Univalent Enriched Categories and the Enriched Rezk Completion

Niels van der Weide

Univalent Foundations

- We work in **univalent foundations (UF)**
- Concretely, we assume the **univalence axiom**:

$$(A = B) \simeq (A \simeq B)$$

- Identity is proof relevant, and we interpret types as spaces

Univalence Principles

Using the univalence axiom we can prove that

- Monoids are **identified** up to **monoid isomorphism**:

$$(M = N) \simeq (M \cong N)$$

- Groups are **identified** up to **group isomorphism**:

$$(G = H) \simeq (G \cong H)$$

- Rings are **identified** up to **ring isomorphism**:

$$(R = S) \simeq (R \cong S)$$

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So: sameness of algebraic structures is given by isomorphism
But **what about categories**?

Category Theory in UF

In UF, we have two notions of categories

- **Strict categories**: identified up to **isomorphism**, i.e.

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

  such that $F(G(x)) = x$ and $G(F(x)) = x$.
- **Univalent categories**: identified up to **adjoint equivalence**, i.e.

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd}$$

  together with natural isomorphisms $F \cdot G \cong \mathrm{id}\ \mathrm{C}$ and $G \cdot F \cong \mathrm{id}\ \mathrm{D}$ for which the triangle equations hold

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Our focus is on **univalent categories**

Why Univalent Categories

In this talk, we are only concerned with **univalent categories**

- In category theory, categories usually are identified up to adjoint equivalence
- The univalent perspective offers an interesting new perspective on category theory

Univalent Categories

Definition

Let $\mathcal{C}$ be a category.

- We have a map idtoiso sending identities $p : x = y$ of objects $x, y : \mathcal{C}$ to isomorphisms $x \cong y$
- $\mathcal{C}$ is **univalent** if idtoiso is an equivalence of types

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So:

- In univalent categories, we have $(x = y) \simeq (x \cong y)$
- Objects of univalent categories are **identified up to isomorphism**
- This follows **common mathematical practice** because properties of objects are invariant up to isomorphism
- Univalent categories are **identified up to adjoint equivalence**

Univalent Category Theory

There are many interesting aspects to univalent category theory

- **Univalence principle**: univalent categories are identified up to adjoint equivalence

[1]Ahrens, Benedikt, Krzysztof Kapulkin, and Michael Shulman. "*Univalent categories and the Rezk completion*.", MSCS.

[2]Ahrens, Benedikt, Paige Randall North, Michael Shulman, and Dimitris Tsementzis. "*The univalence principle*", LICS 2020.

[3]Van der Weide, Niels. "*The Formal Theory of Monads, Univalently.*", FSCD 2023.

Univalent Category Theory

There are many interesting aspects to univalent category theory

- **Univalence principle**: univalent categories are identified up to adjoint equivalence
- Constructively proving that **fully faithful and essentially surjective functors are adjoint equivalences**

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Univalent Category Theory

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- **Univalence principle**: univalent categories are identified up to adjoint equivalence
- Constructively proving that **fully faithful and essentially surjective functors are adjoint equivalences**
- **Rezk completions**: every not necessarily univalent category is weakly equivalent to a univalent one (*weak equivalence: eso and fully faithful*)

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Univalent Category Theory

There are many interesting aspects to univalent category theory

- **Univalence principle**: univalent categories are identified up to adjoint equivalence
- Constructively proving that **fully faithful and essentially surjective functors are adjoint equivalences**
- **Rezk completions**: every not necessarily univalent category is weakly equivalent to a univalent one (*weak equivalence: eso and fully faithful*)
- The usual definition of the Kleisli category does **not** give rise to a univalent category: instead we use the **Rezk completion**

Each of these points have been established in the literature[1] [2] [3]

[1] Ahrens, Benedikt, Krzysztof Kapulkin, and Michael Shulman. "*Univalent categories and the Rezk completion*.", MSCS.

