IIT PALAKKAD

INDIAN INSTITUTE OF TECHNOLOGY PALAKKAD
Department of Computer Science and Engineering

# CS5616 Computational Complexity

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Problem Set – 3 | Total Points – 50
Name: Neeraj Krishna N | Given on 29 Feb
Roll no: 112101033 | Due on 11 Mar

## Instructions

- Use of resources other than class notes and references is forbidden.
- Collaboration is not allowed. Credit will be given for attempts and partial answers.

1. (10 points) (**RE vs co-RE**) Show that the set

$$\{M \mid M \text{ halts on all inputs of length less than } 42\}$$

is recursively enumerable, but is its complement is not.

**Solution:**

Let us construct a turing machine $M'$ for which

$$L(M') = \{M \mid M \text{ halts on all inputs of length less than } 42\}$$

$M'$ = "

On input $M$,

(1) For each $x \in \Sigma^*, |x| < 42$, Run $M$ on $x$

(2) Accept $M$

"

$M \in L$
$\Rightarrow M$ halts on all inputs of length less than 42
$\Rightarrow$step (1) of $M$ terminates and reaches step (2)
$\Rightarrow M'$ accepts $M$
$\Rightarrow M \in L(M')$
$M \notin L$
$\Rightarrow M$ loops on some input of length less than 42
$\Rightarrow$step (1) of $M'$ loops when it reaches such an $x$

$\Rightarrow M'$ loops

$\Rightarrow M'$ does not accept $M$

$\Rightarrow M \notin L(M')$

Hence $L(M') = L$

Suppose complement of $L$ is also recursively enumberable, then $L$ would be decidable since we know

$$L \text{ is decidable} \Leftrightarrow L \text{is r.e and } \overline{L} \text{ is r.e}$$

we will show a reduction $\mathsf{HP} \leq_m L$

The reduction machine $\sigma$ on input $M\#x$ (an instance of $\mathsf{HP}$) outputs $M'$ an instance of $L$

$M' =$ "

On input $y$,

1. Run $M$ on $x$
2. If $M$ halts on $x$, accept $y$

"

we need to prove

$M\#x \in \mathsf{HP} \Leftrightarrow M' \in L$

$M\#x \in \mathsf{HP}$

$\Rightarrow M$ halts on $x$

$\Rightarrow M'$ accepts all $y$, hence $M'$ halts on all inputs of length less than 42

$\Rightarrow M' \in L$

$M\#x \notin \mathsf{HP}$

$\Rightarrow M$ loops on $x$

$\Rightarrow M'$ loops on all $y$, hence $M'$ halts on some inputs of length less than 42

$\Rightarrow M' \notin L$

therefore $M\#x \in \mathsf{HP} \Leftrightarrow M' \in L$

Since we know $\mathsf{HP}$ is undecidable, $L$ is also undecidable, and thus $\overline{L}$ is not r.e

2. (10 points) (**Alternate definition for $\Delta_i$**) Let $A$ be any language. Define $\mathcal{D}^A$ be the class of all languages $L$ such that $L$ is decidable in $A$. Similarly, $\mathcal{SD}^A$ be the class of all $L$ such that $L$ is semi-decidable in $A$ and $\mathsf{co}\mathcal{SD}^A$ be the class of all languages whose complement is in $\mathcal{SD}^A$.
   (a) (5 points) Show that $\mathcal{D}^A = \mathcal{SD}^A \cap \mathsf{co}\mathcal{SD}^A$.

(b) (5 points) For any $i \geq 1$, by definition, $\Delta_i = \Sigma_i \cap \Pi_i$. Show that

$$\Delta_i = \{L \mid \text{ there exists } A \in \Sigma_{i-1} \text{ such that } L \text{ is decidable in A}\}.$$

**Solution:**

(a) Suffices to prove :

(i) $\mathcal{D}^A \subseteq \mathcal{SD}^A \cap co\mathcal{SD}^A$

(ii) $\mathcal{D}^A \supseteq \mathcal{SD}^A \cap co\mathcal{SD}^A$

(a) Let $L \in \mathcal{D}^A$, by definition of $D^A$, $L$ is decidable in $A$, let the total T.M be $M^A$

$L$ is decidable in $A$

$\Rightarrow L$ is r.e in $A$

$\Rightarrow L \in \mathcal{SD}^A$

to prove $\overline{L} \in \mathcal{SD}^A$, we need to construct a T.M $N^A$ such that $L(N^A) = \overline{L}$

