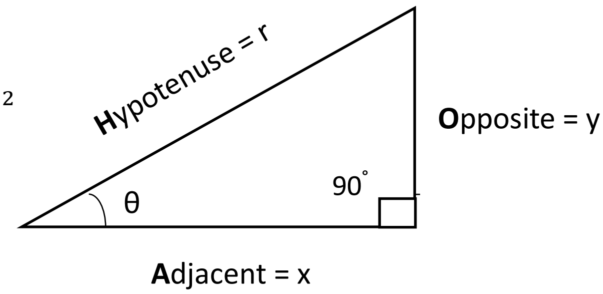
Pythagorean Theorem & Trigonometry

Pythagoras is a mathematical relationship of the angles and side lengths of a triangle, specifically right-angled triangles where one of the angles is 90 degrees and two of the sides are perpendicular joined by the longest side the hypotenuse. We can formulate the mathematical relationships into equations for the lengths of each side and establish relationships between the angles and the side lengths in the following way:

$$(Adjacent)^2 + (Opposite)^2 = (Hypotenuse)^2$$

$$x^2 + y^2 = r^2$$



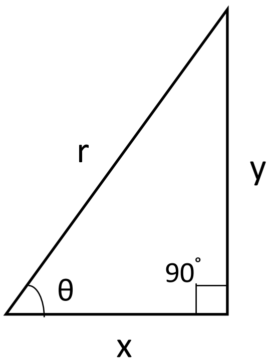


Figure 1. A right-angled triangle with side lengths x, y & r and a right angle 90˚ angle opposite the angle θ.

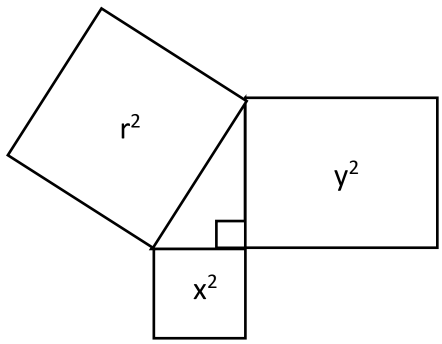


Figure 2. The Pythagoras square law for right angled triangles where ($y^2 = y \times y$), ($x^2 = x \times x$), ($r^2 = r \times r$) and ($x^2 + y^2 = r^2$).

The trigonometric equations have the following form defined by ratios of the side lengths x, y & r for a particular angle θ.

$$\sin(\theta) = \frac{y}{r} \qquad \cos(\theta) = \frac{x}{r} \qquad tan(\theta) = \frac{y}{x}$$

Using these definitions, we can interchange from the x & y values of a function to the θ & r values to establish trigonometric identities. Rearranging the trigonometric identities gives the following equations where x & y are the components values of r:

$$\mathrm{r}\sin(\theta) = y \qquad \mathrm{r}\cos(\theta) = x \qquad x\,tan(\theta) = y$$

We can substitute the component equations for x & y into the Pythagoras equation:

$$x^2 + y^2 = r^2$$

$$\mathrm{r}^2 \sin^2(\theta) + \mathrm{r}^2 \cos^2(\theta) = \mathrm{r}^2$$

The r^2 values in the equation cancel leaving the following trigonometric identity for a circle of any radius where the angle θ is the same for both sin and cos functions:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

We can substitute the component values for x & y to establish the following equation for tan, sin & cos functions.

$$\tan(\theta) = \frac{y}{x} = \frac{r\sin(\theta)}{r\cos(\theta)} = \frac{sin(\theta)}{\cos(\theta)}$$

We can add angles together known as a phase shift, a phase shift increases or decreases the angel by another angle changing the component values of r. For additions of angles there are the following trigonometric relationships in the case of the unit circle where r = 1:

$$\theta = \theta_1 \pm \theta_2$$

$$\sin(\theta_1 \pm \theta_2) = \sin(\theta_1)\cos(\theta_2) \pm \sin(\theta_2)\cos(\theta_1)$$

$$\cos(\theta_1 \pm \theta_2) = \cos(\theta_1)\cos(\theta_2) \mp \sin(\theta_1)\sin(\theta_2)$$

