Newtons Second Law & Periodic Motion

Differential equations have a derivative function of some variable implicit in the equation which can be solved to find the non-derivate function or unique solutions with initial conditions. The equation can include terms of the same variable, and the non-derivative function must satisfy the condition of the equating the derivative and non-derivative terms. This can be achieved by using known functions that are cyclic which means the functions derivate terms are the same as the initial function. Differential equations of this kind are often applied in Physics where phenomenon have a restorative force that is opposite to the displacement of a system, giving it oscillatory motion also known as simple harmonic motion of a spring or pendulum. The equation of motion for a spring is Newtons Second Law F=ma, where the force on a system is proportional to its acceleration and its mass.

In the case of an ideal spring the strength and force of the spring is quantified as the spring constant k, and the equation of motion for the ideal spring is proportional to the displacement x of the spring:

$$F = ma = -kx$$

In differential form where the acceleration is the second derivative of the displacement and first derivative of the velocity the equation is:

Eq 1.
$$\frac{d^2x}{dt^2} = -\frac{k}{m}x$$

For an ideal spring with no resistive force and omitting the mass of the spring then Eq 1. Says that for some function of time for a displacement of x from x=0 there is a second order derivative of the function or an acceleration at the position x proportional to the displacement x dependent on the spring constant k and the mass m.

We can solve differential equations such as these readily with possible solutions such as cyclic functions, this is where the function x in the RHS of the equation is of the same form as the second derivative of the function x on the LHS of the equation.

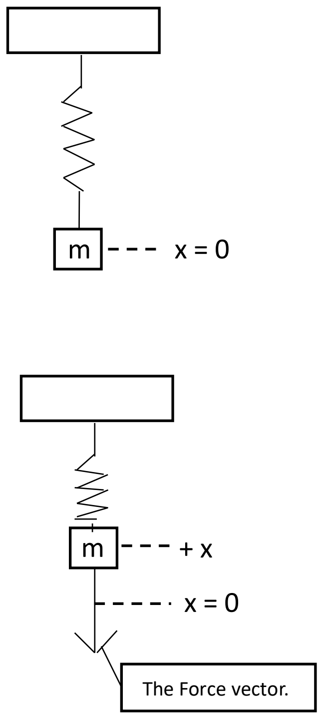


Figure 1. A compressed spring system at displacement +x, with a force vector toward the equilibrium position at x = 0.

$$\frac{d^2(Asin(\omega t))}{dt^2} = -\frac{k}{m}(Asin(\omega t))$$

$$-A\omega^2 \sin(\omega t) = -\frac{k}{m}\left(Asin(\omega t)\right)$$

This equation is true for the following conditions:

$$\omega^2 = \frac{k}{m}$$

The condition for the angular frequency 2πf of the simple harmonic motion differential equation:

$$\omega = 2\pi f = \sqrt{\frac{k}{m}}$$

the Time Period T of the system to complete one oscillation

$$T = 2\pi\sqrt{\frac{m}{k}}$$

The solution of the differential equation shows the possible oscillations of the system from initial conditions of time t = 0. The solution should cover all possible oscillations. We can generalise the function by adding a phase angle so that at t=0 the system could start at a different point in the cycle then displacement at some time is:

$$x(t) = Asin(\omega t + \varphi)$$

$$x(t) = Csin(\omega t) + Bcos(\omega t)$$

Where:

$$C = Acos(\varphi) \quad \& \quad B = Asin(\varphi)$$

At time t=0 the displacement of the system is:

$$x = Asin(\varphi)$$

The phase angle is then:

$$\varphi = sin^{-1}\left(\frac{x}{A}\right)$$

At time t=0 we can consider the initial velocity:

$$\frac{dx}{dt} = v = \omega Acos(\varphi)$$

$$A = \frac{v}{\omega cos(\varphi)}$$

Damped Harmonic Oscillations

For a damped harmonic oscillator there a resistive force of friction that reduces the amplitude of the oscillations and returns the system to equilibrium over some time t. The resistive force is proportional to the velocity of the system and has a damping factor λ determining whether the oscillations are overdamped, underdamped or critically damped where the system returns to equilibrium without any oscillations.

