Introduction to Risk Management

Javier Contreras ©, Luis Baringo ©, UCLM

Department of Electrical Engineering and Automation, Aalto University, Espoo, Finland

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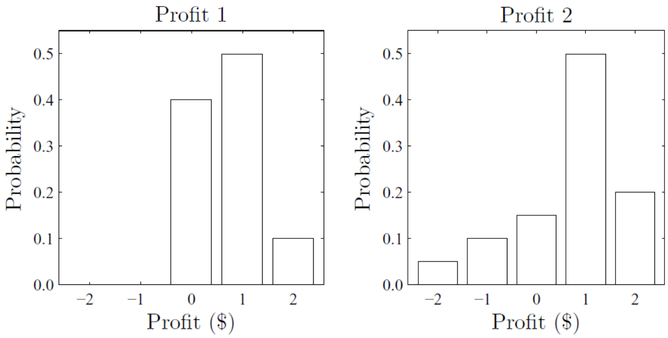
Table of contents

Introduction



Risk-neutral decision making

The general formulation of a two-stage stochastic programming problem is as follows:

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega)} \quad \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \sum_{\omega\in\Omega} \pi(\omega)\,\boldsymbol{q}^{\mathrm{T}}(\omega)\,\boldsymbol{y}(\omega) \tag{1}$$

$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{2}$$

$$\boldsymbol{T}(\omega)\,\boldsymbol{x} + \boldsymbol{W}(\omega)\,\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega)\,, \forall\omega\in\Omega \tag{3}$$

$$\boldsymbol{x}\in X, \boldsymbol{y}(\omega)\in Y, \forall\omega\in\Omega \tag{4}$$

Risk-neutral decision making

Defining

$$f(\boldsymbol{x},\omega) = \boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} \\ + \max_{\boldsymbol{y}(\omega)} \left\{ \boldsymbol{q}^{\mathrm{T}}(\omega)\,\boldsymbol{y}(\omega) : \boldsymbol{T}(\omega)\,\boldsymbol{x} + \boldsymbol{W}(\omega)\,\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega)\,, \boldsymbol{y}(\omega) \in Y \right\}, \tag{5}$$

it is possible to reformulate problem (1)-(4) using the compact form below:

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega)} \quad \mathrm{E}\{f(\boldsymbol{x},\omega)\} \tag{6}$$

$$\text{subject to} \quad \boldsymbol{x} \in X, \forall \omega \in \Omega \tag{7}$$

The objective of problem (1)-(4) is to maximize the expected value of the function $f(\boldsymbol{x},\omega)$

Risk-neutral decision making

Since vectors of variables $\boldsymbol{x}$ and $\boldsymbol{y}(\omega)$ are obtained by maximizing the expected profit without modeling the risk, we denote problem (1)-(4) and the equivalent problem (6)-(7) as **risk-neutral** problems

Example 1

Consider the problem faced by an electricity retailer that seeks to determine the purchases in the futures market in order to maximize the profit resulting from selling energy to a group of clients. In doing so, the retailer buys energy in both the pool and a futures market.
The price of the energy (in \$/MWh) in the pool in each period t is assumed to be unknown and is characterized as a random variable, which is modeled using a set of scenarios, $\lambda^{\mathrm{P}}_{t\omega}$:

Scenario	Period			Scenario	Period		
	1	2	3		1	2	3
1	28.5	38.6	31.4	6	29.2	34.6	31.2
2	27.3	37.5	29.6	7	34.1	36.9	35.4
3	29.4	35.7	31.3	8	33.4	35.4	34.9
4	33.9	35.4	35.1	9	28.4	36.3	32.9
5	34.5	38.9	37.5	10	27.6	38.9	32.1

All scenarios have the same probability π_ω.

