

Univalent Enriched Categories and the Enriched Rezk Completion

Niels van der Weide

Univalent Foundations

- ▶ We work in **univalent foundations (UF)**
- ▶ Concretely, we assume the **univalence axiom**:

$$(A = B) \simeq (A \simeq B)$$

- ▶ Identity is proof relevant, and we interpret types as spaces

Univalence Principles

Using the univalence axiom we can prove that

- ▶ Monoids are **identified** up to **monoid isomorphism**:

$$(M = N) \simeq (M \cong N)$$

- ▶ Groups are **identified** up to **group isomorphism**:

$$(G = H) \simeq (G \cong H)$$

- ▶ Rings are **identified** up to **ring isomorphism**:

$$(R = S) \simeq (R \cong S)$$

So: sameness of algebraic structures is given by isomorphism

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So: sameness of algebraic structures is given by isomorphism
But **what about categories?**

Category Theory in UF

In UF, we have two notions of categories

- **Strict categories**: identified up to **isomorphism**, i.e.

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

such that $F(G(x)) = x$ and $G(F(x)) = x$.

- **Univalent categories**: identified up to **adjoint equivalence**, i.e.

$$\mathcal{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathcal{D}$$

together with natural isomorphisms $F \cdot G \cong \text{id } \mathcal{C}$ and $G \cdot F \cong \text{id } \mathcal{D}$ for which the triangle equations hold

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Our focus is on **univalent categories**

Why Univalent Categories

In this talk, we are only concerned with **univalent categories**

- ▶ In category theory, categories usually are identified up to adjoint equivalence
- ▶ The univalent perspective offers an interesting new perspective on category theory

Univalent Categories

Definition

Let \mathcal{C} be a category.

- ▶ We have a map idtoiso sending identities $p : x = y$ of objects $x, y : \mathcal{C}$ to isomorphisms $x \cong y$
- ▶ \mathcal{C} is **univalent** if idtoiso is an equivalence of types

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So:

- ▶ In univalent categories, we have $(x = y) \simeq (x \cong y)$
- ▶ Objects of univalent categories are **identified up to isomorphism**
- ▶ This follows **common mathematical practice** because properties of objects are invariant up to isomorphism
- ▶ Univalent categories are **identified up to adjoint equivalence**

Univalent Category Theory

There are many interesting aspects to univalent category theory

- **Univalence principle:** univalent categories are identified up to adjoint equivalence

¹Ahrens, Benedikt, Krzysztof Kapulkin, and Michael Shulman. "*Univalent categories and the Rezk completion*.", MSCS.

²Ahrens, Benedikt, Paige Randall North, Michael Shulman, and Dimitris Tsementzis. "*The univalence principle*", LICS 2020.

³Van der Weide, Niels. "*The Formal Theory of Monads, Univalently*", FSCD 2023.

Univalent Category Theory

There are many interesting aspects to univalent category theory

- ▶ **Univalence principle**: univalent categories are identified up to adjoint equivalence
- ▶ Constructively proving that **fully faithful and essentially surjective functors are adjoint equivalences**

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- ▶ **Univalence principle:** univalent categories are identified up to adjoint equivalence
- ▶ Constructively proving that **fully faithful and essentially surjective functors are adjoint equivalences**
- ▶ **Rezk completions:** every not necessarily univalent category is weakly equivalent to a univalent one (*weak equivalence: eso and fully faithful*)

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Univalent Category Theory

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- ▶ **Rezk completions**: every not necessarily univalent category is weakly equivalent to a univalent one (*weak equivalence: eso and fully faithful*)
- ▶ The usual definition of the Kleisli category does **not** give rise to a univalent category: instead we use the **Rezk completion**

Each of these points have been established in the literature^{1 2 3}

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Enriched Categories

We like **enriched categories** ⁴ ⁵

- ▶ An enriched category is a category whose hom-sets are endowed with extra structure
- ▶ For instance: every hom-set could be an abelian group or a DCPO
- ▶ Have found applications in programming languages ⁶, algebraic topology ⁷, higher category theory

