Fourier Series

Suppose we have an arbitrary periodic function f(t) that repeats itself every Time period T with a frequency 1/T. Under certain conditions the function can be expressed as the sum of an infinite number of sin & cosine functions known as a Fourier series. The Fourier equation is defined by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right)$$

Where the amplitudes or coefficients of each term are defined as:

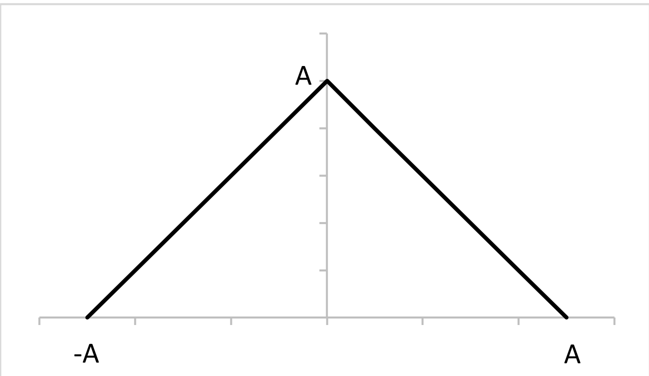
$$a_0 = \frac{2}{T}\int_0^T f(t)dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(\frac{2\pi nt}{T})dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(\frac{2\pi nt}{T})dt$$

The coefficients of the Fourier series integrate an area of the function and factorise each area by the Time period T. The first coefficient a_0 calculates the area of f(t) that is being converted to trigonometric functions, the coefficients a_n (for an even function of f(t)) & b_n (for an odd function of f(t)) calculates the amplitude & components in the series, each term is linearised by the period to a component value for f(t). The component values of f(t) are represented in the Fourier series as the amplitude of sin & cos terms added together to give the value of f(t) for any t from 0 to T. The sum of components is from n=1 to n=∞. The series is normalised by 1/n for each nth term.

In the case of the periodic triangle centred at 0, there is a discontinuity at t=0 so we can split the integral into two functions:



$$f(t) = t + A$$

$$f(t) = -t + A$$

$$a_0 = \frac{1}{A}\left(\int_{-A}^{0}(t+A)\,dt + \int_{0}^{A}(-t+A)\,dt\right)$$

$$a_0 = \frac{1}{A}\left(\left[\frac{t^2}{2}+At\right]_{-A}^{0} + \left[-\frac{t^2}{2}+At\right]_{0}^{A}\right)$$

$$a_0 = \frac{1}{A}\left(-\left(\frac{A^2}{2}-A^2\right)+\left(-\frac{A^2}{2}+A^2\right)\right)$$

$$a_0 = A$$

$$a_n = \frac{2}{T}\int_{0}^{T} f(t)\cos(\frac{2\pi nt}{T})dt$$

$$a_n = \frac{1}{A}\left(\int_{-A}^{0}(t+A)\cos\left(\frac{2\pi nt}{A}\right)dt + \int_{0}^{A}(-t+A)\cos\left(\frac{2\pi nt}{A}\right)dt\right)$$

$$a_n = \frac{1}{A}\left(\left[\frac{A(t+A)\sin\left(\frac{2\pi nt}{A}\right)}{2\pi n}\right]_{-A}^{0} - \left[-\frac{A^2\cos\left(\frac{2\pi nt}{A}\right)}{(2\pi n)^2}\right]_{0}^{A}\right)$$

$$a_n = \frac{1}{A}\left(\frac{A^2\cos(\pi n)}{(2\pi n)^2}-\frac{A^2}{(2\pi n)^2}\right) = -\frac{2A}{(2n-1)^2\pi^2}$$

The equation is 0 for even numbers of n so we can replace n with only odd numbers which can be written as (2n-1). We can substitute the coefficients into the Fourier Series:

$$f(t) = \frac{A}{2}+\sum_{n=1}^{\infty}\left(-\frac{2A}{(2n-1)^2\pi^2}\cos\left(\frac{(2n-1)\pi t}{A}\right)\right)$$