Shear flow:

$$\frac{\partial f(s)}{\partial s} = -t \frac{\partial \sigma_{zz}}{\partial z}$$

$$\sigma_{zz} = E\left[\frac{V_z}{S} - \left(\frac{\bar{x} H_{\bar{x}\bar{y}} - \bar{y} H_{\bar{y}}}{\Delta \bar{H}}\right) M_x - \left(\frac{\bar{x} H_{\bar{x}} - \bar{y} H_{\bar{x}\bar{y}}}{\Delta \bar{H}}\right) M_y\right]$$

$$\frac{\partial f(s)}{\partial s} = -Et\left[\frac{1}{S}\frac{dV_z}{dz} - \left(\frac{\bar{x} H_{\bar{x}\bar{y}} - \bar{y} H_{\bar{y}}}{\Delta \bar{H}}\right)\frac{dM_x}{dz} - \left(\frac{\bar{x} H_{\bar{x}} - \bar{y} H_{\bar{x}\bar{y}}}{\Delta \bar{H}}\right)\frac{dM_y}{dz}\right]$$

sub. in $\frac{dV_z}{dz}$, $\frac{dM_x}{dz}$, $\frac{dM_y}{dz}$

$$\frac{\partial f(s)}{\partial s} = -Et\left[-\frac{1}{S}\left\{P_{z_w} + P_{z_c}\right\} - \left(\bar{x} H_{\bar{x}\bar{y}} - \bar{y} H_{\bar{y}}\right)\left\{\frac{V_y - (P_{zw} + P_{zc})(Y_{cM} - Y_{tc})}{\Delta \bar{H}}\right\} \cdots\right.$$

$$\left. - \left(\bar{x} H_{\bar{x}} - \bar{y} H_{\bar{x}\bar{y}}\right)\left\{-V_x + (P_{zw} + P_{zc})(X_{cM} - X_{tc})\right\}\right]$$

$$\frac{\partial f(s)}{\partial s} = \frac{Et}{S}\left\{P_{z_w} + P_{z_c}\right\} + \left(Et\bar{x} H_{\bar{x}\bar{y}} - Et\bar{y} H_{\bar{y}}\right)\left\{\frac{V_y - (P_{zw} + P_{zc})(Y_{cM} - Y_{tc})}{\Delta \bar{H}}\right\} \cdots$$

$$+ \left(Et\bar{x} H_{\bar{x}} - Et\bar{y} H_{\bar{x}\bar{y}}\right)\left\{\frac{-V_x + (P_{zw} + P_{zc})(X_{cM} - X_{tc})}{\Delta \bar{H}}\right\}$$

- the only variables depending on s: $E(s)$, $t(s)$, $\bar{x}(s) = x(s) - X_{tc}$, $\bar{y}(s) = y(s) - Y_{tc}$

$$f(s) = c + \int_0^s (\,\cdot\,)\, ds$$, c is constant of integration determined from boundary conditions

$$f(s) = c + \left\{\frac{P_{z\omega} + P_{zc}}{S}\right\} \int_0^s Et\,ds \;\ldots$$

$$+ \left\{\frac{V_y - (P_{z\omega} + P_{zc})(y_{cm} - y_{tc})}{\Delta\bar{H}}\right\} \left(H_{\bar{x}\bar{y}} \int_0^s Et\bar{x}\,ds - H_{\bar{y}} \int_0^s Et\bar{y}\,ds \right) \ldots$$

$$+ \left\{\frac{-V_x + (P_{z\omega} + P_{zc})(x_{cm} - x_{tc})}{\Delta\bar{H}}\right\} \left(H_{\bar{x}} \int_0^s Et\bar{x}\,ds - H_{\bar{x}\bar{y}} \int_0^s Et\bar{y}\,ds \right)$$

- introduce the stiffness first moments

$$Q_{\bar{x}}(s) = \int_0^s Et\bar{y}\,ds = \int_0^s Et(y - y_{tc})\,ds$$

$$Q_{\bar{y}}(s) = \int_0^s Et\bar{x}\,ds = \int_0^s Et(x - x_{tc})\,ds$$

↳ these integrals are the stiffness first moments of the cross section from $s=0$ to s, and thus are functions of s

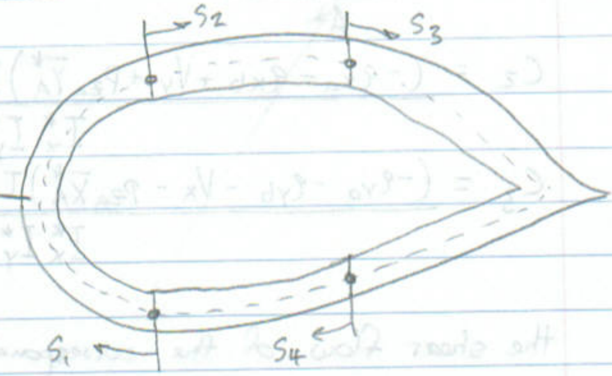
$$\boxed{\begin{aligned} f(s) = c &+ \underbrace{\left\{\frac{P_{z\omega} + P_{zc}}{S}\right\}}_{c_1} \int_0^s Et\,ds \;\ldots \\ &+ \underbrace{\left\{\frac{V_y - (P_{z\omega} + P_{zc})(y_{cm} - y_{tc})}{\Delta\bar{H}}\right\}}_{c_2} \left(H_{\bar{x}\bar{y}}\, Q_{\bar{y}}(s) - H_{\bar{y}}\, Q_{\bar{x}}(s) \right) \ldots \\ &+ \underbrace{\left\{\frac{-V_x + (P_{z\omega} + P_{zc})(x_{cm} - x_{tc})}{\Delta\bar{H}}\right\}}_{c_3} \left(H_{\bar{x}}\, Q_{\bar{y}}(s) - H_{\bar{x}\bar{y}}\, Q_{\bar{x}}(s) \right) \end{aligned}}$$

$$\boxed{f(s) = c + c_1 \int_0^s Et\,ds + \underbrace{\left\{-c_2 H_{\bar{y}} - c_3 H_{\bar{x}\bar{y}}\right\}}_{d_1} Q_{\bar{x}}(s) + \underbrace{\left\{c_2 H_{\bar{x}\bar{y}} + c_3 H_{\bar{x}}\right\}}_{d_2} Q_{\bar{y}}(s)}$$

shear flow - shearing + torsion

example

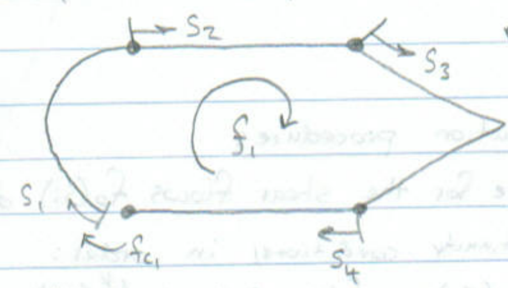
- single cell - 4 segments



each segment of the cell could have different values for E, G, t

replace closed section with an open section (make 1 cut per cell for multi-cellular sections)

curvilinear coords $s_i \in [0, L_i]$



the total shear flow of the closed section is:

$$f(s_i) = f_0(s_i) + f_{c_i}$$

where $f_0(s_i)$ is the shear flow of the corresponding open section and f_{c_i} is a constant which restores the zero axial displacement condition

zero axial displacement implies: $w_{0_i} = \oint_{\text{cell}_i} \frac{f_0(s_i) + f_{c_i}}{Gt} ds_i = 0$

shear flow in the open section is solved by: $f_0(s_i) = f(s_i = 0) - \int_0^s t \frac{\partial \sigma_{zz}}{\partial z} ds_i$

the boundary condition at the cut is known: $f(s_1 = 0) = 0$

by the shear flow continuity condition: $f(s_2 = 0) = f(s_1 = L_1)$

$f(s_3 = 0) = f(s_2 = L_2)$

$f(s_4 = 0) = f(s_3 = L_3)$

$\Rightarrow$ this same analysis can be applied to multi-cellular sections shear flow of the corresponding open section is the solved as:

$$f_o(s_i) = f(s_i = 0) + c_1 \int_0^{L_i} E t \, ds_i + d_1 Q_{\bar{x}}(s_i) + d_2 Q_{\bar{y}}(s_i)$$

see earlier definitions of constants c_1, d_1, and d_2

Solution procedure:

1) solve for the shear flows $f_o(s_i)$ of the open section and enforce continuity conditions in order:

$$f_o(s_1) = f(s_1 = 0) + c_1 \int_0^{L_1} E t \, ds_1 + d_1 Q_{\bar{x}}(s_1) + d_2 Q_{\bar{y}}(s_1)$$
$$f_o(s_2) = f(s_2 = 0) + c_1 \int_0^{L_2} E t \, ds_2 + d_1 Q_{\bar{x}}(s_2) + d_2 Q_{\bar{y}}(s_2)$$
$$f_o(s_3) = f(s_3 = 0) + c_1 \int_0^{L_3} E t \, ds_3 + d_1 Q_{\bar{x}}(s_3) + d_2 Q_{\bar{y}}(s_3)$$
$$f_o(s_4) = f(s_4 = 0) + c_1 \int_0^{L_4} E t \, ds_4 + d_1 Q_{\bar{x}}(s_4) + d_2 Q_{\bar{y}}(s_4)$$

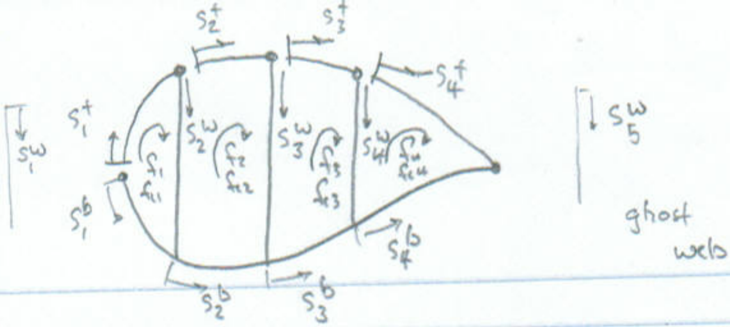
2) solve for the closing shear constant by enforcing zero axial displacement of the open section

$$f_{c1} = \frac{-\oint_{cell,1} \frac{f_o(s)\,ds}{Gt}}{\oint_{cell,1} \frac{1}{Gt}\,ds} = \frac{-\left\{ \int_0^{L_1} \frac{f_o(s_1)\,ds_1}{G_1 t_1} + \int_0^{L_2} \frac{f_o(s_2)\,ds_2}{G_2 t_2} + \int_0^{L_3} \frac{f_o(s_3)\,ds_3}{G_3 t_3} + \int_0^{L_4} \frac{f_o(s_4)\,ds_4}{G_4 t_4} \right\}}{\left\{ \int_0^{L_1} \frac{1}{G_1 t_1} ds_1 + \int_0^{L_2} \frac{1}{G_2 t_2} ds_2 + \int_0^{L_3} \frac{1}{G_3 t_3} ds_3 + \int_0^{L_4} \frac{1}{G_4 t_4} ds_4 \right\}}$$

3) compute the shear flow in the closed section:

$$f_{transverse}(s_i) = f_o(s_i) + f_{c1}$$

⇒ next we will generalize this to multi-cellular cross sections. For multi-cellular sections, solving for the closing shear constant requires the solution of a system of equations



axial displacement of cell 1

$$w_{t_1} = \oint_{cell,1} \frac{f_0(s_i) + f_{c1}}{G_i t_i} ds_i$$

$$w_{t_1} = \int_0^{s_1^t} \frac{f_0(s_1^t) + f_{c1}}{G_1^t t_1^t} ds_1^t + \int_0^{s_2^w} \frac{f_0(s_2^w) + f_{c1} - f_{c2}}{G_2^w t_2^w} ds_2^w - \int_0^{s_1^b} \frac{f_0(s_1^b) - f_{c1}}{G_1^b t_1^b} ds_1^b - \int_0^{s_1^w} f_0(s_1^w)$$

<u>subscripts:</u> t: top
b: bottom
w: web

$\boxed{\text{see file "transverseShearFlow\_4cells.mw" to derive elements of matrices } A, b \text{ for sys. of eqns. } A_{ij} f_{c_i} = b_i}$

$$b_i = -\int_0^{L_i^t} \frac{f_0(s_i^t)}{G_i^t t_i^t} ds_i^t + \int_0^{L_i^b} \frac{f_0(s_i^b)}{G_i^b t_i^b} ds_i^b + \int_0^{L_i^w} \frac{f_0(s_i^w)}{G_i^w t_i^w} ds_i^w - \int_0^{L_{i+1}^w} \frac{f_0(s_{i+1}^w)}{G_{i+1}^w t_{i+1}^w} ds_{i+1}^w$$

$i \in 1, nCells$

$$A_{i,i} = \int_0^{L_i^t} \frac{1}{G_i^t t_i^t} ds_i^t + \int_0^{L_i^b} \frac{1}{G_i^b t_i^b} ds_i^b + \int_0^{L_i^w} \frac{1}{G_i^w t_i^w} ds_i^w + \int_0^{L_{i+1}^w} \frac{1}{G_{i+1}^w t_{i+1}^w} ds_{i+1}^w$$

$i \in 1, nCells-1$

$$A_{i,i+1} = -\int_0^{L^{w_{i+1}}} \frac{1}{G_{i+1}^w t_{i+1}^w} ds_{i+1}^w$$

$i \in 2, nCells$

$$A_{i,i-1} = -\int_0^{L_i^w} \frac{1}{G_i^w t_i^w} ds_i^w$$

this is the general form for any number of cells, configured like the example illustration above

system of equations to solve for the closing shear constants

$$A f_{c_i} = b$$

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \begin{bmatrix} f_{c1} \\ f_{c2} \\ f_{c3} \\ f_{c4} \end{bmatrix} = \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

Maple v15.00 file: transverseShearFlow-4cells.mw

```
> restart:
  with(IntegrationTools):
  with(linalg):
  with(ArrayTools):
```

Define the total relative axial displacements in each cell (note that w_ti = 0)

$$> w_{t1} := \int_0^{L_{t1}} \frac{f_o(s_{t1}) + f_{c1}}{G_{t1}(s_{t1}) \cdot t_{t1}(s_{t1})}\, \mathrm{d}s_{t1} + \int_0^{L_{w2}} \frac{f_o(s_{w2}) + f_{c1} - f_{c2}}{G_{w2}(s_{w2}) \cdot t_{w2}(s_{w2})}\, \mathrm{d}s_{w2} - \left(\int_0^{L_{b1}} \frac{f_o(s_{b1}) - f_{c1}}{G_{b1}(s_{b1}) \cdot t_{b1}(s_{b1})}\, \mathrm{d}s_{b1} \right):$$

$$w_{t2} := \int_0^{L_{t2}} \frac{f_o(s_{t2}) + f_{c2}}{G_{t2}(s_{t2}) \cdot t_{t2}(s_{t2})}\, \mathrm{d}s_{t2} + \int_0^{L_{w3}} \frac{f_o(s_{w3}) + f_{c2} - f_{c3}}{G_{w3}(s_{w3}) \cdot t_{w3}(s_{w3})}\, \mathrm{d}s_{w3} - \left(\int_0^{L_{b2}} \frac{f_o(s_{b2}) - f_{c2}}{G_{b2}(s_{b2}) \cdot t_{b2}(s_{b2})}\, \mathrm{d}s_{b2} + \int_0^{L_{w2}} \frac{f_o(s_{w2}) + f_{c1} - f_{c2}}{G_{w2}(s_{w2}) \cdot t_{w2}(s_{w2})}\, \mathrm{d}s_{w2} \right):$$

$$w_{t3} := \int_0^{L_{t3}} \frac{f_o(s_{t3}) + f_{c3}}{G_{t3}(s_{t3}) \cdot t_{t3}(s_{t3})}\, \mathrm{d}s_{t3} + \int_0^{L_{w4}} \frac{f_o(s_{w4}) + f_{c3} - f_{c4}}{G_{w4}(s_{w4}) \cdot t_{w4}(s_{w4})}\, \mathrm{d}s_{w4} - \left(\int_0^{L_{b3}} \frac{f_o(s_{b3}) - f_{c3}}{G_{b3}(s_{b3}) \cdot t_{b3}(s_{b3})}\, \mathrm{d}s_{b3} + \int_0^{L_{w3}} \frac{f_o(s_{w3}) + f_{c2} - f_{c3}}{G_{w3}(s_{w3}) \cdot t_{w3}(s_{w3})}\, \mathrm{d}s_{w3} \right):$$

$$w_{t4} := \int_0^{L_{t4}} \frac{f_o(s_{t4}) + f_{c4}}{G_{t4}(s_{t4}) \cdot t_{t4}(s_{t4})}\, \mathrm{d}s_{t4} - \left(\int_0^{L_{b4}} \frac{f_o(s_{b4}) - f_{c4}}{G_{b4}(s_{b4}) \cdot t_{b4}(s_{b4})}\, \mathrm{d}s_{b4} + \int_0^{L_{w4}} \frac{f_o(s_{w4}) + f_{c3} - f_{c4}}{G_{w4}(s_{w4}) \cdot t_{w4}(s_{w4})}\, \mathrm{d}s_{w4} \right):$$