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Enriched Categories

We like **enriched categories** [4] [5]

- An enriched category is a category whose hom-sets are endowed with extra structure
- For instance: every hom-set could be an abelian group or a DCPO
- Have found applications in programming languages [6], algebraic topology [7], higher category theory

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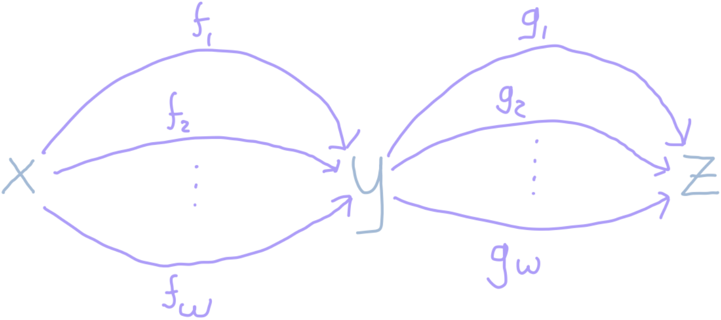
[4] Bénabou Jean. "*Catégories relatives*", C. R. Acad. Sci. Paris, 1965.

[5] Kelly, Max. "*Basic concepts of enriched category theory*", London Math. Soc. Lecture Note Ser., 64, 1982

[6] Power, John. "*Models for the computational λ-calculus*", MFCSIT 2000.

[7] Goerss, Paul G., and John F. Jardine. *Simplicial homotopy theory*.

Enriched Categories Illustrated



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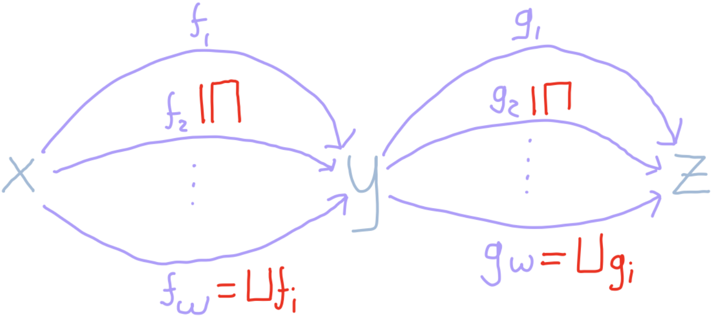

This Paper

This paper studies **univalent enriched categories**, and it contains

- A univalence principle for univalent enriched categories
- A proof that fully faithful and essentially surjective enriched functors are adjoint equivalences
- A construction of the Rezk completion of enriched categories and a proof of the universal mapping property
- Univalent enriched Kleisli categories
- The results are formalized in Coq proof assistant using the UniMath library

Monoidal Categories

Definition

A **monoidal category** V is given by

- an object $\mathsf{I} : \mathsf{V}$ called the **unit**
- an operation $\otimes : \mathsf{V} \to \mathsf{V} \to \mathsf{V}$ called the **tensor**

Unitality and associativity hold up to coherent isomorphism. This means that we have natural isomorphisms $\mathsf{l} : \mathsf{I} \otimes x \cong x$, $\mathsf{r} : x \otimes \mathsf{I} \cong x$, and $\mathsf{a} : x \otimes (y \otimes z) \cong (x \otimes y) \otimes z$ satisfying suitable coherences. In addition, $\otimes$ is required to be functorial.

Enrichments

Suppose that we have

- A monoidal category V with unit I and tensor $\otimes$

Definition

A V-**enrichment** $\mathcal{E}$ of a category C consists of

- a function $\mathcal{E}(-,-) : \mathsf{C} \to \mathsf{C} \to \mathsf{V}$
- for all $x : \mathsf{C}$ a morphism $\mathsf{id}^{\mathsf{e}} : \mathsf{I} \to \mathcal{E}(x, x)$
- for all $x, y, z : \mathsf{C}$ a morphism

$$\mathsf{comp} : \mathcal{E}(y, z) \otimes \mathcal{E}(x, y) \to \mathcal{E}(x, z)$$

- for all $f : x \to y$ a morphism $\overrightarrow{f} : \mathsf{I} \to \mathcal{E}(x, y)$
- for all $f : \mathsf{I} \to \mathcal{E}(x, y)$ a morphism $\overleftarrow{f} : x \to y$