Example 1

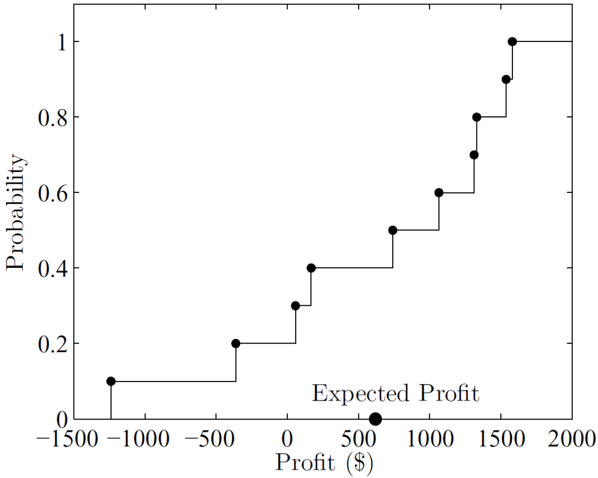
The retailer participates in the futures market by buying energy through three different contracts, $f = 1, 2, 3$:

Contract	Price, λ_f^{F}, [\$/MWh]	Maximum quantity, $X_f^{\max}$ [MW]
1	34	50
2	35	30
3	36	25

The demand of the consumers, P_t^{C} is assumed to be known and equal to 150, 225, and 175 MW in the first, second, and third periods, respectively. The selling price of the energy, λ^{C} to the clients is fixed to \$35/MWh. Determine the power purchased from forward contracts, x_f, and the power purchased from the pool, $y_{t\omega}$, that maximize the retailer's expected profit

Example 1

Profit cdf:



Risk-averse decision making

- The main disadvantage of ignoring risk in risk-neutral problem (1)-(4) is that the optimal values of variables $\boldsymbol{x}$ and $\boldsymbol{y}(\omega)$ may lead to the maximum expected profit at the expense of experiencing very low profits in some unfavorable scenarios
- To avoid such situations, it is advisable to include in the formulation of the problem a term modeling the risk of variability associated with the profit $f(\boldsymbol{x}, \omega)$
- Function assigns to a given random variable representing profit, $r_\omega\{f(\boldsymbol{x}, \omega)\}$, $\forall\omega \in \Omega$, a real number characterizing the risk associated with that profit
- Function $r_\omega\{f(\boldsymbol{x}, \omega)\}$ is referred to as **risk measure**

Risk-averse decision making

Consider the following two-stage stochastic programming problem:

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega)} \quad \mathrm{E}\left\{f\left(\boldsymbol{x},\omega\right)\right\} - \beta r_{\omega}\left\{f\left(\boldsymbol{x},\omega\right)\right\} \tag{8}$$

$$\text{subject to} \quad \boldsymbol{x} \in X \tag{9}$$

where $\beta \in [0, \infty]$ is a weighting parameter used to materialize the tradeoff between expected profit and risk aversion

- If $\beta = 0$, the risk term in the objective function is neglected and the resulting problem becomes the risk-neutral one
- As β increases, the expected profit term becomes less significant with respect to the risk term

Risk-averse decision making

The risk faced by the decision maker can be also controlled by including the risk measure as an additional constraint:

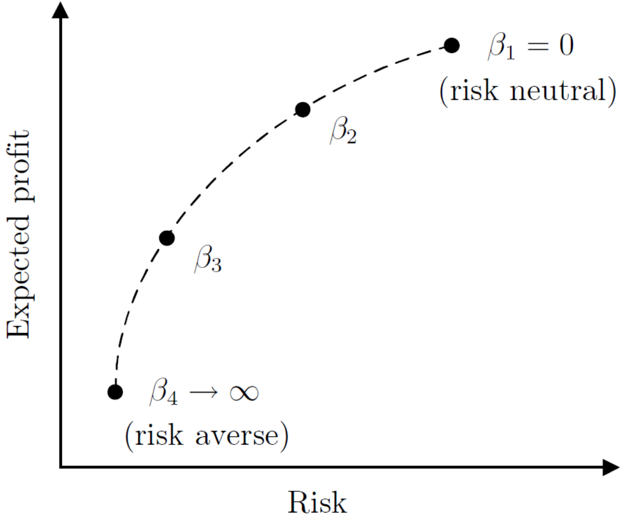
$$
\begin{aligned}
\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega)} \quad & \mathrm{E}\left\{f\left(\boldsymbol{x},\omega\right)\right\} && (10)\\
\text{subject to} \quad & \boldsymbol{x} \in X && (11)\\
& \beta r_{\omega}\left\{f\left(\boldsymbol{x},\omega\right)\right\} \leq \delta && (12)
\end{aligned}
$$