Define the expressions and the unknowns

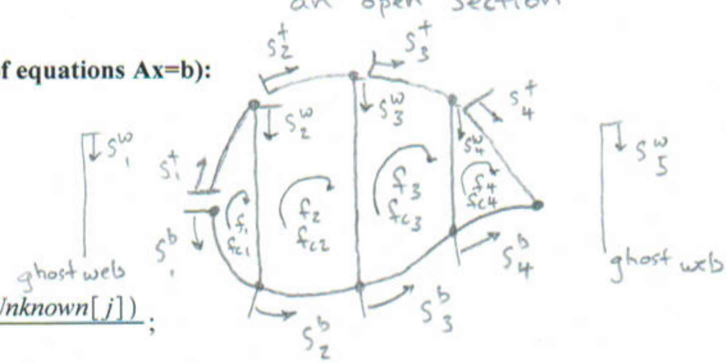
$> Expr := [w_{t1}, w_{t2}, w_{t3}, w_{t4}]:$

$Unknown := [f_{c1}, f_{c2}, f_{c3}, f_{c4}]:$

1 cut per cell to make an open section

Define procedure (returns the matrix A and vector b of the system of equations Ax=b):

```
> shearFlowMatrices := proc(Expr, Unknown)
  local i, j, N, A, b;
  N := nops(Expr);
  A := Matrix(N, N);
  for i from 1 to N do
    for j from 1 to N do
      A(i, j) := select(has, collect(Expand(Expr[i]), Unknown), Unknown[j]) / Unknown[j];
    end do;
  end do;
  b := Matrix(N, 1);
  for i from 1 to N do
    b(i) := -1·remove(has, collect(Expand(Expr[i]), Unknown), Unknown);
  end do;
  return(A, b);
  end proc:
```



Derive the matrix A and vector b:

```
> A, b := shearFlowMatrices(Expr, Unknown) :
```

Display the elements of A and b:

```
> m, n := Size(A) :
  for i from 1 to m do
  for j from 1 to n do
  printf("\nA(%d,%d) = ", i, j);
  print(A(i, j));
  end do
  end do
```

A(1,1) =

$$\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1}$$

A(1,2) =

$$-\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}\right)$$

A(1,3) =

$$0$$

A(1,4) =

$$0$$

A(2,1) =

$$-\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}\right)$$

A(2,2) =

$$\int_0^{L_{t2}} \frac{1}{G_{t2}(s_{t2})\, t_{t2}(s_{t2})}\, ds_{t2} + \int_0^{L_{b2}} \frac{1}{G_{b2}(s_{b2})\, t_{b2}(s_{b2})}\, ds_{b2} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3} + \int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}$$

A(2,3) =

$$-\left(\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\right)$$

A(2,4) =

$$0$$

A(3,1) =

$$0$$

A(3,2) =

$$-\left(\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\right)$$

A(3,3) =

$$\int_0^{L_{t3}} \frac{1}{G_{t3}(s_{t3})\, t_{t3}(s_{t3})}\, ds_{t3} + \int_0^{L_{b3}} \frac{1}{G_{b3}(s_{b3})\, t_{b3}(s_{b3})}\, ds_{b3} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}$$

A(3,4) =

$$-\left(\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\right)$$

A(4,1) =

$$0$$

A(4,2) =

$$0$$

A(4,3) =

$$-\left(\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\right)$$

A(4,4) =

$$\int_0^{L_{t4}} \frac{1}{G_{t4}(s_{t4})\, t_{t4}(s_{t4})}\, ds_{t4} + \int_0^{L_{b4}} \frac{1}{G_{b4}(s_{b4})\, t_{b4}(s_{b4})}\, ds_{b4} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4} \tag{1}$$

> A

$$\Bigg[\Bigg[\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1}, -\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}\right), 0, 0\Bigg], \tag{2}$$

$$\Bigg[-\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}\right), \int_0^{L_{t2}} \frac{1}{G_{t2}(s_{t2})\, t_{t2}(s_{t2})}\, ds_{t2} + \int_0^{L_{b2}} \frac{1}{G_{b2}(s_{b2})\, t_{b2}(s_{b2})}\, ds_{b2} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3} + \int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}, -\left(\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\right), 0\Bigg],$$

$$\Bigg[0, -\left(\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\right), \int_0^{L_{t3}} \frac{1}{G_{t3}(s_{t3})\, t_{t3}(s_{t3})}\, ds_{t3} + \int_0^{L_{b3}} \frac{1}{G_{b3}(s_{b3})\, t_{b3}(s_{b3})}\, ds_{b3} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}, -\left(\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\right)\Bigg],$$

$$\Bigg[0, 0, -\left(\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\right), \int_0^{L_{t4}} \frac{1}{G_{t4}(s_{t4})\, t_{t4}(s_{t4})}\, ds_{t4} + \int_0^{L_{b4}} \frac{1}{G_{b4}(s_{b4})\, t_{b4}(s_{b4})}\, ds_{b4} +$$

$$\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\Bigg]\Bigg]$$

> b

$$\Bigg[\Bigg[-\left(\int_0^{L_{t1}} \frac{f_o(s_{t1})}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1}\right) + \int_0^{L_{b1}} \frac{f_o(s_{b1})}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} - \left(\int_0^{L_{w2}} \frac{f_o(s_{w2})}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}\right)\Bigg],$$

$$\Bigg[-\left(\int_0^{L_{t2}} \frac{f_o(s_{t2})}{G_{t2}(s_{t2})\, t_{t2}(s_{t2})}\, ds_{t2}\right) + \int_0^{L_{b2}} \frac{f_o(s_{b2})}{G_{b2}(s_{b2})\, t_{b2}(s_{b2})}\, ds_{b2} - \left(\int_0^{L_{w3}} \frac{f_o(s_{w3})}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\right) + \int_0^{L_{w2}} \frac{f_o(s_{w2})}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}\Bigg],$$

$$\Bigg[-\left(\int_0^{L_{t3}} \frac{f_o(s_{t3})}{G_{t3}(s_{t3})\, t_{t3}(s_{t3})}\, ds_{t3}\right) + \int_0^{L_{b3}} \frac{f_o(s_{b3})}{G_{b3}(s_{b3})\, t_{b3}(s_{b3})}\, ds_{b3} - \left(\int_0^{L_{w4}} \frac{f_o(s_{w4})}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\right) + \int_0^{L_{w3}} \frac{f_o(s_{w3})}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\Bigg],$$

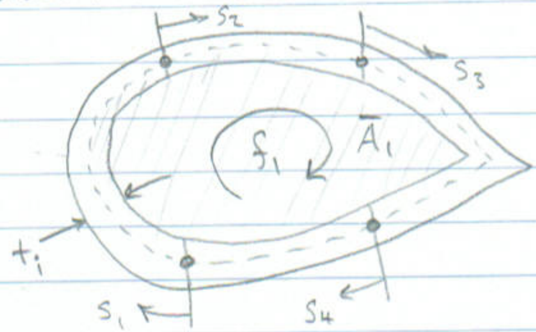
$$\Bigg[-\left(\int_0^{L_{t4}} \frac{f_o(s_{t4})}{G_{t4}(s_{t4})\, t_{t4}(s_{t4})}\, ds_{t4}\right) + \int_0^{L_{b4}} \frac{f_o(s_{b4})}{G_{b4}(s_{b4})\, t_{b4}(s_{b4})}\, ds_{b4} + \int_0^{L_{w4}} \frac{f_o(s_{w4})}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\Bigg]\Bigg] \tag{3}$$

