Vector Equations, Differentiation
& Integration Techniques.

by

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Vectors, The Cartesian Plane & Coordinates (x, y, z)

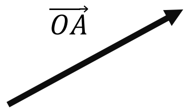

Diagram 1.

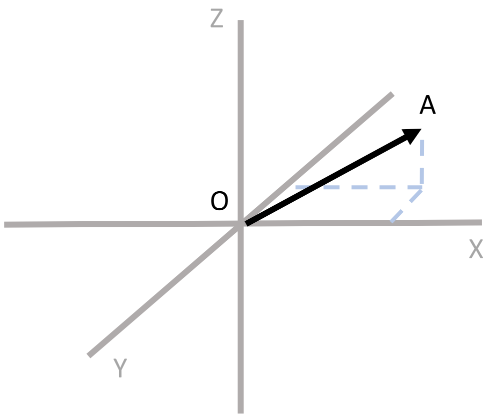

Diagram 2.

Consider the vector $\overrightarrow{OA}$ in diagram 1 and in XYZ coordinates in diagram 2, it starts at the origin (0, 0, 0) and ends at some arbitrary point A with the coordinates (X, Y, Z). The vector from point O to Point A has the following notation in the form of an equation:

$$\overrightarrow{OA} = X\,\hat{\imath} + Y\,\hat{\jmath} + Z\,\hat{k}$$

Where $\hat{\imath}$, $\hat{\jmath}$ & $\hat{k}$ are the cartesian coordinate unit vectors in the direction of the axes (x, y, z) respectively. A unit vector is denoted by the ^ above the letter and has a magnitude of 1 in a specified direction:

$$\hat{\imath} = (1,0,0)$$

$$\hat{\jmath} = (0,1,0)$$

$$\hat{k} = (0,0,1)$$

The magnitude and length of a vector is calculated using Pythagorean theorem in 3 dimensions using the equation:

$$|\overrightarrow{OA}| = \sqrt{X^2 + Y^2 + Z^2}$$

Summing any two vectors $\overrightarrow{OB}$ & $\overrightarrow{BA}$ together results in a third vector OA as shown in diagram 3:

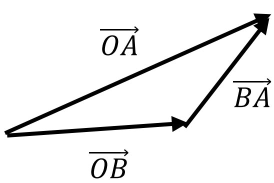

Diagram 3.

In equation form adding the vectors together is a summation of the unit vectors ($\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$) for example if the vectors $\overrightarrow{OB}$ and $\overrightarrow{BA}$ are given:

$$\overrightarrow{OB} = 2\,\hat{\imath} - 3\,\hat{\jmath} + 5\,\hat{k}$$

$$\overrightarrow{BA} = 3\,\hat{\imath} + 5\,\hat{\jmath} - 2\,\hat{k}$$

$$\overrightarrow{OA} = \overrightarrow{OB} + \overrightarrow{BA} = (2 + 3)\,\hat{\imath} + (-3 + 5)\,\hat{\jmath} + (5 - 2)\,\hat{k}$$

$$\overrightarrow{OA} = 5\,\hat{\imath} + 2\,\hat{\jmath} + 3\,\hat{k}$$

<u>Multiplication of Vectors, The Scalar Product & Vector Product</u>

The Scalar product also known as the Dot product of any two vectors results in a number not a vector and can be calculated in the following ways using the vectors D & E that have an angle θ between them as shown in the diagram below:

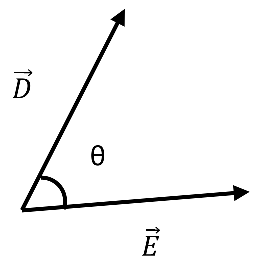

Diagram 4.

Where the vectors D & E are as follows:

$$\overrightarrow{D} = D_x\,\hat{\imath} + D_y\,\hat{\jmath} + D_Z\,\hat{k}$$

$$\overrightarrow{E} = E_x\,\hat{\imath} + E_y\,\hat{\jmath} + E_Z\,\hat{k}$$

The Scalar Dot product is then:

$$\overrightarrow{D} \cdot \overrightarrow{E} = (D_x \times E_x) + (D_y \times E_y) + (D_z \times E_z) = \text{Scalar}$$