where δ represents the maximum risk that the decision maker is willing to take

Risk-averse decision making

- The optimal solution obtained from solving risk-averse problems (8)-(9) or (10)-(12) depends on the value of parameters β/δ
- This optimal solution, in terms of expected profit and risk for a given value of parameters β/δ, defines an efficient point
- A solution with greater expected profit than that of an efficient point can only be obtained at the cost of experiencing a higher risk, and viceversa
- A collection of efficient points (obtained for different values of β/δ) defines an efficient frontier

Efficient frontier



Efficient frontier

- Efficient frontiers are relevant instruments used by decision makers to resolve the tradeoff between expected profit and risk
- Efficient frontiers are composed of a finite set of efficient points (thus, they are not continuous)
- The resulting line from joining efficient points is not necessarily either convex or concave

Risk measures

- Needed for characterizing the risk associated with a given decision
- They enable us to compare two different decisions in terms of the risk involved
- Different risk measures for different applications
- Properties of *coherent risk measures*:
    1. Translation invariance
    2. Subadditivity
    3. Positive homogeneity
    4. Monotonicity

Risk measures

Let us consider two possible random outcomes $f_1(\omega)$ and $f_2(\omega) \in F$ and a risk measure $r_\omega\{f(\omega)\}$:

1. Translation invariance. For all $f_1(\omega) \in F$ and all real numbers a, it holds that $r_\omega\{f_1(\omega) + a\} = r_\omega\{f_1(\omega)\} + a$
2. Subadditivity. For all $f_1(\omega)$, $f_2(\omega) \in F$, it is satisfied that $r_\omega\{f_1(\omega) + f_2(\omega)\} \leq r_\omega\{f_1(\omega)\} + r_\omega\{f_2(\omega)\}$
3. Positive homogeneity. For all $f_1(\omega) \in F$ and all real numbers a, it is verified that $r_\omega\{af_1(\omega)\} = ar_\omega\{f_1(\omega)\}$
4. Monotonicity. For all $f_1(\omega)$, $f_2(\omega) \in F$, if $f_1(\omega) \geq f_2(\omega)$, then $r_\omega\{f_1(\omega)\} \leq r_\omega\{f_2(\omega)\}$

Variance

- The mean-variance model considers that a decision can be characterized by two parameters: the expected return (expected profit/cost) and the variance of this return
- The variance, as a dispersion measure, is used to model the risk faced by the decision maker
- A large variance indicates that there exists a high risk of experiencing a profit different from the expected one
- Considering the profit $f(\boldsymbol{x}, \omega)$, the variance can be formulated as:

$$V(\boldsymbol{x}) = \mathrm{E}_{\omega}\left\{\left(f(\boldsymbol{x}, \omega) - \mathrm{E}_{\omega}\{f(\boldsymbol{x}, \omega)\}\right)^2\right\} \tag{13}$$

Variance

The variance can be incorporated into the risk-neutral problem (1)-(4) as shown below:

$$\begin{aligned} \text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega)} \quad & (1-\beta)\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \sum_{\omega\in\Omega}\pi(\omega)\,\boldsymbol{q}^{\mathrm{T}}(\omega)\,\boldsymbol{y}(\omega)\right) \\ & - \beta\sum_{\omega\in\Omega}\pi(\omega)\left(f(\boldsymbol{x},\omega) - \sum_{\omega'\in\Omega}\pi(\omega')\,f(\boldsymbol{x},\omega')\right)^2 \end{aligned} \quad (14)$$

$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \quad (15)$$

$$\boldsymbol{T}(\omega)\,\boldsymbol{x} + \boldsymbol{W}(\omega)\,\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega)\,, \forall\omega\in\Omega \quad (16)$$

