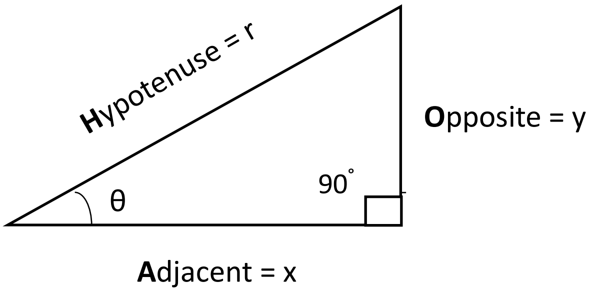
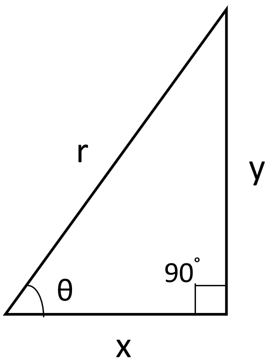
Pythagorean Theorem & Trigonometry

Pythagoras is a mathematical relationship of the angles and side lengths of a triangle, specifically right-angled triangles where one of the angles is 90 degrees and two of the sides therefore perpendicular joined by the longest side the hypotenuse. We can formulate the mathematical relationships into equations for the lengths of each side and establish relationships between the angles and the side lengths.

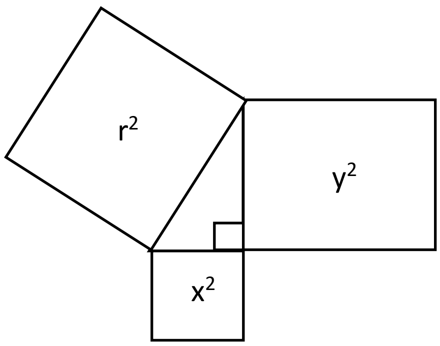
$$(Adjacent)^2 + (Opposite)^2 = (Hypotenuse)^2$$

$$x^2 + y^2 = r^2$$





A right-angled triangle with side lengths x, y & r and a right angle 90˚ angle opposite the angle θ.



The Pythagoras square law for right angled triangles where $(y^2 = y \times y)$, $(x^2 = x \times x)$, $(r^2 = r \times r)$ and $(x^2 + y^2 = r^2)$.

The trigonometric equations have the following form defined by ratios of the side lengths for a certain angle. Using these definitions we can transform from the x & y values of a function to the θ & r values of the function and we can apply relationships for x & y such as the Pythagorean equation to the trigonometric equations and identities.

$$\sin(\theta) = \frac{y}{r} \qquad \cos(\theta) = \frac{x}{r} \qquad tan(\theta) = \frac{y}{x}$$

Rearranging the trigonometric equations by taking the denominator on the RHS to the LHS gives the following expression:

$$\mathrm{r}\sin(\theta) = y \qquad \mathrm{r}\cos(\theta) = x \qquad x\,tan(\theta) = y$$

We can use the Pythagoras formula for a right-angled triangle and substitute the trigonometric identities do find further trigonometric relationships:

$$x^2 + y^2 = r^2$$

$$\mathrm{r}^2 \sin^2(\theta) + \mathrm{r}^2 \cos^2(\theta) = \mathrm{r}^2$$

The r^2 values in the equation cancel leaving the following trigonometric identity for a circle of any radius where the angle θ is the same for both sin and cosine functions:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

We can substitute the trigonometric values for x & y to establish the following relationship between the tan, sin & cos functions.

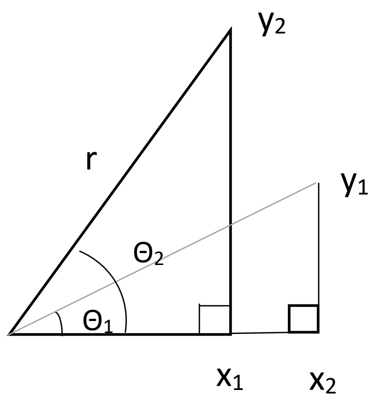
$$\tan(\theta) = \frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \frac{sin(\theta)}{\cos\,(\theta)}$$