>

torsional shear flow

Bauchau § 8.5.2 - torsion of closed section

single cell section



$\bar{A}_1$ = area enclosed by cell 1

"Bredt - Batho" formula:

$$M_z = 2\sum_{i=1}^{N} \bar{A}_i f_i = 2\bar{A}_1 f_1$$

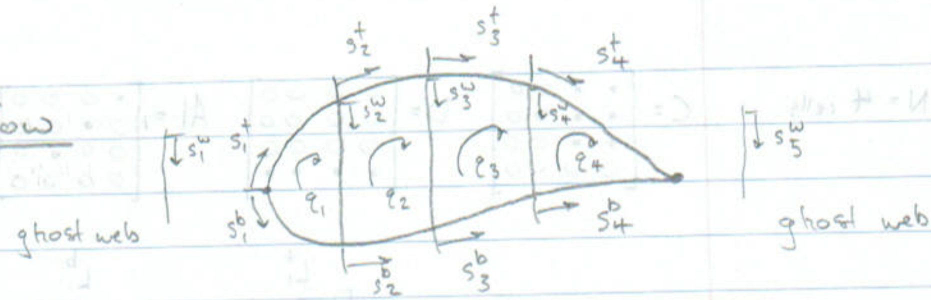
- the shear flow due to torsion is constant

$$f_{i\,\text{torsion}} = \frac{M_z}{2\bar{A}_1}$$

the total shear flow is the summation of the transverse + torsion shear flows

$$f(s_i) = f_{\text{transverse}}(s_i) + f_{i\,\text{torsion}}$$

Torsional shear flow



for N cells, N unknown shear flows (q is constant over the curves s_i)

Bredt-Batho formula: $\sum_{i=1}^{N} 2A_i q_i = M_z$

twist rate: $k_{z_i} = \frac{1}{2A_i} \oint_{cell_i} \frac{q}{Gt} ds$

eqn 1: $\sum A_i q_i - \frac{1}{2} M_z = 0$

eqn 2: $2(k_{z1} - k_{z2}) = 0$ • multiply by 2 for convenience

eqn 3: $2(k_{z1} - k_{z3}) = 0$

eqn 4: $2(k_{z1} - k_{z4}) = 0$

• solve the system $A\vec{q} = \vec{b}$, $A = C + D + A1 + A2 + A3$

$\vec{b} = [0 \quad 0 \quad 0 \quad \frac{1}{2} M_z]^T$

$\vec{q} = [q_1 \quad q_2 \quad q_3 \quad q_4]^T$

$q(s_i^t) = q_i$

$q(s_i^b) = -q_i$

$q(s_i^w) = q_{i-1} - q_i$

torsional stiffness:

$$\text{tor\_stff} = \frac{M_z}{k_{z1}}$$

$N = 4$ cells $\quad C = \begin{bmatrix} \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ \bullet & \bullet & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \bullet & \bullet & \bullet & \bullet \end{bmatrix}$ $A1 = \begin{bmatrix} \bullet & 0 & 0 & 0 \\ 0 & \bullet & 0 & 0 \\ 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $A2 = \begin{bmatrix} 0 & \bullet & 0 & 0 \\ 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $A3 = \begin{bmatrix} 0 & 0 & \bullet & 0 \\ 0 & 0 & 0 & \bullet \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$n = 1:N-1 \longrightarrow C(n,1) = \frac{1}{A_1}\left[\int_0^{L_1^t} \frac{1}{G_1^t t_1^t}\,ds_1^t + \int_0^{L_1^b} \frac{1}{G_1^b t_1^b}\,ds_1^b + \int_0^{L_1^w} \frac{1}{G_1^w t_1^w}\,ds_1^w + \int_0^{L_2^w} \frac{1}{G_2^w t_2^w}\,ds_2^w\right]$$

$$\longrightarrow C(n,2) = -\frac{1}{A_1}\int_0^{L_2^w} \frac{1}{G_2^w t_2^w}\,ds_2^w$$

$$n = 1:N \longrightarrow D(N,n) = A_n$$

$$n = 1:N-1 \longrightarrow A1(n,n) = \frac{1}{A_{n+1}}\int_0^{L_{n+1}^w} \frac{1}{G_{n+1}^w t_{n+1}^w}\,ds_{n+1}^w$$

$$n = 1:N-1 \longrightarrow A2(n,n+1) = \frac{-1}{A_{n+1}}\left[\int_0^{L_{n+1}^t} \frac{1}{G_{n+1}^t t_{n+1}^t}\,ds_{n+1}^t + \int_0^{L_{n+1}^b} \frac{1}{G_{n+1}^b t_{n+1}^b}\,ds_{n+1}^b + \int_0^{L_{n+1}^w} \frac{1}{G_{n+1}^w t_{n+1}^w}\,ds_{n+1}^w + \int_0^{L_{n+2}^w} \frac{1}{G_{n+2}^w t_{n+2}^w}\,ds_{n+2}^w\right]$$

$$n = 1:N-2 \longrightarrow A3(n,n+2) = \frac{1}{A_{n+1}}\int_0^{L_{n+2}^w} \frac{1}{G_{n+2}^w t_{n+2}^w}\,ds_{n+2}^w$$

$$\boxed{A = C + D + A1 + A2 + A3}$$

$$\boxed{A\vec{q} = \vec{b}}$$

To determine the general form of A and b, I used Maple to try and figure out the pattern by solving examples of torsional shear flow for 3, 4, and 5 cells. See Maple files:

torsionShearFlow_3cells.mw

" ——— "_4cells.mw

" ——— "_5cells.mw

Maple v15.00 File: torsionShearFlow_3cells.mw

```
> restart :
  with(IntegrationTools) :
  with(linalg) :
  with(ArrayTools) :
```

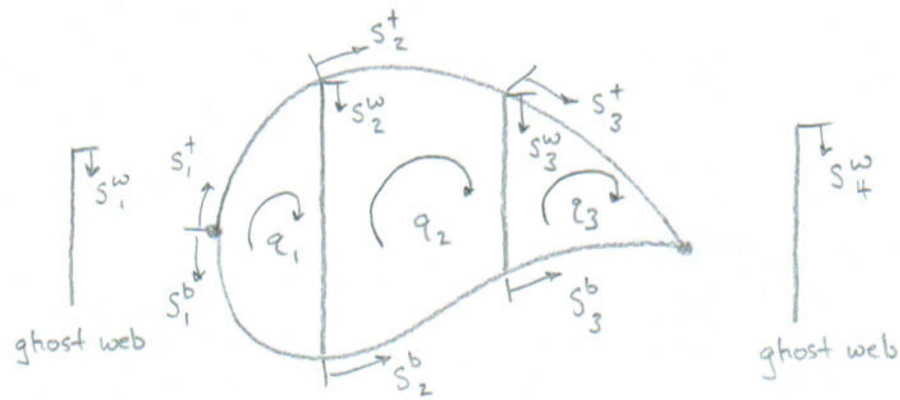
Define the twist rate of each cell, noting that the twist rates of each cell are equal to each other:)

$$> k1 := \frac{1}{2\cdot A1}\cdot\left(\left(\int_0^{L_{t1}} \frac{q1}{G_{t1}(s_{t1})\cdot t_{t1}(s_{t1})}\,\mathrm{d}s_{t1} + \int_0^{L_{w2}} \frac{q1-q2}{G_{w2}(s_{w2})\cdot t_{w2}(s_{w2})}\,\mathrm{d}s_{w2} - \left(\int_0^{L_{b1}} \frac{-q1}{G_{b1}(s_{b1})\cdot t_{b1}(s_{b1})}\,\mathrm{d}s_{b1} + \int_0^{L_{w1}} \frac{q0-q1}{G_{w1}(s_{w1})\cdot t_{w1}(s_{w1})}\,\mathrm{d}s_{w1}\right)\right)\right) :$$

