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# Time Series Analysis

## Fundamental Concepts

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# Contents

# Contents

Fundamental Concepts

# Time Series and Stochastic Processes

# Time series and stochastic processes

**Motivation:** Model time series by a family of random variables.

**Definition:** A *stochastic process* $\{Y_t\}$ is a family of random variables indexed with $t \in \mathcal{T}$.

- Discrete time: $\mathcal{T} = \mathbb{N}$ or $\mathcal{T} = \mathbb{Z}$.
- Continuous time: $\mathcal{T} = [0, \infty)$ or $\mathcal{T} = \mathbb{R}$.

**Here:** Focus on discrete time.

**Distribution:** The complete probabilistic structure of $\{Y_t\}$ is determined by the joint distribution of all finite sets of $Y$'s.

**Of most interest:** First two moments, i.e., means and (co)variances. (In case of *normal* $Y$'s, this determines the full distribution.)

Fundamental Concepts

# Means, Variances, and Covariances

# Means, variances, and covariances

**Definition:** Let $\{Y_t\}$ be a stochastic process with $\mathrm{Var}(Y_t) < \infty$ for all $t$. Then, the *mean function* is defined for $t \in \mathbb{Z}$ as

$$\mu_t \;=\; \mathrm{E}(Y_t).$$

The *autocovariance function* is defined $t, s \in \mathbb{Z}$ as

$$\begin{aligned}
\gamma_{t,s} &= \mathrm{Cov}(Y_t, Y_s) \\
&= \mathrm{E}[(Y_t - \mu_t)\,(Y_s - \mu_s)] \;=\; \mathrm{E}(Y_t Y_s) \;-\; \mu_t \mu_s.
\end{aligned}$$

and the *autocorrelation function* (ACF) as

$$\begin{aligned}
\varrho_{t,s} &= \mathrm{Cor}(Y_t, Y_s) \\
&= \frac{\mathrm{Cov}(Y_t, Y_s)}{\sqrt{\mathrm{Cov}(Y_t)\mathrm{Cov}(Y_s)}} \;=\; \frac{\gamma_{t,s}}{\sqrt{\gamma_t \gamma_s}}.
\end{aligned}$$

# Means, variances, and covariances

**Recall:**

$$
\begin{aligned}
\mathrm{E}(a\,X + b\,Y\ +\ c) &= a\,\mathrm{E}(X) + b\,\mathrm{E}(Y) + c,\\
\mathrm{Var}(a\,X\ +\ b\,Y\ +\ c) &= a^2\,\mathrm{Var}(X) + b^2\,\mathrm{Var}(Y) + 2ab\,\mathrm{Cov}(X,Y),\\
\mathrm{Cov}(a\,X\ +\ b\,Y\ +\ c,\ Z) &= a\,\mathrm{Cov}(X,Z) + b\,\mathrm{Cov}(Y,Z),\\
\mathrm{Cor}(a\,X\ +\ b,\ c\,Y\ +\ d) &= \mathrm{sign}(ac)\,\mathrm{Cor}(X,Y).
\end{aligned}
$$

**Thus:**

$$
\begin{aligned}
\gamma_{t,t} &= \mathrm{Var}(Y_t) & \varrho_{t,t} &= 1,\\
\gamma_{t,s} &= \gamma_{s,t} & \varrho_{t,s} &= \varrho_{s,t},\\
|\gamma_{t,s}| &\le \sqrt{\gamma_{t,t}\gamma_{s,s}} & |\varrho_{t,s}| &\le \sqrt{\varrho_{t,t}\varrho_{s,s}} = 1.
\end{aligned}
$$

# Means, variances, and covariances

**Also:**

$$\text{Cov}\left(\sum_{i=1}^{m} c_i\, Y_{t_i}, \sum_{j=1}^{n} d_j\, Y_{s_j}\right) \;=\; \sum_{i=1}^{m}\sum_{j=1}^{n} c_i d_j\, \text{Cov}(Y_{t_i}, Y_{s_j}),$$

$$\text{Var}\left(\sum_{i=1}^{m} c_i\, Y_{t_i}\right) \;=\; \sum_{i=1}^{m} c_i^2\, \text{Var}(Y_{t_i}) \;+\; 2\,\sum_{i=2}^{m}\sum_{j=1}^{i-1} c_i c_j\, \text{Cov}(Y_{t_i}, Y_{t_j}).$$

