^2_e	E	
Total	s^2-1	s^2_T		

EXERCISE 2.

Problem

Construct the Anova for the following

	C_1	C_2	C_3	C_4	C_5
R_1	$B\,(19)$	$E\,(22)$	$A\,(18)$	$D\,(14)$	$C\,(14)$
R_2	$D\,(16)$	$B\,(19)$	$C\,(16)$	$A\,(19)$	$E\,(18)$
R_3	$A\,(21)$	$D\,(16)$	$E\,(19)$	$C\,(16)$	$B\,(14)$
R_4	$E\,(23)$	$C\,(19)$	$E\,(15)$	$B\,(17)$	$A\,(16)$
R_5	$C\,(21)$	$A\,(18)$	$B\,(22)$	$E\,(15)$	$D\,(18)$

EXERCISE 3.

Problem

Construct the Anova for the following

	C_1	C_2	C_3	C_4	C_5
R_1	$Av\,(19)$	$Aw\,(22)$	$AX\,(18)$	$Ay14$	$A\delta\,(14)$
R_2	$Dw\,(16)$	$EX\,(19)$	$Ay\,(16)$	$B\delta\,(18)$	$Cv\,(18)$
R_3	$BX\,(21)$	$Cy\,(16)$	$D\delta\,(19)$	$Ev\,(16)$	$Aw\,(14)$
R_4	$Ey\,(23)$	$A\delta\,(19)$	$Bv\,(15)$	$Cw\,(17)$	$DX\,(16)$
R_5	$C\delta\,(21)$	$Dv\,(14)$	$Ew\,(22)$	$AX\,(15)$	$By\,(18)$

$Av\,(19) \Rightarrow A$

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11. Split plot design

Here we have$b$ blocks v main treatments. In the 5th main plots of block c there are n subplots and hence the observation of the k^{th} subplot in j^{th} main plot of block i.

Since covariance $\neq$ 0 least squares can only be applied if observations in a main plot are transformed inti independent variibles for instance the observations in j^{th} main plot of block $i$is

$y_{ij} = \begin{pmatrix} y_{ij1} \\ y_{ij2} \\ y_{ijn} \end{pmatrix}$

The transformation is done as follows:
let

$$y_{ij} = \begin{pmatrix} \frac{1}{1n} & d_{11} & d_{12} & \dots & d, n-1 \\ \frac{1}{5n} & d_{21} & d_{22} & \vdots & d_2, n-1 \\ \vdots & \vdots & \vdots & \vdots & \dots \\ \frac{1}{5n} & dn_1 & dn_2 & \vdots & dn, n-1 \end{pmatrix} \begin{Bmatrix} u_{ij} \\ \delta_{ijl} \\ \delta_{ijz} \\ \delta_{ijn-1} \end{Bmatrix}$$

or

$$y_{ij} = D \begin{pmatrix} u_{ij} \\ \delta_{ij} \end{pmatrix}, \delta_{ij}^1 = (\delta_{ijl}, \delta_{ijzl}, \dots, \delta_{ijn-1})$$

Where

$$\sum_{k=1}^{n} d_k\lambda = 0, \lambda = 1, 2, \dots, n-1$$

$$\sum_{k=1}^{n} d_k\lambda^1 = 0, \lambda \mp \lambda^1 = 1, 2, \dots, n-1$$

$$\sum_{k}^{n} d_k\lambda = 1$$

this impies that $D^{-1} = D^T$

$$\Rightarrow D^T y_{ij} = \begin{pmatrix} u_{ij} \\ \delta_{ij} \end{pmatrix}$$

The new variables are:

$$u_{ij} = \frac{1}{5n} \sum_{k}^{n} y_{ijk} \sqrt[2]{n\bar{y}}_{ij}$$

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$$\delta_{ij\lambda} = \sum_k dk\lambda y_{ijk}$$

$$v\begin{pmatrix} u_{ij} \\ \delta_{ij} \end{pmatrix} = D^T v\left(y_{ij}\right) D$$

