

Vector Theory & Surface Area Plots

by

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Vectors, The Cartesian Plane and Coordinates (x, y, z)

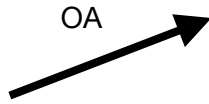


Diagram 1.

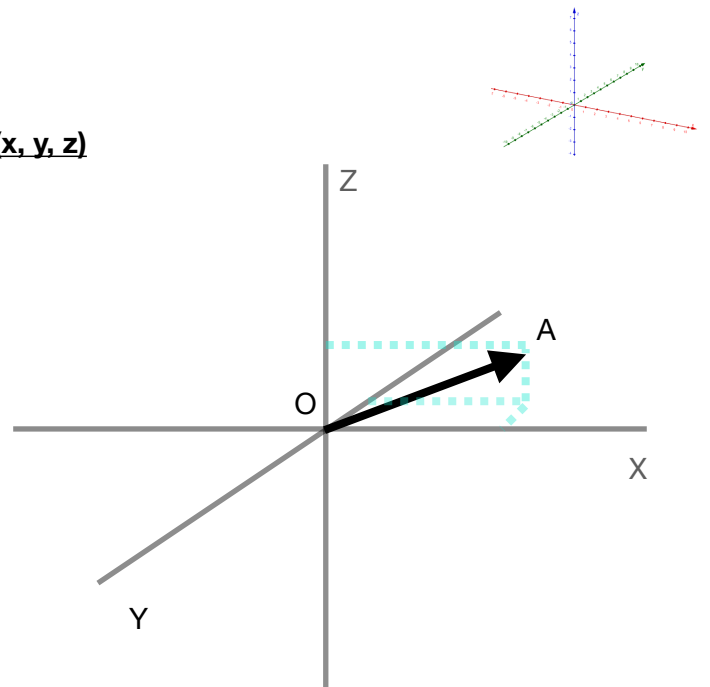


Diagram 2.

Consider the vector OA in diagram 1 and in the XYZ plane in diagram 2, it starts at the origin (0, 0, 0) and ends at some arbitrary point A with the coordinates (X, Y, Z). The vector OA has the following notation in the form of an equation:

$$OA = X\hat{i} + Y\hat{j} + Z\hat{k}$$

(\hat{i} , \hat{j} , \hat{k}) are the unit vectors in the (x, y, z) plane. A unit vector has a magnitude of 1 in a specified direction:

$$\hat{i} = (1, 0, 0)$$

$$\hat{j} = (0, 1, 0)$$

$$\hat{k} = (0, 0, 1)$$

The magnitude of a vector is calculated using pythagorean theorem in 3 dimensions using the equation:

$$|OA| = \sqrt{X^2 + Y^2 + Z^2}$$

Summing any two vectors OB & BA together results in a third vector OA:

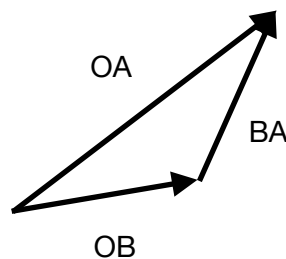


Diagram 3.

In equation form adding the vectors together is a summation of the unit vectors (i, j, k) for example if the vectors OB and BA are given:

$$OB = 2\hat{i} - 3\hat{j} + 5\hat{k}$$

$$BA = 3\hat{i} + 5\hat{j} - 2\hat{k}$$

$$OA = OB + BA = (2 + 3)\hat{i} + (-3 + 5)\hat{j} + (5 - 2)\hat{k}$$

$$OA = 5\hat{i} + 2\hat{j} + 3\hat{k}$$

Multiplication of Vectors, The Scalar Product & Vector Product

The Scalar product also known as the Dot product of any two vectors results in a number not a vector and can be calculated in the following ways using the vectors D & E that have an angle θ between them as shown in the diagram below:

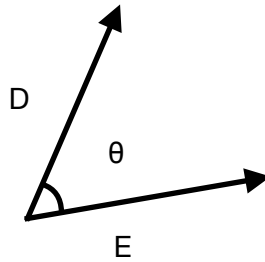


Diagram 4.

