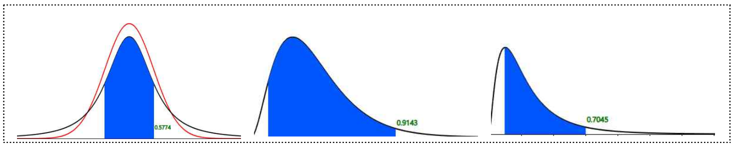
6

Sampling Distribution and Estimation



SECTIONS

CHAPTER OBJECTIVES

The power of modern statistics lies in the ability to predict the characteristics of a population with a small number of samples. The simple random sampling is introduced to collect a sample from the population in Section 6.1.

Section 6.2 describes the distribution of all possible sample means and its application to estimate the population mean.

Section 6.3 describes the distribution of all possible sample variances and its application to estimate the population variance.

Section 6.4 describes the distribution of all possible sample proportions and its application to estimate the population proportion.

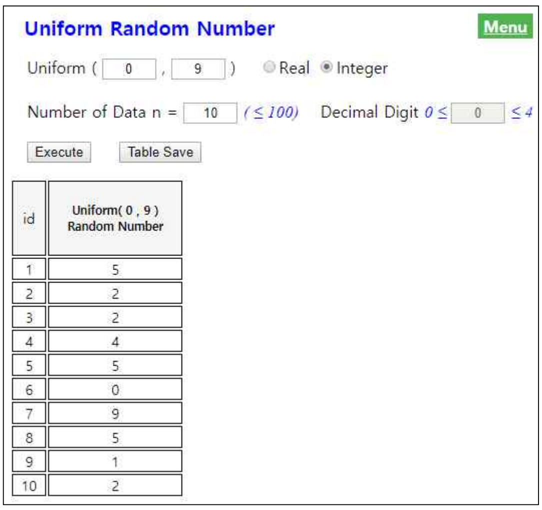
Section 6.5 describes how to determine the sample size to estimate the population mean and population proportion.

6.1 Simple Random Sampling

- Since a population is generally very large, a survey of the entire population takes lots of time and money. Hence, we are trying to estimate characteristics of the population using a set of samples. Estimation of population characteristics using samples is called an **inferential statistics**. However, there may be some difference between the characteristic of the population and the characteristic of the sample. In order to reduce the difference, several methods of sampling have been studied. The most commonly used one is a **simple random sampling** which collects samples with the same probability of all elements of the population being selected.

Definition	**Simple Random Sampling** Samples are collected so that all elements of the population are likely to be selected equally.

- In case of the simple random sampling, there are two possible ways to collect samples. One is to include an element selected once again in the population (**with replacement**), and the other is not to include the selected element back into the population (**without replacement**). However, in practice, almost all sampling is made without replacement.
- Some tools may be needed to ensure that each element of the population is selected equally. We usually use a random number table which is a table of numbers from 0 to 9 without special regularity or partiality. Recently, a random number generator by using a computer which uses the uniform distribution on [0, 1] is widely used to produce a random number. <Figure 6.1.1> shows 10 random numbers from 0 to 9 generated using a random number generator in 『eStatU』.



<Figure 6.1.1> 『eStatU』 Uniform random number

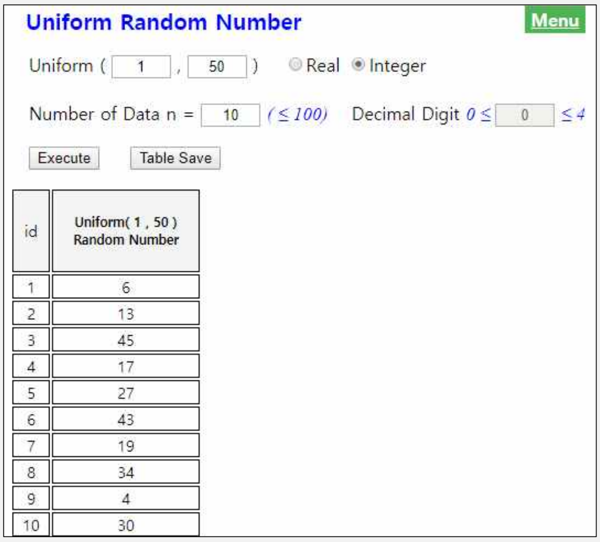
- Consider the following example of simple random sampling using this random number generator.

Example 6.1.1

A class has 50 students. Select three of 50 students as samples without replacement by using 『eStatU』.

Answer

- In order to select three samples randomly from 50 students, a list of students which assigned serial numbers from one to fifty must be made first. In order to collect samples, select ‘Uniform Random Number’ in 『eStatU』 menu. Enter (1 and 50) at the box of ‘Uniform’, check ‘Integer’, enter 10 at the box of ‘Number of Data’ and click the [Execute] button as shown in <Figure 6.1.2>. These are the random numbers to select students, but your computer might generate different numbers, because it depends on computer.



<Figure 6.1.2> 『eStatU』 Uniform(1,50) random number generation

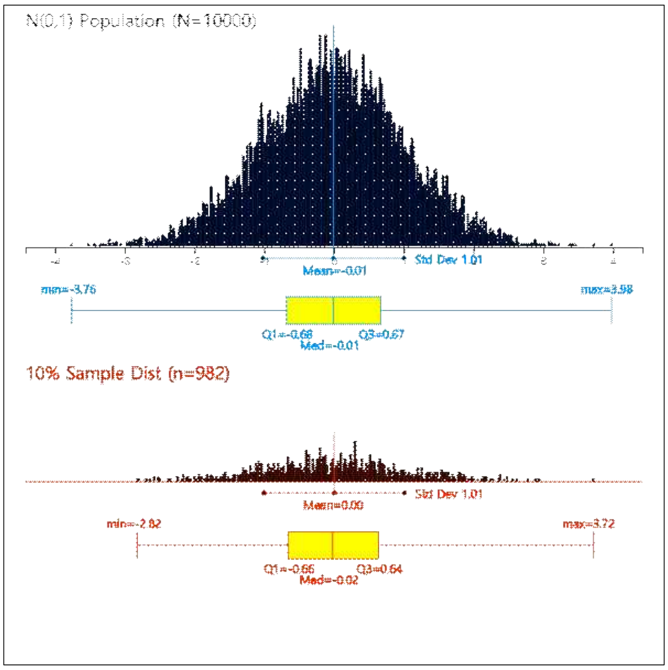
- If you read the three numbers from located above, there are 6, 13 and 45 which are not overlapped and you will choose these student numbers as samples.
- Generated random numbers may have same numbers. If it is a sampling without replacement, discard the following same number.

[Practice 6.1.1]

In a university, a survey was planned for students and the list of students which has an id numbered 1 to 1000 was made. Select ten students randomly using 『eStatU』 without replacement.

6.2 Sampling Distribution of Sample Means and Estimation of the Population Mean

- The purpose of statistical experiments or surveys is to find out some information about a population. Information about the population usually refers to a characteristic value of the population, such as the mean and variance which are called **parameters**. Since it is so difficult or costly to investigate the population parameters, they are usually estimated by using characteristic values of a set of samples such as the sample mean and sample variance. <Figure 6.2.1> is a simulation to show the relationship between population data of size 10,000 and sample data (approximately 10%) using『eStatU』which shows characteristic values of a population are similar to characteristic values of a set of samples.



<Figure 6.2.1> Simulation to show the relationship between population data and sample data

- Characteristic values of a set of samples are called **sample statistic** and the distribution of the sample statistic is called a **sampling distribution**. The sampling distribution identifies the relationship between the sample statistic and the population parameter and it makes possible to estimate and test a population parameter. In this section, let's first look at the sampling distribution of sample means and find out how to estimate the population mean.

6.2.1 Sampling Distribution of Sample Means

- The following example is to find out the sampling distribution of sample means.

Example 6.2.1	Suppose there is a population consisting of five salesman from a company. (Although such a small population does not actually need to be sampled, this is an example to illustrate the sampling distribution of sample means). Consider the number of years of service at this company as a characteristic value of the population and data of five salesman are as follows: 6, 2, 4, 8, 10 1) Obtain the mean and variance of this population. 2) Obtain all possible samples of size two by simple random sampling with replacement from this population and calculate means of each sample. In addition, calculate the mean and variance of all these possible sample means and compare them with the mean and variance of the population. 3) Prepare a frequency distribution of all possible sample means and draw a bar graph. Compare this bar graph with the bar graph of the population distribution.

Example 6.2.1 Answer

1) The mean and variance of the population is μ = 6, σ^2 = 8

2) The number of all possible samples with replacement is 5 × 5 = 25. Table 6.2.1 shows all possible samples and their sample means ($\overline{x}$).

Table 6.2.1 All possible samples of size 2 with replacement from the population and their sample means

sample	$\overline{x}$	sample	$\overline{x}$	sample	$\overline{x}$	sample	$\overline{x}$	sample	$\overline{x}$
2,2	2	4,2	3	6,2	4	8,2	5	10,2	6
2,4	3	4,4	4	6,4	5	8,4	6	10,4	7
2,6	4	4,6	5	6,6	6	8,6	7	10,6	8
2,8	5	4,8	6	6,8	7	8,8	8	10,8	9
2,10	6	4,10	7	6,10	8	8,10	9	10,10	10

- Some of these sample means are exactly the same as the population mean μ = 6, but some others such as 2 or 10 are significantly different. The mean of all possible 25 sample means (indicated by $\mu_{\overline{X}}$) in Table 6.2.1 is also 6 and the variance (indicated by $\sigma^2_{\overline{X}}$) is 4 as follows:

$$\mu_{\overline{X}} = \frac{2+3\times2+4\times3+5\times4+6\times5+7\times4+8\times3+9\times2+10}{25} = 6$$

$$\sigma^2_{\overline{X}} = \frac{(2-6)^2+(3-6)^2\times2+(4-6)^2\times3+(5-6)^2\times4+(6-6)^2\times5+(7-6)^2\times4+(8-6)^2\times3+(9-6)^2\times2+(10-6)^2}{25}$$

$$= 4$$

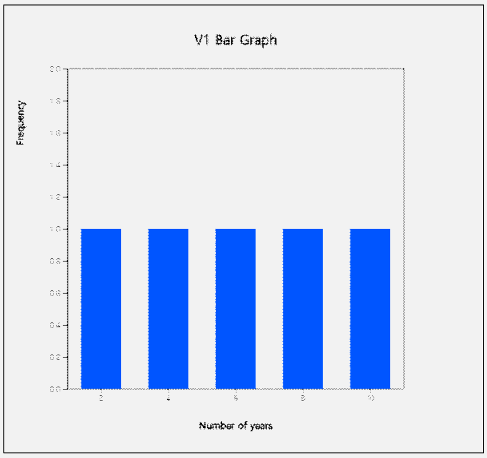
- What can be observed here is that the mean of all 25 possible sample means is the same as the population mean. This fact explained that the sample mean $\overline{x}$ is an **unbiased estimator** of the population mean μ. In addition, the variance of the sample means $\sigma^2_{\overline{X}}$ is the population variance σ^2 divided by the sample size (n=2).

