Homework 5 Solution

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October 8, 2024

This is my GitHub link: Nana Ama's GitHub Link

1 Question 1

The formula

$$f(x) = 1 + \tanh(2x) \tag{1}$$

can be solved analytically as

$$f'(x) = 1 - \tanh^2(2x) \tag{2}$$

Plotting the analytical version and that obtained numerically from python on the range $x = [-4, 4]$, it is seen from Figure 1 that there is little error.

2 Question 2

The integrand of the gamma function

$$\Gamma(x) = \int_0^\infty x^{a-1} \exp^{-x} dx \tag{3}$$

is

$$f(x) = x^{a-1} \exp^{-x} \tag{4}$$

Plotting this function for different values of a on the interval x = [0, 5] yields the graph below.

For a function to be maximum, it's value when its first derivative is set to 0. For the function

$$f(x) = x^{a-1} \exp^{-x} \tag{5}$$

we see that

$$\begin{aligned} f'(x) &= (a-1)x^{a-2} \exp^{-x} - x^{a-1} \exp^{-x} = 0 && (6) \\ f'(x) &= \frac{a-1}{x} x^{a-1} \exp^{-x} - x^{a-1} \exp^{-x} = 0 && (7) \\ &= (\frac{a-1}{x} - 1) x^{a-1} \exp^{-x} = 0 && (8) \end{aligned}$$

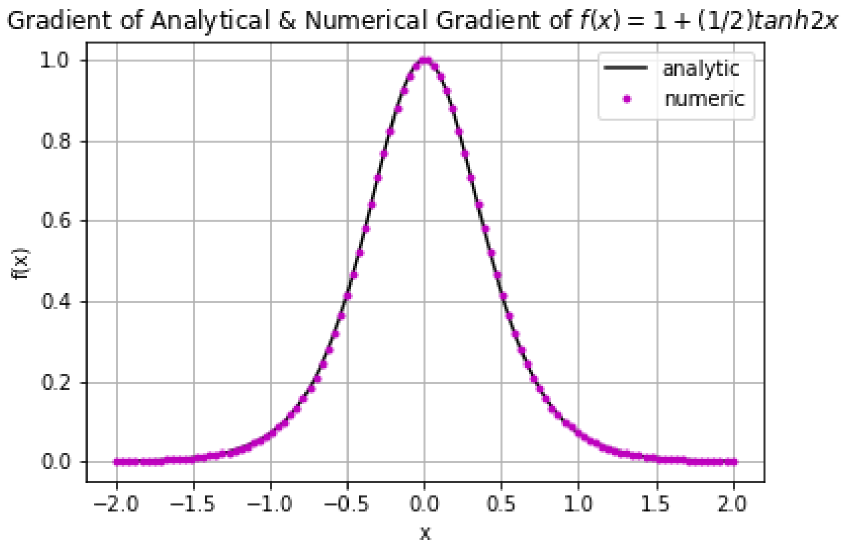

Figure 1: A plot of both the analytical and numerical gradients of the function $f(x) = 1 + \tanh(2x)$

setting the coefficient to 0 yields

$$\frac{a-1}{x} - 1 = 0 \tag{9}$$

$$x = a - 1 \tag{10}$$

For a change in variables $z = \frac{x}{c+x}$, when solved analytically yields the value of $z = \frac{1}{2}$ at $x = c$. Hence, $c = a - 1$. The function is modified to

$$\Gamma(x) = \int_0^\infty \exp{(a-1)(\ln x)} \exp^{-x} dx \tag{11}$$

to prevent underflow and overflow.

Using the modified gamma function expression and change of variables, the function $gamma(a)$ was defined and $gamma(3/2)$ was calculated to have a value of 0.8862269613087213, which is approximates to be 0.866. Also, the values of gamma at a = 3, 6 and 10 were found to be 2.000000000000001, 119.99999999999999, and 362879.99999999994. These values are found to be consistent with the actual values, which are seen to have a formula of $(a-1)!$, with little errors arising from underflow or overflow from Python.