And can also be calculated using:

$$\vec{D} \cdot \vec{E} = |\vec{D}||\vec{E}| \cos(\theta)$$

The Vector product also known as the Cross product of any two vectors results in a third vector $\overrightarrow{C}$ that is perpendicular (at a 90 degree angle) to the two vectors and can be calculated in the following ways:

$$\vec{C} = \overrightarrow{D} \times \overrightarrow{E} = ((D_y \text{ x } E_z) - (E_y \text{ x } D_z))\hat{\imath} - ((D_x \text{ x } E_z) - (E_x \text{ x } D_z))\hat{\jmath} + ((D_x \text{ x } E_y) - (E_x \text{ x } D_y))\hat{k}$$

Where the vector $\vec{C}$ is:

$$\vec{C} = C_x\,\hat{\imath} + C_y\,\hat{\jmath} + C_Z\,\hat{k}$$

The magnitude of the Vector Product is then:

$$|\,\vec{C}\,| = |\,\overrightarrow{D} \times \vec{E}\,| = \sqrt{C_x^2 + C_y^2 + C_z^2}$$

The vector product C is also given by the equation:

$$|\vec{C}\,| = |\,\overrightarrow{D} \times \vec{E}\,| = |D|\,|E|\,\sin(\theta)$$

Where $\hat{n}$ is the unit vector perpendicular to the vectors $\overrightarrow{D}$ & $\vec{E}$:

$$\hat{n} = n_x\,\hat{\imath} + n_y\,\hat{\jmath} + n_z\,\hat{k}$$

Which can also be obtained by dividing the vector by its magnitude using the equation:

$$\hat{n} = \frac{\overrightarrow{(D \times \vec{E})}}{|\overrightarrow{D} \times \vec{E}|} = \frac{\vec{C}}{|\vec{C}|}$$

<u>The Vector Equation Of A Flat Surface in 3D</u>

The vector equation of a flat surface is defined using any two points in the plane. From these the vector from the origin O to a point in the plane and the unit normal (perpendicular) vector to the plane can be determined giving an equation of the plane.

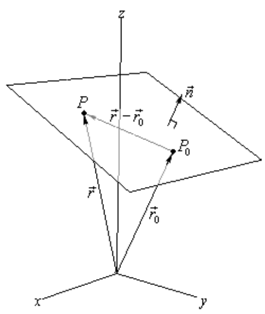

Diagram 5.

If the two points $\overrightarrow{OA}$ and $\overrightarrow{OR}$ in the plane are given by the coordinates (X_1, Y_1, Z_1) & (X_2, Y_2, Z_2), then the vectors from the origin to each point as can be seen in diagram 5 are:

$$\overrightarrow{r} = X_1\,\hat{\imath} + Y_1\,\hat{\jmath} + Z_1\,\hat{k}$$

$$\overrightarrow{a} = X_2\,\hat{\imath} + Y_2\,\hat{\jmath} + Z_2\,\hat{k}$$

The vector in the plane is calculated from:

$$\vec{r} - \vec{a} = (X_1 - X_2)\,\hat{\imath} + (Y_1 - Y_2)\,\hat{\jmath} + (Z_1 - Z_2)\,\hat{k}$$

The normal vector which is at a 90 degree angle from the plane can be used to formulate the equation of the plane using the scalar dot product as the trigonometric function cos(90) is 0 therefore we can write the vector equation of the plane as:

$$\vec{n} \cdot (\vec{r} - \vec{a}) = (\vec{n} \cdot \vec{r}) - (\vec{n} \cdot \vec{a}) = 0$$

If we only know one point in the plane then using the scalar dot product and the cos(90) identity we can write the equation:

$$\vec{n} \cdot (\vec{r} - \vec{a}) = \vec{n} \cdot \frac{\overrightarrow{(r - \vec{a})}}{|\overrightarrow{r - a}|} |\overrightarrow{r - a}| = \vec{n} \cdot \hat{m}\,\lambda = 0$$

Where we have made the unknown vector (r - a) in the flat surface a unit vector denoted as m with an arbitrary coefficient λ which would determine the length of a vector in the plane. We can then write an equation to determine any point (coordinate vector) in the plane where a is a coordinate given or chosen in the surface:

$$\vec{r} = (x\ y\ z) = \lambda\,\hat{m} + \vec{r}_0$$

Vector Equation, Spherical Coordinates and Parametrisation.

The vector equation of a plane derived above can be written in polar coordinates where the unit vectors in the (x y z) direction are changed to (r θ Φ), where r is the magnitude of the vector and θ & Φ are angular parameters between the x, y & z axes. We will now look at the parameterisation of the unit vectors to define curved planes.