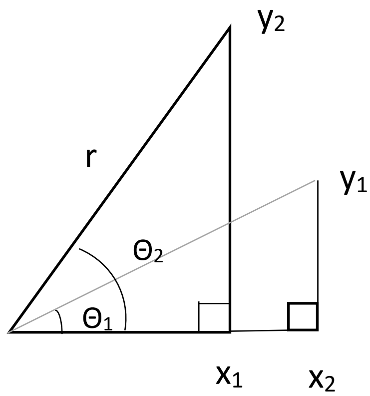


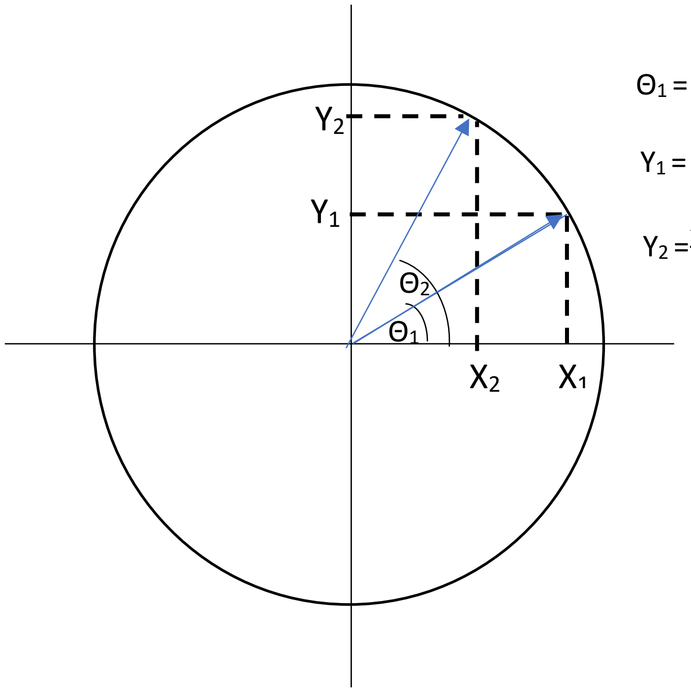
Figure 3.

The Unit Circle

The unit circle is the case where r is a constant r = 1. For angles of θ from 0 to 360° the radius of the unit circle rotates around the origin of the axis (0, 0) to each (x, y) coordinate. The (x, y) values are known as component vectors of the radius vector **r** with magnitude |**r**|.

$$\sin(\theta_1) = \frac{y_1}{r} \qquad \cos(\theta_1) = \frac{x_1}{r} \qquad \tan(\theta_1) = \frac{y_1}{x_1}$$

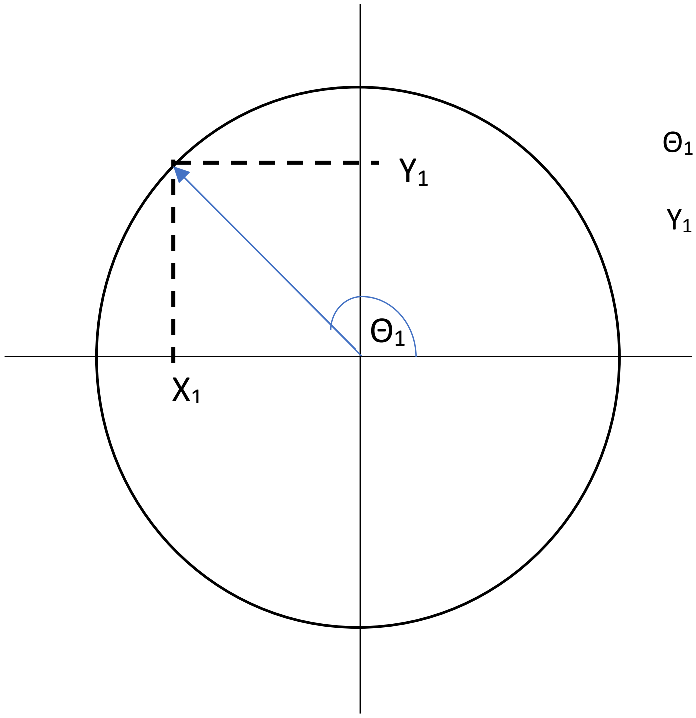
$$\sin(\theta_2) = \frac{y_2}{r} \qquad \cos(\theta_2) = \frac{x_2}{r} \qquad \tan(\theta_2) = \frac{y_2}{x_2}$$



