Stationary Time Series

YENNI ANGRAINI

Outline

- Autoregressive Processes
- The Mixed Autoregressive Moving Average Model
- Stationarity of AR(1) and AR(2)
- Invertibility

Autoregressive Processes

Autoregressive processes are as their name suggests—regressions on themselves. Specifically, a *p*th-order **autoregressive process** $\{Y_t\}$ satisfies the equation

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t$$

The current value of the series Y_t is a linear combination of the p most recent past values of itself plus an "innovation" term e_t that incorporates everything new in the series at time t that is not explained by the past values. Thus, for every t, we assume that e_t is independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$. Yule (1926) carried out the original work on autoregressive processes

The First-Order Autoregressive Processes

AR(1) $Y_t = \phi Y_{t-1} + e_t$

$e_t \sim iid(0, \sigma_e^2)$

we are assuming that Y_t has zero mean.

$E(Y_t) = 0$

we also assume that e_t is independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$.

$E(e_t, Y_{t-1}) = 0, E(e_t, Y_{t-2}) = 0$, ,,,dst

$$E(e_t Y_{t-k}) = E(e_t)E(Y_{t-k}) = 0$$

$$Var(Y_t) = Var(\phi Y_{t-1} + e_t)$$

$$\gamma_0 = \phi^2 \gamma_0 + \sigma_e^2$$

Solving for γ_0 yields

$$\gamma_0 = \frac{\sigma_e^2}{1 - \phi^2}$$

$\phi^2 < 1$ or that $|\phi| < 1$.

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$$E(e_t Y_{t-k}) = E(e_t)E(Y_{t-k}) = 0$$

$$E(Y_{t-k}Y_t) = \phi E(Y_{t-k}Y_{t-1}) + E(e_t Y_{t-k})$$

or

$$\gamma_k = \phi\gamma_{k-1} + E(e_t Y_{t-k})$$

$$\gamma_k = \phi\gamma_{k-1} \qquad \text{for } k = 1, 2, 3, \dots$$

Setting $k = 1$, we get $\gamma_1 = \phi\gamma_0 = \phi\sigma_e^2/(1-\phi^2)$. With $k = 2$, we obtain $\gamma_2 = \phi^2\sigma_e^2/(1-\phi^2)$. Now it is easy to see that in general

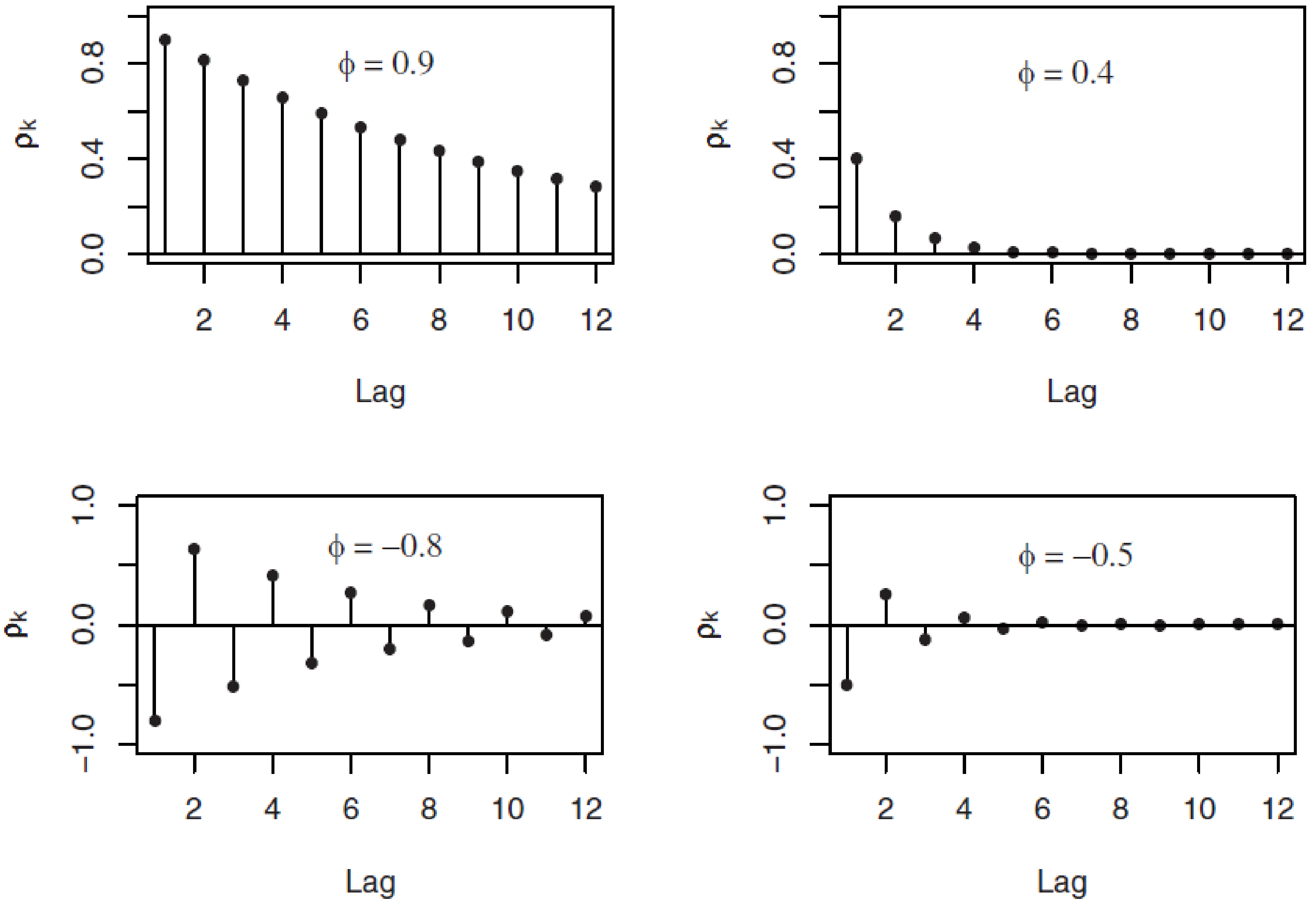
$$\gamma_k = \phi^k \frac{\sigma_e^2}{1-\phi^2}$$

