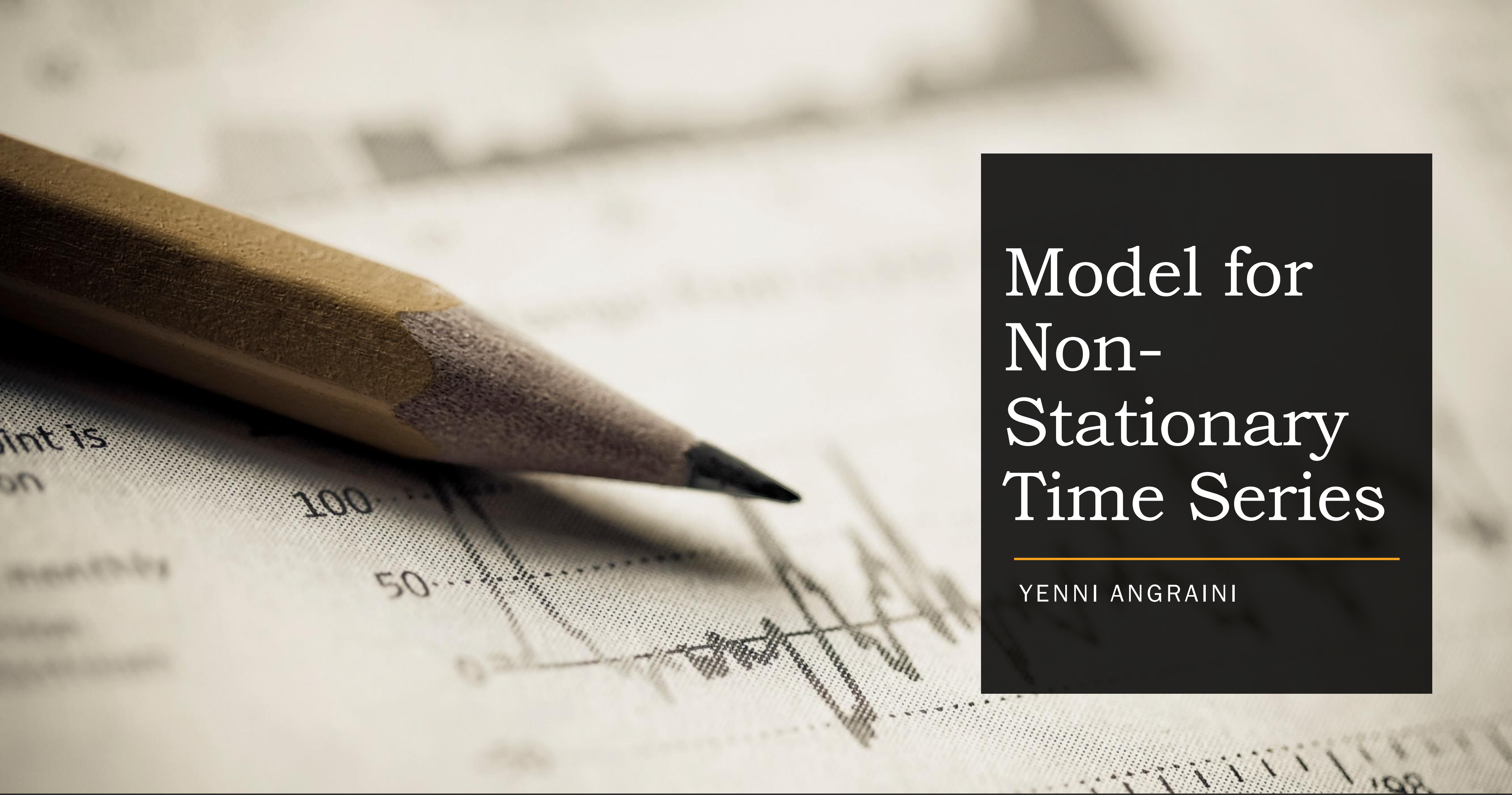

Model for Non-Stationary Time Series

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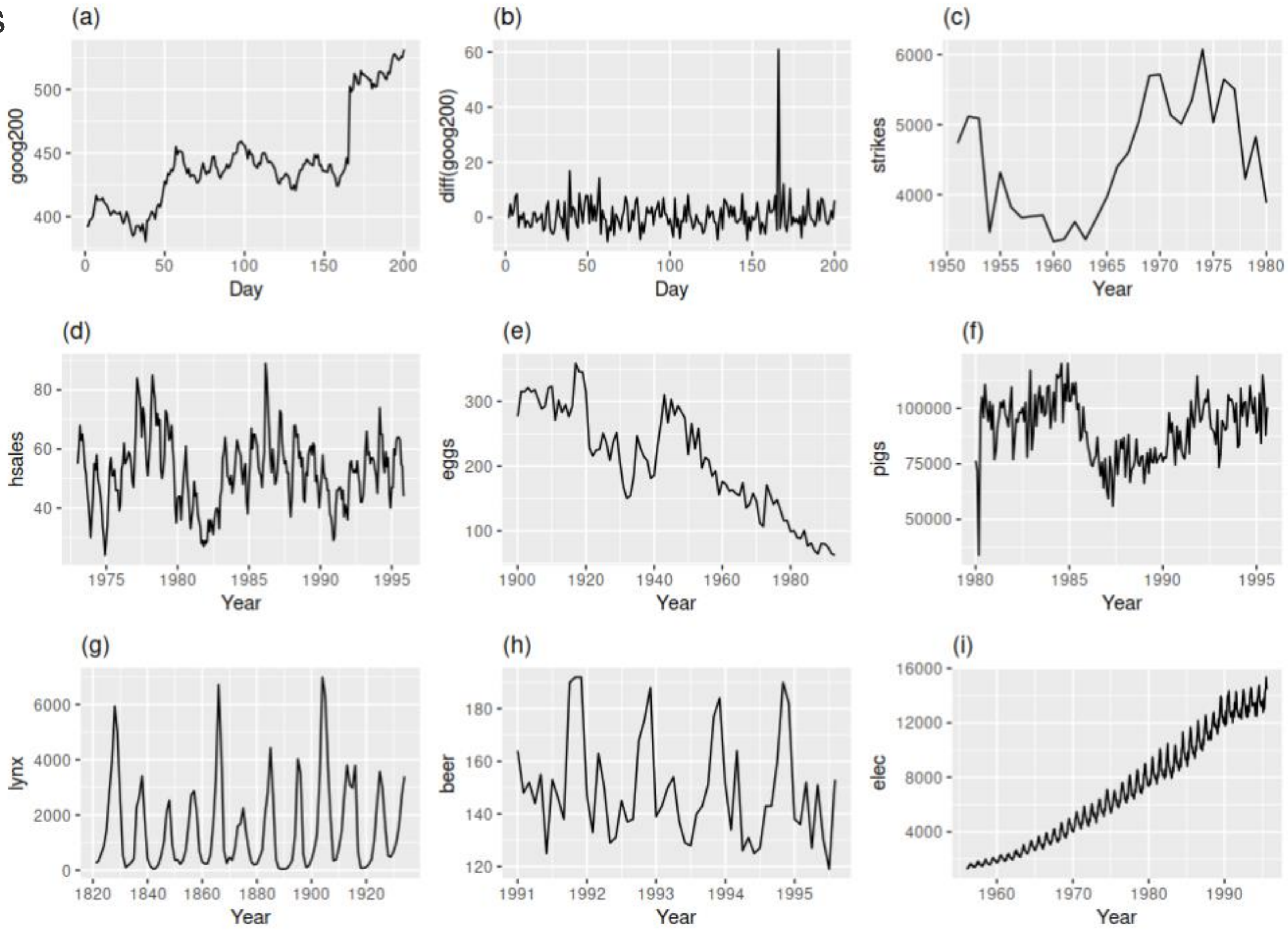
Outline

- Differencing
- ARIMA Model

Which of these series are stationary?

(a) Google stock price for 200 consecutive days;
(b) Daily change in the Google stock price for 200 consecutive days;
(c) Annual number of strikes in the US;
(d) Monthly sales of new one-family houses sold in the US;
(e) Annual price of a dozen eggs in the US (constant dollars);
(f) Monthly total of pigs slaughtered in Victoria, Australia;
(g) Annual total of lynx trapped in the McKenzie River district of north-west Canada;
(h) Monthly Australian beer production;
(i) Monthly Australian electricity production



We normally restrict autoregressive models to stationary data, in which case some constraints on the values of the parameters are required.

- For an AR(1) model: $-1 < \phi_1 < 1$.
- For an AR(2) model: $-1 < \phi_2 < 1, \phi_1 + \phi_2 < 1, \phi_2 - \phi_1 < 1$.

The invertibility constraints for other models are similar to the stationarity constraints.