The differential equation describing this motion has a resultant force & acceleration:

$$F_R = -F_s + F_D$$

$$m\frac{d^2x}{dt^2} = -kx + \lambda\frac{dx}{dt}$$

Eq 2.
$$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + {\omega_0}^2 x = 0$$

In the case where the displacement of the system is 0 the gradient and velocity is maximum and the acceleration is 0 then the equation is:

$$\frac{dx}{dt} = -\frac{k}{\lambda m}x$$

$$\int \frac{1}{x}dx = \int -\frac{k}{\lambda m}dt$$

$$\ln(x) = -\frac{kt}{\lambda m} + c$$

Eq 3.
$$x = e^{-\frac{kt}{\lambda m}+c} = C\, e^{-\gamma t}$$

The equation of motion including a damping term is the resultant force on the mass equal to the second derivative of the displacement function of time. We can apply a similar reasoning as with Eq. 1 to find a solution to the equation by combining the solutions for x(t) in Eq 3. & for the undamped oscillator and substitute it into Eq 2.:

$$x(t) = C\, e^{-\gamma t}\, sin(\omega t + \varphi)$$

$$\frac{d^2(Ce^{-\gamma t}sin(\omega t))}{dt^2} - \lambda\frac{d\left(Ce^{-\gamma t}sin(\omega t)\right)}{dt} + {\omega_0}^2(Ce^{-\gamma t}sin(\omega t)) = 0$$

$$\lambda\frac{d\left(Ce^{-\gamma t}sin(\omega t)\right)}{dt} = -\lambda(-\gamma Ce^{-\gamma t}sin(\omega t) + \omega Ce^{-\gamma t}cos(\omega t))$$

$$\frac{d^2(Ce^{-\gamma t}sin(\omega t)\,)}{dt^2} = (\gamma^2 Ce^{-\gamma t}sin(\omega t) - \gamma\omega Ce^{-\gamma t}\cos(\omega t) - \gamma\omega Ce^{-\gamma t}cos(\omega t) - \omega^2 Ce^{-\gamma t}sin(\omega t))$$

We can equate the coefficients of the sin & cos terms on the LHS & RHS of the equation and cancel out the common coefficients resulting in conditions for the angular frequency:

$$\gamma^2 Ce^{-\gamma t}sin(\omega t) - \omega^2 Ce^{-\gamma t}sin(\omega t) - \lambda\gamma Ce^{-\gamma t}sin(\omega t) + {\omega_0}^2 Ce^{-\gamma t}sin(\omega t) = 0$$

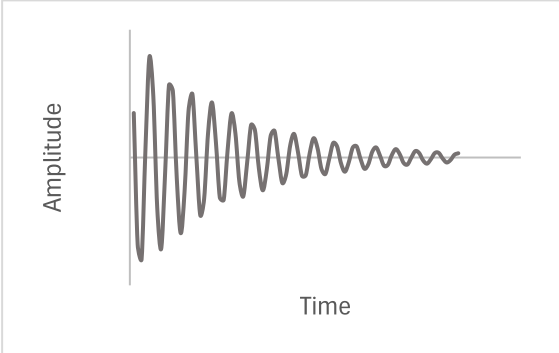
$$-\gamma\omega Ce^{-\gamma t}\cos(\omega t) - \gamma\omega Ce^{-\gamma t}cos(\omega t) - \lambda\omega Ce^{-\gamma t}cos(\omega t) = 0$$

$${\omega_0}^2 + \gamma^2 - \lambda\gamma \;=\; \omega^2$$
$$\lambda = 2\gamma$$

The angular frequency of the damped oscillator is a function of the natural frequency and the damping factor:

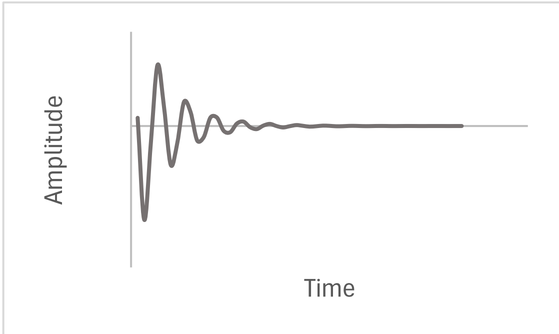
$$\omega^2 = {\omega_0}^2 - \gamma^2$$

We can use graphs of the function for a Damped Harmonic Oscillator for given parameters and initial conditions at t=0.