$$= \begin{pmatrix} \frac{1}{\sqrt{n}} & \frac{1}{\sqrt{n}} & \cdots & \frac{1}{\sqrt{n}} \\ d_{11} & d_{21} & \cdots & dn_1 \\ d_{12} & d_{22} & \cdots & dn_2 \\ \cdots & \cdots & \cdots & \cdots \\ d_1, n-1 & d_2, n-1 & \cdots & dnn-1 \end{pmatrix} \begin{pmatrix} \delta e^2 & p\delta e^2 & \cdots & p\delta e^2 \\ p\delta e^2 & \delta e^2 & \cdots & p\delta e^2 \\ \cdots & \cdots & \cdots & \cdots \\ p\delta e^2 & p\delta e^2 & \cdots & \delta e^2 \end{pmatrix}$$

$$\begin{pmatrix} \frac{1}{5n} & d_{11} & din-1 \\ \cdots & \cdots & \cdots \\ \frac{1}{5n} & dn_1 & dnn-1 \end{pmatrix}$$

$$= \begin{pmatrix} (1+(n-1)\,p)\,\delta e^2 & 0 & 0 \\ 0 & (1+(n-1)\,p)\,\delta e^2 & 0\cdots 0 \\ 1 & \cdots & \cdots \\ 1 & \cdots & \cdots \\ 0 & \cdots & (1+(n-1)\,p)\,\delta e^2 \end{pmatrix}$$

The new variables are independent

$$v\left(uij\right) = \delta e^2\left(1+(n-1)\,p\right)$$

and

$$v\left(\delta_{ij\lambda}\right) = \delta e^2\left(1-p\right)$$

On using least squares estimation we get

$$\hat{m} = \frac{\sum\sum\sum y_{ijk}}{vbn} = iJ$$

$$\alpha_i = \hat{\bar{y}}_i - \bar{y}\hat{B}_j = \bar{y}_j - \bar{y}, \hat{t_k} = \bar{y_k} - \bar{y}$$

$$0\bar{_{jk}} = \bar{y}_i - \bar{y}_j - y_k \bar{+} \bar{y}$$

The sum of squares are:

$$s_b^2 = vn\sum\left(\bar{y}-\bar{y}\right)^2$$

s^2mainplot=bn$\sum_j (y - y)\,2$

$$s^2_{e_1} = n \sum_i \sum_j (\bar{y}_{ij} - \bar{y}_i + \bar{y}_J \ + \bar{y})^2$$

S^2 subplots=$bv\sum_j (\bar{y}_j - \bar{y})^2$
s^2interaction=$b\sum_j \sum_k (\bar{y}_{jk}$-$\bar{y}_j - \bar{y}_k$+$\bar{y})^2$

$$Se_2^2 = \sum_i \sum_j \sum_k \left(y_{y_k} - y_{j_k} + y_j \right) 2$$

$$s_{T^2} = \sum \sum \sum y^2_{ijk} - \frac{G^2}{vbn}$$

12. Factorial design

General factorial design

In factorial designs we consider factors say:M of them denoted by f_1,f_2,$f\,\mathrm{m}$,Where the ? factor ran over at Fiin different levels.

A level could be say a weight or volume of that factor or substance.A treatment is a combination of levels, a level from each factor,hence the total number of possibletreatments

$$v = \pi^m_{i=1} ni$$

As on an illussion suppose one have two factors which have an effort in living a certain disase .Suppose factor one can be applied at 1,2, 3,4, micrograms i.e

F_1	F_2	F_3	F_4
1	1	1	1
2	2	2	2
3		3	3
4		4	
		5	
		6	

Then the $f.f$ number of possible treatments is $4 * 2 * 6 * 3 = 144$,consider

F_1	F_2	F_3	F_4
0	0	0	0
1	1	1	1
2	2	2	2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n_1 - 1$	$n_2 - 1$	$n_3 - 1$	$n_4 - 1$

$$v = n_1 n_2 n_3 n_4$$

Denote a treatment by $a^i b^j c^k d^l$,Where
$i = 0, 1, 2, \ldots, n_1 - 1$
$j = 0, 1, 2 \ldots, n_2 - 1$
$k = 0, 1, 2, \ldots n_3 - 1$
$l = 0, 1, 2, \ldots n_4 - 1$