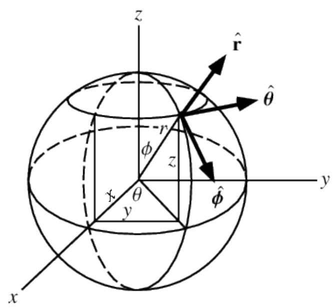

Diagram 6.

$$\vec{r} = (r\ \theta\ \Phi) = \lambda(m_r\,\hat{r} + m_\theta\hat{\theta} + m_\Phi\,\hat{\Phi}) + (r_0\ \theta_0\ \Phi_0)$$

Parametrisation is the substitution of the variables (r θ ϕ) for a function of some other variable t (r(t) θ(t) ϕ(t)), where the functions of t can be selected. For example we can use spherical coordinates where ϕ = z to map an elliptical cylinder.

Eq 1. $$\vec{r} = (r\ \theta\ \Phi) = \lambda(\hat{r} + 2\pi t\,\hat{\theta} + Z\,\hat{k}) \qquad 0 < t < 1$$

Where the coordinates for (x y z) and (r θ Φ) can be interchanged using the following transformations:

$$(r,\ \theta,\ \Phi) = \left(\sqrt{X^2+Y^2+Z^2},\ \arctan\left(\frac{y}{x}\right),\ \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)\right)$$

$$(x,\ y,\ z) = (r\cos(\theta)\sin(\Phi),\ r\sin(\theta)\cos(\Phi),\ r\cos(\Phi))$$

In diagram 7 is the plot using the python programming language of the vector equation Eq. 1 in the XYZ plane. The domain of t is defined and the coordinates calculated to produce the plot.

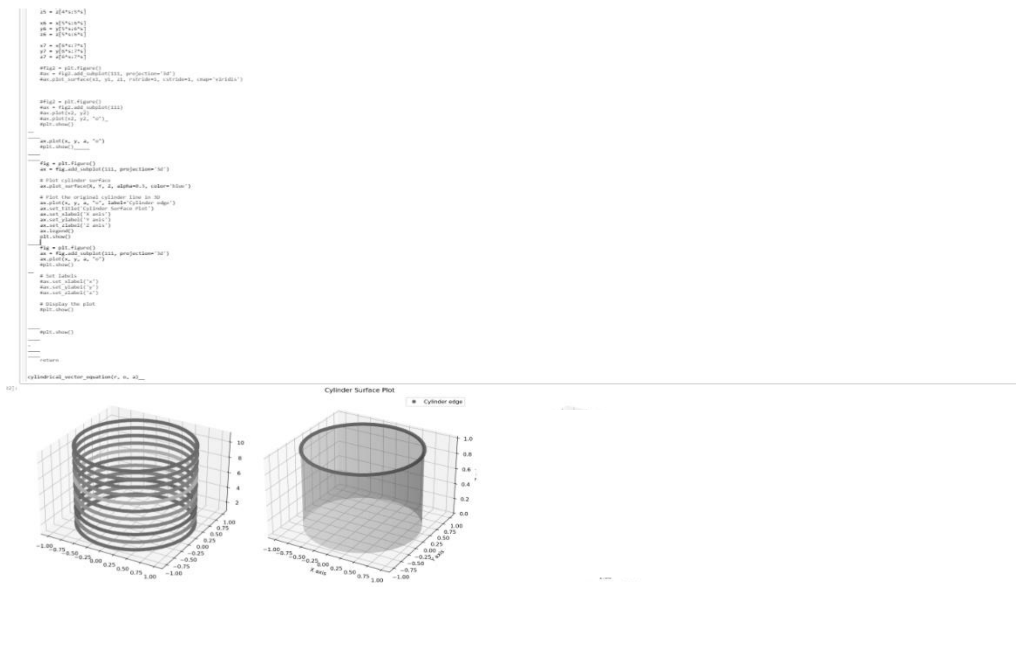

Diagram 7.

The Derivatives & Integrals of Vectors.