# Random walk

**Definition:** Let $\{e_t\}$ with $t \in \mathbb{N}$ be a sequence of independent identically distributed (i.i.d.) random variables with zero mean and variance $\sigma_e^2$. A *random walk* $\{Y_t\}$ is defined as

$$\begin{aligned} Y_t &= \sum_{j=1}^{t} e_j \\ &= Y_{t-1} + e_t, \end{aligned}$$

with initialization $Y_0 = 0$.

**Properties:**

$$\begin{aligned} \mu_t = \mathrm{E}(Y_t) &= \mathrm{E}(e_1 + \ldots + e_t) = 0, \\ \mathrm{Var}(Y_t) &= \mathrm{Var}(e_1 + \ldots + e_t) = t\,\sigma_e^2. \end{aligned}$$

# Random walk

**Furthermore:** For $1 \leq t \leq s$

$$\begin{aligned}\gamma_{t,s} &= \mathrm{Cov}(e_1 + \cdots + e_t, e_1 + \cdots + e_t + e_{t+1} + \cdots + e_s) \\ &= \sum_{i=1}^{s} \sum_{j=1}^{t} \mathrm{Cov}(e_i, e_j) \\ &= t \cdot \sigma_e^2.\end{aligned}$$

Thus,

$$\varrho_{t,s} = \frac{\gamma_{t,s}}{\sqrt{\gamma_{t,t}\gamma_{s,s}}} = \sqrt{\frac{t}{s}}.$$

**Hence:** Values of $Y$ at neighboring time points are more and more strongly and positively correlated as time goes by. Values of $Y$ at distant time points are less and less correlated.

# Random walk

**Remarks:**

- Mean is zero for all time points.
- Nevertheless, due to increasing variance and correlation approaching 1, long excursions away from zero.
- Basic simple approximation to movement of common stock price, or position of small particles in fluid ("Brownian motion").

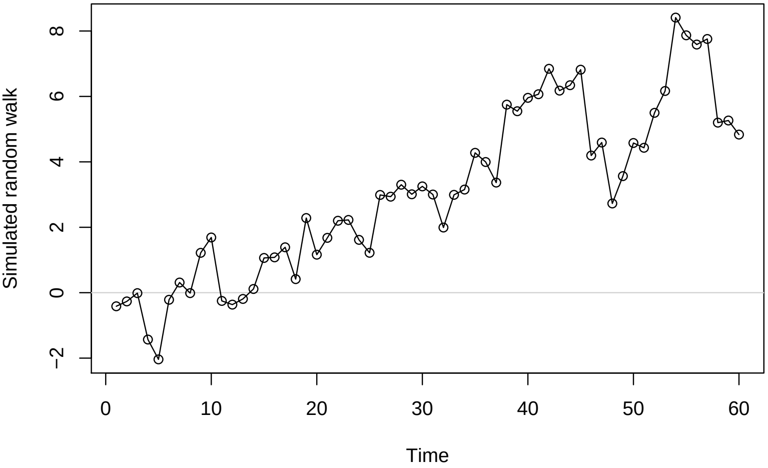
**Illustration:** Straightforward simulation in R.

```
R> e <- rnorm(60)
R> y <- cumsum(c(0, e))
R> y <- ts(y, start = 0)
R> plot(y, type = "o")
```

For exact replication use `set.seed()` prior to simulation.

