Reading Notes for
Handbook of Computational Social Choice

Stupid Icey

Preface

A fucking stupid reading note of *Handbook of Computational Social Choice* (Moulin, 2016) for fucking stupid me, and a lot of thanks to Eric Pacuit, a great teacher whose slides and handouts about Social Choice even makes me feel more stupid!

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Chapter 1

Arrow's Theorem

1.1 Basic Definition

I combined the book with Eric's handout about Arrow's Theorem, which can be found on his website (Pacuit, n.d.).

We fix the following conventions:

- $\mathcal{R}(A)$ the set of all *weak orders* $\succsim$ on A, i.e. the set of all binary relations on A that are complete and transitive;
- $\mathcal{L}(A)$ the set of all *linear orders* $\succsim$ on A, which in addition are antisymmetric;
- $\succ$ the strict part of $\succsim$.

From finite sets of *individuals* (or *voters*, or *agents*) and alternatives (or *candidates*), respectively. Then we can get a *profile*, from which we can construct a function assigning the profile to a *social preference order*.

Definition 1.1 ► Profile

Let $N = \{1, \ldots, n\}$ be a finite set of individuals, and A be a finite set of alternatives. A *profile* $\mathbf{P} = \mathcal{L}(A)^n$, or to say, a function assigning to each $i \in N$ a linear order on A.

For $a, b \in A$, let:

- $\mathbf{P}(a, b) = \{i \in N \mid a \succ_i b\}$;
- $\mathbf{P}_{\restriction\{a,b\}}$ = the function assigning to each $i \in N$ the relation $\succ_i \cap \{a, b\}^2$.

Definition 1.2 ► Social Welfare Function(SWF)

A *social welfare function* (SWF) is a function $f : \mathbf{P} \to \mathcal{R}(A)$.

We call the out come $\mathcal{R}(A)$ as social preference order, and we write $\succsim$ for $f(\succsim_1, \ldots, \succsim_n)$. Noted that here we allow ties in social preference order, but not in the individual preferences.

Then we introduce some properties about SWF, where the first two is considered reasonable by lots of people, while the last is not.

- *weakly Paretian:* For all $a, b \in A$, if $\mathbf{P}(a, b) = N$, i.e. $a \succ_i b$ for all $i \in N$, then $a \succ b$;
- *independent of irrelevant alternatives (IIA):* For all $\mathbf{P}, \mathbf{P}' \in \mathrm{Dom}(f)$ and $a, b \in A$, if $\mathbf{P}_{\restriction\{a,b\}} = \mathbf{P}'_{\restriction\{a,b\}}$, then $a \succ b$ iff $a \succ' b$;
- *dictatorship:* There is an $i^\star \in N$ s.t. for all $a, b \in A$, if $a \succ_{i^\star} b$, then $a \succ b$.

We hope that our SWF is both weakly Paretian and IIA, but not be a dictatorship. There shouldn't be a dictator in our voter-group. However, Arrow's Theorem just tell us that is impossible.

Theorem 1.1 ▸ Arrow's Theorem

When there are three or more alternatives, then every SWF that is weakly Paretian and IIA must be a dictatorship.

1.2 Proof of Arrow's Theorem

Here we first introduce the concept of *decisive coalition*.

Definition 1.3

A coalition $C \subseteq N$ of individuals is called a *decisive coalition* for alternative a versus alternative b, if for all $\mathbf{P} \in \mathrm{Dom}(f)$, $C \subseteq \mathbf{P}(a, b)$ implies $a \succ b$.

We call C *weakly decisive* for a vs. b, if at least $C = P(a, b)$ implies $a \succ b$.

Notice that an SWF is weakly Paretian is the same as to say that the grand coalition N is decisive, and f is dictatorial is the same as to say that there exists a singleton that is decisive.

Sketch of proof: Suppose that $|A| \geq 3$ and let f be any SWF that is weakly Paretian and IIA. Since f is weakly Paretian, the individual-set N is a decisive coalition. First we show that for all weakly decisive coalition for a vs. b, it's also decisive for all pairs of alternatives. Thus N is decisive for all pairs. Then we split N into two nonempty subsets again and again, until we obtain a coalition which is a singleton. We show that every time we split a decisive coalition up, one of the subsets remains decisive. Thus the final singleton we got from spliting N up is decisive, say, a dictator.

Lemma 1.1.1 ▸ Contagion (or Field Expansion)

If C is weakly decisive for a vs. b, then C is decisive for all pairs of alternatives.

Proof ▹ Let $\mathbf{P} \in \mathrm{Dom}(f)$ and C is a coalition s.t. $C \subseteq \mathbf{P}(a', b')$ for arbitrary alternatives a', b' and C is weakly decisive for a vs. b. Our goal is to show that C is decisive for a' vs. b'.

W.L.O.G. let a, b, a', b' be mutually distinct (the other cases are similar). Consider a special profile $\mathbf{P}'$ s.t. $\mathbf{P}'{\restriction_{\{a',b'\}}} = \mathbf{P}{\restriction_{\{a',b'\}}}$, $a' >_i a >_i b >_i b'$ for all $i \in C$, $a' >_j a$, $b >_j b'$ and $b >_j a$ for all $j \in N \backslash C$.

Since C is weakly decisive for a vs. b and $C = \mathbf{P}'(a, b)$, we have $a >_{\mathbf{P}'} b$. Since $\mathbf{P}'(a', a) = \mathbf{P}'(b, b') = N$, from f being weakly Paretian, $a' >_{\mathbf{P}'} a$ and $b >_{\mathbf{P}'} b'$. Since $\mathcal{R}(A)$ is transitive, we have $a' >_{\mathbf{P}'} a >_{\mathbf{P}'} b >_{\mathbf{P}'} b'$.

Since f is IIA and $\mathbf{P}'{\restriction_{\{a',b'\}}} = \mathbf{P}{\restriction_{\{a',b'\}}}$, we have $a' > b'$. Thus C is decisive for a' vs. b'. ⊠

Lemma 1.1.2 ► Splitting (or Group Contraction)

For any $C \subseteq N$ with $|C| \geq 2$ that is decisive, there is nonempty sets $C_1, C_2 \subseteq C$ with $C_1 \cup C_2 = C$ and $C_1 \cap C_2 = \emptyset$ s.t. one of C_1 and C_2 is decisive for all pairs as well.

Proof ▷ Recall that $|A| \geq 3$. Let C be a decisive coalition s.t. $|C| \geq 2$. Consider a profile $\mathbf{P}$ in which everyone ranks alternatives a, b, c in the top three positions. Furthermore, $a >_i b >_i c$ for all $i \in C_1$, $b >_j c >_j a$ for all $j \in C_2$ and $c >_k a >_k b$ for all $k \in N \backslash C$, where $C = C_1 \cup C_2$.

As C is decisive, we have $b > c$. By the completeness of $\mathcal{R}(A)$, either $a > c$ or $c \gtrsim a$.

We conseder two cases.

Case 1: $a > c$: $\mathbf{P}(a, c) = C_1$. Since f is IIA, for any profile $\mathbf{P}' \in \mathrm{Dom}(f)$ s.t. $\mathbf{P}'{\restriction_{\{a,c\}}} = \mathbf{P}{\restriction_{\{a,c\}}}$, we have $C = \mathbf{P}'(a, c)$ implies $a >_{\mathbf{P}'} c$. Thus C is weakly decisive for a vs. b. By Lemma 1.1.1, C is decisive for all pairs.

Case 2: $c \gtrsim a$: By transitivity of $\mathcal{R}(A)$, $b > a$. With $\mathbf{P}(b, a) = C_2$, analogously we can conclude that C_2 is weakly decisive for b vs. a, thus decisive for all pairs. ⊠

1.3 Another Version of Proof

In this part, I will give another version of Arrow's Theorem, which is mentioned by Eric Pacuit. The main difference between the two versions is the part after Lemma 1.1.1.

By Lemma 1.1.1, let $\mathcal{D} = \{C \mid C \text{ is decisive }\}$. Then

- $\mathcal{D} \neq \emptyset$, since $N \in \mathcal{D}$;
- Since N is finite, there is a minimal $C \in \mathcal{D}$, i.e. there is no $C' \in \mathcal{D}$ s.t. $C' \subsetneq C$.(?)

