universität innsbruck

Time Series Analysis

Exercises

Exercise 1

Let $\{Y_t\}$ be stationary with autocovariance function γ_k. Let $\bar{Y} = n^{-1} \sum_{t=1}^{n} Y_t$. Show that

$$
\begin{aligned}
\operatorname{Var}(\bar{Y}) &= \frac{\gamma_0}{n} + \frac{2}{n} \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \gamma_k \\
&= \frac{1}{n} \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) \gamma_k .
\end{aligned}
$$

(Exercise 2.17)

Exercise 2

Let $Y_1 = \theta_0 + e_1$, and then for $t > 1$ define Y_t recursively by $Y_t = \theta_0 + Y_{t-1} + e_t$. Here θ_0 is a constant. The process $\{Y_t\}$ is called a *random walk with drift*.

(a) Show that Y_t may be rewritten as $Y_t = t \cdot \theta_0 + e_t + e_{t-1} + \cdots + e_1$.

(b) Find the mean function for $\{Y_t\}$.

(c) Find the autocovariance function for $\{Y_t\}$.

(Exercise 2.19)

Exercise 3

Let $Y_t = \mu + X_t$ where $\{X_t\}$ has autocorrelation function $\varrho_k = \phi^{|k|}$ for all k with $-1 < \phi < 1$. Show that

$$\mathrm{Var}(\bar{Y}) \approx \frac{1+\phi}{1-\phi}\,\frac{\gamma_0}{n}.$$

Hint: You will need the fact that

$$\sum_{k=0}^{\infty} \phi^k = \frac{1}{1-\phi}$$

(Exercise 3.17)

Exercise 4

Let $Y_t = e_t - \theta e_{t-1}$.

(a) Find the mean function for $\{Y_t\}$.

(b) Find the autocovariance function for $\{Y_t\}$.

(b) Find the autocorrelation function for $\{Y_t\}$.

Exercise 5

Write an R function to simulate Gaussian random walks of length `n` with some `mean` and `sd`.

Employ this function to simulate 100 random walks of length 1000 and explore their empirical mean and variance function.

Repeat the experiment with random walks with drift.

Exercise 6

Conduct exploratory analyses of the UK seatbelt data (`UKDriverDeaths`).

Replicate the result of `decompose()` (approximately) doing all computations "by hand" (i.e., using `filter()`, `rollapply()`, etc.).

Exercise 7

Consider the ACF of an MA(1) process.
Show that

$$\max_{-\infty<\theta<\infty} \varrho_1 \;=\; 0.5, \qquad \min_{-\infty<\theta<\infty} \varrho_1 \;=\; -0.5$$

and that when θ is replaced by $1/\theta$ the ACF does not change.
(Exercises 4.3 and 4.4)

Exercise 8

Consider two MA(2) processes, one with $\theta_1 = \theta_2 = 1/6$ and another one with $\theta_1 = -1$ and $\theta_2 = 6$.

- Show that both processes have the same autocorrelation function.
- How do the roots of the corresponding characteristic polynomials compare?

(Exercise 4.12)

Exercise 9

Consider the AR(1) model $Y_t = \phi Y_{t-1} + e_t$.
Show that if $|\phi| = 1$ the process cannot be stationary.
(Exercise 4.15)

Exercise 10

Write an R function to simulate AR(1) processes of length `n` with some coefficient `ar` and independent Gaussian innovations with some `mean` and `sd`.

Employ this function to simulate 100 AR(1) processes of length 100 and explore their empirical mean and variance function.

What do you observe?

Exercise 11

Identify the following as specific ARIMA models. That is, what are p, d, and q and what are the values of the parameters (the ϕs and θs)?

(a) $Y_t = Y_{t-1} - 0.25Y_{t-2} + e_t - 0.1e_{t-1}$.

(b) $Y_t = 2Y_{t-1} - Y_{t-2} + e_t$.

(c) $Y_t = 0.5Y_{t-1} - 0.5Y_{t-2} + e_t - 0.5e_{t-1} + 0.25e_{t-2}$.

(Exercise 5.1)

Exercise 12

Suppose that

$$\begin{aligned} Y_t &= M_t + e_t \\ M_t &= M_{t-1} + \varepsilon_t \end{aligned}$$

with $\{e_t\}$ and $\{\varepsilon_t\}$ independent white noise processes.