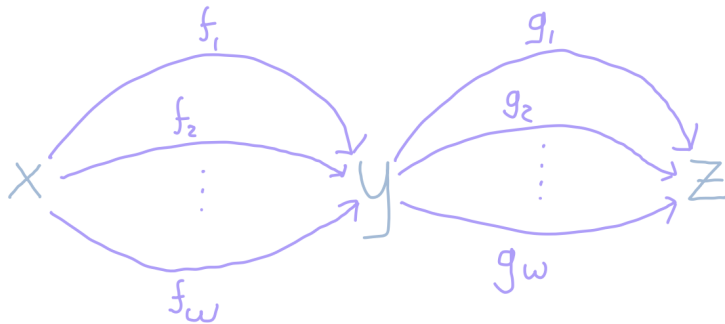
⁴Bénabou Jean. " *Catégories relatives*", C. R. Acad. Sci. Paris, 1965.

⁵Kelly, Max. " *Basic concepts of enriched category theory*", London Math. Soc. Lecture Note Ser., 64, 1982

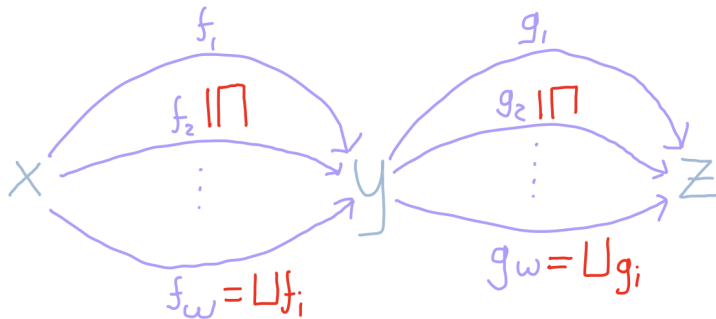
⁶Power, John. " *Models for the computational λ -calculus*", MFCSIT 2000.

⁷Goerss, Paul G., and John F. Jardine. *Simplicial homotopy theory*.

Enriched Categories Illustrated



Enriched Categories Illustrated



This Paper

This paper studies **univalent enriched categories**, and it contains

- ▶ A univalence principle for univalent enriched categories
- ▶ A proof that fully faithful and essentially surjective enriched functors are adjoint equivalences
- ▶ A construction of the Rezk completion of enriched categories and a proof of the universal mapping property
- ▶ Univalent enriched Kleisli categories
- ▶ The results are formalized in Coq proof assistant using the UniMath library

Monoidal Categories

Definition

A **monoidal category** V is given by

- ▶ an object $I : V$ called the **unit**
- ▶ an operation $\otimes : V \rightarrow V \rightarrow V$ called the **tensor**

Unitality and associativity hold up to coherent isomorphism. This means that we have natural isomorphisms $l : I \otimes x \cong x$, $r : x \otimes I \cong x$, and $a : x \otimes (y \otimes z) \cong (x \otimes y) \otimes z$ satisfying suitable coherences. In addition, \otimes is required to be functorial.

Enrichments

Suppose that we have

- ▶ A monoidal category V with unit I and tensor \otimes

Definition

A **V -enrichment** \mathcal{E} of a category C consists of

- ▶ a function $\mathcal{E}(-, -) : C \rightarrow C \rightarrow V$
- ▶ for all $x : C$ a morphism $\text{id}^e : I \rightarrow \mathcal{E}(x, x)$
- ▶ for all $x, y, z : C$ a morphism

$$\text{comp} : \mathcal{E}(y, z) \otimes \mathcal{E}(x, y) \rightarrow \mathcal{E}(x, z)$$

- ▶ for all $f : x \rightarrow y$ a morphism $\overrightarrow{f} : I \rightarrow \mathcal{E}(x, y)$
- ▶ for all $f : I \rightarrow \mathcal{E}(x, y)$ a morphism $\overleftarrow{f} : x \rightarrow y$