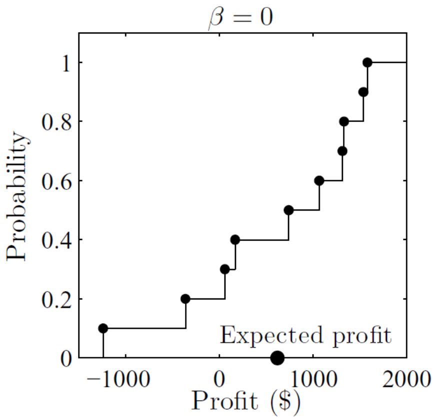
$$\boldsymbol{x}\in X, \boldsymbol{y}(\omega)\in Y, \forall\omega\in\Omega \quad (17)$$

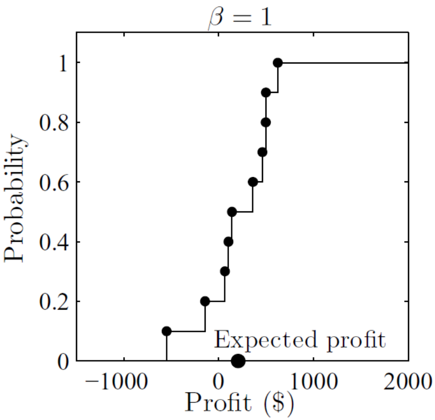
Example 2

Include the variance in the risk-neutral problem of Example 1

Example 2

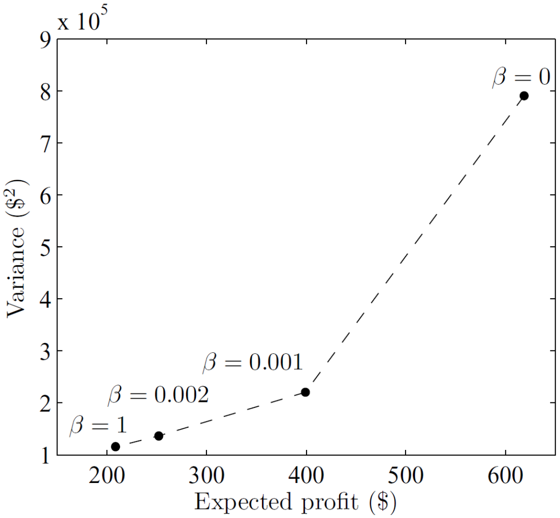
cdfs for $\beta = 0$ and $\beta = 1$:





Example 2

Efficient frontier:



Value-at-risk

- For a given $\alpha \in (0, 1)$, the value-at-risk (VaR), is equal to the largest value η ensuring that the probability of obtaining a profit less than η is lower than $1 - \alpha$
- The VaR$(\alpha, \boldsymbol{x})$ is the $(1 - \alpha)$-quantile of the profit distribution
- Mathematically:

$$\mathrm{VaR}\left(\alpha, \boldsymbol{x}\right) = \max\left\{\eta : P\left(\omega | f\left(\boldsymbol{x}, \omega\right) < \eta\right) \leq 1 - \alpha\right\}, \forall \alpha \in (0, 1) \quad (18)$$

- Note that η is not a given parameter, but the risk measure associated with the random variable representing the profit

Value-at-risk

The VaR($\alpha, \boldsymbol{x}$) can be incorporated into the risk-neutral problem (1)-(4):

$$\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega)} \quad (1-\beta)\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \sum_{\omega\in\Omega}\pi(\omega)\,\boldsymbol{q}^{\mathrm{T}}(\omega)\,\boldsymbol{y}(\omega)\right) + \beta\eta \tag{19}$$

$$\text{subject to} \quad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \tag{20}$$

$$\boldsymbol{T}(\omega)\,\boldsymbol{x} + \boldsymbol{W}(\omega)\,\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega)\,, \forall\omega\in\Omega \tag{21}$$

$$\sum_{\omega\in\Omega}\pi(\omega)\,\theta(\omega) \leq 1-\alpha \tag{22}$$

$$\eta - \left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{q}^{\mathrm{T}}(\omega)\,\boldsymbol{y}(\omega)\right) \leq M\theta(\omega)\,, \forall\omega\in\Omega \tag{23}$$

