Parameter Estimation, Model Diagnostics and Forecasting

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Outline

- Parameter estimation method for AR, MA and ARMA :
  - The method of moments,
  - Least Square Estimation and
  - Maximum Likelihood
- Model Diagnostics : Residual Analysis and Overfitting
- Forecasting
- Illustration

Parameter estimation : The method of moments

- The method of moments is frequently one of the easiest
- The method consists of equating sample moments to corresponding theoretical moments and solving the resulting equations to obtain estimates of any unknown parameters

Parameter estimation :
The method of moments for **Autoregressive Models**

- Consider first the AR(1) case.
- For this process, we have the simple relationship $\rho_1 = \phi$.
- In the method of moments, ρ_1 is equated to r_1, the lag 1 sample autocorrelation.
- Thus we can estimate ϕ by $\hat{\phi} = r_1$

Parameter estimation : The method of moments for **Autoregressive Models**

Now consider the AR(2) case.

The relationships between the parameters ϕ_1 and ϕ_2 and various moments are given by the Yule-Walker equations

$$\rho_1 = \phi_1 + \rho_1\phi_2 \; and \; \rho_2 = \rho_1\phi_1 + \phi_2 \;\rightarrow r_1 = \phi_1 + r_1\phi_2 \; and \; r_2 = r_1\phi_1 + \phi_2$$

which are then solved to obtain $\hat{\phi}_1 = \frac{r_1(1-r_2)}{1-r_1^2} \; and \; \hat{\phi}_2 = \frac{r_2-r_1^2}{1-r_1^2}$

Parameter estimation :
The method of moments for **Autoregressive Models**

The general AR(p) case proceeds similarly. Replace ρ_k by r_k throughout the Yule-Walker equations to obtain

$$\left.\begin{array}{rrrcl} \phi_1 + & r_1\phi_2 + & r_2\phi_3 & + \cdots + r_{p-1}\phi_p & = r_1 \\ r_1\phi_1 + & \phi_2 + & r_1\phi_3 & + \cdots + r_{p-2}\phi_p & = r_2 \\ & & & & \vdots \\ r_{p-1}\phi_1 + & r_{p-2}\phi_2 + & r_{p-3}\phi_3 & + \cdots + \quad \phi_p & = r_p \end{array}\right\}$$

Parameter estimation : The method of moments for **Moving Average Models**

The method of moments is not nearly as convenient when applied to moving average models.

Consider the simple MA(1) case $\rho_1 = -\frac{\theta}{1+\theta^2}$, equating ρ_1 and r_1, we are led to solve a quadratic equation in θ.

$$\hat{\theta} = \frac{-1 + \sqrt{1 - 4r_1^2}}{2r_1}$$

For higher-order MA models, the method of moments quickly gets complicated

Parameter estimation :
The method of moments for **ARMA Models**

The ARMA(1,1) case

$$\rho_k = \frac{(1-\theta\phi)(\phi-\theta)}{1-2\theta\phi+\theta^2}\phi^{k-1} \quad \text{for } k \geq 1$$

Noting that $\rho_2/\rho_1 = \phi$, we can first estimate ϕ as

$$\hat{\phi} = \frac{r_2}{r_1}$$

Having done so, we can then use

$$r_1 = \frac{(1-\theta\hat{\phi})(\hat{\phi}-\theta)}{1-2\theta\hat{\phi}+\theta^2}$$

Note again that a quadratic equation must be solved.

Parameter estimation :
The method of moments for **the Noise Variance**

For the AR(p) models

$$\boxed{\hat{\sigma}_e^2 = (1 - \hat{\phi}_1 r_1 - \hat{\phi}_2 r_2 - \cdots - \hat{\phi}_p r_p) s^2}$$

$$\boxed{s^2 = \frac{1}{n-1} \sum_{t=1}^{n} (Y_t - \bar{Y})^2}$$

In particular, for an AR(1) process,

$$\boxed{\hat{\sigma}_e^2 = (1 - r_1^2) s^2}$$

since $\hat{\phi} = r_1$

Parameter estimation :
The method of moments for **the Noise Variance**

For the MA(q) case

$$\hat{\sigma}_e^2 = \frac{s^2}{1 + \hat{\theta}_1^2 + \hat{\theta}_2^2 + \cdots + \hat{\theta}_q^2}$$

$$s^2 = \frac{1}{n-1} \sum_{t=1}^{n} (Y_t - \bar{Y})^2$$

For the ARMA(1,1) process

$$\hat{\sigma}_e^2 = \frac{1 - \hat{\phi}^2}{1 - 2\hat{\phi}\hat{\theta} + \hat{\theta}^2} s^2$$