Where the vectors D & E are as follows:

$$D = D_i \hat{i} + D_j \hat{j} + D_k \hat{k}$$

$$E = E_i \hat{i} + E_j \hat{j} + E_k \hat{k}$$

The Scalar Dot product is then:

$$D \cdot E = (D_i \times E_i) + (D_j \times E_j) + (D_k \times E_k) = \text{Scalar}$$

And can also be calculated using:

$$D \cdot E = |D| |E| \cos(\theta)$$

The Vector product also known as the Cross product of any two vectors results in a third vector C that is perpendicular (at a 90 degree angle) to the two vectors and can be calculated in the following ways:

$$C = D \times E = ((D_j \times E_k) - (E_j \times D_k)) \hat{i} - ((D_i \times E_k) - (E_i \times D_k)) \hat{j} + ((D_i \times E_j) - (E_i \times D_j)) \hat{k}$$

Where the vector C is:

$$C = C_i \hat{i} + C_j \hat{j} + C_k \hat{k}$$

The magnitude of the Vector Product is then:

$$|C| = |D \times E| = |D| |E| \sin(\theta) = \sqrt{C_i^2 + C_k^2 + C_j^2}$$

The vector product C is also given by the equation:

$$C = D \times E = |D| |E| \sin(\theta) \hat{n}_v$$

Where n is the unit vector perpendicular to the vectors D & E:

$$\hat{n}_v = n_i \hat{i} + n_j \hat{j} + n_k \hat{k}$$

Which can also be obtained using the equation:

$$\hat{n}_v = \frac{(D \times E)}{|D \times E|}$$

The Vector Equation Of A Flat Surface in 3D

The vector equation of a flat surface is defined using any two points in the plane. From these the vector from the origin O to a point in the plane and the unit normal (perpendicular) vector to the plane can be determined giving an equation of the plane.

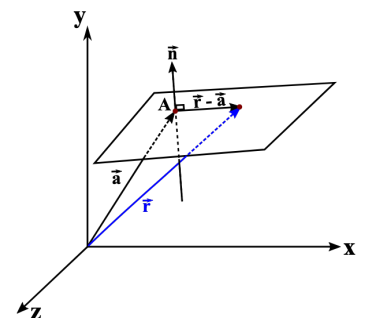


Diagram 5.

If the two points OA and OR in the plane are given by the coordinates (x_1, y_1, z_1) & (x_2, y_2, z_2) , then the vectors from the origin to each point as can be seen in diagram 5 are:

$$r = X_1 \hat{i} + Y_1 \hat{j} + Z_1 \hat{k}$$

$$a = X_2 \hat{i} + Y_2 \hat{j} + Z_2 \hat{k}$$

The vector in the plane is calculated from:

$$r - a = (X_1 - X_2) \hat{i} + (Y_1 - Y_2) \hat{j} + (Z_1 - Z_2) \hat{k}$$

The normal vector which is at a 90 degree angle from the plane can be used to formulate the equation of the plane using the scalar dot product as the trigonometric function $\cos(90)$ is 0 therefore we can write the vector equation of the plane as:

$$n \cdot (r - a) = (n \cdot r) - (n \cdot a) = 0$$

If we only know one point in the plane then using the scalar dot product and the $\cos(90)$ identity we can write the equation:

$$n \cdot (r - a) = n \cdot \frac{(r - a)}{|r - a|} |r - a| = n \cdot \hat{m} \lambda = 0$$

Where we have made the unknown vector $(r - a)$ in the flat surface a unit vector denoted as \hat{m} with an arbitrary coefficient λ which would determine the length of a vector in the plane. We can then write an equation to determine any point (coordinate vector) in the plane where a is a coordinate given or chosen in the surface:

$$r = (x \ y \ z) = \lambda \hat{m} + r_0$$

Vector Equation, Spherical Coordinates and Parametrisation.

