Introduction to Statistics and Data Science using *eStat*

Chapter 7 Testing Hypothesis for Single Population

7.4 Testing Hypothesis with α and β simultaneously

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7.4 Testing Hypothesis with α and β simultaneously

- **Testing hypothesis so far is a conservative decision-making method.**
  ⇨ critical value that reduces the probability of type one error **α**
      (error that rejects the null hypothesis even though it is true).
  ⇨ keep H_0 unless there is sufficient evidence of H_1 which is risky.
  ⇨ probability of type two error (**β**) was not considered.

- **Sometimes it is unclear which should be H_0 and which one should be H_1.**
  ⇨ both **α** and β are important and should be considered simultaneously.
  ⇨ **If the analyst can determine the sample size,**
      testing hypothesis with α and β can be performed.

7.4.1 β and the power of test

[Example 7.4.1] In [Example 7.1.1], calculate the probability of the type 2 error β if the significance level is 5%. Check this result using 『eStatU』.

<Answer>
- Hypothesis is $H_0 : \mu$ = 1500, $H_1 : \mu$ = 1600,
- Population standard deviation is assumed σ = 200, and n = 30.
- Decision rule is as follows if α is 5%.

'If $\bar{X}$ < 1500 + (1.645) $\sqrt{\frac{200^2}{30}}$ = 1560.06, reject H_0'

- Type 2 error (probability of H_0 is true when H_1 is true) is calculated as:
  β = P($\bar{X}$ < 1560.06 | H_1 is true)

  = P(($\bar{X}$ -1600) / $\sqrt{\frac{200^2}{30}}$ < (1560.06-1600)/ $\sqrt{\frac{200^2}{30}}$)

  = P(Z < -1.09) = 0.137

<Answer of Ex 7.4.1> Select 'Testing μ - C, β' at 『eStatU』 menu.
- Enter $\mu_0 = 1500$, $\mu_1 = 1600$, $\sigma = 200$, $\alpha = 0.05$, $n = 30$ and click [Execute] button.
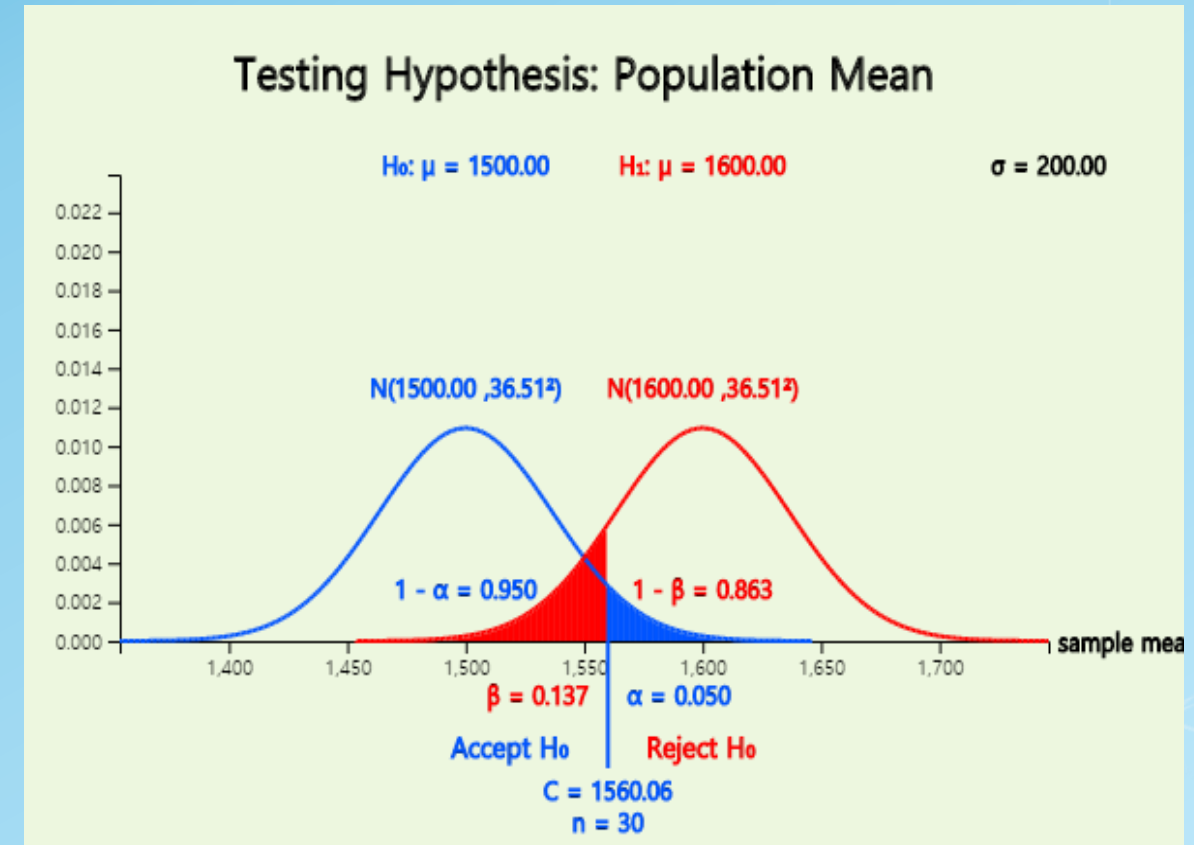- Result of testing hypothesis, critical value C and β, will be shown.

