Logistic function

$$p(x) = \frac{e^{\beta_0 + \beta_1 x}}{1 + e^{\beta_0 + \beta_1 x}}$$

$$\underbrace{\frac{p(x)}{1-p(x)}}_{\text{"odds"}} = e^{\beta_0 + \beta_1 x} \iff \underbrace{\log\left[\frac{p(x)}{1-p(x)}\right]}_{\text{"log-odds" or "logit"}} = \beta_0 + \beta_1 x$$

Statistical Decision Theory

Let $x \in \mathbb{R}^p$ real valued random vector (input)

$Y \in \mathbb{R}$ real valued output vector

We seek $f(x)$ for predicting Y given X

$\Rightarrow$ loss function $L(Y, f(x))$

$$L(Y, f(x)) = (Y - f(x))^2$$

$$f(x) = \underset{c}{\operatorname{argmin}}\ E_{Y|X}\left[(Y-c)^2 \mid X\right]$$

Maximum Likelihood Method

Let $x_1, x_2 \ldots x_n$ joint density $f(x_1, x_2 \ldots x_n | \theta)$

Define likelihood function

$$L(\theta) = L(\theta | x_1, x_2 \ldots x_n) = f(x_1, x_2 \ldots x_n | \theta)$$

Example We want to estimate the probability π of getting TRUE in an experiment

Consider a random draw: T T F T T

$$L(\text{parameter} \mid \text{data}) = L(\pi \mid TTFTT) = \pi^4 (1-\pi)$$

$$= P(TTFTT \mid \pi) = \pi \cdot \pi (1-\pi) \cdot \pi \, \pi$$

π	$L(\pi \mid \text{data})$
0	
0.1	.
0.2	⋮
⋮	⋮
1	

$\Rightarrow$ Value of π most supported by the data

$\hat{\pi} = 0.8$

More generally,

$$L(\pi \mid data) = P(data \mid \pi) = \pi^{x}(1-\pi)^{n-x}$$

n – # of trials

x – # of successes

Find the value of π that maximizes $L(\pi \mid data)$

$$\log L(\pi) = x \log(\pi) + (n-x)\log(1-\pi)$$

Differentiate

$$\frac{d \log L(\pi)}{d\pi} = \frac{x}{\pi} + (n-x)\cdot\frac{1}{1-\pi}\cdot(-1) = \frac{x}{\pi} - \frac{n-x}{1-\pi}$$

Set $\frac{d \log L(\pi)}{d\pi} = 0 \Rightarrow$ Solve $\Rightarrow$ MLE $\hat{\pi} = \frac{x}{n}$

MLE for Multiple Parameters

Ex Simple random sample of n random variables

$$L(\mu, \sigma^2 \mid x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$\ln L(\mu, \sigma^2 \mid x) = -\frac{n}{2}\left(\ln 2\pi + \ln \sigma^2\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial}{\partial \mu} \ln L(\mu, \sigma^2 | x) = \frac{1}{\sigma^2} \sum_{i=1}^{n} (x_i - \mu) = \frac{1}{\sigma^2} n (\bar{X} - \mu)$$

Set $= 0 \Rightarrow \hat{\mu}(x) = \bar{X}$

$$\frac{\partial}{\partial \sigma^2} \ln L(\mu, \sigma^2 | x) = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^{n} (x_i - \mu)^2 =$$

$$= -\frac{n}{2(\sigma^2)^2} \left[\sigma^2 - \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2 \right]$$

Recall $\hat{\mu}(x) = \bar{X}$

Set $= 0 \Rightarrow \hat{\sigma}^2(x) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$