HW 1 - FEM - Nina De La Torre

Part A) 1-D Lagrange Finite-Element Bases

- 1D Mesh: Interval $[0,10]$ with 10 uniform elements, $h=1$

A) For $p=1,2,3$ define the standard C^0 Lagrangian Finite Element space on the uniform partition.

- Lagrange Polynomials equal 1 at their node and 0 @ all others

P=1 (linear)

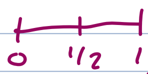
$\xi_1=0,\ \xi_2=1$

(B) • 11 total nodes, each element has 2 nodes (left, right)

(C) • The bases on the reference element $\hat{K}=[0,1]$ is

$$\phi_1(\xi)=1-\xi\ ,\quad \phi_2(\xi)=\xi$$

P=2 (quadratic)



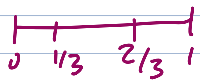
$\xi_1=0,\ \xi_2=1/2,\ \xi_3=1$

(B) • 21 total nodes, each element has 3 nodes (left, midpoint, right)

(C) • The bases on $\hat{K}=[0,1]$ with nodes at 0, 1/2, and 1 are:

$$\phi_1(\xi)=2(\xi-0.5)(\xi-1),\quad \phi_2(\xi)=4\xi(1-\xi),\quad \phi_3(\xi)=2\xi(\xi-0.5)$$

P=3 (cubic)



$\xi_1=0\quad \xi_2=1/3\quad \xi_3=2/3\quad \xi_4=1$

(B) • 31 total nodes, each element has 4 nodes at $\xi_i=0, 1/3, 2/3, 1$.

(C) • The basis functions are:

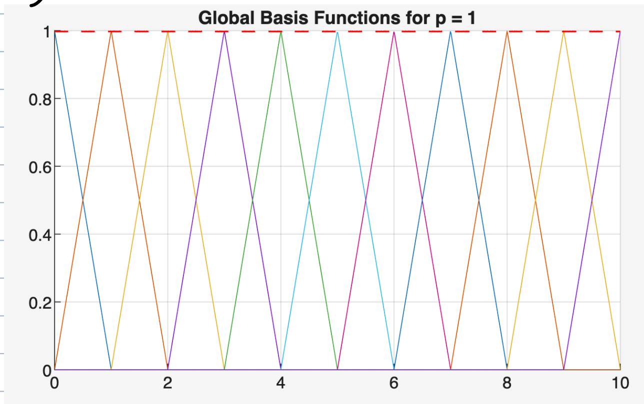
$$\phi_j=\prod_{m\neq j}\frac{\xi-\xi_m}{\xi_j-\xi_m}\ ,\quad j=1,2,3,4$$

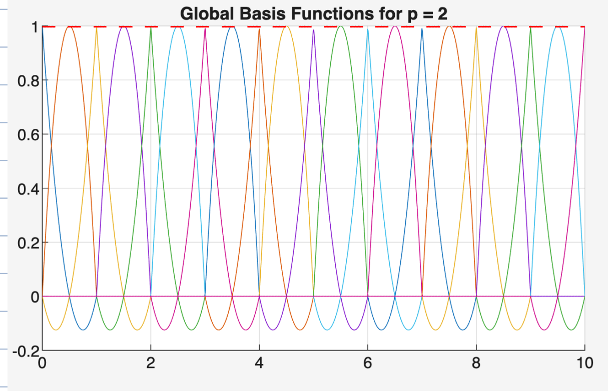
or

$$\phi_1=\left(\frac{\xi-\xi_2}{\xi_1-\xi_2}\right)\left(\frac{\xi-\xi_3}{\xi_1-\xi_3}\right)\left(\frac{\xi-\xi_4}{\xi_1-\xi_4}\right)\qquad \phi_2=\left(\frac{\xi-\xi_1}{\xi_2-\xi_1}\right)\left(\frac{\xi-\xi_3}{\xi_2-\xi_3}\right)\left(\frac{\xi-\xi_4}{\xi_2-\xi_4}\right)$$