Now we prove the following:

Lemma 1.1.3

Let f be an SWF and $\mathcal{D} = \{C \mid C \text{ is decisive }\}$. If $C, C' \in \mathcal{D}$ are minimal, then $C = C'$.

Proof ▷ Let $\mathcal{D}$ be the set of all decisive coalition for SWF f with $C, C' \in \mathcal{D}$ where C, C' are minimal.

Suppose (towards a contradiction) that $C \neq C'$. We show (i) $C \cap C' \neq \emptyset$; (ii) $C \cap C'$ is decisive. Denoted $C \cap C'$ as A.

(i) Suppose (towards a contradiction) that $C \cap C' = \emptyset$. Let $\mathbf{P}$ be a profile s.t. $C \subseteq \mathbf{P}(a,b)$ and $C' \subseteq \mathbf{P}(b,a)$. Since both C and C' is decisive, $a \succ b$ and $b \succ a$, contradict.

(ii) Let $c \neq a$ and $c \neq b$. Suppose $\mathbf{P}$ is a profile where $A \subseteq \mathbf{P}(c,b)$. Consider another profile $\mathbf{P}'$ with $\mathbf{P}'{\restriction_{\{c,b\}}} = \mathbf{P}{\restriction_{\{c,b\}}}$ and the ranking of a, b, c is as follows:

- $c \succ'_i a$ and $b \succ'_i a$ for all $i \in C \backslash A$;
- $a \succ'_j c$ and $a \succ'_j b$ for all $j \in C' \backslash A$;
- $c \succ'_k a \succ'_k b$ for all $k \in A$.

Due to A, A' being decisive, we have $c \succ' a$ and $a \succ' b$, by transitivity, $c \succ' b$. Then we have $c \succ b$, since $\mathbf{P}'{\restriction_{\{c,b\}}} = \mathbf{P}{\restriction_{\{c,b\}}}$ and f is IIA. Thus A is decisive for c vs.b, say, A is decisive for all pairs.

Then $A \in \mathcal{D}$ as well, and $A \subseteq C, C'$ contradict with the assumption that C, C' are minimal. Thus $C = C'$, $\mathcal{D}$ has a unique minimal element. ⊠

Lemma 1.1.4

Let C^* be the unique minimal element of $\mathcal{D}$ and $a, b \in A$. For all $i \in C^*$ and profile $\mathbf{P} \in \mathrm{Dom}(f)$, if $a \succ_i b$, then not $b \succ a$.

Proof ▷ Suppose (towards a contradiction) that there is a $i \in C^*$ and $\mathbf{P} \in \mathrm{Dom}(f)$ s.t. $a \succ_i b$ and $b \succ a$.

Since C^* is decisive, there must be some $C' \subsetneq C^*$ s.t. $C' \neq \emptyset$, $i \notin C'$ and $b \succsim_j a$ for all $j \in C'$. W.l.o.g. let $C' = C^* \backslash \{i\}$. Now we show that C' is decisive for c vs. a, then $C' \in \mathcal{D}$, contradict with C^* is the minimal element by Lemma 1.1.3.

To show that C' is decisive for c vs. a, let $\mathbf{P}'' \in \mathrm{Dom}(f)$ be an arbitrary profile with $C' \subseteq \mathbf{P}''(c,a)$. Consider a profile $\mathbf{P}' \in \mathrm{Dom}(f)$ s.t. $\mathbf{P}'_{\restriction\{a,b\}} = \mathbf{P}_{\restriction\{a,b\}}$ and $\mathbf{P}''_{\restriction\{a,c\}} = \mathbf{P}'_{\restriction\{a,c\}}$. Furthermore, the remaining rankings of a, b, c is:

- $a \succ_i b$ and $c \succ_i b$ for i;
- $c \succ_j a$ and $c \succ_j b$ for all $j \in C'$.

Since $C^* = C' \cup \{i\}$ is decisive and $C^* \subseteq \mathbf{P}'(c,b)$, $c \succ' b$ holds. Moreover, from $\mathbf{P}'_{\restriction\{a,b\}} = \mathbf{P}_{\restriction\{a,b\}}$ and $b \succ a$, we can conclude that $b \succ' a$ by IIA. Thus $c \succ' a$.

Since $\mathbf{P}''_{\restriction\{a,c\}} = \mathbf{P}'_{\restriction\{a,c\}}$, by IIA, $c \succ'' a$. Thus C' is decisive. ⊠

Definition 1.4 ▶ Oligarchy

Suppose that f is an SWF. A set $M \subseteq N$ is an *oligarchy* for f if M is decisive and, for all $\mathbf{P} \in \mathrm{Dom}(f)$, if $a \succ_i b$ for some $i \in M$, then not $b \succ a$.

Theorem 1.2 ▶ Gibbard's Oligarchy Theorem

Assume that $|A| \geq 3$ and N is finite. Then any SWF f satisfying weakly Paretian and IIA has an oligarchy.

Theorem 1.2 is easy to be found. Since $\mathcal{D} \neq \emptyset$, there is a unique minimal element in $\mathcal{D}$, which is the oligarchy.

Now we prove Arrow's Theorem.

Lemma 1.2.1

Assume that $|A| \geq 3$ and N is finite. Let f be an SWF satisfying weakly Paretian and IIA has an oligarchy. Then, if C is an oligarchy of f, then $|C| = 1$, say, f is a dictatorship.

Proof ▷ Suppose (towards a contradiction) that $|C| > 1$. Then we can make a partition of C, i.e. $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$ where C_1, C_2 is nonempty set.
Consider a profile $\mathbf{P}$ in which $a \succ_i b \succ_i c$ for all $i \in C_1$ and $b \succ_j c \succ_j a$ for all $j \in C_2$. Then not $b \succ a$ and not $a \succ c$, which is $a \succsim b$ and $c \succsim a$. By transitivity, $c \succsim b$. However, C is decisive and $C \subseteq \mathbf{P}(b, c)$, which leads us to the conclusion that $b \succ c$, contradict.
Thus $|C| = 1$, f is a dictatorship. ⊠

Chapter 2

Introduction to the Theory of Voting

2.1 Introduction

In this chapter we mainly concentrate on *multicandidate* voting with *ranked* ballots, i.e. each voter submits a linear ordering of the alternatives, and single winners (or several winners, in the event of a tie) as outcomes. A voting rule in this setting is called a social choice function or SCF.

Here we introduce three most prominent results in multicandidate voting:

- **Majority cycles:** Collective peference may have a cycle, i.e. a majority of voters prefer some alternative a to b, a (different) majority prefers b to c, and a third majority prefers c to a;
- **Arrow's Theorem**: Every voting rule for three or more alternatives either violates IIA or is a dictatorship,in which the election outcome depends solely on the ballot of one designated voter;
- **Gibbard-Satterthwaite Theorem (GST)**: Every SCF f other than a dictatorship fails to be strategyproof, i.e. f sometimes provides an incentive for an individual voter i to manipulate the outcome, that is, to misrepresent his or her true preferences over the alternatives by casting an insincere ballot.
  In this case, the voter prefers the alternative that wins when she casts some insincere ballot to the winner that would result from a sincere one. A limitation of GST is that it presumes every election to have a unique winner, which might be problematic. Duggan-Schwartz Theorem provides a solution to it.

2.2 Social Choice Functions: Plurality, Copeland, and Borda

A multicandidate voting with ranked ballots is as follows:

- $N = \{1, 2, \ldots, n\}$ is a finite set of voters.

- $A = \{1, 2, \ldots, m\}$ is a finite set of alternatives, with $m \geq 2$.
- A linear ordering $\succsim_i$ of A stands for the ballot cast by voter i: $\succsim_i$ is transitive, complete and antisymmetric; $x \succ y$ is a shortwrite for $x \succsim y$ holds and $y \succsim x$ fails.
- $\mathcal{L}(A)$ denotes the set of all linear orderings for A.
- A profile $\mathbf{P} = (\succsim_1, \succsim_2, \ldots, \succsim_n) \in \mathcal{L}(A)^n$ is a collection of each voter's ballot.
- $\mathcal{L}(A)^{<\infty}$ stands for $\bigcup_{n \in \mathbb{N}} \mathcal{L}(A)^n$

Notice that $\mathcal{L}(A)$ is actually a *strict* ordering, no voter may express indifference to two alternatives. The ballots mentioned above are called *preference rankings* since $x \succ_i y$ means voter i's (strict) preference for alternative x over alternative y.
Alternatively we might allow *weak* preference rankings (*pre-linear* orderings) as ballots. Thus let R denote a profile of weak preference rankings and similarly to $\mathcal{L}(A)$:

- $\mathcal{R}(A)$ denotes the set of all pre-linear orderings for A.
- $\mathcal{R}(A)^n$ is the set of all profiles of weak rankings for a given A and n.