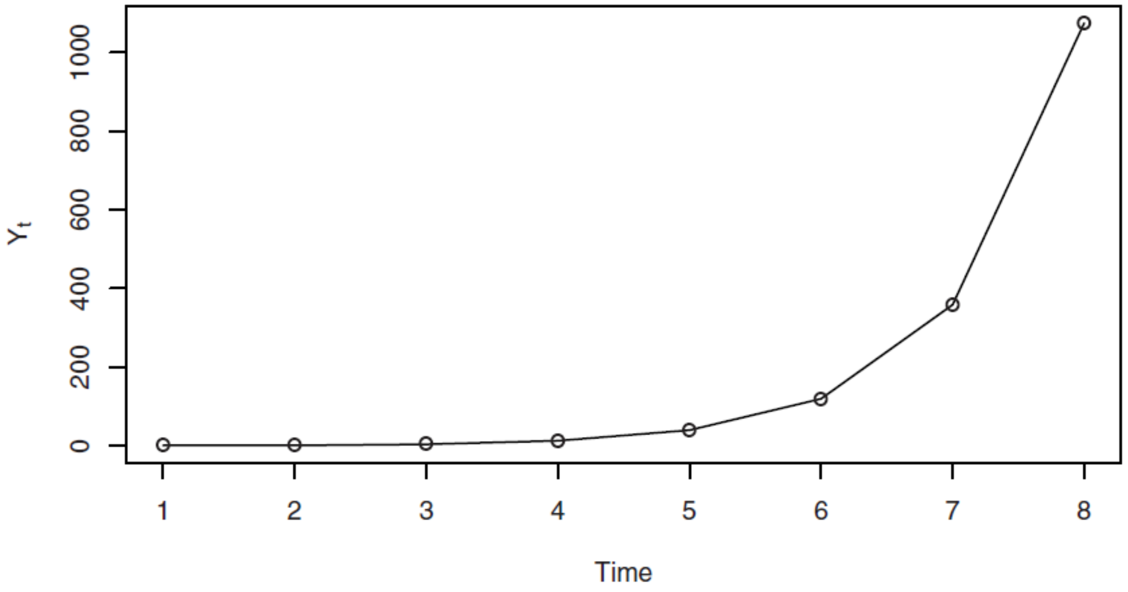
- For an MA(1) model: $-1 < \theta_1 < 1$.
- For an MA(2) model: $-1 < \theta_2 < 1,\ \ \theta_2 + \theta_1 > -1,\ \ \theta_1 - \theta_2 < 1$.

$$Y_t = 3Y_{t-1} + e_t \qquad e_t \sim iid(0, \sigma_e^2)$$

Iterating into the past as we have done before yields

$$Y_t = e_t + 3e_{t-1} + 3^2 e_{t-2} + \cdots + 3^{t-1} e_1 + 3^t Y_0 \qquad Y_0 = 0$$

An Explosive "AR(1)" Series



Simulation of the Explosive "AR(1) Model" $Y_t = 3Y_{t-1} + e_t$

t	1	2	3	4	5	6	7	8
e_t	0.63	−1.25	1.80	1.51	1.56	0.62	0.64	−0.98
Y_t	0.63	0.64	3.72	12.67	39.57	119.33	358.63	1074.91

$$Var(Y_t) = \frac{1}{8}(9^t - 1)\sigma_e^2$$

and

$$Cov(Y_t, Y_{t-k}) = \frac{3^k}{8}(9^{t-k} - 1)\sigma_e^2$$

respectively. Notice that we have

$$Corr(Y_t, Y_{t-k}) = 3^k\left(\frac{9^{t-k} - 1}{9^t - 1}\right) \approx 1 \qquad \text{for large } t \text{ and moderate } k$$

Differencing

$AR(1) : Y_t = \phi Y_{t-1} + e_t \qquad e_t \sim iid(0, \sigma_e^2)$

if $|\phi| \geq 1$, the AR(1) is non stationary model

Suppose $\phi = 1$

$$Y_t = Y_{t-1} + e_t$$
$$Y_t - Y_{t-1} = e_t$$
$$\nabla^1 Y_t = e_t$$

$$E(\nabla^1 Y_t) = E(e_t) = 0$$
$$Var(\nabla^1 Y_t) = Var(e_t) = \sigma_e^2$$

Stationary model

Differencing

Backshift (B) :

$$B(Y_t) = Y_{t-1}$$
$$B^2(Y_t) = Y_{t-2}$$
$$B^k(Y_t) = Y_{t-k}$$

Backward (∇) :

$$\nabla = 1 - B$$
$$\nabla^2 = (1 - B)^2 = (1 - 2B - B^2)$$

$$\nabla Y_t = (1 - B)Y_t = Y_t \; - Y_{t-1}$$
$$\nabla^2 Y_t = (1 - B)^2 Y_t = (1 - 2B - B^2)Y_t = Y_t - 2Y_{t-1} + Y_{t-2}$$

ARIMA Models

A time series $\{Y_t\}$ is said to follow an **integrated autoregressive moving average** model if the dth difference $W_t = \nabla^d Y_t$ is a stationary ARMA process. If $\{W_t\}$ follows an ARMA(p,q) model, we say that $\{Y_t\}$ is an ARIMA(p,d,q) process. Fortunately, for practical purposes, we can usually take $d = 1$ or at most 2.