and thus

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \phi^k \qquad \text{for } k = 1, 2, 3, \dots$$

The First-Order Autoregressive Processes

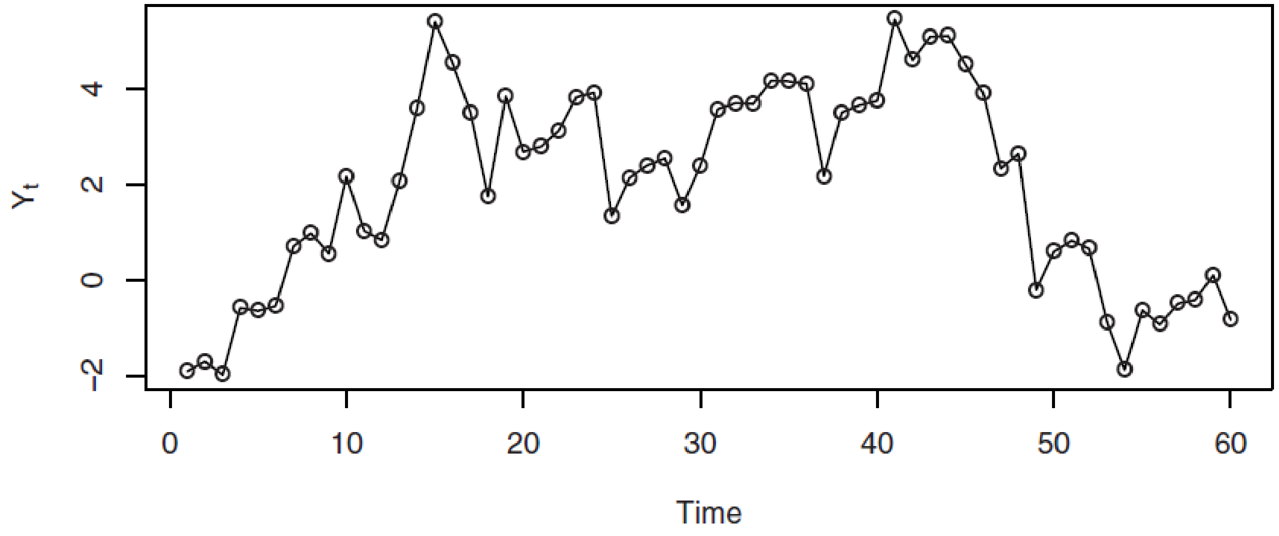
- Since $|\phi| < 1$, the magnitude of the autocorrelation function decreases exponentially as the number of lags, k, increases.
- If $0 < \phi < 1$, all correlations are positive;
- If $-1 < \phi < 0$, the lag 1 autocorrelation is negative ($\rho_1 = \phi$) and the signs of successive autocorrelations alternate from positive to negative, with their magnitudes decreasing exponentially.