Derive the autocorrelation function of ΔY_t.
(Exercise 5.9)

Exercise 13

Simulate an AR(1) time series with $n = 48$ and with $\phi = 0.7$.

(a) Calculate the theoretical autocorrelations at lag 1 and lag 5 for this model.

(b) Calculate the sample autocorrelations at lag 1 and lag 5 and compare the values with their theoretical values.

(c) Repeat the simulation of the series and calculation of r_1 and r_5 many times and form the sampling distributions of r_1 and r_5. How well does the large-sample distribution approximate the variance in your sampling distribution?

(d) Repeat (c) with $n = 250$.

(similar to Exercise 6.20)

Exercise 14

Analyze the `"black"` pepper price series from `PepperPrice` (in package *AER*). Select tentative orders p, d, and q for an ARIMA model.

Exercise 15

Analyze the US gross domestic product (GDP) series `"gdp"` in `USMacroG` (in package *AER*).

- Is GDP (in logs) trend stationary?
- Is GDP growth level stationary?
- Select tentative orders p, d, q for log-GDP or GDP growth, respectively.

Exercise 16

From a series of length 100, we have computed $r_1 = 0.8$, $r_2 = 0.5$, $r_3 = 0.4$, $\bar{Y} = 2$, and a sample variance of 5. If we assume that an AR(2) model with a constant term is appropriate, how can we get (simple) estimates of ϕ_1, ϕ_2, θ_0, and σ_e^2?
(Exercise 7.1)

Exercise 17

If $\{Y_t\}$ satisfies an AR(1) model with ϕ of about 0.7, how long of a series do we need to estimate $\phi = \varrho_1$ with 95% confidence that our estimation error is no more than ± 0.1? (Exercise 7.3)

Exercise 18

Simulate an MA(1) series with $\theta = -0.6$ and $n = 48$.

(a) Find the maximum likelihood estimate of θ.

(b) Repeat part (a) many times with a new simulated series using the same parameters and same sample size.

(c) Form the sampling distribution of the maximum likelihood estimates of θ.

(d) Are the estimates (approximately) unbiased?

(e) Calculate the variance of your sampling distribution and compare it with the large-sample result from slide 7-35.

(f) Repeat(a)–(e) but with error terms from a *t* distribution with 4 degrees of freedom.

(Combination of Exercise 7.11 and 7.19)

Exercise 19

Simulate an AR(2) model with $n = 48$, $\phi_1 = 1.5$, and $\phi_2 = -0.75$.

(a) Fit the correctly specified AR(2) model and look at a time series plot of the residuals. Does the plot support the AR(2) specification?

(b) Display a normal quantile-quantile plot of the standardized residuals. Does the plot support the AR(2) specification?

(c) Display the sample ACF of the residuals. Does the plot support the AR(2) specification?

(d) Calculate the Ljung-Box statistic summing to $K = 12$. Does this statistic support the AR(2) specification?

(Exercise 8.6)

Exercise 20

The data `deere3` (from the *TSA* package) contain 57 consecutive values from a complex machine tool at Deere & Co. The values given are deviations from a target value in units of ten millionths of an inch. The process employs a control mechanism that resets some of the parameters of the machine tool depending on the magnitude of deviation from target of the last item produced. Select an appropriate model for this series and diagnose its fit.
(Similar to Exercise 8.10)

Exercise 21

Suppose that annual sales (in millions of dollars) of the Acme Corporation follow the AR(2) model
$Y_t = 5 + 1.1Y_{t-1} - 0.5Y_{t-2} + e_t$ with $\sigma_e^2 = 2$.

(a) If sales for 2005, 2006, and 2007 were 9, 11, and 10 million dollars, respectively, forecast sales for 2008 and 2009.

(b) Show that $\psi_1 = 1.1$ for this model.

(c) Calculate 95% prediction limits for your forecasts in part (a).

(d) If sales in 2008 turn out to be 12 million dollars, update your forecast for 2009.