In practice, we often use tabular to present a profile. For example, consider $\mathbf{P}_1$ as following:

102	101	100	1
a	b	c	c
b	c	a	b
c	a	b	a

$\mathbf{P}_1$ is not a profile, but a *voting situation*, i.e. a function $s : \mathcal{L}(A) \to \mathbb{N}$. But mane voting rules are blind to the distinction between "profile" and "voting situation", so we use the terms interchangeably.

Next we introduce some basic voting rules.

2.2.1 Plurality

A *plurality ballot* names a single, most-preferred alternative, the *plurality voting* rule selects the alternative(s) with a plurality (greatest number) of votes as the winner(s).
Under plurality voting, we only pay attention to the top-ranked alternatives and ignore the rest of the ranking. For example, in $\mathbf{P}_1$, a is the unique plurality winner, since there's 102 voters on a, even though it's not a majority.

2.2.2 Copeland

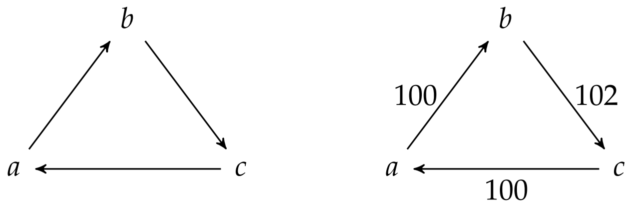
But it's unreasonable to ignore the rest part of the ranking. To make fuller use of the information in the ranking, we use *net preference*:

Definition 2.1 ▸ net preference

$Net_{\mathbf{P}}(a > b) = |\{j \in N \mid a >_j b\}| - |\{j \in N \mid b >_j a\}|$

If the net preference for a over b is strictly positive ($Net_{\mathbf{P}}(a > b) > 0$), then we say a beats b in the *pairwise majority* sense, denoted as $a >^\mu b$ or $a >^\mu_{\mathbf{P}} b$ with profile, where $>^\mu$ is the strict *pairwise majority* relation, which is always complete for an odd number of voters with strict preferences; and $\geq^\mu$ is the weak version.

Also we can use tournament to depict $>^\mu$. Below is an example, where left is the pairwise majority tournament and right is weighted version for $\mathbf{P}_1$.



The next voting rule use *Copeland scores* to select winners. A (symmetric) Copeland score of alternative x is:

Definition 2.2 ▸ (symmetric) Copeland score

$Copeland(x) = |\{y \in A \mid x >^\mu y\}| - |\{y \in A \mid y >^\mu x\}|$

The Copeland rule selects the alternative(s) with highest Copeland score. In $\mathbf{P}_1$ for example, a's Copeland scores is 0, since $a >^\mu b$ and $c >^\mu a$. So as b and c. Thus the winning set of $\mathbf{P}_1$ is $\{a, b, c\}$.

2.2.3 Borda

In the case of Copeland rule, for every alternative a, all we care about is how many times a has been defeated and won. However, as we can notice, the margins of victory or defeat matter as well. If we take these margins into consideration, things will be different.

Given a profile $\mathbf{P}$, the *symmetric Borda score* of an alternative x is[1]:

Definition 2.3 ▸ symmetric Borda score

$Borda^{sym}_{\mathbf{P}}(x) = \sum_{y \in A} Net_{\mathbf{P}}(x > y)$

Under Borda rule, alternative(s) with the highest Borda score is the winner(s). In $\mathbf{P}_1$, the scores of a, b, c are $0, 2, -2$ respectively, thus the winner is b.

A more common asymmetric Borda score is defined via the *vector of scoring weights* (score vector) $\mathbf{w}$:

[1]Noted that the definition here is nonstandard, the standard version will come up later.

Definition 2.4 ▶ asymmetric Borda score

For $|A| = m$. Let $\mathbf{w} = m-1, m-2, m-3, \ldots, 0$. $Borda^{asym}_{\mathbf{P}}(x)$ is the sum of points awarded to x by all voters, where for voter i, if a is top-ranked then a get $(m-1)$ points, if the next then $(m-2)$……, and 0 to the least preferred.

Noted that the two versions of Borda score are affinely equivalent, with $Borda^{asym}_{\mathbf{P}}(x) = n + \frac{1}{2}Borda^{sym}_{\mathbf{P}}(x)$, so they induce the same SCF[2]. Actually, if we make $\mathbf{w} = \{m-1, m-3, m-5, \ldots, -(m-1)\}$, then we replicate the scores from Definition 2.3. An adventage of symmetric approach is that it's well-defined for profiles which is weak preferences, thus the symmetric Copeland and Borda rules can be extended to weak preference cases.

2.2.4 Social Choice Function

Let $C(X)$ denote the set of all nonempty subsets of a set X.

Definition 2.5 ▶ social choice function

- A *social choice function* (SCF) is a map $f : \mathcal{L}(A)^n \to C(A)$ that returns a nonempty set of alternatives for each profile of strict preference.
- If $f(\mathbf{P}) = 1$ then f is *single valued* on $\mathbf{P}$ and sometimes we write $f(\mathbf{P}) = x$ instead of $f(\mathbf{P}) = \{x\}$.
- A *resolute* SCF is one with no tied: it's single valued on all profiles.

Above are fixed electorate SCFs. We can substitude $\mathcal{L}(A)^\infty$ for $\mathcal{L}(A)^n$, from which we get variable electorate SCFs. Notice that the rest of this chapter presumes a fixed electorate, except where explicitly noted otherwise.

In Definition 2.5, the value of the function is a group of winners, while in preceding three rules, we not only concentrate on the winner(s), but also the ”social ranking”—one alternative is ranked over another if it has a higher score. Here we don't restrict the ranking to be strict, which leads to the definition below:

Definition 2.6 ▶ social welfare function

A *social welfare function* (SWF), is a map $f : \mathcal{L}(A)^n \to \mathcal{R}(A)$ that returns a weak ranking of the set of alternatives for each profile of strict preferences.

2.2.5 Strategic Manipulation

Now we take a look at some occasions where voters may have an incentive to cast insincere ballots under such voting rules.

Consider Ali, one of the two $a \succ b \succ c \succ d \succ e$ voters of profile $\mathbf{P}_2$.

[2] Don't understand what 'affinely' mean……

2	3	2
e	d	a
c	e	b
a	b	c
d	c	d
b	a	e

Under Copeland, Ali's least preferred alternative e wins: e's (symmetric) Copeland score is 2, b's is -2 and the other scores are each 0. However, if Ali misrepresents his sincere preferences as a reverse of his ranking, the Copeland winner shifts to d, where $d \succ_{Ali} e$, with a score of 4, the maximum possible.

Definition 2.7

- An SCF f is *single voter manipulable* if for some pair $\mathbf{P}$, $\mathbf{P}'$ of profiles on which f is single valued, and voter i with $\succsim_j' = \succsim_j$ for all $j \neq i$, $f(\mathbf{P}') \succ_i f(\mathbf{P})$;
- f is *single voter strategyproof* if it's not single voter manipulable.

$\succ_i$ stands for the sincere preference, while $\succ_i'$ is the insincere one. An SCF being single voter manipulable means that if voters give an insincere ranking then he'll get a better result.

Under Copeland rule, Ali has to reverse the ballot. And under Borda rule, he can still do such things—by just lifting d to the top position ($d \succ a \succ b \succ c \succ e$).

Plurality voting is not single voter manipulable, since it only care about the top-ranked alternatives. Any voter who wants x rather than y to be the winner won't place y in the top on his sincere ballot, so he cannot lower y's score. But if there's two voters switching there ballot, then Plurality voting can be manipulable as well.

