Intro to Optimization

MUIC Applied Algorithms

Optimization “Blackbox”

- Minimize or Maximize a mathematical function

$$f(\mathbf{x})$$

- Often: subject to some constraints

An Optimist's Dream

An Optimist's Dream

Reality:

* Optimization arises everywhere :)
* General solvers aren't always fast
* Some/many useful variants are NP-hard
* In fact, most optimization problems are really hard!

(Common) Optimization Problem

Minimize $f_0(x)$

- Subject to $f_i(x) \leq 0, \qquad i = 1, \ldots, m$
- and $\qquad g_i(x) = 0, \qquad i = 1, \ldots, p$

where

- $x \in \mathbb{R}^n$ is a vector
- f_0 is the objective function (to be minimized - or maximized)
- $f_1, \ldots, f_m$ are inequality constraint functions
- $g_1, \ldots, g_p$ are equality constraint functions

The Landscape (Very Briefly)

- Unconstrained optimization

$$\min\{f(x) : x \in \mathbb{R}^n\}$$

- Constrained optimization

$$\min\{f(x) : f_i(x) \leq 0, g_j(x) = 0\}$$

- Integer linear programming

$$\min\{f(x) = c^T x : Ax \leq b, x \in \mathbb{Z}^n\}$$

NP-hard

- Linear programming (LP)

$$\min\{f(x) = c^T x : Ax \leq b, x \in \mathbb{R}^n\}$$

- Convex programming

$\min\{f(x) : f_i(x) \leq b_j\}$ where f, f_i are convex functions

"Efficient" Poly-time

Often: relax top to bottom

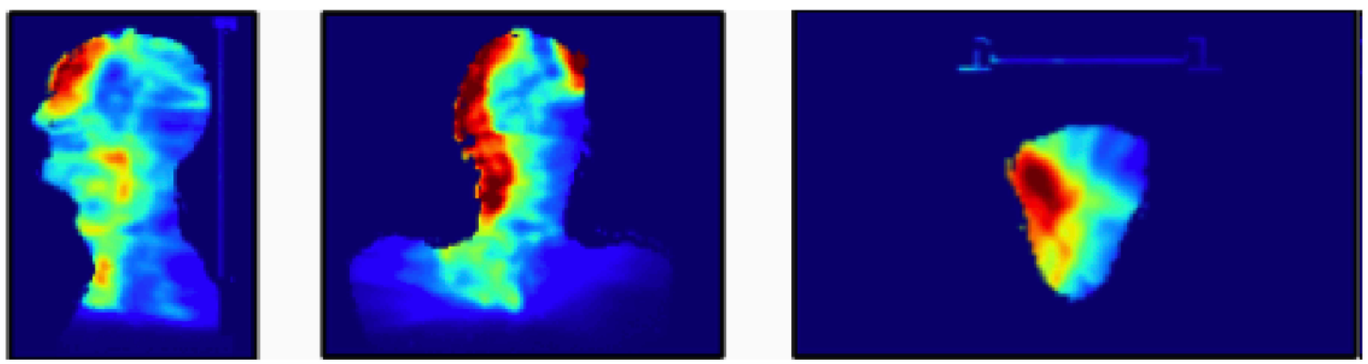
Example: Radiation treatment planning

Physical Modeling

- Radiation beams with intensities $x_j \geq 0$ directed at patient
- Radiation dose y_i received in voxel i
- Overall: $y = Ax$, where A comes from geometry, physics

Goal: Choose x to deliver prescribed radiation dose d_i, so $d_i = 0$ for non-tumor voxels and $d_i > 0$ for tumor voxels

- Ideally $y = d$ but generally not possible
- Typical setup: $n \approx 10^3, m \approx 10^6$ (a few seconds on the GPU)



A. Fu, B. Ungun, L. Xing, and S. Boyd
Optimization and Engineering, vol. 20 (1): 277-300, March 2019.

Example: Image Reconstruction

512×512 grayscale image ($n \approx 300000$ variables)

https://www.cvxpy.org/examples/applications/tv_inpainting.html

Example: Machine learning classifiers

- Support vector machine
- Boosting (turn a collection of unimpressive classifiers into a better overall classifier)
- Etc.

Cheapest Wholesome “Meal”

Food	Carrot, Raw	White Cabbage, Raw	Cucumber, Pickled	Required per dish
Vitamin A [mg/kg]	35	0.5	0.5	0.5 mg
Vitamin C [mg/kg]	60	300	10	15 mg
Dietary Fiber [g/kg]	30	20	10	4 g
price [€/kg]	0.75	0.5	0.15*	—

*Residual accounting price of the inventory, most likely unsaleable.

Activities

1. Use scipy to solve:

Maximize $7x_1 - x_2 + 5x_3$

Subject to:

$$x_1 + x_2 + x_3 \leq 8$$
$$3x_1 - x2 + 2x_3 \leq 3$$
$$2x_1 + 5x_2 - x_3 \leq -7$$
$$x \geq 0$$

2. Find the dual of the (primal) program on the left.

3. Use scipy to solve the dual program. How do their objective values compare?