$$k2 := \frac{1}{2\cdot A2}\cdot\left(\left(\int_0^{L_{t2}} \frac{q2}{G_{t2}(s_{t2})\cdot t_{t2}(s_{t2})}\,\mathrm{d}s_{t2} + \int_0^{L_{w3}} \frac{q2-q3}{G_{w3}(s_{w3})\cdot t_{w3}(s_{w3})}\,\mathrm{d}s_{w3} - \left(\int_0^{L_{b2}} \frac{-q2}{G_{b2}(s_{b2})\cdot t_{b2}(s_{b2})}\,\mathrm{d}s_{b2} + \int_0^{L_{w2}} \frac{q1-q2}{G_{w2}(s_{w2})\cdot t_{w2}(s_{w2})}\,\mathrm{d}s_{w2}\right)\right)\right) :$$

$$k3 := \frac{1}{2\cdot A3}\cdot\left(\left(\int_0^{L_{t3}} \frac{q3}{G_{t3}(s_{t3})\cdot t_{t3}(s_{t3})}\,\mathrm{d}s_{t3} + \int_0^{L_{w4}} \frac{q3-q4}{G_{w4}(s_{w4})\cdot t_{w4}(s_{w4})}\,\mathrm{d}s_{w4} - \left(\int_0^{L_{b3}} \frac{-q3}{G_{b3}(s_{b3})\cdot t_{b3}(s_{b3})}\,\mathrm{d}s_{b3} + \int_0^{L_{w3}} \frac{q2-q3}{G_{w3}(s_{w3})\cdot t_{w3}(s_{w3})}\,\mathrm{d}s_{w3}\right)\right)\right) :$$

$$expr_1 := 2\cdot(k1 - k2) :$$
$$expr_2 := 2\cdot(\mathrm{k1} - \mathrm{k3}) :$$
$$expr_3 := A1\cdot q1 + A2\cdot q2 + A3\cdot q3 - \frac{Mz}{2} :$$



Define the expressions and the unknowns

```
> Expr := [expr1, expr2, expr3] :
  Unknown := [q1, q2, q3] :
```

Define procedure (returns the matrix A and vector b of the system of equations Ax=b):

```
> shearFlowMatrices := proc(Expr, Unknown)
  local i, j, N, A, b;
  N := nops(Expr);
  A := Matrix(N, N);
  for i from 1 to N do
    for j from 1 to N do
      A(i, j) := select(has, collect(Expand(Expr[i]), Unknown), Unknown[j]) / Unknown[j];
    end do;
  end do;
  b := Matrix(N, 1);
  for i from 1 to N do
    b(i) := -1·remove(has, collect(Expand(Expr[i]), Unknown), Unknown);
  end do;
  return(A, b);
  end proc:
```

Derive the matrix A and vector b:

```
> A, b := shearFlowMatrices(Expr, Unknown) :
```

Display the elements of A and b:

```
> m, n := Size(A) :
  for i from 1 to m do
  for j from 1 to n do
  printf("\nA(%d,%d) = ", i, j);
  print(A(i,j));
  end do
  end do
```

A(1,1) =

$$\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right) + \frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A2}$$

A(1,2) =

$$-\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1} - \frac{1}{A2}\left(\int_0^{L_{t2}} \frac{1}{G_{t2}(s_{t2})\, t_{t2}(s_{t2})}\, ds_{t2} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3} + \int_0^{L_{b2}} \frac{1}{G_{b2}(s_{b2})\, t_{b2}(s_{b2})}\, ds_{b2} + \int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}\right)$$

A(1,3) =

$$\frac{\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}}{A2}$$

A(2,1) =

$$\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right)$$

A(2,2) =

$$-\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1} + \frac{\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}}{A3}$$

```
A(2,3) =
```

$$-\frac{1}{A3}\left(\int_0^{L_{t3}} \frac{1}{G_{t3}(s_{t3})\, t_{t3}(s_{t3})}\, ds_{t3} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4} + \int_0^{L_{b3}} \frac{1}{G_{b3}(s_{b3})\, t_{b3}(s_{b3})}\, ds_{b3} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\right)$$

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A(3,1) =
```

$$A1$$

```
A(3,2) =
```

$$A2$$

```
A(3,3) =
```

$$A3 \tag{1}$$

> A

$$\left[\left[\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right) + \frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A2}, -\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1} - \frac{1}{A2}\left(\int_0^{L_{t2}} \frac{1}{G_{t2}(s_{t2})\, t_{t2}(s_{t2})}\, ds_{t2} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3} + \int_0^{L_{b2}} \frac{1}{G_{b2}(s_{b2})\, t_{b2}(s_{b2})}\, ds_{b2} + \int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}\right), \frac{\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}}{A2}\right], \tag{2}$$

$$\left[\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right), -\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1} + \frac{\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}}{A3}, -\frac{1}{A3}\left(\int_0^{L_{t3}} \frac{1}{G_{t3}(s_{t3})\, t_{t3}(s_{t3})}\, ds_{t3} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4} + \int_0^{L_{b3}} \frac{1}{G_{b3}(s_{b3})\, t_{b3}(s_{b3})}\, ds_{b3} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\right)\right],$$

$$\left[\left[A1, A2, A3\right]\right]$$

> b

$$\left[\begin{array}{c} \dfrac{q0\left(\displaystyle\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\right)}{A1} \\ \dfrac{q0\left(\displaystyle\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\right)}{A1} - \dfrac{q4\left(\displaystyle\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, \mathrm{d}s_{w4}\right)}{A3} \\ \dfrac{1}{2}\, Mz \end{array}\right] \tag{3}$$

>

Maple v15.00 file: torsionShearFlow_4cells.mw

```
> restart:
  with(IntegrationTools):
  with(linalg):
  with(ArrayTools):
```

Define the twist rate of each cell, noting that the twist rates of each cell are equal to each other:)

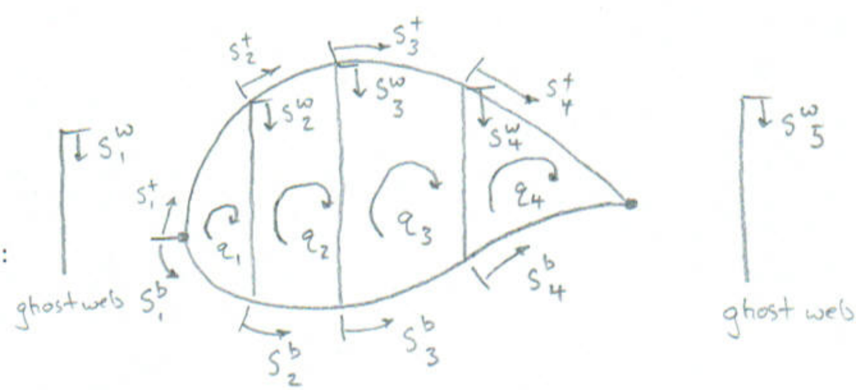
$$> k1 := \frac{1}{2 \cdot A1} \cdot \left(\int_0^{L_{t1}} \frac{q1}{G_{t1}(s_{t1}) \cdot t_{t1}(s_{t1})} \, ds_{t1} + \int_0^{L_{w2}} \frac{q1 - q2}{G_{w2}(s_{w2}) \cdot t_{w2}(s_{w2})} \, ds_{w2} - \left(\int_0^{L_{b1}} \frac{-q1}{G_{b1}(s_{b1}) \cdot t_{b1}(s_{b1})} \, ds_{b1} + \int_0^{L_{w1}} \frac{q0 - q1}{G_{w1}(s_{w1}) \cdot t_{w1}(s_{w1})} \, ds_{w1} \right) \right) :$$

$$k2 := \frac{1}{2 \cdot A2} \cdot \left(\int_0^{L_{t2}} \frac{q2}{G_{t2}(s_{t2}) \cdot t_{t2}(s_{t2})} \, ds_{t2} + \int_0^{L_{w3}} \frac{q2 - q3}{G_{w3}(s_{w3}) \cdot t_{w3}(s_{w3})} \, ds_{w3} - \left(\int_0^{L_{b2}} \frac{-q2}{G_{b2}(s_{b2}) \cdot t_{b2}(s_{b2})} \, ds_{b2} + \int_0^{L_{w2}} \frac{q1 - q2}{G_{w2}(s_{w2}) \cdot t_{w2}(s_{w2})} \, ds_{w2} \right) \right) :$$

$$k3 := \frac{1}{2 \cdot A3} \cdot \left(\int_0^{L_{t3}} \frac{q3}{G_{t3}(s_{t3}) \cdot t_{t3}(s_{t3})} \, ds_{t3} + \int_0^{L_{w4}} \frac{q3 - q4}{G_{w4}(s_{w4}) \cdot t_{w4}(s_{w4})} \, ds_{w4} - \left(\int_0^{L_{b3}} \frac{-q3}{G_{b3}(s_{b3}) \cdot t_{b3}(s_{b3})} \, ds_{b3} + \int_0^{L_{w3}} \frac{q2 - q3}{G_{w3}(s_{w3}) \cdot t_{w3}(s_{w3})} \, ds_{w3} \right) \right) :$$

$$k4 := \frac{1}{2 \cdot A4} \cdot \left(\int_0^{L_{t4}} \frac{q4}{G_{t4}(s_{t4}) \cdot t_{t4}(s_{t4})} \, ds_{t4} + \int_0^{L_{w5}} \frac{q4 - q5}{G_{w5}(s_{w5}) \cdot t_{w5}(s_{w5})} \, ds_{w5} - \left(\int_0^{L_{b4}} \frac{-q4}{G_{b4}(s_{b4}) \cdot t_{b4}(s_{b4})} \, ds_{b4} + \int_0^{L_{w4}} \frac{q3 - q4}{G_{w4}(s_{w4}) \cdot t_{w4}(s_{w4})} \, ds_{w4} \right) \right) :$$

$$expr_1 := 2 \cdot (k1 - k2) :$$
$$expr_2 := 2 \cdot (k1 - k3) :$$
$$expr_3 := 2 \cdot (k1 - k4) :$$
$$expr_4 := A1 \cdot q1 + A2 \cdot q2 + A3 \cdot q3 + A4 \cdot q4 - \frac{Mz}{2} :$$



Define the expressions and the unknowns

$$> Expr := [expr_1, expr_2, expr_3, expr_4] :$$
$$Unknown := [q1, q2, q3, q4] :$$

Define procedure (returns the matrix A and vector b of the system of equations Ax=b):