Numerical example for method of moments parameter estimation

	True Parameters			Method-of-Moments Estimates			
Model	θ	ϕ_1	ϕ_2	θ	ϕ_1	ϕ_2	***n***
MA(1)	–0.9			–0.554			120
MA(1)	0.9			0.719			120
MA(1)	–0.9			NA†			60
MA(1)	0.5			–0.314			60
AR(1)		0.9			0.831		60
AR(1)		0.4			0.470		60
AR(2)		1.5	–0.75		1.472	–0.767	120

† No method-of-moments estimate exists since $r_1 = 0.544$ for this simulation.

Parameter estimation :
Least Squares Estimation for **Autoregressive Models**

Consider the first-order case where

$$Y_t - \mu = \phi(Y_{t-1} - \mu) + e_t$$

AR(1)

We can view this as a regression model with predictor variable Y_{t-1} and response variable Y_t. Least squares estimation then proceeds by minimizing the sum of squares of the differences

$$(Y_t - \mu) - \phi(Y_{t-1} - \mu)$$

Since only $Y_1, Y_2, \ldots, Y_n$ are observed, we can only sum from $t = 2$ to $t = n$. Let

$$S_c(\phi, \mu) = \sum_{t=2}^{n} [(Y_t - \mu) - \phi(Y_{t-1} - \mu)]^2$$

Conditional sum-of squares function

Parameter estimation :
Least Squares Estimation for **Autoregressive Models**

Consider the equation $\partial S_c / \partial \mu = 0$. We have

$$\frac{\partial S_c}{\partial \mu} = \sum_{t=2}^{n} 2[(Y_t - \mu) - \phi(Y_{t-1} - \mu)](-1 + \phi) = 0$$

or, simplifying and solving for μ,

$$\boxed{\mu = \frac{1}{(n-1)(1-\phi)}\left[\sum_{t=2}^{n} Y_t - \phi \sum_{t=2}^{n} Y_{t-1}\right]}$$

Parameter estimation :
Least Squares Estimation for **Autoregressive Models**

Now, for large n,

$$\frac{1}{n-1}\sum_{t=2}^{n} Y_t \approx \frac{1}{n-1}\sum_{t=2}^{n} Y_{t-1} \approx \bar{Y}$$

$$\hat{\mu} \approx \frac{1}{1-\phi}(\bar{Y} - \phi\bar{Y}) = \bar{Y}$$

We sometimes say, except for end effects, $\hat{\mu} = \bar{Y}$.

Parameter estimation : Least Squares Estimation for **Autoregressive Models**

Consider now the minimization of $S_c(\phi, \bar{Y})$ with respect to ϕ. We have

$$\frac{\partial S_c(\phi, \bar{Y})}{\partial \phi} = \sum_{t=2}^{n} 2[(Y_t - \bar{Y}) - \phi(Y_{t-1} - \bar{Y})](Y_{t-1} - \bar{Y})$$

Setting this equal to zero and solving for ϕ yields

$$\boxed{\hat{\phi} = \frac{\sum_{t=2}^{n} (Y_t - \bar{Y})(Y_{t-1} - \bar{Y})}{\sum_{t=2}^{n} (Y_{t-1} - \bar{Y})^2}}$$

Parameter estimation :
Least Squares Estimation for **Moving Average Models**

Consider now the least-squares estimation of θ in the MA(1) model:

$$Y_t = e_t - \theta e_{t-1}$$

$$Y_t = -\theta Y_{t-1} - \theta^2 Y_{t-2} - \theta^3 Y_{t-3} - \cdots + e_t$$

an autoregressive model but of infinite order. Thus least squares can be meaningfully carried out by choosing a value of θ that minimizes

Parameter estimation :
Least Squares Estimation for **Moving Average Models**

$$S_c(\theta) = \sum(e_t)^2 = \sum[Y_t + \theta Y_{t-1} + \theta^2 Y_{t-2} + \theta^3 Y_{t-3} + \cdots]^2$$

where, implicitly, $e_t = e_t(\theta)$ is a function of the observed series and the unknown parameter θ.