The vector equation of a plane derived above can be written in spherical polar coordinates where unit vectors contain the variables in the transformation matrices. We will now look at the parameterisation of the unit vectors to define curved planes.

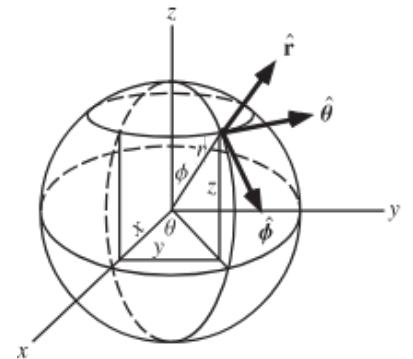


Diagram 6.

$$\mathbf{v} = (r \ \theta \ \phi) = \lambda \ (m_r \ \hat{r} + m_\theta \ \hat{\theta} + m_\phi \ \hat{\phi}) + (r_0 \ \theta_0 \ \phi_0)$$

Parametrisation is the substitution of the variables (r θ ϕ) for a function of some other variable t (r(t) θ(t) ϕ(t)), where the functions of t can be selected. For example we can use spherical coordinates where ϕ = z to map an elliptical cylinder.

$$\text{Eq. 1.} \quad \mathbf{v} = (r \ \theta \ \phi) = (A(\cos(2\pi t) \ \hat{r} + 2\pi t \ \hat{\theta} + Z \ \hat{\phi}) + (r_0 \ \theta_0 \ \phi_0), \quad 0 < t < 1$$

Where the coordinate transformations to spherical coordinates are:

$$(r, \theta, \phi) = (\sqrt{x^2 + y^2 + z^2}, \arctan(\frac{y}{x}), \arctan(\frac{\sqrt{x^2 + y^2}}{z}))$$

$$(x, y, z) = (r \cos \theta \sin \phi, r \sin \theta \sin \phi, r \cos \phi)$$

In diagram 7 is the plot using the python programming language of the vector equation Eq. 1 in the XYZ plane. The domain of t is defined and the coordinates calculated to produce the plot.

