IIT PALAKKAD

Indian Institute of Technology Palakkad
Department of Computer Science and Engineering

CS5616 Computational Complexity

January – May 2024

Problem Set – 3
Name: Neeraj Krishna N
Roll no: 112101033

Total Points – 50
Given on 29 Feb
Due on 11 Mar

Instructions

- Use of resources other than class notes and references is forbidden.
- Collaboration is not allowed. Credit will be given for attempts and partial answers.

1. (10 points) (**RE vs co-RE**) Show that the set

$$\{M \mid M \text{ halts on all inputs of length less than } 42\}$$

is recursively enumerable, but is its complement is not.

Solution:

Let us construct a turing machine M' for which

$$L(M') = \{M \mid M \text{ halts on all inputs of length less than } 42\}$$

M' = "

On input M,

(1) For each $x \in \Sigma^*, |x| < 42$, Run M on x

(2) Accept M

"

$M \in L$
$\Rightarrow M$ halts on all inputs of length less than 42
$\Rightarrow$step (1) of M terminates and reaches step (2)
$\Rightarrow M'$ accepts M
$\Rightarrow M \in L(M')$
$M \notin L$
$\Rightarrow M$ loops on some input of length less than 42
$\Rightarrow$step (1) of M' loops when it reaches such an x

$\Rightarrow M'$ loops

$\Rightarrow M'$ does not accept M

$\Rightarrow M \notin L(M')$

Hence $L(M') = L$

Suppose complement of L is also recursively enumberable, then L would be decidable since we know

$$L \text{ is decidable} \Leftrightarrow L \text{is r.e and } \overline{L} \text{ is r.e}$$

we will show a reduction $\mathsf{HP} \leq_m L$

The reduction machine σ on input $M\#x$ (an instance of HP) outputs M' an instance of L

$M' =$ "

On input y,

1. Run M on x
2. If M halts on x, accept y

"

we need to prove

$M\#x \in \mathsf{HP} \Leftrightarrow M' \in L$

$M\#x \in \mathsf{HP}$

$\Rightarrow M$ halts on x

$\Rightarrow M'$ accepts all y, hence M' halts on all inputs of length less than 42

$\Rightarrow M' \in L$

$M\#x \notin \mathsf{HP}$

$\Rightarrow M$ loops on x

$\Rightarrow M'$ loops on all y, hence M' halts on some inputs of length less than 42

$\Rightarrow M' \notin L$

therefore $M\#x \in \mathsf{HP} \Leftrightarrow M' \in L$

Since we know HP is undecidable, L is also undecidable, and thus $\overline{L}$ is not r.e

2. (10 points) (**Alternate definition for Δ_i**) Let A be any language. Define $\mathcal{D}^A$ be the class of all languages L such that L is decidable in A. Similarly, $\mathcal{SD}^A$ be the class of all L such that L is semi-decidable in A and $\mathsf{co}\mathcal{SD}^A$ be the class of all languages whose complement is in $\mathcal{SD}^A$.

   (a) (5 points) Show that $\mathcal{D}^A = \mathcal{SD}^A \cap \mathsf{co}\mathcal{SD}^A$.

(b) (5 points) For any $i \geq 1$, by definition, $\Delta_i = \Sigma_i \cap \Pi_i$. Show that

$$\Delta_i = \{L \mid \text{ there exists } A \in \Sigma_{i-1} \text{ such that } L \text{ is decidable in A}\}.$$

Solution:

(a)

(b)

3. (10 points) (**Closure properties of** Σ_n, Π_n). Fix any $i \geq 1$. Show that Σ_i as well as Π_i are closed under intersection and union.

Solution:

4. (20 points) (**Rice's theorem**) Identify if the following are (0) properties of SD languages, (1) non-trivial properties and (2) non-monotone properties. If (1) / (2) is true, apply Rice's theorems suitably and give your conclusions. A direct use of diagonalisation or reductions does not fetch any credit.
   (a) (4 points) given a Turing machine M, $L(M)$ is not regular ?
   (b) (4 points) given a Turing machine M, does there exist a non-empty regular set L' such that $L' \subseteq L(M)$?
   (c) (4 points) given M, does M represent a DFA that accepts some string with equal number of 0s and 1s ?
   (d) (4 points) given a Turing machine M, is $L(M) \in \Pi_{42}$?

Solution:

(a)

(b)

(c)

(d)