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Time Series Analysis

Models for Stationary Time Series

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Contents

Contents

Models for Stationary Time Series

General Linear Processes

General linear processes

Remarks: In the following

- $\{Y_t\}$ – observed series,
- $\{e_t\}$ – unobserved white noise, mean 0 and variance σ_e^2,
- $\{e_t\}$ – i.i.d. assumed for simplicity (although most results also hold for weakly stationary white noise).

Definition: A *general linear process* $\{Y_t\}$ is a weighted linear combination of present and past white noise terms

$$\begin{aligned} Y_t &= e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \dots \\ &= \sum_{i=0}^{\infty} \psi_i e_{t-i}, \end{aligned}$$

where $\psi_0 = 1$ and

$$\sum_{i=0}^{\infty} \psi_i^2 < \infty.$$

General linear processes

Example: $\psi_i = \phi^i$ with $-1 < \phi < 1$. Then

$$
\begin{aligned}
Y_t &= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots \\
\mathrm{E}(Y_t) &= \mathrm{E}(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots) = 0 \\
\mathrm{Var}(Y_t) &= \mathrm{Var}(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \dots) \\
&= \mathrm{Var}(e_t) + \phi^2 \mathrm{Var}(e_{t-1}) + \phi^4 \mathrm{Var}(e_{t-2}) + \dots \\
&= \sigma_e^2 \, (1 + \phi^2 + \phi^4 + \dots) = \frac{\sigma_e^2}{1 - \phi^2} \\
\mathrm{Cov}(Y_t, Y_{t-1}) &= \mathrm{Cov}(e_t + \phi e_{t-1} + \dots, e_{t-1} + \phi e_{t-2} + \dots) \\
&= \mathrm{Cov}(\phi e_{t-1}, e_{t-1}) + \mathrm{Cov}(\phi^2 e_{t-2}, \phi e_{t-2}) + \dots \\
&= \phi \, \sigma_e^2 + \phi^3 \sigma_e^2 + \phi^5 \sigma_e^2 + \dots \\
&= \phi \, \frac{\sigma_e^2}{1 - \phi^2} \\
\mathrm{Cor}(Y_t, Y_{t-1}) &= \phi
\end{aligned}
$$

General linear processes

Similarly: $\mathrm{Cor}(Y_t, Y_{t-k}) = \phi^k$.

More generally: For general linear processes

$$\begin{aligned} \mathrm{E}(Y_t) &= 0 \\ \mathrm{Var}(Y_t) &= \sigma_e^2 \sum_{i=0}^{\infty} \psi_i^2 \\ \mathrm{Cov}(Y_t, Y_{t-k}) &= \sigma_e^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k} \end{aligned}$$

Remark: Zero mean does not influence covariance properties. Hence, assumed for discussing theory.

Models for Stationary Time Series

Moving Average Processes

Moving average processes

Definition: A *moving average of order q*, MA(q) for short, is defined as

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \ldots - \theta_q e_{t-q}.$$

Remarks:

- Special case of general linear process with finite number of non-zero weights.
- First considered by Slutsky (1927) and Wold (1938).
- Sometimes defined with $+$ instead of $-$ signs.
- Not unified across software packages: R employs $+$ signs!

Moving average processes: MA(1)

Example: MA(1) with $\theta_1 = \theta$.

$$
\begin{aligned}
Y_t &= e_t - \theta e_{t-1} \\
\mathrm{E}(Y_t) &= 0 \\
\mathrm{Var}(Y_t) &= \sigma_e^2\,(1+\theta^2) \\
\mathrm{Cov}(Y_t, Y_{t-1}) &= \mathrm{Cov}(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}) \\
&= \mathrm{Cov}(-\theta e_{t-1}, e_{t-1}) = -\theta\,\sigma_e^2 \\
\mathrm{Cov}(Y_t, Y_{t-2}) &= \mathrm{Cov}(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}) \\
&= 0 \\
\mathrm{Cov}(Y_t, Y_{t-k}) &= 0 \quad (k > 1)
\end{aligned}
$$

Moving average processes: MA(1)

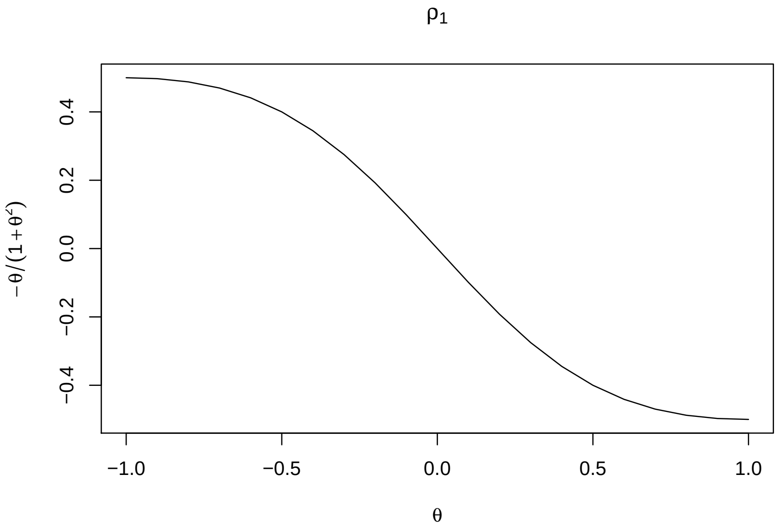
In summary:

$$
\begin{aligned}
\mu_t &= 0 \\
\gamma_0 &= \sigma_e^2 \, (1 + \theta^2) \\
\gamma_1 &= -\theta \, \sigma_e^2 \\
\varrho_1 &= -\frac{\theta}{1 + \theta^2} \\
\varrho_k &= \gamma_k \; = \; 0 \quad (k > 1)
\end{aligned}
$$

