Enriched Categories in Univalent Foundations

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What are enriched categories?

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- **Category**: we have objects and between objects, we have a **set** of morphisms
- **Enriched category**: we take the previous definition, but what if we replace set by partial order, abelian group, dcpo, or *an object of an arbitrary monoidal category*?

So: enriched categories are categories whose homsets are endowed with extra structure

Motivation

Applications in mathematics:

- Simplicial homotopy theory [1]
- Strict n-categories can be defined using enriched categories
- Homological algebra [2]

Applications in computer science:

- Interpreting general recursion in categories [3]
- Models for the computational λ-calculus [4]
- Models for typed PCF with general recursion [5]
- Enriched effect calculus [6]

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[1] Goerss, Paul G., and John F. Jardine. Simplicial homotopy theory.

[2] Weibel, Charles A. An introduction to homological algebra.

[3] Wand, Mitchell. "Fixed-point constructions in order-enriched categories."

[4] Power, John. "Models for the computational λ-calculus."

[5] Plotkin, Gordon, and John Power. "Adequacy for algebraic effects."

[6] Egger, Jeff, Rasmus Ejlers Møgelberg, and Alex Simpson. "The enriched effect calculus: syntax and semantics."

Enriched Categories in Univalent Foundations

According to the title, this talk will be about enriched categories in univalent foundations.

More specifically, we discuss the following

- What is a univalent enriched category?
- The univalent bicategory of univalent enriched categories

The theorems/definitions in this talk are formalized in UniMath[7].

[7] `https://github.com/UniMath/UniMath`

Overview of the Construction

Goal: the univalent bicategory of univalent enriched categories

[8] Ahrens, Benedikt, and Peter LeFanu Lumsdaine. "Displayed categories.
[9] Ahrens, Benedikt, et al. "Bicategories in univalent foundations.

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Main idea: a univalent enriched category is a univalent category with an enrichment

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Technique: displayed bicategories[8] [9]

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Technique: displayed bicategories[8] [9]
This talk: we discuss

- Short recap: what are univalent categories
- Enrichments for categories
- Brief overview of the construction with displayed bicategories

[8]Ahrens, Benedikt, and Peter LeFanu Lumsdaine. "Displayed categories.
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Recall: Univalence for Categories

Definition

Let C be a category.

- For all objects x, y, we have a map $\mathbf{idtoiso}_{x,y} : x = y \to x \cong y$ sending equalities to isomorphism (*defined using path induction*)
- A category is called **univalent**[10] if for all x, y the map $\mathbf{idtoiso}_{x,y}$ is an equivalence of types.

Note: I deviate from the terminology in the HoTT book where category is used for univalent precategories

[10] Ahrens, Benedikt, Krzysztof Kapulkin, and Michael Shulman. "Univalent categories and the Rezk completion.

Enrichments: Definition

Suppose that we have

- A monoidal category $\mathcal{V}$ with unit $\mathbb{1}$ and tensor $\otimes$

Definition

A $\mathcal{V}$-**enrichment** E of a category C consists of

- a function $\mathsf{E}(-,-) : \mathsf{C} \to \mathsf{C} \to \mathcal{V}$;

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- for $x : \mathsf{C}$ a morphism $\mathsf{Id} : \mathbb{1} \to \mathsf{E}(x,x)$ in $\mathcal{V}$;
- for $x, y, z : \mathsf{C}$ a morphism $\mathsf{Comp} : \mathsf{E}(y,z) \otimes \mathsf{E}(x,y) \to \mathsf{E}(y,z)$ in $\mathcal{V}$;

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- for $x, y, z : \mathsf{C}$ a morphism $\mathsf{Comp} : \mathsf{E}(y, z) \otimes \mathsf{E}(x, y) \to \mathsf{E}(y, z)$ in $\mathcal{V}$;
- functions $\mathsf{FromArr} : \mathsf{C}(x, y) \to \mathcal{V}(\mathbb{1}, \mathsf{E}(x, y))$ and $\mathsf{ToArr} : \mathcal{V}(\mathbb{1}, \mathsf{E}(x, y)) \to \mathsf{C}(x, y)$ for all $x, y : \mathsf{C}$

We require the usual axioms and that FromArr and ToArr are inverses.