Obviously voting rules with ties or single voter manipulable is inappropriate, thus we have to deal with such problem. That's what we do in the later section.

2.3 Axioms I: Anonymity, Neutrality, and the Pareto Property

From now on, we switch to the *axiomatic method* to identify voting rules. Axioms of SCFs can be loosely divided into three groups:

- *the First Group:* Axioms here represent the minimal demands, which has been seen as uncontroversial;
- *the Second Group (or Middling Strength):* They are satisfied by some interesting SCFs, but the cost is high, since it rules out many attractive voting rules;
- *the Third Group:* They are the strongest, including IIA and strategyproofness, in that they tend to rule out all reasonable voting rules.

In this section we mainly discuss five axioms from the first group. Let f be an SCF.

Definition 2.8

- **Anonymous:** f is anonymous if each pair of voters plays interchangeable roles: $f(\mathbf{P}) = f(\mathbf{P}^\star)$ holds if for $i, j \in N$, $\succsim_i^\star = \succsim_j$, $\succsim_j^\star = \succsim_i$, and $\succsim_k^\star = \succsim_k$ for all $k \neq i, j$.
- **Dictatorial:** f is dictatorial if for some $i \in N$, i act as dictator, i.e. for all profile $\mathbf{P}$ and $a \in A$, if $a \succ_i b$ holds for all $b \in A$, then $f(\mathbf{P}) = a$.
- **Neutral:** f is neutral if each pair of alternatives are interchangeable in the following sence: whenever a profile $\mathbf{P}^\dagger$ is obtained from another $\mathbf{P}$ by swapping the positions of the two alternatives x and y in every ballot, the outcome $f(\mathbf{P}^\dagger)$ is obtained from $f(\mathbf{P})$ via a similar swap.
- **Imposed:** f is imposed if for no profile $\mathbf{P}$ does $f(\mathbf{P}) = \{x\}$.
- Given a profile $\mathbf{P}$ and $x, y \in A$, we say that x *Pareto dominates* y if every voter ranks x over y; and y is called being Pareto dominated if such an x exists.
- **Pareto Principle:** f is Pareto (Pareto optimal, or Paretian) if $f(\mathbf{P})$ never contains a Pareto dominated alternative.

Noted that *anonymity* and *neutrality* are strong forms of equal treatment of voters, *nondictatoriality* serves as a particularly weak version of anonymous, and *nonimposition* serves as a particularly weak version of neutrality. We also have Pareto implies nonimposition.

The voting rules mentioned above (Plurality, Copeland and Borda) are anonymous, neutral and Pareto, while reverse Borda[3] is not Pareto. Although being uncontroversial, these three axioms do leads to unintended consequences, which is mentioned in Moulin et al., 2015.

Proposition 2.1

Let $m \geq 2$ be the number of alternatives and n be the number of voters. If n is divisible by any integer r with $1 < r \leq m$, then no neutral, anonymous and Pareto SCF is resolute (single-valued).

Proof ▷ For $m \geq 3$ (with $A = \{a, b, c, x_1, \ldots, x_{m-3}\}$) and $n = 3k$. Consider a profile $\mathbf{P}$ as following.

[3]"Reverse Borda" SCF: elect the alternative(s) having the lowest Borda score.

k	k	k
a	c	b
b	a	c
c	b	a
x_1	x_1	x_1
$\vdots$	$\vdots$	$\vdots$

We show that $f(\mathbf{P}) = \{a, b, c\}$. Since f is Pareto, $f(\mathbf{P}) \subseteq \{a, b, c\}$. W.l.o.g. $a \in f(\mathbf{P})$. First we swap the position of a and b, then b and c, finally we'll get a profile $\mathbf{P'}$ which remains the same as $\mathbf{P}$. Thus we have $c \in f(\mathbf{P})$ since f is neutral. Analogously we can prove that $b \in f(\mathbf{P})$.
Using the same method we can extend the conclusion above to the case in Proposition 2.1. ⊠

Proposition 2.1 tell us that we have to deal with ties in the outcome. In Moulin, 2016 he come up with four method trying to solve the problem:

1. Use a fixed ordering of the alternatives (or a designated voter) to break all ties.
2. Use a randomized mechanism to break all ties.
3. Deal with set-valued outcomes directly.
4. Ignore or suppress the issue (assume no ties exist).

2.4 Voting Rules I: Condorcet Extensions, Scoring Rules, and Run-Offs

Introduce two axioms here:

Definition 2.9

- *monotonicity*: if x is the winner and one voter switches his ballot from y to x, then x is still a winner;
- *positive responsiveness*: if x is a winner and one voter switches her ballot from y to x, then x becomes the unique winner.

Now we concentrate on the *Majority rule*, which suits for the case when the number of alternatives is 2. Majority rule, which selects the alternative with more votes as winner, is anonymous, neutral and resolute when the number of voters is odd. In additional, the characterizition of it requires *monotonicity* or *positive responsiveness*.

Proposition 2.2 ► May's Theorem

For two alternatives and an odd number of voters, majority rule is the unique resolute, anonymous, neutral, and monotonic SCF. For two alternatives and any number of voters, it is the unique anonymous, neutral, and positively responsive SCF.

Proof ▷ Obviously majority rule satisfies these properties. For uniqueness, consider any other rules that selects the alternative with fewer votes as the winner. Let $A = \{a, b\}$ and $N = \{1, \ldots, n\}$ with n being odd. W.l.o.g. $|\{i \in N \mid a >_i b\}| = k_1$ and $|\{j \in N \mid b >_j a\}| = k_2$ where $k_1 + k_2 = n$ and $k_1 < k_2$. According to the rule, a is the winner. Then we switch enough ballots to form a new profile where $|\{i \in N \mid a >_i b\}| = k_2$ and $|\{j \in N \mid b >_j a\}| = k_1$. b will be the winner by neutrality, while monotonicity implies a is still the winner, from which we get a contradiction. Analogously we can prove the other case. ⊠

Proposition 2.2 tells us that majority rule is the best voting rule when $|A| = 2$. Since all other SCFs considered so far can be reduced to majority rule in the case of two alternatives, we can also say that these SCFs can be seen as "majority rule for 3 of more alternatives". But for SCFs with a full domain ($\mathrm{Dom}(f) = \mathcal{L}(A)^{<\infty}$), there is no complitely satisfactory extension of May's Theorem to the case of $|A| \geq 3$.

There's another rule which is considered deserving:

Definition 2.10 ▸ Condorcet winner

A *Condorcet winner* for a profile $\mathbf{P}$ is an alternative x that defeats every other alternative in the strict pairwise majority sense: $x >^{\mu}_{\mathbf{P}} y$ for all $y \neq x$[a].
Pairwise Majority Rule (PMR) declares the winning alternative to be the Condorcet winner, and is undefined when a profile has no Condorcet winner.

[a] The weak version is $x \geq^{\mu}_{\mathbf{P}} y$ for all $y \neq x$

Whenever a Condorcet winner exists, it must be unique. But with $|A| \geq 3$, it might forms *majority cycles* which rule them out, then no PMR winner exists. The majority cycle is known as *Condorcet's voting paradox*, which reminds us that $>^{\mu}$ is intranstive.

Notice that PMR is an SCF with *restricted domain*, since it has the possibility that no winner exists. Our interest here is with full SCFs that agree with PMR on its domain:

Definition 2.11

- *Condorcet domain:* $\mathcal{D}_{\mathit{Condorcet}} = \{\mathbf{P} \mid \mathbf{P} \text{ has a Condorcet winner}\}$;
- An SCF f is *Condorcet extension (consistent):* if for all $\mathbf{P} \in \mathcal{D}_{\mathit{Condorcet}}$, f selects the Condorcet winner alone.

Theorem 2.1 ▸ Campbell-Kelly Theorem

Consider SCFs with domain $\mathcal{D}_{\mathit{Condorcet}}$ for three or more alternatives. Pairwise Majority Rule is resolute, anonymous, neutral, and strategyproof; for an odd number of voters, it is the unique such rule.

If we restrict f's domain to $\mathcal{D}_{\mathit{Condorcet}}$, then Theorem 2.1 can be seen as "May's Theorem for three or more alternatives", specially for strategyproof, when $|A| = 2$ it can be shown that monotonicity is equivalent to strategyproof.