Remarks:

- ϱ_1 is maximal for $|\theta| = 1$.
- $\varrho_1(\theta) = \varrho_1(1/\theta)$.
- Thus, θ cannot be uniquely determined from ϱ_1, unless $-1 \leq \theta \leq 1$.

Moving average processes: MA(1)

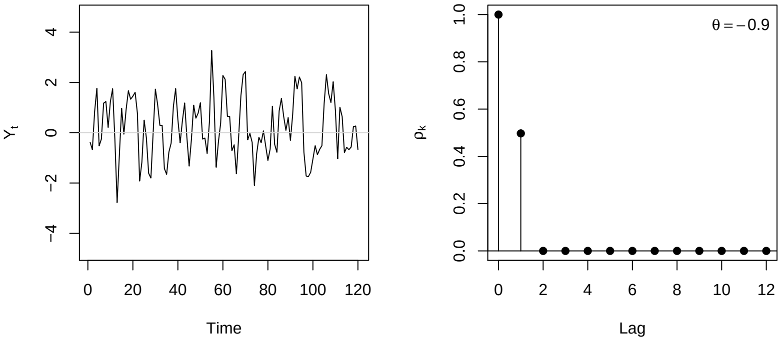


Moving average processes: MA(1)

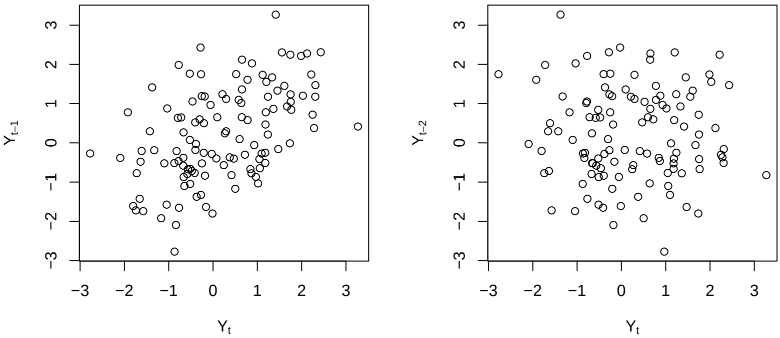
Illustration:

- Simulated MA(1) processes with standard normal white noise.
- Employ $\theta = -0.9$ and $\theta = 0.9$.
- In R: `arima.sim()` with reversed signs.
- Display simulated series, theoretical ACF, scatterplots of Y_t against Y_{t-1} and Y_{t-2}.

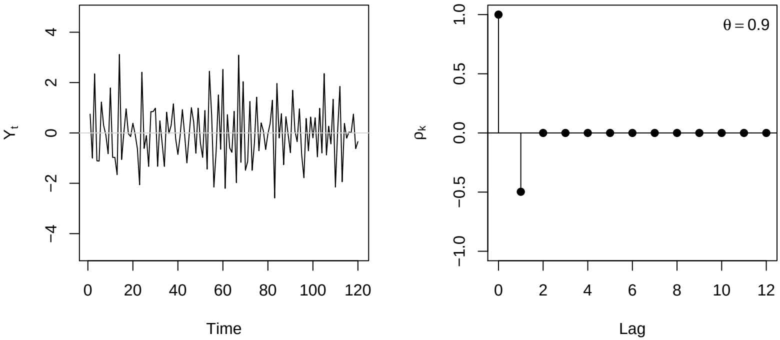
Moving average processes: MA(1)



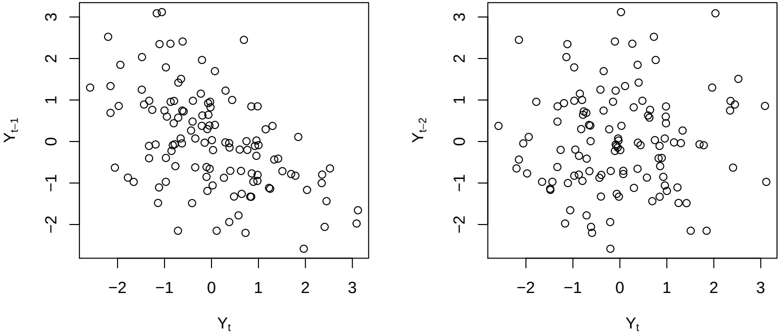
Moving average processes: MA(1)



Moving average processes: MA(1)



Moving average processes: MA(1)



Moving average processes: MA(2)

Example: MA(2).

