Machine Learning 1 - Week 1

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Problem 1

a.

Marginal distributions of $p(x)$:

$p(x_1) = 0.1 + 0.05 + 0.01 = 0.16$
$p(x_2) = 0.02 + 0.1 + 0.05 = 0.17$
$p(x_3) = 0.03 + 0.05 + 0.03 = 0.11$
$p(x_4) = 0.1 + 0.07 + 0.05 = 0.22$
$p(x_5) = 0.1 + 0.2 + 0.04 = 0.34$

Marginal distributions of $p(y)$:

$p(y_1) = 0.01 + 0.02 + 0.03 + 0.1 + 0.1 = 0.26$
$p(y_2) = 0.05 + 0.1 + 0.05 + 0.07 + 0.2 = 0.47$
$p(y_3) = 0.1 + 0.05 + 0.03 + 0.05 + 0.04 = 0.27$

b.

The conditional distributions of $p(x|Y = y_1)$:

$p(x_1|y_1) = \frac{0.01}{0.26} = \frac{1}{26}$

$p(x_2|y_1) = \frac{0.02}{0.26} = \frac{1}{13}$

$p(x_3|y_1) = \frac{0.03}{0.26} = \frac{3}{26}$

$p(x_4|y_1) = \frac{0.1}{0.26} = \frac{5}{13}$

$p(x_5|y_1) = \frac{0.1}{0.26} = \frac{5}{13}$

The conditional distributions of $p(x|Y = y_3)$:

$p(x_1|y_3) = \frac{0.1}{0.27} = \frac{10}{27}$

$p(x_2|y_3) = \frac{0.05}{0.27} = \frac{5}{27}$

$p(x_3|y_3) = \frac{0.03}{0.27} = \frac{1}{9}$

$p(x_4|y_3) = \frac{0.05}{0.27} = \frac{5}{27}$

$p(x_5|y_3) = \frac{0.04}{0.27} = \frac{4}{27}$

Problem 2

Prove

$$E_X[X] = E_Y\Bigg[E_X[x|y]\Bigg]$$

$E_X[x|y]$: the expected value of x under the conditional distribution p(x,y)

We will start with:

$$\begin{aligned}
\textbf{RHS} &= E_Y\Bigg[E_X[x|y]\Bigg] \\
&= E_Y\Bigg[\sum_x x \times p(X=x|Y=y) \times p(Y=y)\Bigg] \\
&= \sum_x \sum_y x \times p(X=x|Y=y) \times p(Y=y) \\
&= \sum_x x \sum_y p(X=x|Y=y) \times p(Y=y) \\
&= \sum_x p(X=x) \\
&= E_X[x] = \textbf{LHS}
\end{aligned} \tag{1}$$

The equation has been proved.

Problem 3

Let A: "Người được phỏng vấn sử dụng sản phẩm X"
Let B: "Người được phỏng vấn sử dụng sản phẩm Y"

As provided,

$$\begin{gathered}
\mathrm{p}(A) = 0.207 \\
\mathrm{p}(B) = 0.5 \\
\mathrm{p}(A|B) = 0.365
\end{gathered}$$

a.

The probability of that random interviewed person uses both X and Y *is*

$$p(A, B) = p(B) \times p(A|B) = 0.5 \times 0.365 = 0.1825$$

b.

The probability of that random interviewed person uses Y, known that doesn't use X *is*

$$p(B|\bar{A}) = \frac{p(\bar{A}|B) \times p(B)}{p(\bar{A})} = \frac{p(\bar{A}|B) \times 0,5}{1 - p(\bar{A})} \tag{2}$$

Meanwhile, we have:

$$\begin{aligned} p(\bar{A}|B) &= \frac{p(\bar{A}B)}{p(B)} \\ &= \frac{p(B) - p(AB)}{p(B)} \\ &= \frac{0.5 - 0.1825}{0.5} \\ &= 0.635 \end{aligned} \tag{3}$$

Substitute (3) to (2), we get:

$$\begin{aligned} p(B|\bar{A}) &= \frac{0.635 \times 0.5}{1 - 0.207} \\ &= 0.4004 \end{aligned} \tag{4}$$

Result: $p(B|\bar{A}) = 0.4004$

Problem 4

Prove the relationship:

$$V_X = E_X[x^2] - (E_X[x])^2$$

As you know, variance is defined as the expected squared difference between a random variable and the mean a.k.a expected value:

$$Var(X) = E[(X - \mu)^2]$$

Then,

$$
\begin{aligned}
V_X &= E_X[(x-\mu)^2] \\
&= E_X[(x-E_X[x])^2] \\
&= E_X[(x-E_X[x]) \times (x-E_X[x])] \\
&= E_X[x^2 - 2 \times x \times E_X[x] + (E_X[x])^2] \\
&= E_X[x^2] - 2 \times E_x[x \times E_X[x]] + (E_X[x])^2 \\
&= E_X[x^2] - 2 \times E_X[x] \times E_X[x] + (E_X[x])^2 \\
&= E_X[x^2] - (E_X[x])^2
\end{aligned} \tag{5}
$$

We have proved the **LHS** = **RHS**

Problem 5

Assumption:
- one door has car
- the two remaining have goat
- these doors are equally likely
- the contestant has no idea which one has car but Monty knows which is which
- assuming that you, the contestant want the car not the goat

First, **don't switch case**

Let S: "*success*"
$\Longrightarrow$ p(S)= $\frac{1}{3}$

So whatever door you choose, the probability of getting the car is $\frac{1}{3}$ if you don't switch.

Second, **switch case**

Assume you will always switch, then Let **S'**: "success when *switched*"
Let D_j: 'Door j has car' (j = 1,2,3)

By sum rule:

$$
\begin{aligned}
p(S') &= p(S'|D_1) \times p(D_1) + p(S'|D_2) \times p(D_2) + p(S'|D_3) \times p(D_3) \\
&= p(S'|D_1) \times \frac{1}{3} + p(S'|D_2) \times \frac{1}{3} + p(S'|D_3) \times \frac{1}{3} \\
&= 0 + 1 \times \frac{1}{3} + 1 \times \frac{1}{3} \\
&= \frac{2}{3}
\end{aligned} \tag{6}
$$

In first case of D_j, $p(S'|D_1) = 0$ because, when we initially pick D_1, The car is in D_1 (conditionally), Monty Hall knows where the car is, he's gonna open D_2 or D_3 to reveal a goat, **then we switch**, we're **failed**.

Next, $p(S'|D_2) = 1$ because, when we initially pick D_1, The car is in D_2 (conditionally), Monty Hall knows where the car is, he's gonna open D_3 to reveal a goat, **then we switch to** D_2, we **succeed**.

Lastly, $p(S'|D_3) = 1$ because, when we initially pick D_1, The car is in D_3 (conditionally), Monty Hall knows where the car is, he's gonna open D_2 to reveal a goat, **then we switch to** D_3, we **succeed**.

Comparing of $notswitching$ to $switching$, $p(S) < p(S')$ ($\frac{1}{3} < \frac{2}{3}$), the chance of success when we switch is bigger, ***you should switch if you want to get a car instead of a goat.***