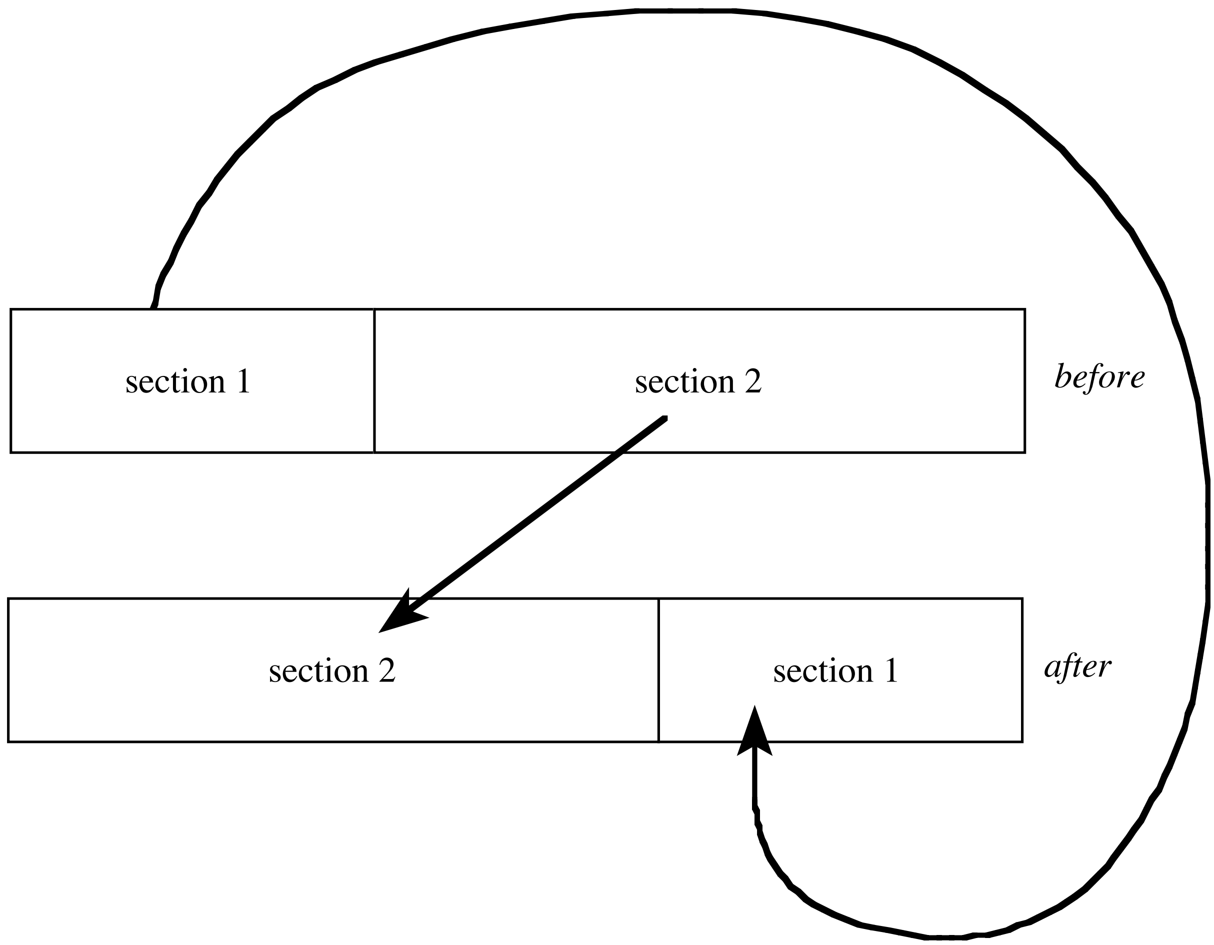
Saving space: a circular shift algorithm.

Consider the problem of shifting the elements of a large array ‘circularly’ by some significant distance.



I emphasise size and distance, because this is fundamentally a problem about *space*, and it becomes interesting only when we have a large problem.