Testing μ - C, β Menu

[Hypothesis] $H_o : \mu = \mu_o$ [1500] $H_1 : \mu = \mu_1$ [1600]

Population Standard Deviation $\sigma =$ [200]

Type 1 Error $\alpha =$ [0.050]   Sample Size $n =$ [30]

[Execute] μ_1 $\mu_0 - 3\sigma$ ——————|——————— $\mu_0 + 3\sigma$

Testing Hypothesis: Population Mean

Ho: μ = 1500.00   H1: μ = 1600.00   σ = 200.00

N(1500.00 ,36.51²)   N(1600.00 ,36.51²)

1 – α = 0.950   1 – β = 0.863

β = 0.137   α = 0.050

Accept Ho | Reject Ho

C = 1560.06
n = 30

[Example 7.4.2] In [Example 7.1.1], if the null hypothesis is not changed, but the alternative hypothesis is changed as follows:

H_0: μ = 1500, H_1: μ = 1580

1) Calculate the probability of the type 2 error if the significance level is 5%.
2) Check this result using 『eStatU』.

\<Answer\>

1) Although the alternative hypothesis has been changed to H_1: μ = 1580, decision rule will not be changed because H_1 is the ype as H_1: μ > 1580.

   'If $\overline{X}$ < 1500 + (1.645) $\sqrt{\dfrac{200^2}{30}}$ = 1560.06, reject H_0'

- Hence, the probability of type 2 error is as follows:

   β = P($\overline{X}$ < 1560.06 | H_1 is true)

   = P(($\overline{X}$ -1580) / $\sqrt{\dfrac{200^2}{30}}$ < (1560.06-1580)/ $\sqrt{\dfrac{200^2}{30}}$)

   = P(Z < -0.546) = 0.293

7.4.1   β and the power of test

- **Discriminating ability of two hypothesis is compared by using the power of a test.**
  Power = 1 - (Probability of the type 2 error) = 1 - β
  Large power increases discriminating ability of the test.

- **Power of a test can be obtained for any μ_1 of $H_1 : \mu = \mu_1$. It means that the power is a function over the value of and it is called a power function.**

- **A function of the probability that the null hypothesis is correct when the null hypothesis is true is called an operating characteristic function.**
  Operating characteristic function = 1 - α

[Example 7.4.3] In [Example 7.1.1], calculate the power of the following alternative hypothesis. Use α = 0.05. By using this, approximate the power function.

1) $H_1 : \mu$ = 1500   2) $H_1 : \mu$ = 1510   3) $H_1 : \mu$ = 1520
4) $H_1 : \mu$ = 1530   5) $H_1 : \mu$ = 1540   6) $H_1 : \mu$ = 1550
7) $H_1 : \mu$ = 1560   8) $H_1 : \mu$ = 1570   9) $H_1 : \mu$ = 1580
10) $H_1 : \mu$ = 1590   11) $H_1 : \mu$ = 1600   12) $H_1 : \mu$ = 1610

<Answer>

1) Although the alternative hypothesis are different, decision rule will not be changed because H_1 is the ype as $H_1 : \mu$ > 1580.

