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Time Series Analysis

Seasonal Models

Seasonal Models

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Seasonal ARIMA models

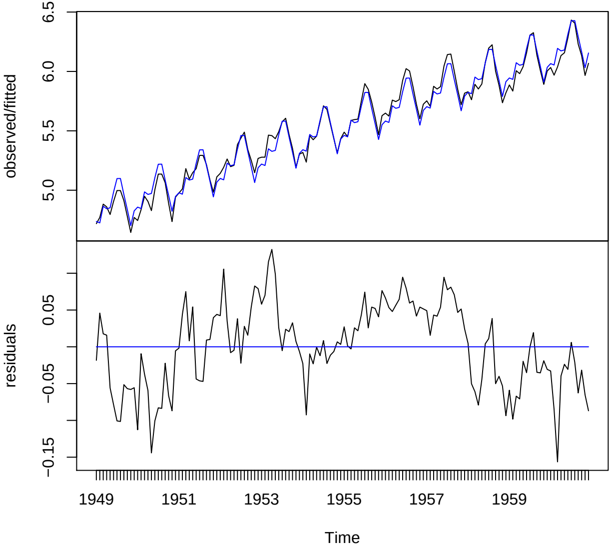
So far:

- Included seasonal patterns in deterministic models but not ARIMA models.
- In deterministic models, correlations are not exploited.
- Associated residuals are often highly autocorrelated.
- Need for stochastic seasonal models.

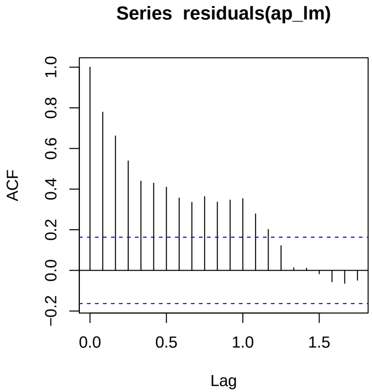
Illustration: Deterministic linear trend plus seasonal effects for monthly airline passenger numbers from 1949(1) to 1960(12).

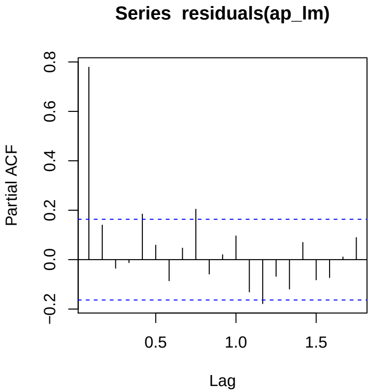
```
R> data("AirPassengers", package = "datasets")
R> ap <- log(AirPassengers)
R> ap_lm <- dynlm(ap ~ trend(ap) + season(ap))
```

Seasonal ARIMA models



Seasonal ARIMA models





Seasonal ARIMA models

Residuals are not white noise and may even be nonstationary.

```
R> Box.test(residuals(ap_lm), type = "Ljung-Box")

        Box-Ljung test

data:  residuals(ap_lm)
X-squared = 89, df = 1, p-value <2e-16

R> kpss.test(residuals(ap_lm))

        KPSS Test for Level Stationarity

data:  residuals(ap_lm)
KPSS Level = 0.39, Truncation lag parameter = 4, p-value
= 0.08
```

Seasonal ARIMA models

Idea: Introduce AR and/or MA effects at seasonal lag s, e.g., $s = 4$ for quarterly data or $s = 12$ for monthly data.

Example: MA(12) model with only one non-zero coefficient.

$$Y_t \; = \; e_t \; - \; \Theta e_{t-12}$$

Properties: Stationary with nonzero autocorrelation only at lag 12.

$$\begin{aligned} \mathrm{Cov}(Y_t, Y_{t-1}) &= \mathrm{Cov}(e_t - \Theta e_{t-12}, e_{t-1} - \Theta e_{t-13}) \; = \; 0 \\ \mathrm{Cov}(Y_t, Y_{t-12}) &= \mathrm{Cov}(e_t - \Theta e_{t-12}, e_{t-12} - \Theta e_{t-24}) \; = \; -\Theta \sigma_e^2 \end{aligned}$$