Consider then an ARIMA(p,1,q) process. With $W_t = Y_t - Y_{t-1}$, we have

$$W_t = \phi_1 W_{t-1} + \phi_2 W_{t-2} + \cdots + \phi_p W_{t-p} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

or, in terms of the observed series,

$$Y_t - Y_{t-1} = \phi_1(Y_{t-1} - Y_{t-2}) + \phi_2(Y_{t-2} - Y_{t-3}) + \cdots + \phi_p(Y_{t-p} - Y_{t-p-1}) + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

ARIMA Models

$$Y_t - Y_{t-1} = \phi_1(Y_{t-1} - Y_{t-2}) + \phi_2(Y_{t-2} - Y_{t-3}) + \cdots + \phi_p(Y_{t-p} - Y_{t-p-1})$$
$$+ e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

which we may rewrite as

$$Y_t = (1 + \phi_1)Y_{t-1} + (\phi_2 - \phi_1)Y_{t-2} + (\phi_3 - \phi_2)Y_{t-3} + \cdots$$
$$+ (\phi_p - \phi_{p-1})Y_{t-p} - \phi_p Y_{t-p-1} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2} - \cdots - \theta_q e_{t-q}$$

We call this the **difference equation form** of the model.
Notice that it appears to be an ARMA(p + 1,q) process

The IMA(1,1) Model

$d = 1, q = 1$

$Y_t = Y_{t-1} + e_t - \theta e_{t-1}$ → Y_t is Non-stationary process

$Y_t - Y_{t-1} = e_t - \theta e_{t-1}$

$W_t = e_t - \theta e_{t-1}$ → W_t is stationary process

The IMA(2,2) Model

$d = 2, q = 2$

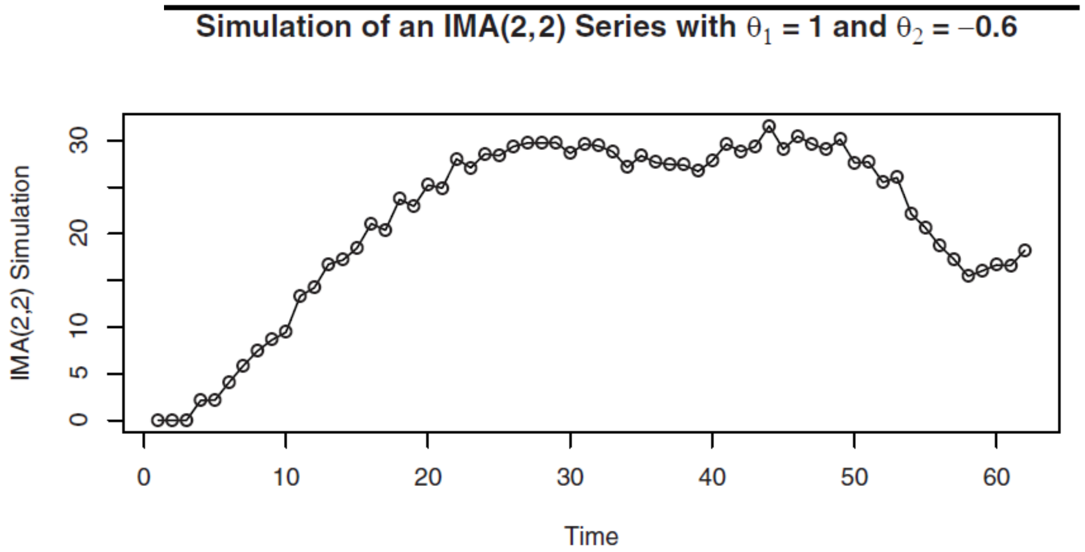
$Y_t = 2Y_{t-1} - Y_{t-2} + e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$ $\longrightarrow$ Y_t is Non-stationary process

