MS-E2121 - Linear optimization **DEADLINE:**
Homework 2 **14.02.2021 @23:55**

Please submit your answers to this page before the given deadline. Late submissions will be penalised by 5 points per day after the deadline.

Problem 2.1: Caratheodory theorem

Let $A_1, ..., A_n$ be a collection of vectors in $\mathbb{R}^m$.

(a) Let

$$C = \{\sum_{i=1}^{n} \lambda_i A_i : \lambda_1, ..., \lambda_n \geq 0\}$$

Show that any element of C can be expressed in the form $\sum_{i=1}^{n} \lambda_i A_i$, with $\lambda_i \geq 0$, and with at most m of the coefficients λ_i being nonzero.

Hint: Consider the polyhedron

$$\Lambda = \{(\lambda_1, ..., \lambda_n) \in \mathbb{R}^n : \sum_{i=1}^{n} \lambda_i A_i = y \ , \ \lambda_1, ..., \lambda_n \geq 0\}$$

and the properties of a basic feasible solution.

(b) Let P be the convex hull of the vectors A_i:

$$P = \{\sum_{i=1}^{n} \lambda_i A_i : \sum_{i=1}^{n} \lambda_i = 1 \ , \ \lambda_1, ..., \lambda_n \geq 0\}$$

Show that any element of P can be expressed in the form $\sum_{i=1}^{n} \lambda_i A_i$, where $\sum_{i=1}^{n} \lambda_i = 1$ and $\lambda_i \geq 0$ for all i with at most $m+1$ of the coefficients λ_i being nonzero.

Problem 2.2: Local minima of convex functions

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function (that is, $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$, $\lambda \in [0,1]$) and let $S \subset \mathbb{R}^n$ be a convex set. Let $\mathbf{x}^*$ be an element of S. Suppose that $\mathbf{x}^*$ is a local optimum for the problem of minimizing $f(\mathbf{x})$ over S; that is, there exists some $\epsilon > 0$ such that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$ for which $\|\mathbf{x} - \mathbf{x}^*\| \leq \epsilon$. Prove that $\mathbf{x}^*$ is globally optimal; that is, $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in S$.

Problem 2.3: Optimality conditions

Consider the problem of minimizing $\mathbf{c}^\top \mathbf{x}$ over a polyhedron P. Prove the following:

(a) A feasible solution $\mathbf{x}$ is optimal if and only if $\mathbf{c}^\top \mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.
(b) A feasible solution $\mathbf{x}$ is unique optimal solution if and only if $\mathbf{c}^\top \mathbf{d} > 0$ for every nonzero feasible direction $\mathbf{d}$ at $\mathbf{x}$.

Problem 2.4: Conditions for unique optima

Let $\mathbf{x}$ be a basic feasible solution associated with some basis matrix $\mathbf{B}$. Prove the following:

(a) If the reduced cost of every nonbasic variable is positive, then $\mathbf{x}$ is the unique optimal solution.
(b) If $\mathbf{x}$ is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.