$\Theta_1 = 30^\circ$, $\Theta_2 = 60^\circ$

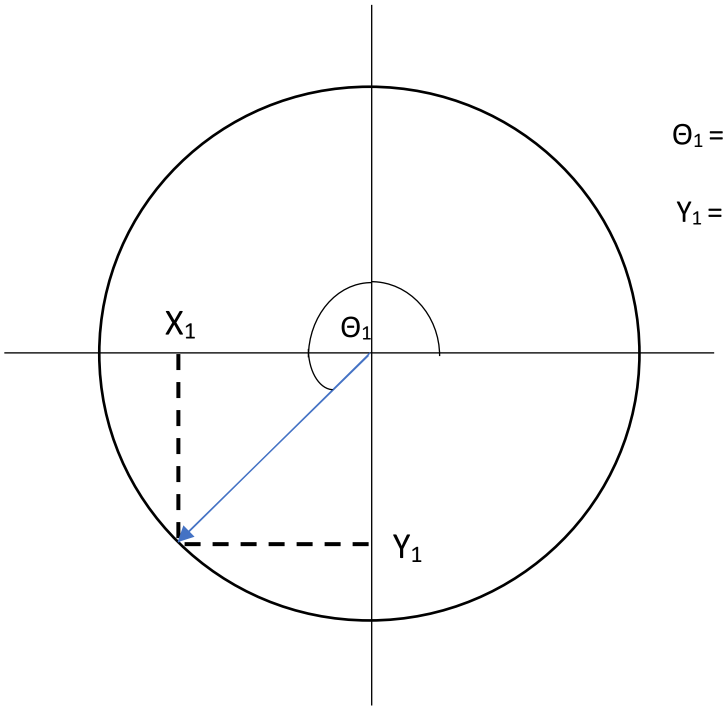
$Y_1 = \frac{1}{2}$, $X_1 = \frac{\sqrt{3}}{2}$

$Y_2 = \frac{\sqrt{3}}{2}$, $X_2 = \frac{1}{2}$



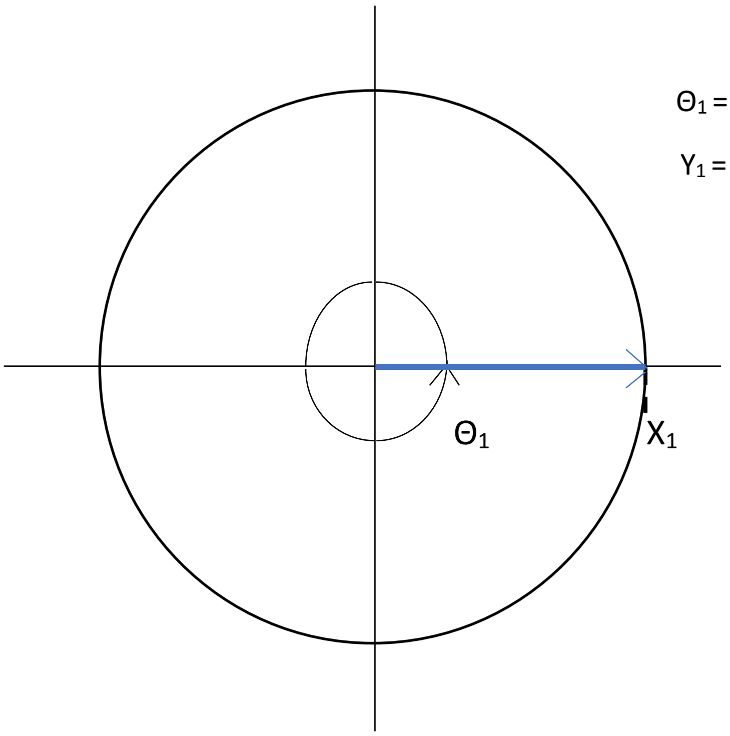
$\Theta_1 = 135^\circ$

$Y_1 = \frac{\sqrt{2}}{2}$, $X_1 = -\frac{\sqrt{2}}{2}$



$\Theta_1 = 225^{\circ}$

$Y_1 = -\frac{\sqrt{2}}{2}$, $X_1 = -\frac{\sqrt{2}}{2}$



$\Theta_1 = 360^{\circ}$

$Y_1 = 0$, $X_1 = 1$

Trigonometric Function Data For A Constant Value of r = 1:

Angle θ (Radians)	Angle θ (Degrees)	Y = SIN(θ)	X = COS(θ)
0	0	0	1
0.1	5.29578	0.09983342	0.99500417
0.2	11.02578	0.19866933	0.98006658
0.3	16.75578	0.29552021	0.95533649
0.4	22.48578	0.38941834	0.92106099
0.5	28.21578	0.47942554	0.87758256
0.6	33.94578	0.56464247	0.82533561
0.7	39.67578	0.64421769	0.76484219
0.8	45.40578	0.71735609	0.69670671
0.9	51.13578	0.78332691	0.62160997
1	56.86578	0.84147098	0.54030231
1.1	62.59578	0.89120736	0.45359612
1.2	68.32578	0.93203909	0.36235775
1.3	74.05578	0.96355819	0.26749883
1.4	79.78578	0.98544973	0.16996714
1.5	85.51578	0.99749499	0.0707372
1.6	91.24578	0.9995736	-0.0291995
1.7	96.97578	0.99166481	-0.1288445
1.8	102.7058	0.97384763	-0.2272021
1.9	108.4358	0.94630009	-0.3232896
2	114.1658	0.90929743	-0.4161468
2.1	119.8958	0.86320937	-0.5048461
2.2	125.6258	0.8084964	-0.5885011
2.3	131.3558	0.74570521	-0.666276
2.4	137.0858	0.67546318	-0.7373937
2.5	142.8158	0.59847214	-0.8011436
2.6	148.5458	0.51550137	-0.8568888
2.7	154.2758	0.42737988	-0.9040721
2.8	160.0058	0.33498815	-0.9422223
2.9	165.7358	0.23924933	-0.9709582
3	171.4658	0.14112001	-0.9899925
3.1	177.1958	0.04158066	-0.9991352
3.2	182.9258	-0.0583741	-0.9982948
3.3	188.6558	-0.1577457	-0.9874798
3.4	194.3858	-0.2555411	-0.9667982
3.5	200.1158	-0.3507832	-0.9364567
3.6	205.8458	-0.4425204	-0.8967584
3.7	211.5758	-0.5298361	-0.8481
3.8	217.3058	-0.6118579	-0.7909677
3.9	223.0358	-0.6877662	-0.7259323

4	228.7658	-0.7568025	-0.6536436
4.1	234.4958	-0.8182771	-0.5748239
4.2	240.2258	-0.8715758	-0.4902608
4.3	245.9558	-0.9161659	-0.4007992
4.4	251.6858	-0.9516021	-0.3073329
4.5	257.4158	-0.9775301	-0.2107958
4.6	263.1458	-0.993691	-0.1121525
4.7	268.8758	-0.9999233	-0.0123887
4.8	274.6058	-0.9961646	0.08749898
4.9	280.3358	-0.9824526	0.18651237
5	286.0658	-0.9589243	0.28366219
5.1	291.7958	-0.9258147	0.37797774
5.2	297.5258	-0.8834547	0.46851667
5.3	303.2558	-0.8322674	0.55437434
5.4	308.9858	-0.7727645	0.63469288
5.5	314.7158	-0.7055403	0.70866977
5.6	320.4458	-0.6312666	0.77556588
5.7	326.1758	-0.5506855	0.83471278
5.8	331.9058	-0.4646022	0.88551952
5.9	337.6358	-0.3738767	0.92747843
6	343.3658	-0.2794155	0.96017029
6.1	349.0958	-0.1821625	0.98326844
6.2	354.8258	-0.0830894	0.9965421
6.3	360.5558	0.0168139	0.99985864

Table 1. is a table of data of the trigonometric functions sin(θ) & cos(θ) where r = 1. The angle θ in column 1 is in Radians from 0 to 6.3 in increments of 0.1. In column 2 θ is in degrees from 0° to 360° in increments of 5.29578°, 0.1 Radians is equivalent to 5.29578° and 6.3 Radians is equivalent to 360.5558°. Radians is a measure of angle similarly to degrees with a conversion constant of π since the circumference of the unit circle is 2π. In columns 3, 4 & 5 are the values of the x & y coordinates which are the sin and cos values at each angle.

$360° = 2\pi$ radians

Plots of Data From Table 1.