- It is clear from this equation that the least squares problem is *nonlinear* in the parameters.
- We will not be able to minimize $Sc(\theta)$ by taking a derivative with respect to θ, setting it to zero, and solving→ numerical optimization

Parameter estimation : Maximum Likelihood

- For any set of observations, Y_1, Y_2,..., Y_n, time series or not, the likelihood function L is defined to be the joint probability density of obtaining the data actually observed
- It is considered as a function of the unknown parameters in the model with the observed data held fixed
- For ARIMA models, L will be a function of the ϕ's, θ's, μ, and σ_e^2 given the observations Y_1, Y_2,..., Y_n.
- The maximum likelihood estimators are then defined as those values of the parameters for which the data actually observed are *most likely*, that is, the values that maximize the likelihood function.

Parameter estimation : Maximum Likelihood AR(1)

The likelihood function for an AR(1) model is given by

$$L(\phi, \mu, \sigma_e^2) = (2\pi\sigma_e^2)^{-\frac{n}{2}}(1-\phi^2)^{1/2} exp\left[-\frac{1}{2\sigma_e^2} S(\phi, \mu)\right]$$

Where

$$S(\phi, \mu) = \sum_{t=2}^{n} [(Y_t - \mu) - \phi(Y_{t-1} - \mu)]^2 + (1-\phi^2)(Y_1 - \mu)$$

Unconditional sum-of squares function

The log-likelihood function for an AR(1) model is given by

$$l(\phi, \mu, \sigma_e^2) = -\frac{n}{2}\log(2\pi) - -\frac{n}{2}\log(\sigma_e^2) + \frac{1}{2}\log(1-\phi^2) - \frac{1}{2\sigma_e^2} S(\phi, \mu)$$

Properties of the Estimates

For large n, the estimators are approximately unbiased and normally distributed. The variances and correlations are as follows:

$$\text{AR(1)}: \; Var(\hat{\phi}) \approx \frac{1-\phi^2}{n}$$

$$\text{AR(2)}: \begin{cases} Var(\hat{\phi}_1) \approx Var(\hat{\phi}_2) \approx \dfrac{1-\phi_2^2}{n} \\ Corr(\hat{\phi}_1, \hat{\phi}_2) \approx -\dfrac{\phi_1}{1-\phi_2} = -\rho_1 \end{cases}$$

$$\text{MA(1)}: \; Var(\hat{\theta}) \approx \frac{1-\theta^2}{n}$$

$$\text{MA(2)}: \begin{cases} Var(\hat{\theta}_1) \approx Var(\hat{\theta}_2) \approx \dfrac{1-\theta_2^2}{n} \\ Corr(\hat{\theta}_1, \hat{\theta}_2) \approx -\dfrac{\theta_1}{1-\theta_2} \end{cases}$$

$$\text{ARMA(1,1)}: \begin{cases} Var(\hat{\phi}) \approx \left[\dfrac{1-\phi^2}{n}\right]\left[\dfrac{1-\phi\theta}{\phi-\theta}\right]^2 \\ Var(\hat{\theta}) \approx \left[\dfrac{1-\theta^2}{n}\right]\left[\dfrac{1-\phi\theta}{\phi-\theta}\right]^2 \\ Corr(\hat{\phi}, \hat{\theta}) \approx \dfrac{\sqrt{(1-\phi^2)(1-\theta^2)}}{1-\phi\theta} \end{cases}$$

Illustrations of Parameter Estimation

Parameter Estimation for Simulated AR(1) Models

Parameter ϕ	Method-of-Moments Estimate	Conditional SS Estimate	Unconditional SS Estimate	Maximum Likelihood Estimate	n
0.9	0.831	0.857	0.911	0.892	60
0.4	0.470	0.473	0.473	0.465	60

$$\sqrt{\hat{Var}(\hat{\phi})} \approx \sqrt{\frac{1-\hat{\phi}^2}{n}} = \sqrt{\frac{1-(0.831)^2}{60}} \approx 0.07$$

