Stationary Time Series

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General Linear Processes

A *general linear process*, $\{Y_t\}$, is one that can be represented as a weighted linear combination of present and past white noise terms as

$$Y_t = e_t + \psi_1 e_{t-1} + \psi_2 e_{t-2} + \cdots$$

$$\sum_{i=1}^{\infty} \psi_i^2 < \infty$$

We should also note that since $\{e_t\}$ is unobservable, there is no loss in the generality of $\sum_{i=1}^{\infty} \psi_1^2 < \infty$

if we assume that the coefficient on e_t is 1; effectively, $\psi_0 = 1$.

General Linear Processes

An important nontrivial example to which we will return often is the case where the ψ's form an exponentially decaying sequence

$$\psi_j = \phi^j$$

where ϕ is a number strictly between -1 and $+1$. Then

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots$$

For this example,

$$E(Y_t) = E(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots) = 0$$

- $\{e_t\}$ represent an unobserved white noise series, that is, a sequence of identically distributed, zero-mean, independent random variables.
- The assumption of independence could be replaced by the weaker assumption that the $\{e_t\}$ are uncorrelated random variables

General Linear Processes -example

$$
\begin{aligned}
Var(Y_t) &= Var(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots) \\
&= Var(e_t) + \phi^2 Var(e_{t-1}) + \phi^4 Var(e_{t-2}) + \cdots \\
&= \sigma_e^2 (1 + \phi^2 + \phi^4 + \cdots) \\
&= \frac{\sigma_e^2}{1 - \phi^2} \quad \text{(by summing a geometric series)}
\end{aligned}
$$

General Linear Processes -example

$$
\begin{aligned}
Cov(Y_t, Y_{t-1}) &= Cov(e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots, e_{t-1} + \phi e_{t-2} + \phi^2 e_{t-3} + \cdots) \\
&= Cov(\phi e_{t-1}, e_{t-1}) + Cov(\phi^2 e_{t-2}, \phi e_{t-2}) + \cdots \\
&= \phi\sigma_e^2 + \phi^3\sigma_e^2 + \phi^5\sigma_e^2 + \cdots \\
&= \phi\sigma_e^2(1 + \phi^2 + \phi^4 + \cdots) \\
&= \frac{\phi\sigma_e^2}{1-\phi^2} \quad \text{(again summing a geometric series)}
\end{aligned}
$$

$$Cov(Y_t, Y_{t-k}) = \frac{\phi^k\sigma_e^2}{1-\phi^2}$$

$$Corr(Y_t, Y_{t-1}) = \left[\frac{\phi\sigma_e^2}{1-\phi^2}\right] \Big/ \left[\frac{\sigma_e^2}{1-\phi^2}\right] = \phi$$

$$Corr(Y_t, Y_{t-k}) = \phi^k$$

Moving Average Processes

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

A **moving average of order *q*** and abbreviate the name to MA(q)

The First-Order Moving Average Processes

MA(1)

$$Y_t = e_t - \theta_1 e_{t-1}$$

$e_t \sim iid(0, \sigma_e^2)$

$$E(Y_t) = E(e_t - \theta_1 e_{t-1}) = 0$$

$$Var(Y_t) = Var(e_t - \theta_1 e_{t-1}) = \sigma_e^2(1 + \theta^2)$$

$$\begin{aligned} Cov(Y_t, Y_{t-1}) &= Cov(e_t - \theta e_{t-1}, e_{t-1} - \theta e_{t-2}) \\ &= Cov(-\theta e_{t-1}, e_{t-1}) = -\theta\sigma_e^2 \end{aligned}$$

$$\begin{aligned} Cov(Y_t, Y_{t-2}) &= Cov(e_t - \theta e_{t-1}, e_{t-2} - \theta e_{t-3}) \\ &= 0 \end{aligned}$$