We require that $\overleftrightarrow{\overrightarrow{f}} = f$ and that $\overleftrightarrow{\overleftarrow{f}} = f$, and that these operations preserve identity and composition. Associativity and unitality are given in the next slides

Enrichments: Unitality Axioms

$$\begin{array}{ccc} \mathbf{1} \otimes \mathcal{E}(x, y) & \xrightarrow{\text{id}^e(y) \otimes \text{id}} & \mathcal{E}(y, y) \otimes \mathcal{E}(x, y) \\ & \searrow \text{l}_{\mathcal{E}(x, y)} & \downarrow \text{comp}(x, y, y) \\ & & \mathcal{E}(x, y) \end{array}$$

$$\begin{array}{ccc} \mathcal{E}(x, y) \otimes \mathbf{1} & \xrightarrow{\text{id} \otimes \text{id}^e(x)} & \mathcal{E}(x, y) \otimes \mathcal{E}(x, x) \\ & \searrow \text{r}_{\mathcal{E}(x, y)} & \downarrow \text{comp}(x, x, y) \\ & & \mathcal{E}(x, y) \end{array}$$

Enrichments: Associativity Axiom

$$\begin{array}{ccc} (\mathcal{E}(y, z) \otimes \mathcal{E}(x, y)) \otimes \mathcal{E}(w, x) & \xrightarrow{a} & \mathcal{E}(y, z) \otimes (\mathcal{E}(x, y) \otimes \mathcal{E}(w, x)) \\ \downarrow \text{comp}(x, y, z) \otimes \text{id} & & \downarrow \text{id} \otimes \text{comp}(w, x, y) \\ & & \mathcal{E}(y, z) \otimes \mathcal{E}(w, y) \\ & & \downarrow \text{comp}(w, y, z) \\ \mathcal{E}(x, z) \otimes \mathcal{E}(w, x) & \xrightarrow{\text{comp}(w, x, z)} & \mathcal{E}(w, z) \end{array}$$

Examples of Enrichments

Examples of enrichments:

- ▶ Every category has a unique set-enrichment
- ▶ The category of DCPOs is enriched over DCPOs
- ▶ More general, every cartesian closed category is enriched over itself
- ▶ Even more general, every symmetric monoidal closed category is enriched over itself

Enrichments and the Underlying Category

Some standard facts from enriched category theory

- ▶ We have 2-categories EnrichCat_V and Cat
- ▶ We have a pseudofunctor F from EnrichCat_V to Cat that sends an enriched category E to its underlying category E_0 (objects: same as in E , morphisms $I \rightarrow E(x, y)$)

Idea:

- ▶ a V -enrichment of C is an object in the fiber of C along F .
- ▶ the definition on the previous slide formalizes this idea.

For this reason, our definition is equivalent to the usual one.

Univalent Enriched Category Theory

Now we discuss

1. a notion of univalent enriched category and a univalence principle for them
2. essentially surjective fully faithful functors are adjoint equivalences
3. the Rezk completion of enriched categories

In addition, our proof techniques arise from **bicategory theory**

Univalent Enriched Categories

Definition

A **univalent V -enriched category** is a univalent category together with a V -enrichment.

If we would use Kelly's definition: a V -enriched category is univalent if its underlying category is univalent. This agrees with completeness of enriched ∞ -categories⁸.

⁸David Gepner and Rune Haugseng. " *Enriched ∞ -categories via non-symmetric ∞ -operads*", Adv. Math. 2015.