$$\theta(\omega)\,\{0,1\}\,, \forall\omega\in\Omega \tag{24}$$

$$\boldsymbol{x}\in X, \boldsymbol{y}(\omega)\in Y, \forall\omega\in\Omega \tag{25}$$

where η is a variable whose optimal value is equal to the VaR($\alpha, \boldsymbol{x}$), $\theta(\omega)$ is a binary variable that is equal to 1 if the profit in scenario ω is less than η (and equal to 0 otherwise) and M is a large enough constant

Value-at-risk

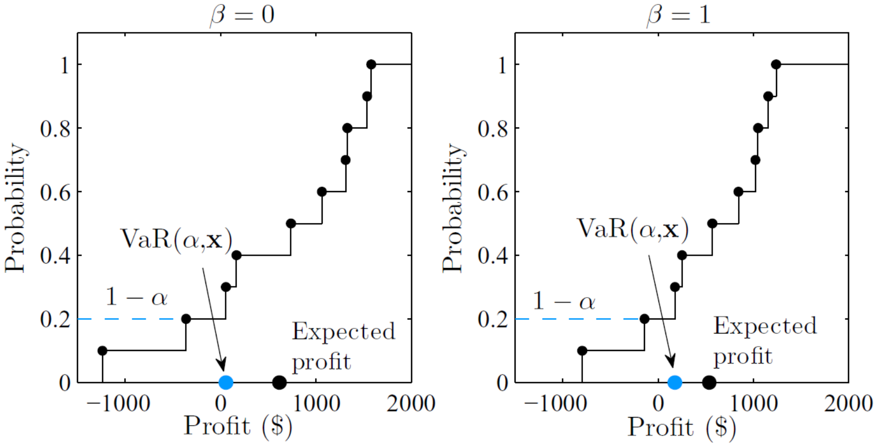
- A serious shortcoming of the VaR is that it gives no information about the profit distribution beyond its value
- Thus, a *fat tail* appearing in profit distributions is not detected by the VaR

Example 3

Include the VaR in the risk-neutral problem of Example 1

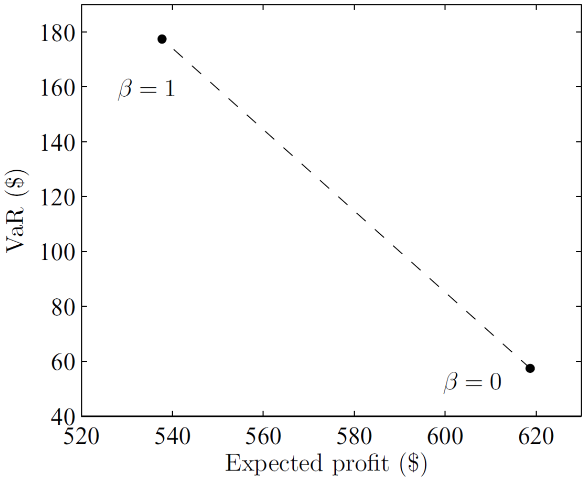
Example 3

cdfs for $\beta = 0$ and $\beta = 1$:



Example 3

Efficient frontier:



Conditional value-at-risk

- For a given $\alpha \in (0, 1)$, the conditional value-at-risk (CVaR) is defined as the expected value of the profit smaller than the $(1 - \alpha)$-quantile of the profit distribution
- If all profit scenarios are equiprobable, the $\text{CVaR}(\alpha, \boldsymbol{x})$ is computed as the expected profit in the $(1 - \alpha) \times 100\%$ worst scenarios