Enrichments: Idea

Some standard facts from enriched category theory[11]

- We have 2-categories $\mathcal{V}\mathsf{Cat}$ and Cat
- We have a pseudofunctor from $\mathcal{V}\mathsf{Cat}$ to Cat that sends an enriched category E to its **underlying category** E_0 (*objects: same as in* E, *morphisms* $\mathbb{1} \to \mathsf{E}(x, y)$)

Idea:

- a $\mathcal{V}$-**enrichment** of C is an object in the fiber of C along this pseudofunctor.
- the definition on the previous slide formulates this idea.

[11] Kelly, Max. Basic concepts of enriched category theory.

[12] McDermott, Dylan, and Tarmo Uustalu. "What makes a strong monad?.

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Note: other definitions of enrichments have also been given[12]

[11] Kelly, Max. Basic concepts of enriched category theory.

[12] McDermott, Dylan, and Tarmo Uustalu. "What makes a strong monad?.

Univalent Enriched Categories

A **univalent $\mathcal{V}$-enriched category** is a univalent category together with a $\mathcal{V}$-enrichment.

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Comments:

- One might wonder: should univalence interact with enrichment?
- For example, for bicategories we have a local and a global univalence condition.
- However, bicategories are instances of **weak enrichments** (over bicategories).
- We look at a stricter notion, namely enrichments over monoidal categories.

The Univalent Bicategory of Univalent Enriched Categories

Overview of the construction:

- We have the bicategory UnivCat of univalent categories
- We define a displayed bicategory $\mathcal{V}\mathsf{UnivCat}_{\mathsf{disp}}$ over UnivCat whose objects over C are $\mathcal{V}$-enrichments over C

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Theorem

If $\mathcal{V}$ is univalent, then $\mathcal{V}\mathsf{UnivCat}$ is a univalent bicategory.

Change of Base

Suppose, we have

- A lax monoidal functor $F : \mathcal{V} \to \mathcal{W}$
- A $\mathcal{V}$-enriched category E

Then we define a $\mathcal{W}$-enriched category E_F

- The objects of E_F are objects of E
- For $x, y : \mathsf{E}$ we define $\mathsf{E}_F(x, y)$ to be $F(\mathsf{E}(x, y))$
- Composition and identity: from E

Change of Base and Univalence

Note:

- We have a functor $! : \mathcal{V} \to \mathbf{1}$ to the terminal monoidal category
- So: every $\mathcal{V}$-enriched category gives rise to a $\mathbf{1}$-enriched category

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Instantiate this to Set:

- Set is Set-enriched
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What does the **underlying category** of $\mathrm{Set}_!$ look like?

- Objects: sets
- Morphisms: inhabitants of unit type

This is not univalent at all.

Change of Base in our setting

Suppose, we have

- A **fully faithful** and **strong** monoidal functor $F : \mathcal{V} \to \mathcal{W}$
- A category C with a enrichment E over $\mathcal{V}$

Then we define a $\mathcal{W}$-enrichment $\mathsf{E}_F(x, y)$ of C

- For $x, y : \mathsf{E}$ we define $\mathsf{E}_F(x, y)$ to be $F(\mathsf{E}(x, y))$
- Composition and identity: from E

What's included in the formalization so far

- The univalent bicategory of univalent enriched categories
- Limits and colimits in enriched categories
- Enriched monads, and a construction of Eilenberg-Moore objects in the bicategory of enriched categories
- Various examples: self-enriched categories, change of base, the opposite
- Characterization of enrichments over structured sets

What's included in the formalization so far

- The univalent bicategory of univalent enriched categories
- Limits and colimits in enriched categories
- Enriched monads, and a construction of Eilenberg-Moore objects in the bicategory of enriched categories
- Various examples: self-enriched categories, change of base, the opposite
- Characterization of enrichments over structured sets (*in the literature, often simplified definitions of enriched categories are used (eg for posets/abelian groups). We define a general notion of structured set and we characterize enrichments over structured sets via a similar simplified definition*)

Conclusion

Main take-aways of this talk:

- Enriched categories are nice and useful
- Univalence for enriched categories: the underlying category is univalent
- We showed: the bicategory of univalent enriched categories is again univalent
- Some interesting peculiarities happen with univalent enriched categories (change of base)