The Univalence Principle

- ▶ **Theorem:** identity of enriched categories corresponds to enriched adjoint equivalence
- ▶ We formulate that using **univalent bicategories**
- ▶ More specifically, we show that the bicategory of univalent enriched categories is univalent

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- ▶ Global univalence: identity of enriched categories corresponds to enriched adjoint equivalence
- ▶ Local univalence: identity of enriched functors corresponds to enriched natural isomorphism

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- ▶ Global univalence: identity of enriched categories corresponds to enriched adjoint equivalence
- ▶ Local univalence: identity of enriched functors corresponds to enriched natural isomorphism
- ▶ **Method:** displayed bicategories ⁹

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Displayed Bicategories

- ▶ **Main idea:** a displayed bicategory over a bicategory B represents structure/properties to be added to the objects, 1-cells, and 2-cells of B
- ▶ Displayed bicategories allow for **modular proofs of univalence**
- ▶ Every displayed bicategory gives rise to a bicategory by taking the total bicategory
- ▶ In essence, this generalizes taking Σ -types to bicategories

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Enriched categories as a displayed bicategory over \mathbf{Cat} :

- ▶ Objects over C : V -enrichments for C
- ▶ 1-cells over $F : C \rightarrow C'$ from \mathcal{E} to \mathcal{E}' : V -enrichments for F
- ▶ 2-cells over $\tau : F \Rightarrow G$ from \mathcal{F} to \mathcal{G} : proofs that τ is V -enriched

Univalent Enriched Category Theory

1. ~~a notion of univalent enriched category and a univalence principle for them~~
2. essentially surjective fully faithful functors are adjoint equivalences
3. the Rezk completion of enriched categories

Adjoint Equivalences

- ▶ **Key theorem about equivalences:** essentially surjective fully faithful functors are adjoint equivalences
- ▶ Usually, proving this requires the axiom of choice.

Proof sketch:

- ▶ Suppose $F : C \rightarrow D$ is fully faithful and essentially surjective
- ▶ To define an inverse, we need to find a preimage for every $y : D$
- ▶ Such preimages are only unique **up to isomorphism** (by fully faithfulness)
- ▶ However, if C is univalent, then the preimages are unique

Orthogonal Factorization Systems

- ▶ We construct the orthogonal factorization system of eso functors and fully faithful functors¹⁰
- ▶ Orthogonality means that we can solve the following lifting problems:

$$\begin{array}{ccc} C_1 & \xrightarrow{G_1} & C_2 \\ F_1 \downarrow & \nearrow \text{dashed} & \downarrow F_2 \\ D_1 & \xrightarrow{G_2} & D_2 \end{array}$$

where F_1 is eso and F_2 is fully faithful

- ▶ By taking $F_1 = F_2$ and $G_1 = G_2 = \text{id}$, we get the desired theorem

¹⁰Fosco Loregian and Emily Riehl. " *Categorical notions of fibration* ", Expo. Math. 2019.

Univalent Enriched Category Theory

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2. ~~essentially surjective fully faithful functors are adjoint equivalences~~
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The Rezk Completion

- ▶ Every category C is weakly equivalent to a univalent one
- ▶ This is known as the **Rezk completion**
- ▶ Universal property: left biadjoint of $\text{UnivCat} \rightarrow \text{Cat}$

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The Rezk completion of C is constructed as follows:

- ▶ Take the image of $C \rightarrow [C^{\text{op}}, \text{Set}]$
- ▶ Essentially surjective: by definition
- ▶ Fully faithful: by the Yoneda lemma
- ▶ Univalence: since Set is univalent

The **Enriched** Rezk Completion

Every **enriched** category \mathcal{E} is weakly equivalent to a univalent one

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Every **enriched** category \mathcal{E} is weakly equivalent to a univalent one
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- ▶ Essentially surjective: by definition
- ▶ Fully faithful: by the Yoneda lemma
- ▶ Univalence: since \mathbf{V} is univalent

We can also prove the universal property
Here we need:

- ▶ \mathbf{V} is symmetric (opposite enriched categories)
- ▶ \mathbf{V} is closed (self enriched categories)
- ▶ \mathbf{V} is complete (enriched functor categories)

Conclusion

- ▶ We developed enriched categories in univalent foundations
- ▶ We proved a univalence principle and we showed that weak equivalences between univalent categories are adjoint equivalences
- ▶ We also constructed a Rezk completion and proved its universal property (useful for Kleisli categories)
- ▶ Key techniques: enrichments, displayed bicategories