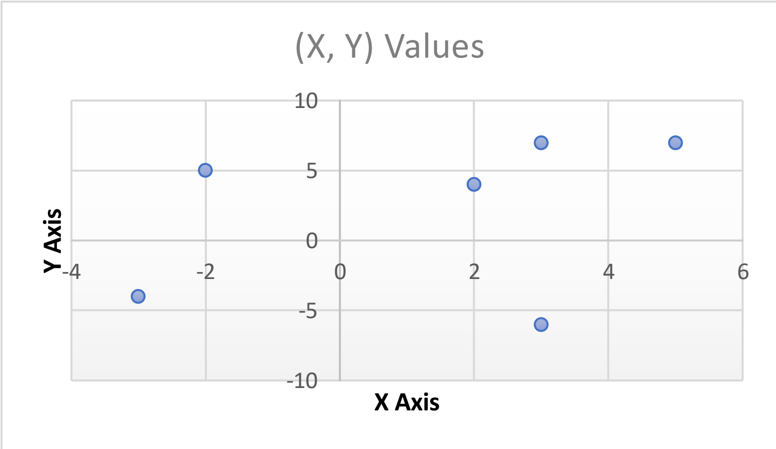
Functions In The Cartesian Plane

The cartesian plane is a coordinate system with perpendicular axes x & y, where the x and y set of natural numbers select a point in the plane, denoted by (x, y), for example (5, 7), (2, 4), (-2, 5), (3, 7), (-3, -4) & (3, -6) as shown in the table and graph below.

X	Y
5	7
2	4
-2	5
3	7
-3	-4
3	-6

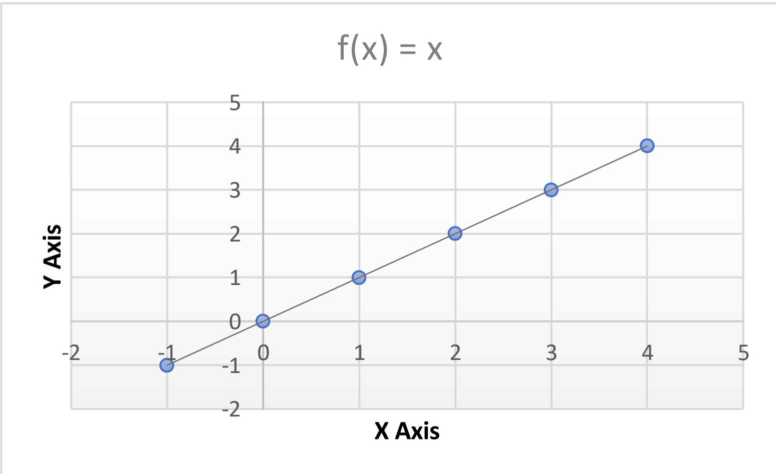


Another form of identifying points in the X-Y plane or listing a set of data is using a Function or Equation. The value of the function or the y variable is determined by the x value. This has the notation f(x) where the array of x & y values can form a solution to f(x):

$$y = f(x)$$

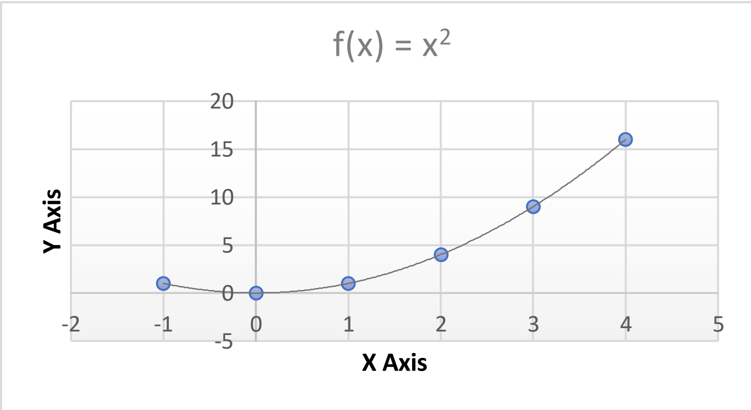
In the case where f(x) is x then the y value is equal to x as is shown in the table and graph below. The precision of the numbers and data can also be displayed in the table and chart such as 2.73 or 1.587 which shows how many decimal places there are.