$$Cov(Y_t, Y_{t-k}) = 0 \text{ whenever } k \geq 2$$

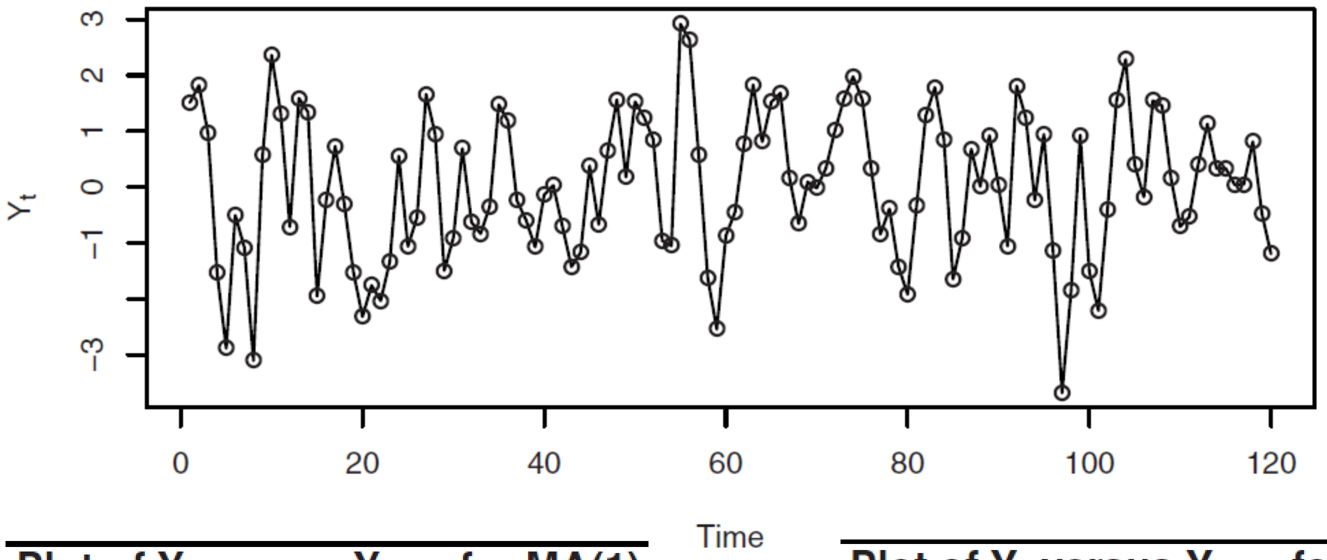
The First-Order Moving Average Processes

In summary, for an MA(1) model $Y_t = e_t - \theta e_{t-1}$,

$$\left.\begin{aligned} E(Y_t) &= 0 \\ \gamma_0 &= Var(Y_t) = \sigma_e^2(1+\theta^2) \\ \gamma_1 &= -\theta\sigma_e^2 \\ \rho_1 &= (-\theta)/(1+\theta^2) \\ \gamma_k &= \rho_k = 0 \quad \text{for } k \geq 2 \end{aligned}\right\}$$

MA(1)