**Example:** Simulated random walk `rwalk` from *TSA*.

# Random walk



# Moving average

**Definition:** Let $\{e_t\}$ be a sequence of independent identically distributed (i.i.d.) random variables with zero mean and variance $\sigma_e^2$. A *moving average* $\{Y_t\}$ (of order 1, with equal weights) is defined as

$$Y_t = \frac{e_t + e_{t-1}}{2}.$$

**Properties:**

$$\begin{aligned}
\mu_t = \mathrm{E}(Y_t) &= \mathrm{E}\left(\frac{e_t + e_{t-1}}{2}\right) = 0, \\
\mathrm{Var}(Y_t) &= \mathrm{Var}\left(\frac{e_t + e_{t-1}}{2}\right) \\
&= \frac{\mathrm{Var}(e_t) + \mathrm{Var}(e_{t-1})}{4} = 0.5\,\sigma_e^2.
\end{aligned}$$

# Moving average

**Furthermore:**

$$\begin{aligned}\mathrm{Cov}(Y_t, Y_{t-1}) &= \mathrm{Cov}\left(\frac{e_t + e_{t-1}}{2}, \frac{e_{t-1} + e_{t-2}}{2}\right), \\ &= \frac{\mathrm{Cov}(e_{t-1}, e_{t-1})}{4} = 0.25\ \sigma_e^2.\end{aligned}$$

All $\mathrm{Cov}(Y_t, Y_{t-k})$ with $k > 1$ are zero.

**Thus:**

$$\gamma_{t,s} = \begin{cases} 0.50\ \sigma_e^2 & \text{for } |t-s| = 0 \\ 0.25\ \sigma_e^2 & \text{for } |t-s| = 1 \\ 0 & \text{for } |t-s| > 1 \end{cases}$$

# Moving average

**Similarly:**

$$\varrho_{t,s} = \begin{cases} 1 & \text{for } |t-s| = 0 \\ 0.5 & \text{for } |t-s| = 1 \\ 0 & \text{for } |t-s| > 1 \end{cases}$$

**Interpretation:**

- Values of $Y$ precisely one time unit apart have exactly the same correlation no matter where they occur in time.
- More generally, $\varrho_{t,t-k}$ is the same for all values of $t$.

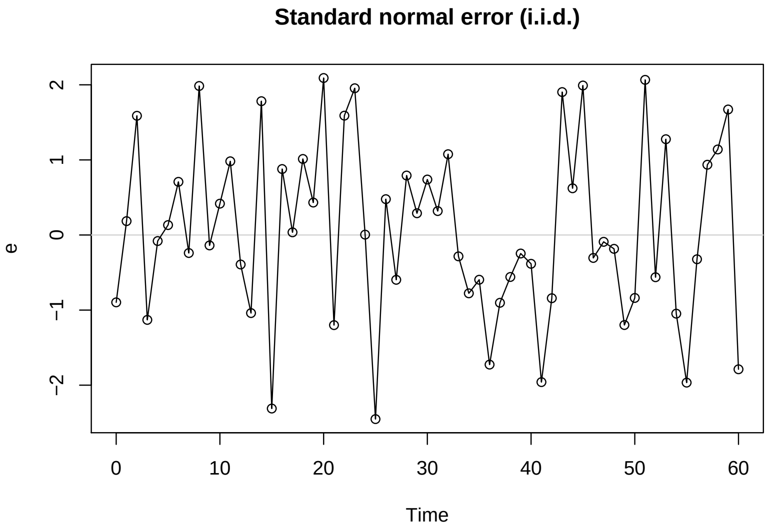
**Illustration:** Simulation in R.

```
R> e <- rnorm(61)
R> y <- rep(0, 60)
R> for(i in 1:60) y[i] <- 0.5 * e[i] + 0.5 * e[i+1]
R> y <- ts(y, start = 1)
```

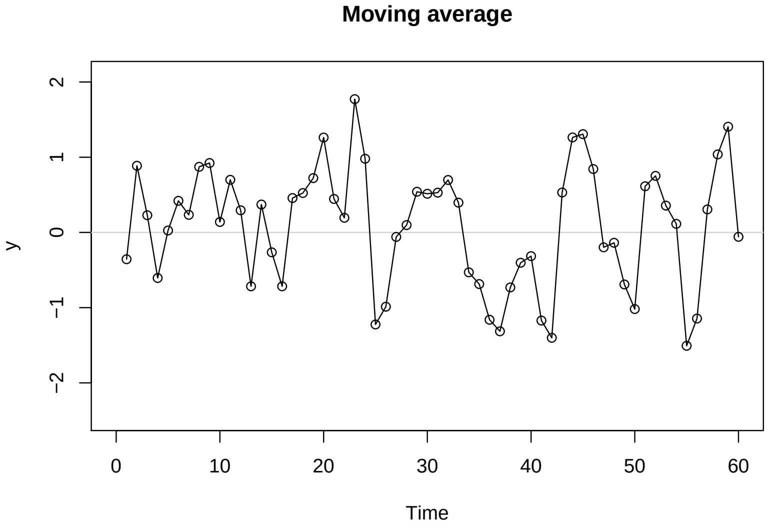
Or in a single line using `filter()`.

```
R> y <- filter(rnorm(61), c(0.5, 0.5))
```