```
> shearFlowMatrices := proc(Expr, Unknown)
  local i, j, N, A, b;
  N := nops(Expr);
  A := Matrix(N, N);
  for i from 1 to N do
    for j from 1 to N do
      A(i, j) := select(has, collect(Expand(Expr[i]), Unknown), Unknown[j]) / Unknown[j];
```

```
    end do;
  end do;
  b := Matrix(N, 1);
  for i from 1 to N do
    b(i) := -1·remove(has, collect(Expand(Expr[i]), Unknown), Unknown);
  end do;
  return(A, b);
  end proc:
```

Derive the matrix A and vector b:

```
> A, b := shearFlowMatrices(Expr, Unknown) :
```

Display the elements of A and b:

```
> m, n := Size(A) :
  for i from 1 to m do
  for j from 1 to n do
  printf("\nA(%d,%d) = ", i, j);
  print(A(i,j));
  end do
  end do
```

A(1,1) =

$$\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, \mathrm{d}s_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, \mathrm{d}s_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\right) + \frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}}{A2}$$

A(1,2) =

$$-\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}}{A1} - \frac{1}{A2}\left(\int_0^{L_{t2}} \frac{1}{G_{t2}(s_{t2})\, t_{t2}(s_{t2})}\, \mathrm{d}s_{t2} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, \mathrm{d}s_{w3} + \int_0^{L_{b2}} \frac{1}{G_{b2}(s_{b2})\, t_{b2}(s_{b2})}\, \mathrm{d}s_{b2} + \int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}\right)$$

A(1,3) =

$$\frac{\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, \mathrm{d}s_{w3}}{A2}$$

A(1,4) =

$$0$$

A(2,1) =

$$\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, \mathrm{d}s_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, \mathrm{d}s_{t1} +\right.$$

$$\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\Bigg)$$

A(2,2) =

$$-\frac{\displaystyle\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1} + \frac{\displaystyle\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}}{A3}$$

A(2,3) =

$$-\frac{1}{A3}\left(\int_0^{L_{t3}} \frac{1}{G_{t3}(s_{t3})\, t_{t3}(s_{t3})}\, ds_{t3} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4} + \int_0^{L_{b3}} \frac{1}{G_{b3}(s_{b3})\, t_{b3}(s_{b3})}\, ds_{b3} + \right.$$

$$\left.\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\right)$$

A(2,4) =

$$\frac{\displaystyle\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}}{A3}$$

A(3,1) =

$$\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \right.$$

$$\left.\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right)$$

A(3,2) =

$$-\frac{\displaystyle\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1}$$

A(3,3) =

$$\frac{\displaystyle\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}}{A4}$$

A(3,4) =

$$-\frac{1}{A4}\left(\int_0^{L_{t4}} \frac{1}{G_{t4}(s_{t4})\, t_{t4}(s_{t4})}\, ds_{t4} + \int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5} + \int_0^{L_{b4}} \frac{1}{G_{b4}(s_{b4})\, t_{b4}(s_{b4})}\, ds_{b4} + \right.$$

$$\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, \mathrm{d}s_{w4}\Bigg)$$

```
A(4,1) =
```

$$A1$$

```
A(4,2) =
```

$$A2$$

```
A(4,3) =
```

$$A3$$

```
A(4,4) =
```

$$A4 \tag{1}$$

> A

$$\Bigg[\Bigg[\frac{1}{A1}\Bigg(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, \mathrm{d}s_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, \mathrm{d}s_{t1} + \tag{2}$$

$$\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\Bigg) + \frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}}{A2}, -\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}}{A1} - \frac{1}{A2}\Bigg($$

$$\int_0^{L_{t2}} \frac{1}{G_{t2}(s_{t2})\, t_{t2}(s_{t2})}\, \mathrm{d}s_{t2} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, \mathrm{d}s_{w3} + \int_0^{L_{b2}} \frac{1}{G_{b2}(s_{b2})\, t_{b2}(s_{b2})}\, \mathrm{d}s_{b2} +$$

$$\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}\Bigg), \frac{\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, \mathrm{d}s_{w3}}{A2}, 0\Bigg],$$

$$\Bigg[\frac{1}{A1}\Bigg(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, \mathrm{d}s_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, \mathrm{d}s_{t1} +$$

$$\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\Bigg), -\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}}{A1} + \frac{\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, \mathrm{d}s_{w3}}{A3}, -\frac{1}{A3}\Bigg($$

$$\int_0^{L_{t3}} \frac{1}{G_{t3}(s_{t3})\, t_{t3}(s_{t3})}\, \mathrm{d}s_{t3} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, \mathrm{d}s_{w4} + \int_0^{L_{b3}} \frac{1}{G_{b3}(s_{b3})\, t_{b3}(s_{b3})}\, \mathrm{d}s_{b3} +$$

$$\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\Bigg), \frac{\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}}{A3}\Bigg],$$

$$\Bigg[\frac{1}{A1}\Bigg(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} +$$

$$\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\Bigg), -\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1}, \frac{\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}}{A4}, -\frac{1}{A4}\Bigg($$

$$\int_0^{L_{t4}} \frac{1}{G_{t4}(s_{t4})\, t_{t4}(s_{t4})}\, ds_{t4} + \int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5} + \int_0^{L_{b4}} \frac{1}{G_{b4}(s_{b4})\, t_{b4}(s_{b4})}\, ds_{b4} +$$

$$\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\Bigg)\Bigg],$$

$$\Big[A1, A2, A3, A4\Big]\Big]$$

> b

$$\begin{bmatrix} \dfrac{q0\left(\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right)}{A1} \\ \dfrac{q0\left(\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right)}{A1} \\ \dfrac{q0\left(\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right)}{A1} - \dfrac{q5\left(\int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5}\right)}{A4} \\ \frac{1}{2}\, Mz \end{bmatrix} \tag{3}$$

>

Maple v15.00 File: torsionShearFlow_5cells.mw