Opposite in Figure 4. is the plot of x and y data in table 1 in columns 3 & 4, which shows the coordinates of the equation:

$$x^2 + y^2 = r^2$$

$$r^2 \cos^2(\theta) + r^2 \sin^2(\theta) = r^2$$

For the unit circle where r = 1:

$$x^2 + y^2 = 1$$

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

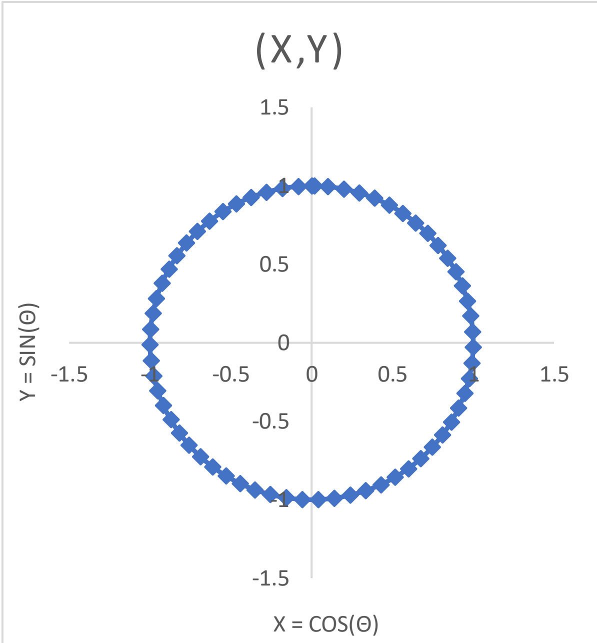


Figure 4.

Opposite in Figure 5. is the plot of Y values and θ values from columns 2 & 3, which shows the y component of the radius vector r. As the angle increases between the r vector and the x axis the y value changes in the following way as seen in the graph:

$$y = \sin\theta$$

The y value starts at 0 when x is maximum, increases to a maximum y value when x is 0 then decreases to 0 at -x and to -y when x is 0 again. Then returns to y = 0 at θ = 360

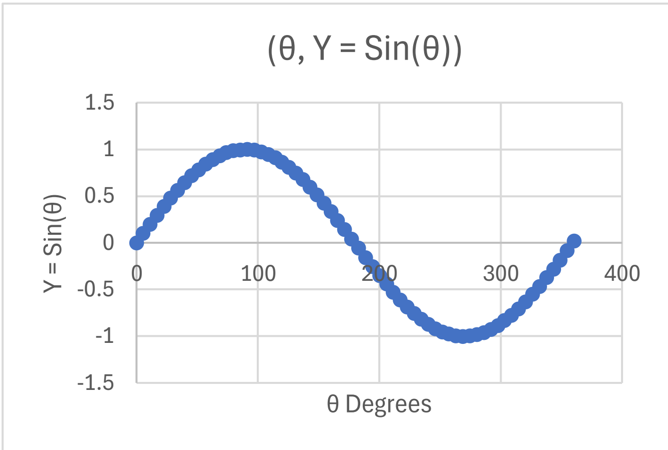


Figure 5.

In Figure 6. is the plot of the X value and θ value from columns 2 and 4, which shows how the x component of the radial vector r changes with the angle θ:

$$x = \cos\theta$$

The function starts at a r = 1 when y = 0 where the vector is in the x direction then decreases to 1 in the negative x direction and back to 1 at 360.

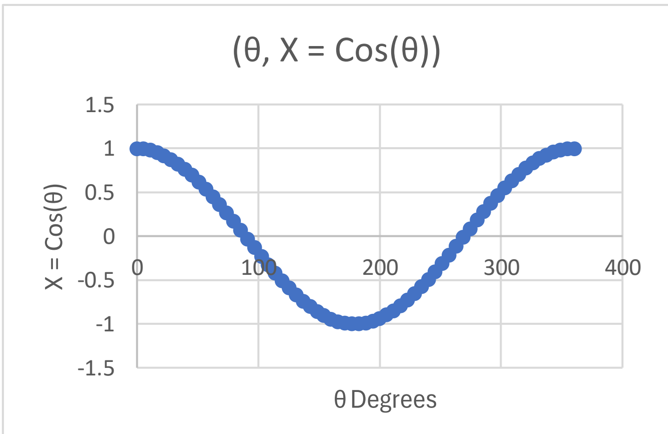


Figure 6.