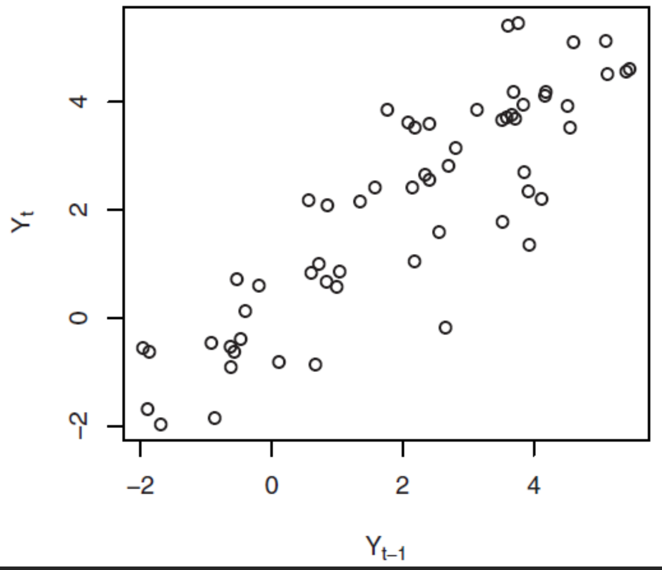
AR(1)

$$Y_t = \phi Y_{t-1} + e_t$$

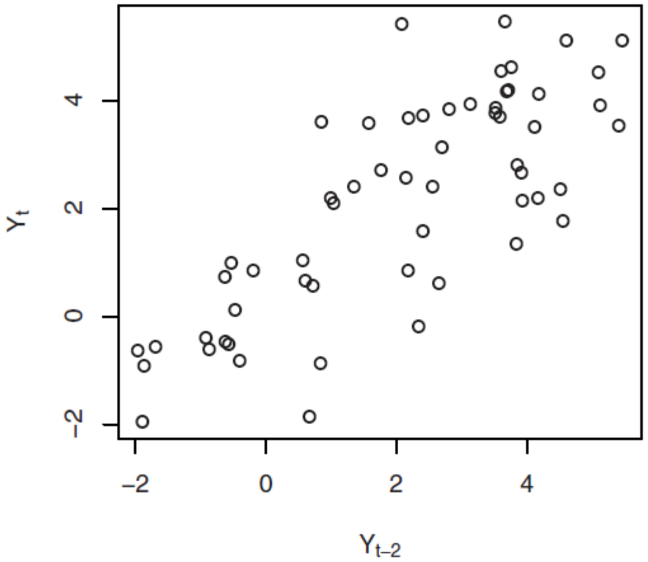
Time Plot of an AR(1) Series with $\phi = 0.9$



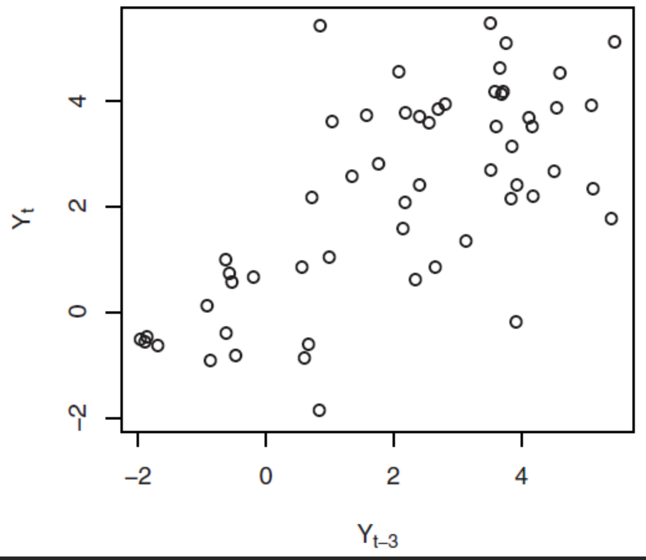
Plot of Y_t versus Y_{t-1} for AR(1)



Plot of Y_t versus Y_{t-2} for AR(1)



Plot of Y_t versus Y_{t-3} for AR(1)



The General Linear Process Version of the AR(1) Model

The General Linear Process Version of the AR(1) Model can be written as follows :

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \cdots + \phi^{k-1} e_{t-k+1} + \phi^k Y_{t-k}$$

Assuming $|\phi| < 1$ and letting k increase without bound, it seems reasonable (this is almost a rigorous proof) that we should obtain the infinite series representation

$$Y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots$$

The Second-Order Autoregressive Processes

AR(2) $$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$$

$$e_t \sim iid(0, \sigma_e^2)$$

we also assume that e_t is independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$

$$E(e_t, Y_{t-1}) = 0, E(e_t, Y_{t-2}) = 0, ,,,\text{dst}$$

How to get the variance for AR(2) model ?

