Semantics of two-dimensional type theory

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2-Dimensional interpretations of type theory

There are many interpretations of type theory that are 2-dimensional in a certain sense

- ▶ The groupoid interpretation by Hofmann and Streicher ¹
- ► The two-dimensional models by Garner²

Interpreted in something like groupoids

¹Hofmann, Martin, and Thomas Streicher. "The groupoid interpretation of type theory." *Twenty-five years of constructive type theory (Venice, 1995)* 36 (1998): 83-111.

²Garner, Richard. "Two-dimensional models of type theory." *Mathematical structures in computer science* 19.4 (2009): 687-736.

Directed type theory

But directed variants have also been considered

- ► An interpretation with directed definitional equality³
- ► A syntactical framework for directed type theory⁴
- ► An interpretation with directed identity types⁵

Interpreted in something like categories

³Licata, Daniel R., and Robert Harper. "2-dimensional directed type theory." *Electronic Notes in Theoretical Computer Science* 276 (2011): 263-289.

⁴Nuyts, Andreas. Towards a directed homotopy type theory based on 4 kinds of variance. Master's thesis, KU Leuven, 2015.

⁵North, Paige Randall. "Towards a directed homotopy type theory." *Electronic Notes in Theoretical Computer Science* 347 (2019): 223-239.

A framework is missing

Problem:

- ► Garner gave a general notion of 2-dimensional comprehension category, but this only works for **undirected** type theory
- The interpretations of directed type theory are ad hoc

Goal of this talk:

find categorical framework in which one can interpret various flavors of 2-dimensional type theory

The work in this talk has been formalized using UniMath.

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- Use bicategories instead of categories
- Define comprehension bicategories.
- For that, we need a bicategorical notion of fibration
- ▶ We use ingredients by Hermida⁶ and by Buckley⁷
- Find suitable instances of comprehension bicategories

⁶Hermida, Claudio. "Some properties of Fib as a fibred 2-category." *Journal of Pure and Applied Algebra* 134.1 (1999): 83-109.

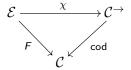
⁷Buckley, Mitchell. "Fibred 2-categories and bicategories." *Journal of Pure and Applied Algebra* 218.6 (2014): 1034-1074.

Comprehension categories

Type theory can be interpreted in **comprehension categories**.

Definition

A comprehension category is a strictly commuting triangle



where F is a Grothendieck fibration and where χ preserves cartesian cells.

Fibrations of bicategories

► The notion of fibration of bicategories has a **global** condition and **local** condition.

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- ► For both the local condition, we need an appropriate notion of cartesian 2-cells.

Fibrations of bicategories

- ► The notion of fibration of bicategories has a **global** condition and **local** condition.
- ► For both the global condition, we need an appropriate notion of cartesian 1-cells.
- For both the local condition, we need an appropriate notion of cartesian 2-cells.
- We also require that cartesian 2-cells are closed under horizontal composition.

The global condition

A pseudofunctor satisfying the global condition is called a **global fibration**.

The global condition basically says the following: given

- ightharpoonup contexts Γ_1 and Γ_2
- ▶ a substitution $s : \Gamma_1 \to \Gamma_2$
- ightharpoonup a type A in context Γ_2

we get a type A[s] in context Γ_1 .

The local condition

We think of 2-cells $\tau: s_1 \Rightarrow s_2$ as reductions from s_1 to s_2 A pseudofunctor satisfying the local condition is called a **local opfibration**.

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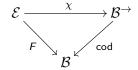
Local condition: given

- ightharpoonup contexts Γ_1, Γ_2
- ▶ substitutions $s_1, s_2 : \Gamma_1 \to \Gamma_2$
- ightharpoonup a reduction $au: s_1 \Rightarrow s_2$
- \triangleright a type A in context Γ_2
- ightharpoonup a term t of type $A[s_1]$

we get a a term of type $A[s_2]$.

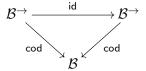
Comprehension bicategories

A comprehension bicategory is a *strictly* commuting triangle



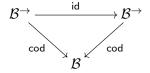
where χ preserves cartesian cells and where ${\it F}$ is a global fibration and a local opfibration.

Given a **locally groupoidal** bicategory $\mathcal B$ with pullbacks, take



⁸Garner, Richard. "Two-dimensional models of type theory." *Mathematical structures in computer science* 19.4 (2009): 687-736.

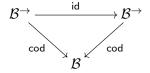
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This resembles the models by Garner⁸

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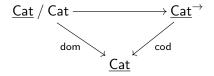
Given a **locally groupoidal** bicategory ${\cal B}$ with pullbacks, take



This resembles the models by Garner⁸
This does **not** work for arbitrary bicategories.

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The work by North⁹ and by Licata and Harper¹⁰ is encapsulated in the following comprehension bicategory



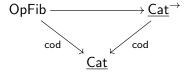
Here:

- <u>Cat</u> is the bicategory of categories
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⁹North, Paige Randall. "Towards a directed homotopy type theory." *Electronic Notes in Theoretical Computer Science* 347 (2019): 223-239.

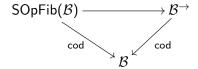
¹⁰Licata, Daniel R., and Robert Harper. "2-dimensional directed type theory." *Electronic Notes in Theoretical Computer Science* 276 (2011): 263-289.

We have the following comprehension bicategory



Categorification of the previous one

Given a bicategory ${\cal B}$ with pullbacks, take



Here $\mathsf{SOpFib}(\mathcal{B})$ is the bicategory of Street opfibrations in \mathcal{B} . Directed version of Garner's stuff.

Remark on the formalization

- ► The notion of comprehension bicategory and fibration have been formalized using UniMath.
- ► Here we make use of **displayed bicategories**¹¹

 $^{^{11}\}mbox{Ahrens, B., et al (2022)}.$ Bicategories in univalent foundations. Mathematical Structures in Computer Science, 1-38.

Conclusion

- ▶ We defined a notion of comprehension bicategory
- ► This is a suitable framework in which one can interpret (directed) type theory: we proved soundness
- ► There are general instances of this definition (internal Street fibrations)

Further work: look at type formers, completeness