$$
\begin{aligned}
Y_t &= e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} \\
\gamma_0 &= \mathrm{Var}(Y_t) = \mathrm{Var}(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) \\
&= (1 + \theta_1^2 + \theta_2^2)\, \sigma_e^2 \\
\gamma_1 &= \mathrm{Cov}(Y_t, Y_{t-1}) \\
&= \mathrm{Cov}(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) \\
&= \mathrm{Cov}(-\theta_1 e_{t-1}, e_{t-1}) + \mathrm{Cov}(-\theta_2 e_{t-2}, -\theta_1 e_{t-2}) \\
&= [-\theta_1 + (-\theta_1)(-\theta_2)]\, \sigma_e^2 \\
&= (-\theta_1 + \theta_1\theta_2)\, \sigma_e^2 \\
\gamma_2 &= \mathrm{Cov}(Y_t, Y_{t-2}) \\
&= \mathrm{Cov}(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}) \\
&= \mathrm{Cov}(-\theta_2 e_{t-2}, e_{t-2}) \\
&= -\theta_2\, \sigma_e^2
\end{aligned}
$$

Moving average processes: MA(2)

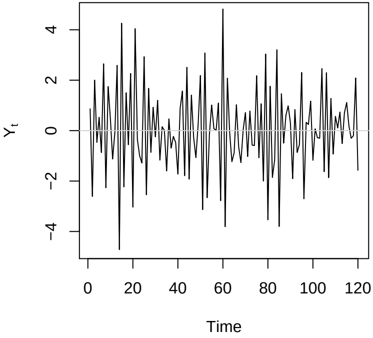
Thus:

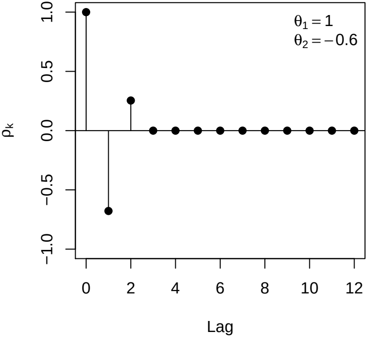
$$\begin{aligned}
\varrho_1 &= \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} \\
\varrho_2 &= \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2} \\
\varrho_k &= 0 \quad (k > 2)
\end{aligned}$$

Illustration: MA(2) $Y_t = e_t - e_{t-1} + 0.6e_{t-2}$, i.e., with $\theta_1 = 1$ and $\theta_2 = -0.6$.

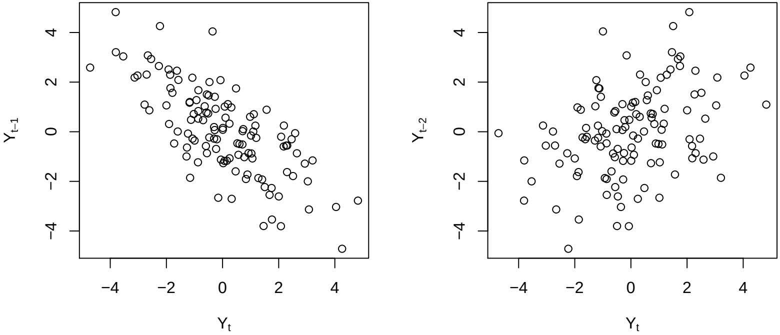
$$\begin{aligned}
\varrho_1 &= \frac{-1.6}{2.36} \approx -0.678 \\
\varrho_2 &= \frac{0.6}{2.36} \approx 0.254
\end{aligned}$$

Moving average processes: MA(2)





Moving average processes: MA(2)



Moving average processes: MA(q)

More generally: MA(q) $Y_t = e_t - \theta_1 e_{t-1} - \cdots - \theta_q e_{t-q}$.

Variance:

$$\gamma_0 \;=\; \left(1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2\right) \sigma_e^2.$$

Autocorrelation for $k = 1, \ldots, q$:

$$\varrho_k \;=\; \frac{-\theta_k + \theta_1 \theta_{k+1} + \theta_2 \theta_{k+2} + \cdots + \theta_{q-k} \theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2}$$

Autocorrelation for $k > q$: $\varrho_k = 0$.

Remark: Very flexible parametrization of ACFs that "cut off" after lag q.

Models for Stationary Time Series

Autoregressive Processes

Autoregressive processes

Definition: An *autoregressive process of order p*, AR(p) for short, is defined as

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \ldots + \phi_p Y_{t-p} + e_t.$$

Remarks:

- Y_t is linear combination of p previous values plus "innovation".
- For each t, e_t is assumed to be independent of $Y_{t-1}, Y_{t-2}, \ldots$
- First studied by Yule (1926).

Autoregressive processes: AR(1)

Example: AR(1) with $\phi_1 = \phi$, assumed to be stationary.

$$\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
\mathrm{Var}(Y_t) &= \mathrm{Var}(\phi Y_{t-1} + e_t) \\
&= \phi^2 \mathrm{Var}(Y_{t-1}) + \mathrm{Var}(e_t) \\
&= \phi^2 \mathrm{Var}(Y_t) + \sigma_e^2 \\
&= \frac{\sigma_e^2}{1-\phi^2}
\end{aligned}$$

which implies $\phi^2 < 1$ or equivalently $|\phi| < 1$. Furthermore

$$\begin{aligned}
\mathrm{Cov}(Y_t, Y_{t-k}) &= \mathrm{Cov}(\phi Y_{t-1} + e_t, Y_{t-k}) \\
&= \phi\, \mathrm{Cov}(Y_{t-1}, Y_{t-k}) + \mathrm{Cov}(e_t, Y_{t-k}) \\
&= \phi\, \mathrm{Cov}(Y_t, Y_{t-k+1})
\end{aligned}$$

