The Wave Equation

Travelling waves such as Sound waves or Electromagnetic waves can be considered as sinusoidal wavefunctions propagating in some direction with some frequency and wavelength. The speed of the waves v also known as the dispersion relation is defined by the wavelength & frequency:

$$v = \frac{\lambda}{T} = \lambda f$$

The units of wavelength are meters, and the speed of the wave is meters per second (ms^{-1}). The wavelength as angular definition called the wavenumber given by:

$$k = \frac{2\pi}{\lambda}$$

We can then define the wave travelling in the x direction as a function of time & distance in a sinusoidal form as:

$$f(t,x) = Asin(\omega t \pm kx)$$

The function f(t,x) for a time t is defined over a distance x and for a distance x is defined over some time t.

For a changing f(t,x) or time varying f(t,x) the function has derivatives of t and derivatives of x. Both are orthogonal to f(t,x) so are partial derivatives and for a constant velocity v the spatial component is not a function of time. The second derivatives of time and space are then related in the following way:

Eq. 1
$$\frac{\partial^2 f(t,x)}{\partial x^2} = C\frac{\partial^2 f(t,x)}{\partial t^2}$$

$$\frac{\partial^2 (Asin(\omega t + kx))}{\partial x^2} = -\beta^2 Asin(\omega t + kx)$$

$$\frac{\partial^2 (Asin(\omega t + kx))}{\partial x^2} = -\omega^2 CAsin(\omega t + kx)$$

Substituting the derivatives into Eq.1 and cancelling terms:

$$k^2 = \omega^2 C$$

$$C = \frac{k^2}{\omega^2} = \left(\frac{2\pi}{\lambda}\frac{1}{2\pi f}\right)^2 = \left(\frac{1}{\lambda f}\right)^2 = \frac{1}{v^2}$$

So the derivatives are related by the speed of the wave:

$$\frac{\partial^2 f(t,x)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f(t,x)}{\partial t^2}$$

Standing Waves

Standing waves are most evident in musical instruments where there are fixed ends or one fixed end of a vibrating medium resulting in displacements or pressure variations contained in the instrument such as a guitar string or clarinet. In the case of a vibrating string such as a Piano or Harp, both ends are fixed and so the function f(t,x) is 0 at x = 0 and x = L, the length of the string. The standing waves occur when a wave travels in one direction on the string reflects and travels in the opposite direction, the waves constructively & destructively interfere. We can formulate the oscillations in the following way:

$$f(t,x) = \; Acos(\omega t + kx) + Acos(\omega t - kx)$$

$$f(t,x) = (Asin(\omega t)\sin{(kx)} - Acos(kx)\cos(\omega t)) + (Asin(\omega t)\sin{(kx)} + Acos(kx)\cos(\omega t))$$

$$f(t,x) = 2Asin(\omega t)\sin{(kx)}$$

Applying the Boundary conditions then for x = L and for any t f(t,x)=0:

$$0 = \sin{(kL)}$$

This condition applies for $\sin{(n\pi)}$ for even numbers of n, therefore we can say that for the value k and in terms of the wavelength λ:

$$kL = \mathrm{n\pi} \qquad \& \qquad \lambda = \frac{\mathrm{2L}}{n}, \qquad n = 1, 2, 3, 4.5 \ldots.$$

This formula relates the possible wavelengths or harmonics possible on the vibrating system, dependent on the boundary conditions and the speed v of the waves.