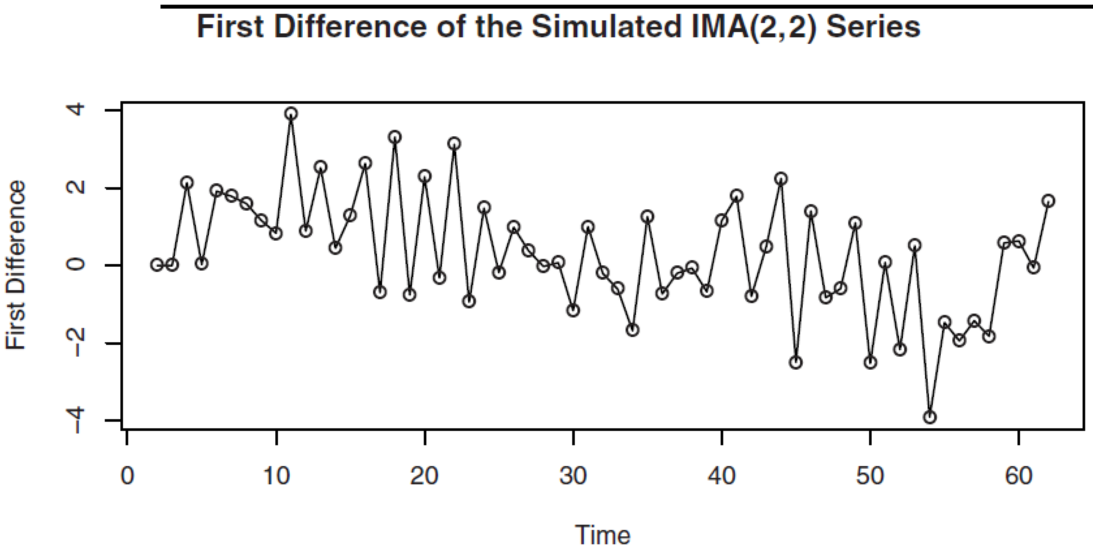
$Y_t - 2Y_{t-1} + Y_{t-2} = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$

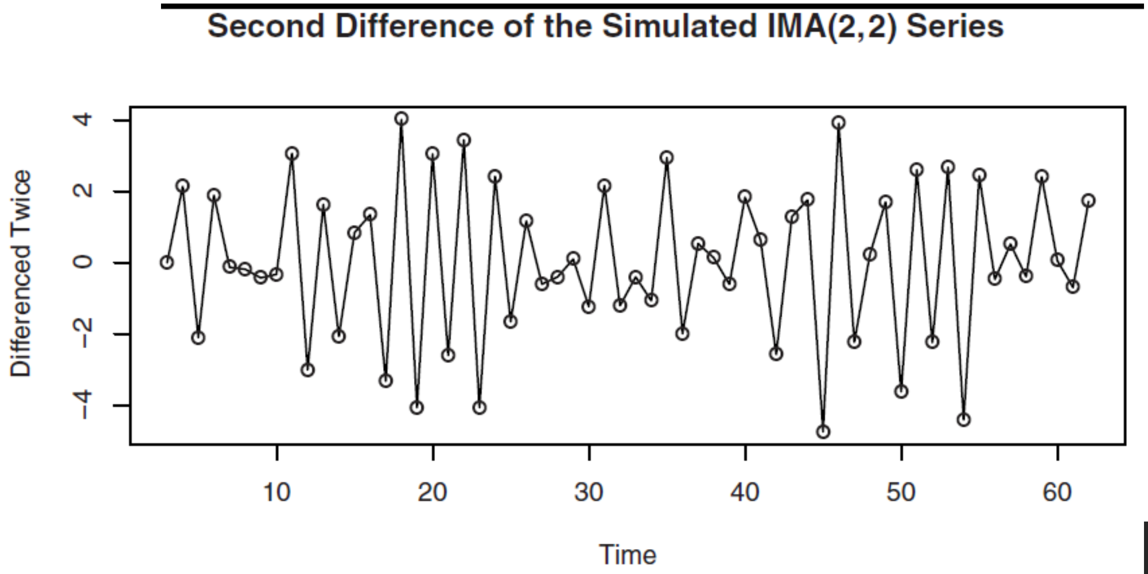
$\nabla^2 Y_t = e_t - \theta e_{t-1} - \theta_2 e_{t-2}$

$W_t = e_t - \theta e_{t-1} - \theta_2 e_{t-2}$ $\longrightarrow$ W_t is stationary process