3 Question 3

Singular Value Decomposition can be used to make function fits to data. Here, a series of polynomials were tried and fitted against the original plot. In addition to that, a set of sines and cosines,

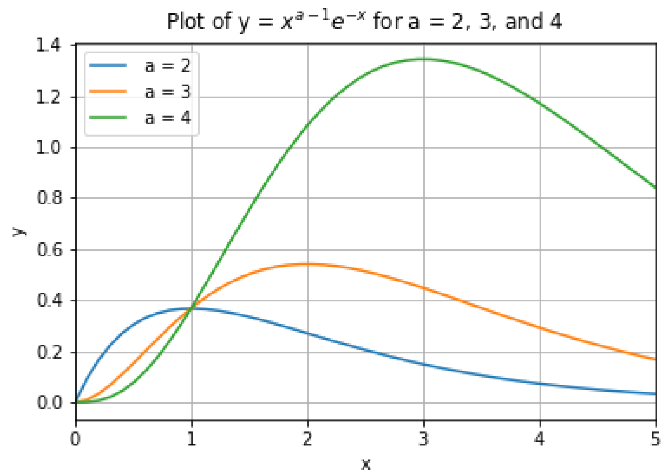

Figure 2: A plot of the integrand $f(x) = x^{a-1} \exp^{-x}$ for a = 2, 3, and 4

forming a harmonic series, were also plotted against the data. The condition numbers (w) were calculated in each case. The closer the condition number is to 1, the better the fit.
A plot of the original data obtained from the signal.dat file was plotted as shown in Figurefig:signal data
A plot of the residuals in Figure 4 show little difference in the original plot.

A third order polynomial was fitted against the plot and plotted in Figure 5. The condition number was found to be 9.080721776290792e+24 of order 24 which is not appropriate.

A higher order polynomial of order 11 was tested and plotted as shown in Figure 6. The condition number was calculated as 6.633791010480661e+93, which is considerably worse than that of the third order polynomial fit.

It was revealed that higher order polynomials gave greater condition numbers and hence, worse fits. Any higher order polynomial returned an error with an infinite condition number. Hence, a plot of a first order polynomial was made as shown in Figure 7 which revealed a condition number of 145696560.95949677 of order 10, which although bad, is better than the previous values.
Clearly, polynomials are not a good fit for the signal data.

A set of cosines and sines were then used to fit the data as shown in Figure 8. The harmonic sequence returned a condition number of 1.6822417943999717, which is proven to be the best choice for data of this type.