Proof ▷ Clearly, restricted to $\mathcal{D}_{Condorcet}$, PMR is resolute, anonymous and neutral. For strategyproof, conside a voter i with the sincere ballot being $y \succ_i x$ for alternatives x, y, and the Condorcet winner is x. Then however i changes his ballot, x remains to be the winner.
For uniqueness…

Note that Condorcet extension isn't necessary when choosing a voting rule. Borda is not a Condorcet extension actually, consider a profile $\mathbf{P}$ with $|N| = 5$ and $A = \{a, b, c\}$, if three voters rank $a \succ b \succ c$ and two rank $b \succ c \succ a$…. And Copeland rule is a Condorcet extension.

Condorcet extensions form the first class of voting rules. The second class is *scoring rules*:

Definition 2.12

- A *score vector* $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ consists of real number *scoring weights*.
- $\mathbf{w}$ is *proper* if $w_1 \geq w_2 \geq \cdots \geq w_m$ and $w_1 > w_m$.
- Every score vector induces a *scoring rule*.
- A *proper scoring rule* is one induced by a proper score vector, in which each voter awards w_1 points to their top-ranked alternative, w_2 points to their second-ranked, and so on. All points awarded to a given alternative are summed, and the winner is the alternative(s) with greatest sum.

The third class consists *multiround rules*, which is based on the idea that less popular alternatives in one round be dropped from all ballots in the next round (with each ballot then ranking the remaining alternatives in the same relative order that they had in the initial version of that ballot); these rounds continue until some surviving alternative achieves majority support (or until only one is left standing).

2.5 An Informational Basis for Voting Rules: Fishburn's Classification

Fishburn divided SCFs into three classes according to the information required in them:

- *C1 functions*: SCFs which need only the information from the tournament;
- *C2 functions*: SCFs which need the additional information in the weighted tournament;
- *C3 functions*: SCFs which are neither C1 nor C2, plurality for example.

2.6 Axioms II: Reinforcement and Monotonicity Properties

In this part, we mainly discuss the second group of the axioms, which are about profile changes when adding more voters or changing several voters' ballots. Hence we need variable electorate context and voting situation: for $s, t : \mathcal{L}(A) \to \mathbb{Z}^{+4}$. For simplicity, we fix the following conventions:

- $s + t$ stands for putting s and t together in one voter-set;
- ks ($k \in \mathbb{N}$) stands for replacing each individual voter of s with k "clones".

Definition 2.13 ▸ Reinforcement (aka Consistency)

An SCF f is reinforcing if $f(s) \cap f(t) \neq \emptyset \Rightarrow f(s+t) = f(s) \cap f(t)$.

Intuitively, reinforcement requires that the common winning alternatives chosen by two disjoint sets of voters (if there exists) be exactly those chosen by the union of these sets. Specially if $f(s) = f(t)$ then $f(s+t) = f(s) = f(t)$.

There's also a weak form of reinforcement:

Definition 2.14 ▸ Homogeneity

$f(ks) = f(s)$ for all $k \in \mathbb{N}$.

Noted that all scoring rules are reinforcing, since if some alternatives get the highest score both in s and t, then they must have the highest score in $s + t$. Analogously we can apply the same argument to *compound scoring rules*.

Definition 2.15 ▸ Compound Scoring Rules

A voting rule is a *compound scoring rule* if any ties resulting from a first score vector $\mathbf{w}_1$ may be broken by score differences arising from a second such vector $\mathbf{w_2}$, with a possible third vector used to break ties that still remain, and so on; any finite number $j \geq 1$ of score vectors may be used.

A *simple scoring rule* is the scoring rule which is not compound.

Notice that on a domain that is restricted by fixing an upper bound on the number of voters, every such compound rule is equivalent to some simple scoring rule[5].

Theorem 2.2 ▸ See Smith (1973) and Young (1975)

The anonymous, neutral, and reinforcing SCFs are exactly the compound scoring rules.

Proof ▹ 886 ⊠

[4] s, t can be regarded as two group of voters with different ballots.

[5] Why??????Anyone can tell me why…?????????

Proposition 2.3

All Condorcet extension SCFs for three or more alternatives violate reinforcement.

Proof ▷ Consider voting situation s and t with 3 alternatives below:

2	2	2
a	c	b
b	a	c
c	b	a

2	1
b	a
a	b
c	c

Let f be any Condorcet extension. Since b is the Condorcet winner in t, we have $f(t) = \{b\}$. Also a is the Condorcet winner in $(s + t)$, thus $f(s + t) = \{a\}$. If f is reinforcing then $a \in f(t)$ should hold, which leads to a contradiction.

If we assume f to be Pareto, then it's easy to extend our construction to a general form, just by adding other alternatives x_i behind a, b, c in the voting situation. If not, we have to find another more complicated voting situation as a counterexample. ⊠

Intuitively, for a winner a, if we move a from under some alternatives to over them in some voters' preference, without changing the relative order of other pair of alternatives excluded a, then a still should be the winner (*simple lift*). That's what *monotonicity* said.

Definition 2.16

A resolute SCF f satisfies *monotonicity* (aka weak monotonicity) if whenever a profile $\mathbf{P}$ is modified to $\mathbf{P}'$ by having one voter i switch $\succsim_i$ to $\succsim_i'$ by lifting the winning alternative $x = f(\mathbf{P})$ simply, $f(\mathbf{P}') = f(\mathbf{P})$.

Notice that here we require that $f(\mathbf{P})$ is single-valued, if not, then the definition only cares about the cases when $|f(\mathbf{P})| = 1$.

Suppose f is a voting rule which select the alternative(s) with highest score, and lifting alternatives x never lower x's score or raise y's score for $y \neq x$. Then f is always monotonic. It follows that all proper scoring rules are monotonic. However, each of scoring run-off rule is not monotonic[6]. I omitted the proof here since it seems to do with some knowledge of computation…

Remark ▷ Every resolute SCF f which violate monotonicity implies that there's an opportunity for voter i manipulate f by simple lifting or dropping an alternative:

Let $\succ_i \mapsto \succ_i'$ by a simple lift of he winning alternative a which makes b win and a lose. If $b \succ_i a$, then the voter with sincere ballot $\succ_i$ would gain by casting the insincere ballot $\succ_i'$ and vice versa.

Thus monotonicity is a weak form of strategyproofness. Below are some relavent definitions:

[6]For scoring run-off rules, see the last part of Section 2.4.

Definition 2.17

A resolute SCF f satisfies:

- *Strategyproofness* if whenever a profile $\mathbf{P}$ is modified to $\mathbf{P}'$ by having one voter i switch $\succsim_i$ to $\succsim_i'$, $f(\mathbf{P}) \succsim_i f(\mathbf{P}')$.
- *Maskin monotonicity* (aka *strong monotonicity*) if whenever a profile $\mathbf{P}$ is modified to $\mathbf{P}'$ by having one voter i switch $\succsim_i$ to a ballot $\succsim_i'$ satisfying for all y, $f(\mathbf{P}) \succsim_i y \Rightarrow f(\mathbf{P}) \succsim_i' y$ then $f(\mathbf{P}') = f(\mathbf{P})$[a].
- *Down monotonicity* if whenever a profile $\mathbf{P}$ is modified to $\mathbf{P}'$ by having one voter i switch $\succsim_i$ to $\succsim_i'$ by dropping a losing alternative $b \neq f(\mathbf{P})$ simply, $f(\mathbf{P}') = f(\mathbf{P})$.
- *One-way monotonicity* if whenever a profile $\mathbf{P}$ is modified to $\mathbf{P}'$ by having one voter i switch $\succsim_i$ to $\succsim_i'$, $f(\mathbf{P}) \succsim_i f(\mathbf{P}')$ or $f(\mathbf{P}') \succsim_i f(\mathbf{P})$[b]
- *Half-way monotonicity* if whenever a profile $\mathbf{P}$ is modified to $\mathbf{P}'$ by having one voter i switch $\succsim_i$ to $\succsim_i^{rev}$, $f(\mathbf{P}) \succsim_i f(\mathbf{P}')$, where $\succsim^{rev}$ denotes the reverse of $\succsim$.
- *Participation* (the absence of no show paradoxes) if whenever a profile $\mathbf{P}$ is modified to $\mathbf{P}'$ by adding one voter i with ballot $\succsim_i$ to the electorate, $f(\mathbf{P}') \succsim_i f(\mathbf{P})$.