The top waveform in Figure 2 shows an underdamped oscillator where the amplitude of the decays exponentially and the frequencies remain the same, it has a decay factor of 0.4 with the function:

$$x(t) = 15\, e^{-0.4t}\, cos(12t)$$



The lower waveform shows the graph of an overdamped oscillator where the oscillations stop much sooner with few oscillations, it has a decay factor of 2 with the function:

$$x(t) = 15\, e^{-2t}\, cos(12t)$$

Figure 2.

Forced Damped Harmonic Oscillator

The forced damped harmonic oscillator has another term for an applied force to the system which increases the amplitude, the applied force can be periodic or a pulse depending on the type of constraints. The oscillating system has a natural frequency determined by boundary conditions and the restorative force constant. The applied force can have a frequency which changes the frequency of the oscillations from the natural frequency giving it more modes of vibration. In the case where the applied frequency and natural frequency constructively, the system undergoes resonance which is a maximum amplitude of the system. This equation has the following form:

Eq. 3 $$\frac{d^2x}{dt^2} + \lambda\frac{dx}{dt} + {\omega_0}^2 x = Fcos(\omega_{\mathrm{A}} t)$$

The solution to the forced Harmonic Oscillator has two parts, the damped harmonic solution without the force term and the solution with the force term. For the solution with the force term in Eq. 3, we can substitute a solution:

$$x(t) = Asin(\omega_{\mathrm{A}} t) + Bcos(\omega_{\mathrm{A}} t)$$

$$\frac{d^2(\ Asin(\omega_A t) + Bcos(\omega_A t))}{dt^2} + \lambda \frac{d\ (Asin(\omega_A t) + Bcos(\omega_A t))}{dt} + \omega_0{}^2\ (Asin(\omega_A t) + Bcos(\omega_A t))\ \ = Fcos(\omega_A t)$$

$$\lambda \frac{d\ (Asin(\omega_A t) + Bcos(\omega_A t))}{dt} =\ \lambda(-\omega_A B\ sin(\omega_A t) + \omega_A A\ cos(\omega_A t))$$

$$\frac{d^2(\ Asin(\omega_A t) + Bcos(\omega_A t)\ )}{dt^2} = (-\omega_A{}^2 B\ cos(\omega_A t) - \omega_A{}^2 Asin(\omega_A t))$$

We can equate the coefficients of the terms on the LHS & RHS omitting common factors:

$$\left(\left(-\omega_A{}^2 A - \lambda\omega_A B + A\omega_0{}^2\right) sin(\omega t) + \left(-\omega_A{}^2 B + \lambda\omega_A A + B\omega_0{}^2\right) cos(\omega t)\right) = Fcos(\omega_A t)$$

$$\lambda\omega_A A + B(\omega_0{}^2\ - \omega_A{}^2) = F$$

$$A = \frac{\lambda\omega_A B}{(\omega_0{}^2 - \omega_A{}^2)}$$

$$B = \frac{(\omega_0{}^2 - \omega_A{}^2)F}{(\omega_0{}^2\ - \omega_A{}^2)^2 + (\lambda\omega_A)^2}$$

$$A = \frac{(\lambda\omega_A)F}{(\omega_0{}^2\ - \omega_A{}^2)^2 + (\lambda\omega_A)^2}$$