Autoregressive processes: AR(1)

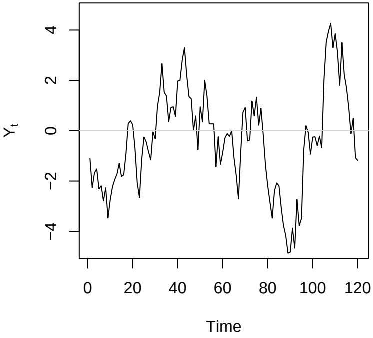
Therefore:

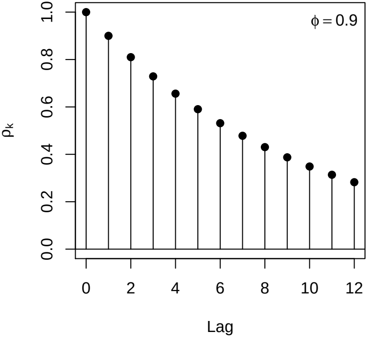
$$
\begin{aligned}
\gamma_k &= \phi\,\gamma_{k-1} \\
&= \phi^k \gamma_0 \\
&= \phi^k\,\frac{\sigma_e^2}{1-\phi^2} \\
\varrho_k &= \phi^k
\end{aligned}
$$

Illustration:

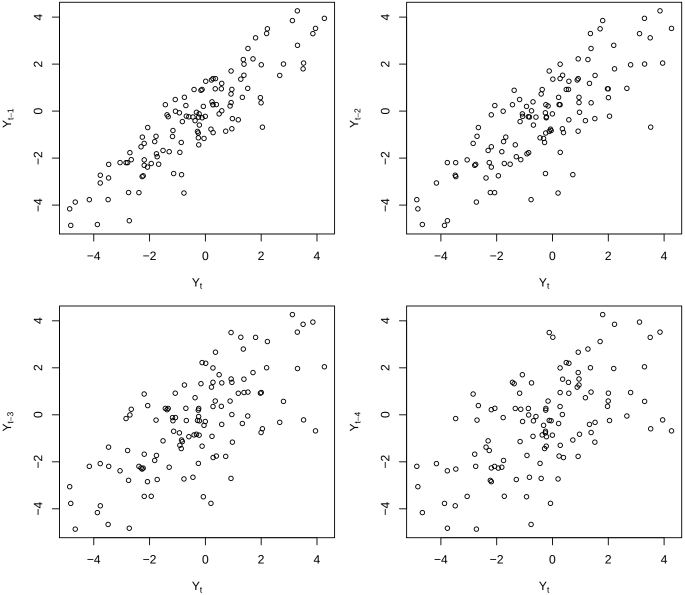
- Simulated AR(1) processes with standard normal white noise.
- Employ $\phi = 0.9, 0.7, 0.4, -0.5, -0.8$.
- In R: `arima.sim()`.
- Display simulated series, theoretical ACF, scatterplots of Y_t against $Y_{t-1}, Y_{t-2}, \ldots$.

Autoregressive processes: AR(1)

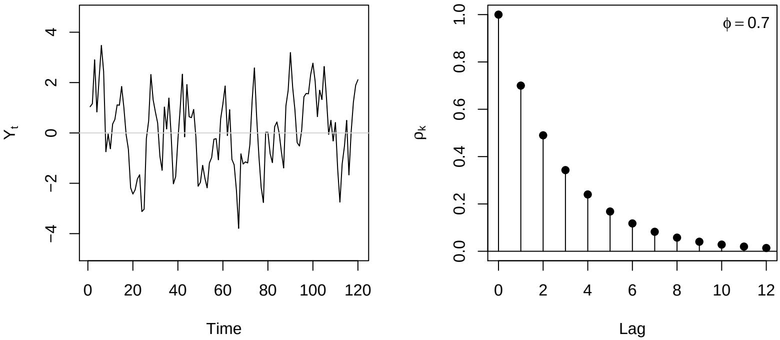




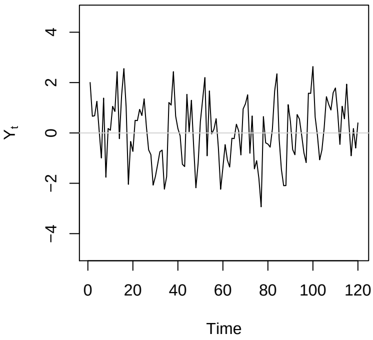
Autoregressive processes: AR(1)

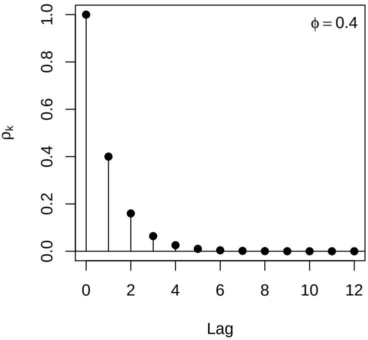


Autoregressive processes: AR(1)