Time Plot of an MA(1) Process with θ = −0.9



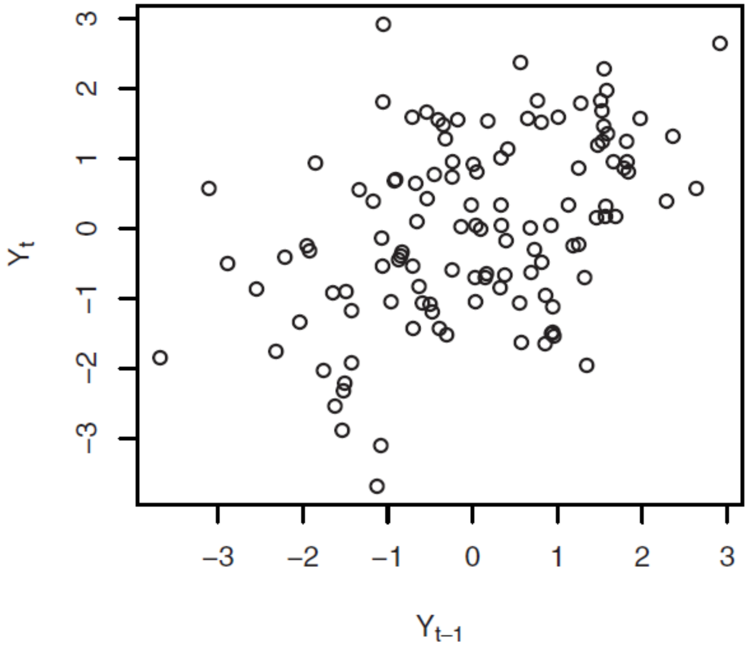
$$Y_t = e_t + 0.9e_{t-1}$$

$$\gamma_1 = -\theta\sigma_e^2$$

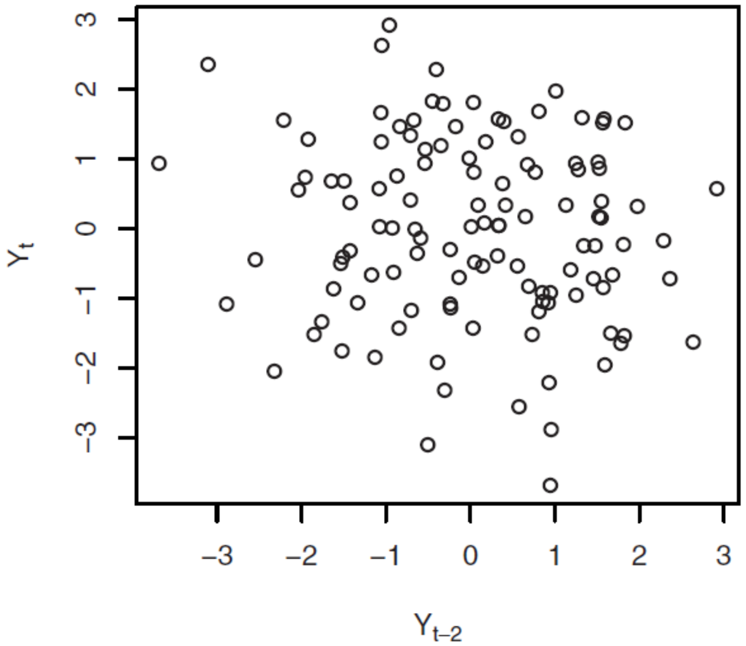
$$\rho_1 = (-\theta)/(1+\theta^2)$$

$$\gamma_k = \rho_k = 0 \quad \text{for } k \geq 2$$

Plot of Y_t versus Y_{t-1} for MA(1)



Plot of Y_t versus Y_{t-2} for MA(1)



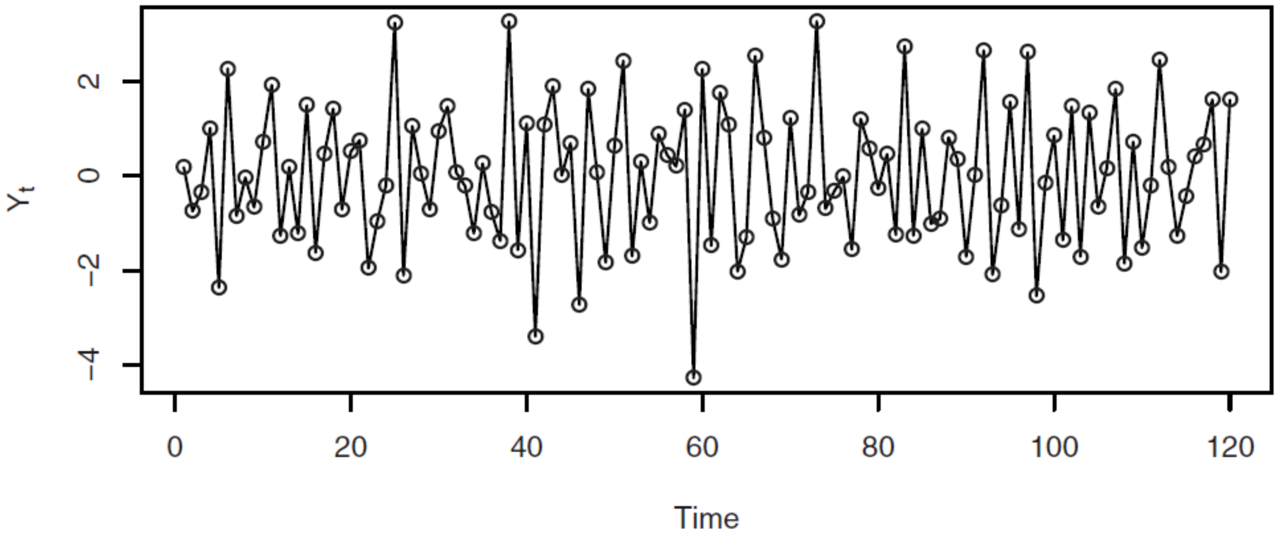
$$\gamma_1 = 0.9\sigma_e^2$$

$$\rho_1 = \frac{0.9}{(1+(-0.9)^2)}$$

$$\rho_2 = 0$$

MA(1)