A & B are the amplitude functions of the force amplitude for the different phases of the sine and cos terms:

$$x(t) = \frac{(\lambda\omega_A)F}{(\omega_0{}^2\ - \omega_A{}^2)^2 + (\lambda\omega_A)^2}\ sin(\omega_A t) + \frac{(\omega_0{}^2 - \omega_A{}^2)F}{(\omega_0{}^2\ - \omega_A{}^2)^2 + (\lambda\omega_A)^2} cos(\omega_A t)$$

The solution of the homogenous equation plus the inhomogeneous equation are a linear sum and a solution to the Eq.3 the coefficients of the force term and the non-forced term are determined by the initial conditions and the angular frequencies of the system plus the driving frequency:

$$x(t) = C\ e^{-\gamma t}\ sin(\omega t + \varphi) + (Asin(\omega_A t) + Bcos(\omega_A t))$$

The natural frequency of the system defined by its boundary conditions and the driving force frequency determine the vibrational modes of the system. In figure 3 is a plot of the function:

$$12e^{-0.5t}\sin(20.3t) + (6.75\cos(1.3t))$$

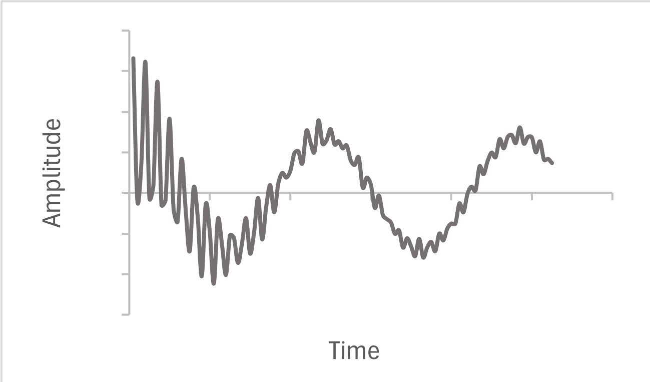


Figure 3.

<u>Complex Exponentials</u>

We can use complex exponential notation to find determine solutions to Second Order Differential equations by changing the trigonometric functions to the functions in Euler's formula. In Eq. 3 For a generalised periodic force where:

$$R\left\{\frac{d^2x_R(t)}{dt^2} + \lambda\frac{dx_R(t)}{dt} + {\omega_0}^2 x_R(t)\right\} = R\left\{Fe^{-(i\omega)t}\right\}$$

Substituting a possible solution for $x_R(t)$:

$$x_R(t) = Ce^{-(i\omega)t}$$

$$C((i\omega)^2 - \lambda(i\omega) + {\omega_0}^2) \;=\; F$$

$$C = \frac{F}{({\omega_0}^2 - \omega^2 - \lambda(i\omega))}$$

$$C = \frac{F}{({\omega_0}^2 - \omega^2 - \lambda(i\omega))} + \frac{({\omega_0}^2 - \omega^2 + \lambda(i\omega))}{({\omega_0}^2 - \omega^2 + \lambda(i\omega))} = \frac{({\omega_0}^2 - \omega^2) + i\lambda\omega}{(({\omega_0}^2 - \omega^2)^2) + (\lambda^2\omega^2))}F$$

For a complex number the amplitude function C in terms of its components is:

$$C = A + iB$$

$$A = \frac{({\omega_0}^2 - \omega^2)F}{(({\omega_0}^2 - \omega^2)^2) + (\lambda^2\omega^2))} \quad \& \quad B = \frac{\lambda\omega F}{(({\omega_0}^2 - \omega^2)^2) + (\lambda^2\omega^2))}$$

The real part of the solution for $x_R(t)$:

$$R\left\{Ce^{-(i\omega)t}\right\} = \; R\{(A + iB)(\cos(\omega t) - isin(\omega t))\}$$

Coupled Oscillator For A Spring Mass System.

We can analyse the dynamics of a system by the conservation of energy from potential energy to kinetic energy, for a closed system this value remains constant and applies to oscillatory systems where the energy is stored as a mechanical force such as tension. Using the mechanical definitions for potential energy PE & Kinetic energy KE the Lagrange equations is:

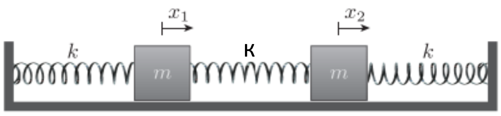


Figure 1.