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \qquad \text{for } k = 1, 2, 3, \ldots$$

or, dividing through by γ_0,

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \qquad \text{for } k = 1, 2, 3, \ldots$$

These Equations are usually called
the **Yule-Walker equations**

Setting $k = 1$ and using $\rho_0 = 1$ and $\rho_{-1} = \rho_1$, we get and so

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}$$

Using the now known values for ρ_1 (and ρ_0) and ρ_k, for k = 2 we can obtain

$$\rho_2 = \phi_1 \rho_1 + \phi_2 \rho_0$$

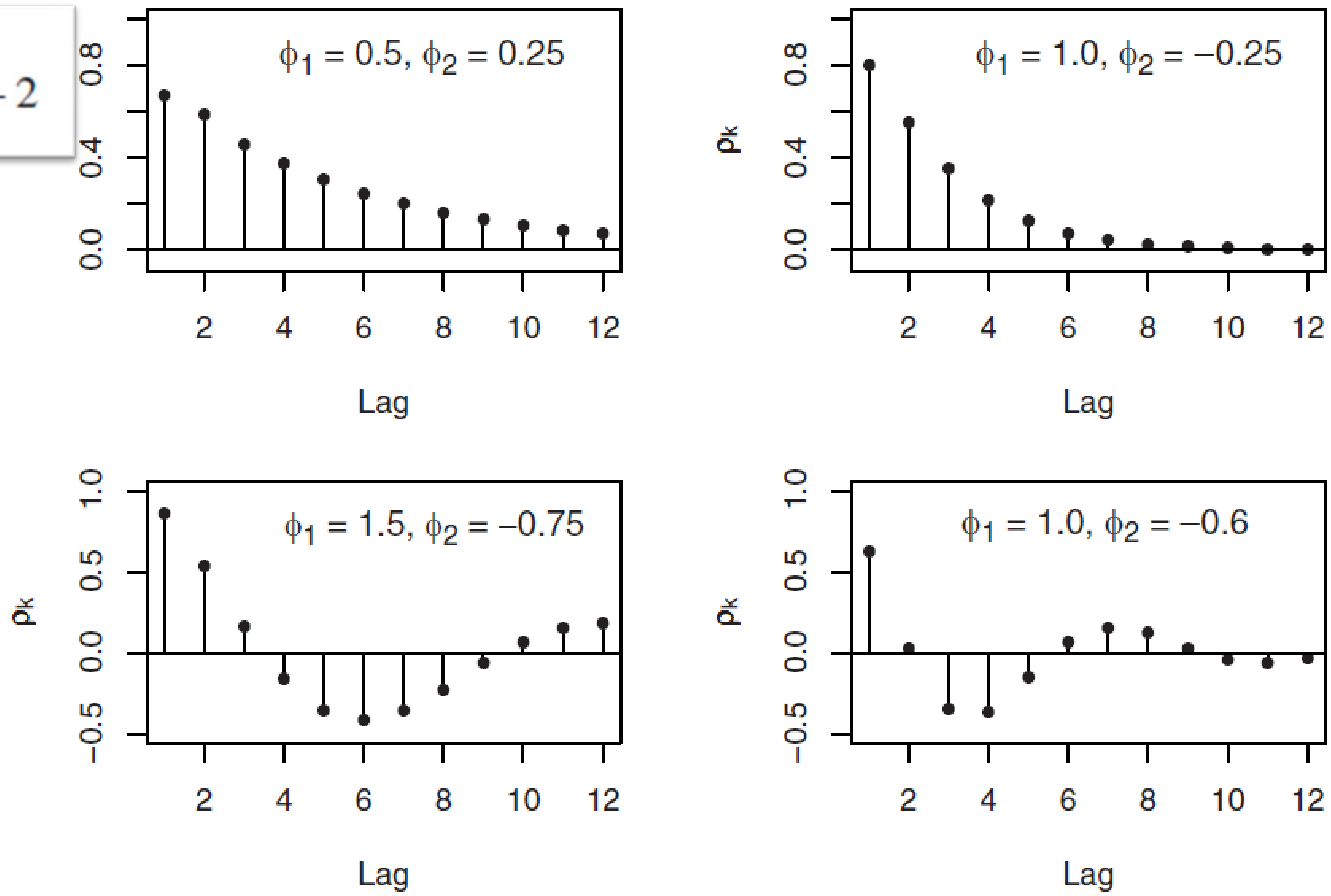
$$= \frac{\phi_2(1 - \phi_2) + \phi_1^2}{1 - \phi_2}$$