We can add angles together known as a phase shift, a phase shift increases or decreases the angel by another angle changing the component values of r. For additions of angles there are the following trigonometric relationships in the case of the unit circle where r = 1:

$$\theta = \theta_1 \pm \theta_2$$

$$\sin(\theta_1 \pm \theta_2) = \sin(\theta_1)\cos(\theta_2) \pm \sin(\theta_2)\cos(\theta_1)$$

$$\cos(\theta_1 \pm \theta_2) = \cos(\theta_1)\cos(\theta_2) \mp \sin(\theta_1)\sin(\theta_2)$$

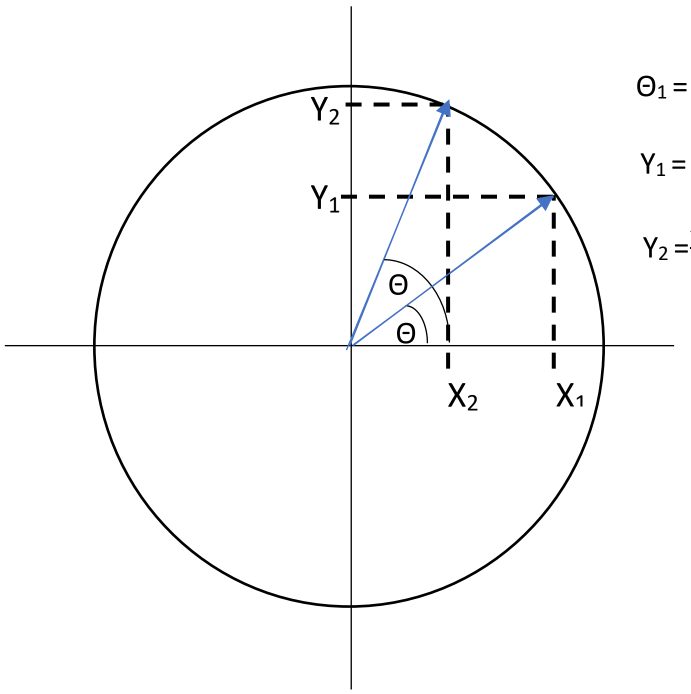


The Unit Circle

The unit circle is the case where r is a constant r = 1. For angles of θ from 0 to 360° the radius of the unit circle rotates around the origin of the axis (0, 0) and plots (x, y) values of the trigonometric functions where with r = 1. The (x, y) values are known as component vectors of the radius vector **r** with magnitude |**r**|.

$$\sin(\theta_1) = \frac{y_1}{r} \qquad \cos(\theta_1) = \frac{x_1}{r} \qquad \tan(\theta_1) = \frac{y_1}{x_1}$$

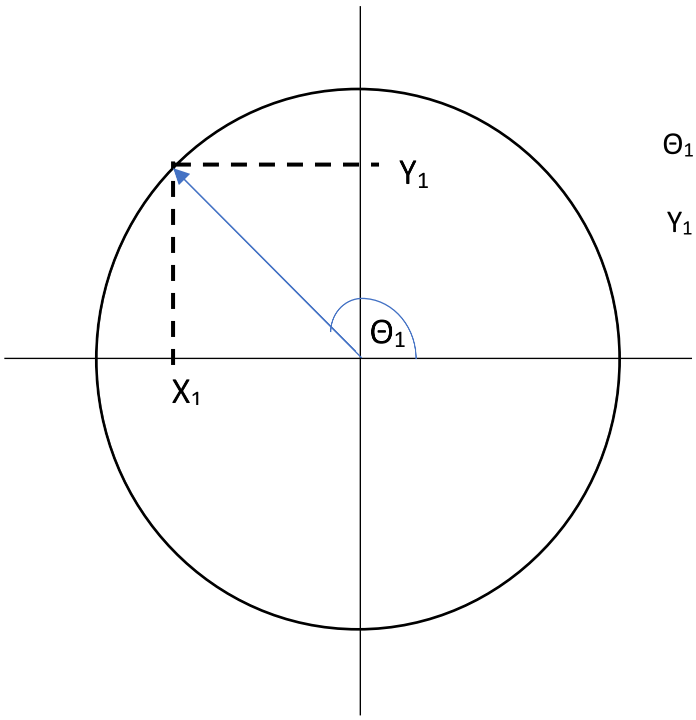
$$\sin(\theta_2) = \frac{y_2}{r} \qquad \cos(\theta_2) = \frac{x_2}{r} \qquad \tan(\theta_2) = \frac{y_2}{x_2}$$