Opposite is the plot of the function tan(θ) which shows the relationship between the x and y data in columns 3 & 4 and the graphs of sin(θ) and cos(θ) where:

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{y}{x}$$

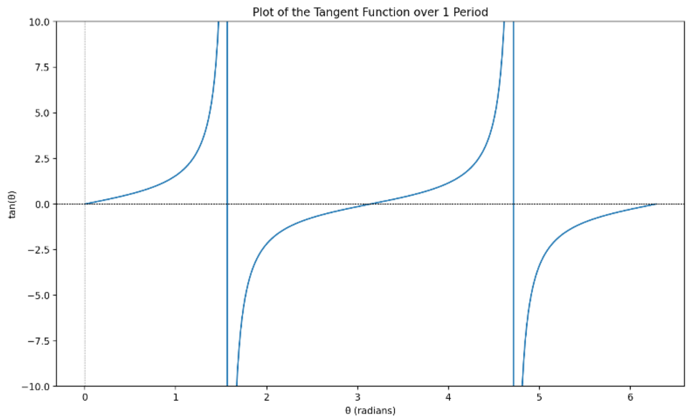


Figure 7.

At 0 degrees sin(θ) is 0 and cos(θ) is 1, as θ increases to 90° cos(θ) decreases to 0, sin(θ) increases to 1 and tan(θ) increases to a positive infinity then switches to a negative infinity, as θ increases to 180° cos(θ) decreases to -1, sin(θ) decreases to 0 and tan(θ) returns to 0. From 180° to 360° the tan(θ) repeats itself.

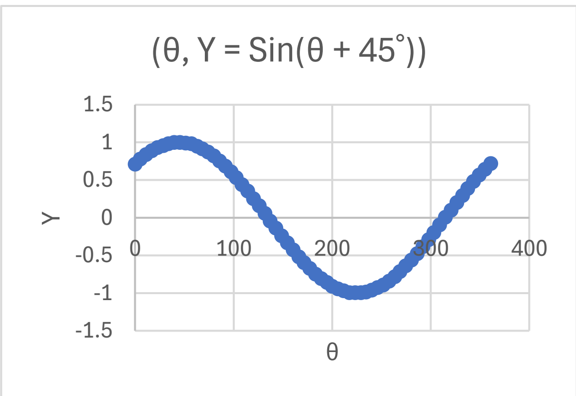
The trigonometric functions repeat the same values and are known as cyclic functions, which are useful in differential equations where the derivatives contain the same function.

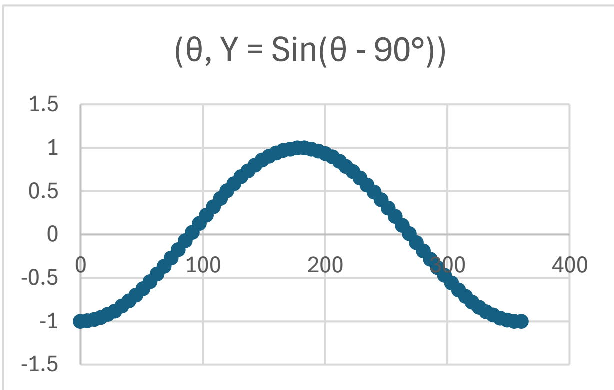
Phase Shifts

A phase shift is where the angle θ of the trigonometric function is shifted by some angle $\pm\varphi$ which changes the coordinate values of x & y on the circle for a given value of θ. We can use some equations and graphs to visualise these:

$$y = \sin(\theta + \varphi)$$

$$x = \cos(\theta + \varphi)$$





In the example above where y = Sin(θ+45°) each value θ is shifted by 45° and the waveform at θ = 0 has a different value of sin(45°) where y = 0.707 whereas for sin(0) y = 0. In the second waveform where y = sin(θ+90°) each θ value is shifted by 90° and the waveform is the same as the cos(θ) function:

$$y = \sin(\theta - 90°) = -\cos(\theta)$$

Arcsin, Arccos & Arctan Functions

The Arcsine functions are the inverse trigonometric equations where the angle θ is a function of the coordinates x & y as opposed to the coordinates as a function of x & y.

$$\theta = sin^{-1}\left(\frac{y}{r}\right) = \text{Arcsin}\left(\frac{y}{r}\right)$$

$$\theta = cos^{-1}\left(\frac{x}{r}\right) = \text{Arccos}\left(\frac{x}{r}\right)$$

$$\theta = tan^{-1}\left(\frac{x}{r}\right) = \text{Arctan}\left(\frac{x}{r}\right)$$

Reciprocal Functions

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\text{cosec}(\theta) = \frac{1}{\sin(\theta)}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$