12.1. Treatment contrast

A treatment contrast is given by:$\sum_i \sum_j \sum_k Z_L P_{ijkl}$where$\sum_i \sum_j \sum_k \sum_l p_{ijkl} = 0$

A treatment contrast is said 2 b systemic - certain factors if the coefficients $p_{ijkl}$are independent of those particular factors e.g
$\sum \sum \sum \sum p_{ijkl} a^i b^j c^k d^l$
$\sum \sum \sum \sum p_{ijkl} a^i b^j c^k d^l$ then the format contrast is symetric with respect to F_2, F_3, F_4
Let
$Q_{a=\left(a^0+a^1+\ldots+a^{n_1-1}\right)}$
Q_b=$(b^0 + b^1 f - - - b^{n_2-1})$
Q_C=$(c^0 + c^1 + \ldots + c^{n_3-1})$
Q_d=$(d^0 + d^1 + \ldots + d^{n_4-1})$
then:
$\sum_i \sum_j \sum_k \sum_l a^i b^j c^k d^l = \sum_i p_i a_i Q_b Q_c Q d$
Let G be sum of all treatments,then $G = \sum \sum \sum \sum a^i b^j c^k d^l = \sum \sum \sum \sum a^i b^j c^k d^l = \sum_i a \sum_j b \sum_k c \sum_l d^l = QaQbQcQd$

- **Sysmmeric Factorial Designs**

In symmetric factorial designs there are m factors - $F_1, F_2, F_m$where each occurs at s different levels.

F_1	F_2	F_m
0	0	0
1	0	1
2	2	2
$\vdots$	$\vdots$	$\vdots$
$s-1$	$s-1$	$s-1$

Each facor has$s - 1$ d.f,there are $\begin{pmatrix} m \\ 1 \end{pmatrix}$factors,=$m$

Any two factors have $(s-1)(s-1)$=$(s-1)^2$d.f There are $\left(\frac{m}{2}\right)^2$factors interactions,For 3 factors we have $(S-1)^3$with $\begin{pmatrix} m \\ c \\ 3 \end{pmatrix}$three factors interaction.

This gives:
$\begin{pmatrix} m \\ 1 \end{pmatrix}(s-1)+\begin{pmatrix} m \\ 2 \end{pmatrix}(s-1)^2+.....+\begin{pmatrix} m \\ m \end{pmatrix}(s-1)^m=\sum_{i=1}^{m}\begin{pmatrix} m \\ i \end{pmatrix}(s-i)^i-1$

$=\sum_{i=1}^{m}\begin{pmatrix} m \\ i \end{pmatrix}(s-i)^i 1^i - 1 = (s-1+1)^m - 1$

s^m-1 degree of freedom.

12.2. CORRESPONDENCE WITH EG(m,s)

Suppose there are m factors,$F_{1,}F_2,\ldots F_m$each having s levels,$0,1,......s-1$ where s is prime.A treatment can be denoted by:

$(x_1, x_2, \ldots x_m)$,(Where $x_i = u, i, \ldots, s-1$ or $X_i \in GF(s)$

Thus obviously,a pt will be equivalent to a treatment.Also there are S^mpoints in $EG(m,s)$.Now consider a pencil:

A pencil will be denied by:

p$(a_{1,}a_2,\ldots,a_m)$,ai,$\in GF(s)$not all Zero.

A flat in this pencil will be of the form.

$a_1x_1 + a_2x_2 + a_mx_m = \alpha_i, \alpha_i \in GF(s)$.

There are s flats.

$a_1x_1 + \ldots + a_mx_m = \alpha_0$

$a_1x_1 + +a_mx_m = \alpha_1$

$a_1x_1 + +a_mx_m = \alpha_{s-1}$

Each flat carries s^{m-1}treatments. it is called an $m-1$ flat.