3) Table 6.2.2 shows the frequency distribution of sample means in Table 6.2.1. The frequency distribution of all possible sample means is called a **sampling distribution of sample means** when n = 2. <Figure 6.2.2> shows a bar graph of the population distribution and <Figure 6.2.3> shows the distribution of all possible sample means. As shown in the table, the population mean is 6 and some of sample means are the same or close to the population mean, but some of sample means are much more different from 6. However, you can see that all possible sample means are concentrated around the population mean 6 and, as discussed in 2), the average of all 25 sample means is 6. Also the distribution of all possible sample means is symmetrical about the population mean 6.

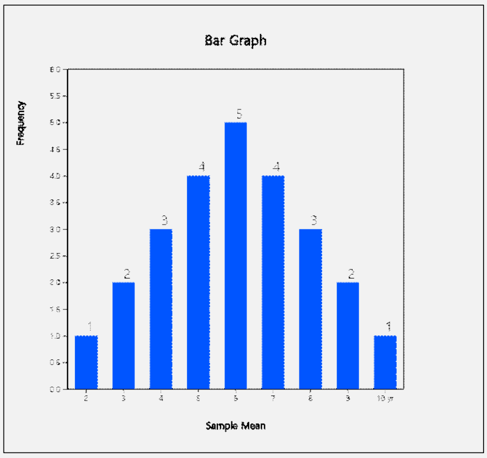
Table 6.2.2 Frequency table of sample means

Sample Mean	Frequency	Relative Frequency
2	1	0.04
3	2	0.08
4	3	0.12
5	4	0.16
6	5	0.20
7	4	0.16
8	3	0.12
9	2	0.08
10	1	0.04
	25	1.00

Example 6.2.1 Answer (continued)



<Figure 6.2.2> Population distribution



<Figure 6.2.3> Sampling distribution of $\overline{x}$'s

Definition

Parameter, Estimator and Estimate

The population mean is a single value, but there are many possible sample means. The population mean μ is called a **parameter,** which is a characteristic value of the population, and the sample mean is a random variable that can have many different values and is usually expressed with a capital letter such as $\overline{X}$ which is called an **estimator** of the parameter μ. An observed sample mean, marked $\overline{x}$ with a lowercase letter, is called an **estimate** of μ.

An estimator of the population variance σ^2 is the sample variance S^2 and its observed value which is an estimate of S^2 is denoted as s^2.

- The relationship between the population mean and all possible sample means in [Example 6.2.1] is observed even if the population has a different shape of distribution. If the population is very large, it is not possible to find all possible samples as shown in [Example 6.2.1] and to find a distribution of sample means. Therefore, the following theoretical research has been developed.

- If a population is normally distributed with $N(\mu, \sigma^2)$, the distribution of all possible sample means is exactly a normal distribution such as $N(\mu, \sigma^2/n)$. If the population is an infinite population with the mean μ and variance σ^2, then the distribution of all possible sample means is approximately a normal distribution such as $N(\mu, \sigma^2/n)$ if the sample size is large enough. This is referred to as the **Central Limit Theorem**, which is a key theory in statistics, specifically summarized as follows:

Theorem 6.2.1

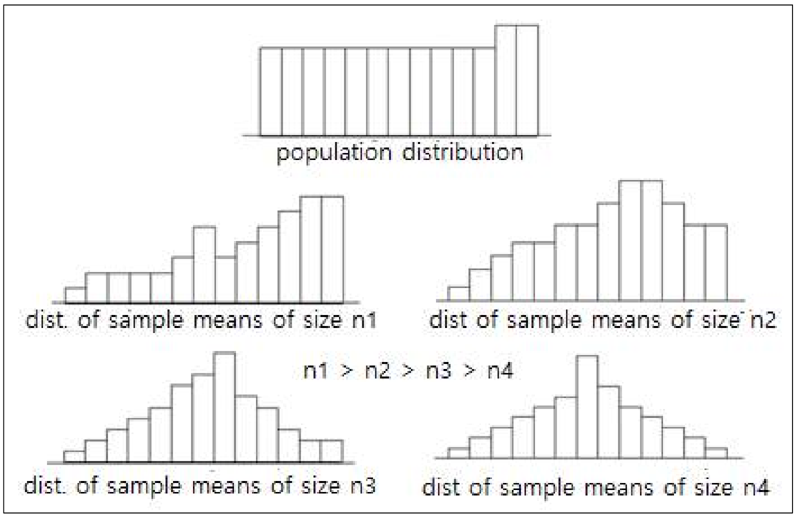
Central Limit Theorem

Suppose a population is not a normal distribution and its mean and variance are μ and σ^2. If we select samples of size n with replacement, the distribution of all possible sample means has following characteristics:

1) The average of all possible sample means, $\mu_{\overline{X}}$, is equal to the population mean μ. (i.e., $\mu_{\overline{X}}=\mu$)
2) The variance of all possible sample means, $\sigma^2_{\overline{X}}$, is the population variance divided by n. (i.e., $\sigma^2_{\overline{X}} = \frac{\sigma^2}{n}$)
3) The distribution of all possible sample means is approximately a normal distribution.

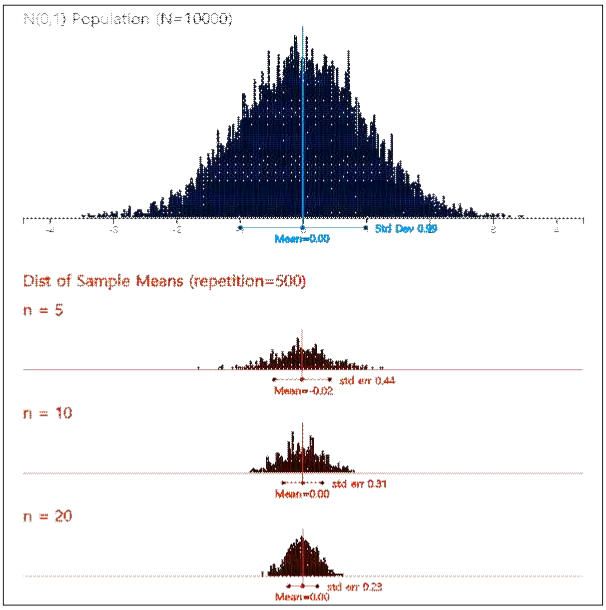
The above facts can be briefly written as $\overline{X} \sim N(\mu, \frac{\sigma^2}{n})$.

- The central limit theorem is very important as a theory underlying modern statistics. <Figure 6.2.4> shows that, although a population is skewed from its mean, the distribution of sample means is closer to normal as the sample size increases.



<Figure 6.2.4> Sampling distribution of sample means with different sample sizes when population is not a normal distribution

- <Figure 6.2.5> is a simulation of the central limit theorem using 『eStatU』



<Figure 6.2.5> 『eStatU』 Simulation of the central limit theorem

6.2.2 Estimation of the Population Mean

- When a sample survey is conducted, only one set of samples is selected from the population to estimate a characteristic value of the population, such as the population mean. In general, we consider the sample mean of selected samples as an estimate of the population mean. Do you think this sample mean can estimate the population mean well even if it is only one set of samples?
- This is a basic question on the estimation that everyone can think about at least once. The sampling distribution of all possible sample means which we studied in the previous section is the answer to this question. That is, whatever a population distribution is, if the sample size is large enough, all possible sample means are clustered around the population mean in the form of a normal distribution. Therefore, the sample mean obtained from one set of samples is usually close to the population mean. Even in the worst case, the difference between the population mean and sample mean, known as an error, is not so significant, and it is possible to estimate the population mean by using the sample mean. The larger the sample size, the more sample means are concentrated around the population mean based on the central limit theorem and Hence, we can reduces the error of the estimation.

A. Point Estimation of the Population Mean

- A value of one observed sample mean is called a **point estimate** of the population mean.
- In general, the sample statistic used to estimate a population parameter must have good characteristics, so that the estimate can be accurate. If the average of all possible sample statistics is equal to the population parameter, then the sample statistic has a good chance to estimate the population parameter and it is called an **unbiased estimator**. In the previous section we found that a sample

mean is an unbiased estimator of the population mean.

- If the value of a sample statistic becomes closer and closer to the population parameter when the sample size grows, the sample statistic is called a **consistent estimator**. The variance of all possible sample means is closer to zero as the sample size increases by the central limit theorem studied in the previous section, so the sample mean is closer to the population mean. Therefore, the sample mean is a consistent estimator of the population mean.
- If a sample statistic has the least variance when there are several unbiased estimators for the population parameter, it is called an **efficient estimator**. The sample mean is also an efficient estimator. Consequently, a sample mean has all good characteristics necessary to estimate the population mean.

Definition

Point Estimate, Unbiased, Consistent, and Efficient Estimator

A value of one observed sample mean is called a **point estimate** of the population mean.

If the average value of all possible sample statistics is equal to the population parameter, then the sample statistic is called an **unbiased estimator** of the population parameter. The sample mean is an unbiased estimator of the population mean.

When a sample size grows, if the value of the sample statistic becomes closer and closer to the population parameter, the sample statistic is called a **consistent estimator**. The sample mean is a consistent estimator of the population mean.

If a sample statistic has the least variance when there are several unbiased estimators for the population parameter, it is called an **efficient estimator**. The sample mean is an efficient estimator.

