Noether's Theorem for Classical and Quantum Mechanics

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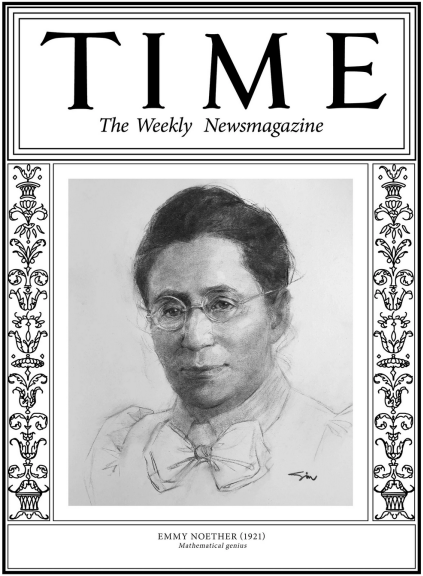
Emmy Noether



Noether's Theorem

"Associated to every symmetry of a system, there is a conserved quantity"

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We shall soon see the definition of the terms observables and generators, and how Noether's theorem can be expressed algebraically.

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Noether's theorem is most commonly proved using techniques from topology and analysis, and relies heavily on the fact that the laws of motion derive from a Lagrangian.
However, if we start with an algebraic setting that satisfies certain conditions (that are easily met for ordinary classical mechanics and complex quantum mechanics), Noether's theorem is almost a tautology.

Algebraic structures

Definition: A *Lie algebra* is a vector space L together with an operation called the Lie bracket
$[\ ,\ ] : L \times L \to L$, that is alternating($[a, a] = 0$),
bilinear($[a, b + c] = [a, b] + [a, c]$) and satisfies the Jacobi identity
$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 \ \forall a, b, c \in L$

Definition: An *Associative algebra* A is an algebraic structure with compatible operations of associative addition, associative multiplication, and a scalar multiplication by elements in some field K. The addition and multiplication operations together give A the structure of a ring; the addition and scalar multiplication operations together give A the structure of a vector space over K.

Algebraic structures

Definition: A *Poisson algebra* is a vector space over a field K equipped with two bilinear products: $\cdot$ and $\{\ ,\ \}$ having the following properties:

- The product $\cdot$ forms an associative algebra over the field K
- The Poisson bracket $\{\ ,\ \}$ forms a Lie Algebra
- $\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$

Definition: A *complex *-algebra* is an associative algebra S over $\mathbb{C}$ with an operation $* : S \to S$ s.t. $(\alpha a)^* = \overline{\alpha} a^* \ \forall \alpha \in \mathbb{C}, a \in S$ where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$

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$$\Longleftrightarrow$$

The generator b generates transformations which leave the observable a fixed

Classical Mechanics

The observables obey **Hamilton's equations**, and form a **Poisson algebra**.

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It is also easy to see that the maps F_t^o act as symmetry generators for the Poisson algebra:

$$F_t^o(\alpha a + \beta b) = F_t^o(\alpha a) + F_t^o(\beta b)$$
$$F_t^o(a \cdot b) = F_t^o(a) \cdot F_t^o(b)$$
$$F_t^o(\{a, b\}) = \{F_t^o(a), F_t^o(b)\}$$

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$$\frac{d}{dt} F_t^o(b) = \{o, F_t^o(b)\} \tag{1}$$

The Main Realisation

When we take the observable o to be the Hamiltonian (h) of the system, F_t^h describes the time evolution of the observables.

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$$F_t^h(b) = b \ \forall t \in \mathbb{R} \iff F_t^b(h) = h \ \forall t \in \mathbb{R} \tag{4}$$

Proof

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and form a **complex *- algebra**. Here, the Poisson bracket $\{\ ,\ \}$ is replaced by the commutator bracket $[\ ,\ ]$, and we get a similar proof for Noether's theorem.

Why does Noether's Theorem hold?

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Thus, the fact that nature follows abstract algebraic structures that allow simple manipulations, forms the heart of Noether's theorem.
There are examples of mechanical theories which do not adhere to these structures, and as a consequence, no analogue of Noether's theorem exists for them.

Why use algebra?

A lot of the assumptions we made require topological justification. However, if we *begin* by postulating that the existence of one-parameter groups, Noether's theorem is almost self-evident.

The Faustian Bargain

"Algebra is the offer made by the devil to the mathematician. The devil says: I will give you this powerful machine, it will answer any question you like. All you need to do is give me your soul: give up geometry and you will have this marvelous machine."

— Michael Francis Atiyah