Remember that two pencils $p(a_1, a_2 \ldots, a_m)$ and $P(pa_1, pa_2, \ldots pa_m)$ are equal for $p \in GF(s)$. Hence the f.f. number of pencils is $\frac{S^m-1}{s-1}$.Remember that any two flats of a given pencil have no common pt, but a pair of flat where each flat comes from a different pencil share s^{m-2}treatment. Such a pair is said to form an m-2 flat.

example

Suppose $s=2, m=3$

treatments=8

t_1	(0,	0,	0)		p_1	(1,	0	0)
t_2	(1,	0,	0)		p_2	(0	1	0)
t_3	(0,	1,	1)		p_3	(0,	0,	1)
t_4	(0,	0,	1)		p_4	(1,	1,	0)
t_5	(1,	1,	0)		p_5	(1,	0,	1)
t_6	(1,	0,	1)		p_6	(0,	1,	1)
t_7	(0,	1,	1)		p_7	(1,	1,	1)
t_8	(1,	1,	1)					

flats of pencil $p(1,0,0)$

$a_1x_1 = \alpha_j, \alpha_j \in GF(s) = GF(2) = (0,1), a_1 = 1$ or $x_1 = 0$ (m-1) flat.$(3-1)$ flat s^{m-1} treatment $s_2(t_1, t_3, t_4, t_7)$.

$x = 1(m-1)$flat $(t2, t5, t6, t8)$flats of pencil $p(1,0,1)$

$$x_1 + x_3 = 0,\ (t_1, t_3, t_6, t_8)$$

$$x_1 + x_3 = 1,\ (t_2, t_4, t_5, t_7)$$

Two flats one from p(1,0,0) and one from p(1,0,1)

$\left\{ \begin{array}{c} x_1 = 1 \\ x_1 + x_3 = 0 \end{array} \right\}(m-2)$ flat with $s^{m-1} = s^{m-2}$ treatments.

$(t1, t3)$

There are 2X2=4 such pairs,

$\left\{ \begin{array}{c} x_1 = 0 \\ x_1 + x_3 = 0 \end{array} \right\}, \left\{ \begin{array}{c} x_1 = 1 \\ x_1 + x_3 = 0 \end{array} \right\} \left\{ \begin{array}{c} x_1 = 0 \\ x_1 + x_3 = 1 \end{array} \right\} \left\{ \begin{array}{c} x_1 = 1 \\ x_1 + x_3 = 1 \end{array} \right\}$

each with two treatments.

Theorem

Treatment contrasts belonging for different pencils are orthogonal.

Proof

$p(a_1, a_2, \ldots, a_m), p(b_1, b_2, \ldots, b_m)$denote the flats of those two pencils as $\sum_0, \sum_1, \sum \ldots$and $\sum_0, \sum_1, \sum_{s-1}$ let $(\sum_i)$and$(\sum_i)$be the sums of treatments in those flats. Then a treatment contrast of $p(a)$ therefore $(\sum_i) - \left(\sum_j\right)$ and that of $p(b)$: $\left(\sum_i^{'}\right) - \left(\sum_j^{'}\right)$.

Recall that t.contrasts are orthogonal if the sum of products of coefficient is zero . We note that $\sum_i$and $\sum_i^{'}$have sm-2 common treatment. Hence the sum of products of coefficient from both flats is

$$\left[\left(\sum_i\right) - \sum_j\right]\left[\left(\sum_i^{'}\right) - \left(\sum_j^{'}\right)\right] = 1{-}sm{-}2{-}1.sm{-}2{-}1.sm{-}2{+}1.sm{-}2 = 0$$

e.g n example,flats of pencil p(1,0,0)

$$(, t_1 t_3, t_4, t_7) \sum_0$$

$$(t_2, t_3, t_6, t_8)\left(\sum_1\right)$$

flats pencil $p(1,0,1)$

$$(t_1, t_3, t_6, t_8) \sum_0$$

$$(t_2, t_4, t_5, t_7) \sum_{1}$$

$$\left(\sum 0\right) - \sum 1 = t_1 - t_2 + t_3 + t_4 - t_5 - t_6 + t_7 - t_8$$

$$\left(\sum_{0}^{1}\right) - \left(\sum_{1}^{1}\right) = t_1 - t_2 + t_3 - t_4 - t_5 + t_6 - t_7 + t_8$$

Theorem

Contrast of main effects and 2 fact int t.contrast are always orthogonal.