The IMA(2,2) Model







The ARI(1,1) Model

$p = 1, d = 1$

$\nabla Y_t = \phi \nabla Y_{t-1} + e_t$ → ∇Y_t is stationary process

$Y_t - Y_{t-1} = \phi(Y_{t-1} - Y_{t-2}) + e_t$

$Y_t = (1 + \phi)Y_{t-1} - \phi Y_{t-2} + e_t$ → Y_t is Non-stationary process

ARIMA Models

ARIMA(0, 0, 1) : $Y_t = e_t - \theta e_{t-1}$

ARIMA (1,1,0) : $\nabla Y_t = \phi \nabla Y_{t-1} + e_t$

ARIMA(0, 1, 1) : $\nabla Y_t = e_t - \theta e_{t-1}$

ARIMA (0,2,2) : $\nabla^2 Y_t = e_t - \theta_1 e_{t-1} - \theta_2 e_{t-2}$

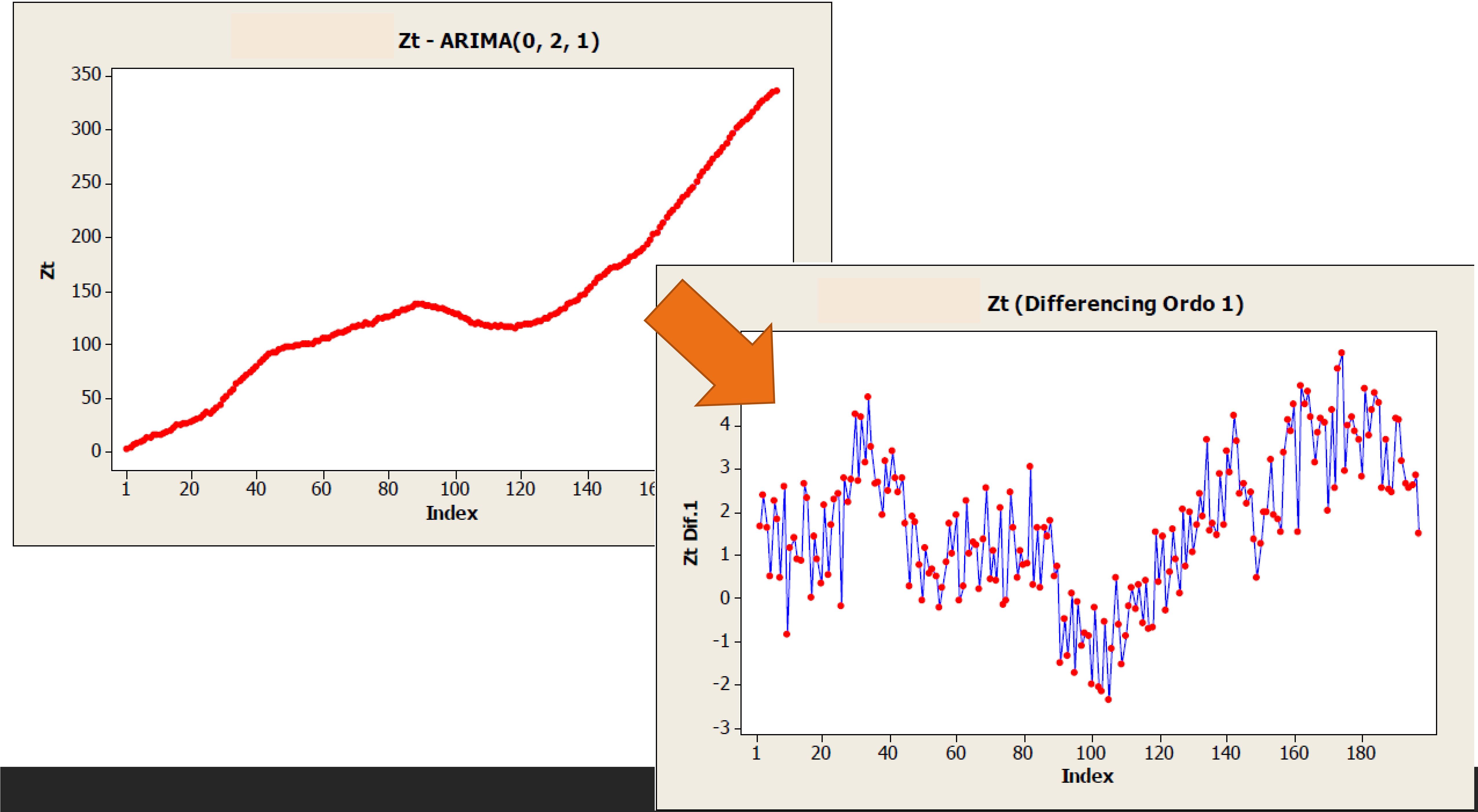
ARIMA(1, 0, 0) : $Y_t = \phi Y_{t-1} + e_t$