X	Y
-1	-1
0	0
1	1
2	2
3	3
4	4

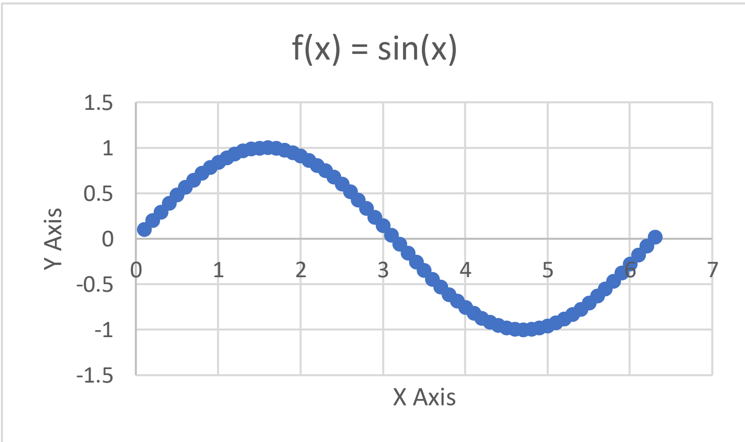


We can analyse the data and the function of x as polynomial where the variable x has an exponent n which could be 1, 2, 3… etc. For y = x the exponent of x is 1 (x^1). The possible functions of x such as Polynomial, Trigonometric, exponential or Logarithmic give information about the variability of f(x). Below are the tables and graphs of the function x^2, the trigonometric function cos(x) and the exponential function e^x.

X	Y
-1	1
0	0
1	1
2	4
3	9
4	16



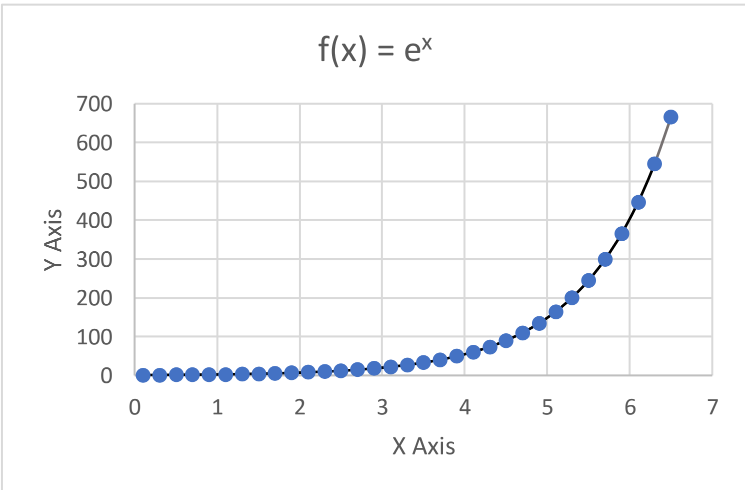
X	Y
0 < x < 6.3	-1 < y < 1



The x value increases in increments of 0.1 from 0 to 6.3 radians.

The sin(x) function varies by a ratio of y/r, for a unit circle r = 1 and x is the angle in radians.

X	Y
0.1 < x < 6.5	1.1 < y < 665.1



The letter e is Eulers number approximately equal to 2.72.

The exponential function increases in increasing increments and has the logarithmic inverse function.

The Gradient, Derivatives & Integrals of Functions

The derivative is a calculation of the gradient of the function, where the gradient is a change in the y value divided by a change in the x value. Below is the gradient shown on a graph and in an equation. In the case of a linear function for example y = 3x the gradient is a constant and equal to the coefficient of x which is 3.

As can be seen in figure 1 for a linear function with constant gradient the change in the x value Δx is:

$$\Delta\mathrm{x} = x_2 - x_1$$

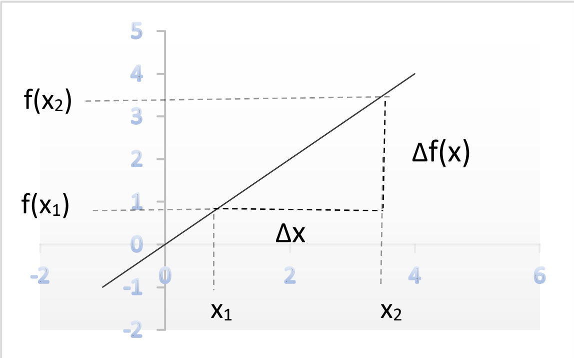


Figure 1. A linear function of x with constant gradient.