In the case of the coupled oscillator, we can find the resultant force on each mass due the compression or extension to formulate the equations of motion.

$$F_{x1} = m\ddot{x}_1 = -kx_1 + \kappa x_2 - \kappa x_1 = -(k + \kappa)x_1 + \kappa x_2$$

$$F_{x2} = m\ddot{x}_2 = -kx_2 - \kappa x_2 + \kappa x_1 = -(k+\kappa)x_2 + \kappa x_1$$

$$\ddot{y}_S = F_{x1} + F_{x2} = \ddot{x}_1 + \ddot{x}_2 = -k(x_1 + x_2)$$

$$\ddot{y}_F = F_{x1} - F_{x2} = \ddot{x}_1 - \ddot{x}_2 = -(k + 2\kappa)(x_1 - x_2)$$

In the case of $\ddot{y}_S$ we have the function:

$$\omega_S = \sqrt{\frac{k}{m}}$$

$$x_1 + x_2 = A_s cos(\omega_S t + \varphi_s)$$

For $\ddot{y}_F$:

$$\omega_F = \sqrt{\frac{k + 2\kappa}{m}}$$

$$x_1 - x_2 = A_F cos(\omega_F t + \varphi_F)$$

The solutions for $\ddot{y}_S$ & $\ddot{y}_F$ are the normal modes of oscillations of the system where a general solution is a linear combination of the two modes in the form:

$$x_1 = \frac{1}{2}((x_1 + x_2) + (x_1 - x_2)) = \frac{1}{2}(A_s cos(\omega_S t + \varphi_S) + A_F cos(\omega_F t + \varphi_F))$$

$$x_2 = \frac{1}{2}((x_1 + x_2) - (x_1 - x_2)) = \frac{1}{2}(A_s cos(\omega_S t + \varphi_S) - A_F cos(\omega_F t + \varphi_F))$$

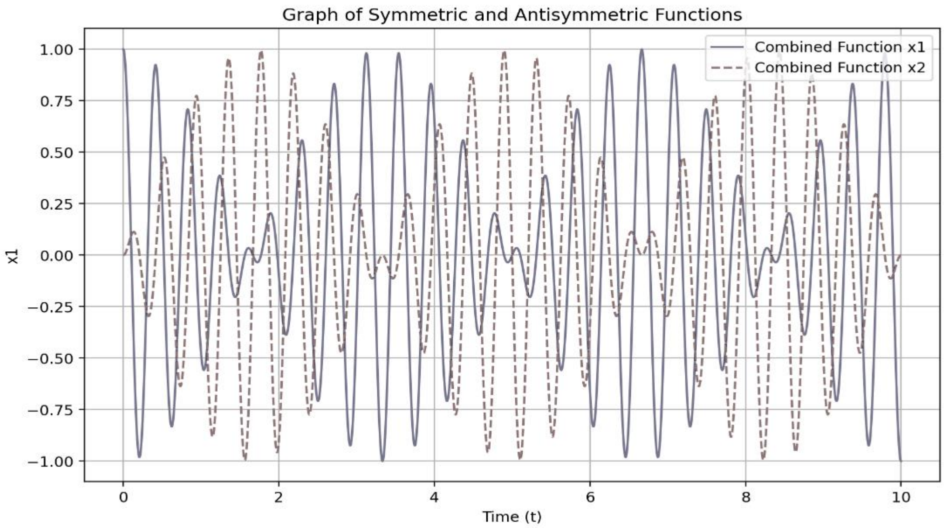


Figure 2. A plot of $\frac{1}{2}(A_s cos(\omega_S t + \varphi_s) \pm A_F cos(\omega_F t + \varphi_F))$ for $\omega_S = 4.5\pi$ $\omega_f = 5.1\pi$