Theorem 6.2.2

Point estimation of the population mean μ

$\Rightarrow$ Sample mean $\overline{x}$ ($\overline{x}$ is an unbiased, consistent, efficient estimator of μ)

B. Interval Estimation of the Population Mean – Known Population Variance

- In contrast to the point estimate for the population mean, estimating by using an interval is called an **interval estimation**. If the population is a normal distribution with the mean μ and variance σ^2, the distribution of all possible sample means is a normal distribution with the mean μ and variance σ^2/n, so the probability that one sample mean will be included in the interval $\mu \pm 1.96\sigma/\sqrt{n}$ is 95% as follows:

$$P(\mu - 1.96\frac{\sigma}{\sqrt{n}} \le \overline{X} \le \mu + 1.96\frac{\sigma}{\sqrt{n}}) = 0.95$$

We can rewrite this formula as follows:

$$P(\overline{X} - 1.96\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}) = 0.95$$

- Assuming σ is known, the meaning of the above formula is that 95% of intervals obtained by applying the formula $\left[\overline{X} - 1.96\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$ for all possible sample means include the population mean. The formula of this interval is referred to as the 95% confidence interval of the population mean.

$$\left[\overline{X} - 1.96\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + 1.96\frac{\sigma}{\sqrt{n}}\right]$$

- Generally, since $\overline{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$, the standardized random variable of $\overline{X}$, $Z = \frac{\overline{X} - \mu}{\sigma/\sqrt{n}}$, follows the standard normal distribution $N(0,1)$. Therefore, the following probability of the standard normal random variable Z is $1-\alpha$.

$$P\left(-z_{\alpha/2} < \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} < z_{\alpha/2}\right) = P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1-\alpha$$

This formula can be written as follows:

$$P(\mu - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \overline{\mathrm{X}} \le \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1-\alpha$$

This formula can also be written as follows:

$$P(\overline{\mathrm{X}} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \overline{\mathrm{X}} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = 1-\alpha$$

- The confidence interval for the population mean μ is as follows if the population is normally distributed and the population variance σ^2 is known.

Theorem 6.2.3

100(1-α)% Confidence Interval for Population Mean μ

Assume a population is a normal distribution and the population variance σ^2 is known.

$$\left[\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

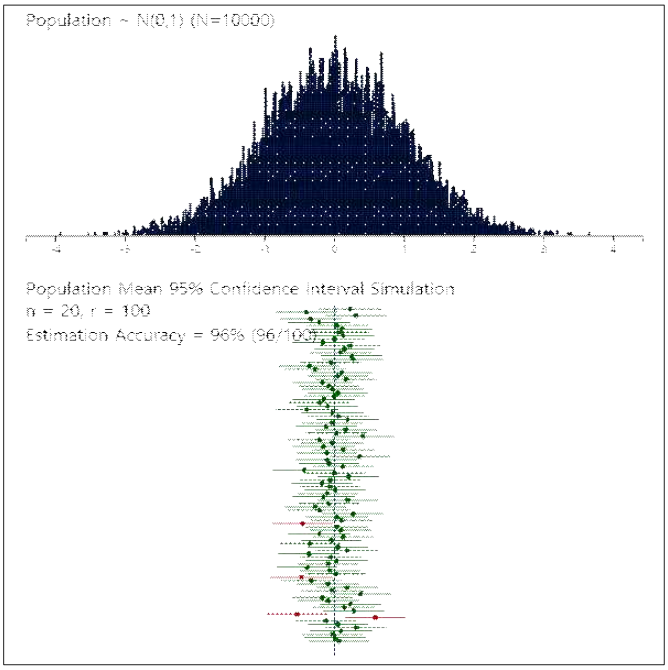
- 100(1-α)% here is called a **confidence level**, which refers to the probability of intervals that will include the population mean among all possible intervals calculated by the confidence interval formula. Usually, we use 0.01 or 0.05 for α. z_α is the $100(1-\alpha)$ percentile of the standard normal distribution. In other words, if Z is the random variable which follows the standard normal distribution, the probability that Z is greater than z_α is α, i.e.,

$$P(Z > z_\alpha) = \alpha.$$

For example,

$z_{0.025}$ = 1.96
$z_{0.95}$ = -1.645
$z_{0.005}$ = 2.575.

- <Figure 6.2.6> shows a simulation of the 95% confidence interval for the population mean by extracting 100 sets of samples with the sample size $n=20$ from a population of 10,000 numbers which follow the standard normal distribution N(0,1). In this case, 96 of the 100 confidence intervals contain the population mean 0. This result might be different on your computer, because the program use a random number generator which depends on computer. Whenever we repeat these experiments, the result may also vary slightly.



<Figure 6.2.6> 『eStatU』 Simulation of the 95% confidence interval

Example 6.2.2	The average starting salary per month of college graduates was 275 (unit: 10,000 KRW) after a simple random sampling of 100 college graduates this year. Assume that the starting salary for all college graduates is a normal distribution and its standard deviation is 50. 1) What is the point estimate the average starting salary of all college graduates. 2) Estimate a 95% confidence interval of the average starting salary of college graduates. 3) Estimate a 99% confidence interval of the average starting salary of college graduates. Compare the width of this interval to the 95% confidence interval. 4) If the sample size is increased to 400 and its average is the same, estimate a 95% confidence interval of the average starting salary for all college graduates. Compare the width of the interval to question 2).
Answer	1) Point estimation of the average starting salary is the sample mean which is 275 (unit: 10,000 KRW). 2) Since the 95% confidence interval implies α = 0.05, z value is as follows: $z_{\alpha/2} \;=\; z_{0.05/2} = z_{0.025} \;=\; 1.96$ Therefore, the 95% confidence interval is as follows: $\left(\overline{X} - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \;,\; \overline{X} + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right)$ ⇔ (275 − 1.96(5/10), 275 + 1.96(5/10)) ⇔ (274.02, 275.98)

Example 6.2.2 Answer (continued)

3) Since the 99% confidence interval implies α = 0.01, z value is as follows:

$$z_{\alpha/2} = z_{0.01/2} = z_{0.005} = 2.575$$

Therefore, the 99% confidence interval is as follows:

$$(275 - 2.575(5/10) , 275 + 2.575(5/10))$$
$$\Leftrightarrow (273.71, 276.29)$$

Therefore, if the confidence level is increasing, the width of the confidence interval becomes wider.

4) If the sample size is 400, the 95% confidence interval is as follows:

$$(275 - 1.96(5/20) , 275 + 1.96(5/20))$$
$$\Leftrightarrow (274.51, 275.49)$$

Therefore, if the sample size is increasing, the width of the confidence interval becomes narrower which is more accurate.

[Practice 6.2.1]

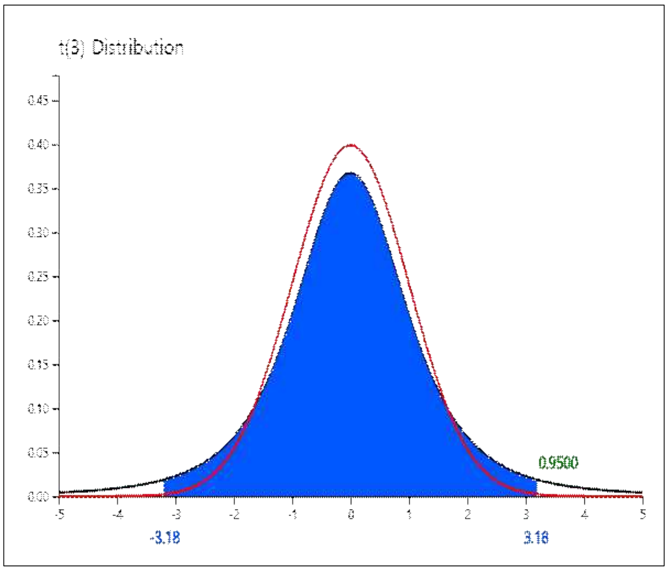
The quality manager of a large manufacturer wants to know the average weight of raw materials. 25 samples were collected by simple random sampling and their sample mean was 60 kg. Assume the population standard deviation is 5 kg. Use 『eStatU』 to answer the followings.

1) What is the point estimation of the population mean weight of raw materials.
2) Estimate a 95% confidence interval of the population mean weight of raw materials.
3) Estimate a 99% confidence interval of the population mean weight of raw materials. Compare the width of this interval to the 95% confidence interval.
4) If the sample size is increased to 100 and its average is the same, estimate a 95% confidence interval of the population mean weight of raw materials. Compare the width of the interval to question 2).

C. Interval Estimation of the Population Mean – Unknown Population Variance

- One problem in estimating the unknown population mean by using the formula of the confidence interval in the previous section is that either the population variance may not be known or the population is not normally distributed. If the sample size is large enough, a confidence interval of the population mean can be obtained approximately using the sample variance instead of the population variance on the previous formula. However, if the sample size is small and the sample variance is used, a confidence interval based on the t distribution should be used.
- The t distribution was studied by a statistician W. S. Gosset, who worked for a brewer in Ireland, and published his study result in 1907 under the alias Student. So t distribution is often referred to as Student's t distribution. The t distribution is not just a single distribution, but it is a family of distributions under a parameter called a degree of freedom, 1,2, ... , 30, ... and denoted as t_1, t_2, ... , t_{30}, ...
- The shape of the t distribution is symmetrical about zero (y axis), similar to the standard normal distribution, but it has a tail that is flat and longer than the standard normal distribution. <Figure 6.2.7> shows the standard normal distribution

N(0,1) and t distribution with 3 degrees of freedom at the same time by using the t distribution module of 『eStatU』.



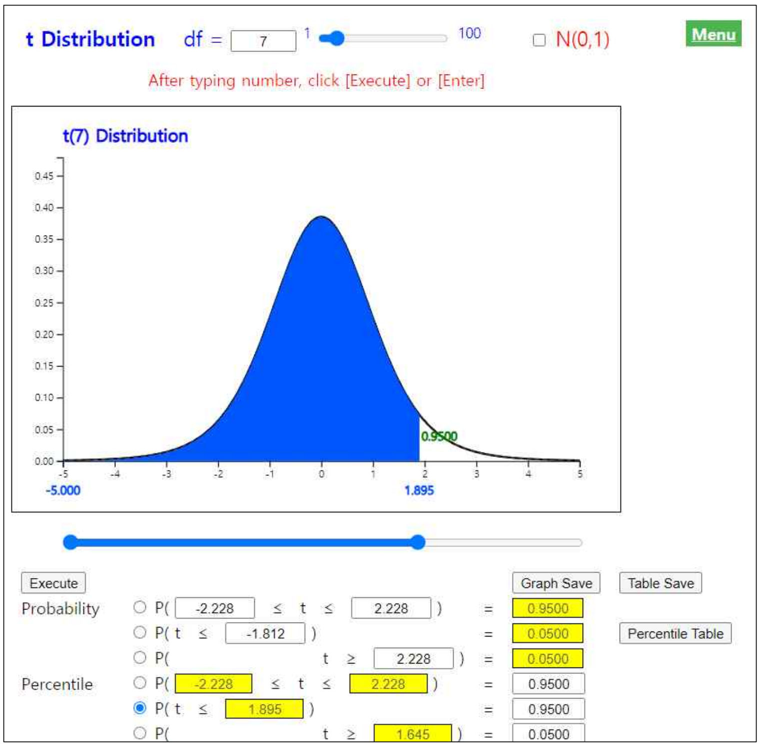
<Figure 6.2.7> Comparison of t_3 and N(0,1)