$\Theta_1 = 30^\circ$, $\Theta_2 = 60^\circ$

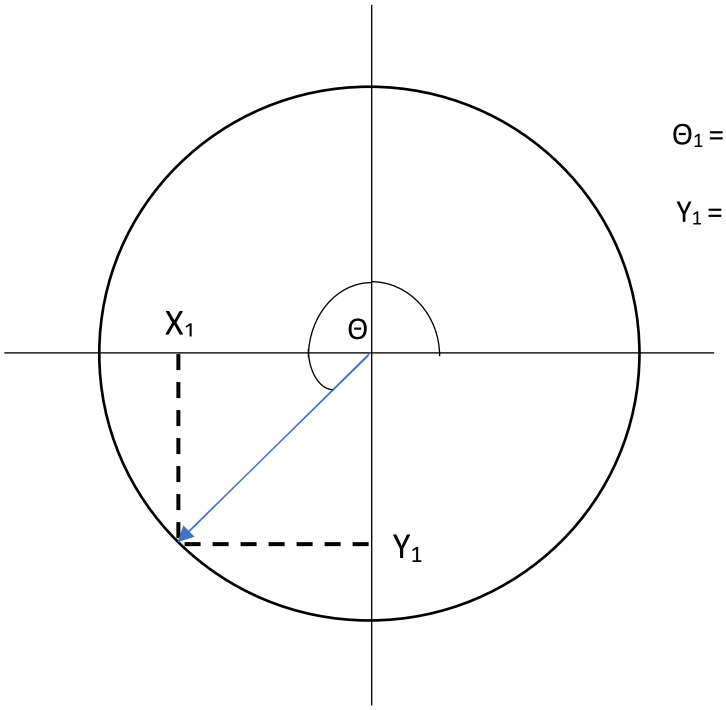
$Y_1 = \frac{1}{2}$, $X_1 = \frac{\sqrt{3}}{2}$

$Y_2 = \frac{\sqrt{3}}{2}$, $X_2 = \frac{1}{2}$



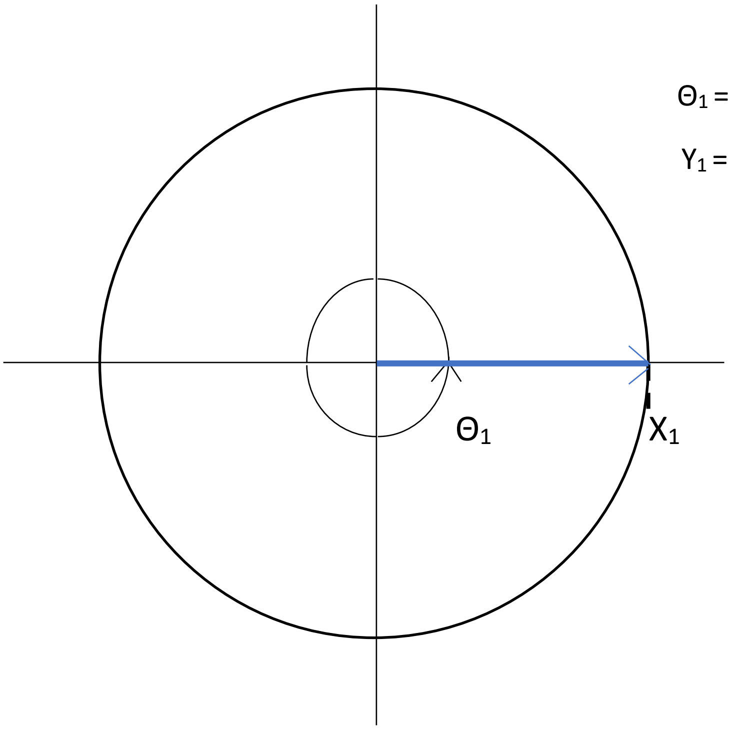
$\Theta_1 = 135^\circ$

$Y_1 = \frac{\sqrt{2}}{2}$, $X_1 = -\frac{\sqrt{2}}{2}$



$\Theta_1 = 225^\circ$

$Y_1 = -\frac{\sqrt{2}}{2}\,,\quad X_1 = -\frac{\sqrt{2}}{2}$



$\Theta_1 = 360^\circ$

$Y_1 = 0\,,\quad X_1 = 1$

Trigonometric Function Data For A Constant Value of r = 1:

Angle θ (Radians)	Angle θ (Degrees)	Y = SIN(θ)	X = COS(θ)
0	0	0	1
0.1	5.29578	0.09983342	0.99500417
0.2	11.02578	0.19866933	0.98006658
0.3	16.75578	0.29552021	0.95533649
0.4	22.48578	0.38941834	0.92106099
0.5	28.21578	0.47942554	0.87758256
0.6	33.94578	0.56464247	0.82533561
0.7	39.67578	0.64421769	0.76484219
0.8	45.40578	0.71735609	0.69670671
0.9	51.13578	0.78332691	0.62160997
1	56.86578	0.84147098	0.54030231
1.1	62.59578	0.89120736	0.45359612
1.2	68.32578	0.93203909	0.36235775
1.3	74.05578	0.96355819	0.26749883
1.4	79.78578	0.98544973	0.16996714
1.5	85.51578	0.99749499	0.0707372
1.6	91.24578	0.9995736	-0.0291995
1.7	96.97578	0.99166481	-0.1288445
1.8	102.7058	0.97384763	-0.2272021
1.9	108.4358	0.94630009	-0.3232896
2	114.1658	0.90929743	-0.4161468
2.1	119.8958	0.86320937	-0.5048461
2.2	125.6258	0.8084964	-0.5885011
2.3	131.3558	0.74570521	-0.666276
2.4	137.0858	0.67546318	-0.7373937
2.5	142.8158	0.59847214	-0.8011436
2.6	148.5458	0.51550137	-0.8568888
2.7	154.2758	0.42737988	-0.9040721
2.8	160.0058	0.33498815	-0.9422223
2.9	165.7358	0.23924933	-0.9709582
3	171.4658	0.14112001	-0.9899925
3.1	177.1958	0.04158066	-0.9991352
3.2	182.9258	-0.0583741	-0.9982948
3.3	188.6558	-0.1577457	-0.9874798
3.4	194.3858	-0.2555411	-0.9667982
3.5	200.1158	-0.3507832	-0.9364567
3.6	205.8458	-0.4425204	-0.8967584
3.7	211.5758	-0.5298361	-0.8481
3.8	217.3058	-0.6118579	-0.7909677
3.9	223.0358	-0.6877662	-0.7259323

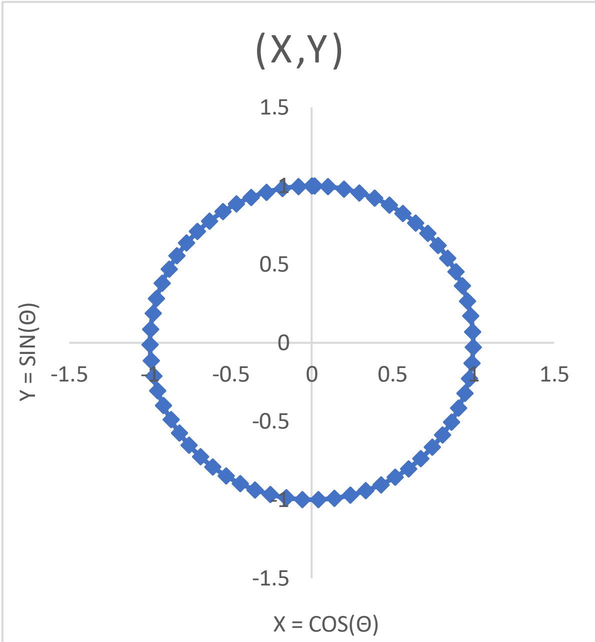
4	228.7658	-0.7568025	-0.6536436
4.1	234.4958	-0.8182771	-0.5748239
4.2	240.2258	-0.8715758	-0.4902608
4.3	245.9558	-0.9161659	-0.4007992
4.4	251.6858	-0.9516021	-0.3073329
4.5	257.4158	-0.9775301	-0.2107958
4.6	263.1458	-0.993691	-0.1121525
4.7	268.8758	-0.9999233	-0.0123887
4.8	274.6058	-0.9961646	0.08749898
4.9	280.3358	-0.9824526	0.18651237
5	286.0658	-0.9589243	0.28366219
5.1	291.7958	-0.9258147	0.37797774
5.2	297.5258	-0.8834547	0.46851667
5.3	303.2558	-0.8322674	0.55437434
5.4	308.9858	-0.7727645	0.63469288
5.5	314.7158	-0.7055403	0.70866977
5.6	320.4458	-0.6312666	0.77556588
5.7	326.1758	-0.5506855	0.83471278
5.8	331.9058	-0.4646022	0.88551952
5.9	337.6358	-0.3738767	0.92747843
6	343.3658	-0.2794155	0.96017029
6.1	349.0958	-0.1821625	0.98326844
6.2	354.8258	-0.0830894	0.9965421
6.3	360.5558	0.0168139	0.99985864