The derivative of a vector equation returns a gradient vector that is an infinitesimal change of the equation divided by an infinitesimal change in it's parameter and is in the direction of one data point to the next. The small change in the equation or parameter is denoted by the letter δ. The unit vectors in polar coordinates change with the coordinate value and therefore are functions of (r θ Φ), which requires the product rule of differentiation and the unit vector identities for the vectors velocity components.

$$\left(\frac{\delta \vec{r}_r}{\delta t}\right) = \vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\,\hat{\theta}$$

And in Spherical polar coordinates the velocity vector is the following.

$$\left(\frac{\delta \vec{r}_r}{\delta t}\right) = \vec{v} = \dot{r}\,\hat{r} + r\,\dot{\theta}\sin(\varphi)\,\hat{\theta} + r\,\dot{\varphi}\,\hat{\varphi}$$

In the case of a circle with unit radius and a linear function of θ the derivative would return a vector that changes by a constant in the θ direction. Eq. 1 is used to calculate the first derivative:

$$\left(\frac{\delta \vec{r}_r}{\delta t}\right)_\theta = 2\pi\,\hat{\theta}$$

The result is a vector which only changes in the radial direction and changes in the direction toward and away from the origin, as the variable t changes, the θ unit vector is a constant and the ϕ unit vector is now zero.

Computing the integral is a sum of multiplications of an equation by changes in its parameters. For example, the vector equation v plots a line of data points in 3D space, if we integrate small sections of the line dv from data point to data point the length of the line can be calculated as shown below:

$$\int_0^r d\vec{r} = \int_0^r \left|\left(\frac{\delta \vec{r}_r}{\delta t}\right)_\theta\right| dt = \int_0^1 2\pi\,dt = [2\pi - 0]_0^1 = 2\pi$$

Surface Area & Volume Integrals

The surface area integral is a summation of infinitesimal sub-surfaces δs of an arbitrary larger surface defined by a vector equation. In this example we will use a cylinder given by the following vector equation where t & z are parameters. The orthogonal surface vectors are identified using the gradient vectors and a sub-surface area calculated by the vector product and integrated over the domains of the parameters.

$$\vec{r} = (\mathrm{r}\ \theta\ \Phi) = (\hat{r} + 2\pi t\,\hat{\theta} + 2\,\mathrm{z}\,\hat{\mathrm{z}}), \quad (0 < \mathrm{t} < 1) \;\; \& \;\; (0 < \mathrm{z} < 3)$$

$$\int_0^r d\vec{s} = \iint_0^t \left| \left(\frac{\delta \vec{r}_r}{\delta t}\right)_\theta X \frac{\delta \vec{r}}{\delta z} \right| dt\,dz = \iint_0^t \left|\frac{\delta \vec{r}}{\delta t}\right| \left|\frac{\delta \vec{r}}{\delta z}\right| dt\,dz$$

$$\int_0^r d\vec{s} = \iint_0^1 (2\pi)(2)\,dt\,dz = \int_0^3 4\pi\,dz = 12\pi\ \mathrm{m}^2$$

The same method can be used to determine the volume of the cylinder by adding a radial parameter to the equation which then plots the circumference at different values of r.

$$\vec{r} = (\mathrm{r}\ \theta\ \Phi) = (r\,\hat{r} + 2\pi t\,\hat{\theta} + 2\,\mathrm{z}\,\hat{\mathrm{z}}), \quad (0 < \mathrm{t} < 1) \;\; \& \;\; (0 < \mathrm{z} < 3)$$

The surface integral uses the circumference multiplied in the z direction and can be used for the volume integral by adding a multiplication of dr. The integral is then adding small surface volumes $d\vec{s}\ dr$ together over the range of r (0 > r > 1), which is the following:

$$\left(\frac{\delta \vec{r}_r}{\delta t}\right)_\theta = 2\pi r\,\hat{\theta}$$

$$\iint_0^r d\vec{s}\,dr = \iiint_0^r \left| \left(\frac{\delta \vec{r}_r}{\delta t}\right)_\theta X \frac{\delta \vec{r}}{\delta z} \right| dt\,dz\,dr = \iiint_0^r \left|\frac{\delta \vec{r}}{\delta t}\right| \left|\frac{\delta \vec{r}}{\delta z}\right| dt\,dz\,dr$$

$$\iint_0^r d\vec{s}\,dr = \iiint_0^r (2\pi r)(2) dt\,dz\,dr = 12\pi \int_0^1 r\,dr = \left[\frac{12\pi\,r^2}{2}\right]_0^1 = 6\pi\ \mathrm{m}^3$$

The Del Vector

The del vector or gradient vector has implicit derivatives of the gradient in each direction where the vector equation is a function of the coordinates. It is symbolised by the letter ∇ and is used as a multiplication of vector equations. In the XYZ coordinate system it is given by:

$$\nabla = \frac{\delta}{\delta x}\,\hat{\imath} + \frac{\delta}{\delta y}\,\hat{\jmath} + \frac{\delta}{\delta z}\,\hat{k}$$

