IIT PALAKKAD

Indian Institute of Technology Palakkad
Department of Computer Science and Engineering
CS5616 Computational Complexity
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Problem Set – 2
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Total Points – 50
Given on 09 Feb
Due on 16 Feb

Instructions

- Use of resources other than class notes and references is forbidden.
- Collaboration is not allowed. Credit will be given for attempts and partial answers.

1. (10 points) [**Properties of $\leq_m$**] Show that $\leq_m$ relation is reflexive and transitive over languages in Σ^*. Is $\leq_m$ symmetric ? Argue.

Solution:

1. To prove $\leq_m$ is reflexive :

   Let L be a language, we have to prove $L \leq_m L$. Let M be a turing machine which accepts L

   Now we have to prove that if L is decidable, then L is decidable, which is trivial

2. To prove $\leq_m$ is transitive :

   Let L_1, L_2, L_3 be 3 languages such that $L_1 \leq_m L_2$ and $L_2 \leq_m L_3$, we have to prove that $L_1 \leq_m L_3$

   Let M_{12} be the reduction machine which converts an instance x of L_1 to an equivalent instance y of L_2 and M_{23} be the reduction machine which converts an instance y of L_2 to an equivalent instance z of L_3 (by definition of reduction machines both M_{12} and M_{23} are total)

   We will construct a reduction machine M_{13} which converts an instance x of L_1 to an equivalent instance z of L_3

   $M_{13} =$

   "

   On input x,

   (a) Run M_{12} on x and let y denote the output
   (b) Run M_{23} on y and let z denote the output
   (c) Return z

   "

   This machine M_{13} is total since both M_{12} and M_{23} are total

Let M_3 be a turing machine which accepts L_3

Now to prove $x \in L_1 \Leftrightarrow z \in L_3$

$x \in L_1 \Leftrightarrow y \in L_2$ since M_{12} converts instance x of L_1 to an equivalent instance y of L_2

and

$y \in L_2 \Leftrightarrow y \in L_3$ since M_{23} converts instance y of L_2 to an equivalent instance z of L_3

Hence

$x \in L_1 \Leftrightarrow z \in L_3$

Therefore the machine M_{13} converts an instance x of L_1 to an equivalent instance z of L_3. Hence this is the required reduction machine

3. $\leq_m$ is not symmetric

   Example : Let $L = 0^*$, we know that L is decidable, Let M be the total turing machine which accepts L

   We also know that $HP \not\leq_m L$, since then it would be the case that HP is decidable which is not the case

   To prove $L \leq_m HP$

   Let the reduction machine be M_σ which on input x outputs $\langle M', x\rangle$ where desciption of M' is as follows

   $M' =$

   "

   On input y,

   (a) Run M on x
   (b) If M accepts, halt and accepts
   (c) If M rejects, loop

   "

   we will prove that $x \in L \Leftrightarrow y \in HP$

   $x \in L$

   $\Rightarrow M$ accepts x

   $\Rightarrow M'$ halts on every input y, in particular x

   $\Rightarrow \langle M', x\rangle \in HP$

   $x \notin L$

   $\Rightarrow M$ will reject every input x (since M is total)

   $\Rightarrow M'$ will loop on every input y, in particular x

   $\Rightarrow \langle M', x\rangle \notin HP$

   Hence we have showed two languages L and HP such that

   $L \leq_m HP$ but $HP \not\leq_m L$

   Therefore, $\leq_m$ is not symmetric

2. (20 points) [**Reduction by containment !**] Let $L_1, L_2 \subseteq \Sigma^*$ where $L_1 \subseteq L_2$. Consider the statements **(1)** "$L_2 \leq_m L_1$" **(2)** "$L_1 \leq_m L_2$"

   (a) (5 points) Give a pair of languages where **(1)** is true. Provide appropriate justification if no such pair exists.

   (b) (5 points) Give a pair of languages where **(2)** is true. Provide appropriate justification if no such pair exists.

   (c) (10 points) Prove or disprove the following statements (a), (b).

      (a) "for any L_1, L_2 with $L_1 \subseteq L_2$, **(1)** is true."

      (b) "for any L_1, L_2 with $L_1 \subseteq L_2$, **(2)** is true."

Solution:

(a)

(b)

(c)

3. (10 points) [**Proof of Rice's theorem 1**] In the proof of Rice's theorem for showing undecidability done in class, we assumed that the non-trivial property $\mathcal{P}$ is false for $\emptyset$. Explain how can this assumption be removed. [*Hint: Use* $\overline{\mathsf{HP}}$ *!*]

Solution:

4. (10 points) [**Rice's theorem ?**] Define the language $\mathsf{TOTAL} = \{M \mid M \text{ halts on all inputs}\}$.

   (a) Describe the language $\overline{\mathsf{TOTAL}}$ in a way similar to the language TOTAL.

   (b) Show that the languages TOTAL as well as $\overline{\mathsf{TOTAL}}$ are not semi-decidable.

Solution:

(a) $\overline{\mathsf{TOTAL}} = \{M \mid \exists y \in \Sigma^*, M \text{ does not halt on } y\}$

(b)