- The t distribution is closer to the standard normal distribution as degrees of freedom increase above 30. This is why a confidence interval can be obtained approximately by using the standard normal distribution if the sample size is greater than 30. Denote $t_{n\,;\,\alpha}$ as the $100(1-\alpha)\%$ percentile of the t distribution with n degrees of freedom. For example, $t_{7\,;\,0.05}$ is the $100(1-0.05) = 95\%$ percentile of the t distribution and its value is 1.895 as <Figure 6.2.8>. In the standard normal distribution, this value was 1.645. Since the t distribution is symmetrical, $t_{n\,;\,\alpha} \;=\; -t_{n\,;\,1-\alpha}$.
- In order to find a percentile value of the t_7 distribution using 『eStatU』, click on 't distribution' in the main menu of 『eStatU』 and then set the degree of freedom (df) to 7, and set the probability value in the fifth option below the t distribution graph to 0.95, then $t_{7\,;\,0.05}$ = 1.895 will appear as <Figure 6.2.8>.
- Consider the interval estimation of the population mean when you do not know the population variance, but assume that the population is a normal distribution. If $X_1, X_2, \cdots, X_n$ is the random sample of size n from the normal population, then it can be shown that the distribution of $\frac{\overline{X}-\mu}{S/\sqrt{n}}$, where σ is replaced with S, is the t distribution with n-1 degrees of freedom.

$$\frac{\overline{X}-\mu}{S/\sqrt{n}} \;\sim\; t_{n-1}$$

Hence, the probability of the (1- α)% interval is as follows:

$$P\left[-t_{n-1\,;\,\alpha/2} < \frac{\overline{X}-\mu}{S/\sqrt{n}} < t_{n-1\,;\,\alpha/2}\right] \;=\; 1-\alpha$$



<Figure 6.2.8> Upper 5 percentile of t_7 distribution $t_{7,0.05}$ = 1.8946

- The left hand side of the above formula can be summarized as the confidence interval for the population mean when the population variance is unknown as follows:

Theorem 6.2.4

100(1-α)% Confidence Interval for Population Mean μ

Assume a population is a normal distribution and the population variance σ^2 is unknown.

$$\left[\overline{X} - t_{n-1\,;\,\alpha/2} \cdot \frac{S}{\sqrt{n}} \;,\; \overline{X} + t_{n-1\,;\,\alpha/2} \cdot \frac{S}{\sqrt{n}} \right]$$

where n is the sample size and S is the sample standard deviation.

Example 6.2.3

Suppose we do not know the population variance in Example 6.2.2. If the sample size is 25 and the sample standard deviation is 50 (unit: 10,000 KRW), estimate the mean of the starting salary of college graduates at the 95% confidence level.

Answer

- Since we do not know the population variance, t distribution should be used for interval estimation of the population variance.
- Since $t_{n-1\,;\,\alpha/2} = t_{25-1\,;\,0.05/2} = t_{24\,;\,0.025} = 2.0639$, the 95% confidence interval of the population mean is as follows:

$$\left(\overline{X} - t_{n-1\,;\,\alpha/2} \cdot \frac{S}{\sqrt{n}} \;,\; \overline{X} + t_{n-1\,;\,\alpha/2} \cdot \frac{S}{\sqrt{n}} \right)$$

⇔ (275 − 2.0639(5/5) , 275 + 2.0639(5/5))
⇔ (272.9361, 277.0639)

Note that the smaller the sample size, the wider the interval width.

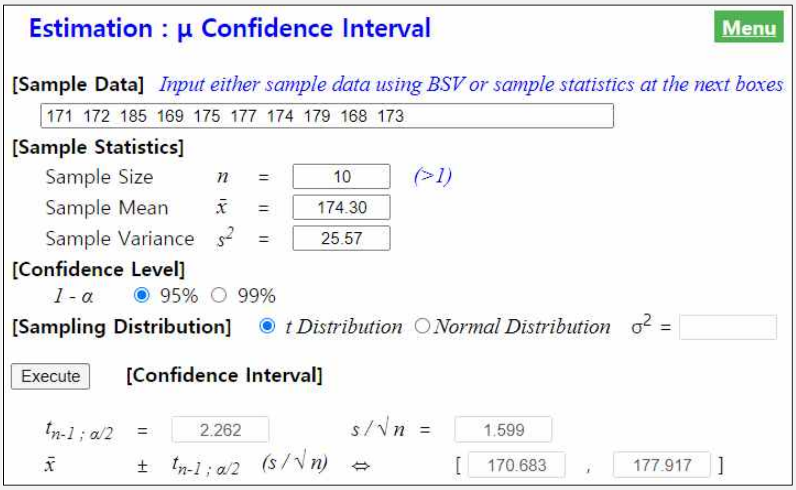
Example 6.2.4

The following data shows a simple random sampling of 10 new male students in a college this year to investigate their heights. Use 『eStatU』 to make a 95% confidence interval of the height of the college freshmen.

171 172 185 169 175 177 174 179 168 173

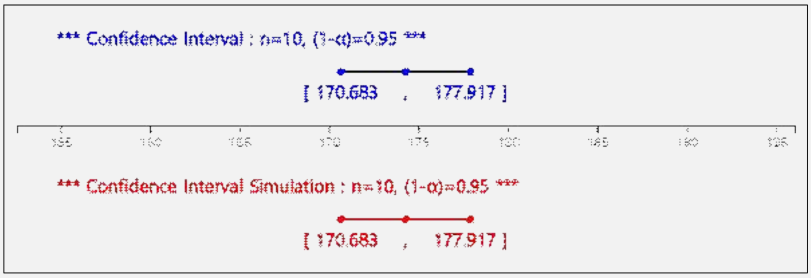
Answer

- Click 'Estimation : μ Confidence Interval' on the menu of 『eStatU』 and enter data at the [Sample Data] box as <Figure 6.2.9>. Then the confidence intervals [170.68, 177.92] are calculated using the t_9 distribution.



<Figure 6.2.9> 『eStatU』 Estimation for population mean

- In this module of 『eStatU』, a simulation experiment to investigate the size of the confidence interval can be done by changing the sample size n and the confidence level 1 - α as in <Figure 6.2.10>. If you increase n, the interval size becomes narrower. If you increase 1 - α, the interval size becomes wider.



<Figure 6.2.10> Simulation experiment to investigate the size of the confidence interval

- In this 『eStatU』 module, confidence intervals can be obtained by entering the sample sizes, sample mean, and sample variance without entering data.

[Practice 6.2.2]

In [Practice 6.2.1], suppose you do not know the population standard deviation and the sample standard deviation is 5 kg. Answer the same questions in [Practice 6.2.1] using 『eStatU』.

6.3 Sampling Distribution of Sample Variances and Estimation of the Population Variance

- If we know the relationship between the population variance and the sample variance, it is possible to estimate the population variance. In this section, the distribution of all possible sample variances is discussed in section 6.3.1 and the estimation of the population variance using the sample variance is discussed in section 6.3.2.

6.3.1 Sampling Distribution of Sample Variances

- Consider the following example to understand the sampling distribution of sample variances.

Example 6.3.1

Let's consider the data again in Example 6.2.1 which is the number of years of service for five salespeople in a company.

6, 2, 4, 8, 10

1) Calculate the population variance.
2) Find all possible samples of size 2 with replacement and calculate the sample variance of each sample. In addition, calculate the average and variance of all sample variances and compare them to the population variance.
3) Find the frequency distribution of all possible sample variances and draw its bar graph.

Answer

1) The population mean is μ = 6 and the population variance is σ^2 = 8.

2) All possible samples of size 2 with replacement from the population and the sample variance of each sample are as in Table 6.3.1.

Table 6.3.1 All possible samples of size 2 with replacement from the population and the sample variance of each sample

Sample	s^2	Sample	s^2	Sample	s^2	Sample	s^2	Sample	s^2
2,2	0	4,2	2	6,2	8	8,2	18	10,2	32
2,4	2	4,4	0	6,4	2	8,4	8	10,4	18
2,6	8	4,6	2	6,6	0	8,6	2	10,6	8
2,8	18	4,8	8	6,8	2	8,8	0	10,8	2
2,10	32	4,10	18	6,10	8	8,10	2	10,10	0

- As discussed in [Example 6.2.1], the sample variance is also a random variable that can have many values, so it is denoted as S^2 and the observed sample variance as s^2. Table 6.3.1 shows that some of these sample variances are exactly the same as the population variance σ^2 = 8, others such as 0 or 32 are significantly different from 8. The average of all possible sample variances, denoted as μ_{S^2}, is as follows:

$$\mu_{S^2} = \frac{0\times 5+2\times 8+8\times 6+18\times 4+32\times 2}{25} = 8$$

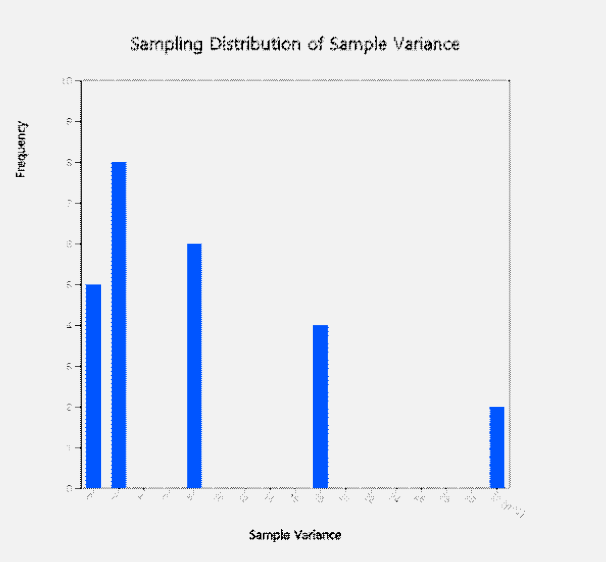
Note that the average of all possible sample variances is the same as the population variance which means the sample variance is an unbiased estimate of the population variance.

Example 6.3.1 Answer (continued)

3) Table 6.3.2 shows the frequency distribution of all possible sample variances and <Figure 6.3.1> is its bar graph. This is called the sampling distribution of sample variances. The fact that can be observed in this Figure is that there are many small sample variances and there are few large sample variances. In addition, the average of all sample variances, μ_{S^2}, is equal to the population variance σ^2. In other words, the sample variance is an unbiased estimate of the population variance.