Seasonal ARIMA models

Definition: Seasonal MA(Q) model of order Q with seasonal period s.

$$Y_t = e_t - \Theta_1 e_{t-s} - \Theta_2 e_{t-2s} - \ldots - \Theta_Q e_{t-Qs}$$

Characteristic polynomial:

$$\Theta(x) = 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \ldots - \Theta_Q x^{Qs}$$

Autocorrelation: Zero except at seasonal lags $s, 2s, \ldots, Qs$.

$$\rho_{ks} = \frac{\Theta_k + \Theta_1\Theta_{k+1} + \cdots + \Theta_{Q-k}\Theta_Q}{1 + \Theta_1^2 + \cdots + \Theta_Q^2} \qquad \text{for } k = 1, \ldots, Q$$

Invertibility condition: All roots of $\Theta(x) = 0$ must exceed 1 in absolute value (i.e., lie outside the complex unit circle).

Seasonal ARIMA models

Analogously: AR(12) model with only one nonzero coefficient.

$$Y_t = \Phi Y_{t-12} + e_t$$

Properties: Stationary with nonzero autocorrelation only at seasonal lags 12, 24, 36, ...

$$\begin{aligned}
\rho_k &= \Phi \rho_{k-12} \qquad \text{for } k \geq 1 \\
\rho_{12} &= \Phi \rho_0 = \Phi \\
\rho_{24} &= \Phi \rho_{12} = \Phi^2 \\
\rho_{12k} &= \Phi^k \qquad \text{for } k \geq 1 \\
\rho_1 &= \Phi \rho_{11} \\
\rho_{11} &= \Phi \rho_1
\end{aligned}$$

and hence $\rho_1 = \rho_{11} = 0$.

Seasonal ARIMA models

Definition: Seasonal AR(P) model of order P with seasonal period s.

$$Y_t = \Phi_1 Y_{t-s} + \Phi_2 Y_{t-2s} + \ldots + \Phi_P Y_{t-Ps} + e_t$$

Characteristic polynomial:

$$\Phi(x) = 1 - \Phi_1 x^s - \Phi_2 x^{2s} - \ldots - \Phi_P x^{Ps}$$

Autocorrelation: Zero except at seasonal lags $s, 2s, 3s, \ldots$.
E.g., for seasonal AR(1)

$$\rho_{ks} = \Phi^k \qquad \text{for } k = 1, 2, \ldots$$

Stationarity condition: All roots of $\Phi(x) = 0$ must exceed 1 in absolute value (i.e., lie outside the complex unit circle).

Seasonal Models

Multiplicative Seasonal ARMA Models

Multiplicative seasonal ARMA models

Idea: Combine autocorrelations at seasonal lags with those at neighboring lags.

Conceivable approaches: Illustrated for MA model with one seasonal and one nonseasonal coefficient.

- *Subset MA:* Consider subset of MA(12) with nonzero coefficients at lags 1 and 12.

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_{12} e_{t-12}$$

- *Multiplicative seasonal MA:* Consider MA characteristic polynomial

$$(1 - \theta x)(1 - \Theta x^{12}) = 1 - \theta x - \Theta x^{12} + \theta\Theta x^{13}$$

  leading to the model

$$Y_t = e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$$

The latter is more commonly used in practice.

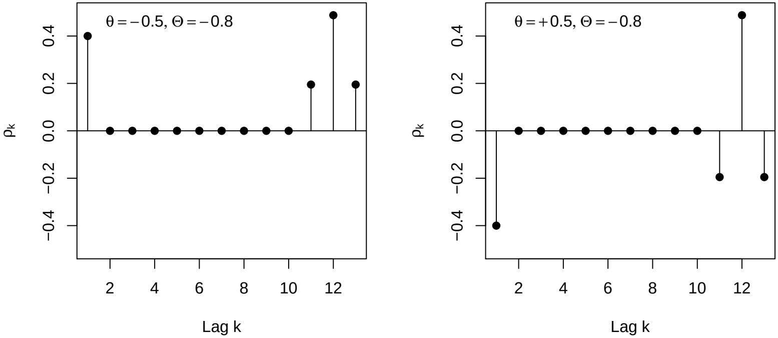
Multiplicative seasonal ARMA models

Properties: Stationary with nonzero autocorrelations only at lags 1, 11, 12, 13.