$$\sqrt{\hat{Var}(\hat{\phi})} = \sqrt{\frac{1-(0.470)^2}{60}} \approx 0.11$$

Illustrations of Parameter Estimation

Parameter Estimation for a Simulated AR(2) Model

Parameters	Method-of-Moments Estimates	Conditional SS Estimates	Unconditional SS Estimates	Maximum Likelihood Estimate	n
$\phi_1 = 1.5$	1.472	1.5137	1.5183	1.5061	120
$\phi_2 = -0.75$	–0.767	–0.8050	–0.8093	–0.7965	120

$$\sqrt{\hat{Var}(\hat{\phi}_1)} \approx \sqrt{\hat{Var}(\hat{\phi}_2)} \approx \sqrt{\frac{1-\phi_2^2}{n}} = \sqrt{\frac{1-(0.75)^2}{120}} \approx 0.06$$

Illustrations of Parameter Estimation

Parameter Estimation for a Simulated ARMA(1,1) Model

Parameters	Method-of-Moments Estimates	Conditional SS Estimates	Unconditional SS Estimates	Maximum Likelihood Estimate	n
$\phi = 0.6$	0.637	0.5586	0.5691	0.5647	100
$\theta = -0.3$	−0.2066	−0.3669	−0.3618	−0.3557	100

Illustrations of Parameter Estimation

Parameter Estimation for the Color Property Series

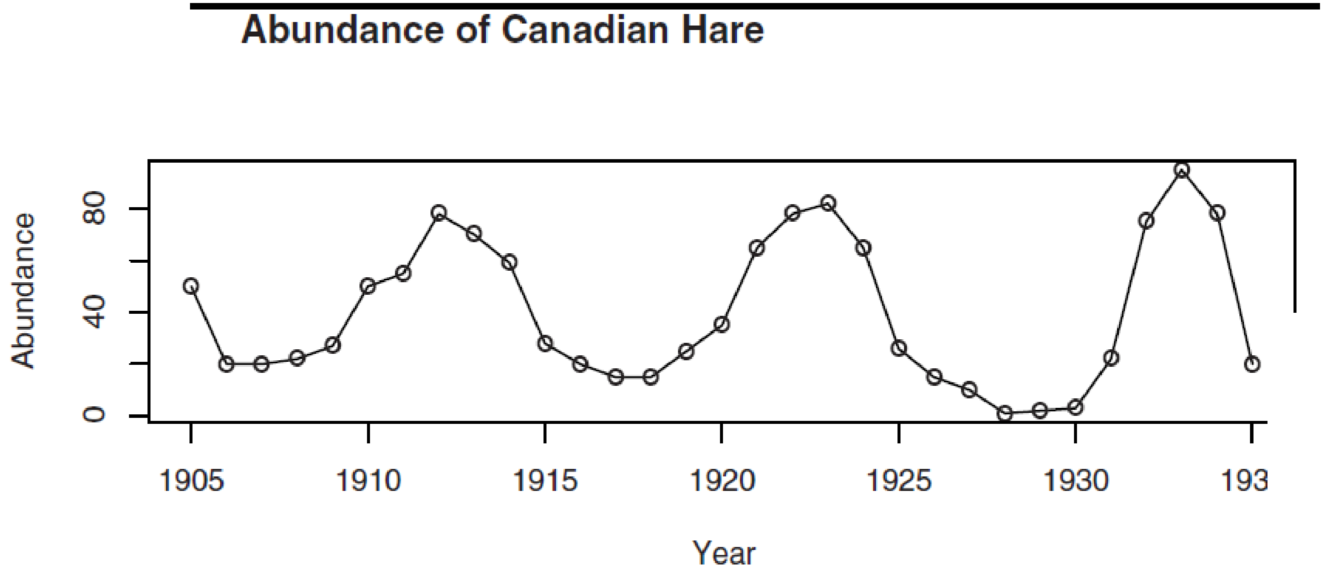
Parameter	Method-of-Moments Estimate	Conditional SS Estimate	Unconditional SS Estimate	Maximum Likelihood Estimate	n
ϕ	0.5282	0.5549	0.5890	0.5703	35