And the change in the value of the function is:

$$\Delta\mathrm{f(x)} = f(x_2) - f(x_1)$$

The gradient of the function from x_1 to x_2 is then:

$$\frac{\Delta\mathrm{f(x)}}{\Delta\mathrm{x}} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

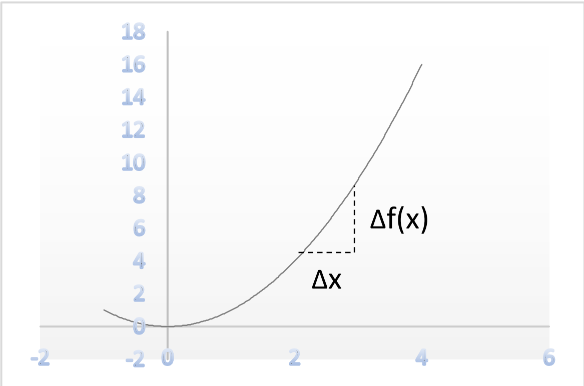


Figure 2. A quadratic function (x^2) of x with a varying gradient.

In figure 2 is shown the plot of a non-linear function where the gradient is changing and to obtain an accurate value the derivative is used. The differential uses small changes or infinitesimal changes in the value of x where Δx approaches 0 and has the notation dx (dx ≈ 0) and the change in the value of the function $\Delta\mathrm{f(x)} \to d\mathrm{f(x)}$. The gradient for some function of x in general form is then:

$$\frac{\mathrm{df(x)}}{\mathrm{dx}} = \frac{f(x_1 + dx) - f(x_1)}{(x_1 + dx) - x_1} = \frac{f(x_1 + dx) - f(x_1)}{dx}$$

In the case of figure 2 the function is x^2 and we can use the general equation to find the gradient function of an nth order polynomial x^n:

$$\frac{\mathrm{df(x)}}{\mathrm{dx}} = \frac{\mathrm{d}(\mathrm{x}^n)}{\mathrm{dx}} = \frac{(x+dx)^n - x^n}{(x+dx)-x}$$

$$\frac{\mathrm{df(x)}}{\mathrm{dx}} = \frac{(x^n + \; nx^{n-1}(dx) + nx(dx^{n-1}) \; + dx^n) - x^n}{(dx)} = \frac{nx^{n-1}(dx) \; + dx^n}{(dx)}$$

The (dx^n) terms in the equation are approximately 0 and can be discarded leaving a polynomial term with a reduced exponent giving the following generalised derivative of a polynomial:

$$\frac{\mathrm{df(x)}}{\mathrm{dx}} = nx^{n-1}$$

The trigonometric and exponential functions identified previously have generalised derivative rules and are known as cyclic functions as the derivative function contains the same function with a different coefficient these are listed in the table below. You can find lists of derivatives of various other functions.

f(x)	$\frac{\mathrm{df(x)}}{\mathrm{dx}}$
sin(nx)	ncos(nx)
cos(nx)	-nsin(nx)
-sin(nx)	-ncos(nx)
-cos(nx)	nsin(nx)
e^{nx}	ne^{nx}
a^x	$a^x \ln(a)$
x^n	nx^{n-1}
$(1 + f(x))^m$	$\frac{\mathrm{df(x)}}{\mathrm{dx}}$ m $(1+f(x))^{m-1}$

Table 1. Showing the differential functions.

Riemann Sums & Integration.

The Riemann sum is an approximate calculation of the area between the function f(x) and the independent variable x axis by splitting the area into rectangles where the area of one rectangle is the width x height. The width of a rectangle is:

$$\text{width} = \Delta\text{x} = x_2 - x_1$$

The height of the rectangle is $f(x_1)$ and the area is then:

$$\text{Area } = \text{f}(\text{x}_1)\Delta x$$

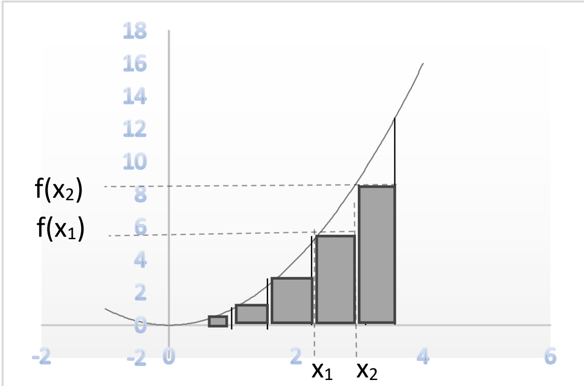


Figure 3. Rectangle segments of the function f(x).