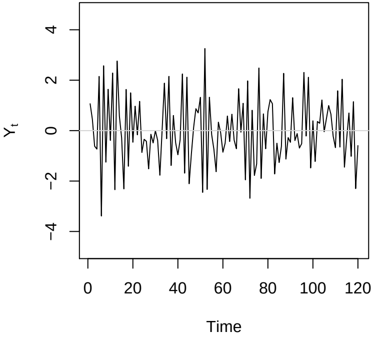


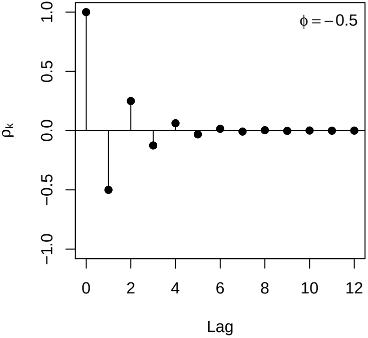
Autoregressive processes: AR(1)



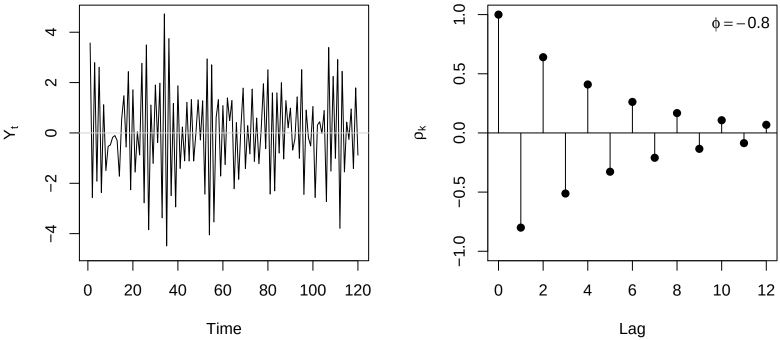


Autoregressive processes: AR(1)





Autoregressive processes: AR(1)



Autoregressive processes: AR(1)

Remarks:

- ACF decays exponentially.
- Sign of ACF switches for negative ϕ.
- Large positive ϕ leads to smooth series, theoretical mean zero is rarely crossed.
- Large negative ϕ leads to jagged series, theoretical mean zero is often crossed.

Autoregressive processes: AR(1)

Substitution:

$$
\begin{aligned}
Y_t &= \phi Y_{t-1} + e_t \\
&= \phi(\phi Y_{t-2} + e_{t-1}) + e_t \\
&= e_t + \phi e_{t-1} + \phi^2 Y_{t-2} \\
&= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \ldots + \phi^{k-1} e_{t-k+1} + \phi^k Y_{t-k} \\
&= e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \ldots
\end{aligned}
$$

Thus: If $|\phi| < 1$, AR(1) process is general linear process with $\psi_i = \phi^i$. This could also be employed to derive ACF.

Stationarity condition: In fact, AR(1) process with $\sigma_e^2 > 0$ is stationary if and only if $|\phi| < 1$.

Remark: For $\phi = 1$, process is random walk (already shown to be nonstationary).

Autoregressive processes: AR(2)

Example: AR(2).

$$Y_t \;=\; \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

Definition: The *AR(2) characteristic polynomial* is

$$\phi(x) = 1 - \phi_1 x - \phi_2 x^2.$$

The corresponding *AR(2) characteristic equation* is

$$1 - \phi_1 x - \phi_2 x^2 = 0.$$

Recall: Quadratic equation has always two roots, possibly complex.

Autoregressive processes: AR(2)

Stationarity condition: AR(2) process is stationary if and only if the roots of the AR characteristic equation exceed 1 in absolute value (modulus).

Equivalently: All roots must lie outside the unit circle in the complex plane.

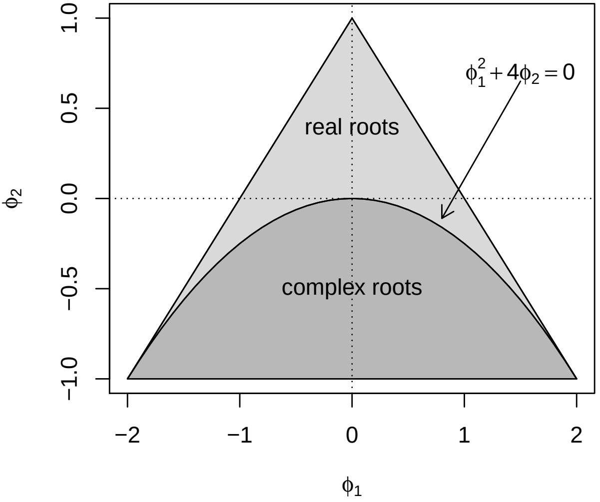
Roots:

$$\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}$$

It can be shown that this exceed 1 in absolute value if and only if:

$$\begin{aligned} \phi_1 + \phi_2 &< 1 \\ \phi_2 - \phi_1 &< 1 \\ |\phi_2| &< 1 \end{aligned}$$

Autoregressive processes: AR(2)



Autoregressive processes: AR(2)

Analogously to AR(1): Recursion holds for ACF.

$$\begin{aligned}\gamma_k &= \phi_1\gamma_{k-1} + \phi_2\gamma_{k-2} \\ \varrho_k &= \phi_1\varrho_{k-1} + \phi_2\varrho_{k-2}\end{aligned}$$

These are known as the *Yule-Walker equations*, especially for $k = 1, 2$.