the standard error of the estimates is about

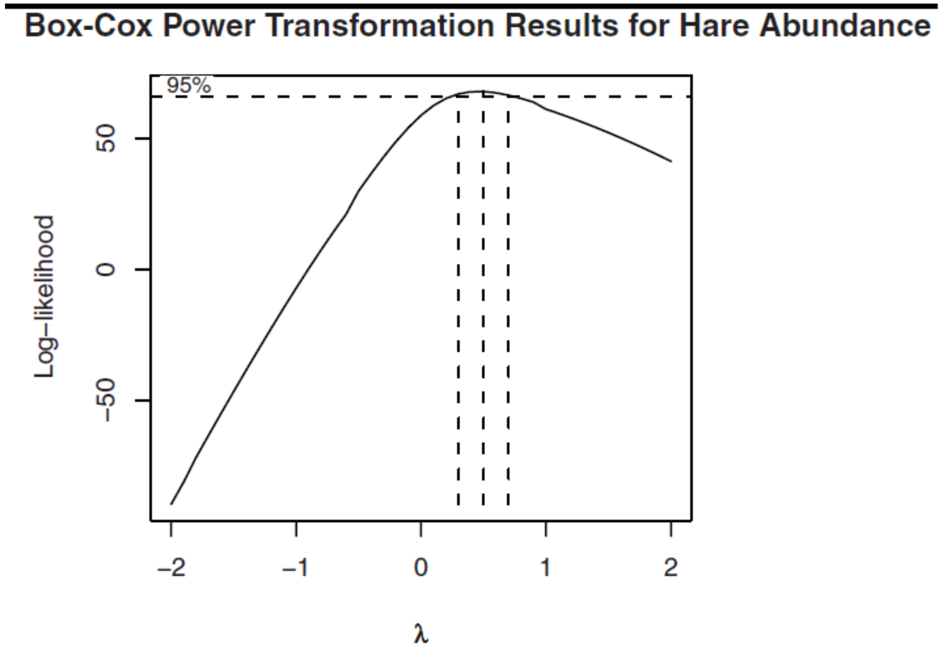
$$\sqrt{\hat{Var}(\hat{\phi})} \approx \sqrt{\frac{1-(0.57)^2}{35}} \approx 0.14$$