$N^A =$ "

On input $x$,

i. Run $M^A$ on $x$, (using the oracle $A$ to answer queries of the form $x \in A$?)
ii. If $M^A$ accepts, reject
iii. If $M^A$ rejects, accept

"

The above machine is total since every step is total including the step (i) since $M^A$ is total

$x \in \overline{L}$

$\Leftrightarrow x \notin L$

$\Leftrightarrow M^A$ rejects $x$

$\Leftrightarrow N^A$ accepts $x$

$\Leftrightarrow x \in L(N^A)$

Hence $L(N^A) = \overline{L}$

and thus $\overline{L}$ is decidable in $A$

$\Rightarrow \overline{L}$ is r.e in $A$

$\Rightarrow \overline{L} \in \mathcal{SD}^A$

$\Rightarrow L \in co\mathcal{SD}^A$

and hence proved that $L \in \mathcal{D}^A \Rightarrow L \in \mathcal{SD}^A \cap co\mathcal{SD}^A$

$\Rightarrow \mathcal{D}^A \subseteq \mathcal{SD}^A \cap co\mathcal{SD}^A$

(b) Let $L \in \mathcal{SD}^A \cap co\mathcal{SD}^A$

$\Rightarrow L \in \mathcal{SD}^A$ and $\overline{L} \in \mathcal{SD}^A$

$\Rightarrow$there exists an OTM $N_1^A$ accepting $L$ and and also $N_2^A$ accepting $\overline{L}$
now let us construct an OTM $N^A$ such that it accepts $L$ and is total

$N^A =$ "

On input $x$,

i. Run $N_1^A$ and $N_2^A$ on $x$ with time sharing, and also whenever a query of the form $x \in A$ is made, use the oracle $A$
ii. If $N_1^A$ accepts, accept
iii. If $N_2^A$ accepts, reject

"

$\forall x \in \Sigma^*, x \in L \vee x \in \overline{L}$ but not both

$\Rightarrow$either $N_1^A$ accepts $x$ or $N_2^A$ accepts $x$

Since for each input, one of the machines is guaranteed to accept, the machine $N^A$ halts on all inputs and hence is total

To prove $L(N^A) = L$

$x \in L$

$\Rightarrow N_1^A$ accepts $x$

$\Rightarrow N^A$ accepts $x$

$\Rightarrow x \in L(N^A)$

$x \notin L$

$\Rightarrow x \in \overline{L}$

$\Rightarrow N_2^A$ accepts $x$

$\Rightarrow N^A$ rejects $x$

$\Rightarrow x \notin L(N^A)$

Therefore $L = L(N^A)$ and hence proved that $\mathcal{SD}^A \cap co\mathcal{SD}^A \subseteq \mathcal{D}^A$

Hence $D^A = \mathcal{SD}^A \cap co\mathcal{SD}^A$

(b) To prove :

$$\Delta_i = \{L \mid \exists A \in \Sigma_{i-1} \text{ such that } L \text{ is decidable in } A\}$$

or

$$\Sigma_i \cap \Pi_i = \{L \mid \exists A \in \Sigma_{i-1} \text{ such that } L \text{ is decidable in } A\}$$

Let $L$ be such that $\exists A \in \Sigma_{i-1}$ such that $L$ is decidable in $A$

$\Rightarrow L \in \{L \mid \exists A \in \Sigma_{i-1} \text{ such that } L \text{ is decidable in } A\}$

$\Rightarrow L$ is semidecidable in $A$, (decidable languages are trivially semidecidable)

$\Rightarrow L \in \Sigma_i$

From the previous question we proved that $D^A = \mathcal{SD}^A \cap co\mathcal{SD}^A$

$\Rightarrow$Since $L$ is decidable in $A$, both $L$ and $\overline{L}$ are semidecidable in $A$

$\Rightarrow L \in \Pi_i$

hence $L \in \Sigma_i$ and $L \in \Pi_i$

$\Rightarrow L \in \Sigma_i \cap \Pi_i$

Now let $L \in \Sigma_i \cap \Pi_i$, we have to prove that $L \in \Delta_i$

$L \in \Sigma_i$

$\Rightarrow \exists A \in \Sigma_{i-1}$ such that $L$ is semidecidable in $A$

$\Rightarrow \exists$ an OTM $N^A$ accepting $L$

$L \in \Pi_i$

$\Rightarrow \exists B \in \Sigma_{i-1}$ such that $\overline{L}$ is semidecidable in $B$

$\Rightarrow \exists$ an OTM $N^B$ accepting $\overline{L}$

3. (10 points) (**Closure properties of** $\Sigma_n, \Pi_n$). Fix any $i \geq 1$. Show that $\Sigma_i$ as well as $\Pi_i$ are closed under intersection and union.