[a]See Maskin (1999). The original definition is based on not only resolute SCF: f is Maskin monotonic if $\forall a \in f(\mathbf{P})$, if $\forall i \in N,\ \forall b \in A,\ a \succsim_i b \Rightarrow a \succsim_i' b$, then $a \in f(\mathbf{P}')$. It can be understood as, if a does not fall below any alternatives that it was not below before, then a will still be the winner.

[b]See Sanver and Zwicker (2009): It asserts that whenever one identification represents a successful manipulation, the other represents a failure.

Participation is a corresponding version of strategyproofness for no-show paradox. No-show paradox, first come up by Fishburn and Brams (1983), also mentioned by Sanver and Zwicker (2009), shows that, one additional participating voter shows up to cast her vote, and the winner is then an alternative that is strictly inferior (according to the preferences of the participating voter) to the alternative who would have won had she not shown up. Thus the paradox implies an opportunity to manipulate by abstaining.

Proposition 2.4

For resolute SCFs:

1. Strategyproofness $\Rightarrow$ Maskin monotonicity $\Leftrightarrow$ Down monotonicity $\Rightarrow$ monotonicity
2. Strategyproofness $\Rightarrow$ One-way monotonicity $\Rightarrow$ Half-way monotonicity
3. Participation $\Rightarrow$ monotonicity

Proof ▷ The proof of *Item I* and *Item II* is straightforward, except for the first arrow in Item I. Consider an SCF f which violates Maskin monotonicity. Let $a, b \in A$, $\mathbf{P}$ be a profile where $f(\mathbf{P}) = \{a\}$ and for voter i, $b \succsim_i a$. Now $\mathbf{P}'$ is modified from $\mathbf{P}$ by changing $b \succsim_i a$ to $a \succsim_i b$, and $f(\mathbf{P}') \neq \{a\}$. If $f(\mathbf{P}') = \{b\}$ or any other alternatives c with $c \succsim_i a$, then a voter with sincere preference $\succsim_i$ would gain a better result by casting the insincere ballot $\succsim_i'$. Otherwise a voter with sincere preference $\succsim_i'$ can change his ballot to $\succsim$. Thus f is not strategyproofness.
For *Item III*, see Sanver and Zwicker (2009). Note that the original proof aims at proving partic-

ipation $\Rightarrow$ one-way monotonicity, but failed.
Let $\mathbf{P} \in \mathrm{Dom}(f)$ be a profile and s, t be two preference given by two voters s and t seperately. Assume that f is an SCF satisfying participation with $f(\mathbf{P} \wedge s) = a$ and $f(\mathbf{P} \wedge t) = b$, we show that $a \gtrsim_s b$ or $b \gtrsim_t a$. Consider 3 cases:

- *Case 1:* $f(\mathbf{P}) = a$ or $f(\mathbf{P}) = b$. If $f(\mathbf{P}) = a$, since $f(\mathbf{P} \wedge t) = b$, $b \gtrsim_t a$ by participation; if $f(\mathbf{P}) = b$, then similarly $a \gtrsim_s b$.
- *Case 2:* $f(\mathbf{P} \wedge s \wedge t) = a$ or $f(\mathbf{P} \wedge s \wedge t) = b$. It's analogous to the preceding case.
- *Case 3:* $f(\mathbf{P}) = x$ with $x \notin \{s, t\}$ and $f(\mathbf{P} \wedge s \wedge t) = y$ with $y \notin \{s, t\}$. By participation we have $a \gtrsim_s x$, $b \gtrsim_t x$, $y \gtrsim_s b$ and $y \gtrsim_t a$. If $\gtrsim_s = \gtrsim_t^{rev}$ then $a \gtrsim_s x$ and $x \gtrsim_s b$, thus $a \gtrsim_s b$ as desired. ⊠

Note that the definition of participation here is a typical form for fixed-electorate SCF, while as a property of variable-electorate SCFs, participation cannot follow from strategyproofness.

Theorem 2.3

Let f be any resolute Condorcet extension for four or more alternatives. Then

1. f violates participation (if f is a variable-electorate SCF) and
2. f violates half-way monotonicity (if f is a fixed-electorate SCF for sufficiently large n).

Proof ▷ For Item I, see Moulin (1988).
Before showing the counterexample, recall Definition 2.1, we claim that for any profile $\mathbf{P}$ and $a, b \in A$,

$$\text{if } m_b < Net_{\mathbf{P}}(b > a), \text{ then } f(\mathbf{P}) \neq a \tag{2.1}$$

where $m_b := \sup_{a \neq b} Net_{\mathbf{P}}(a > b)$.
Let f be an Condorcet extension and $a, b \in A$ with $m_b < Net_{\mathbf{P}}(b > a)$. Suppose (toward a contradiction) that $f(\mathbf{P}) = a$. Then $b \notin f(\mathbf{P})$, which means $a >^{\mu} b$, hence $m_b > Net_{\mathbf{P}}(a > b) > 0$. Since $Net_{\mathbf{P}}(b > a) < 0$, $m_b < Net_{\mathbf{P}}(b > a)$ cannot hold, contradict.
Consider a voting situation as follows[7]:

3	3	5	4
a	a	d	b
d	d	b	c
c	b	c	a
b	c	a	d

Note that this does not have a Condorcet winner, however, using 2.1, we can get to the conclusion that $f(\mathbf{P}) = a$. Now we add 4 more voters with ballot $c >_i a >_i b >_i d$ where i is one of the

[7] if $|A| \geq 4$, then put all other alternatives below.

4 voters.
The new voting situation is:

3	3	5	4	4
a	a	d	b	c
d	d	b	c	a
c	b	c	a	b
b	c	a	d	d

Using 2.1 again, we can conclude that $f(\mathbf{P}') \neq c$ and $f(\mathbf{P}') \neq a$, thus $f(\mathbf{P}) \succsim_i f(\mathbf{P}')$, contradict.

For Item II: See Sanver and Zwicker (2009). We use the following definition to help us complete the proof.

Definition 2.18

f is *weakly coalitional half-way monotonic* if whenever a set of voters having identical strict ranking $\succsim_i$ all simultaneously change their votes to $\succsim_i^{rev}$, and the effect is to switch the winner from a alone to b alone, it must be that $a \succsim_i b$.

First we prove that *half-way monotonic* implies *weakly coalitional half-way monotonic*, then claim that for $|A| \geq 4$, every Condorcet extension violates weakly coalitional half-way monotonic.
It's easy to show that half-way monotonic implies weakly coalitional half-way monotonic. Suppose the set of voters $|I| = j$ and their ballot is $\succsim_i$, we can fix an enumeration $\mathbf{P} = \mathbf{P}_0, \mathbf{P}_1, \ldots, \mathbf{P}_j = \mathbf{P}'$, where $\mathbf{P}$ is the original profile and $\mathbf{P}_i$ stands for the profile with i voters changing his ballot to a reversed form. Then $\mathbf{P}'$ is the final profile where all of the voters in the set have changed his ballot. By half-way monotonic, $a = f(\mathbf{P}) \succsim_i f(\mathbf{P}_1) \succsim_i \cdots \succsim_i f(\mathbf{P}') = b$, hence the proposition holds.
For the second part, we fix the following conventions:

- C denotes the profile, for $m \geq 4$ alternatives, in which each possible ranking occurs exactly once. Note that C has $m!$ voters.
- $k = k(\mathbf{P})$ is the maximum integer j such that each of the $m!$ rankings occurs at least j times in $\mathbf{P}$; $k(\mathbf{P})$ stands for the "number of copies of C contained in $\mathbf{P}$".
- $n^* = n^*(\mathbf{P}) := |N| - (m!)k(\mathbf{P})$ is the number of voters remaining once one ignores the cpoies of C.
- $\mathbf{P}$ satisfies *Condition M* if $2k(\mathbf{P}) \geq n^*(\mathbf{P}) + 2$; satisfying Condition M means that $\mathbf{P}$ contains "enough" copies of C relative to the number of voters who would remain if all copies of C were removed.
- The *Simpson score* of an alternative a for profile $\mathbf{P}$ is $t^*(a) = \text{Min}\{Net_{\mathbf{P}}(a > x) \mid x \neq a\}$.