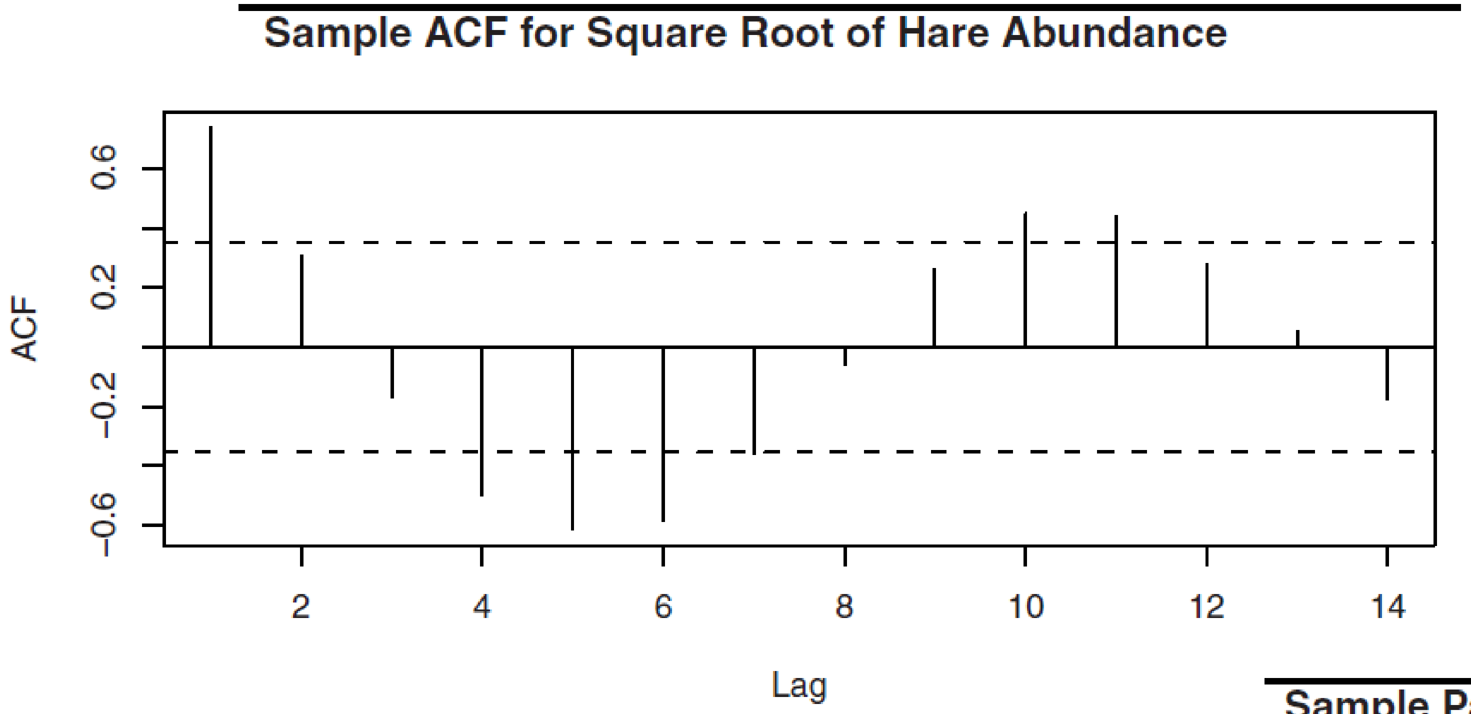
Illustrations of Parameter Estimation

The Annual Abundance of Canadian Hare Series

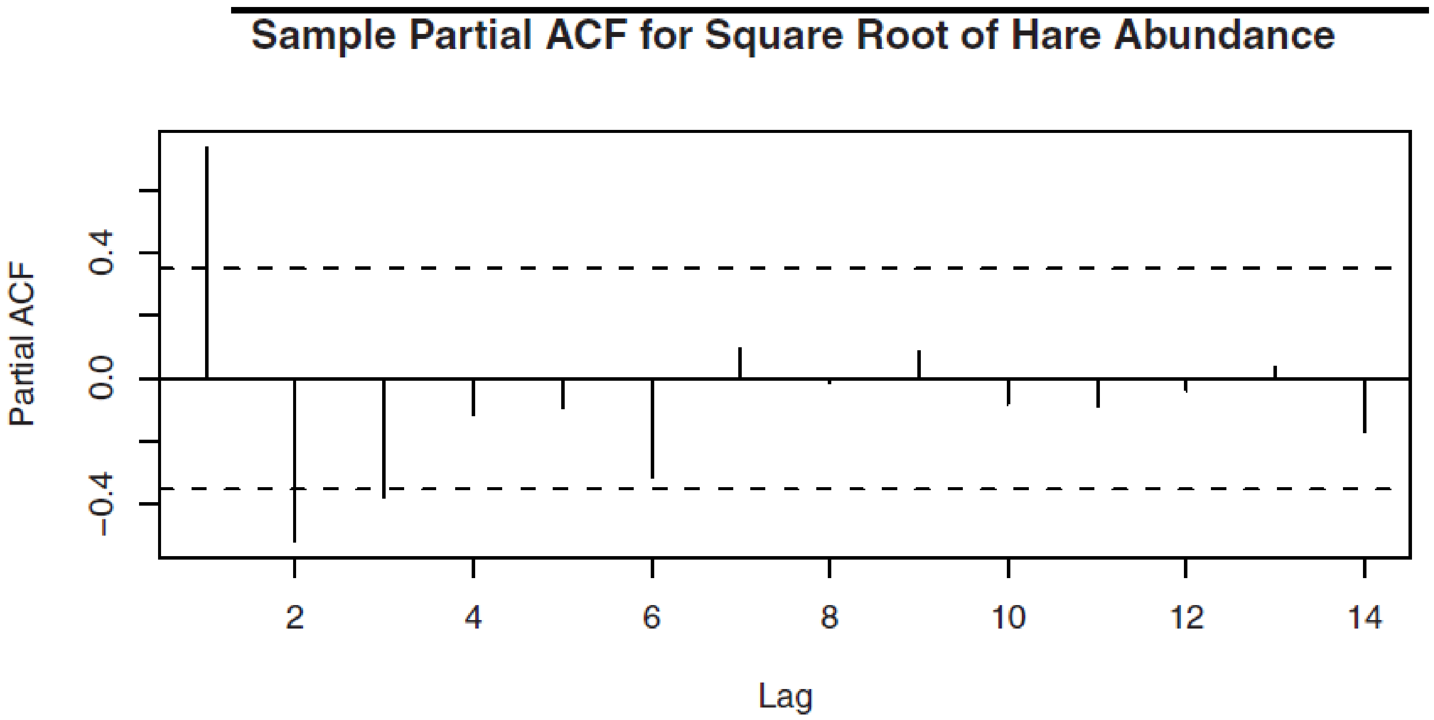


λ	Transformed Data
-2	y^{-2}
-1	y^{-1}
-0.5	$1/\sqrt{y}$
0	$\ln(y)$
$\boxed{0.5}$	$\boxed{\sqrt{y}}$
1	y
2	y^2





- → ACF tails off
- → PACF cuts off after lag 3



Maximum Likelihood Estimates from R Software: Hare Series

Coefficients:	ar1	ar2	ar3	Intercept[†]