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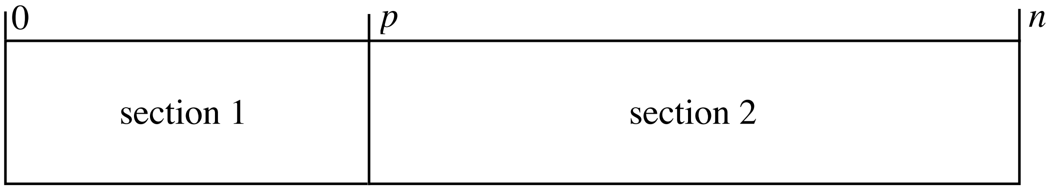
I presume that we have some position p at which the array should be divided:



The predicate calculus specification is straightforward:

$$\forall i \begin{pmatrix} (0 \le i \le p \to A'[n-p+i] = A[i]) \wedge \\ (p \le i < n \to A'[i-p] = A[i]) \end{pmatrix}$$

This is not at all a mysterious program to write, if we have a spare array to hand:

```
type[] B = new type(A.length);
for (i=0; i<p; i++) B[A.length-p+i]=A[i];
for (i=p; i<n; i++) B[i-p]=A[i];
for (i=0; i<A.length; i++) A[i]=B[i];
```

This program is $O(N)$ in time and $O(N)$ in space.

But we may not always have that much space.

The space problems can be reduced a little:

```
type[] B = new type(p);
for (i=0; i<p; i++) B[i]=A[i];
for (i=p; i<n; i++) A[i-p]=A[i];
for (i=0; i<p; i++) A[A.length-p+i]=B[i];
```

Now it's $O(p)$ in space, and a little quicker in execution (less copying). We have a better bound on the time: it's $O(N+p)$.

But we still have a program which uses too much space: in the worst case p can be close to N.

We might reduce the worst case space usage to $N/2$, but this program will always have a space problem.

There is a better way.

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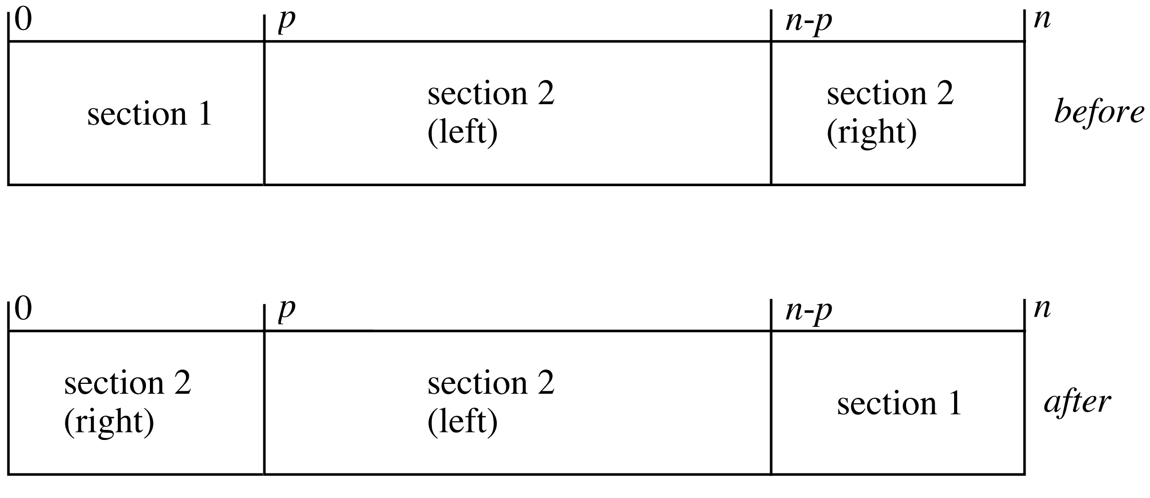
Trading speed for space.

I shall abandon, for a while, the search for a faster solution.

We can save space by moving things around more often.

Suppose that $p \leq n - p$: that is, the left section is the smaller.

Then we might begin by swapping $A_{0..p-1}$ with $A_{n-p..n-1}$:



We can do that work using only one extra variable (to implement the swap operation):

```
for (i=0; i<p; i++) A[i]<->A[n-p+i];
```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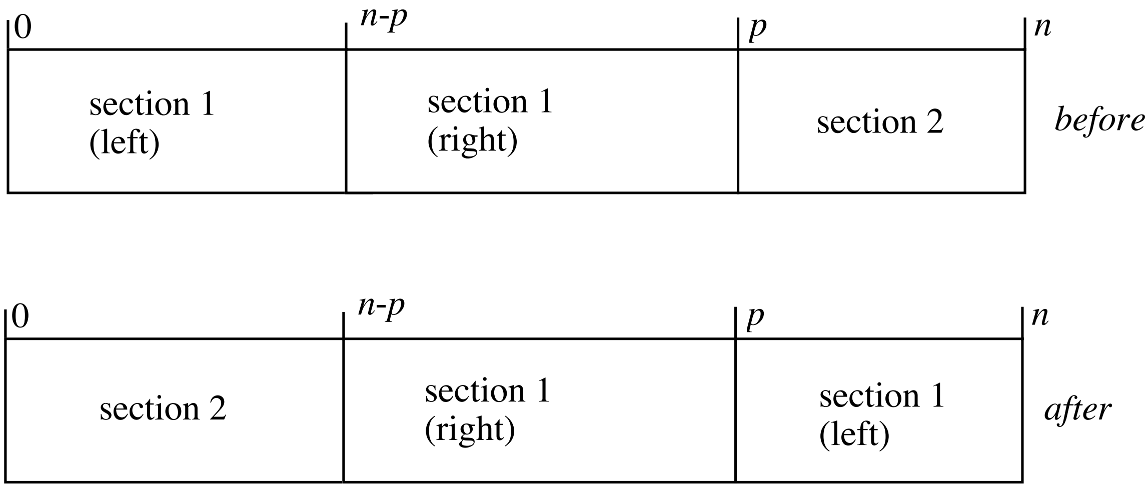
Now of course if section 2 is the smaller, it won't work because of overlap: but then we can do something very similar to swap section 2 into place:




```
for (i=0; i<n-p; i++) A[i]<->A[p+i];
```

In either case we have reduced the problem to that of reordering the left and right parts of section 2 (first case) or section 1 (second case) – clearly a case for repetition.

Here's the whole program. Amazingly enough the end-limits *m* and *n* vary, but the boundary *p* always stays in the same place!

```
for (m=0, n=A.length; m!=p && n!=p; ) {
  if (p-m<=n-p) { // shift section 1
    for (i=0; i<p-m; i++) A[i+m]<->A[n-p+i];
    n=n-(p-m); // section 1 is in place
  }
  else { // shift section 2
    for (i=0; i<n-p; i++) A[i+m]<->A[p+i];
    m=m+(n-p); // section 2 is in place
  }
}
```

This program doesn't use much space – $O(1)$, because of the variables i, m, n and p, plus the variable needed for the swaps – but it does a lot too much work.

Each swap takes three assignments; each time we shift a section into place we put a similarly-sized section in the wrong place (unless p divides the interval $m..n$ exactly in half).

We can do better ...

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A perfect circular shift.

What should move into A_0? Why, A_p. And what should move into A_p? Why, A_{2p} ... and so on, till we fall off the end of the array because $j \times p > n$.

We don't have to stop there. A_i should be replaced by A_{i+p}, if that's within the array, or else by A_{i+p-n} – because it is a circular shift! And so on, till we get back to A_0 again.

In a complicated multi-way exchange you only need on temporary variable! Here's a bit of program which does the job:

```
type t=A[0];
for (i=0, j=p;
     j!=0;
     i=j, j = j+p<n ? j+p : j+p-n)
  A[i]=A[j];
A[i]=t;
```

This program moves quite a bit of the array around, and it only uses variables *i*, *j* and *t*.

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But if p divides n exactly this program won't do the whole problem: if $p = n \div 2$ it only exchanges A_0 and A_p; if $p = n \div 3$ it only rotates A_0, A_p and A_{2p}; and so on.

And if p and n have factors in common this program won't solve the whole problem. In fact if the greatest common divisor of p and n is q then this program will move exactly $n \div q$ things. But then the nice thing is that we can use the same idea, starting again with A_1 ...

Here's the complete program:

```
for (m=0, count=0; count!=n; m++) {
  type t=A[m];
  for (i=m, j=m+p;
       j!=m;
       i=j, j = j+p<n ? j+p : j+p-n, count++)
    A[i]=A[j];
  A[i]=t; count++;
}
```

That program only uses variables i, j and t; it does $O(n)$ assignments; it does the 'extra' assignment `t=A[m]` only $\gcd(n,p)$ times.

If $p = n \div 2$ then it has no advantage over the earlier segment-swapping program, but in all other cases it does a lot less work.

A proof that it works is remarkably difficult ...