Time Plot of an MA(1) Process with $\theta = +0.9$



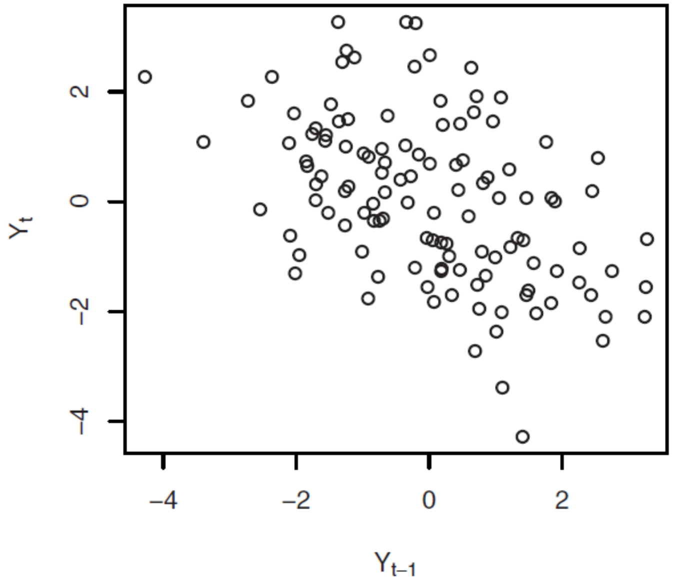
$$Y_t = e_t - 0.9e_{t-1}$$

$$\gamma_1 = -\theta\sigma_e^2$$

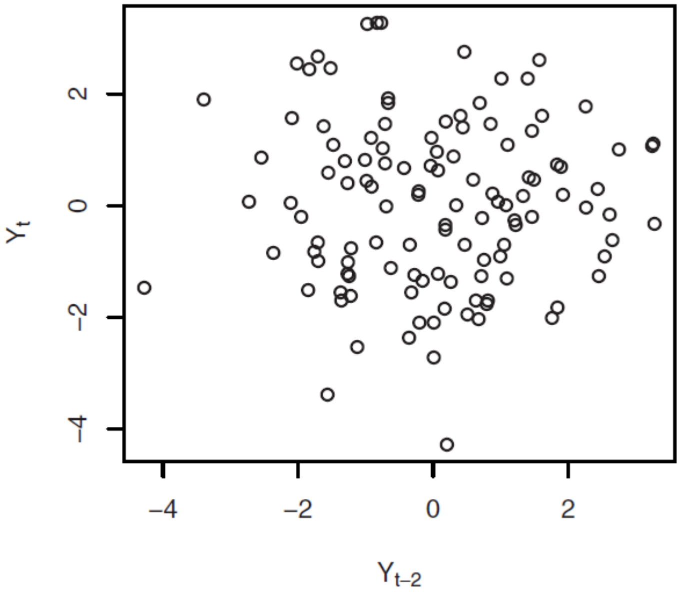
$$\rho_1 = (-\theta)/(1+\theta^2)$$

$$\gamma_k = \rho_k = 0 \quad \text{for } k \geq 2$$

Plot of Y_t versus Y_{t-1} for MA(1)



Plot of Y_t versus Y_{t-2} for MA(1)



$$\gamma_1 = -0.9\sigma_e^2$$

$$\rho_1 = \frac{-0.9}{(1+0.9^2)}$$

$$\rho_2 = 0$$

The Second-Order Moving Average Process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

$$\gamma_0 = Var(Y_t) = Var(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}) = (1 + \theta_1^2 + \theta_2^2)\sigma_e^2$$

$$\begin{aligned}
\gamma_1 &= Cov(Y_t, Y_{t-1}) = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-1} - \theta_1 e_{t-2} - \theta_2 e_{t-3}) \\
&= Cov(-\theta_1 e_{t-1}, e_{t-1}) + Cov(-\theta_1 e_{t-2}, -\theta_2 e_{t-2}) \\
&= [-\theta_1 + (-\theta_1)(-\theta_2)]\sigma_e^2 \\
&= (-\theta_1 + \theta_1\theta_2)\sigma_e^2
\end{aligned}$$

$$\begin{aligned}
\gamma_2 &= Cov(Y_t, Y_{t-2}) = Cov(e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}, e_{t-2} - \theta_1 e_{t-3} - \theta_2 e_{t-4}) \\
&= Cov(-\theta_2 e_{t-2}, e_{t-2}) \\
&= -\theta_2 \sigma_e^2
\end{aligned}$$

