L7: Intro to Optimization + LP Basics

→ Intro: Slides

Mathematical Optimization: " a program
$$\begin{cases} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le b_i \end{cases}$$
maximize is the same

$f_0 : \mathbb{R}^n \to \mathbb{R}$ "objective function"

$f_i : \mathbb{R}^n \to \mathbb{R}$ "constraints"

→ where $x \in \mathbb{R}^n$

→ Goal: find a / the solution (aka, optimal point) $x^* \in \mathbb{R}^n$ that has the smallest f_0 satisfying the constraints.

Example:

Food	Carrot, Raw (x_1)	White Cabbage, Raw (x_2)	Cucumber, Pickled (x_3)	Required per dish
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg/kg]	60	300	10	15 mg
Dietary Fiber [g/kg]	30	20	10	4 g
price [€/kg]	0.75	0.5	0.15*	—

*Residual accounting price of the inventory, most likely unsaleable.

Minimize $\underbrace{0.75x_1 + 0.5x_2 + 0.15x_3}_{f_0(x)}$

Subj to: $x_1, x_2, x_3 \geq 0$

(Vit A) $35x_1 + 0.5x_2 + 0.5x_3 \geq 0.5$

(Vit C) $60x_1 + 300x_2 + 10x_3 \geq 15$

(Diet Fiber) $30x_1 + 20x_2 + 10x_3 \geq 4$

Linear Programming: → Objective & constraint functions are linear.

Hence: Minimize $c^T x$ subj. to $Ax \le b$

$c \in \mathbb{R}^n$, $x \in \mathbb{R}^m$, A: $m \times n$, $b \in \mathbb{R}^m$

→ Simple example & geometric view

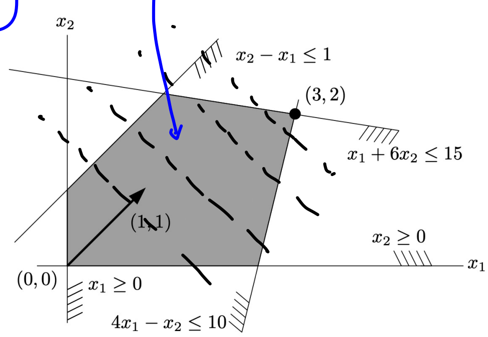
(A) Maximize $x_1 + x_2$

Subj to
$$x_2 - x_1 \le 1$$
$$x_1 + 6x_2 \le 15$$
$$4x_1 - x_2 \le 10$$
$$x \geq 0.$$

(B) Maximize $\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{c}^T x$

Subj to:
$$\begin{pmatrix} -1 & 1 \\ 1 & 6 \\ 4 & -1 \end{pmatrix} x \le \begin{pmatrix} 1 \\ 15 \\ 10 \end{pmatrix}$$
$$x \geq 0$$

(C) Feasible set



Geometrically: → a polytope in $\mathbb{R}^n$

→ Each inequality constraint is a half-space → feasible set = $\bigcap$ half spaces

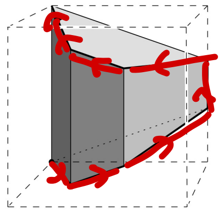
→ Optimal sol^n is at a corner (vertex)

→ # corners can be exponential in n

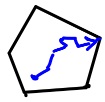
How (not) to solve an LP?

1. Try all corners (vtxes) — a corner = intersection of $(n-1)$ constraints
   → exponential time. Meh

* 2. Simplex (also: tableau) [Dantzig, 1947]
   → moves around the surface of the polytope → $O(n(n+m))$ per iteration
   dense matrix
   ∃ sparse mat. optimization
   → Up to exponential # of iterations
   → Generally "great" in practice.



3. Ellipsoid (Khachian, 1979)
   → first-known poly-time LP algo → not exactly practical.



* 4. Interior Point: moves inside the polytope in some controlled way.
   → Projective (Karmakar, 1984), Affine (Dikin, 1967), Log-Barrier (Fiacco '68, Frisch '77, etc.)

In practice: simplex & interior point. • No clear winner "it depends" • Most solver packages have both
→ CPLEX, Gurobi, scipy.

(Linear Programming) Duality.

Consider the following LP:

$$
\begin{aligned}
\text{Maximize} \quad & 2x_1 + 3x_2 \\
\text{Subj to} \quad & 4x_1 + 8x_2 \le 12 \quad (y_1)\\
& 2x_1 + x_2 \le 3 \quad (y_2)\\
& 3x_1 + 2x_2 \le 4 \quad (y_3)\\
& x \ge 0.
\end{aligned}
$$

(P) Primal.

Scrap: find upperbound w/o solving the LP

- $2x_1 + 3x_2 \le 4x_1 + 8x_2 \le \boxed{12}$ ∵ $x \ge 0$ ⇒ $OPT \le 12$
- $2x_1 + 3x_2 \le \frac{1}{2}(4x_1 + 8x_2) \le \frac{1}{2}\cdot 12 = \boxed{6}$
- $2x_1 + 3x_2 = \frac{1}{3}\left([4x_1 + 8x_2) + (2x_1 + x_2)\right] \le \frac{1}{3}(12+3) = \boxed{5}$

Q: How to find the "best" upperbound? A: Write another LP.

→ Suppose we multiply the constraints by y_1, y_2, y_3 resp. Then:

$$
\begin{aligned}
4\cdot y_1 + 2\cdot y_2 + 3\cdot y_3 &\ge 2 \qquad y \ge 0 \quad (\text{o/w. the signs will flip})\\
8 y_1 + y_2 + 2\cdot y_3 &\ge 3
\end{aligned}
$$

Minimize $12\cdot y_1 + 3y_2 + 4\cdot y_3$

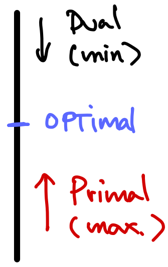
D(ual)

The trick:

(P) Maximize $c^T x$
Subj to $Ax \le b,\ x \ge 0$

(D) Minimize $y^T b$
Subj to $A^T y \ge c$

Observations & Properties



- If x is feasible for primal and y, for dual: $\boxed{c^T x \le b^T y}$
  (meet all constraints)
- Example: max. flow = min cut
- Dualization recipe — no need to memorize. look it up. (Only for ref.)

Dualization Recipe

	Primal linear program	Dual linear program
Variables	$x_1, x_2, \ldots, x_n$	$y_1, y_2, \ldots, y_m$
Matrix	A	A^T
Right-hand side	$\mathbf{b}$	$\mathbf{c}$
Objective function	$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
Constraints	ith constraint has $\le$	$y_i \ge 0$
	$\ge$	$y_i \le 0$
	$=$	$y_i \in \mathbb{R}$
	$x_j \ge 0$	jth constraint has $\ge$
	$x_j \le 0$	$\le$
	$x_j \in \mathbb{R}$	$=$