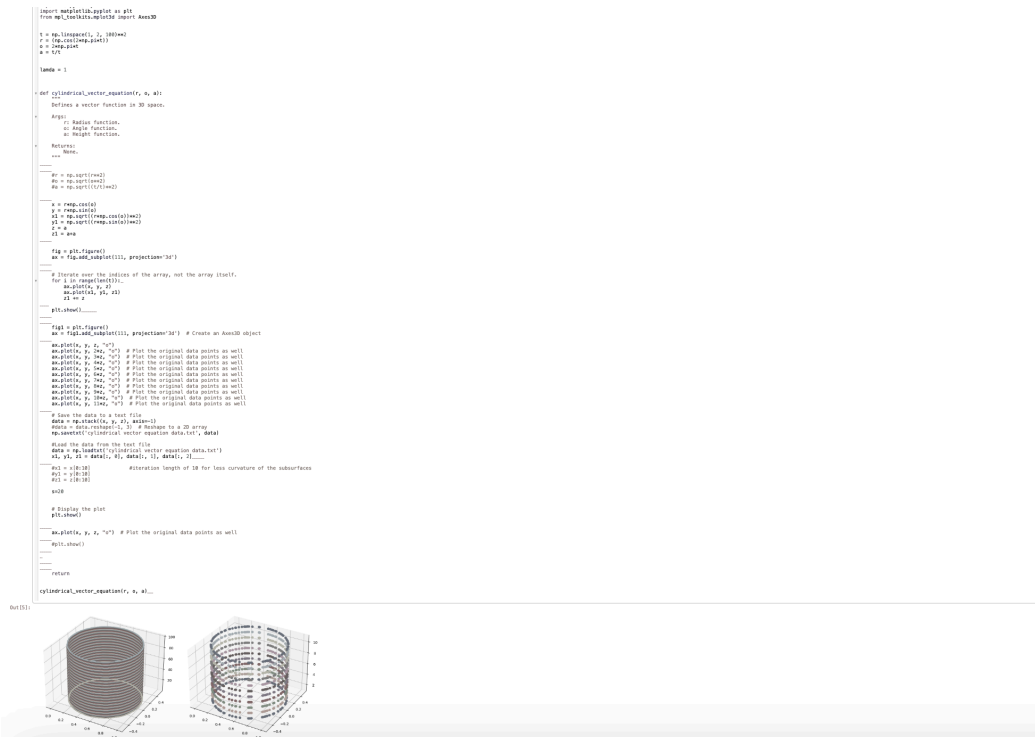


Diagram 7.

The Derivatives & Integrals of Vectors.

The derivative of a vector equation returns a gradient vector that is the change of the equation with it's parameter and is in the direction of one data point to the next. In the case of a circle with constant radius the derivative would return a vector perpendicular to the radial vector as it only changes in the θ and/or ϕ direction. Eq. 1 is used to calculate the first derivative:

$$\frac{\delta v}{\delta t} = \left(\frac{\delta r}{\delta t} \frac{\delta \theta}{\delta t} \frac{\delta \phi}{\delta t} \right) = (-2\pi A \sin(2\pi t) \hat{r} + 2\pi \hat{\theta})$$

The result is a vector which only changes in the radial direction and changes in the direction toward and away from the origin, as the variable t changes, the θ unit vector is a constant and the ϕ unit vector is now zero.

Computing the integral, which is a sum of multiplications of an equation by changes in its parameters. For example the vector equation v plots a line of data points in 3D space, if we integrate small sections of the line dv from data point to data point the length of the line can be calculated as shown below:

$$\int_0^v \delta v = \int_0^t \frac{\delta v}{\delta t} \delta t = \int_0^t (-2\pi A \sin(2\pi t) \hat{r} + 2\pi \hat{\theta}) \delta t$$

$$\int_0^v \delta v = [A \cos(2\pi t) \hat{r} + 2\pi t \hat{\theta}]_0^1 = |2\pi \hat{\theta}| = 2\pi$$

The Del Vector

The del vector or gradient vector has implicit derivatives and is symbolised by the letter ∇ . It is used as a multiplication of vector equations. In the XYZ coordinate system it is given by:

$$\nabla = \frac{\delta}{\delta x} \hat{i} + \frac{\delta}{\delta y} \hat{j} + \frac{\delta}{\delta z} \hat{k}$$

And in spherical polar coordinates:

$$\nabla = \frac{\delta}{\delta r} \hat{r} + \frac{1}{r} \frac{\delta}{\delta \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\delta}{\delta \phi} \hat{\phi}$$

The del vector is used as an operator on non vector equations (scalar fields) & vector equations (vector fields) in the following ways using the multiplication methods identified above. Given the scalar field V and the vector field \mathbf{V} these are:

$$\nabla V = \frac{\delta V}{\delta r} \hat{r} + \frac{1}{r} \frac{\delta V}{\delta \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\delta V}{\delta \phi} \hat{\phi} = A \text{ vector field}$$

$$\nabla \cdot \mathbf{V} = \left(\frac{1}{r^2} \frac{\delta(r^2 \mathbf{V}_r)}{\delta r} + \frac{1}{r \sin(\phi)} \frac{\delta \mathbf{V}_\theta}{\delta \theta} + \frac{1}{r \sin \phi} \frac{\delta (\sin(\phi) \mathbf{V}_\phi)}{\delta \phi} \right) = A \text{ scalar field}$$