**Solution:**

**Claim 1** : $\Sigma_i$ closed under union $\Rightarrow \Pi_i$ closed under intersection

**Claim 2** : $\Sigma_i$ closed under intersection $\Rightarrow \Pi_i$ closed under union

Proof of Claim 1 :

$\Sigma_i$ closed under union

$\Rightarrow \forall L_1, L_2 \in \Sigma_i, L_1 \cup L_2 \in \Sigma_i$

$\Rightarrow \forall L_1, L_2 \in \Sigma_i, \overline{(\overline{L_1} \cap \overline{L_2})} \in \Sigma_i$, by De Morgan's Law

$\Rightarrow \forall \overline{L_1}, \overline{L_2} \in \Pi_i, (\overline{L_1} \cap \overline{L_2}) \in \Pi_i$

$\Rightarrow \forall L_1', L_2' \in \Pi_i, (L_1' \cap L_2') \in \Pi_i$

$\Rightarrow \Pi_i$ closed under intersection

Proof of Claim 2 :

$\Sigma_i$ closed under intersection

$\Rightarrow \forall L_1, L_2 \in \Sigma_i, L_1 \cap L_2 \in \Sigma_i$

$\Rightarrow \forall L_1, L_2 \in \Sigma_i, \overline{(\overline{L_1} \cup \overline{L_2})} \in \Sigma_i$, by De Morgan's Law

$\Rightarrow \forall \overline{L_1}, \overline{L_2} \in \Pi_i, (\overline{L_1} \cup \overline{L_2}) \in \Pi_i$

$\Rightarrow \forall L_1', L_2' \in \Pi_i, (L_1' \cup L_2') \in \Pi_i$

$\Rightarrow \Pi_i$ closed under union

Suffices to prove

$\Sigma_i$ is closed under intersection and union

We'll argue using induction

**Base case** : $i = 0$ $\Sigma_0 = \Delta_0 = \Delta_1$ which is the set of decidable languages and we already know that decidable languages are closed under union and intersection

**Strong Induction Hypothesis** The below statement is true for all $i \leq n$

$$\Sigma_i \text{ is closed under intersection and union}$$

The below statement also follows from Claim 1 and claim 2

$$\Pi_i \text{ is closed under intersection and union}$$

Therefore we can consider both statements as Strong induction Hypothesis

**To prove for** $i = n + 1$

Let $L_1, L_2 \in \Sigma_{n+1}$

$\Rightarrow \exists A \in \Pi_n, A \subseteq \Sigma^* \times \Sigma^*, x \in L_1 \Leftrightarrow \exists y_1, (x, y_1) \in A$ and

$\exists B \in \Pi_n, B \subseteq \Sigma^* \times \Sigma^*, x \in L_2 \Leftrightarrow \exists y_2, (x, y_2) \in B$

$\Rightarrow \exists (A \cup B) \in \Pi_n$ (by induction hypothesis) such that

$x \in L_1 \cup L_2 \Leftrightarrow \exists y, (x, y) \in (A \cup B)$

$\Rightarrow L_1 \cup L_2 \in \Pi_{n+1}$

and thus proved $\Sigma_{n+1}$ is closed under union which proves $\Pi_{n+1}$ is closed under intersection from the claims

Let $L_1, L_2 \in \Sigma_{n+1}$

$\Rightarrow \exists A \in \Pi_n, A \subseteq \Sigma^* \times \Sigma^*, x \in L_1 \Leftrightarrow \exists y_1, (x, y_1) \in A$ and

$\exists B \in \Pi_n, B \subseteq \Sigma^* \times \Sigma^*, x \in L_2 \Leftrightarrow \exists y_2, (x, y_2) \in B$