Claim Let f be a Condorcet extension satisfying weak coalitional half-way monotonic. Let $a, b \in A$ and $\mathbf{P}$ be a profile with $|A| \geq 4$ that meets the following three conditions:

- Condition M,

- the Simpson score $t^*(b)$ is even and $t^*(b) \geq 0$, and
- $Net(b > a) > -t^*(b) + 2$.

Then f does not elect a at $\mathbf{P}$. *Proof.* Let $a, b \in A$ be as stated. Noted that $|t^*(b)| \leq n$ and copies of C hace no effect on the value of $t^*(b)$, hence we have $|t^*(b)| \leq n^*$, hence $-t^*(b) + 2 \leq n^* + 2 \leq 2k$ from Condition M.

Let $r = \frac{-t^*(b)+2}{2}$, which is a strictly positive integer. Choose a specific ballot $\succsim_i$ as $x_0 \succsim_i x_1 \succsim_i \cdots \succsim_i b \succsim_i a$. By Condition M, there is at least r voters with ballot $\succsim_i$[8]. Now let $\mathbf{P}'$ be obtained from $\mathbf{P}$ via having r voters with $\succsim_i$ all change their votes to $\succsim_i^{rev}$. Then we have the following changes:

- For all $x \in A$ with $b \neq x \neq a$, $Net_{\mathbf{P}'}(b > x) = Net_{\mathbf{P}'}(b > x) + 2r = Net_{\mathbf{P}}(b > x) + (-t^*(b)) + 2$. Since $Net_{\mathbf{P}}(b > x) \geq \mathrm{Min}\{Net_{\mathbf{P}}(b > x)\} = t^*(b)$, $Net_{\mathbf{P}'}(b > x) > 0$.
- $Net_{\mathbf{P}'}(b > a) = Net_{\mathbf{P}}(b > a) - 2r = Net_{\mathbf{P}}(b > a) + t^*(b) - 2 > 0$.

Thus b is the Condorcet winner for $\mathbf{P}'$ and a is not. If a was a winner for $\mathbf{P}$, then f no longer be weakly coalitional half-way monotonic. Therefore a is not the Condorcet winner for $\mathbf{P}$.

Now we go back to the main proof: Consider such a profile $\mathbf{P}$:

6	6	10	8
a	a	d	b
d	d	b	c
c	b	c	a
b	c	a	d

In this profile, $n^*(\mathbf{P}) = |N| = 30$. Let $\mathbf{P}'$ be obtained from $\mathbf{P}$ by adding 28 copies of C, then $n^*(\mathbf{P}') = n^*(\mathbf{P}) = 30$.

Using the *claim* above we can show that b, c, d cannot be the winner, thus $f(\mathbf{P}) = a$. Now we construct a new profile $\mathbf{P}''$ via having 4 voters in $\mathbf{P}$ change their ballot from $\succsim_i := d \succsim_i b \succsim_i a \succsim_i c$ to $\succsim_i^{rev}$. Using the *claim* again we have $a, c \notin f(\mathbf{P})$, contradict with the fact that f is weakly coalitional half-way monotonic.

Thus we can draw a conclusion that f violates half-way monotonicity, since f even violates the weak form of it. ⊠

Corollary 2.3.1

Let f be a resolute SCF for four or more alternatives and sufficiently large odd n. If f is neutral and anonymous on $\mathcal{D}_{Condorcet}$, then either f fails to be strategyproof on $\mathcal{D}_{Condorcet}$,or f violates half-way monotonicity.

Proof ▷ Recall Theorem 2.1, we know that PMR is the unique resolute, anonymous, neutral and strategyproof SCF on $\mathcal{D}_{Condorcet}$ for odd n. Thus for f satisfying the conditions above, if f is

[8] Why,,,?

a strategyproof on $\mathcal{D}_{Condorcet}$, then f must be a Condorcet extension. By Theorem 2.3, f violates half-way monotonicity. ⊠

2.7 Voting Rules II: Kemeny and Dodgson

John Kemeny defined a neutral, anonymous, and reinforcing Condorcet extension that escapes the limitation in Proposition 2.3 by using the *social preference function*, whose outcome is a set of one or more rankings. Here is his basic method.

Definition 2.19

- The *Kendall tau metric* d_K measures the distance between two linear orderings $\succ$, $\succ^\star$ by counting pairs of alternatives on which they disagree, i.e.

$$d_K(\succ,\ \succ^\star) = |\{(a,b) \in A^2 \mid a \succ b \text{ and } b \succ^\star a\}|.$$

- d on profiles is an extension of d on a ballot, where

$$d(\mathbf{P},\mathbf{P}') = \sum_{i=1}^{n} d(\succ,\succ')$$

- For each $\succ$ define the unanimous profile $U^{\succ}$ by $U_i^{\succ} = \succ$ for all i.
- For any profile $\mathbf{P}$, the *Kemeny Rule* returns the ranking(s) $\succ$ minimizing $d_K(\mathbf{P}, U^{\succ})$.

$d_K(\succ,\succ^\star)$ gives the minimum number of sequential inversions needed to convert $\succ$ to $\succ^\star$ and analogously we get what $d_K(\mathbf{P}, U^{\succ})$ means. Noted that Kemeny Rule is a Condorcet extension: If a is $\mathbf{P}$'s Condorcet winner and $\succ$ does not put a on top, then lifting a simply to the top would strictly decrease $d_K(\mathbf{P}, U^{\succ})$, thus $d_K(\mathbf{P}, U^{\succ})$ is not minimal and all rankings in the Kemeny outcome place a on top, from which we have Kemeny Rule is a Condorcet extension.

Kemeny Rule is also a scoring rule, although not in the sense of Definition 2.12. Consider a definition in the following:

Definition 2.20 ▶ ranking score rule

- A *ranking score function* $\mathbf{W} : \mathcal{L}(A) \times \mathcal{L}(A) \to \mathbb{R}$ assigns a real number scoring weight $\mathbf{W}(\succ^\star, \succ)$ to each pair of rankings.
- Any such function induces a *ranking scoring rule*, in which a voter with ranking $\succ_i$ awards $\mathbf{W}(\succ_i, \succ)$ points to each ranking $\succ \in \mathcal{L}(A)$. All points awarded to a given ranking are summed, and the winner is the ranking(s) with greatest sum.

The definition above extends the range of scoring rule to include Kemeny Rule. The scoring weight $\mathbf{W}(\succ^\star, \succ) = \frac{m(m-1)}{2} - d_K(\succ^\star, \succ)$.

It's obvious that Kemeny Rule is neutral and anonymous. For Reinforcement, since the outcome is no longer a winning set, but a set of rankings, the reinforcement requirement have actually been weakened, where $f(s+t)$ denote *sets of rankings*. Consider the requirement of reinforcement:

$$f(s) \cap f(t) \neq \emptyset \Rightarrow f(s+t) = f(s) \cap f(t)$$

but for the situation likely to Proposition 2.3, $f_{Kem}(s) = \{a \succ b \succ c,\ b \succ c \succ a,\ c \succ a \succ b\}$, $f_{Kem} = \{b \succ a \succ c\}$ and $f(s) \cap f(t)$ is empty.

Theorem 2.4 ► (Young & Levenglick, 1978)

Among social preference functions Kemeny's rule is the unique neutral and reinforcing Condorcet extension.

Another famous voting rule which is often compared to Kemeny's rule is proposed by Charles Dodgson.

Definition 2.21

For any profile $\mathbf{P}$ the Dodgson rule returns the Condorcet winner(s) for the profile(s) $\mathbf{P}' \in \mathcal{D}_{Condorcet}$ minimizing $d_K(\mathbf{P}, \mathbf{P}')$ among all $P' \in \mathcal{D}_{Condorcet}$.