(Similar to Exercise 9.2)

Exercise 22

Simulate an ARMA(1, 1) process with $\phi = 0.7$, $\theta = -0.5$, and $\mu = 100$. Simulate 50 values but set aside the last 10 values to compare forecasts with actual values.

(a) Using the first 40 values of the series, find the values for the maximum likelihood estimates of ϕ, θ, and μ.

(b) Using this model, forecast the next ten values of the series. Plot the series together with the forecasts. Place a horizontal line at the estimate of the process mean.

(c) Compare the ten forecasts with the actual values that you set aside.

(d) Plot the forecasts together with 95% forecast limits. Do the actual values fall within the forecast limits?

(e) Repeat (a) through (d) with a new simulated series using the same parameters and same sample size.

(Exercise 9.13)

Exercise 23

Based on quarterly data, a seasonal model of the form

$$Y_t = Y_{t-4} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

has been fit to a certain time series.

(a) Find the first four ψ-weights for this model.

(b) Suppose that $\theta_1 = 0.5$, $\theta_2 = -0.25$, and $\sigma_e = 1$. Find forecasts for the next four quarters if data for the last four quarters are

Quarter	I	II	III	IV
Series	25	20	25	40
Residual	2	1	2	3

(c) Find 95% prediction intervals for the forecasts in part (b).

(Exercise 10.1)

Exercise 24

Consider the time series `UKNonDurables` (in *AER*) of consumption of nondurables in the UK for the time period from 1955 Q1 to 1970 Q4.

- Conduct an exploratory analysis of season and trend and describe the patterns you find.
- Find an ARIMA model for a suitably transformed version of the time series and assess its goodness of fit.
- Produce point and interval estimates for the remaining time series and compare them with the actual observations.

Exercise 25

Reconsider the US gross domestic product (GDP) series `"gdp"` in `USMacroG` (in package *AER*).

- Conduct the usual residual analysis of a suitable ARIMA model for the data. What are the conclusions?
- Consider the squared residuals and conduct a structural change analysis (testing and dating). What are the conclusions?
- How can the results be interpreted with respect to the "Great Moderation"?

Exercise 26

Lütkepohl, Teräsvirta, Wolters (1999, *Journal of Applied Econometrics*) investigate the stability of the German M1 money demand using a so-called error correction model based on quarterly data from 1960(1)–1996(3) for the following macroeconomic variables:

- m_t: log-real per capita stock M1 (`m1`),
- y_t: log-real per capita GDP (`gnp`),
- p_t: log GNP price index (`price`),
- R_t: long-term interest (`interest`, in decimals).

As m_t and y_t are clearly non-stationary but appear to be cointegrated, the following error correction model for money demand is used:

$$
\begin{aligned}
\Delta m_t \quad = \quad & \beta_1 \cdot \Delta y_{t-2} + \beta_2 \cdot \Delta R_t + \beta_3 \cdot \Delta R_{t-1} + \beta_4 \cdot \Delta p_t + \\
& \beta_5 \cdot m_{t-1} + \beta_6 \cdot y_{t-1} + \beta_7 \cdot R_{t-1} + \\
& \beta_8 + \beta_9 \cdot q_{1t} + \beta_9 \cdot q_{2t} + \beta_9 \cdot q_{3t} + \varepsilon_t
\end{aligned}
$$

Exercise 26

The variables q_{1t}–q_{3t} are dummy variables for Q1–Q3.

- Estimate the model using OLS for the full data set.
- Carry out different structural change tests. Is there evidence for an instability in the money demand relationship?
- Try to estimate a model with breakpoints. Did the money demand relationship change and, if so, where? How can this be interpreted?

Exercise 27

Consider the time series `d.ibm6298wmx` (in *FinTS*, accompanying Tsay, 2005), providing daily returns for IBM from 1962-07-03 to 1998-12-31 (in the first column).

- Fit an AR(2)-GARCH$(1, 1)$ model as suggested by Tsay (2005, pp. 294) and provide the model equations with the estimated parameters.
- Forecast the mean and standard deviation of the return for the next trading day. (Also do this "by hand" by extracting the lagged observations, errors, volatilities and plugging them into the model equations.)
- Which return for the next trading day can you expect at least with 5% confidence? Which with 1% confidence?
- What are the associated values at risk for a portfolio with a value of USD 1,000,000?