Semantics of two-dimensional type theory

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2-Dimensional interpretations of type theory

There are many interpretations of type theory that are 2-dimensional in a certain sense

- The groupoid interpretation by Hofmann and Streicher [1]
- The two-dimensional models by Garner[2]

Interpreted in something like **groupoids**

[1] Hofmann, Martin, and Thomas Streicher. "The groupoid interpretation of type theory." *Twenty-five years of constructive type theory (Venice, 1995)* 36 (1998): 83-111.

[2] Garner, Richard. "Two-dimensional models of type theory." *Mathematical structures in computer science* 19.4 (2009): 687-736.

Directed type theory

But directed variants have also been considered

- An interpretation with directed definitional equality[3]
- A syntactical framework for directed type theory[4]
- An interpretation with directed identity types[5]

Interpreted in something like **categories**

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[3] Licata, Daniel R., and Robert Harper. "2-dimensional directed type theory." *Electronic Notes in Theoretical Computer Science* 276 (2011): 263-289.

[4] Nuyts, Andreas. Towards a directed homotopy type theory based on 4 kinds of variance. Master's thesis, KU Leuven, 2015.

[5] North, Paige Randall. "Towards a directed homotopy type theory." *Electronic Notes in Theoretical Computer Science* 347 (2019): 223-239.

A framework is missing

Problem:

- Garner gave a general notion of 2-dimensional comprehension category, but this only works for **undirected** type theory
- The interpretations of directed type theory are ad hoc

Goal of this talk:

find categorical framework in which one can interpret various flavors of 2-dimensional type theory

The work in this talk has been formalized using UniMath.

Idea

- Use bicategories instead of categories
- Define **comprehension bicategories**.
- For that, we need a bicategorical notion of fibration
- We use ingredients by Hermida[6] and by Buckley[7]
- Find suitable instances of comprehension bicategories

[6] Hermida, Claudio. "Some properties of Fib as a fibred 2-category." *Journal of Pure and Applied Algebra* 134.1 (1999): 83-109.

[7] Buckley, Mitchell. "Fibred 2-categories and bicategories." *Journal of Pure and Applied Algebra* 218.6 (2014): 1034-1074.

Comprehension categories

Type theory can be interpreted in **comprehension categories**.

Definition

A **comprehension category** is a *strictly* commuting triangle

$$\begin{tikzcd} \mathcal{E} \arrow[rr, "\chi"] \arrow[dr, "F"'] & & \mathcal{C}^{\to} \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{C} & \end{tikzcd}$$

where F is a Grothendieck fibration and where χ preserves cartesian cells.

Fibrations of bicategories

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Fibrations of bicategories

- The notion of fibration of bicategories has a **global** condition and **local** condition.
- For both the global condition, we need an appropriate notion of cartesian 1-cells.
- For both the local condition, we need an appropriate notion of cartesian 2-cells.
- We also require that cartesian 2-cells are closed under horizontal composition.

The global condition

A pseudofunctor satisfying the global condition is called a **global fibration**.

The global condition basically says the following: given

- contexts Γ_1 and Γ_2
- a substitution $s : \Gamma_1 \to \Gamma_2$
- a type A in context Γ_2

we get a type $A[s]$ in context Γ_1.

The local condition

We think of 2-cells $\tau : s_1 \Rightarrow s_2$ as reductions from s_1 to s_2
A pseudofunctor satisfying the local condition is called a **local opfibration**.

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Local condition: given

- contexts Γ_1, Γ_2
- substitutions $s_1, s_2 : \Gamma_1 \to \Gamma_2$
- a reduction $\tau : s_1 \Rightarrow s_2$
- a type A in context Γ_2
- a term t of type $A[s_1]$

we get a a term of type $A[s_2]$.

Comprehension bicategories

A **comprehension bicategory** is a *strictly* commuting triangle

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "\chi"] \arrow[dr, "F"'] & & \mathcal{B}^{\rightarrow} \arrow[dl, "\mathsf{cod}"] \\
& \mathcal{B} &
\end{tikzcd}$$

where χ preserves cartesian cells and where F is a global fibration and a local opfibration.

Examples of comprehension bicategories

Given a **locally groupoidal** bicategory $\mathcal{B}$ with pullbacks, take

$$\begin{tikzcd} \mathcal{B}^{\rightarrow} \arrow[rr, "\mathrm{id}"] \arrow[dr, "\mathrm{cod}"'] & & \mathcal{B}^{\rightarrow} \arrow[dl, "\mathrm{cod}"] \\ & \mathcal{B} & \end{tikzcd}$$

[8] Garner, Richard. "Two-dimensional models of type theory." *Mathematical structures in computer science* 19.4 (2009): 687-736.

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This resembles the models by Garner[8]

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This resembles the models by Garner[8]
This does **not** work for arbitrary bicategories.

[8] Garner, Richard. "Two-dimensional models of type theory." *Mathematical structures in computer science* 19.4 (2009): 687-736.

Examples of comprehension bicategories

The work by North[9] and by Licata and Harper[10] is encapsulated in the following comprehension bicategory

$$\begin{tikzcd} \underline{\mathrm{Cat}} \,/\, \mathrm{Cat} \arrow[rr] \arrow[dr, "\mathrm{dom}"'] & & \underline{\mathrm{Cat}}^{\to} \arrow[dl, "\mathrm{cod}"] \\ & \underline{\mathrm{Cat}} & \end{tikzcd}$$

Here:

- $\underline{\mathrm{Cat}}$ is the bicategory of categories
- Cat is the category of categories

[9]North, Paige Randall. "Towards a directed homotopy type theory." *Electronic Notes in Theoretical Computer Science* 347 (2019): 223-239.

[10]Licata, Daniel R., and Robert Harper. "2-dimensional directed type theory." *Electronic Notes in Theoretical Computer Science* 276 (2011): 263-289.

Examples of comprehension bicategories

We have the following comprehension bicategory

$$\begin{tikzcd} \mathsf{OpFib} \arrow[rr] \arrow[dr, "\mathsf{cod}"'] & & \underline{\mathsf{Cat}}^{\rightarrow} \arrow[dl, "\mathsf{cod}"] \\ & \underline{\mathsf{Cat}} & \end{tikzcd}$$

Categorification of the previous one

Examples of comprehension bicategories

Given a bicategory $\mathcal{B}$ with pullbacks, take

$$\begin{tikzcd} \mathsf{SOpFib}(\mathcal{B}) \arrow[rr] \arrow[dr, "\mathsf{cod}"'] & & \mathcal{B}^{\rightarrow} \arrow[dl, "\mathsf{cod}"] \\ & \mathcal{B} & \end{tikzcd}$$

Here $\mathsf{SOpFib}(\mathcal{B})$ is the bicategory of Street opfibrations in $\mathcal{B}$.
Directed version of Garner's stuff.

Remark on the formalization

- The notion of comprehension bicategory and fibration have been formalized using UniMath.
- Here we make use of **displayed bicategories**[11]

[11] Ahrens, B., et al (2022). Bicategories in univalent foundations. *Mathematical Structures in Computer Science*, 1-38.

Conclusion

- We defined a notion of **comprehension bicategory**
- This is a suitable framework in which one can interpret (directed) type theory: we proved **soundness**
- There are general instances of this definition (internal Street fibrations)

Further work: look at type formers, completeness