We can then calculate the area from x = b to x = a using a summation series of rectangles at different values of x where Δx is the same for all x values:

$$Area = \sum_{a}^{b} \text{f}(\text{x}_a)\Delta\text{x} = (\text{f}(\text{x}_a) + \text{f}(\text{x}_{a+1}) + \text{f}(\text{x}_{a+2}) + f(\text{x}_{a+3}) + \text{f}(\text{x}_b))\,\Delta\text{x}$$

In the case where the graphed function is of the speed on the y axes and time on the x axes then the derivative of speed/time would represent an acceleration and the Riemann sum of the area speed x time would represent a distance.

As can be seen in figure 3 between the Riemann rectangles and the curve x^2, there are white right-angled triangle spaces. They are not accounted for in the calculation of the area under the curve, the integral method accounts for this by using infinitesimal widths where $\Delta x \rightarrow 0 = dx$ and the number of rectangles increases, this makes the calculation accurate in non-linear functions.

$$Area = \sum_{a}^{b} \text{f}(\text{x}_a)\text{dx} = (\text{f}(\text{x}_a) + \text{f}(\text{x}_a + dx) + \text{f}(\text{x}_a + 2dx) + \cdots + \text{f}(\text{x}_b))\,\text{dx}$$

We can assume the function f(x) is equal to the derivative of some function u(x) such that:

$$\text{f}(\text{x}) = \frac{\text{du}(\text{x})}{\text{dx}}$$

$$Area = \sum_{a}^{b} \frac{\mathrm{du}(\mathrm{x}_a)}{dx}\mathrm{dx} = \frac{\left(\mathrm{du}(\mathrm{x}_a) + \mathrm{du}(\mathrm{x}_a + dx) + \mathrm{du}(\mathrm{x}_a + 2dx) + \cdots + \mathrm{du}(\mathrm{x}_b)\right)}{dx} dx$$

$$\sum_{a}^{b} \frac{\mathrm{du}(\mathrm{x}_a)}{dx}\mathrm{dx} = ((\mathrm{u}(\mathrm{x}_a + dx) - \mathrm{u}(\mathrm{x}_a)) + (\mathrm{u}(\mathrm{x}_a + 2dx) - \mathrm{u}(\mathrm{x}_a + dx)) + (\mathrm{u}(\mathrm{x}_a + 3dx) - \mathrm{u}(\mathrm{x}_a + 2dx)) + \cdots + (\mathrm{u}(\mathrm{x}_b) - \mathrm{u}(\mathrm{x}_b - dx))$$

The terms in the summation in between the range of the summation cancel leaving the following expression.

$$\sum_{a}^{b} \frac{\mathrm{du}(\mathrm{x}_a)}{dx}\mathrm{dx} = \mathrm{u}(\mathrm{x}_b) - \mathrm{u}(\mathrm{x}_a)$$

The integral is then taking the limits of the summation where $\Delta x \to 0$ and the number of terms in the summation $n \to \infty$, so as the value of the areas become smaller number of areas in the summation increases.

$$A = \int_{a}^{b} \mathrm{f}(\mathrm{x})\mathrm{dx} = \int_{a}^{b} \frac{\mathrm{du}(\mathrm{x}_a)}{dx}\mathrm{dx} \quad = \mathrm{u}(\mathrm{x}_b) - \mathrm{u}(\mathrm{x}_a)$$

Below is the same table showing that the integral of the function f(x) is the differential of some function g(x).

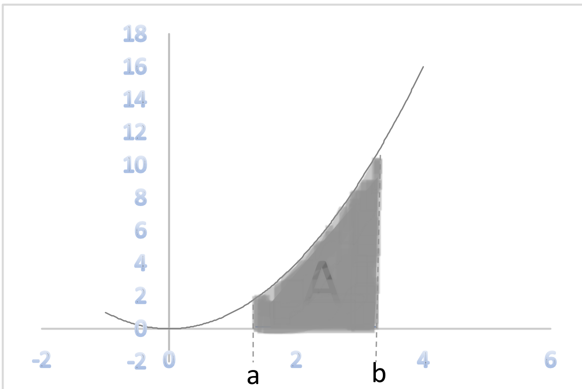


Figure 4. Integral area where the number of dx ≈ 0 & the number of Riemann terms n ≈ ∞.

u(x)	f(x)
sin(nx)	ncos(nx)
cos(nx)	-nsin(nx)
-sin(nx)	-ncos(nx)
-cos(nx)	nsin(nx)
e^{nx}	ne^{nx}
a^x	$a^x \ln(a)$
x^n	nx^{n-1}

Table 2. Showing the integrals of f(x) and g(x) is the Anti-derivative.