Next paragraph is cited from Moulin (2016): "Both Kemeny and Dodgson may be interpreted as minimizing a distance to "consensus." They use the same metric on rankings, but different notions of consensus: unanimity for Kemeny versus membership in $\mathcal{D}_{Condorcet}$ for Dodgson. It is not difficult to see that every preference function that can be defined by minimizing distance to unanimity is a ranking scoring rule,75 hence is reinforcing in the preference function sense. We can convert a preference function into an SCF by selecting all top-ranked alternatives from winning rankings, but this may transform a reinforcing preference function into an non-reinforcing SCF—as happens for Kemeny. The conversion preserves homogeneity, however, so every distance-from-unanimity minimizer is homogeneous as a social choice function. In this light, the inhomogeneity of Dodgson argues an advantage for unanimity over $\mathcal{D}_{Condorcet}$ as a consensus notion."

2.8 Strategyproofness: Impossibilities

In this section, we go into Gibbard-Satterthwaite Theorem, which claims that Any resolute, nonimposed, and strategyproof SCF for three or more alternatives must be a dictatorship. Conversely, we can find that every resolute, nonimposed, and nondictatorial SCF for three or more alternatives is manipulable.

Theorem 2.5 ▶ Gibbard-Satterthwaite Theorem

Any resolute, nonimposed, and strategyproof SCF for three or more alternatives must be a dictatorship.

Begin with the following definition:

Definition 2.22

Let f be a resolute social choice function for $m \geq 3$ alternatives, $a, b \in A$ be two distinct alternatives and $X \subseteq N$ be a set of voters. Then we say that X can use a to block b, notated $X_{a>b}$, if for every profile $\mathbf{P}$ wherein each voter in X ranks a over b, $f(\mathbf{P}) \neq b$;
X is a *dictating set* if $X_{z>w}$ holds for every choice $z \neq w$ of distinct alternatives.

Lemma 2.5.1 ▶ Push-Down Lemma

Let $a, b, c_1, c_2, \ldots, c_{m-2}$ enumerate the $m \geq 3$ alternatives in A, f be a resolute and down monotonic SCF for A, and $\mathbf{P}$ be any profile with $f(\mathbf{P}) = a$. Then there exists a profile $\mathbf{P}^\star$ with $f(\mathbf{P}^\star) = a$ such that:

- For every voter i with $a >_i b$: $>_i^\star = a > b > c_1 > \cdots > c_{m-2}$;
- For every voter i with $b >_i a$: $>_i^\star = b > a > c_1 > \cdots > c_{m-2}$;

Proof ▷ The proof is quite intuitive, due to f being down monotonic, $\mathbf{P}^\star$ can be formed just by dropping simply $c_1, \ldots, c_{m-2}$ to the bottom of each ranking one by one. ⊠

Lemma 2.5.2

Let f be a resolute and down monotonic SCF. If there exists a profile $\mathbf{P}$ for which every voter in X has a over b, every voter in $N \backslash X$ has b over a, and $f(\mathbf{P}) = a$, then $X_{a>b}$.

Proof ▷ Suppose (towards a contradiction) that there is a $\mathbf{P}$ as stated and not $X_{a>b}$. Choose a second profile $\mathbf{P}'$ s.t. each voter in X rank a above b and $f(\mathbf{P}') = b$. If any $\mathbf{P}'$ voters in $N \backslash X$ have a over b, then let them one-at-a-time drop a simply below b to get another new profile $\mathbf{P}''$. Due to f being down monotonic, $f(\mathbf{P}'') = b$. Applying Lemma 2.5.1 to both $\mathbf{P}$ and $\mathbf{P}''$, we get $\mathbf{P}^\star$ with $f(\mathbf{P}^\star) = a$ and $\mathbf{P}''^\star$ with $f(\mathbf{P}''^\star) = b$. However, $\mathbf{P}^\star = \mathbf{P}''^\star$, contradict. ⊠

Lemma 2.5.3

Let f be a resolute, Pareto, and down monotonic SCF for three or more alternatives. Assume $X_{a>b}$, with $X = Y \cup Z$ split into disjoint subsets Y and Z. Let c be any alternative distinct from a and b. Then $Y_{a>c}$ or $Z_{c>b}$.

Proof ▷ Consider a profile below:

$$\begin{array}{ccc} Y & Z & N \backslash X \\ \hline a & c & b \\ b & a & c \\ c & b & a \\ \vdots & \vdots & \vdots \end{array}$$

Since f is Pareto, $f(\mathbf{P}) \in \{a, b, c\}$. According to the assumption that $X_{a>b}$, $f(\mathbf{P}) \neq b$. If $f(\mathbf{P}) = a$, then by Lemma 2.5.2, $Y_{a>c}$. Otherwise, $Z_{c>b}$. ⊠

Lemma 2.5.4

Let f be a resolute, Pareto, and down monotonic SCF for three or more alternatives. Assume $X_{a>b}$.Let c be any alternative distinct from a and b. Then (i) $X_{a>c}$ and (ii) $X_{c>b}$.

Proof ▷ Noted that none of conditions in Lemma 2.5.3 rules out the possibilities of Y or Z being empty. And Pareto implies that $\emptyset_{z>w}$ is impossible. Thus let $X = Y$ and $Z = \emptyset$ in last proof, then we have $X_{a>c}$. Analogously let $Y = \emptyset$, then $X_{c>b}$. ⊠

Lemma 2.5.5

Let f be a resolute, Pareto, and down monotonic SCF for three or more alternatives. Assume $X_{a>b}$. Then X is a dictating set.

Proof ▷ Let $a, b, y \in A$ and $X_{a>b}$ hold. We show that $X_{y>z}$ holds for every $z \neq y$. Consider several cases below:

Case 1: $y = a$. By Lemma 2.5.4 (i), we already have for all z distinct from a, $X_{a>z}$.

Case 2: $y \notin \{a, b\}$. By Lemma 2.5.4, $X_{y>b}$. Applying Lemma 2.5.4 to $X_{y>b}$ again, we then have for every z dsitinct from y and b, $X_{y>z}$.

Case 3: $y = b$. First $X_{a>c}$ holds for c distinct from a and y. Then $X_{y>c}$ since y is distinct from a and c. Finally we get $X_{y>z}$ from all $z \neq b$. ⊠

Lemma 2.5.6 ▸ Splitting Lemma

Let f be a resolute, Pareto, and down monotonic SCF for three or more alternatives. If a dictating set $X = Y \cup Z$ is split into disjoint subsets Y and Z, then either Y is a dictating set, or Z is.

Proof ▷ Since $X_{a>b}$, by Lemma 2.5.3, either $Y_{a>c}$ or $Z_{c>b}$. Using Lemma 2.5.5, either Y is a dictating set of Z is. ⊠

Lemma 2.5.7 ▶ Adjustment Lemma

Let f be any resolute, nonimposed SCF (but no longer assume f is Pareto). If f is down monotonic then it is Pareto.

Proof ▷ Suppose ($\lightning$) that f is not Pareto, i.e. there is a profile $\mathbf{P}$ s.t. every voter ranks b over a while $f(\mathbf{P}) = a$. Choose a second profile $\mathbf{P}'$ with $f(\mathbf{P}') = b$. Due to f being nonimposed, such a profile nust exist. Now if any voter in $\mathbf{P}'$ ranks a over b, then let him one-at-a-time drop a simply below b. By down monotonicity, the resulting profile $\mathbf{P}''$ satisfies $f(\mathbf{P}'') = b$. Applying Lemma 2.5.1 to $\mathbf{P}$ and $\mathbf{P}''$ to obtain $\mathbf{P}^\star$ and $\mathbf{P}''^\star$ with $\mathbf{P}^\star = \mathbf{P}''^\star$, while $f(\mathbf{P}^\star) = a$ and $f(\mathbf{P}''^\star) = b$ ⊠

Proof (for Gibbard-Satterthwaite Theorem) ▷ Let f be an SCF satisfying resolute, nonimposed, and strategyproof. Recall Proposition 2.4, f is down monotonic as well. Applying Lemma 2.5.7 we know f is Pareto, thus N itself is a dictating set. Using Lemma 2.5.6 repeatedly until a singleton has been left, then by Lemma 2.5.4 and Lemma 2.5.5 the singleton is a dictating set. ⊠

There is also a well known variant reformulating Theorem 2.5 in terms of monotonicity.

Theorem 2.6

Any resolute, nonimposed, and Maskin monotonic SCF for three or more alternatives must be a dictatorship.

The proof is easy, since Maskin monotonicity implies down monotonicity (Proposition 2.4).

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