Autocorrelation Functions for Several AR(2) Models

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2}$$

$$\rho_1 = \frac{\phi_1}{1 - \phi_2}$$

$$\rho_0 = 1$$



The Mixed Autoregressive Moving Average Model

If we assume that the series is partly autoregressive and partly moving average, we obtain a quite general time series model. In general, if

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

we say that $\{Y_t\}$ is a mixed autoregressive moving average process of orders p and q, respectively; we abbreviate the name to ARMA(p,q)

The ARMA(1,1) Model

The defining equation can be written

$$Y_t = \phi Y_{t-1} + e_t - \theta e_{t-1}$$

To derive Yule-Walker type equations, we first note that

$$\begin{aligned} E(e_t Y_t) &= E[e_t(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\ &= \sigma_e^2 \end{aligned}$$

$$\begin{aligned} E(e_{t-1} Y_t) &= E[e_{t-1}(\phi Y_{t-1} + e_t - \theta e_{t-1})] \\ &= \phi\sigma_e^2 - \theta\sigma_e^2 \\ &= (\phi - \theta)\sigma_e^2 \end{aligned}$$

$$\left.\begin{aligned} \gamma_0 &= \phi\gamma_1 + [1 - \theta(\phi - \theta)]\sigma_e^2 \\ \gamma_1 &= \phi\gamma_0 - \theta\sigma_e^2 \\ \gamma_k &= \phi\gamma_{k-1} \quad \text{for } k \ge 2 \end{aligned}\right\}$$

Solving the first two equations yields

$$\gamma_0 = \frac{(1 - 2\phi\theta + \theta^2)}{1 - \phi^2}\sigma_e^2$$

and solving the simple recursion gives

$$\rho_k = \frac{(1 - \theta\phi)(\phi - \theta)}{1 - 2\theta\phi + \theta^2}\phi^{k-1} \quad \text{for } k \ge 1$$

Stationarity of AR(1)

- It can be shown that, subject to the restriction that e_t be independent of $Y_{t-1}, Y_{t-2}, Y_{t-3}, \ldots$ and that $\sigma_e^2 > 0$,
- The solution of the AR(1) defining recursion $Y_t = \phi Y_{t-1} + e_t$ will be stationary if and only $|\phi| < 1$.
- The requirement $|\phi| < 1$ is usually called the stationary condition for the AR(1) process.

Stationarity of AR(2)

The AR(2) process is called the stationary if and only if three conditions are satisfied:

$$\phi_1 + \phi_2 < 1, \qquad \phi_2 - \phi_1 < 1, \qquad \text{and} \qquad |\phi_2| < 1$$

As with the AR(1) model, we call these the stationarity conditions for the AR(2) model.

Invertibility (MA(q) → AR(p))

An MA(1) model: $Y_t = e_t - \theta e_{t-1}$

First rewriting this as $e_t = Y_t + \theta e_{t-1}$ and then replacing t by $t - 1$ and substituting for e_{t-1} above, we get

$$\begin{aligned} e_t &= Y_t + \theta(Y_{t-1} + \theta e_{t-2}) \\ &= Y_t + \theta Y_{t-1} + \theta^2 e_{t-2} \end{aligned}$$

If $|\theta| < 1$, we may continue this substitution "infinitely" into the past and obtain the expression

$$e_t = Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \cdots$$

or

$$Y_t = (-\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \cdots) + e_t$$

- If $|\theta| < 1$, we see that the MA(1) model can be inverted into an infinite-order autoregressive model.
- We say that the MA(1) model is invertible if and only if $|\theta| < 1$.

Invertibility

- Invertibility refers to the fact that the moving average (MA) models can be written as an autoregressive (AR) model
- Or more generally, if ARMA models can be written as AR models, we say that the time series model is invertible.

Assignment

1. Find the autocovariance function and the autocorrelation function for AR (3)
2. Sketch the autocorrelation function for AR (2) model as follows :
   - $\phi_1 = 0.6$ and $\phi_2 = 0.3$
   - $\phi_1 = -0.4$ and $\phi_2 = 0.5$
   - $\phi_1 = 1.2$ and $\phi_2 = -0.7$
3. Find the autocovariance function and the autocorrelation function for ARMA(1,2)
4. Find the autocovariance function and the autocorrelation function for ARMA(2,1)
5. Find the invertibility condition for MA(2) process

Thanks