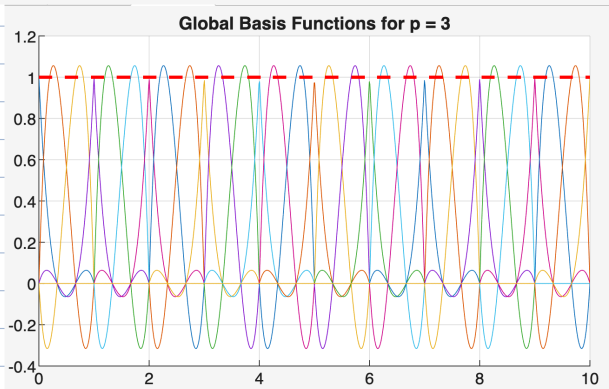
$$\phi_3=\left(\frac{\xi-\xi_1}{\xi_3-\xi_1}\right)\left(\frac{\xi-\xi_2}{\xi_3-\xi_2}\right)\left(\frac{\xi-\xi_4}{\xi_3-\xi_4}\right)\qquad \phi_4=\left(\frac{\xi-\xi_1}{\xi_4-\xi_1}\right)\left(\frac{\xi-\xi_2}{\xi_4-\xi_2}\right)\left(\frac{\xi-\xi_3}{\xi_4-\xi_3}\right)$$

D) Plot all global basis functions $\{\phi_i(x)\}$ over $[0,10]$





See Appendix for code



E) Verify Partition of Unity

```matlab
% Part E: check the partition of Unity
sum_phi = sum(phi,1);
plot(xx,sum_phi,'r--',LineWidth=2);
```

The red dotted line shows $\sum \phi_i(x)$ and is constant at 1

F) Confirm Local Support, nonnegativity, & C^0 continuity at element interfaces

Nonnegativity

- Notice that the Basis functions for p=1 are always non-negative

Local Support

- Notice that each basis function is only nonzero over a few elements

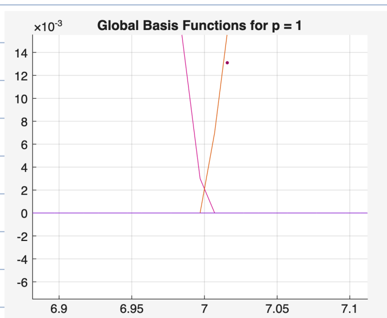
```
Basis 1 is nonzero on [0.00, 0.99]
Basis 2 is nonzero on [0.01, 1.99]
Basis 3 is nonzero on [1.00, 2.99]
Basis 4 is nonzero on [2.00, 3.99]
Basis 5 is nonzero on [3.00, 4.99]
Basis 6 is nonzero on [4.00, 6.00]
Basis 7 is nonzero on [5.01, 7.00]
Basis 8 is nonzero on [6.01, 8.00]
Basis 9 is nonzero on [7.01, 9.00]
Basis 10 is nonzero on [8.01, 9.99]
Basis 11 is nonzero on [9.01, 10.00]
```

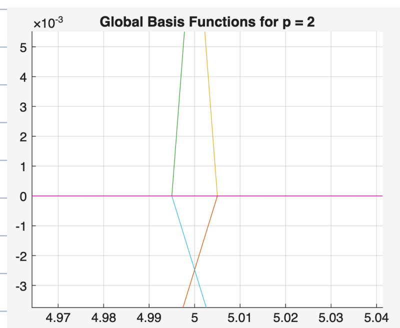
```
Basis 1 is nonzero on [0.00, 0.99]
Basis 2 is nonzero on [0.01, 0.66]
Basis 3 is nonzero on [0.34, 0.99]
Basis 4 is nonzero on [0.01, 1.99]
Basis 5 is nonzero on [1.00, 1.66]
Basis 6 is nonzero on [1.34, 1.99]
Basis 7 is nonzero on [1.00, 2.99]
Basis 8 is nonzero on [2.00, 2.66]
Basis 9 is nonzero on [2.34, 2.99]
Basis 10 is nonzero on [2.00, 3.99]
Basis 11 is nonzero on [3.00, 3.66]
Basis 12 is nonzero on [3.34, 3.99]
Basis 13 is nonzero on [3.00, 4.99]
Basis 14 is nonzero on [4.00, 4.66]
Basis 15 is nonzero on [4.33, 4.99]
Basis 16 is nonzero on [4.00, 6.00]
Basis 17 is nonzero on [5.01, 5.67]
Basis 18 is nonzero on [5.34, 6.00]
Basis 19 is nonzero on [5.01, 7.00]
Basis 20 is nonzero on [6.01, 6.66]
Basis 21 is nonzero on [6.34, 7.00]
Basis 22 is nonzero on [6.01, 8.00]
Basis 23 is nonzero on [7.01, 7.66]
Basis 24 is nonzero on [7.34, 8.00]
Basis 25 is nonzero on [7.01, 9.00]
Basis 26 is nonzero on [8.01, 8.66]
Basis 27 is nonzero on [8.34, 9.00]
Basis 28 is nonzero on [8.01, 9.99]
Basis 29 is nonzero on [9.01, 9.66]
Basis 30 is nonzero on [9.34, 9.99]
Basis 31 is nonzero on [9.01, 10.00]
```