Using $\varrho_0 = 1$ and $\varrho_{-k} = \varrho_k$, recursion can be initialized:

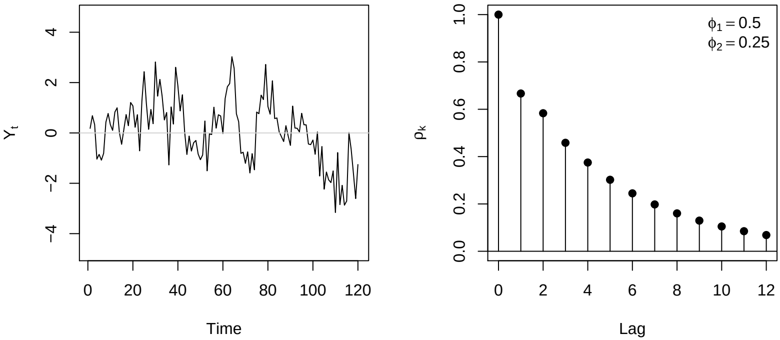
$$\begin{aligned}\varrho_1 &= \phi_1 + \phi_2\varrho_1 \\ &= \frac{\phi_1}{1-\phi_2} \\ \varrho_2 &= \phi_1\varrho_1 + \phi_2 \\ &= \frac{\phi_1^2 + \phi_2(1-\phi_2)}{1-\phi_2}\end{aligned}$$

Autoregressive processes: AR(2)

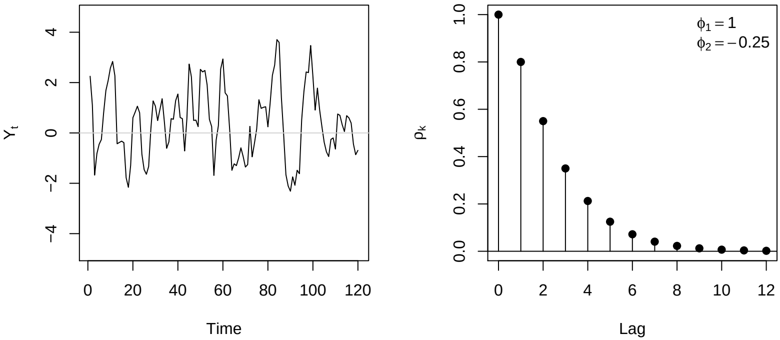
Remarks:

- ACF can assume wide variety of shapes.
- Magnitude of ϱ_k decays exponentially fast in lag k.
- In R: `ARMAacf()` for numerical computation.

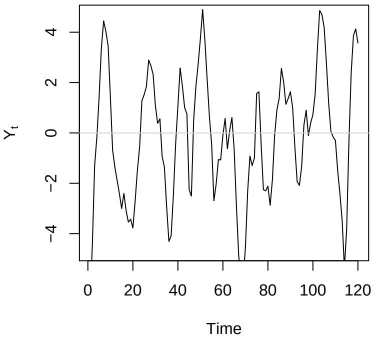
Autoregressive processes: AR(2)

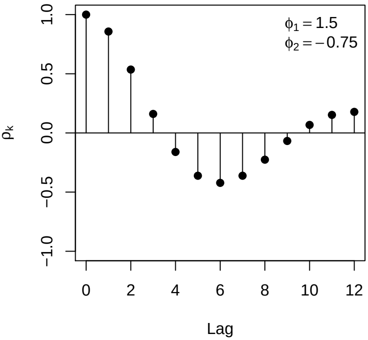


Autoregressive processes: AR(2)

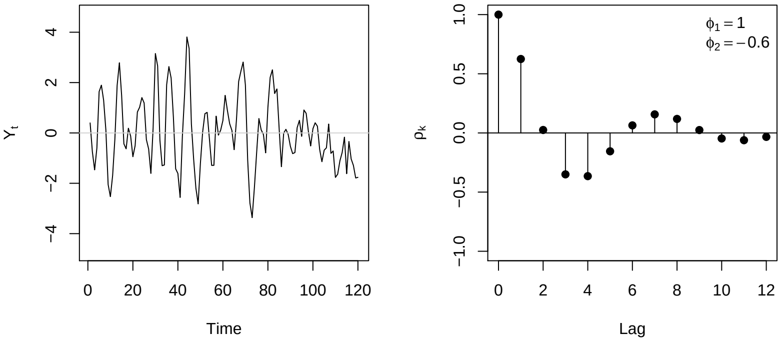


Autoregressive processes: AR(2)





Autoregressive processes: AR(2)



Autoregressive processes: AR(2)

Similarly: Solve for variance using recursive equations.

$$
\begin{aligned}
\gamma_0 &= (\phi_1^2 + \phi_2^2)\gamma_0 \ + \ 2\phi_1\phi_2\gamma_1 \ + \ \sigma_e^2 \\
&= \frac{(1-\phi_2)\sigma_e^2}{(1-\phi_2)(1-\phi_1^2-\phi_2^2) - 2\phi_2\phi_1^2} \\
&= \frac{1-\phi_2}{1+\phi_2}\,\frac{\sigma_e^2}{(1-\phi_2)^2 - \phi_1^2}
\end{aligned}
$$

Furthermore: General linear process coefficients ψ_j can be found by recursive substitution of $Y_{t-1}, Y_{t-2}, \ldots$.