Second Order Derivatives, Double & Triple Integrals

For the first derivative we have the equation:

$$\frac{dy}{dx} = \frac{df(x)}{dx} = f'(x)$$

We can find the second derivative of the function f(x) by using the same process on f'(x):

$$\frac{d^2y}{dx^2} = \frac{d^2f(x)}{dx^2} = f''(x)$$

We can use an example from the table of derivatives of cyclic functions where the function is sin(nx):

$$\frac{dy}{dx} = \frac{d(\sin(nx))}{dx} = ncos(nx)$$

The second derivative is then:

$$\frac{d^2y}{dx^2} = \frac{d^2(\sin(nx))}{dx^2} = -n^2\sin\,(nx)$$

The Integral method can be multiplied to 2 & 3 dimensions by making the area integral a function of another integral. In the previous integral we have multiplied the y dimension by the x dimension f(x)dx to obtain an area, we can multiply the xy area in the z direction f(x,y)dxdy to obtain a volume or compute the integral of volumes dV = dxdydz and the volumes of revolution method to multiply an area dAdx.

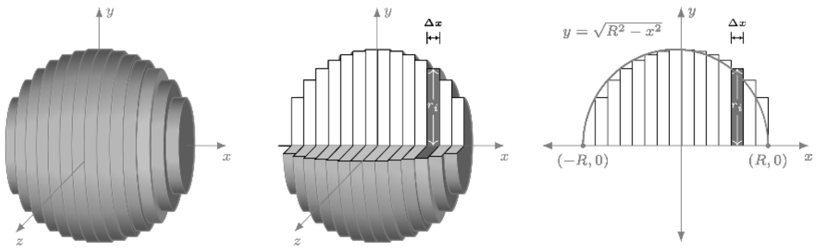
$$V = \iint f(x,y)dxdy$$

$$V = \iiint dxdydz$$

We can use the equation of a circle in xy coordinates as an example:

$$x^2 + y^2 = r^2$$

We can compute this integral in xyz coordinates by simplifying the equation using the volumes of revolution method to simplify the integration function to the known formula for the area of a circle πr^2 around the axis:



$$Area\ of\ Circles = f(x) = \pi y^2 = \pi(r^2 - x^2)$$

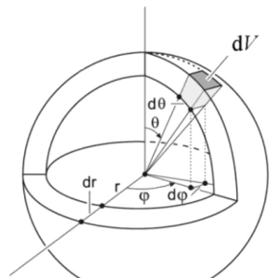
$$V = \int \pi y^2\, dx = \int_{-r}^{r} \pi(r^2 - x^2)\, dx = \left[\pi(xr^2 - \tfrac{x^3}{3})\right]_{-r}^{r} = \frac{4\pi r^3}{3}$$

We can also use the following relationships for x, y & z in spherical polar coordinates to compute the volume of a sphere where:

$$x = r\cos\theta\sin\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\varphi$$

Given the following integral where $r^2\sin\varphi$ is the coordinate transformation function from xyz coordinates to rθφ polar coordinates.

$$\oint dV = \iiint_0^r r^2 \sin\varphi\ \mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi$$



This computes to the volume of a sphere with radius r:

$$\frac{r^3}{3}\iint_0^{2\pi} \sin\varphi\ \mathrm{d}\theta\mathrm{d}\varphi = 2\pi\frac{r^3}{3}\int_{-\pi}^{\pi} \sin\varphi\ \mathrm{d}\varphi = (2\pi)\left(\frac{r^3}{3}\right)(2) = \frac{4\pi r^3}{3}$$

Product Rule of Differentiation & Integration

The Product rule of differentiation applies to the multiplication of functions of the same variable in the following way where u & v are some functions of x multiplied together:

$$\frac{\mathrm{d(uv)}}{\mathrm{dx}} = \frac{\mathrm{du}}{\mathrm{dx}}v + \frac{\mathrm{dv}}{\mathrm{dx}}u$$