$\Rightarrow \exists (A \cap B) \in \Pi_n$ (by induction hypothesis) such that

$x \in L_1 \cap L_2 \Leftrightarrow \exists y, (x, y) \in (A \cap B)$

$\Rightarrow L_1 \cap L_2 \in \Pi_{n+1}$

and thus proved $\Sigma_{n+1}$ is closed under intersection which proves $\Pi_{n+1}$ is closed under union from the claims

therefore we have proved that if the statement holds for all $i \leq n$, it holds for $i = n + 1$

and thus by principle of mathematical induction it is proved that

$$\forall i \in \mathbb{N}, \Sigma_i \text{ and } \Pi_i \text{ are closed under union and intersection}$$

4. (20 points) (**Rice's theorem**) Identify if the following are (0) properties of SD languages, (1) non-trivial properties and (2) non-monotone properties. If (1) / (2) is true, apply Rice's theorems suitably and give your conclusions. A direct use of diagonalisation or reductions does not fetch any credit.

   (a) (4 points) given a Turing machine $M$, $L(M)$ is not regular ?

   (b) (4 points) given a Turing machine $M$, does there exist a non-empty regular set $L'$ such that $L' \subseteq L(M)$ ?

   (c) (4 points) given $M$, does $M$ represent a DFA that accepts some string with equal number of 0s and 1s ?

   (d) (4 points) given a Turing machine $M$, is $L(M) \in \Pi_{42}$ ?

**Solution:**

(a) $L = \{M \mid L(M) \text{ is not regular}\}$

is a property of SD language

If there are 2 machines $M_1$ and $M_2$ accepting the same language, then it cannot be the case that $L(M_1)$ is regular and $L(M_2)$ is not regular

This is non trivial since there are turing machines which accept regular languages and turing machines which accept non-regular languages

This is also non-monotone since, there exists $L_1, L_2$ such that $L_1 \subseteq L_2$ and $\mathcal{P}(L_1) = 1$ and $\mathcal{P}(L_2) = 0$

$L_2 = \Sigma^*$ and $L_1 = \{0^n 1^n \mid n \geq 0\}$

here $L_1$ is not regular which satisfies the property whereas $L_2$ is regular and a strict superset of $L_1$ which does not satisfy the property

Since the property is non-trivial, by Rice's Theorem 1, $L$ is not decidable

Since the property is non-monotone, by Rice's Theorem 2, $L$ is not semi-decidable

(b) $L = \{M \mid \exists L' \subseteq \Sigma^*, L' \neq \phi \text{ and} L' \subseteq L(M)\}$

This is equivalent to saying $L = \{M \mid L(M) \text{ is not empty}\}$ because

$L(M)$ is not empty

$\Rightarrow \exists L' \subseteq L(M)$ which is $L(M)$ itself, such that $L' \subseteq L(M)$ i.e $L' = L(M)$

$\exists L' \subseteq \Sigma^*, L' \neq \phi \text{ and} L' \subseteq L(M)$

$\Rightarrow L(M)$ is not empty

$L(M)$ is not empty is a property of SD languages since if there exists 2 machines $M_1$ and $M_2$ accepting the same language, it cannot be the case that

$L(M_1) = \phi$ and $L(M_2) \neq \phi$ or vice-versa

This is a non-trivial property, since there are turing machines accepting empty languages and there are turing machines accepting non-empty languages

This is a monotone property because

$\forall L_1, L_2 \in \Sigma^*, \mathcal{P}(L_1) = 1 \Rightarrow \mathcal{P}(L_2) = 1$

because if $L_1$ is non-empty, any superset of $L_1$ will be non-empty

Since, this property is non-trivial, by Rice's theorem 1, the given language is undecidable

We cannot conclude anything about semi-decidability from Rice's Theorem 2 because the language is not non-monotone

(c) The language is

(d) The language is

$L = \{M \mid L(M) \in \Pi_{42}\}$

This is a property of language since it cannot be the case that there exists $M_1, M_2$ accepting the same language but $L(M_1) \in \Pi_{42}$ and $L(M_2) \notin \Pi_{42}$ or vice-versa

This is non-trivial because there exists turing machines accepting $L \in \Pi_{42}$ and turing machines accepting languages $L \notin \Pi_{42}$

Hence this is undecidable by Rice's Theorem 1

This is also non-monotone because, let $L_1 \in \Pi_{42}$ and $L_2 \in \Pi_{43} \setminus \Pi_{42}$

$\mathcal{P}(L_1) = 1$ and $\mathcal{P}(L_1 \cup L_2) = 0$ since union of $L_1$ and $L_2$ cannot be in $\Pi_{42}$ since $L_2$ is in $\Pi_{43} \setminus \Pi_{42}$