$$\begin{aligned}
\gamma_0 &= (1+\theta^2)(1+\Theta^2)\sigma_e^2 \\
\rho_1 &= -\frac{\theta}{1+\theta^2} \\
\rho_{11} = \rho_{13} &= \frac{\theta\Theta}{(1+\theta^2)(1+\Theta^2)} \\
\rho_{12} &= -\frac{\Theta}{1+\Theta^2}
\end{aligned}$$

Example: ACF for $\Theta = -0.8$ and $\theta = \pm 0.5$.

Multiplicative seasonal ARMA models



Multiplicative seasonal ARMA models

Definition: *Multiplicative seasonal ARMA model* with orders p and q and seasonal orders P and Q at seasonal lag s, ARMA(p, q)(P, Q)$_s$ for short.

$$\phi(B)\Phi(B)Y_t = \theta(B)\Theta(B)e_t$$

with polynomials

$$\begin{aligned}
\phi(x) &= 1 - \phi_1 x^1 - \phi_2 x^2 - \ldots - \phi_p x^p \\
\Phi(x) &= 1 - \Phi_1 x^s - \Phi_2 x^{2s} - \ldots - \Phi_P x^{Ps} \\
\theta(x) &= 1 - \theta_1 x^1 - \theta_2 x^2 - \ldots - \theta_q x^q \\
\Theta(x) &= 1 - \Theta_1 x^s - \Theta_2 x^{2s} - \ldots - \Theta_Q x^{Qs}
\end{aligned}$$

Note: A mean or intercept can be included as before.

Multiplicative seasonal ARMA models

Remarks:

- ARMA$(p, q)(P, Q)_s$ is a special case of ARMA$(p + Ps,$ $q + Qs)$.
- Coefficients are not completely free but determined by $p + P + q + Q$ parameters.
- Typically, much more parsimonious, especially if s is large.
- Sometimes also called SARMA.

Special case: ARMA$(0, 1)(1, 0)_{12}$.

$$Y_t = \Phi Y_{t-12} + e_t - \theta e_{t-1}$$

which has

$$\begin{aligned} \gamma_1 &= \Phi\gamma_{11} - \theta\sigma_e^2 \\ \gamma_k &= \Phi\gamma_{k-12} \qquad \text{for } k \geq 2 \end{aligned}$$

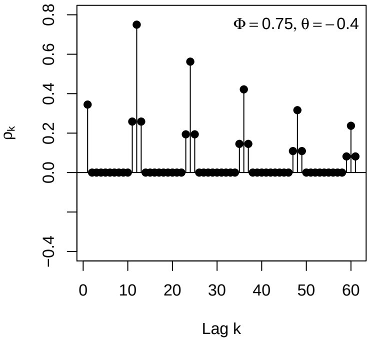
Multiplicative seasonal ARMA models

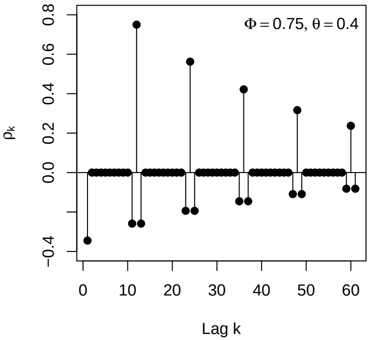
Hence:

$$
\begin{aligned}
\gamma_0 &= \frac{1+\theta^2}{1-\Phi^2}\,\sigma_e^2 \\
\rho_{12k} &= \Phi^k \qquad \text{for } k \geq 1 \\
\rho_{12k-1} = \rho_{12k+1} &= -\frac{\theta}{1+\theta^2}\Phi^k \qquad \text{for } k = 0, 1, 2, \ldots
\end{aligned}
$$

Example: ACF for $\Phi = 0.75$ and $\theta = \pm 0.4$.

