Mechanical Wave Equation

Travelling waves such as Sound waves or Electromagnetic waves can be considered as sinusoidal wavefunctions propagating in some direction with some frequency and wavelength. The speed of the waves v also known as the dispersion relation is defined by the wavelength & time period:

$$v = \frac{\lambda}{T} = \lambda f$$

The units of wavelength are meters, and the speed of the wave is meters per second ($ms^{-1}$). The wavelengths angular definition is the wavenumber given by:

$$\beta = \frac{2\pi}{\lambda}$$

We can then define the wave travelling in the x direction as a function of time & distance in a sinusoidal form as:

$$f(t,x) = Asin(\omega t \pm \beta x)$$

The function f(t,x) for a time t is defined over a distance x and for a distance x is defined over some time t.

For a changing f(t,x) or time varying f(t,x) the function has derivatives of t and derivatives of x. Both are orthogonal to f(t,x) so are partial derivatives and for a constant velocity v the spatial component is not a function of time. The second derivatives of time and space are then related in the following way:

Eq. 1

$$\frac{\partial^2 f(t,x)}{\partial x^2} = C\frac{\partial^2 f(t,x)}{\partial t^2}$$

$$\frac{\partial^2 (Asin(\omega t + \beta x))}{\partial x^2} = -\beta^2 Asin(\omega t + \beta x)$$

$$\frac{\partial^2 (Asin(\omega t + \beta x))}{\partial x^2} = -\omega^2 CAsin(\omega t + \beta x)$$

Substituting the derivatives Eq.1 and cancelling terms:

$$\beta^2 = \omega^2 C$$

$$C = \frac{\beta^2}{\omega^2} = \left(\frac{2\pi}{\lambda}\frac{1}{2\pi f}\right)^2 = \left(\frac{1}{\lambda f}\right)^2 = \frac{1}{v^2}$$

So the derivatives are related by the speed of the wave:

$$\frac{\partial^2 f(t,x)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 f(t,x)}{\partial t^2}$$

Standing Waves

Standing waves are most evident in musical instruments where there a fixed ends or one fixed of a vibrating medium resulting in displacements or pressure variations in the medium such as a guitar string. This is a source of sound waves where the vibrations travel through the air or the walls and into the ear canals vibrating the ear drum.

In the case of a vibrating string such as a Piano or Harp both ends are fixed and so the function f(t,x) is 0 at x = 0 and x = L, the length of the string. The standing waves occur when a wave travels with some speed v on the string & reflects, travelling in the opposite direction, the waves constructively & destructively interfere. We can formulate the oscillations in the following way:

$$f(t,x) = \ Asin(\omega t + \beta x) + Asin(\omega t - \beta x)$$

$$f(t,x) = (Asin(\omega t)\cos{(\beta x)} + Asin(\beta x)\cos(\omega t)) + (Asin(\omega t)\cos{(\beta x)} - Asin(\beta x)\cos(\omega t))$$

$$f(t,x) = 2Asin(\omega t)\cos{(\beta x)}$$

Applying the Boundary conditions then for x = L and for any t f(t,x)=0:

$$0 = \cos{(\beta L)}$$

This condition applies for cos(2nπ) for even numbers of n, therefore we can say that for the value β and in terms of the wavelength λ:

$$\beta = \frac{2\mathrm{n}\pi}{L} \qquad \& \qquad \lambda = \frac{\mathrm{L}}{n}$$

This formula relates the possible wavelengths or harmonics possible on the vibrating system, dependent on the Boundary conditions and the speed v of the waves in medium.