We require that $\overleftarrow{\overrightarrow{f}} = f$ and that $\overrightarrow{\overleftarrow{f}} = f$, and that these operations preserve identity and composition. Associativity and unitality are given in the next slides

Enrichments: Unitality Axioms

$$\begin{tikzcd}
\mathsf{I} \otimes \mathcal{E}(x,y) \arrow[r, "\mathsf{id}^{\mathsf{e}}(y) \otimes \mathsf{id}"] \arrow[dr, "\mathsf{l}_{\mathcal{E}(x,y)}"'] & \mathcal{E}(y,y) \otimes \mathcal{E}(x,y) \arrow[d, "\mathsf{comp}(x,y,y)"] \\
& \mathcal{E}(x,y)
\end{tikzcd}$$

$$\begin{tikzcd}
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\end{tikzcd}$$

Enrichments: Associativity Axiom

$$\begin{tikzcd}
(\mathcal{E}(y,z) \otimes \mathcal{E}(x,y)) \otimes \mathcal{E}(w,x) \arrow[r, "\mathsf{a}"] \arrow[dd, "\mathsf{comp}(x,y,z)\otimes\mathsf{id}"'] & \mathcal{E}(y,z) \otimes (\mathcal{E}(x,y) \otimes \mathcal{E}(w,x)) \arrow[d, "\mathsf{id}\otimes\mathsf{comp}(w,x,y)"] \\
& \mathcal{E}(y,z) \otimes \mathcal{E}(w,y) \arrow[d, "\mathsf{comp}(w,y,z)"] \\
\mathcal{E}(x,z) \otimes \mathcal{E}(w,x) \arrow[r, "\mathsf{comp}(w,x,z)"'] & \mathcal{E}(w,z)
\end{tikzcd}$$

Examples of Enrichments

Examples of enrichments:

- Every category has a unique set-enrichment
- The category of DCPOs is enriched over DCPOs
- More general, every cartesian closed category is enriched over itself
- Even more general, every symmetric monoidal closed category is enriched over itself

Enrichments and the Underlying Category

Some standard facts from enriched category theory

- We have 2-categories $\mathsf{EnrichCat}_{\mathsf{V}}$ and Cat
- We have a pseudofunctor F from $\mathsf{EnrichCat}_{\mathsf{V}}$ to Cat that sends an enriched category E to its underlying category E_0 (objects: same as in E, morphisms $\mathsf{I} \to \mathsf{E}(x, y)$)

Idea:

- a V-enrichment of C is an object in the fiber of C along F.
- the definition on the previous slide formalizes this idea.

For this reason, our definition is equivalent to the usual one.

Univalent Enriched Category Theory

Now we discuss

1. a notion of univalent enriched category and a univalence principle for them
2. essentially surjective fully faithful functors are adjoint equivalences
3. the Rezk completion of enriched categories

In addition, our proof techniques arise from **bicategory theory**

Univalent Enriched Categories

Definition

A **univalent** V-**enriched category** is a univalent category together with a V-enrichment.

If we would use Kelly's definition: a V-enriched category is univalent if its underlying category is univalent. This agrees with completeness of enriched ∞-categories [8].

[8] David Gepner and Rune Haugseng. "*Enriched ∞-categories via non-symmetric ∞-operads*", Adv. Math. 2015.

The Univalence Principle

- **Theorem**: identity of enriched categories corresponds to enriched adjoint equivalence
- We formulate that using **univalent bicategories**
- More specifically, we show that the bicategory of univalent enriched categories is univalent

[9] Benedikt Ahrens, Dan Frumin, Marco Maggesi, Niccolò Veltri, and Niels van der Weide. "*Bicategories in univalent foundations.*", MSCS.

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- Global univalence: identity of enriched categories corresponds to enriched adjoint equivalence
- Local univalence: identity of enriched functors corresponds to enriched natural isomorphism

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- Local univalence: identity of enriched functors corresponds to enriched natural isomorphism
- **Method**: displayed bicategories [9]

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[9] Benedikt Ahrens, Dan Frumin, Marco Maggesi, Niccolò Veltri, and Niels van der Weide. "*Bicategories in univalent foundations.*", MSCS.