We can use an example where u = x^3 and v= e^x which gives the following function:

$$uv = \mathrm{x}^3\mathrm{e}^x$$

$$\frac{\mathrm{du}}{\mathrm{dx}} = 3\mathrm{x}^2 \qquad and \qquad \frac{\mathrm{dv}}{\mathrm{dx}} = \mathrm{e}^x$$

$$\frac{\mathrm{d(uv)}}{\mathrm{dx}} = 3\mathrm{x}^2\mathrm{e}^x + \mathrm{x}^3\mathrm{e}^x = \mathrm{e}^x\,(3\mathrm{x}^2 + \mathrm{x}^3)$$

The integration by parts rule is the integral of two functions of x multiplied together such as uv. We can prove that the by parts integral is the antiderivative of the derivative using the function uv in the previous example:

Assuming v is the integral of the differential function of v where:

$$v = \int \frac{\mathrm{dv}}{\mathrm{dx}}dx \qquad u = (3\mathrm{x}^2 + \mathrm{x}^3), \quad \frac{\mathrm{dv}}{\mathrm{dx}} = \mathrm{e}^x$$

$$\int u\frac{\mathrm{dv}}{\mathrm{dx}}\,dx = \mathrm{u}v - \int v\frac{\mathrm{du}}{\mathrm{dx}}\,dx$$

$$\int \mathrm{e}^x(3\mathrm{x}^2 + \mathrm{x}^3)\,dx = \mathrm{e}^x(3\mathrm{x}^2 + \mathrm{x}^3) - \int (6\mathrm{x} + 3\mathrm{x}^2)\mathrm{e}^x\,dx$$

$$\int \mathrm{e}^x(3\mathrm{x}^2 + \mathrm{x}^3)\,dx = \mathrm{e}^x\left(3\mathrm{x}^2 + \mathrm{x}^3\right) + ((6\mathrm{x} + 3\mathrm{x}^2)\mathrm{e}^x - \int (6 + 6x)\mathrm{e}^x\,dx$$

$$\int \mathrm{e}^x(3\mathrm{x}^2 + \mathrm{x}^3)\,dx = \mathrm{e}^x(\mathrm{x}^3 + 6) - \int 6\mathrm{e}^x\,dx = \mathrm{e}^x(\mathrm{x}^3 + 6) - 6\mathrm{e}^x = \mathrm{x}^3\mathrm{e}^x$$

Line Integrals

A line integral can used to determine the length of an arbitrary curve or for a calculation of component forces like the energy transfer of a force on a moving system.

To compute the line integral we can sum the x & y components of the function and using the infinitesimal limits get an accurate result.

A small segment of the line dr is:

$$\Delta r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

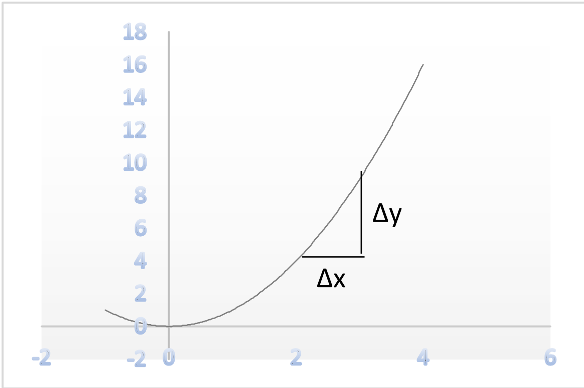


Figure 1

We can then sum the line segments and take the integral limits from some point on the curve a and some point b:

$$\int_a^b dr = \int_a^b \sqrt{(dx)^2 + (dy)^2} = \int_a^b dx \sqrt{1 + (\frac{dy}{dx})^2}$$

We can then use the example of the function x^2 where:

$$\frac{dy}{dx} = 2x$$

Giving the following integral:

$$\int_a^b dx\sqrt{1 + (2x)^2} = \int_a^b (1 + 4x^2)^{\frac{1}{2}}\, dx$$

$$\int_a^b (1 + 4x^2)^{\frac{1}{2}}\, dx = \left[\frac{1}{12x}(1 + 4x^2)^{\frac{3}{2}}\right]_a^b$$

This can be extended to 3 dimensions which in cartesian coordinates is:

$$dr = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = dx\sqrt{1 + (\frac{dy}{dx})^2 + (\frac{dz}{dx})^2}$$