Table 1. is a table of data of the trigonometric functions sin(θ) & cos(θ) where r = 1. The angle θ in column 1 is in Radians from 0 to 6.3 in increments of 0.1. In column 2 θ is in degrees from 0˚ to 360˚ in increments of 5.29578˚, 0.1 Radians is equivalent to 5.29578˚ and 6.3 Radians is equivalent to 360.5558˚. Radians is a measure of angle similarly to degrees with a conversion constant of π since the circumference of the unit circle is 2π. In columns 3, 4 & 5 are the values of the x & y coordinates which are the sin and cos values at each angle.

Opposite is the plot of x and y data in columns 3 & 4, which shows the relationship between the equations:

$$x^2 + y^2 = r^2$$

$$r^2 \cos^2(\theta) + r^2 \sin^2(\theta) = r^2$$



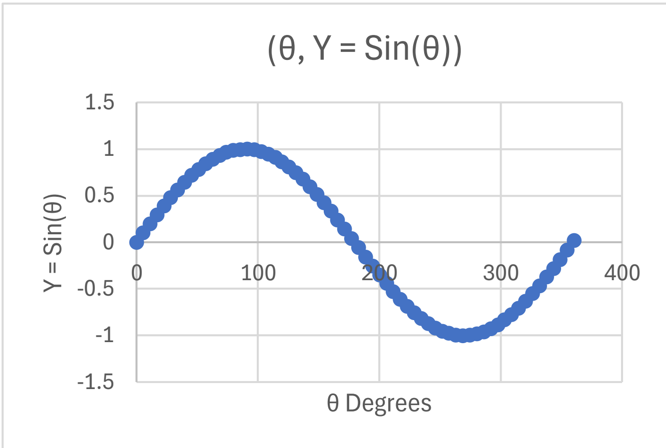
For the unit circle where r = 1:

$$x^2 + y^2 = 1$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

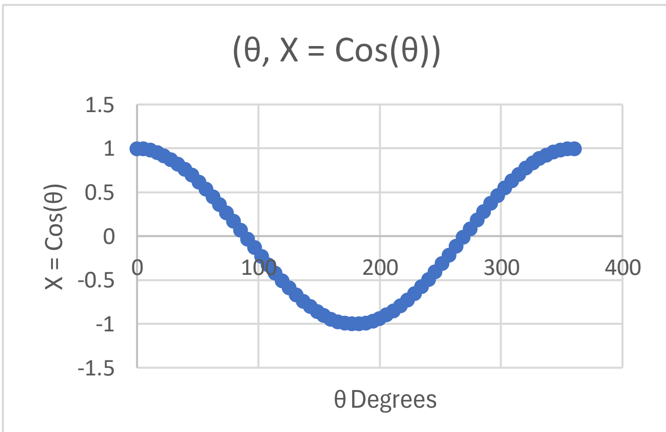
Opposite in the blue dots is the plot of Y values and θ values in columns 2 & 3, which shows the y component values for the radius vector r. As the angle increases between the r vector and the x axis the y value changes in the following way as seen in the graph:

$$y = \sin\theta$$



In the second waveform is a plot of the X value and θ value in column 2 and 4, which shows hoe x component of the radial vector r change with the angle between the r vector and the x axes.