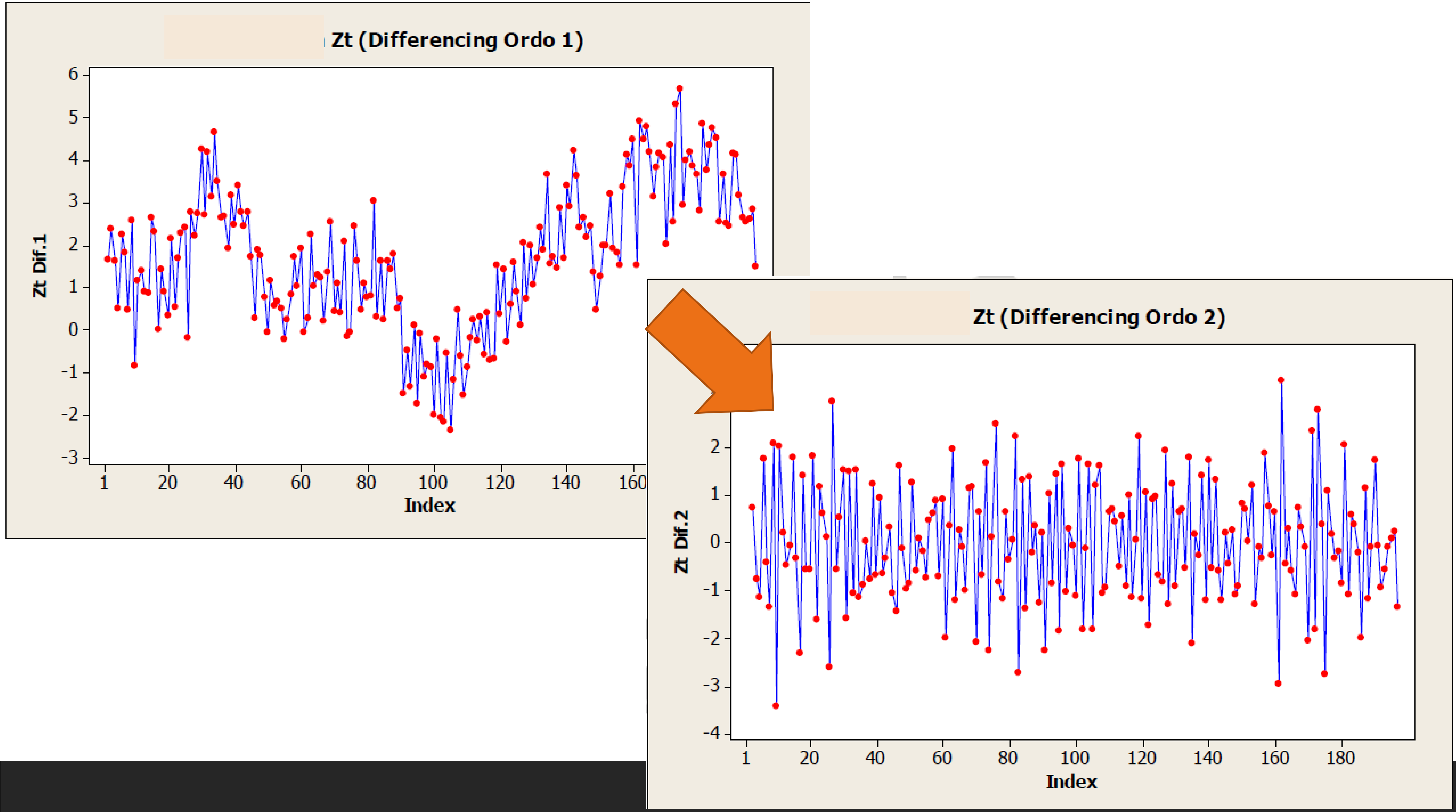
ARIMA (1,1,1) : $\nabla Y_t = \phi \nabla Y_{t-1} + e_t - \theta_1 e_{t-1}$

Exercise

Consider the process $Y_t = Y_{t-1} + e_t + \frac{1}{4} e_{t-1}$

Is the process invertible?
Is the process stationary?