Table 6.3.2 Frequency table of sample variances

s^2	Frequency	Relative Frequency
0	5	0.20
2	8	0.32
8	6	0.24
18	4	0.16
32	2	0.08
	25	1.00



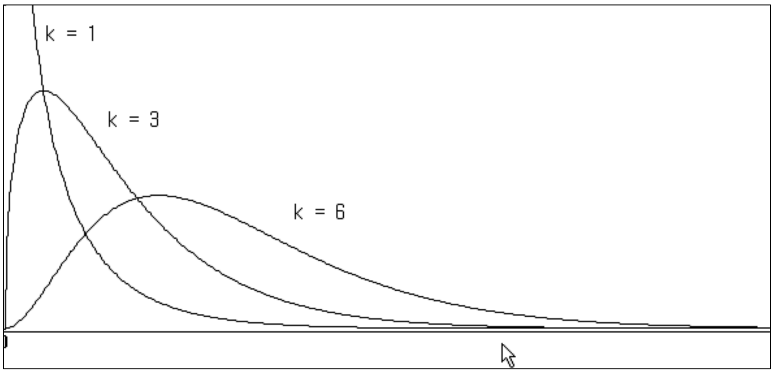
<Figure 6.3.1> Sampling distribution of sample variances

- As observed in the above example, the sampling distribution of the sample variances is an asymmetric distribution with many small sample variances and a few large sample variances. In general, when the population is normally distributed and the population variance is σ^2, all possible sample variances scaled by a constant follow the chi-square χ^2 distribution. More accurately, the sample statistic

$$\frac{(n-1)S^2}{\sigma^2}$$

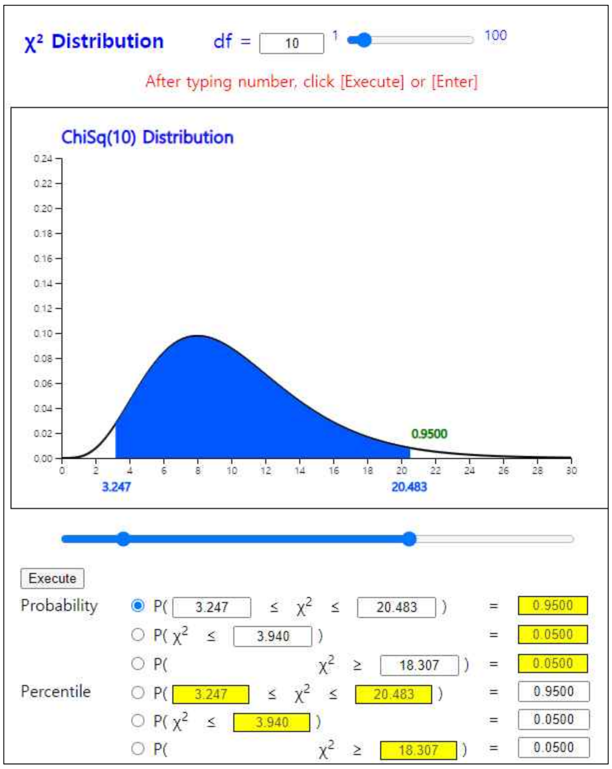
follows a chi-square distribution with n-1 degrees of freedom.

- This chi-square distribution is a family of distributions depending on the degree of freedom, such as χ^2_1, χ^2_2, ... , χ^2_{27}, ... etc. The chi-square distribution is an asymmetrical distribution as <Figure 6.3.2>. If the degree of freedom is small, the shape of the chi-square distribution is much skewed to the right.



<Figure 6.3.2> Chi-square distributions for different degrees of freedoms

- A cumulated probability and a percentile of the chi-square distribution can be calculated by using 『eStatU』 as in <Figure 6.3.3>.



<Figure 6.3.3> 『eStatU』 Chi-square distribution

- The sampling distribution of all sample variances is summarized as follows:

Theorem 6.3.1

The sampling distribution of sample variances

When the population is normally distributed and the sample of size n is selected randomly with replacement, the distribution of all sample variances multiplied by a specific constant follows the chi-square distribution with n-1 degrees of freedom as follows:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

6.3.2 Estimation of the Population Variance

- Examples of estimating the population variance are as follows:

  - Two companies supply bolts to an automobile maker. Bolts are defective if they are too large or too small in diameter. The automobile maker wants to know the variance of bolt diameters supplied by each bolt company and use them as data for selecting the better one.
  - In order to evaluate the difficulty of the college entrance exam conducted this year, the variance of exam scores is calculated and compared with the variance of exam scores in previous year.

- In order to estimate the population variance, the sampling distribution of all possible sample variances should be used. As discussed in [Example 6.3.1], for an infinite population, the mean of all possible sample variances is the population variance. That is, the sample variance S^2 is an unbiased estimator of the population variance σ^2. Therefore, the sample variance is used to estimate the population variance. In addition, estimation of the population standard deviation σ uses the sample standard deviation S, but it should be noted that the sample standard deviation is not an unbiased estimator of the population standard deviation. However, as the sample size increases, there is no significant error in using S as an estimator of σ.

Theorem 6.3.2

Point estimation of the population variance σ^2

⇒ Sample variance S^2 (S^2 is an unbiased estimator of σ^2)

Point estimation of the population standard deviation σ

⇒ Sample standard deviation S (S is not an unbiased estimator of σ)

- In the previous section, when a population was normally distributed, we found that the sample variance multiplied by a constant, $(n-1)S^2/\sigma^2$, follows the chi-square distribution with n-1 degrees of freedom. Using this, we can find the confidence interval of the population variance σ^2 as follows:

Theorem 6.3.3

100(1-α)% Confidence interval of the population variance σ^2

Assume that a population is normally distributed.

$$\left[\frac{(n-1)S^2}{\chi^2_{n-1\,;\,\alpha/2}} \;,\; \frac{(n-1)S^2}{\chi^2_{n-1\,;\,1-\alpha/2}} \right]$$

where S^2 is the sample variance, $\chi^2_{k\,;\,p}$ is the 100(1-p) percentile of the chi-square distribution with k degrees of freedom.

100(1-α)% Confidence interval of the population standard deviation σ

Assume a population is normally distributed and the sample size is large.

$$\left[\sqrt{\frac{(n-1)S^2}{\chi^2_{n-1\,;\,\alpha/2}}} \;,\; \sqrt{\frac{(n-1)S^2}{\chi^2_{n-1\,;\,1-\alpha/2}}} \right]$$

Example 6.3.2

A survey for the starting salary of 25 college graduates this year shows the sample standard deviation is 5 (unit 10000won). Find point estimation and 95% confidence interval of the population variance and the population standard deviation of the starting salary. Assume that the population is normally distributed.

Answer

- The point estimate of the population variance for the starting salary of college graduate is the sample variance, so s^2 = 5^2 = 25. Since the point estimate of the population standard deviation is the sample standard deviation, so s = 5.
- The 95% confidence interval of the population variance is as follows:

$$\left[\frac{(n-1)\,S^2}{\chi^2_{25-1\,;\,0.05/2}} \;,\; \frac{(n-1)\,S^2}{\chi^2_{25-1\,;\,1-0.05/2}} \right]$$

$$\Leftrightarrow \quad \left[\frac{(25-1)5^2}{39.364} \;,\; \frac{(25-1)5^2}{12.401} \right]$$

$$\Leftrightarrow \quad [\ 15.242\ ,\ 48.383\ ]$$

- The 95% confidence interval of the population standard deviation is as follows:

$$[\sqrt{15.242}\ ,\ \sqrt{48.383}\,] \quad \Leftrightarrow \quad [3.904\ ,\ 6.956]$$

Example 6.3.3

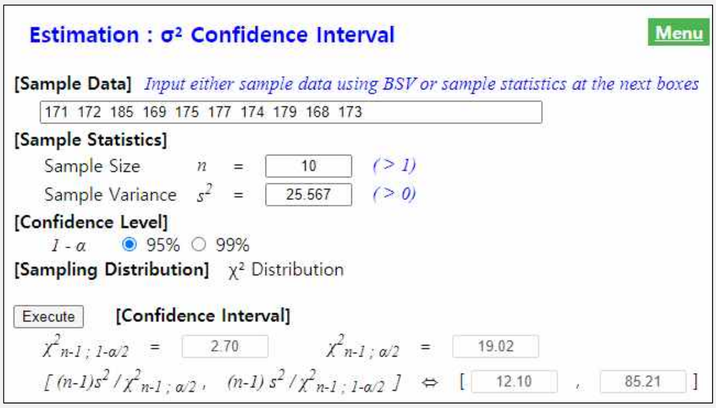
The height data of 10 male samples from freshmen in a college is as follows:

171 172 185 169 175 177 174 179 168 173

Using 『eStatU』, find a 95% confidence interval for estimating the population variance of college freshmen.

Answer

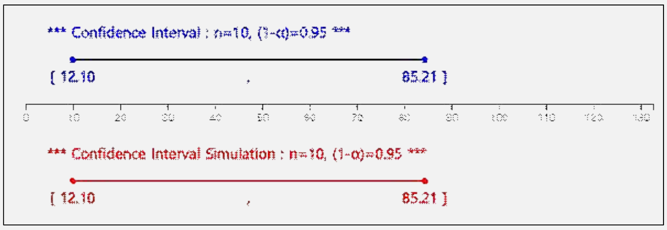
- On the menu of 『eStatU』, click 'Estimation : σ^2 Confidence Interval' and enter data in the [Sample Data] box as shown in <Figure 6.3.4>. The system will show the confidence interval [12.10, 85.21] immediately by using the chi-square distribution.



<Figure 6.3.4> 『eStatU』 Estimation of population variance

- In this module of 『eStatU』, a simulation experiment to investigate the size of the confidence interval can be done by changing the sample size n and the confidence level 1 - α as in <Figure 6.3.5>. If you increase n, the interval size becomes narrower. If you increase 1 - α, the interval size becomes wider.

Example 6.3.3 Answer (continued)



<Figure 6.3.5> 『eStatU』 simulation for the confidence interval of the population variance

- You can enter only the sample size and sample variance to calculate the confidence interval in this 『eStatU』 module without entering data.

[Practice 6.3.1]

A team of health researchers wants to measure the average amount of oxygen consumed after a certain standard exercise for men between ages of 17 and 21. Data of a simple random sampling of 10 persons are as follows:

2.87 2.05 2.90 2.41 2.93 2.94 2.26 2.21 2.20 2.88

Use 『eStatU』 to find a 95% confidence interval of the population variance of oxygen consumed.

6.4 Sampling Distribution of Sample Proportions and Estimation of the Population Proportion

- If we know the relationship between the population proportion and sample proportion, it is possible to estimate the population proportion. In this section, the distribution of all possible sample proportions is discussed in section 6.4.1 and the estimation of the population proportion using the sample proportion is discussed in section 6.4.2.