Multiplicative seasonal ARMA models





Seasonal Models

Nonstationary Seasonal ARIMA Models

Nonstationary seasonal ARIMA models

Idea: Analogous to differencing in nonstationary series, seasonal nonstationarity may be captured by *seasonal differences*.

$$\begin{aligned}\Delta_s &= 1 - B^s \\ \Delta_s Y_t &= Y_t - Y_{t-s}\end{aligned}$$

The resulting series is then of length $n - s$.

Example: Seasonal random walk $\{S_t\}$ plus independent noise.

$$\begin{aligned}Y_t &= S_t + e_t \\ S_t &= S_{t-s} + \varepsilon_t\end{aligned}$$

If $\sigma_\varepsilon \ll \sigma_e$, $\{S_t\}$ captures a slowly changing seasonal component.

Nonstationary seasonal ARIMA models

Nonstationarity: Y_t inherits nonstationarity from S_t. This can be removed via seasonal differences.

$$\begin{aligned} \Delta_s Y_t &= S_t + e_t - S_{t-s} - e_{t-s} \\ &= \varepsilon_t + e_t - e_{t-s} \end{aligned}$$

which is stationary and has the ACF of a seasonal MA$(1)_s$ model (depending on ratio of noise variances).

Analogously: Consider an additional nonseasonal stochastic trend.

$$\begin{aligned} Y_t &= M_t + S_t + e_t \\ S_t &= S_{t-s} + \varepsilon_t \\ M_t &= M_{t-1} + \xi_t \end{aligned}$$

with $\{e_t\}$, $\{\varepsilon_t\}$, $\{\xi_t\}$ mutually independent white noise series.

Nonstationary seasonal ARIMA models

Nonstationarity: Can be removed by taking both seasonal and nonseasonal differences.

$$\begin{aligned}\Delta\Delta_s Y_t &= \Delta(M_t - M_{t-s} + \varepsilon_t + e_t - e_{t-s}) \\ &= (\xi_t + \varepsilon_t + e_t) - (\varepsilon_{t-1} + e_{t-1}) - (\xi_{t-s} + e_{t-s}) + e_{t-s-1}\end{aligned}$$

This process is stationary and has nonzero autocorrelations only at lags 1, $s-1$, s, $s+1$. ACF is equivalent to seasonal ARMA$(0, 1)(0, 1)_s$.

Definition: *Multiplicative seasonal ARIMA model* with orders p, d, and q and seasonal orders P, D, and Q at seasonal lag s, denoted ARIMA$(p, d, q)(P, D, Q)_s$ for short.

$$\phi(B)\Phi(B)(1-B)^d(1-B^s)^D Y_t = \theta(B)\Theta(B)e_t$$

Note: This can capture the structure of a wide range of empirical time series, often with just a few parameters.

Seasonal Models

Model Specification, Fitting, and Checking

Model specification, fitting, and checking

Approach: Employ the same ideas as for nonseasonal models.

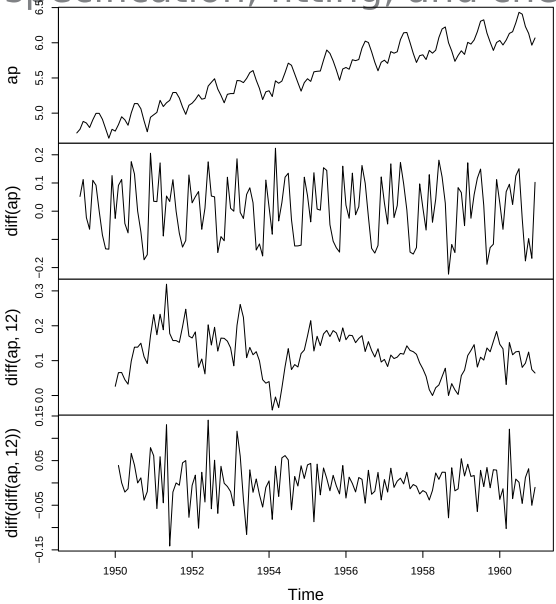
- *Model specification:* Inspect time series plot, ACF and PACF of suitably differenced series. Can also be accompanied by stationarity or (seasonal) unit root tests.
- *Model fitting:* Fit a preliminary (seasonal) ARIMA model.
- *Diagnostic checking:* Inspect (standardized) residuals (series, QQ plot, ACF, Ljung-Box tests, . . .).