	1.0519	–0.2292	–0.3931	5.6923
s.e.	0.1877	0.2942	0.1915	0.3371

```
sigma^2 estimated as 1.066: log-likelihood = -46.54, AIC = 101.08
```

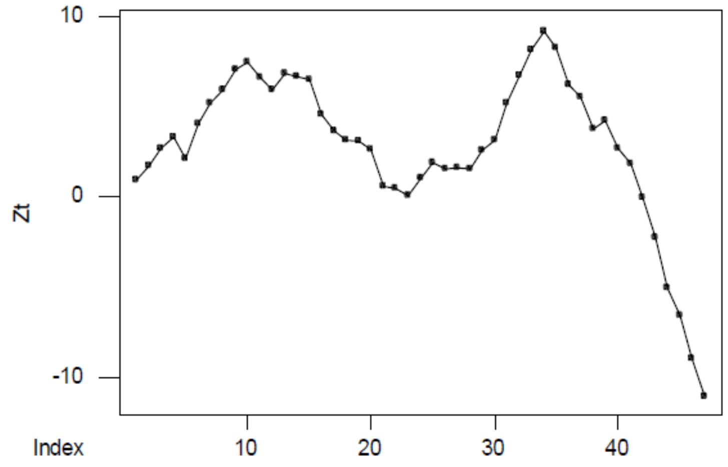
[†] The intercept here is the estimate of the process mean μ—not of θ_0.

$$t_{ar1} = \frac{1.0519}{0.1877} = 5.604$$

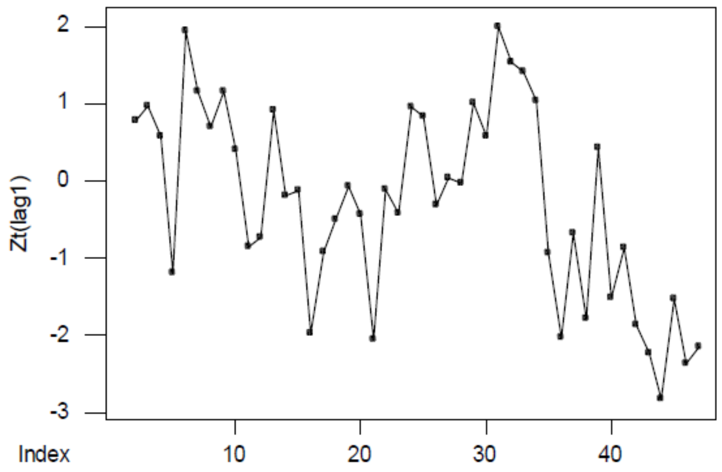
$$\boxed{t_{ar2} = -\frac{0.2292}{0.2941} = -0.779}$$

