Newtons Second Law & Periodic Motion

Differential equations have a derivative function of some variable implicit in the equation which can be solved to find the non-derivate function or unique solutions with initial conditions. The equation can include terms of the same variable, and the non-derivative function must satisfy the condition of the equating the derivative and non-derivative terms. This can be achieved by using known functions that are cyclic which means the functions derivate terms are the same as the initial function. Differential equations of this kind are often applied in Physics where phenomenon have a restorative force that is opposite to the displacement of a system, giving it oscillatory motion also known as simple harmonic motion of a spring or pendulum. The equation of motion for a spring is Newtons Second Law F=ma, where the force on a system is proportional to its acceleration and its mass.

In the case of an ideal spring the strength and force of the spring is quantified as the spring constant k, and the equation of motion for the ideal spring is proportional to the displacement x of the spring:

$$F = ma = -kx$$

In differential form where the acceleration is the second derivative of the displacement and first derivative of the velocity the equation is:

Eq 1.
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

For an ideal spring with no resistive force and omitting the mass of the spring then Eq 1. Says that for some function of time for a displacement of x from x=0 there is a second order derivative of the function or an acceleration at the position x proportional to the displacement x dependent on the spring constant k and the mass m.

We can solve differential equations such as these readily with possible solutions such as cyclic functions, this is where the function x in the RHS of the equation is of the same form as the second derivative of the function x on the LHS of the equation.

$$\frac{d^2(Asin(\omega t))}{dt^2} = -\frac{k}{m}(Asin(\omega t))$$

$$-A\omega^2 \sin(\omega t) = -\frac{k}{m}\left(Asin(\omega t)\right)$$

This equation is true for the following conditions:

$$\omega^2 = \frac{k}{m}$$

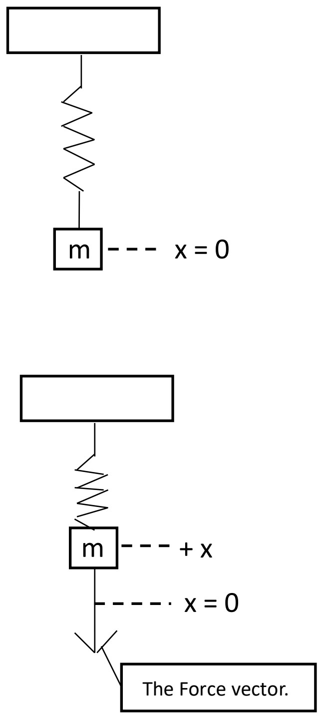

Figure 1. A compressed spring system at displacement +x, with a force vector toward the equilibrium position at x = 0.

The condition for the angular frequency 2πf of the simple harmonic motion differential equation:

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

the Time Period T of the system to complete one oscillation

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The solution of the differential equation shows the possible oscillations of the system from initial conditions of time t = 0 to some time t. The solution should cover all possible oscillations. We can generalise the function by adding a phase angle so that at t=0 the system could start at a different point in the cycle then displacement at some time is:

$$x(t) = Asin(\omega t + \varphi)$$

$$x(t) = Csin(\omega t) + Bcos(\omega t)$$

Where:

$$C = Acos(\varphi) \quad \& \quad B = Asin(\varphi)$$

At time t=0 the displacement of the system is:

$$x = Asin(\varphi)$$

The phase angle is then:

$$\varphi = sin^{-1}\left(\frac{x}{A}\right)$$

At time t=0 we can consider the initial velocity:

$$\frac{dx}{dt} = v = \omega Acos(\varphi)$$

$$A = \frac{v}{\omega cos(\varphi)}$$

Damped Harmonic Oscillations

For a damped harmonic oscillator there a resistive force of friction that reduces the amplitude of the oscillations and returns the system to equilibrium over some time t. The resistive force is proportional to the velocity of the system and has a damping factor λ determining whether the oscillations are overdamped, underdamped or critically damped where the system returns to equilibrium without any oscillations. The differential equation describing this motion has a resultant acceleration:

$$F_R = -F_s + F_D$$

$$m \frac{d^2x}{dt^2} = -kx + \lambda \frac{dx}{dt}$$

Eq 2.
$$\frac{d^2x}{dt^2} + \lambda \frac{dx}{dt} + {\omega_0}^2 x = 0$$

In the case where the displacement of the system is 0 the gradient and velocity is maximum and the acceleration is 0 then:

$$\frac{dx}{dt} = -\frac{k}{\lambda m}x$$

$$\int \frac{1}{x} dx = \int -\frac{k}{\lambda m} dt$$

$$\ln(x) = -\frac{kt}{\lambda m} + c$$

Eq 3.
$$x = e^{-\frac{kt}{\lambda m}+c} = C\, e^{-\beta t}$$

The equation of motion including a damping term is the resultant force and acceleration of the mass equal to the second derivative of the displacement function of time. We can apply a similar reasoning as with Eq. 1 to find a solution to the equation, in the case where:

$$x(t) = Asin(\omega t)$$

$$\frac{d^2(Asin(\omega t))}{dt^2} - \lambda \frac{d(Asin(\omega t))}{dt} + {\omega_0}^2(Asin(\omega t)) = 0$$

$$-\omega^2 Asin(\omega t) - \lambda\omega A\cos(\omega t) + {\omega_0}^2 Asin(\omega t) = 0$$

$$-\lambda\omega\cos(\omega t) + {\omega_0}^2 sin(\omega t) = \omega^2 Asin(\omega t)$$

We can equate the coefficients of the sin terms on the LHS & RHS of the equation:

$$\omega_0 = \omega$$

We can combine the solutions for x(t) in Eq 3. & the undamped oscillator:

$$x(t) = C\, e^{-\beta t}\, sin(\omega t + \varphi)$$

We can use graphs of the function for a Damped Harmonic Oscillator for given parameters and initial conditions at t=0.

Figure 2 shows a graph of an underdamped oscillator where the amplitude of the decays exponentially and the frequencies remain the same, it has a decay factor of 0.4 with the function:

$$x(t) = 15\, e^{-0.4t}\, cos(12t)$$

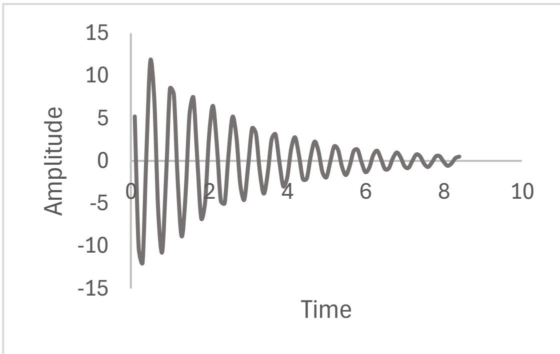

Figure 2.

Figure 3 shows the graph of an overdamped oscillator where the oscillations slow down more quickly and stop with few oscillations, it has a decay factor of 2 with the function:

$$x(t) = 15\, e^{-2t}\, cos(12t)$$

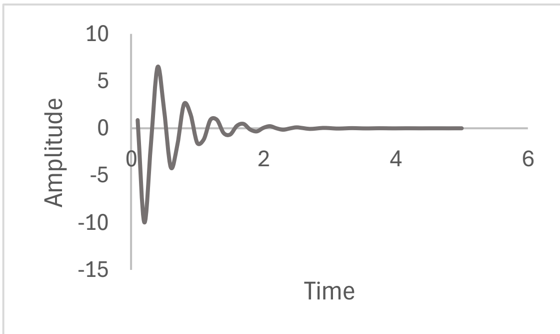

Figure 3.