Proof

a t.c. is of this form:-$\sum_i \sum_j \sum_k \sum_l p_{ijkl} a^i b^j c^k d^l = 0 \Rightarrow \sum_i \sum_j \sum_k \sum_l p_{ijkl} = 0$ main effects $\sum_i \sum_j \sum_k \sum_l p_{ijkl} = \sum p_i a^1$but $\sum_i p_i = 0$.since its at c

also

$$\sum_i \sum_j p_j Q_c Q_d$$

$$\Rightarrow \sum \sum p_{ij} = 0, \sum p_{ij} = 0$$

hence $\sum_i p_i Q_b Q_c Q_d$-main effect.$\sum_i \sum_j p_{ij} Q_c Q_d$-two factor one.
orthogonal because $\sum_i \sum_j p_i p_{ij} = \sum_i p_i \sum_i p_{ij} = \sum_i p_i o = 0$

12.3. Confounded factorial designs.

Consider a design with m factors, where each factor has got s levers (s^m treatments) sometimes the number of treatments sm is so large , such that when they are placed in a block, the block looses homogeinity under those circumstances, it is advisable to divide the treatment into groups, where a group is given to a block.

The design which results from this is referred to as a confounded factorial design.

This is due to the fact that some treatment contrasts the same as block contrasts and hence they cannot be estimated.

Usually we use the flats of a pencil to group the treatments e.g pencil a $p(a)$ having s flats , each with s^{m-1} treatments.

12.4. Fractional factorial designs

Sometimes even when to confound with s^k blocks , each with s^{k-m} treatments, then number sm is usually quite large.

Under this circumstances, some treatments are simply dropped from the design. When they are dropped, we remain with a fraction of the t.t, number of treatment and the design is referred to as fractional factorial design.

suppose we take a fraction $\frac{1}{s}k$ fraction = number of treatments in the design si s^{m-k}.

Those treatments can be obtained from an $m-k$ flat of pencils $p(a_1), p(a_2), \ldots p(a_k)$.

$$a_{11}x_1 + a_{12}x_2 + +a_{1m}x_m = \alpha_{i1}$$

$$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m = a_{i2}$$

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{km}x_m = \alpha_{ik}$$

when a_{ij} is a constant.

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Solutions to Exercises

Exercise 1. v=4, b=3, G=134 $s_t^2 = \frac{1}{3}(10^2 + 20^2 + 50^2 + 44^2) - \frac{134^2}{12} = 315.67$ $s_b^2 = \frac{1}{4}(42^2 + 42^2 + 50) - \frac{134^2}{12} = 10.67$

source	df	ss	mss	F
Treatments	3	315.67	105.22	11.83
Blocks	2	10.67	5.33	0.5996
Error	6	53.33	8.89	
Total	11	379.67		

$$F_{0.95(3,6)} = 4.76$$

,

$$F_{0.95(2,6)} = 5.14$$

Exercise 1

Exercise 2. Totals $R_i = 87$ $R_2 = 87$ $R_3 = 86$ $R_4 = 90$ $R_5 = 90$ $G = 440$ $\subset_1 = 100$, $\subset_2 = 90$, $\subset_3 = 90$, $\subset_4 = 80$, $\subset_5 = 80$, $G = 440$ Total treatments $T_A = 87$, $T_B = 91$, $T_\subset = 79$, $T_\in = 97$ $G = 440$ $s_r^2 = \frac{87^2}{5} + \frac{87^2}{5} + \frac{86^2}{5} + \frac{90^2}{5} + \frac{90^2}{5}$-$\frac{440^2}{5^2} = 9208.4$ $S_c^2 = \frac{100^2}{5} + \frac{90^2}{5} + \frac{90^2}{5} + \frac{80^2}{5} + \frac{80^2}{5} - \frac{440^2}{5} = 7782.4$

Exercise 2

Exercise 3. check from the above exercise Exercise 3