# Moving average



# Moving average



Fundamental Concepts

# Stationarity

# Stationarity

**Motivation:** Need reasonable simplifying assumptions about stochastic processes for making statistical inference.

**Idea:** The probability properties of a process (or at least important aspects of it) remain constant over time.

**Definition:** A stochastic process is *strictly stationary* if

$$(Y_{t_1}, \ldots, Y_{t_n}) \overset{d}{=} (Y_{t_1-k}, \ldots, Y_{t_n-k})$$

for all $(t_1, \ldots, t_n) \in \mathbb{Z}^n$ and all lags $k \in \mathbb{Z}$.

**Hence:**

- For $n = 1$, this yields that $Y_t$ and $Y_{t-k}$ have the same marginal distribution for all $k$. In particular, mean $\mu_t = \mu_0$ and variance $\gamma_{t,t} = \gamma_{0,0}$ are constant over time.
- Setting $n = 2$, this implies that $\mathrm{Cov}(Y_t, Y_s) = \mathrm{Cov}(Y_{t-k}, Y_{s-k})$.

# Stationarity

For $k = s$ and $k = t$, respectively,

$$\begin{aligned}\gamma_{t,s} &= \mathrm{Cov}(Y_{t-s}, Y_0) \\ &= \mathrm{Cov}(Y_0, Y_{s-t}) \\ &= \mathrm{Cov}(Y_0, Y_{|t-s|}) = \gamma_{0,|t-s|}\end{aligned}$$

i.e., covariance only depends on time lag between $t$ and $s$.

**Notation:** Can be simplified for stationary processes.

$$\begin{aligned}\gamma_k &= \mathrm{Cov}(Y_t, Y_{t-k}), \\ \varrho_k &= \mathrm{Cor}(Y_t, Y_{t-k}) = \frac{\gamma_k}{\gamma_0}.\end{aligned}$$

And thus

$$\begin{aligned}\gamma_0 &= \mathrm{Var}(Y_t), & \varrho_0 &= 1, \\ \gamma_k &= \gamma_{-k}, & \varrho_k &= \varrho_{-k}, \\ |\gamma_k| &\leq \gamma_0, & |\varrho_k| &\leq 1.\end{aligned}$$

# Stationarity

**Definition:** A stochastic process is *(weakly) stationary* if

- The mean function is constant over time: $\mu_t = \mu_0$.
- $\gamma_{t,t-k} = \gamma_{0,k}$ for all $t$ and $k$.

**Remarks:**

- Stationarity typically refers to weak stationarity.
- For processes defined by multivariate normal distributions, weak and strict stationarity coincide.
- For stationary processes, typically consider $\gamma_k$ and $\varrho_k$ for $k \geq 0$.

# Stationarity

**Definition:** A stochastic process $\{e_t\}$ is called (weak) *white noise* if expectation and variance are constant
$\mathrm{E}(e_t) = \mu_t = \mu_0$, $\mathrm{Var}(e_t) = \sigma_e^2$ for all $t$ and
$\mathrm{Cov}(e_t, e_{t-k}) = \gamma_k = 0$ for all $k \neq 0$.

**Remark:**

- If the $e_t$ are also i.i.d., then called strong white noise.
- Name is in analogy to white light where all frequencies enter equally.
- Typically assume that $\mu_t = 0$.
- Basic building block for other stochastic processes (e.g., random walk or moving average etc.)