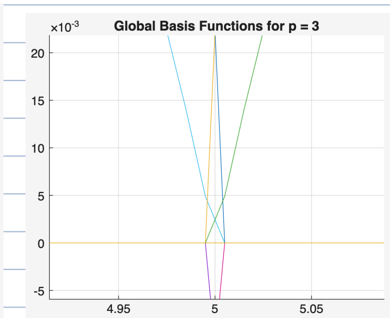
```
Basis 1 is nonzero on [0.00, 0.49]
Basis 2 is nonzero on [0.01, 0.99]
Basis 3 is nonzero on [0.50, 1.49]
Basis 4 is nonzero on [1.00, 1.99]
Basis 5 is nonzero on [1.50, 2.49]
Basis 6 is nonzero on [2.00, 2.99]
Basis 7 is nonzero on [2.50, 3.49]
Basis 8 is nonzero on [3.00, 3.99]
Basis 9 is nonzero on [3.50, 4.49]
Basis 10 is nonzero on [4.00, 4.99]
Basis 11 is nonzero on [4.50, 5.50]
Basis 12 is nonzero on [5.01, 6.00]
Basis 13 is nonzero on [5.51, 6.50]
Basis 14 is nonzero on [6.01, 7.00]
Basis 15 is nonzero on [6.51, 7.50]
Basis 16 is nonzero on [7.01, 8.00]
Basis 17 is nonzero on [7.51, 8.50]
Basis 18 is nonzero on [8.01, 9.00]
Basis 19 is nonzero on [8.51, 9.50]
Basis 20 is nonzero on [9.01, 9.99]
Basis 21 is nonzero on [9.51, 10.00]
```