```
> restart :
  with(IntegrationTools) :
  with(linalg) :
  with(ArrayTools) :
```

Define the twist rate of each cell, noting that the twist rates of each cell are equal to each other:)

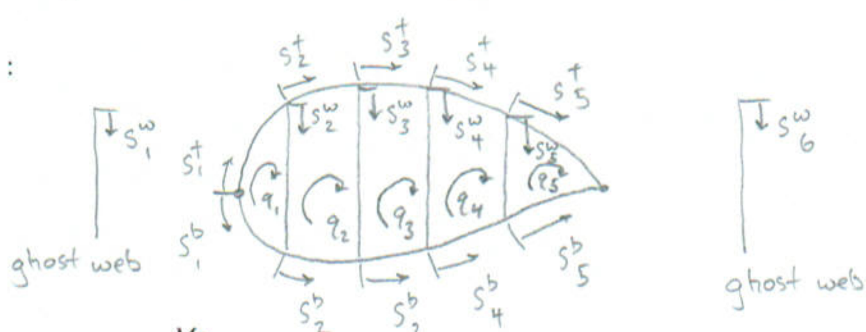
$$> k1 := \frac{1}{2\cdot A1}\cdot\left(\left(\int_0^{L_{t1}} \frac{q1}{G_{t1}(s_{t1})\cdot t_{t1}(s_{t1})}\,\mathrm{d}s_{t1} + \int_0^{L_{w2}} \frac{q1-q2}{G_{w2}(s_{w2})\cdot t_{w2}(s_{w2})}\,\mathrm{d}s_{w2} - \left(\int_0^{L_{b1}} \frac{-q1}{G_{b1}(s_{b1})\cdot t_{b1}(s_{b1})}\,\mathrm{d}s_{b1} + \int_0^{L_{w1}} \frac{q0-q1}{G_{w1}(s_{w1})\cdot t_{w1}(s_{w1})}\,\mathrm{d}s_{w1}\right)\right)\right):$$

$$k2 := \frac{1}{2\cdot A2}\cdot\left(\left(\int_0^{L_{t2}} \frac{q2}{G_{t2}(s_{t2})\cdot t_{t2}(s_{t2})}\,\mathrm{d}s_{t2} + \int_0^{L_{w3}} \frac{q2-q3}{G_{w3}(s_{w3})\cdot t_{w3}(s_{w3})}\,\mathrm{d}s_{w3} - \left(\int_0^{L_{b2}} \frac{-q2}{G_{b2}(s_{b2})\cdot t_{b2}(s_{b2})}\,\mathrm{d}s_{b2} + \int_0^{L_{w2}} \frac{q1-q2}{G_{w2}(s_{w2})\cdot t_{w2}(s_{w2})}\,\mathrm{d}s_{w2}\right)\right)\right):$$

$$k3 := \frac{1}{2\cdot A3}\cdot\left(\left(\int_0^{L_{t3}} \frac{q3}{G_{t3}(s_{t3})\cdot t_{t3}(s_{t3})}\,\mathrm{d}s_{t3} + \int_0^{L_{w4}} \frac{q3-q4}{G_{w4}(s_{w4})\cdot t_{w4}(s_{w4})}\,\mathrm{d}s_{w4} - \left(\int_0^{L_{b3}} \frac{-q3}{G_{b3}(s_{b3})\cdot t_{b3}(s_{b3})}\,\mathrm{d}s_{b3} + \int_0^{L_{w3}} \frac{q2-q3}{G_{w3}(s_{w3})\cdot t_{w3}(s_{w3})}\,\mathrm{d}s_{w3}\right)\right)\right):$$

$$k4 := \frac{1}{2\cdot A4}\cdot\left(\left(\int_0^{L_{t4}} \frac{q4}{G_{t4}(s_{t4})\cdot t_{t4}(s_{t4})}\,\mathrm{d}s_{t4} + \int_0^{L_{w5}} \frac{q4-q5}{G_{w5}(s_{w5})\cdot t_{w5}(s_{w5})}\,\mathrm{d}s_{w5} - \left(\int_0^{L_{b4}} \frac{-q4}{G_{b4}(s_{b4})\cdot t_{b4}(s_{b4})}\,\mathrm{d}s_{b4} + \int_0^{L_{w4}} \frac{q3-q4}{G_{w4}(s_{w4})\cdot t_{w4}(s_{w4})}\,\mathrm{d}s_{w4}\right)\right)\right):$$

$$k5 := \frac{1}{2\cdot A5}\cdot\left(\left(\int_0^{L_{t5}} \frac{q5}{G_{t5}(s_{t5})\cdot t_{t5}(s_{t5})}\,\mathrm{d}s_{t5} + \int_0^{L_{w6}} \frac{q5-q6}{G_{w6}(s_{w6})\cdot t_{w6}(s_{w6})}\,\mathrm{d}s_{w6} - \left(\int_0^{L_{b5}} \frac{-q5}{G_{b5}(s_{b5})\cdot t_{b5}(s_{b5})}\,\mathrm{d}s_{b5} + \int_0^{L_{w5}} \frac{q4-q5}{G_{w5}(s_{w5})\cdot t_{w5}(s_{w5})}\,\mathrm{d}s_{w5}\right)\right)\right):$$



$$expr_1 := 2\cdot(k1 - k2):$$
$$expr_2 := 2\cdot(\mathrm{k1} - \mathrm{k3}):$$
$$expr_3 := 2\cdot(\mathrm{k1} - \mathrm{k4}):$$
$$expr_4 := 2\cdot(\mathrm{k1} - \mathrm{k5}):$$
$$expr_5 := A1\cdot q1 + A2\cdot q2 + A3\cdot q3 + A4\cdot q4 + A5\cdot q5 - \frac{Mz}{2}:$$

Define the expressions and the unknowns

$$> Expr := [expr_1, expr_2, expr_3, expr_4, expr_5]:$$
$$Unknown := [q1, q2, q3, q4, q5]:$$

Define procedure (returns the matrix A and vector b of the system of equations Ax=b):

```
> shearFlowMatrices := proc(Expr, Unknown)
  local i, j, N, A, b;
  N := nops(Expr);
  A := Matrix(N, N);
  for i from 1 to N do
    for j from 1 to N do
      A(i, j) := select(has, collect(Expand(Expr[i]), Unknown), Unknown[j]) / Unknown[j];
    end do;
  end do;
  b := Matrix(N, 1);
  for i from 1 to N do
    b(i) := -1·remove(has, collect(Expand(Expr[i]), Unknown), Unknown);
  end do;
  return(A, b);
  end proc:
```

Derive the matrix A and vector b:

```
> A, b := shearFlowMatrices(Expr, Unknown) :
```

Display the elements of A and b:

```
> m, n := Size(A) :
  for i from 1 to m do
  for j from 1 to n do
  printf("\nA(%d,%d) = ", i, j);
  print(A(i, j));
  end do
  end do
```

A(1,1) =

$$\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, \mathrm{d}s_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, \mathrm{d}s_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\right) + \frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}}{A2}$$

A(1,2) =

$$-\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}}{A1} - \frac{1}{A2}\left(\int_0^{L_{t2}} \frac{1}{G_{t2}(s_{t2})\, t_{t2}(s_{t2})}\, \mathrm{d}s_{t2} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, \mathrm{d}s_{w3} + \int_0^{L_{b2}} \frac{1}{G_{b2}(s_{b2})\, t_{b2}(s_{b2})}\, \mathrm{d}s_{b2} + \int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, \mathrm{d}s_{w2}\right)$$

A(1,3) =

$$\frac{\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, \mathrm{d}s_{w3}}{A2}$$

A(1,4) =

$$0$$

A(1,5) =

$$0$$

A(2,1) =

$$\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right)$$

A(2,2) =

$$-\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1} + \frac{\int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}}{A3}$$

A(2,3) =

$$-\frac{1}{A3}\left(\int_0^{L_{t3}} \frac{1}{G_{t3}(s_{t3})\, t_{t3}(s_{t3})}\, ds_{t3} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4} + \int_0^{L_{b3}} \frac{1}{G_{b3}(s_{b3})\, t_{b3}(s_{b3})}\, ds_{b3} + \int_0^{L_{w3}} \frac{1}{G_{w3}(s_{w3})\, t_{w3}(s_{w3})}\, ds_{w3}\right)$$

A(2,4) =

$$\frac{\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}}{A3}$$

A(2,5) =

$$0$$

A(3,1) =

$$\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right)$$

A(3,2) =

$$-\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1}$$

A(3,3) =

$$\frac{\displaystyle\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}}{A4}$$

A(3,4) =

$$-\frac{1}{A4}\left(\int_0^{L_{t4}} \frac{1}{G_{t4}(s_{t4})\, t_{t4}(s_{t4})}\, ds_{t4} + \int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5} + \int_0^{L_{b4}} \frac{1}{G_{b4}(s_{b4})\, t_{b4}(s_{b4})}\, ds_{b4} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\right)$$

A(3,5) =

$$\frac{\displaystyle\int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5}}{A4}$$

A(4,1) =

$$\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right)$$

A(4,2) =

$$-\frac{\displaystyle\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1}$$

A(4,3) =

$$0$$

A(4,4) =

$$\frac{\displaystyle\int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5}}{A5}$$

A(4,5) =

$$-\frac{1}{A5}\left(\int_0^{L_{t5}} \frac{1}{G_{t5}(s_{t5})\, t_{t5}(s_{t5})}\, ds_{t5} + \int_0^{L_{w6}} \frac{1}{G_{w6}(s_{w6})\, t_{w6}(s_{w6})}\, ds_{w6} + \int_0^{L_{b5}} \frac{1}{G_{b5}(s_{b5})\, t_{b5}(s_{b5})}\, ds_{b5} + \int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5}\right)$$

A(5,1) =

$$A1$$

A(5,2) =

$$A2$$

A(5,3) =

$$A3$$

A(5,4) =

$$A4$$

A(5,5) =

$$A5 \tag{1}$$

> A

$$\Bigg[\Bigg[\frac{1}{A1}\Bigg(\int_0^{L_{w2}}\frac{1}{G_{w2}(s_{w2})\,t_{w2}(s_{w2})}\,\mathrm{d}s_{w2}+\int_0^{L_{b1}}\frac{1}{G_{b1}(s_{b1})\,t_{b1}(s_{b1})}\,\mathrm{d}s_{b1}+\int_0^{L_{t1}}\frac{1}{G_{t1}(s_{t1})\,t_{t1}(s_{t1})}\,\mathrm{d}s_{t1}+\int_0^{L_{w1}}\frac{1}{G_{w1}(s_{w1})\,t_{w1}(s_{w1})}\,\mathrm{d}s_{w1}\Bigg)+\frac{\int_0^{L_{w2}}\frac{1}{G_{w2}(s_{w2})\,t_{w2}(s_{w2})}\,\mathrm{d}s_{w2}}{A2}, -\frac{\int_0^{L_{w2}}\frac{1}{G_{w2}(s_{w2})\,t_{w2}(s_{w2})}\,\mathrm{d}s_{w2}}{A1}-\frac{1}{A2}\Bigg(\int_0^{L_{t2}}\frac{1}{G_{t2}(s_{t2})\,t_{t2}(s_{t2})}\,\mathrm{d}s_{t2}+\int_0^{L_{w3}}\frac{1}{G_{w3}(s_{w3})\,t_{w3}(s_{w3})}\,\mathrm{d}s_{w3}+\int_0^{L_{b2}}\frac{1}{G_{b2}(s_{b2})\,t_{b2}(s_{b2})}\,\mathrm{d}s_{b2}+\int_0^{L_{w2}}\frac{1}{G_{w2}(s_{w2})\,t_{w2}(s_{w2})}\,\mathrm{d}s_{w2}\Bigg), \frac{\int_0^{L_{w3}}\frac{1}{G_{w3}(s_{w3})\,t_{w3}(s_{w3})}\,\mathrm{d}s_{w3}}{A2}, 0, 0\Bigg], \tag{2}$$

$$\Bigg[\frac{1}{A1}\Bigg(\int_0^{L_{w2}}\frac{1}{G_{w2}(s_{w2})\,t_{w2}(s_{w2})}\,\mathrm{d}s_{w2}+\int_0^{L_{b1}}\frac{1}{G_{b1}(s_{b1})\,t_{b1}(s_{b1})}\,\mathrm{d}s_{b1}+\int_0^{L_{t1}}\frac{1}{G_{t1}(s_{t1})\,t_{t1}(s_{t1})}\,\mathrm{d}s_{t1}+\int_0^{L_{w1}}\frac{1}{G_{w1}(s_{w1})\,t_{w1}(s_{w1})}\,\mathrm{d}s_{w1}\Bigg), -\frac{\int_0^{L_{w2}}\frac{1}{G_{w2}(s_{w2})\,t_{w2}(s_{w2})}\,\mathrm{d}s_{w2}}{A1}+\frac{\int_0^{L_{w3}}\frac{1}{G_{w3}(s_{w3})\,t_{w3}(s_{w3})}\,\mathrm{d}s_{w3}}{A3}, -\frac{1}{A3}\Bigg(\int_0^{L_{t3}}\frac{1}{G_{t3}(s_{t3})\,t_{t3}(s_{t3})}\,\mathrm{d}s_{t3}+\int_0^{L_{w4}}\frac{1}{G_{w4}(s_{w4})\,t_{w4}(s_{w4})}\,\mathrm{d}s_{w4}+\int_0^{L_{b3}}\frac{1}{G_{b3}(s_{b3})\,t_{b3}(s_{b3})}\,\mathrm{d}s_{b3}+\int_0^{L_{w3}}\frac{1}{G_{w3}(s_{w3})\,t_{w3}(s_{w3})}\,\mathrm{d}s_{w3}\Bigg), \frac{\int_0^{L_{w4}}\frac{1}{G_{w4}(s_{w4})\,t_{w4}(s_{w4})}\,\mathrm{d}s_{w4}}{A3}, 0\Bigg],$$

$$\left[\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right), -\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1}, \frac{\int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}}{A4}, -\frac{1}{A4}\left(\int_0^{L_{t4}} \frac{1}{G_{t4}(s_{t4})\, t_{t4}(s_{t4})}\, ds_{t4} + \int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5} + \int_0^{L_{b4}} \frac{1}{G_{b4}(s_{b4})\, t_{b4}(s_{b4})}\, ds_{b4} + \int_0^{L_{w4}} \frac{1}{G_{w4}(s_{w4})\, t_{w4}(s_{w4})}\, ds_{w4}\right), \frac{\int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5}}{A4}\right],$$

$$\left[\frac{1}{A1}\left(\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2} + \int_0^{L_{b1}} \frac{1}{G_{b1}(s_{b1})\, t_{b1}(s_{b1})}\, ds_{b1} + \int_0^{L_{t1}} \frac{1}{G_{t1}(s_{t1})\, t_{t1}(s_{t1})}\, ds_{t1} + \int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, ds_{w1}\right), -\frac{\int_0^{L_{w2}} \frac{1}{G_{w2}(s_{w2})\, t_{w2}(s_{w2})}\, ds_{w2}}{A1}, 0, \frac{\int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5}}{A5}, -\frac{1}{A5}\left(\int_0^{L_{t5}} \frac{1}{G_{t5}(s_{t5})\, t_{t5}(s_{t5})}\, ds_{t5} + \int_0^{L_{w6}} \frac{1}{G_{w6}(s_{w6})\, t_{w6}(s_{w6})}\, ds_{w6} + \int_0^{L_{b5}} \frac{1}{G_{b5}(s_{b5})\, t_{b5}(s_{b5})}\, ds_{b5} + \int_0^{L_{w5}} \frac{1}{G_{w5}(s_{w5})\, t_{w5}(s_{w5})}\, ds_{w5}\right)\right],$$

$$\left[A1, A2, A3, A4, A5\right]\Bigg]$$

> b

(3)

$$\begin{bmatrix} \dfrac{q0\left(\displaystyle\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\right)}{A1} \\ \dfrac{q0\left(\displaystyle\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\right)}{A1} \\ \dfrac{q0\left(\displaystyle\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\right)}{A1} \\ \dfrac{q0\left(\displaystyle\int_0^{L_{w1}} \frac{1}{G_{w1}(s_{w1})\, t_{w1}(s_{w1})}\, \mathrm{d}s_{w1}\right)}{A1} - \dfrac{q6\left(\displaystyle\int_0^{L_{w6}} \frac{1}{G_{w6}(s_{w6})\, t_{w6}(s_{w6})}\, \mathrm{d}s_{w6}\right)}{A5} \\ \dfrac{1}{2}\, Mz \end{bmatrix} \qquad \textbf{(3)}$$