The Second-Order Moving Average Process

$$Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$$

$$\rho_1 = \frac{-\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho_2 = \frac{-\theta_2}{1 + \theta_1^2 + \theta_2^2}$$

$$\rho_k = 0 \quad \text{for } k = 3, 4, \ldots$$

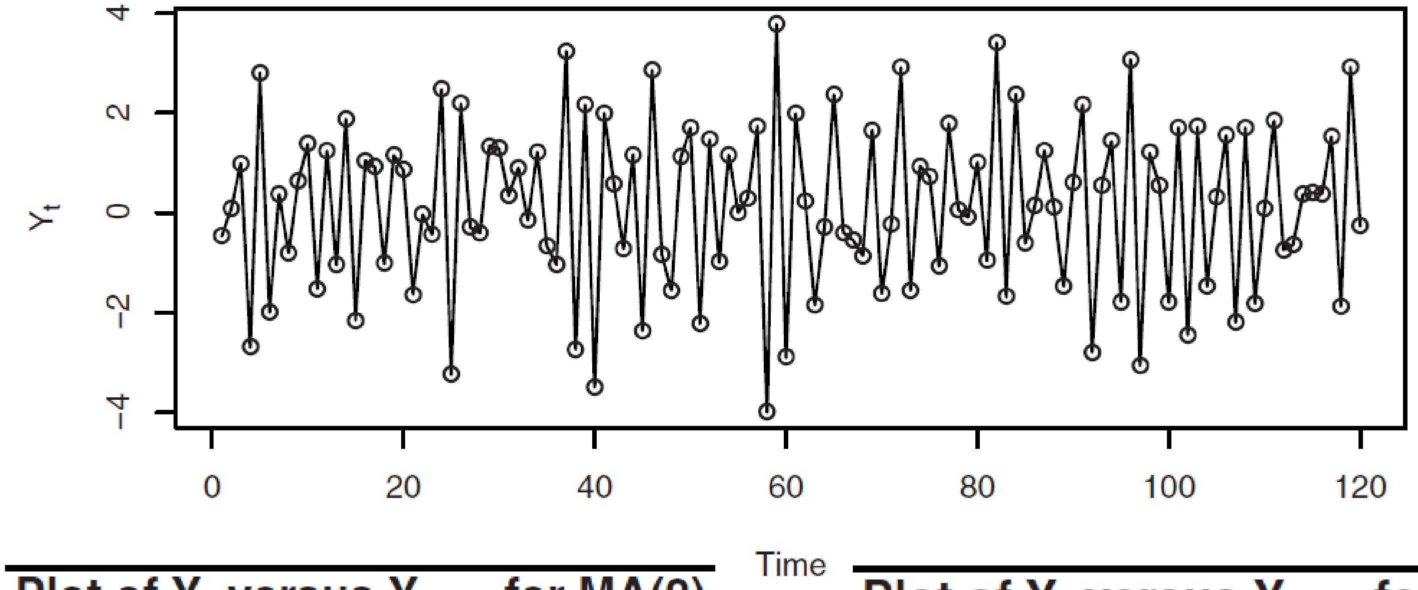
MA(2)

$$Y_t = e_t - e_{t-1} + 0.6e_{t-2}$$

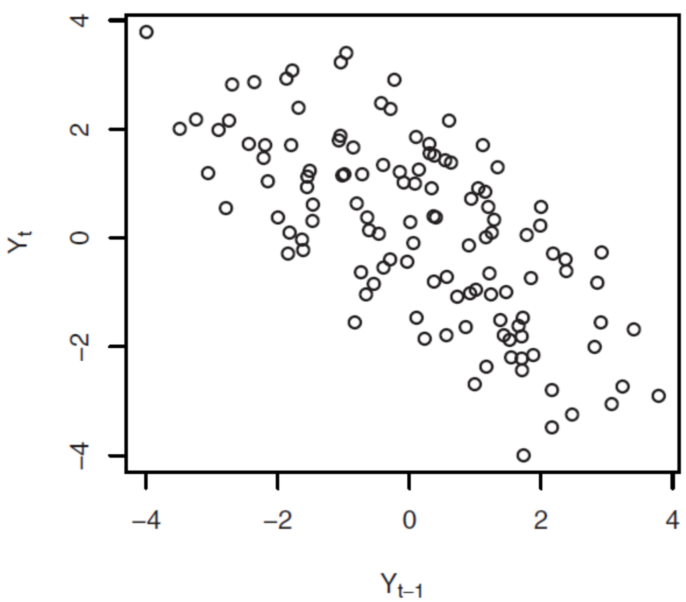
$$\rho_1 = \frac{-1 + (1)(-0.6)}{1 + (1)^2 + (-0.6)^2} = \frac{-1.6}{2.36} = -0.678$$