Continuity at element interfaces

- we can zoom in at element interfaces to inspect continuity

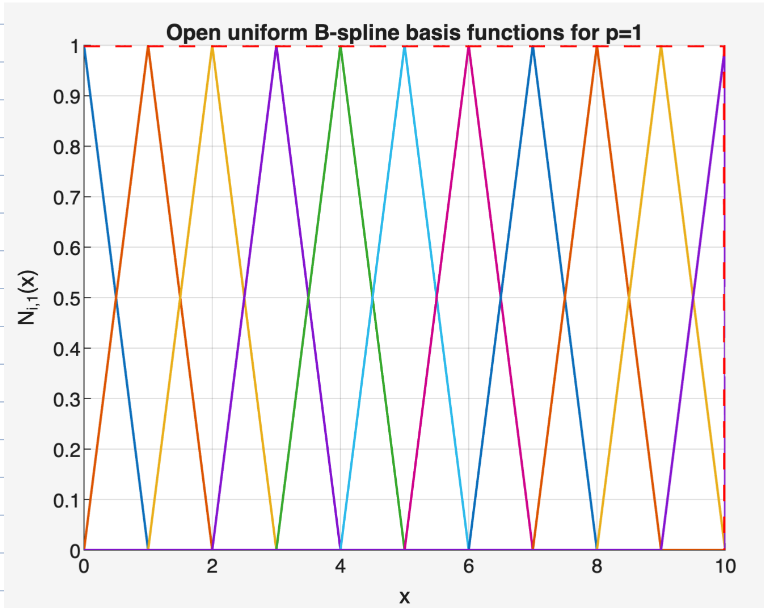


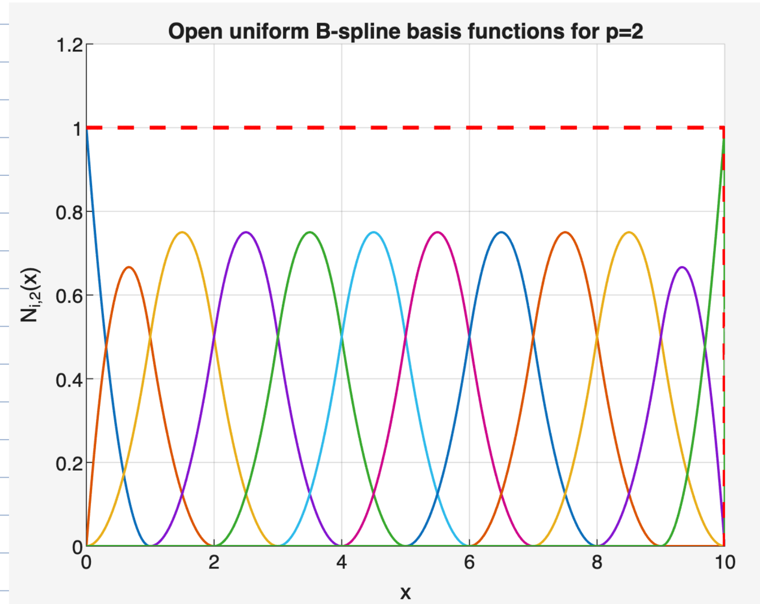


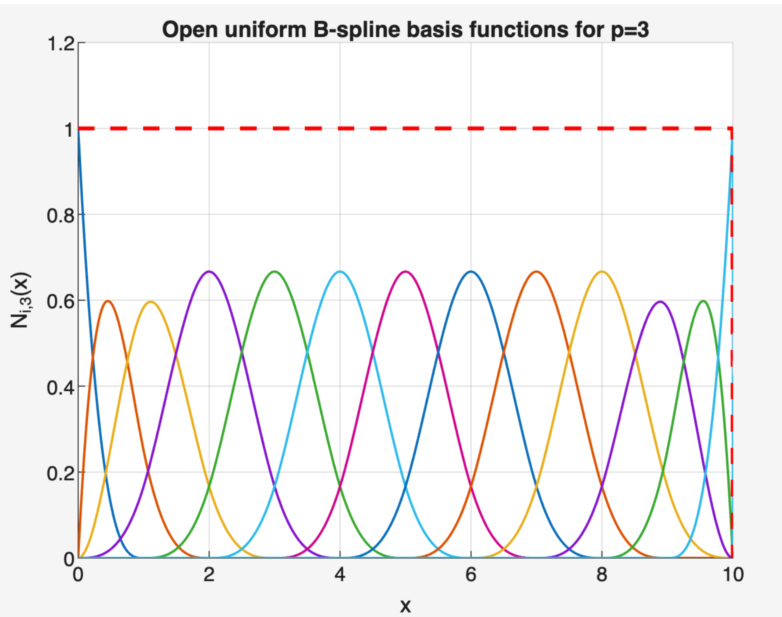


Part B) 1D Open, Maximally Smooth B-Splines

- $p = 1, 2, 3$ with uniform knot vector. Construct using Cox-de Boor recursion.
- Plots of $N_{i,1}(x)$, $N_{i,2}(x)$, & $N_{i,3}(x)$







The red dotted line shows the sum of the bases, i.e., the partition of unity. Notice that it is constant at one.

E) Compare to Lagrange: Support size, C^{p-1} vs. C^0 continuity, & smoothness.

Comparison Table

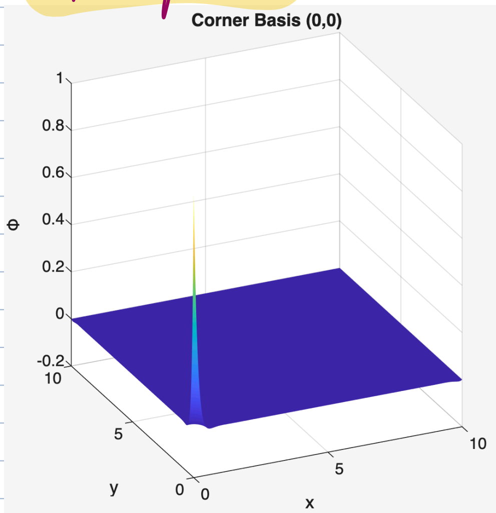
	B-spline	Lagrangian
Support size	Every B-spline basis of degree p has compact support spanning exactly p+1 knot intervals	Lagrange basis support span for basis functions vary more and are typically smaller and non-uniform (can be under 1 element length)
Continuity C^{p-1} vs. C^0	Open knot vector & interior knots have C^{p-1} continuity at interior knot locations.	Global space is C^0 continuous, first derivatives can jump, and function values can be continuous at shared nodes.
smoothness	B-splines are typically "smoother" (since they have a higher degree of continuity (C^{p-1}))	Lagrangian basis are typically "less smooth" as they only have C^0 continuity.