'If $\overline{X}$ < 1500 + (1.645) $\sqrt{\dfrac{200^2}{30}}$ = 1560.06, reject H_0 '
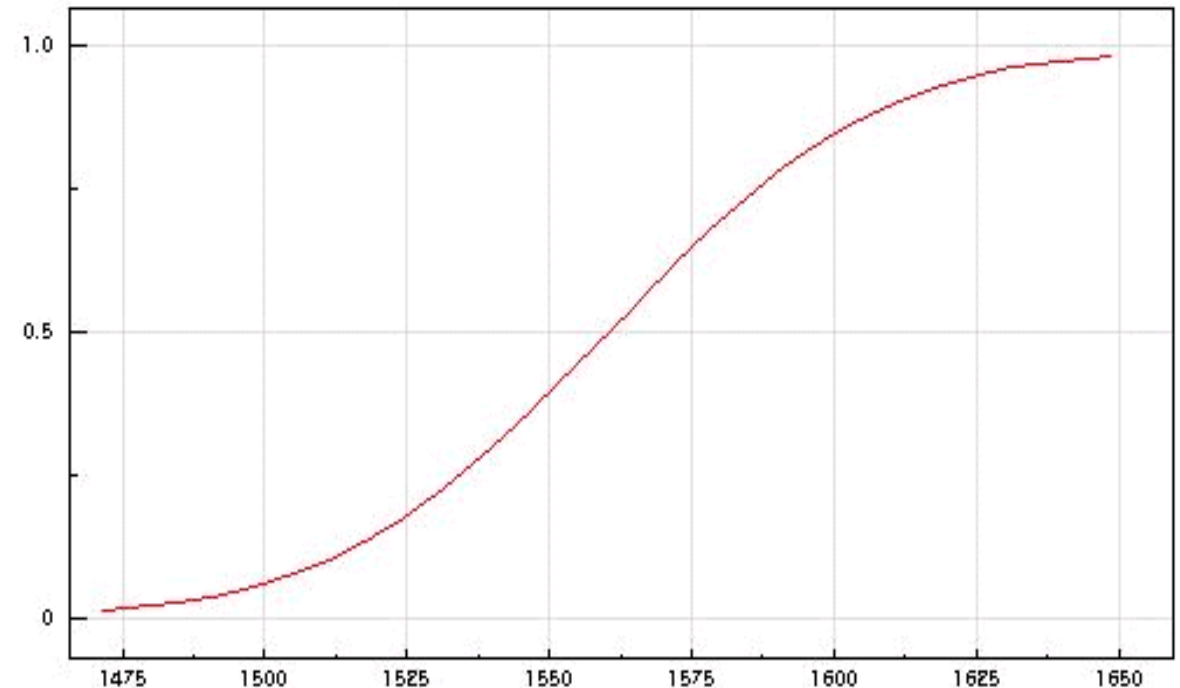
- Hence, if we calculate the probability of the type 2 error as [Example 7.4.2], the power of each test is as follows:

<Answer of Ex 7.4.3>

Alternative Hypothesis	β	Power = 1 - β
1) $H_1 : \mu = 1500$	0.95	0.05
2) $H_1 : \mu = 1510$	0.91	0.09
3) $H_1 : \mu = 1520$	0.86	0.14
4) $H_1 : \mu = 1530$	0.79	0.21
5) $H_1 : \mu = 1540$	0.71	0.29
6) $H_1 : \mu = 1550$	0.61	0.39
7) $H_1 : \mu = 1560$	0.50	0.50
8) $H_1 : \mu = 1570$	0.39	0.61
9) $H_1 : \mu = 1580$	0.29	0.71
10) $H_1 : \mu = 1590$	0.21	0.79
11) $H_1 : \mu = 1600$	0.14	0.86
12) $H_1 : \mu = 1610$	0.09	0.91

- **Power function**

7.4.2 Testing Hypothesis with α and β

[Example 7.4.4] Consider the testing hypothesis on the bulb life such as H_0: μ = 1500, H_1 : μ = 1570. Find sample size n and decision rule which satisfies α = 5% and β = 10%. Assume σ = 200 hours.

<Answer>
- Let n be the sample size and C be the critical value of a decision rule.
- Probability of the type 1 error α and type 2 error β are defined as follows:

   α = P($\bar{X} > C$ | H_0 is true)
   β = P($\bar{X} < C$ | H_1 is true)

\<Answer of Example 7.4.4\>

- If H_0 is true, $\bar{X} \sim N(1500, \frac{200^2}{n})$ and if H_1 is true, $\bar{X} \sim N(1570, \frac{200^2}{n})$. If α = 0.05 and β = 0.10, then $z_{0.05}$ = 1.645 and $z_{0.10}$ = -1.280. Hence n and C should satisfy both of the following equations.

$$C = 1500 + 1.645 \times \frac{200}{\sqrt{n}}$$

$$C = 1570 - 1.280 \times \frac{200}{\sqrt{n}}$$

- By solving two system of equations, solution is n = 69.8, C = 1539.4. i.e., the sample size is 70 approximately and the decision rule is as follows:

    'If $\bar{X}$ > 1539.4, then reject H_0'
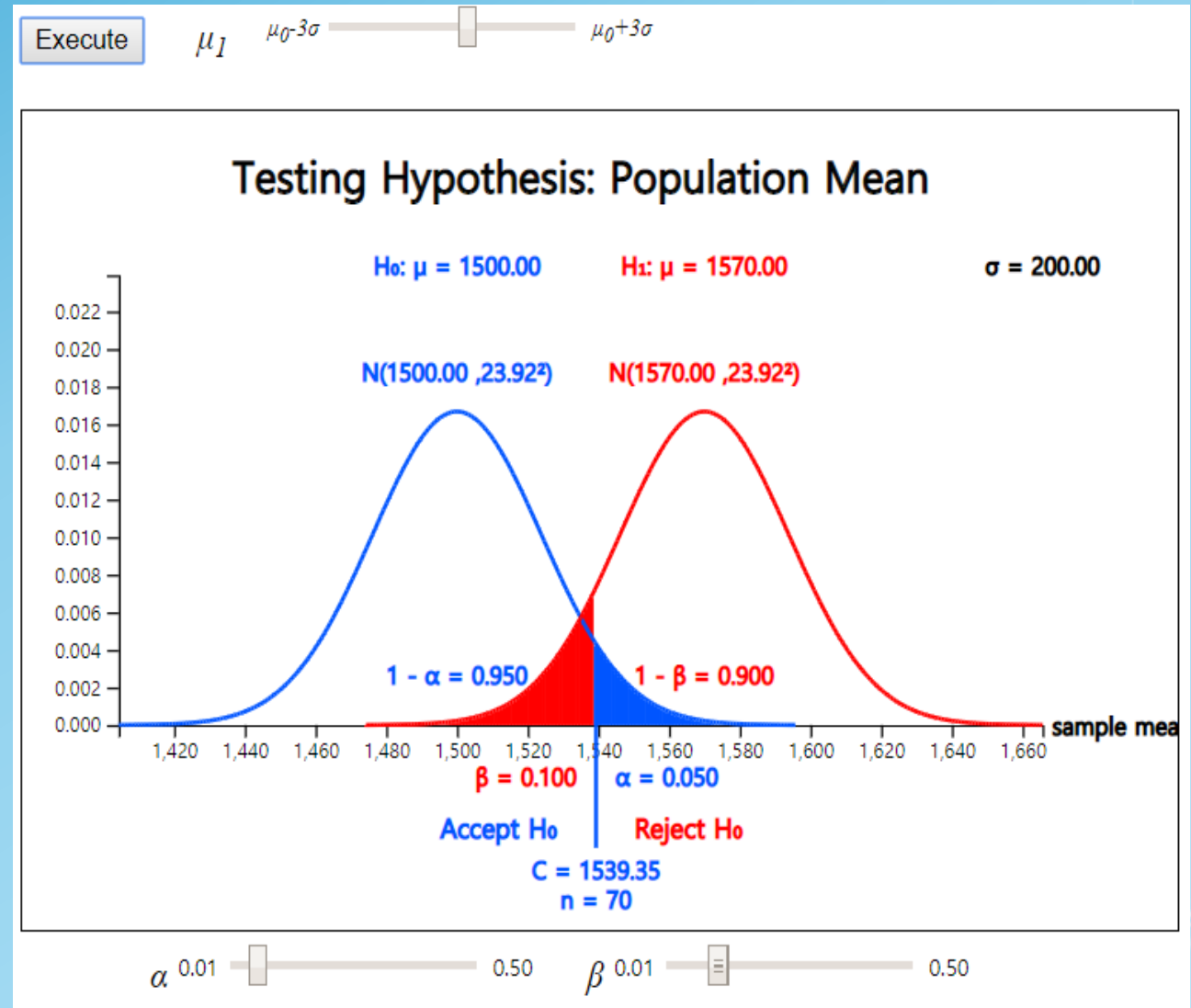
7.4 Testing Hypothesis with α and β simultaneously

<Answer of Ex 7.4.4>

Testing μ - C, n   Menu

[Hypothesis] $H_o : \mu = \mu_o$ [1500] $H_1 : \mu = \mu_1$ [1570]

Population Standard Deviation $\sigma =$ [200]

Type 1 Error $\alpha =$ [0.050]   Type 2 Error $\beta =$ [0.100]

Execute μ_1 $\mu_0 - 3\sigma$ ——————|—————— $\mu_0 + 3\sigma$

Testing Hypothesis: Population Mean

$H_0: \mu = 1500.00$ $H_1: \mu = 1570.00$ $\sigma = 200.00$

$N(1500.00, 23.92^2)$ $N(1570.00, 23.92^2)$

0.022
0.020
0.018
0.016
0.014
0.012
0.010
0.008
0.006
0.004
0.002
0.000

$1 - \alpha = 0.950$ $1 - \beta = 0.900$

1,420 1,440 1,460 1,480 1,500 1,520 1,540 1,560 1,580 1,600 1,620 1,640 1,660   sample mea

$\beta = 0.100$ | $\alpha = 0.050$

Accept H₀ | Reject H₀

C = 1539.35
n = 70

α 0.01 —|———— 0.50 β 0.01 ——|——— 0.50

Thank you