And in spherical polar coordinates:

$$\nabla = \frac{\delta}{\delta r}\,\hat{r} + \frac{1}{r}\frac{\delta}{\delta \theta}\,\hat{\theta} + \frac{1}{r\sin(\varphi)}\frac{\delta}{\delta \Phi}\,\hat{\Phi}$$

The del vector is used as an operator on non vector equations (scalar fields) & vector equations (vector fields) in the following ways using the multiplication methods identified above. Given the scalar field V and the vector field **V** these are then:

$$\nabla\,\mathrm{V} = \frac{\delta V}{\delta r}\,\hat{r} + \frac{1}{r}\frac{\delta V}{\delta \theta}\,\hat{\theta} + \frac{1}{r\sin(\Phi)}\frac{\delta V}{\delta \Phi}\,\hat{\Phi} = A\,Vector\,Field$$

$$\nabla\cdot\mathbf{V} = \frac{1}{r^2}\frac{\delta(r^2\boldsymbol{V}_r)}{\delta r}\,\hat{r} + \frac{1}{rsin(\varphi)}\frac{\delta \boldsymbol{V}_\theta}{\delta \theta}\,\hat{\theta} + \frac{1}{r\sin(\varphi)}\frac{\delta(\sin(\varphi)\boldsymbol{V}_\varphi)}{\delta \Phi}\,\hat{\Phi} = A\,Scalar\,Field$$

$$\nabla\times\mathbf{V} = \frac{1}{rsin(\varphi)}(\frac{\delta}{\delta\varphi}\left(\boldsymbol{V}_\theta\sin(\varphi) - \frac{\delta\boldsymbol{V}_\varphi}{\delta\theta}\right)\hat{r} + \frac{1}{r}\left(\frac{\delta}{\delta r}(r\boldsymbol{V}_\varphi) - \frac{\delta\boldsymbol{V}_r}{\delta\varphi}\right)\hat{\theta} + \left(\frac{1}{r\sin(\varphi)}\frac{\delta(\boldsymbol{V}_r)}{\delta\theta} - \frac{1}{r}\frac{\delta}{\delta r}(r\boldsymbol{V}_\theta)\right)\hat{\Phi} = A\,Vector\,Field$$

Relavent Formulae

Unit vector transformations are:

$$\hat{r} = sin\theta cos\phi\ \hat{x}\ +\ sin\theta sin\phi\ \hat{y}\ +\ cos\theta\ \hat{z}$$

$$\hat{\theta} = -\,sin(\theta)\ \hat{x} + cos(\theta))\ \hat{y}$$

$$\hat{\phi} = cos\theta cos\phi\ \hat{x}\ +\ cos\theta sin\phi\ \hat{y}\ -\ sin\theta\ \hat{z}$$

Coordinate transformations are:

$$(r, \theta, \phi) = (\sqrt{x^2 + y^2 + z^2}\ , \arctan(\frac{y}{x}), \arctan(\frac{\sqrt{x^2 + y^2}}{z}))$$

$$(x, y, z) = (\ rcos\theta sin\phi,\ rsin\theta sin\phi,\ rcos\phi\ )$$

$$(x, y, z) = (rcos\theta,\ rsin\theta,\ z)$$

Del Vector Multiplications:

$$\nabla \cdot \mathbf{V} = (\frac{1}{r^2}\frac{\delta(r^2\mathbf{V}_r)}{\delta r}\ +\ \frac{1}{rsin(\phi)}\frac{\delta \mathbf{V}_\theta}{\delta\theta}\ +\ \frac{1}{rsin\phi}\frac{\delta\ (sin(\phi)\ V_\phi)}{\delta\phi}\ )\ =\ A\ scalar\ field$$

$$\nabla\ X\ \mathbf{V} = \frac{1}{rsin(\phi)}(\frac{\delta}{\delta\phi}\ (\mathbf{V}_\theta sin(\phi))\ -\ \frac{\delta\mathbf{V}_\phi}{\delta\theta})\ \hat{r}\ +\ \frac{1}{r}(\frac{\delta}{\delta r}(r\mathbf{V}_\phi) - \frac{\delta\mathbf{V}_r}{\delta\phi})\ \hat{\theta}\ +\ (\frac{1}{rsin\phi}\ \frac{\delta\mathbf{V}_r}{\delta\theta} - \frac{1}{r}\frac{\delta}{\delta r}(r\mathbf{V}_\theta))\ \hat{\phi}\ = A\ vector\ field$$