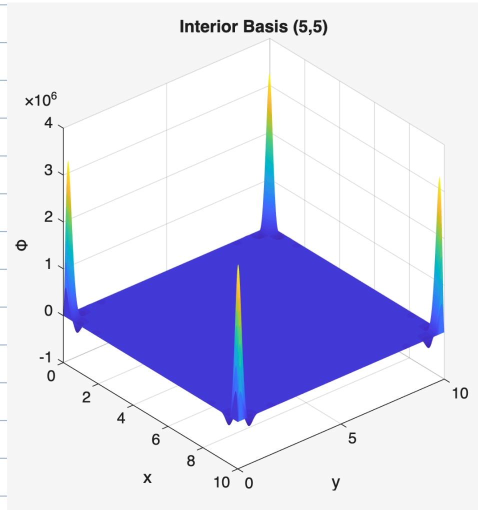
Part c) 2D Tensor-Product Bases & Elevation Plots

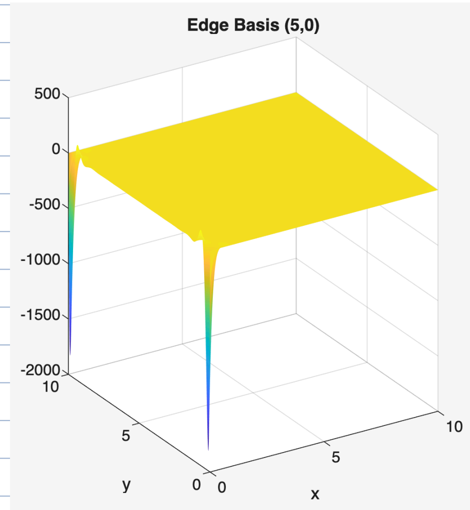
① 2D Lagrange Bases with tensor product form $\phi_{ij}^{(p)}(x,y) = \phi_i^{(p)}(x)\,\phi_j^{(p)}(y)$

plots of elevation surfaces

For $p=2$





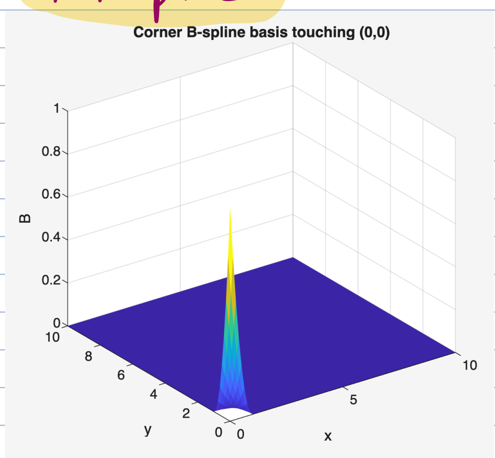


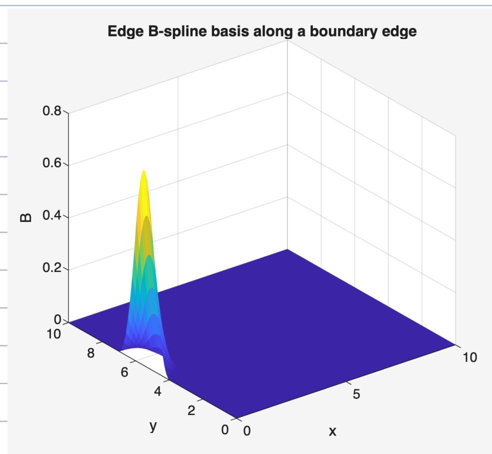
- For the support in 2D, the corner basis has support outside of a small box $[0,h]\times[0,h]$. The interior basis has support on the nodes near $(5,5)$ and the edge basis has support away from the edge of the boundary.
- The tensor product preserves C^0 continuity so the 2D bases are also C^0 continuous with possibly discontinuous derivatives.

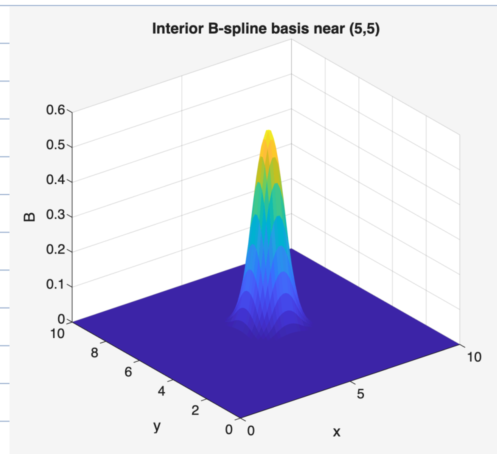
② 2D B-Spline Bases with tensor product form: $B_{ij}^{(p)}(x,y) = N_{i,p}(x)\,N_{j,p}(y)$

plots of elevation surfaces

For $p=2$







- In 1D, the B-spline supports were always over $p+1$ elements. Now in 2D, the tensor product just multiplies the supports so each 2D basis function has support over $(p+1)\times(p+1)$ elements.
- Also in 2D, tensor products preserve the C^{p-1} continuity accross the surface.