$$\nabla \times \mathbf{V} = \frac{1}{r \sin(\phi)} \left(\frac{\delta}{\delta \phi} (\mathbf{V}_\theta \sin(\phi)) - \frac{\delta \mathbf{V}_\phi}{\delta \theta} \right) \hat{r} + \frac{1}{r} \left(\frac{\delta}{\delta r} (r \mathbf{V}_\phi) - \frac{\delta \mathbf{V}_r}{\delta \phi} \right) \hat{\theta} + \left(\frac{1}{r \sin \phi} \frac{\delta \mathbf{V}_r}{\delta \theta} - \frac{1}{r} \frac{\delta}{\delta r} (r \mathbf{V}_\theta) \right) \hat{\phi} = A \text{ vector field}$$

Surface Area Integrals

The surface area integral is a summation of subsurfaces of an arbitrary larger surface defined by a vector equation. In this example we will use a cylinder given by the following equation where t & z are parameters:

$$\mathbf{v} = (r \ \theta \ \phi) = (\hat{r} + 2\pi t \ \hat{\theta} + 2z \ \hat{z}) \quad (0 < t < 1) \quad (0 < z < 3)$$

$$\frac{\delta \mathbf{v}}{\delta t} = 2\pi \ \hat{\theta} \quad 0 < t < 1$$

$$\frac{\delta \mathbf{v}}{\delta z} = 2 \ \hat{z} \quad 0 < z < 3$$

$$\int \delta s = \int_0^3 \int_0^1 \left| \frac{\delta \mathbf{v}}{\delta t} \right| \times \left| \frac{\delta \mathbf{v}}{\delta z} \right| \delta t \delta z = \int_0^3 \int_0^1 \left| \frac{\delta \mathbf{v}}{\delta t} \right| \left| \frac{\delta \mathbf{v}}{\delta z} \right| \delta t \delta z$$

$$\int \delta s = \int_0^3 \int_0^1 4\pi \ \delta t \delta z = 12\pi$$

Relavent Formulae

Unit vector transformations are:

$$\hat{r} = \sin\theta\cos\phi \hat{x} + \sin\theta\sin\phi \hat{y} + \cos\theta \hat{z}$$

$$\hat{\theta} = -\sin(\theta) \hat{x} + \cos(\theta) \hat{y}$$

$$\hat{\phi} = \cos\theta\cos\phi \hat{x} + \cos\theta\sin\phi \hat{y} - \sin\theta \hat{z}$$

Coordinate transformations are:

$$(r, \theta, \phi) = (\sqrt{x^2 + y^2 + z^2}, \arctan(\frac{y}{x}), \arctan(\frac{\sqrt{x^2 + y^2}}{z}))$$

$$(x, y, z) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\phi)$$

$$(x, y, z) = (r\cos\theta, r\sin\theta, z)$$

Del Vector Multiplications:

$$\nabla \cdot \mathbf{V} = (\frac{1}{r^2} \frac{\delta(r^2 \mathbf{V}_r)}{\delta r} + \frac{1}{r\sin(\phi)} \frac{\delta \mathbf{V}_\theta}{\delta \theta} + \frac{1}{r\sin\phi} \frac{\delta (\sin(\phi) \mathbf{V}_\phi)}{\delta \phi}) = A \text{ scalar field}$$

$$\nabla \times \mathbf{V} = \frac{1}{r\sin(\phi)} (\frac{\delta}{\delta \phi} (\mathbf{V}_\theta \sin(\phi)) - \frac{\delta \mathbf{V}_\phi}{\delta \theta}) \hat{r} + \frac{1}{r} (\frac{\delta}{\delta r} (r \mathbf{V}_\phi) - \frac{\delta \mathbf{V}_r}{\delta \phi}) \hat{\theta} + (\frac{1}{r\sin\phi} \frac{\delta \mathbf{V}_r}{\delta \theta} - \frac{1}{r} \frac{\delta}{\delta r} (r \mathbf{V}_\theta)) \hat{\phi} = A \text{ vector field}$$