$$t_{ar3} = -\frac{0.3931}{0.1915} = -2.053$$

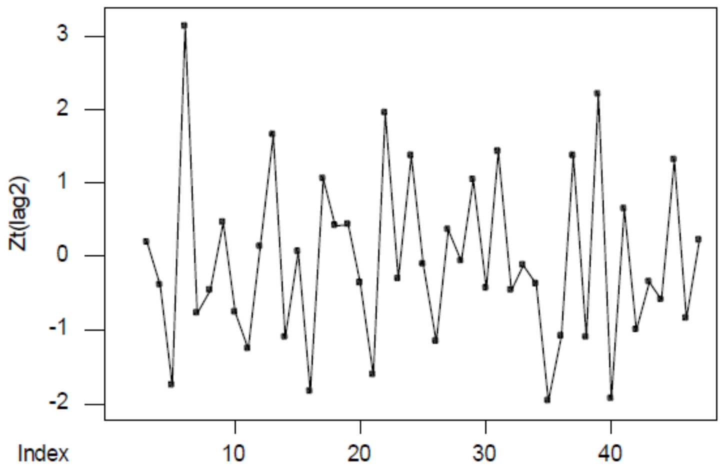
Illustrations of Parameter Estimation



Zt : Data Asal

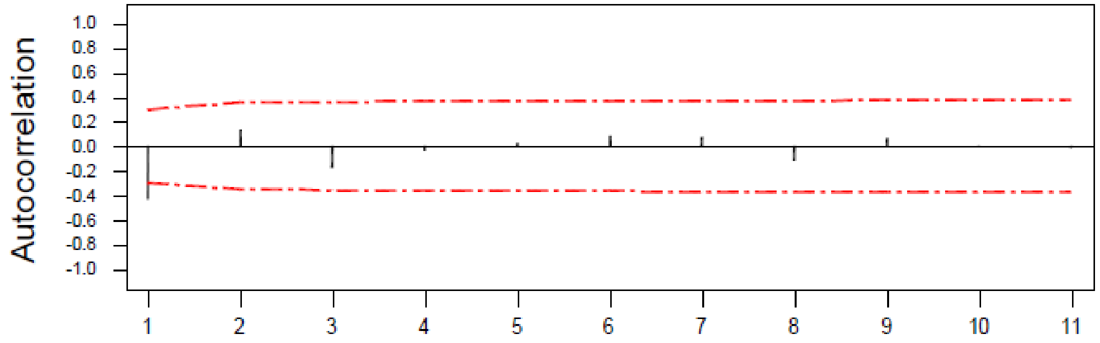


Zt : Data Setelah Differencing Ordo-1

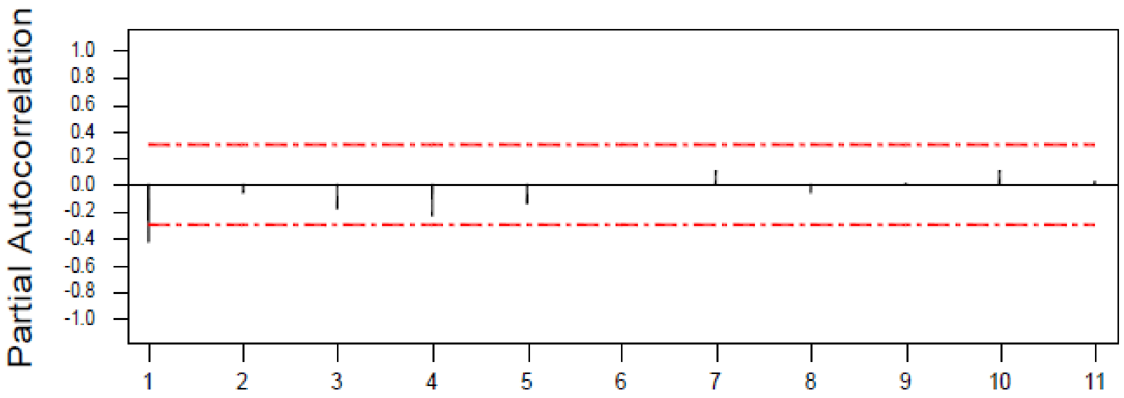


Zt : Data Setelah Differencing Ordo-2

Autocorrelation Function for Zt(lag2)



Partial Autocorrelation Function for Zt(lag2)



ARIMA(0,2,1) & **ARIMA(1,2,0)**

ARIMA(0, 2, 1)

Final Estimates of Parameters

Type	Coef	SE Coef	T	P
MA 1	-0.4393	0.1371	-3.20	$\boxed{0.003}$
Constant	-0.0995	0.1581	-0.63	0.533

ARIMA(1, 2, 0)

Final Estimates of Parameters

Type	Coef	SE Coef	T	P
AR 1	0.5958	0.1225	4.86	$\boxed{0.000}$
Constant	-0.06673	0.06299	-1.06	0.295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