Complex Numbers & Complex Coordinates

A complex number has the form:

$$X + iY$$

Where i = $\sqrt{-1}$ is the imaginary unit of the y axis & X & Y are the components of the coordinate vector (X, iY). We can write the complex number in polar coordinates using the trigonometric functions:

$$X + iY = \mathrm{r}\cos(\theta) + \ i\,rsin(\theta)$$

Where the radial vector or amplitude is:

$$r = \sqrt{X^2 + Y^2}$$

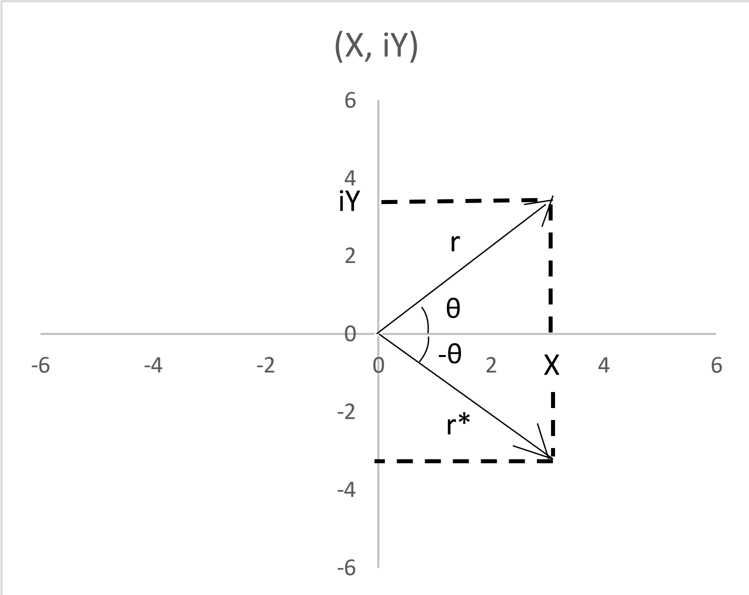

Figure 1. X component is symmetric for -θ and the Y component asymmetric.

We can use the complex number to provide information such as the magnitude of the vector r and the phase angle θ. It also enables the use of different notation and functions that are more easily integrable in areas like Fourier Analysis and Spherical Harmonics.

It is useful to consider the following properties of functions. The symmetry and asymmetry can be understood as odd or even where a function symmetric about the axis is an even function and a function asymmetric is an odd function. For y = x if x is negative then y is negative -y = -x with the same magnitude and so y = x is an odd function. If y = x^2 then y = $(-x)^2$ = x^2 and so the negative x values the same as the positive x values which is an even function. We can apply this to the trigonometric functions:

$$x = \cos(-\theta) = \cos\,(\theta) \qquad \text{Even Function}$$

$$y = \sin(-\theta) = -\sin(\theta) \qquad \text{Odd Function}$$

The conjugate of r is r* and the magnitude of the complex vector is r^2 = rr* and using the identity i^2 = -1, then:

$$If \quad r = r\cos(\theta) + irsin(\theta) \quad then \quad r^* = \mathrm{r}\cos(\theta) - irsin(\theta)$$

$$rr^* = (X + iY)(X - iY) = X^2 - i^2Y^2 = X^2 + Y^2 = r^2$$

Eulers Formula:

$$Ce^{i\theta} = \mathrm{C}(\cos(\theta) + \ isin(\theta))$$

& the complex conjugate:

$$Ce^{-i\theta} = \mathrm{C}(\cos(\theta) - \ isin(\theta))$$

From the above identities are the following expressions for the trigonometric functions in terms of complex exponentials:

$$e^{i\theta} \ \pm \ e^{-i\theta} = \ (\cos(\theta) + \ i\ sin(\theta)) \pm \ (\cos(\theta) - \ i\ sin(\theta))$$

$$e^{i\theta} \ + \ e^{-i\theta} = \ 2\cos(\theta)$$

$$\cos(\theta) = \frac{e^{i\theta} \ + \ e^{-i\theta}}{2}$$

&

$$\sin(\theta) = \frac{e^{i\theta} \ - \ e^{-i\theta}}{2i}$$

Series Approximation:

We can approximate the values of the functions in Eulers formula with a series of exponents of x using the Taylor Series to prove the Euler Identity:

$$\cos(\theta) \approx 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots$$

$$\sin(\theta) \approx \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{i\theta} \approx 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots$$

$$e^{i\theta} \approx 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)$$

$$e^{i\theta} \ \approx \ \cos(\theta) + i\sin(\theta)$$

$$e^{-i\theta} \ \approx \ \cos(\theta) - i\sin(\theta)$$