$$\rho_2 = \frac{0.6}{2.36} = 0.254$$

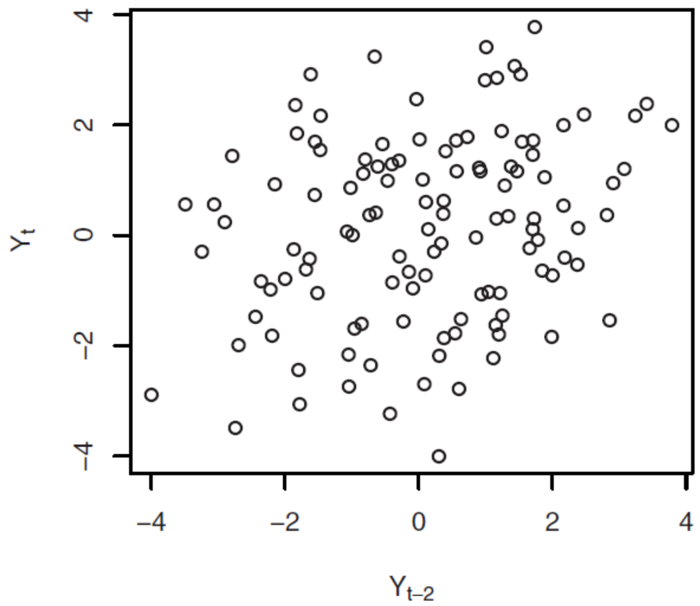
Time Plot of an MA(2) Process with $\theta_1 = 1$ and $\theta_2 = -0.6$



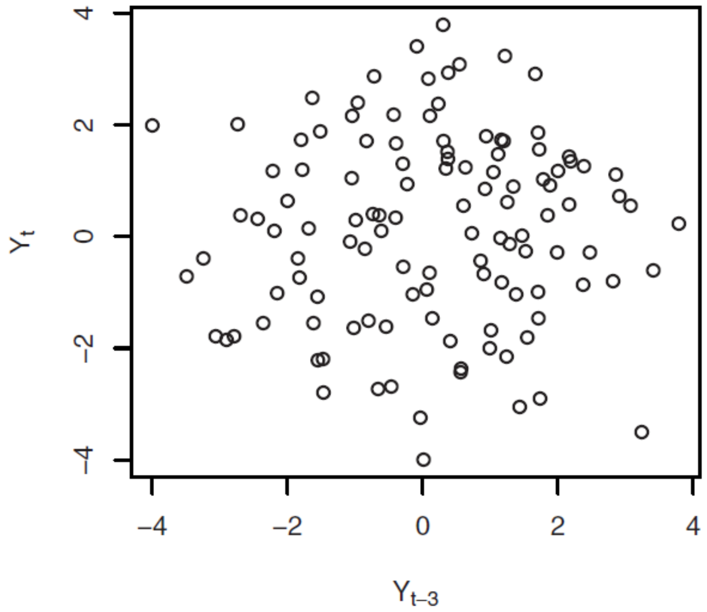
Plot of Y_t versus Y_{t-1} for MA(2)



Plot of Y_t versus Y_{t-2} for MA(2)



Plot of Y_t versus Y_{t-3} for MA(2)



The General MA(q) Process

For the general MA(q) process $Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$, similar calculations show that

$$\gamma_0 = (1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2)\sigma_e^2$$

and

$$\rho_k = \begin{cases} \dfrac{-\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \cdots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \cdots + \theta_q^2} & \text{for } k = 1, 2, \ldots, q \\ 0 & \text{for } k > q \end{cases}$$

Thanks