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_1 \ - \ \phi_1\psi_0 &= 0 \\
\psi_j \ - \ \phi_1\psi_{j-1} \ - \ \phi_2\psi_{j-2} &= 0 \quad (j \geq 2)
\end{aligned}
$$

yielding $\psi_0 = 1, \psi_1 = \phi_1, \psi_2 = \phi_1^2 + \phi_2, \ldots$

Autoregressive processes: AR(p)

General AR(p) process:

Model equation:

$$Y_t \;=\; \phi_1 Y_{t-1} \;+\; \phi_2 Y_{t-2} \;+\; \ldots \;+\; \phi_p Y_{t-p} \;+\; e_t.$$

AR characteristic polynomial:

$$\phi(x) \;=\; 1 \;-\; \phi_1 x \;-\; \phi_2 x^2 \;-\; \ldots \;-\; \phi_p x^p.$$

AR characteristic equation:

$$1 \;-\; \phi_1 x \;-\; \phi_2 x^2 \;-\; \ldots \;-\; \phi_p x^p \;=\; 0.$$

Stationarity condition: All p roots of the AR characteristic polynomial must exceed 1 in absolute value (modulus), i.e., lie outside the complex unit circle. Necessary conditions:

$$\begin{aligned} \phi_1 + \phi_2 + \cdots + \phi_p &< 1 \\ |\phi_p| &< 1 \end{aligned}$$

Autoregressive processes: AR(p)

Assuming stationarity and zero means, recursion holds for $k \geq 1$:

$$\varrho_k = \phi_1 \varrho_{k-1} + \phi_2 \varrho_{k-2} + \ldots + \phi_p \varrho_{k-p}.$$

For $k = 1, \ldots, p$ and using $\varrho_0 = 1$ and $\varrho_{-k} = \varrho_k$, this yields the *Yule-Walker equations*:

$$\begin{aligned}
\varrho_1 &= \phi_1 + \phi_2 \varrho_1 + \phi_3 \varrho_2 + \ldots + \phi_p \varrho_{p-1} \\
\varrho_2 &= \phi_1 \varrho_1 + \phi_2 + \phi_3 \varrho_1 + \ldots + \phi_p \varrho_{p-2} \\
&\vdots \\
\varrho_p &= \phi_1 \varrho_{p-1} + \phi_2 \varrho_{p-2} + \phi_3 \varrho_{p-3} + \ldots + \phi_p
\end{aligned}$$

This can be solved for $\varrho_1, \ldots, \varrho_p$ and ϱ_k for $k > p$ can be obtained recursively.

Autoregressive processes: AR(p)

Variance can be obtained using $\text{Cov}(Y_t, e_t) = \sigma_e^2$ and $\varrho_k = \gamma_k / \gamma_0$:

$$\begin{aligned}
\gamma_0 &= \text{Cov}(Y_t, Y_t) \\
&= \text{Cov}(Y_t, \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t) \\
&= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \cdots + \phi_p \gamma_p + \sigma_e^2 \\
&= \frac{\sigma_e^2}{1 - \phi_1 \varrho_1 - \phi_2 \varrho_2 - \cdots - \phi_p \varrho_p}
\end{aligned}$$

Remarks:

- ϱ_k can be shown to be linear combination of exponentially decaying terms (corresponding to real roots) and damped sine waves (corresponding to complex roots).
- Assuming stationarity, general linear process form can again be found recursively.

Models for Stationary Time Series

The Mixed Autoregressive Moving Average Model

Autoregressive moving average model

Idea: Combine autoregressive process with moving average innovations.

Definition: An *autoregressive moving average process of orders p and q*, ARMA(p, q) for short, is defined as

$$Y_t = \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \ldots - \theta_q e_{t-q}.$$

Remark: There should not be common factors in autoregressive and moving average polynomials. Otherwise the model actually has orders lower than p and q. E.g., for ARMA(1, 1) this means $\phi \neq \theta$.

ARMA(1, 1)

Example: ARMA(1, 1) with $\phi_1 = \phi$ and $\theta_1 = \theta$.

$$\begin{aligned}
Y_t &= \phi Y_{t-1} \;+\; e_t \;-\; \theta e_{t-1} \\
\mathrm{Cov}(Y_t, e_t) &= \mathrm{E}(Y_t e_t) \\
&= \mathrm{E}[e_t\,(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\
&= \sigma_e^2 \\
\mathrm{Cov}(Y_t, e_{t-1}) &= \mathrm{E}[e_{t-1}\,(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\
&= \phi\sigma_e^2 \;-\; \theta\sigma_e^2 \\
&= (\phi - \theta)\,\sigma_e^2
\end{aligned}$$

Set up Yule-Walker type equations:

$$\begin{aligned}
\gamma_0 &= \phi\gamma_1 \;+\; \sigma_e^2 \;-\; \theta(\phi - \theta)\sigma_e^2 \\
\gamma_1 &= \phi\gamma_0 \;-\; \theta\sigma_e^2 \\
\gamma_k &= \phi\gamma_{k-1} \quad (k \geq 2)
\end{aligned}$$

ARMA(1, 1)

Solving the first two equations gives

$$\gamma_0 \;=\; \frac{1 - 2\phi\theta + \theta^2}{1 - \phi^2}\,\sigma_e^2.$$

Solving the recursion then yields for $k \geq 1$:

$$\varrho_k \;=\; \frac{(1 - \phi\theta)(\phi - \theta)}{1 - 2\phi\theta + \theta^2}\,\phi^{k-1}.$$

ACF decays exponentially with damping factor ϕ. Decay starts from ϱ_1 which also depends on θ.

