Noether's Theorem for Classical and Quantum Mechanics

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Abstract

It is well-known that the symmetries present in a physical system have interesting consequences on the dynamics of the observables associated with that system. Most commonly, when talking about symmetries one encounters Noether's theorem, which when simply stated, implies that symmetries lead to conserved quantities. There are various proofs of this theorem in the context of different physical theories. In this paper, we shall examine the precise algebraic statement of the theorem in the context of Classical mechanics and Quantum mechanics. We shall see how group theory and abstract algebra form a natural setting for studying these symmetries.

1 Introduction

We begin by highlighting the difference between observables and generators. An observable is a real valued quantity whose value depends on the state that the system is in. The set of all observables is most often a group. Generators on the other hand, generate symmetries of the system (which are observables themselves), and thus form a group too.

An observable can be thought of as an object that 'acts on' the state of the system to pluck out a real value, whereas a generator is something that gives rise to these objects that perform this task. In most cases, observables act as generators themselves. In particular, Noether's theorem inherently requires the interchangeability of observables and generators.

In classical mechanics, we work with a structure known as a **Poisson algebra**, whose elements are observables as well as symmetry generators. In quantum mechanics however, observables are operators on a Hilbert space that are Hermitian, i.e $\hat{O}^* = \hat{O}$, whereas generators are skew-Hermitian, with $\hat{O}^* = -\hat{O}$. To ensure interchangeability of observables and generators, we use a **complex *-algebra**. To turn observables into symmetry generators, we multiply by i, and vice-versa.

2 Mathematical Preliminaries

Definition: A *Lie algebra* is a vector space L together with an operation called the Lie bracket
$[\ ,\ ] : L \text{ x } L \rightarrow L$, that is alternating($[a,a] = 0$), bilinear($[a, b+c] = [a,b] + [a,c]$) and satisfies the Jacobi identity $[a,[b,c] + [b,[c,a]] + [c,[a,b]] = 0 \ \forall a,b,c \in L$

Definition: An *Associative algebra* A is an algebraic structure with compatible operations of associative addition, associative multiplication, and a scalar multiplication by elements in some field K. The addition and multiplication operations together give A the structure of a ring; the addition and scalar multiplication operations together give A the structure of a vector space over K.

Definition: A *Poisson algebra* is a vector space over a field K equipped with two bilinear products: $\cdot$ and $\{\ ,\ \}$ having the following properties:

- The product $\cdot$ forms an associative algebra over the field K
- The Poisson bracket $\{\ ,\ \}$ forms a Lie Algebra
- $\{x, y \cdot z\} = \{x, y\} \cdot z + y \cdot \{x, z\}$

Definition: A *complex *-algebra* is an associative algebra S over $\mathbb{C}$ with an operation $* : S \rightarrow S$ s.t. $(\alpha a)^* = \overline{\alpha} a^* \ \forall \alpha \in \mathbb{C}, a \in S$ where $\overline{z}$ is the complex conjugate of $z \in \mathbb{C}$

3 Classical mechanics

We see that in classical mechanics, the observables obey Hamilton's equations, and thus end up forming a **Poisson algebra**. Let us call this Poisson Algebra P, and consider some observable $o \in P$.

We define an important **one-parameter group** associated with this observable (the parameter being time $t \in \mathbb{R}$) as follows: $F^o_t : P \rightarrow P$ at every time $t \in \mathbb{R}$ associates one observable to another observable, such that $F^o_{t=0}(b) = b$ and $F^o_{t_1}(F^o_{t_2}(b)) = F^o_{t_1+t_2}(b)$.

It is straightforward to see that the set of all F^o_t form a **group**. The group operation is composition, which automatically leads to closure under the group operation. Associativity follows from the fact that $F^o_{t_1}(F^o_{t_2}(b)) = F^o_{t_1+t_2}(b)$. The identity element for each group element is $F^o_{t=0}$, and the inverse of every F^o_t is F^o_{-t}. It is also easy to see that the maps F^o_t act as symmetry generators for the Poisson algebra:

$$F^o_t(\alpha a + \beta b) = F^o_t(\alpha a) + F^o_t(\beta b)$$
$$F^o_t(a \cdot b) = F^o_t(a) \cdot F^o_t(b)$$
$$F^o_t(\{a, b\}) = \{F^o_t(a), F^o_t(b)\}$$

For our case, we start with the assumption that for each observable o, there exists a unique one-parameter group of maps $F_t^o : P \to P$ such that the following equation, known as Hamilton's equation, is satisfied:

$$\frac{d}{dt} F_t^o(b) = \{o, F_t^o(b)\} \tag{1}$$

where $\{a, b\}$ is the Poisson bracket associated with the Poisson algebra.

It is not necessary for such a one-parameter group to exist for a given system with observables. However, under certain conditions (which can be described analytically), we can be assured that there uniquely exists such a one-parameter group associated with each observable.

Once we have this assumption, and we equip the space of observables with a Poisson bracket, the proof of Noether's theorem is actually tautologous. The fact that our physical world is described by Classical mechanics, which happens to agree with these two principles, is the true reason behind why Noether's theorem holds.