6.4.1 Sampling Distribution of Sample Proportions

- Consider the sampling distribution of all possible sample proportions by using the following example.

Example 6.4.1

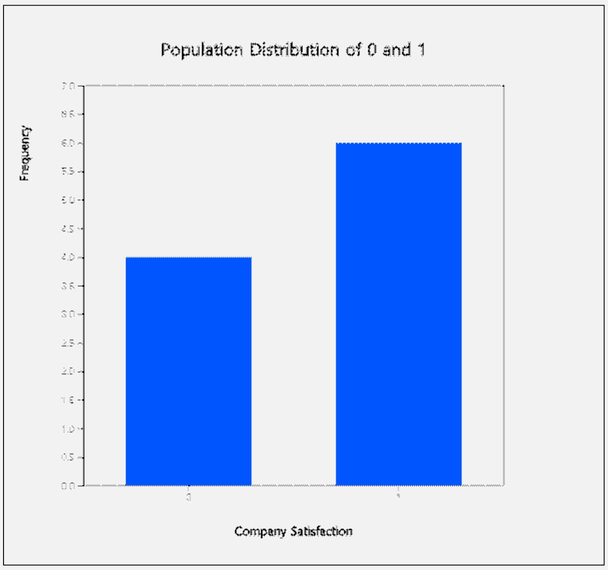
Consider a population consists of only 10 employees of a company. When employees' satisfaction level for the company is investigated and the satisfaction is expressed as 1, the complaint is 0 as follows:

1 0 1 1 0 1 1 0 0 1

That is, the population proportion p of the satisfaction is 0.6 (6 ‘1’s out of 10). Consider all possible samples of size 5 with replacement to obtain a sampling distribution of sample proportions. (It is to illustrate the sample variance of sample proportions, although it is not necessary to extract samples from such a small population.)

Example 6.4.1 Answer

- The population proportion is p = 0.6 and the distribution of '1' and '0' in the population consisting of 10 persons is as <Figure 6.4.1>.



<Figure 6.4.1> Population distribution of employees' satisfaction, no for 0 or yes for 1

- Total number of possible samples of size 5 with replacement is 10×10×10×10×10 = 100000 and the number of cases of possible sample proportions are as in Table 6.4.1.

Table 6.4.1 All possible sample cases

Sample Case	Number of cases
all unsatisfactory (0,0,0,0,0)	${}_5C_0$×4×4×4×4×4 = 1024
1 satisfactory (0,0,0,0,1)	${}_5C_1$×4×4×4×4×6 = 7680
2 satisfactory (0,0,0,1,1)	${}_5C_2$×4×4×4×6×6 = 23040
3 satisfactory (0,0,1,1,1)	${}_5C_3$×4×4×6×6×6 = 34560
4 satisfactory (0,1,1,1,1)	${}_5C_4$×4×6×6×6×6 = 25920
5 satisfactory (1,1,1,1,1)	${}_5C_5$×6×6×6×6×6 = 7776
	Total 100000

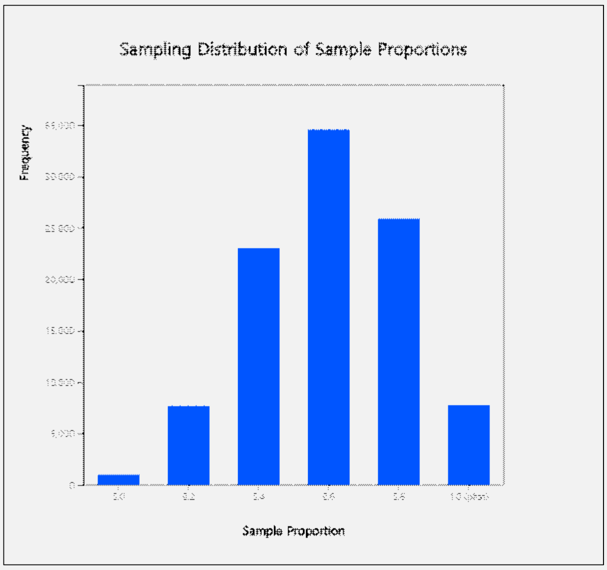
- If the sample proportion $(\hat{p})$ is obtained from each case, the sampling distribution of all possible sample proportions is as Table 6.4.2. The cases of three satisfactory (the sample proportion of 0.6) are most likely.

Table 6.4.2. Sampling Distribution of Sample Proportions

Sample case	$\hat{p}$	Frequency	Relative Frequency
all unsatisfactory	0.0	1024	0.01024
1 satisfactory	0.2	7680	0.07680
2 satisfactory	0.4	23040	0.23040
3 satisfactory	0.6	34560	0.34560
4 satisfactory	0.8	25920	0.25920
5 satisfactory	1.0	7776	0.07776
	Total	100000	1.0

Example 6.4.1 Answer (continued)

- <Figure 6.4.2> shows the sampling distribution of all possible sample proportions. If the sample size is larger, sample proportions are symmetrical around the population proportion 0.6 which is similar to that of sample means, and is close to a normal distribution.



<Figure 6.4.2> Sampling distribution of sample proportions

- When the sample size is large, the sampling distribution of all possible sample means in general is as follows:

Theorem 6.4.1

Sampling distribution of sample proportions

Assume the population is infinite and the population proportion is p. If $\hat{p}$ is the sample proportion and the sample size n is large, the sampling distribution of all possible sample proportions is approximately a normal distribution with the mean p and variance $p(1-p)/n$.

$$\hat{p} \sim N(p, \frac{p(1-p)}{n})$$

- If the population size is N which is finite and the sampling is without replacement, the variance of $\hat{p}$ becomes $\frac{p(1-p)}{n} \frac{N-n}{N-1}$. The term $\frac{N-n}{N-1}$ is called the **finite population correction factor**.

Example 6.4.2

Let's say 3% of semiconductors made in a semiconductor factory are defective. When 300 samples were taken without replacement, the sample proportion for defective products was 2%. Find out where this sample proportion is located among all possible sample proportions. What is the probability that the sample proportion is greater than 2%?

Example 6.4.2 Answer	◆ Since the sampling distribution of the sample proportions is approximately normal distribution, $\hat{p} \sim N\left(0.03, \frac{0.03(1-0.03)}{300}\right)$, the probability can be calculated as follows: $P(\hat{p}$ > 0.02) = $P(Z$ > (0.02-0.03)/0.00985) = $P(Z$ > -1.02) = 1 - $P(Z \leq$ -1.02) = 1 - 0.1539 = 0.8461

6.4.2 Estimation of the Population Proportion

- Some practical examples to estimate the population proportion are as follows:

  - What is the approval rating of a particular political party in this year's election?
  - What is the percentage of the current unemployment rate?
  - What percentage of defective products we would have if we imported 10,000 car accessories?

- Assume that the proportion of a population is p. As discussed in Section 6.1, since the sample proportion, $\hat{p}$, meets all the criteria of a good estimator when estimating the population proportion p, the sample proportion is used to estimate the population proportion. The sampling distribution of all possible sample proportions is approximately a normal distribution with the mean p and variance $p(1-p)/n$ and the standard error of the sample proportions is $\sqrt{p(1-p)/n}$. But, since the population proportion p is unknown, we use $\sqrt{\hat{p}(1-\hat{p})/n}$ as an estimate of the standard error of the sample proportions.

Theorem 6.4.2	**Point estimate of the population proportion p** **⇒ The sample proportion $\hat{p}$.** The sample proportion ($\hat{p}$) is an unbiased, efficient and consistent estimator of the population proportion p and the estimate of the standard error of $\hat{p}$ is $\sqrt{\hat{p}(1-\hat{p})/n}$

- From the fact that the distribution of the sample proportion $\hat{p}$ is approximated to a normal distribution, interval estimation of the population proportion p can be done as follows:

Theorem 6.4.3	**Interval estimation of the population proportion** If the population proportion is p, 100(1-α)% confidence interval of p when the sample size n is large is as follows: $\left[\hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$ Criteria of large sample size n are $n\hat{p} > 5$, $n(1-\hat{p}) > 5$.

Example 6.4.3

A student running for the president of a student body in a university had a survey of 200 students to find out his approval ratings, and found that 120 students supported him. Find a point estimate of his approval rating in the population, and find a 95% confidence interval. Check the interval estimation using 『eStatU』.

Answer

- The point estimate of the population approval rating is the sample proportion.

$$\hat{p} = \frac{120}{200} = 0.6$$

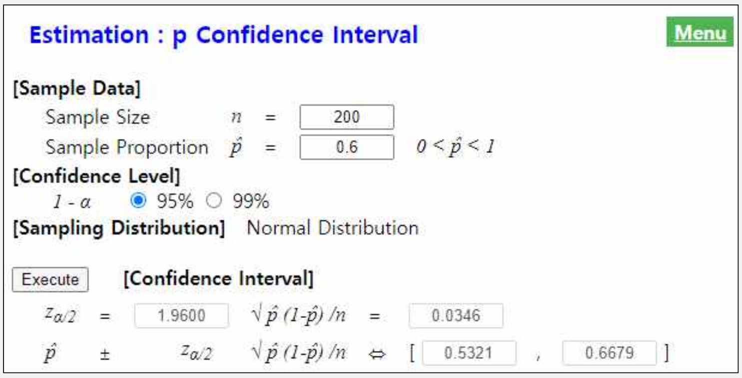
- The 95% confidence interval is as follows:

$$[\hat{p} - z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \;,\; \hat{p} + z_{\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\,]$$

$$\Leftrightarrow \quad [0.6 - 1.96 \sqrt{\frac{0.6(1-0.6)}{200}} \;,\; 0.6 + 1.96 \sqrt{\frac{0.6(1-0.6)}{200}}\,]$$

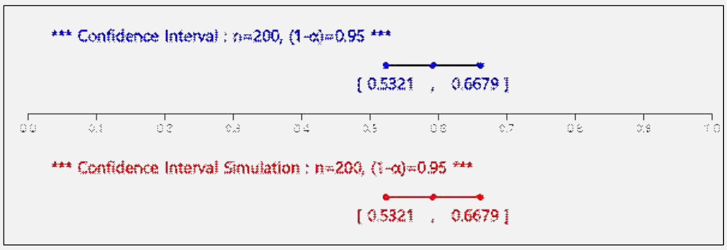
$$\Leftrightarrow \quad [0.532 \;,\; 0.668]$$

- In 『eStatU』, enter data in the [Sample Data] box as shown in <Figure 6.4.3> and click the [Execute] button to calculate the confidence interval [0.5321, 0.6679] using the normal distribution.