Zt - ARIMA(0, 2, 1)
Zt
350
300
250
200
150
100
50
0
1
20
40
60
80
100
120
140
Index
Zt (Differencing Ordo 1)
Zt Dif.1
4
3
2
1
0
-1
-2
-3
1
20
40
60
80
100
120
140
160
180
Index

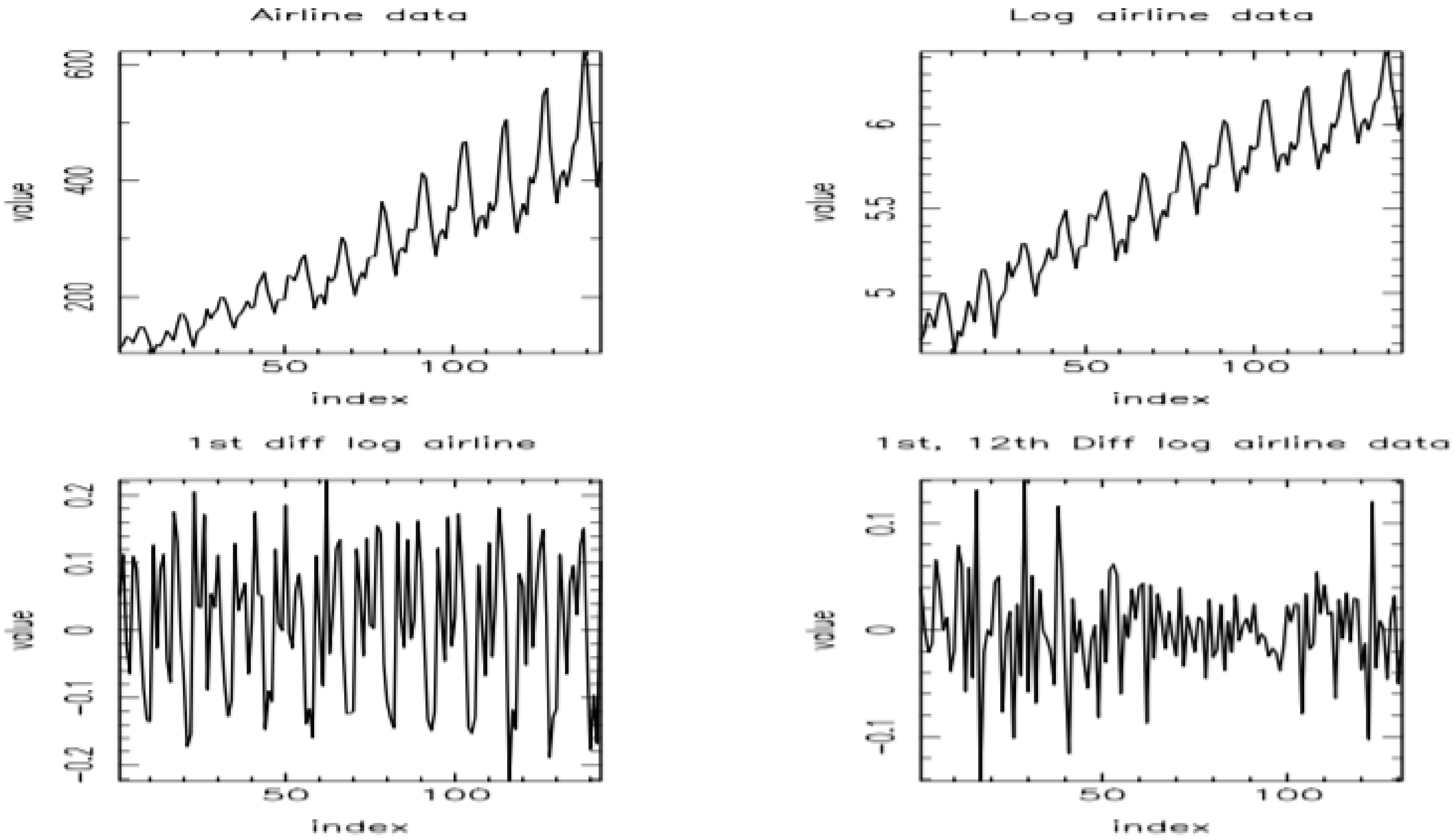


Zt (Differencing Ordo 1)
Zt Dif.1
Index
Zt (Differencing Ordo 2)
Zt Dif.2
Index

Other Transformations

- We have seen how differencing can be a useful transformation for achieving stationarity.
- However, the other transformation is also a useful method for achieving stationarity.

Common Box-Cox Transformations	
Lambda	Suitable Transformation
-2	$Y^{-2} = 1/Y^2$
-1	$Y^{-1} = 1/Y^1$
-0.5	$Y^{-0.5} = 1/(Sqrt(Y))$
0	log(Y)
0.5	$Y^{0.5} = Sqrt(Y)$
1	$Y^1 = Y$
2	Y^2


Airline data
Log airline data
1st diff log airline
1st, 12th Diff log airline data
value
index
200
400
600
5
5.5
6
-0.2
-0.1
0
0.1
0.2
50
100

The next meetings

- Model Specification
- Parameter estimation
- Model Diagnostic
- Forecasting

Thanks