In R:

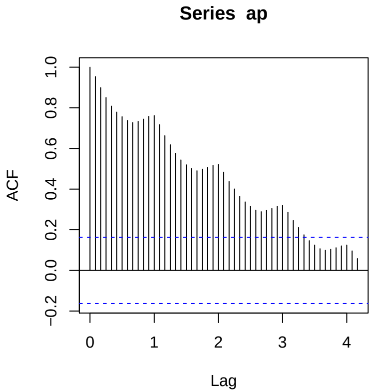
- `diff(y, s)` can also compute seasonal differences.
- `arima()` can also include `seasonal` orders.
- `tsdiag()` can be applied as before.

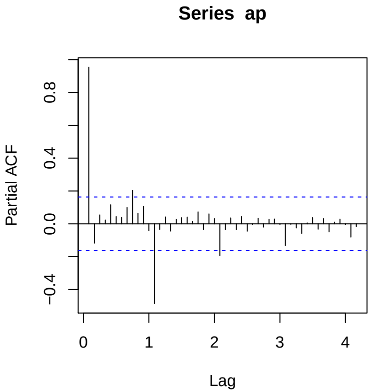
Illustration: Revisit `AirPassengers` data.

Model specification, fitting, and checking

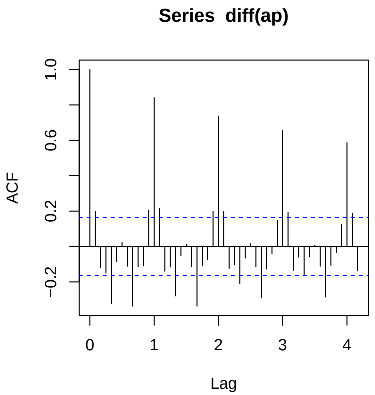


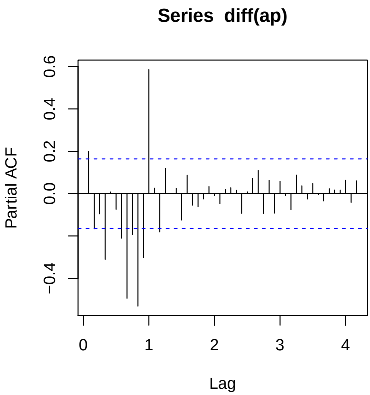
Model specification, fitting, and checking