<Figure 6.4.3> 『eStatU』 Estimation of the population proportion

- In this module of 『eStatU』, a simulation experiment to investigate the size of the confidence interval can be done by changing the sample size n and the confidence level 1 - α as in <Figure 6.4.4>. If you increase n, the interval size becomes narrower. If you increase 1 - α, the interval size becomes wider.



<Figure 6.4.4> Simulation for estimating the population proportion

[Practice 6.4.1]

200 workers were selected to investigate the causes of worker turnover. Of the 200, 140 said they moved because they could not reconcile with their superiors. Find a 95% confidence interval in the mobile rate for those who have transferred using 『eStatU』.

6.5 Determination of the Sample Size

- Until now, we have studied the estimation of a population parameter using a given sample. However, it is often necessary to first determine how large a sample should be before obtaining such a sample. This problem is closely related to the precision of the estimate. We could see in the previous section that the width of the confidence interval was reduced if the sample size is larger. However, as sampling is costly, it usually determines the minimum sample size required to achieve this precision that would satisfy the researcher.

6.5.1 Determination of the Sample Size to Estimate the Population Mean

- The 100(1-α)% confidence interval of a population mean μ is as follows as explained in section 6.2.

$$\left[\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \;,\; \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

The term $z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ which is the half size of the confidence interval is called the **maximum allowable error bound** to estimate the population mean and is denoted as d.

- Therefore, if the maximum allowable error bound d and the confidence level $1-\alpha$ are given, the sample size can be determined by solving the following equation.

$$z_{\alpha/2}\frac{\sigma}{\sqrt{n}} = d \quad \Rightarrow \quad n = [\frac{z_{\alpha/2}\,\sigma}{d}]^2$$

Theorem 6.5.1	**Determination of the sample size to estimate the population mean** If the maximum allowable error bound d and the confidence level $1-\alpha$ are given, the sample size n can be determined as follows: $n = [\frac{z_{\alpha/2}\,\sigma}{d}]^2$

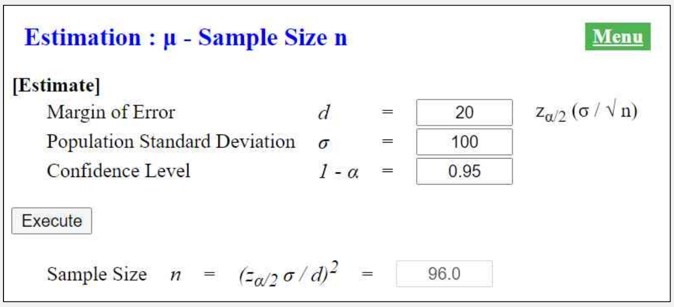
- Since the population standard deviation σ in the above equation is usually unknown, the estimated value from past experience or from data obtained by a preliminary survey is used. The estimation of the population standard deviation σ through a preliminary survey can be done by using the range as follows:

$$\hat{\sigma} = \frac{range}{4} = \frac{\max - \min}{4}$$

Example 6.5.1	The expected life of light bulbs produced at a plant has a standard deviation 100 hours. In order to estimate the average life of a bulb at the 95% confidence level, determine the sample size to be within 20 hours of the allowable error bound. Check this calculation using 『eStatU』.
Answer	◆ $n = [\frac{z_{\alpha/2}\,\sigma}{d}]^2 = (\frac{1.96 \times 100}{20})^2 = 9.8^2 = 96.04$ Hence, the sample size is 96 approximately.

Example 6.5.1 Answer (continued)

- In 『eStatU』, select the menu 'Estimation : μ - Sample Size n'. Enter the margin of Error d = 20, the population standard deviation σ = 100 and confidence level $1-\alpha$ = 0.95 as in <Figure 6.5.1>, then click the [Execute] button to calculate the sample size n.



<Figure 6.5.1> Determination of sample size for estimating μ

[Practice 6.5.1]

A company with 10,000 workers wants to estimate the average time it takes to commute. The surveyor wants to estimate that the maximum margin error bound is less than 5 minutes at the 95% confidence level. If the estimate of the population variance obtained from the preliminary study was σ^2 = 25, what size of the sample should be extracted?

6.5.2 Determination of the Sample Size to Estimate the Population Proportion

- As explained in section 6.4, the 100(1-α)% confidence interval of a population proportion p is as follows:

$$\left[\ \hat{p} - z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\ ,\ \hat{p} + z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\ \right]$$

  The term $z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ which is the half size of the confidence interval is called the **maximum allowable error bound** to estimate the population proportion and is denoted as d.
- Therefore, if the maximum allowable error bound d and the confidence level $1-\alpha$ are given, the sample size can be determined by solving the following equation.

$$z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = d \qquad \Rightarrow \qquad n = \hat{p}(1-\hat{p})(\frac{z_{\alpha/2}}{d})^2$$

Theorem 6.5.2

Determination of the sample size to estimate the population proportion.

If the maximum allowable error bound d and the confidence level $1-\alpha$ are given, the sample size n can be determined as follows:

$$n = \hat{p}(1-\hat{p})(\frac{z_{\alpha/2}}{d})^2$$

- In the above equation, the value of the sample proportion $\hat{p}$ is usually used by the value of past experience or preliminary investigation. If there is no information about the population proportion, the value of 0.5 is usually used.

Example 6.5.2

For this year presidential election, a survey for estimating the 95% confidence interval of a candidate's approval rating is conducted. Determine the size of the sample to be within 2.5% of the maximum allowable error bound. Check this calculation using 『eStatU』.

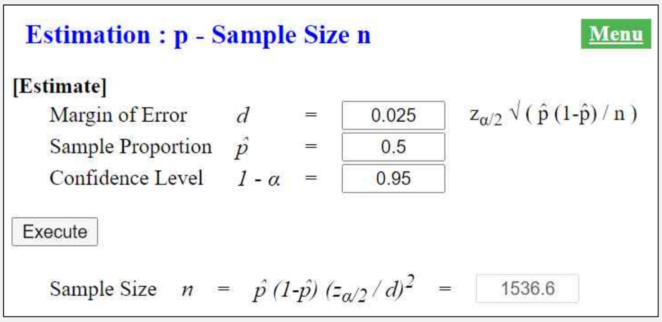
Answer

- Since we do not have any information on the population proportion, assume $\hat{p}$ = 0.5. Then the sample size n can be calculated as follows:

$$n = \hat{p}(1-\hat{p})(\frac{z_{\alpha/2}}{d})^2$$

$$n = 0.5(1-0.5)\frac{1.96^2}{0.025^2} = 1536.6$$

- Therefore, samples must be extracted with at least 1537 persons to limit the error bound to 2.5%. Various opinion polls conducted nationwide often show that the size of the sample is around 1,500 which implies that the margin of error does not exceed 3 percent.
- In 『eStatU』, select the menu 'Estimation : p - Sample Size n'. Enter the margin of Error d = 0.025, the estimate of the sample proportion $\hat{p}$ = 0.5 (since we don't know $\hat{p}$, it is an estimate), and the confidence level $1-\alpha$ = 0.95 as in <Figure 6.5.2>, then click the [Execute] button to calculate the sample size n.



<Figure 6.5.2> Determination of sample size for estimating p

[Practice 6.5.2]

When a university opens an online lecture this semester, it tries to estimate the percentage of students enrolled in this online class as 95 percent confidence intervals. How many students should be surveyed to keep the maximum allowable error bound within 0.02?

Exercise

6.1 There are 70 workers in a factory. Ten persons will be selected by simple random sampling to investigate the production amount per week. Use the random number generator in 『eStatU』 for sampling.

6.2 List all cases in which three of the five people A, B, C, D are sampled with replacement.

6.3 A telephone response service produces a report of the call time at the end of each call. Nine reports were collected by simple random sampling and their sample mean of the call time was 1.2 minutes. If the population follows a normal distribution with a standard deviation of 0.6 minutes, find a 99% confidence interval of the population mean.

6.4 The quality manager of a large manufacturer wants to know the average weight of 5,500 raw materials. 250 samples were collected by simple random sampling and their sample mean is 65 kg. If the population standard deviation is 15 kg, find a 95% confidence interval for the unknown population mean.

6.5 A physical health researcher wants to measure the average amount of oxygen consumed after a standard exercise for men between the ages of 17 and 21. Studies show that the population variance is 0.0512. The result of a simple random sampling of 25 persons are as follows:

2.87 2.05 2.90 2.41 2.93 2.94 2.26 2.21 2.20 2.88
2.51 2.51 2.56 2.59 2.52 2.51 2.50 2.58 2.52 2.58
2.44 2.48 2.43 2.46 2.46 (Liter/min)

Obtain a 95% confidence interval of the population mean when oxygen consumption follows a normal distribution.

6.6 An industrial psychologist wants to know the average age of female workers in a particular population. The average age at which 60 samples from the population by simple random sampling was 23.67. Obtain a 99% confidence interval of the population mean when the population standard deviation is 15.

6.7 In a study to determine whether a machine can use the flexible plastic hose, an engineer tries to estimate the average pressure the hose receives. The engineer measured the pressure nine times at intervals of 24 hours. The mean and standard deviation of the samples are 362, 45 respectively, and the pressure is approximately normal. Obtain a 99% confidence interval for the average pressure.

6.8 Sixteen radio stations were collected by simple random sampling to estimate the cost of radio broadcasting for 30 seconds of insertion advertisement. The sample mean is 15 million US$ and the sample variance is 8. Obtain a 95% confidence interval for the population mean when the advertising costs for all radio stations follow a normal distribution.

6.9 The tension of a thread used to make a piece of cloth is examined. Ten samples of this thread were collected by simple random sampling and tested for tension, and the sample variance is 4. What assumptions do you need for the interval estimation of the population variance? Find a 95% confidence interval of the population variance σ^2.

6.10 A production manager wants to know the time required to finish a specific work in the product process. A sample of 25 observations by simple random sampling was collected for analysis and the sample variance is 0.32.

1) Obtain a 95% confidence interval of the population variance σ^2.
2) Obtain a 99% confidence interval of the population variance σ^2.
3) Obtain a 90% confidence interval of the population variance σ^2.
4) What assumptions do you need to obtain a valid confidence interval?

6.11 An ecologist seeks to measure the amount of certain pollutants that contain 15 samples of water from the river where the factory area is located. If the amount of contaminants is normally distributed and $\sum(x_i - \bar{x})^2$ = 508.06, obtain a 95% confidence interval of the population variance.