Conditional value-at-risk

The $\text{CVaR}(\alpha, \boldsymbol{x})$ can be incorporated into the risk-neutral problem (1)-(4):

$$\begin{aligned}
\text{Maximize}_{\boldsymbol{x},\boldsymbol{y}(\omega)} \quad & (1-\beta)\left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \sum_{\omega\in\Omega}\pi(\omega)\,\boldsymbol{q}^{\mathrm{T}}(\omega)\,\boldsymbol{y}(\omega)\right) \\
& + \beta\left(\eta - \frac{1}{1-\alpha}\sum_{\omega\in\Omega}\pi(\omega)\,s(\omega)\right) && (26) \\
\text{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} && (27) \\
& \boldsymbol{T}(\omega)\,\boldsymbol{x} + \boldsymbol{W}(\omega)\,\boldsymbol{y}(\omega) = \boldsymbol{h}(\omega), \forall\omega\in\Omega && (28) \\
& \eta - \left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{q}^{\mathrm{T}}(\omega)\,\boldsymbol{y}(\omega)\right) \le s(\omega), \forall\omega\in\Omega && (29) \\
& s(\omega) \ge 0, \forall\omega\in\Omega && (30) \\
& \boldsymbol{x}\in X, \boldsymbol{y}(\omega)\in Y, \forall\omega\in\Omega && (31)
\end{aligned}$$

where η is an auxiliary variable and $s(\omega)$ is a continuous variable equal to the maximum of $\eta - \left(\boldsymbol{c}^{\mathrm{T}}\boldsymbol{x} + \boldsymbol{q}^{\mathrm{T}}(\omega)\,\boldsymbol{y}(\omega)\right)$ and 0

Conditional value-at-risk

The advantages of the CVaR with respect to the VaR are twofold:

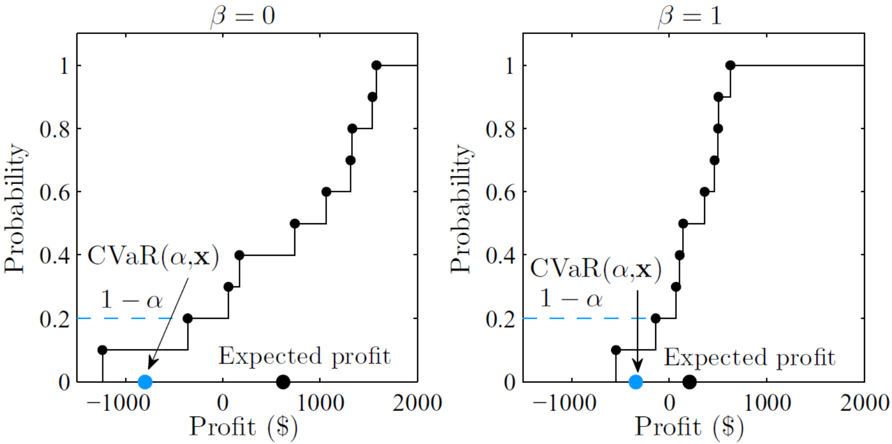
1. No binary variables are needed for its calculation
2. It is able to quantify *fat tails* beyond the VaR

Example 4

Include the CVaR in the risk-neutral problem of Example 1

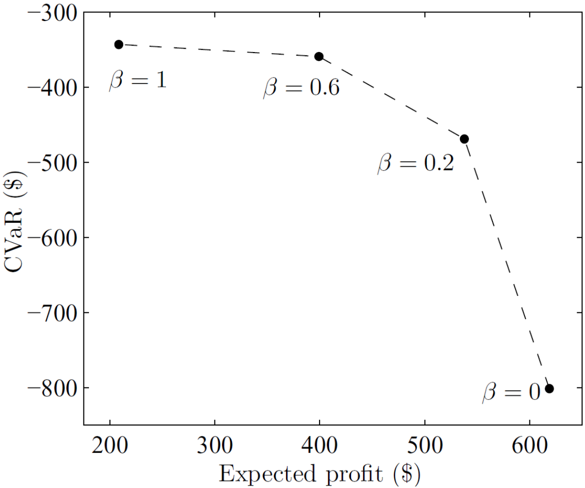
Example 4

cdfs for $\beta = 0$ and $\beta = 1$:



Example 4

Efficient frontier:



References

1. A.J. Conejo, M. Carrión, J.M. Morales. Decision-making under uncertainty in electricity markets. Springer, New York, 2010