ARMA(1, 1)

General linear process form can again by found by substitution:

$$Y_t \;=\; e_t \;+\; (\phi - \theta) \sum_{j=1}^{\infty} \phi^{j-1} e_{t-j},$$

i.e., coefficients are $\psi_j = (\phi - \theta)\, \phi^{j-1}$ for $j \geq 1$.

Stationarity condition: Roots of AR characteristic equation $1 - \phi x = 0$ must exceed 1 in absolute value, i.e., $|\phi| < 1$.

ARMA(p, q)

More generally: Consider ARMA(p, q).

Stationarity condition: All roots of AR characteristic equation $\phi(x) = 0$ exceed unity in absolute value (modulus).

General linear process form: $Y_t = \sum_{j=0}^{\infty} \psi_j e_{t-j}$ with coefficients ψ_j following the recursion

$$
\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= -\theta_1 + \phi_1 \\
\psi_2 &= -\theta_2 + \phi_2 + \phi_1\psi_1 \\
&\vdots \\
\psi_j &= -\theta_j + \phi_p\psi_{j-p} + \phi_{p-1}\psi_{j-p+1} + \cdots + \phi_1\psi_{j-1}
\end{aligned}
$$

where $\psi_j = 0$ for $j < 0$ and $\theta_j = 0$ for $j > q$. The mapping $j \to \psi_j$ for $j = 0, 1, 2, \ldots$ is also known as *impulse response function*.

ARMA(p, q)

Thus: The autocovariance can be written via

$$
\begin{aligned}
\mathrm{Cov}(Y_t, e_{t-k}) &= \mathrm{Cov}\left(\sum_{j=0}^{\infty} \psi_j e_{t-j}, e_{t-k}\right) \\
&= \psi_k \sigma_e^2 \\
\gamma_k &= \mathrm{Cov}(Y_t, Y_{t-k}) \\
&= \mathrm{Cov}\left[\left(\sum_{j=1}^{p} \phi_j Y_{t-j} \;-\; \sum_{j=0}^{q} \theta_j e_{t-j}\right), Y_{t-k}\right] \\
&= \sum_{j=1}^{p} \phi_j \gamma_{k-j} \;-\; \sigma_e^2 \sum_{j=k}^{q} \theta_j \psi_{j-k}
\end{aligned}
$$

where $\theta_0 = -1$ and the second sum is absent for $k > q$.

ARMA(p, q)

Recursion: For computing ACF.

1. Compute ψs from ϕs and θs.
2. Solve linear equations for $\gamma_0, \gamma_1, \ldots, \gamma_p$.
3. For $k > p$ use recursion for γ_k.
4. Obtain ACF as $\varrho_k = \gamma_k / \gamma_0$.

In R:

- Algorithm is basis for `ARMAacf()`.
- For step 1 only, `ARMAtoMA()` can be used.

Models for Stationary Time Series

Invertibility

Invertibility

Problem: For MA(1) process, ACF for coefficient θ is the same as for $1/\theta$. Similar issues in higher order MA(q) process.

Thus: Parameters cannot be inferred uniquely from ACF.

Solution: Related to another (seemingly unrelated) question.

Observation: AR(p) process can be thought of MA(∞) process. (By means of the general linear process representation.)

Question: Can MA(q) processes be written as AR(∞) processes?

Invertibility

Example: For MA(1) process $Y_t = e_t - \theta e_{t-1}$. If $|\theta| < 1$, an infinite recursive substitution leads to

$$\begin{aligned}
e_t &= Y_t + \theta e_{t-1} \\
&= Y_t + \theta(Y_{t-1} + \theta e_{t-2}) \\
&= Y_t + \theta Y_{t-1} + \theta^2 e_{t-2} \\
&= Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \dots \\
Y_t &= -\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \dots + e_t \\
&= \sum_{j=1}^{\infty} \pi_j Y_{t-j} + e_t
\end{aligned}$$

where $\pi_j = -\theta^j$.

Definition: An MA process is called *invertible* if such an AR(∞) representation can be found.

Thus: If $|\theta| < 1$ the MA(1) process is invertible.

Invertibility

More generally: For MA(q) and ARMA(p, q) processes, the MA characteristic polynomial is defined as

$$\theta(x) = 1 - \theta_1 x - \theta_2 x^2 - \ldots - \theta_q x^q.$$

The corresponding MA characteristic equation is

$$1 - \theta_1 x - \theta_2 x^2 - \ldots - \theta_q x^q = 0.$$

Invertibility condition: The MA(q) model is invertible if and only if all roots of the MA characteristic equation exceed 1 in absolute value (modulus).

Furthermore: It may be shown that there is only one set of parameter values that yield an invertible MA process with a given ACF.

Hence: Restrict attention to stationary and invertible ARMA(p, q) processes.