6.12 The internal diameter of 12 ball bearings made in a manufacturing process was measured by simple random sampling as follows:

3.01, 3.05, 2.99, 2.99, 3.00, 3.02, 2.98, 2.99, 2.97, 2.97, 3.02, 3.01

If diameters follow a normal distribution, obtain a 99% confidence interval of the population variance.

6.13 A researcher wants to know whether office workers who change jobs due to the monotony of work. 400 samples are collected by simple random sampling from office workers who recently changed jobs. Of them, 200 said they changed jobs, because of the monotony of their jobs. Find a 95% confidence interval of the population proportion who have changed jobs, because they are monotonous.

6.14 A manager of a production company would like to know the ratio of workers who remember the safety management prints distributed to all workers last week. 300 workers collected by simple random sampling and a test was conducted to check whether they remembered the contents of the printed material. Seventy-five of those who took the exam passed it. Obtain a 95% confidence interval in the population proportion of workers who remember the safety management.

6.15 200 workers were selected to investigate the causes of worker's turnover. Of the 200, 140 said they moved, because they could not reconcile with their superiors. For this reason, find a 95% confidence interval in the mobile rate for those who have transferred.

6.16 A manufacturer guarantees a defect rate of less than 5% for a customer company that regularly purchases the product. The customer company collected 200 of the purchased products by simple random sampling and inspected the samples to confirm the manufacturer's claims. If 19 defective products of the 200 samples were found, what would be the 95% confidence interval for the defect rate?

6.17 A candidate came from the Democratic Party in the mayoral election of a metropolitan city. Polls showed 152 out of 284 samples collected by simple random sampling support the candidate.

1) Find a 95 percent confidence interval in the approval rating of the candidate.
2) Find a 99 percent confidence interval in the approval rating of the candidate.
3) To what extent do you think the election of the candidate of the party is certain?
4) Explain whether the sampling method of the poll will affect the result of 3) above.

6.18 In question above, if 1,368 out of 2,556 samples collected by simple random sampling were found to support the candidate, answer the questions 1), 2), and 3) and compare the result.

6.19 The advertising manager of a company is trying to put the new product on a one-minute commercial for a TV Saturday evening program. He found from the TV station that the one-minute ad was priced as follows:

Price $= 6{,}500 + 350{,}000\,\hat{p}$ (Unit: US$)

where $\hat{p}$ is an estimate of the nationwide ratings of the program. The station relies on a ratings survey of the M service station, which has installed and operated devices in 1,500 homes nationwide. According to records from the M Station, 360 households watched the program last Saturday.

1) What's the price of this one-minute commercial?
2) What would you decide if the advertising manager had decided to buy this commercial time only if he could believe that the actual viewing rate was 0.2 or higher with 95 percent confidence level?

6.20 In order to know the average strength of a plastic product produced by a company, how many experiments do you need to do to have the maximum allowable error bound within 20 (unit: psi) with a 99% confidence level? Previous experience suggests that the estimate for σ^2 is 4900.

6.21 A company with 2,500 workers wants to estimate the average time it takes to commute. The surveyor wants to estimate that the maximum allowable error bound is less than 5 minutes at the 95% confidence level. If the estimate of the population variance obtained from the preliminary study was σ^2 = 25, what size of the sample should be extracted?

6.22 The average IQ of an employee at a company is estimated to be within 5 points of the maximum allowable error bound at a 95% confidence level. Past experience shows that the IQ of this group follows a normal distribution with a variance of 100. How many samples should be taken?

6.23 When a university opens a lecture Saturday, it tries to estimate the percentage of students enrolled in the class as a 95 percent confidence interval. How many students should be surveyed to keep the maximum allowable error bound within 0.05?

6.24 Of all manufacturers, we would like to know the percentage of companies that require a doctor's diagnosis from workers when they are absent due to illness for more than three days. How many samples do you need to estimate the maximum allowable error bound within 0.05% at the 95% confidence level?

6.25 I want to know what percentage of households in a region where at least one member of the family sees advertisements in newspapers. I want the maximum allowable error bound is 0.04 at the 95% confidence level. When a preliminary survey of 20 households found that at least 35 percent of the responding households saw an advertisement, what should the number of sample households be?

6.26 The consumer research group believes that TV tube life is normally distributed with two years of standard deviation. How many TV sets should be tested with 95% confidence level if the maximum allowable error bound is two years? How many TV sets should be tested with 95% confidence level if the maximum allowable error bound is one year?

6.27 If you want to reduce the length of the confidence interval by half, should you double the size of the sample? Explain.

6.28 A school district had 100 samples by simple random sampling of the 5th grade elementary school students and took a math test with the sample mean of 74.3 and the sample standard deviation of 9.2.

1) Find a 99% confidence interval in the average score of all fifth graders in this school district when they take the test.
2) If this sample mean 74.3 is used as an estimate of the population mean of the math test for all fifth graders in this school district, what is the maximum size of the error bound at the 95% confidence level?

6.29 The average weight of 50 cans of peaches which are collected by simple random sampling is 16.1 grams and their standard deviation is 0.4 grams. If the sample mean of 16.1 grams is used as an estimate of the population mean weight of all peach cans, what is the probability that the estimated error bound is less than 0.1 grams?

Multiple Choice Exercise

6.1 Which distribution is used to estimate the population mean in a small sample with a small number of samples?

① Normal distribution ② Exponential distribution
③ t distribution ④ F distribution

6.2 When the population size is infinite and variance is σ^2, what is the variance of all possible sample means? (n is the sample size)

① $\sigma_{\bar{x}}{}^2 = \sigma^2$ ② $\sigma_{\bar{x}}{}^2 = n\sigma^2$
③ $\sigma_{\bar{x}}{}^2 = \dfrac{\sigma^2}{n}$ ④ $\sigma_{\bar{x}}{}^2 = \dfrac{n\sigma^2}{n-1}$

6.3 The following is a description about estimation. Which explanation is wrong?

① In statistics, an estimation is a quantitative estimate using a sample of the characteristics of a population.
② There are two kinds of estimation, one is the point estimation and the other is the interval estimation.
③ An estimator which is unbiased, consistent and efficient is desirable.
④ The population standard deviation is an estimator of the sample standard deviation.

6.4 The weight of products produced by a company follows a normal distribution. In order to estimate the average weight, 49 products collected by simple random sampling and examined for their weights, resulting in an average of 6200 grams and a standard deviation of 140 grams. What is the 95% confidence interval for the average weight of this product?

① 6200 ± 32.8 ② 6200 ± 39.2
③ 6200 ± 52.6 ④ 6200 ± 77.4

6.5 Population variance of a normal population was σ^2. A sample of size n was taken from this population to obtain a sample mean $\bar{x}$. What is the 99% confidence interval for the population mean μ?

① $\bar{x} \pm 1.96\sigma / \sqrt{n}$ ② $\bar{x} \pm 1.96\sigma / \sqrt{n-1}$
③ $\bar{x} \pm 2.58\sigma / \sqrt{n}$ ④ $\bar{x} \pm 2.58\sigma^2 / \sqrt{n}$

6.6 What is the quantitative measure of the characteristic of a population?

① parameter ② variation
③ represent ④ statistic

6.7 400 voters are randomly selected from a city and polled. 240 voters answered ‘yes’ for an issue and the rest were against it. Obtain a 95% confidence interval for the population proportion of ‘yes’.

① $0.55 \le p \le 0.65$ ② $0.5 \le p \le 0.7$
③ $0.45 \le p \le 0.75$ ④ $0.4 \le p \le 0.8$

6.8 What is the quantitative measure of the characteristic of a sample?

① probability ② statistic
③ parameter ④ variable

6.9 What is the most appropriate sample statistic to estimate the population variance when we collect n samples by simple random sampling from a normal population?

① standard deviation ② sample variance
③ sum of squares ④ $1/n$ of sum of squares

6.10 If the measured values of a sample collected randomly are $x_1,\ x_2,\ \ldots, x_n$ and their sample mean is $\overline{x}$, which of the following is an unbiased estimator of the population variance?

① $\frac{1}{n}\sum|x_i-\overline{x}|$ ② $\frac{1}{n}(\sum x_i^{\,2}-(n-1)x^2)$
③ $\frac{1}{n}(\sum x_i^{\,2}-n\overline{x}^{\,2})$ ④ $\frac{1}{n-1}\sum(x_i-\overline{x})^2$

6.11 Out of 2,000 products made by a company, 400 were randomly extracted to measure the weight and the sample mean was $\overline{x}=25.0$ and sample standard deviation was $s=4.99$. Obtain a 95% confidence interval of the population mean?

① $24.5 \le \mu \le 25.5$ ② $24 \le \mu \le 26$
③ $24.3 \le \mu \le 25.7$ ④ $23.6 \le \mu \le 26.4$

6.12 In a parliamentary election, 60 voters are selected randomly in a district and 36 of them support the candidate A. Obtain a 95% confidence interval that support the candidate A?

① $0.576 \le p \le 0.760$ ② $0.576 \le p \le 0.824$
③ $0.440 \le p \le 0.760$ ④ $0.440 \le p \le 0.824$

6.13 The average monthly income of 5,000 households which were randomly selected in a city was 3600 US$ and the standard deviation was 500 US$. Obtain a 95% confidence interval of the city's monthly income.

① $3600 \pm 1.96 \times \frac{500}{\sqrt{5000}}$ ② $3600 \pm 2.58 \times \frac{500}{\sqrt{5000}}$
③ $3600 \pm 1.64 \times \frac{500}{\sqrt{5000}}$ ④ $3600 \pm 2.33 \times \frac{500}{\sqrt{5000}}$

6.14 A sample of size 49 was taken from a population with a population standard deviation $\sigma = 5$. If the sample mean is $\overline{x}=15.2$, what is a 95% confidence interval for the population mean μ?

① (13.8, 16.6) ② (13.4, 17.0)
③ (13.6, 16.8) ④ (13.2, 17.2)

6.15 When a population follows a normal distribution and a random sample of size 25 was selected, the sample mean was 14.6 and sample standard deviation was 5. What if we get a 95% confidence interval for the population mean?

① 14.6±2.787
② 14.6±2.056
③ 14.6±2.064
④ 14.6±2.779

t distribution table		
significance level / degrees of freedom	0.05	0.01
24	2.064	2.797
25	2.060	2.787
26	2.056	2.779

(Answers)
6.1 ③, 6.2 ③, 6.3 ④, 6.4 ②, 6.5 ①, 6.6 ①, 6.7 ①, 6.8 ②, 6.9 ②, 6.10 ④,
6.11 ①, 6.12 ③, 6.13 ①, 6.14 ①, 6.15 ③