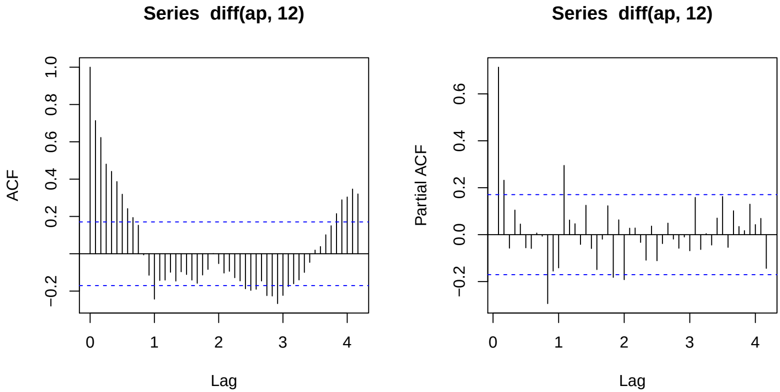


Model specification, fitting, and checking

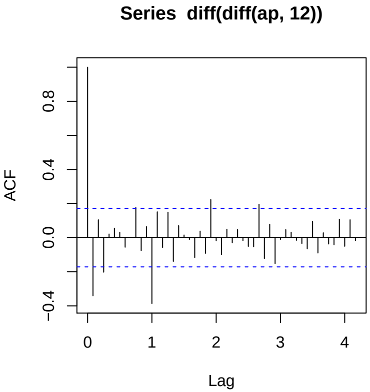


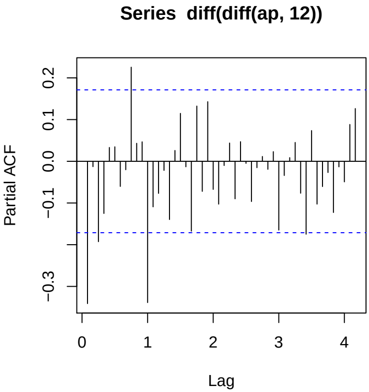


Model specification, fitting, and checking



Model specification, fitting, and checking





Model specification, fitting, and checking

Model fitting: Employ seasonal ARIMA$(0, 1, 1)(0, 1, 1)_{12}$.

```
R> fit <- arima(ap, c(0, 1, 1), seasonal = list(order = c(0, 1 ,1)))
R> fit
```

```
Call:
arima(x = ap, order = c(0, 1, 1), seasonal = list(order = c(0, 1, 1)))

Coefficients:
         ma1     sma1
      -0.402  -0.557
s.e.   0.090   0.073

sigma^2 estimated as 0.00135:  log likelihood = 244.7,  aic = -483.4
```

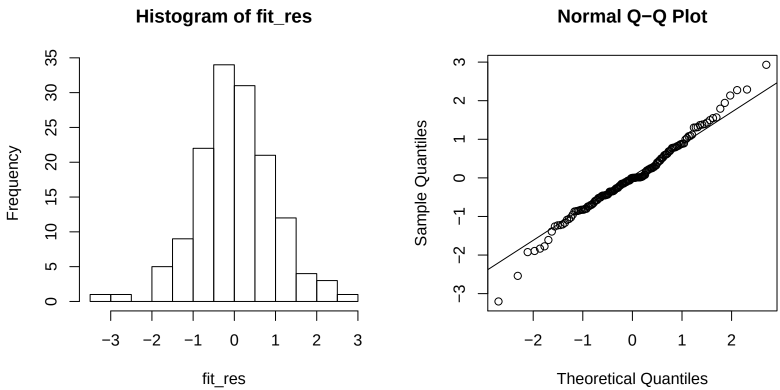
Model specification, fitting, and checking

Model fitting: Employ seasonal ARIMA$(0, 1, 1)(0, 1, 1)_{12}$.

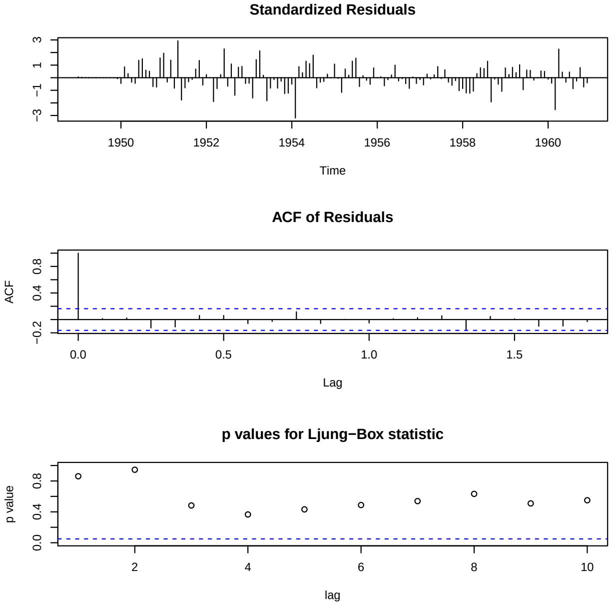
```
R> library("forecast")
R> fit <- Arima(ap, c(0, 1, 1), seasonal = list(order = c(0, 1 ,1)))
R> fit
```

```
Series: ap
ARIMA(0,1,1)(0,1,1)[12]

Coefficients:
         ma1     sma1
      -0.402  -0.557
s.e.   0.090   0.073

sigma^2 estimated as 0.00137:  log likelihood=244.7
AIC=-483.4   AICc=-483.2   BIC=-474.8
```

Model specification, fitting, and checking



Model specification, fitting, and checking



Model specification, fitting, and checking

Furthermore: The same model would be selected by an information-criteria-based approach, e.g., `auto.arima(ap, d = 1, D = 1, approx = FALSE, stepwise = FALSE)` or `auto.arima(ap, d = 1, D = 1, approx = FALSE)` etc.