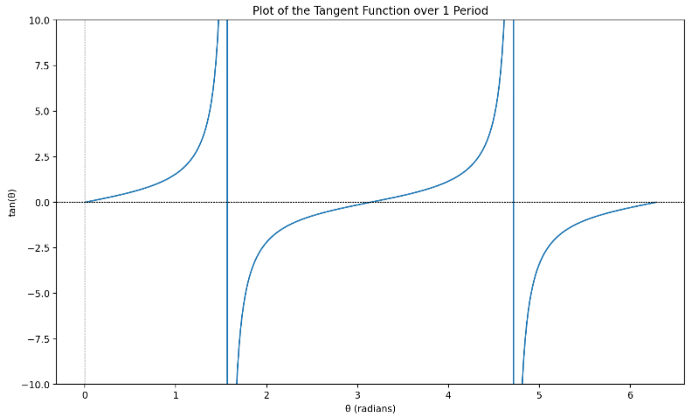
$$x = \cos\theta$$



We can think of the angle increasing over time with a frequency f or period T and plotting the circles circumference point by point. The sin and cos functions graphs show how the x and y radial coordinates or components change over time as the radial line processes. In this case we would use an equation which contains the frequency or the period which is the time it takes to complete 1 full circle of 360°:

$$\theta = \omega t = 2\pi f t = \frac{2\pi t}{T}$$

Opposite is the plot of the function tan(θ) which shows the relationship between the x and y data in columns 3 & 4 and the graphs of sin(θ) and cos(θ) where:



$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x}$$

At 0 sin(θ) is 0 and cos(θ) is 1, as θ increases to 90˚ cos(θ) decreases to 0 and tan(θ) increases to a positive infinity then switches to a negative infinity, as θ increases to 180˚ cos(θ) decreases to -1, sin(θ) decreases to 0 and tan(θ) returns to 0. From 180˚ to 360˚ the tan(θ) repeats itself.

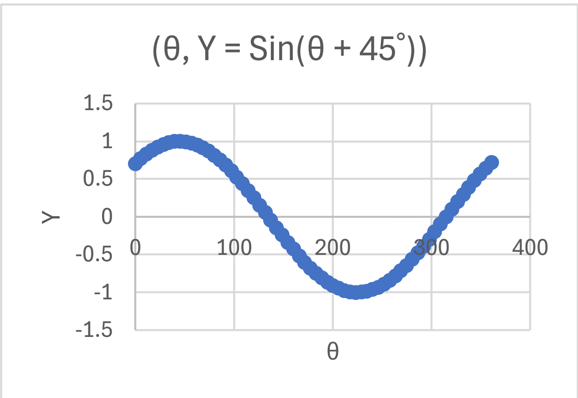
The trigonometric functions repeat the same values and are known as cyclic functions, which are useful in differential equations where the derivatives contain the same function.

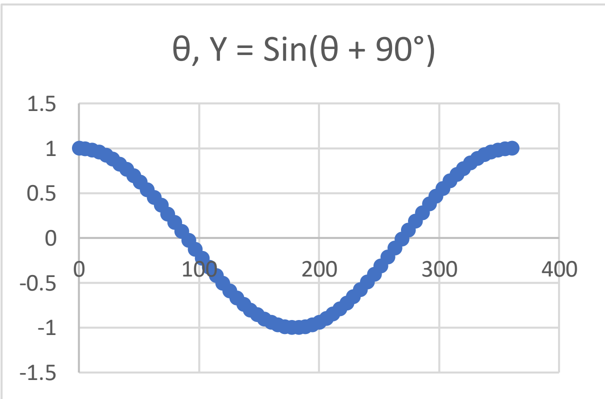
Phase Shifts

A phase shift is where the angle θ of the trigonometric function is shifted by some angle $\pm\,\varphi$ which changes the coordinate values of x & y on the circle for a given value of θ. We can use some equations and graphs to visualise these:

$$y = \sin(\theta + \varphi)$$

$$x = \cos(\theta + \varphi)$$





In the example above where y = Sin(θ+45°) each value θ is shifted by 45° and the waveform at θ = 0 has a different value of sin(45°) where y = 0.707 whereas for sin(0) y = 0. In the second waveform where y = sin(θ+90°) each θ value is shifted by 90° and the waveform is the same as the cos(θ) function:

$$y = \sin(\theta + 90°) = \cos(\theta)$$