In particular, when we take the observable o to be the Hamiltonian (h) of the system, F_t^h describes the time evolution of the observables. We say an observable $b \in P$ is a conserved quantity if

$$F_t^h(b) = b \quad \forall t \in \mathbb{R} \tag{2}$$

and b generates symmetries of the Hamiltonian if

$$F_t^b(h) = h \quad \forall t \in \mathbb{R} \tag{3}$$

Noether's theorem, in this setting, then becomes

$$F_t^h(b) = b \ \forall t \in \mathbb{R} \iff F_t^b(h) = h \ \forall t \in \mathbb{R} \tag{4}$$

Proof:

$$\begin{aligned} & F_t^h(b) = b \ \ \forall t \in \mathbb{R} \\ \iff & \tfrac{d}{dt} F_t^h(b) = 0 \ \ \forall t \in \mathbb{R} \\ \iff & \{h, F_t^h(b)\} = 0 \ \ \forall t \in \mathbb{R} \\ \iff & \{h, b\} = 0 \ \ \forall t \in \mathbb{R} \end{aligned}$$

Proof that $\{h, F_t^h(b)\} = 0 \Rightarrow \{h, b\} = 0$: Since the equation must hold for $t = 0$, we get $\{h, b\} = 0$

Proof that $\{h, b\} = 0 \Rightarrow \{h, F_t^h(b)\} = 0$: By the uniqueness of F_t^h, which satisfies Hamilton's equation (1), we get that $F_t^h(b) = b$ satisfies (1) too, because $\frac{db}{dt} = 0 = \{h, b\}$, and thus $F_t^h(b) \equiv b$ and $\{h, b\} = \{h, F_t^h(b)\} = 0$

Now, we repeat the above procedure for h and b interchanged to obtain that

$$\begin{aligned} & F_t^b(h) = h \quad \forall t \in \mathbb{R} \\ \iff & \{b,h\} = 0 \ \ \forall t \in \mathbb{R} \end{aligned}$$

Once we exploit the antisymmetry of the Poisson bracket, i.e $\{b,h\} = -\{h,b\}$, we are done. $\square$

4 Quantum mechanics

Operators which correspond to observables in quantum mechanics form a **complex *-algebra**. This time, we assume that there exists a **one-parameter family** with the following properties: associated to every operator $o \in S$ where S is the complex *-algebra, we have a unique solution $F_t^o : S \to S$ of Heisenberg's equation:

$$\frac{d}{dt} F_t^o(b) = [io, F_t^o(b)] \tag{5}$$

where $[a,b]$ is the commutator bracket defined as $[a,b] := ab - ba$, and we have as before $F_{t=0}^o(b) = b$ and $F_{t_1}^o(F_{t_2}^o(b)) = F_{t_1+t_2}^o(b)$. Again, F_t^o forms a group since

As before, we take o to be the Hamiltonian (h) of the system and F_t^h describes the time evolution of the operators. Using similar definitions of an observable b being a conserved quantity and of b generating symmetries of the Hamiltonian h, we get the same form of Noether's theorem:

$$F_t^h(b) = b \ \forall t \in \mathbb{R} \iff F_t^b(h) = h \ \forall t \in \mathbb{R} \tag{6}$$

Proof: We follow precisely the same procedure as in the classical case:

$$\begin{aligned} F_t^h(b) = b \quad \forall t \in \mathbb{R} \iff [h,b] = 0 \ \ \forall t \in \mathbb{R} \\ F_t^b(h) = h \quad \forall t \in \mathbb{R} \iff [b,h] = 0 \ \ \forall t \in \mathbb{R} \end{aligned}$$

We then use the antisymmetry of the commutator $[b,h] = -[h,b]$ to conclude the proof. $\square$

5 Examples

Example 1: Let us consider a classical mechanical problem that is time-translationally invariant. This implies that the Hamiltonian has no explicit time-dependence. Let us try to find the associated conserved quantity.

For this system, we have $\{h,t\} = \sum_{i=1}^n \frac{\partial h}{\partial p_i}\frac{\partial t}{\partial q_i} - \frac{\partial h}{\partial q_i}\frac{\partial t}{\partial p_i} = 0$, which tells us that the energy of the system is conserved. This also allows us to conclude that the hamiltonian is the generator of time-translations. For this example, we used the form of the Poisson bracket and had to carry out an explicit computation. However, once we naturally identify the generator to the observable, Noether's theorem becomes very simple to apply.

Example 2: Let us consider a classical mechanical problem that has space-translational symmetry. Let us try to find the associated conserved quantity.

For this situation, we can simply start with the observation that momentum is the generator of space-translations. Thus we can directly conclude that momentum will be the conserved quantity. We can explicitly evaluate the Poisson bracket $\{p, x\}$ to verify this. This makes evident how we can avoid computations by using Noether's theorem, which utilises the tools of group theory and the underlying structure of the Poisson algebra.

Example 3: Let us consider a quantum mechanical problem that has rotational symmetry. Let us try to find the associated conserved quantity.

Since angular momentum is the generator of rotations, we can simply conclude that it must be the conserved quantity. To verify this formally, we take the Hamiltonian to be radially symmetric, i.e $\hat{H}(r, \theta, \phi) = \hat{H}(r)$, which allows us to compute $[\hat{H}, \hat{L}] = 0$, where $\hat{L}$ is the angular momentum operator.

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