Summary:

- ARIMA(0, 1, 1)(0, 1, 1)$_{12}$ model leads to satisfactory fit and has appealing simple interpretation.
- The model for this data was popularized by Box & Jenkins (1976) and is hence also called *airline model*.
- It is also found to yield satisfactory fits for many other time series.

Forecasting Seasonal Models

Forecasting seasonal models

Again: The same strategies for computing forecasts and associated standard errors can be used as in the nonseasonal case.

In R: As before, `predict()` or `forecast()`.

Example: Seasonal AR$(1)_{12}$.

$$\begin{aligned} Y_t &= \Phi Y_{t-12} + e_t \\ \widehat{Y}_t(\ell) &= \Phi \widehat{Y}_t(\ell - 12) \\ &= \Phi^{k+1} Y_{t+r-11} \end{aligned}$$

where k is the integer part and $r/12$ is the fractional part of $(\ell - 1)/12$.
The MA(∞) weights are nonzero only for multiple lags of 12.

$$\psi_j = \begin{cases} \Phi^{j/12} & \text{for } j = 0, 12, 24, \ldots \\ 0 & \text{otherwise} \end{cases}$$

Forecasting seasonal models

Hence: With k defined as before.

$$\mathrm{Var}(e_t(\ell)) = \left(\frac{1-\Phi^{2k+2}}{1-\Phi^2}\right)\sigma_e^2$$

Example: Seasonal MA$(1)_{12}$ with intercept.

$$\begin{aligned}
Y_t &= \theta_0 + e_t - \Theta e_{t-12} \\
\widehat{Y}_t(1) &= \theta_0 - \Theta e_{t-11} \\
\widehat{Y}_t(2) &= \theta_0 - \Theta e_{t-10} \\
&\vdots \\
\widehat{Y}_t(12) &= \theta_0 - \Theta e_t \\
\widehat{Y}_t(\ell) &= \theta_0 \qquad \text{for } \ell > 12
\end{aligned}$$

Forecasting seasonal models

Thus: The MA(∞) weights are simply $\psi_0 = 1$, $\psi_{12} = \Theta$, and $\psi_j = 0$ otherwise. This yields

$$\text{Var}(e_t(\ell)) \;=\; \begin{cases} \sigma_e^2 & 1 \le \ell \le 12 \\ (1+\Theta^2)\sigma_e^2 & \ell > 12 \end{cases}$$

Example: ARIMA(0, 1, 1)(0, 1, 1)$_{12}$ (airline model).

$$Y_t \;=\; Y_{t-1} + Y_{t-12} - Y_{t-13} + e_t - \theta e_{t-1} - \Theta e_{t-12} + \theta\Theta e_{t-13}$$

Forecasting seasonal models

Then, the forecasts combine properties of AR and MA forecasts:

$$
\begin{array}{rcllllll}
\widehat{Y}_t(1) & = & Y_t & +Y_{t-11} & -Y_{t-12} & -\theta e_t & -\Theta e_{t-11} & +\theta\Theta e_{t-12} \\
\widehat{Y}_t(2) & = & \widehat{Y}_t(1) & +Y_{t-10} & -Y_{t-11} & & -\Theta e_{t-10} & +\theta\Theta e_{t-11} \\
 & \vdots & & & & & & \\
\widehat{Y}_t(12) & = & \widehat{Y}_t(11) & +Y_t & -Y_{t-1} & & -\Theta e_t & +\theta\Theta e_{t-1} \\
\widehat{Y}_t(13) & = & \widehat{Y}_t(12) & +\widehat{Y}_t(1) & -Y_t & & & +\theta\Theta e_t
\end{array}
$$

and

$$\widehat{Y}_t(\ell) = \widehat{Y}_t(\ell-1) + \widehat{Y}_t(\ell-12) - \widehat{Y}_t(\ell-13) \qquad \text{for } \ell > 13$$

Forecasting seasonal models