Displayed Bicategories

- **Main idea**: a displayed bicategory over a bicategory B represents structure/properties to be added to the objects, 1-cells, and 2-cells of B
- Displayed bicategories allow for **modular proofs of univalence**
- Every displayed bicategory gives rise to a bicategory by taking the total bicategory
- In essence, this generalizes taking $\sum$-types to bicategories

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Enriched categories as a displayed bicategory over Cat:

- Objects over C: V-enrichments for C
- 1-cells over $F : \mathsf{C} \to \mathsf{C}'$ from $\mathcal{E}$ to $\mathcal{E}'$: V-enrichments for F
- 2-cells over $\tau : F \Rightarrow G$ from $\mathcal{F}$ to $\mathcal{G}$: proofs that τ is V-enriched

Univalent Enriched Category Theory

1. ~~a notion of univalent enriched category and a univalence principle for them~~
2. essentially surjective fully faithful functors are adjoint equivalences
3. the Rezk completion of enriched categories

Adjoint Equivalences

- **Key theorem about equivalences**: essentially surjective fully faithful functors are adjoint equivalences
- Usually, proving this requires the axiom of choice.

Proof sketch:

- Suppose $F : \mathsf{C} \to \mathsf{D}$ is fully faithful and essentially surjective
- To define an inverse, we need to find a preimage for every $y : \mathsf{D}$
- Such preimages are only unique **up to isomorphism** (by fully faithfulness)
- However, if C is univalent, then the preimages are unique

Orthogonal Factorization Systems

- We construct the orthogonal factorization system of eso functors and fully faithful functors [10]
- Orthogonality means that we can solve the following lifting problems:

$$\begin{tikzcd}
\mathsf{C}_1 \arrow[r, "G_1"] \arrow[d, "F_1"'] & \mathsf{C}_2 \arrow[d, "F_2"] \\
\mathsf{D}_1 \arrow[r, "G_2"'] \arrow[ur, dashed] & \mathsf{D}_2
\end{tikzcd}$$

  where F_1 is eso and F_2 is fully faithful
- By taking $F_1 = F_2$ and $G_1 = G_2 = \mathrm{id}$, we get the desired theorem

[10] Fosco Loregian and Emily Riehl. "*Categorical notions of fibration*", Expo. Math. 2019.

Univalent Enriched Category Theory

1. ~~a notion of univalent enriched category and a univalence principle for them~~
2. ~~essentially surjective fully faithful functors are adjoint equivalences~~
3. the Rezk completion of enriched categories

The Rezk Completion

- Every category C is weakly equivalent to a univalent one
- This is known as the **Rezk completion**
- Universal property: left biadjoint of UnivCat $\rightarrow$ Cat

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The Rezk completion of C is constructed as follows:

- Take the image of $\mathsf{C} \rightarrow [\mathsf{C}^{\mathsf{op}}, \mathsf{Set}]$
- Essentially surjective: by definition
- Fully faithful: by the Yoneda lemma
- Univalence: since Set is univalent

The **Enriched** Rezk Completion

Every **enriched** category $\mathcal{E}$ is weakly equivalent to a univalent one

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Every **enriched** category $\mathcal{E}$ is weakly equivalent to a univalent one
The Rezk completion of $\mathcal{E}$ is constructed as follows:

- Take the image of $\mathcal{E} \to [\mathcal{E}^{\mathsf{op}}, \mathsf{V}]$
- Essentially surjective: by definition
- Fully faithful: by the Yoneda lemma
- Univalence: since V is univalent

We can also prove the universal property
Here we need:

- V is symmetric (opposite enriched categories)
- V is closed (self enriched categories)
- V is complete (enriched functor categories)

Conclusion

- We developed enriched categories in univalent foundations
- We proved a univalence principle and we showed that weak equivalences between univalent categories are adjoint equivalences
- We also constructed a Rezk completion and proved its universal property (useful for Kleisli categories)
- Key techniques: enrichments, displayed