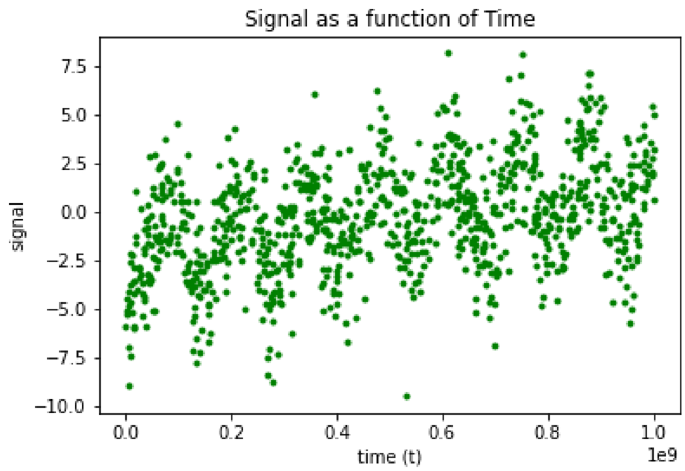

Figure 3: A plot of signal as a function of time

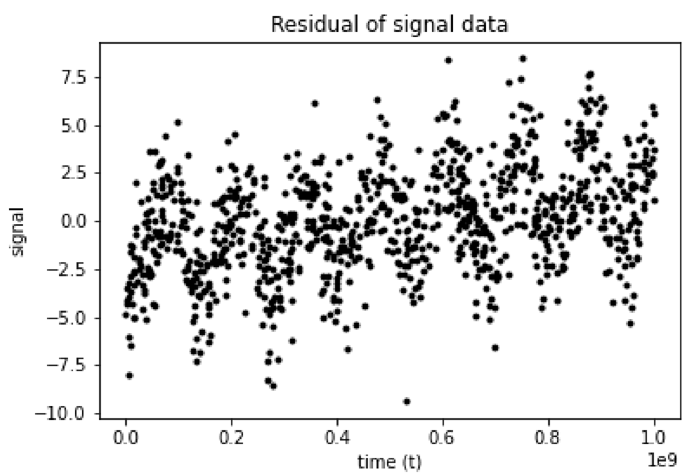

Figure 4: A plot of residual signal

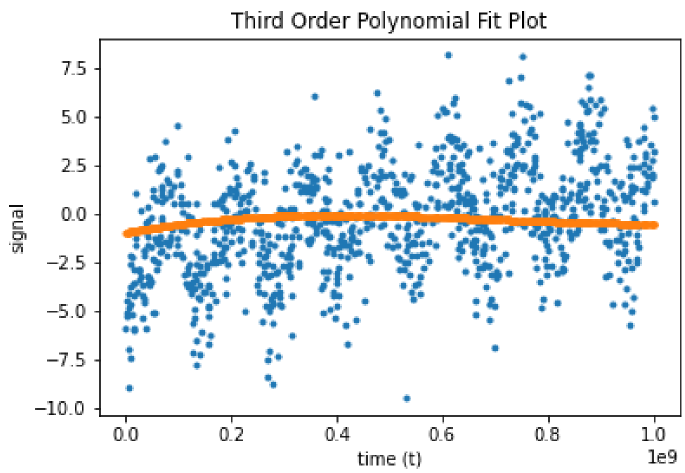

Figure 5: A plot of third order polynomial fit against the original data plot

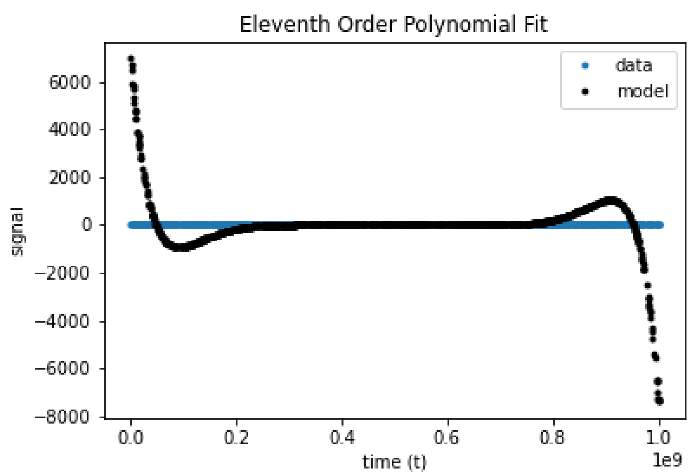

Figure 6: A plot of an eleventh order polynomial fit against the original data plot

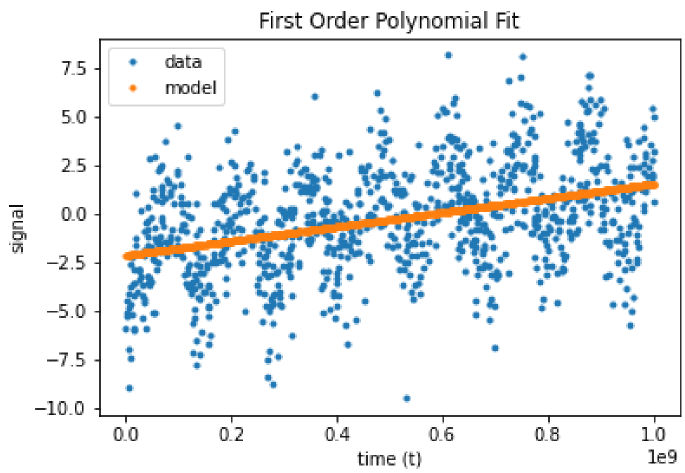

Figure 7: A plot of a first order polynomial fit against the original data plot

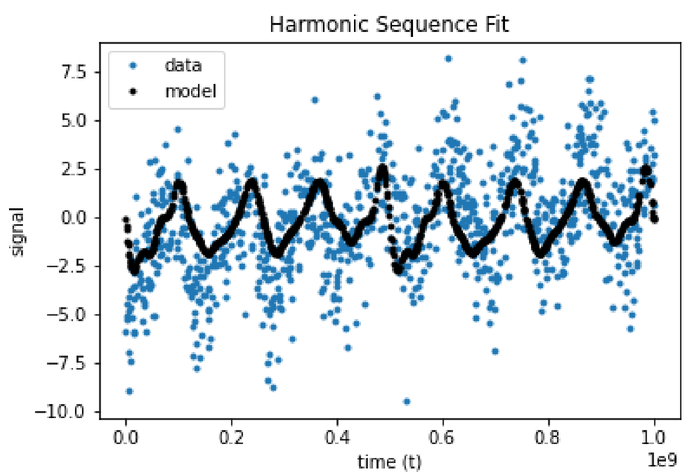

Figure 8: A plot of a harmonic sequence fit against the original data plot