

## Fourier Series

Suppose we have an arbitrary periodic function  $f(t)$  that repeats itself every Time period  $T$  with a frequency  $1/T$ . Under certain conditions the function can be expressed as the sum of an infinite number of sin & cosine functions known as a Fourier series. The Fourier equation is defined by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right) \right)$$

Where the amplitudes or coefficients of each term are defined as:

$$a_0 = \frac{2}{T} \int_0^T f(t) dt$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2n\pi t}{T}\right) dt$$

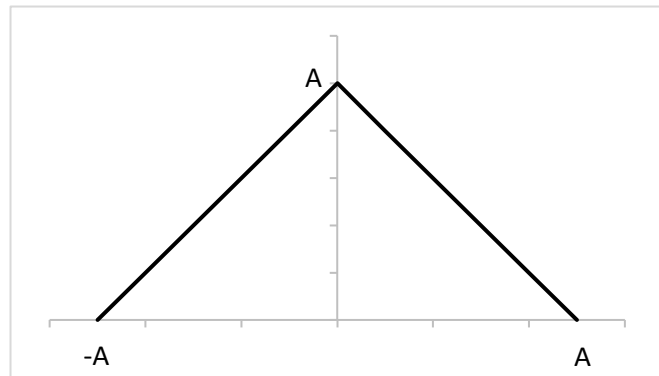
$$b_n = \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2n\pi t}{T}\right) dt$$

The coefficients of the Fourier series integrate an area of the function and factorise each area by the Time period  $T$ . The first coefficient  $a_0$  calculates the area of  $f(t)$  that is being converted to trigonometric functions, the coefficients  $a_n$  (for an even function of  $f(t)$ ) &  $b_n$  (for an odd function of  $f(t)$ ) calculates the amplitude & components in the series, each term is linearised by the period to a component value for  $f(t)$ . The component values of  $f(t)$  are represented in the Fourier series as the amplitude of sin & cos terms added together to give the value of  $f(t)$  for any  $t$  from 0 to  $T$ . The sum of components is from  $n=1$  to  $n=\infty$ . The series is normalised by  $1/n$  for each  $n$ th term.

In the case of the periodic triangle centred at 0, there is a discontinuity at  $t=0$  so we can split the integral into two functions:

$$f(t) = t + A$$

$$f(t) = -t + A$$



$$a_0 = \frac{1}{A} \left( \int_{-A}^0 (t + A) dt + \int_0^A (-t + A) dt \right)$$

$$a_0 = \frac{1}{A} \left( \left[ \frac{t^2}{2} + At \right]_{-A}^0 + \left[ -\frac{t^2}{2} + At \right]_0^A \right)$$

$$a_0 = \frac{1}{A} \left( -\left( \frac{A^2}{2} - A^2 \right) + \left( -\frac{A^2}{2} + A^2 \right) \right)$$

$$a_0 = A$$

$$a_n = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2\pi nt}{T}\right) dt$$

$$a_n = \frac{1}{A} \left( \int_{-A}^0 (t + A) \cos\left(\frac{\pi nt}{A}\right) dt + \int_0^A (-t + A) \cos\left(\frac{\pi nt}{A}\right) dt \right)$$

$$a_n = \frac{1}{A} \left( \left[ \frac{A(t + A) \sin\left(\frac{\pi nt}{A}\right)}{2\pi n} \right]_{-A}^0 - \left[ -\frac{A^2 \cos\left(\frac{\pi nt}{A}\right)}{(2\pi n)^2} \right]_0^A \right)$$

$$a_n = \frac{1}{A} \left( \frac{A^2 \cos(\pi n)}{(2\pi n)^2} - \frac{A^2}{(2\pi n)^2} \right) = -\frac{2A}{(2n-1)^2 \pi^2}$$

The equation is 0 for even numbers of n so we can replace n with only odd numbers which can be written as (2n-1). We can substitute the coefficients into the Fourier Series:

$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \left( -\frac{2A}{(2n-1)^2 \pi^2} \cos\left(\frac{(2n-1)\pi t}{A}\right) \right)$$