

1 Probability and statistics

1.1 Working with probability distributions

- Given probability distribution \mathbb{P} , sample space Ω , and event $A \subseteq \Omega$:

- $\mathbb{P} \geq 0 \quad \forall A$ (probabilities are nonzero)
- $\mathbb{P}[\Omega] = 1$ (probabilities sum to 1)
- $\mathbb{P}[\emptyset] = 0$ (probability of empty set is 0)
- $\mathbb{P}\left[\bigcup_{i=1}^{\infty} A_i\right] = \sum_{i=1}^{\infty} \mathbb{P}[A_i] = 1$

- Probabilities are independent when the joint probability is equal to the product of the marginal probabilities.

$$A \perp\!\!\!\perp B \iff \mathbb{P}[A \cap B] = \mathbb{P}[A] \mathbb{P}[B]$$

- The conditional probability of A given B is the joint probability of A and B divided by the probability of just B .

$$\mathbb{P}[A | B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$$

- The Probability Mass Function (PMF) is used to describe the behavior of *discrete* probability distributions.

$$f_X(x) = \mathbb{P}[X = x]$$

- The Probability Density Function (PDF) is the equivalent for *continuous* distributions. We use the PDF to determine the probability that random variable X is between A and B .

$$\mathbb{P}[a \leq X \leq b] = \int_a^b f(x) dx$$

- The Cumulative Distribution Function (CDF) is the integral of the PDF and we use it to determine the probability that random variable X is less than or equal to x . It maps $\mathbb{R} \rightarrow [0, 1]$ and is monotonically non-decreasing. The left and right limits are 0 and 1 ($\lim_{x \rightarrow -\infty} = 0$ and $\lim_{x \rightarrow \infty} = 1$).

$$F_X(x) = \mathbb{P}[X \leq x]$$

1.1.1 Notes on the normal distribution

- The normal distribution is a function of mean μ and variance σ^2
- The simplest case is the **standard normal distribution**, $Z \sim \mathcal{N}(0, 1)$, which reduces to:

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

- Interestingly, others have defined even simpler standard normals. Gauss proposed $\sigma^2 = \frac{1}{2}$, which reduces to:

$$\phi(x) = \frac{e^{-x^2}}{\sqrt{\pi}}$$

- Stigler proposed a formulation with $\sigma^2 = \frac{1}{2\pi}$, leading to:

$$\phi(x) = e^{-\pi x^2}$$

- We can convert any normally distributed variable X to a *standard normal* by subtracting the mean and dividing by the standard deviation.

$$Z = \frac{X - \mu}{\sigma}$$

- 68-95-99.7 rule:** the percentage of values that lie within 1, 2, and 3 standard deviations of the mean of a normal distribution are 68.27%, 95.45%, and 99.73% respectively. A $\mu \pm 3\sigma$ deviation should occur at a frequency of about 1 in 370.

- The Gauss Error Function gives the probability of a RV $Z \sim \mathcal{N}(0, 1/2)$ falling in the range $[-x, x]$:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}$$

1.1.2 Notes on the uniform distribution

- The continuous uniform distribution is a function of the minimum and maximum values a and b with mean and median equal to $\frac{a+b}{2}$
- The **standard uniform** is a random variable $\sim \mathcal{U}(0, 1)$
- The PDF of a uniform distribution is a horizontal line from a to b

1.1.3 Notes on binomial distribution

- Discrete distribution $\mathcal{B}(n, p)$ for the number of successes in a sequence of n Bernoulli trials with probability of success p .

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1.2 Common distributions

| | Type | $F_X(x)$ | $f_X(x)$ | $\mathbb{E}[X]$ | $\mathbb{V}[X]$ | $M_X(s)$ |
|---------------------|-------------------|---|--|--|--|--|
| Uniform | <i>Discrete</i> | $\begin{cases} 0 & x < a \\ \frac{\lfloor x \rfloor - a + 1}{b - a} & a \leq x \leq b \\ 1 & x > b \end{cases}$ | $\frac{I(a \leq x \leq b)}{b - a + 1}$ | $\frac{a + b}{2}$ | $\frac{(b - a + 1)^2 - 1}{12}$ | $\frac{e^{as} - e^{-(b+1)s}}{s(b - a)}$ |
| Bernoulli | <i>Discrete</i> | $(1 - p)^{1-x}$ | $p^x (1 - p)^{1-x}$ | p | $p(1 - p)$ | $1 - p + pe^s$ |
| Binomial | <i>Discrete</i> | $I_{1-p}(n - x, x + 1)$ | $\binom{n}{x} p^x (1 - p)^{n-x}$ | np | $np(1 - p)$ | $(1 - p + pe^s)^n$ |
| Multinomial | <i>Discrete</i> | | $\frac{n!}{x_1! \dots x_k!} p_1^{x_1} \dots p_k^{x_k} \quad \sum_{i=1}^k x_i = n$ | $\begin{pmatrix} np_1 \\ \vdots \\ np_k \end{pmatrix}$ | $\begin{pmatrix} np_1(1 - p_1) & -np_1p_2 \\ -np_2p_1 & \ddots \end{pmatrix}$ | $\left(\sum_{i=0}^k p_i e^{s_i} \right)^n$ |
| Poisson | <i>Discrete</i> | $e^{-\lambda} \sum_{i=0}^x \frac{\lambda^i}{i!}$ | $\frac{\lambda^x e^{-\lambda}}{x!}$ | λ | λ | $e^{\lambda(e^s - 1)}$ |
| Uniform | <i>Continuous</i> | $\begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a < x < b \\ 1 & x > b \end{cases}$ | $\frac{I(a < x < b)}{b - a}$ | $\frac{a + b}{2}$ | $\frac{(b - a)^2}{12}$ | $\frac{e^{sb} - e^{sa}}{s(b - a)}$ |
| Normal | <i>Continuous</i> | $\Phi(x) = \int_{-\infty}^x \phi(t) dt$ | $\phi(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \left\{ -\frac{(x - \mu)^2}{2\sigma^2} \right\}$ | μ | σ^2 | $\exp \left\{ \mu s + \frac{\sigma^2 s^2}{2} \right\}$ |
| Log-Normal | <i>Continuous</i> | $\frac{1}{2} + \frac{1}{2} \operatorname{erf} \left[\frac{\ln x - \mu}{\sqrt{2\sigma^2}} \right]$ | $\frac{1}{x\sqrt{2\pi\sigma^2}} \exp \left\{ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right\}$ | $e^{\mu + \sigma^2/2}$ | $(e^{\sigma^2} - 1)e^{2\mu + \sigma^2}$ | |
| Multivariate Normal | <i>Continuous</i> | | $(2\pi)^{-k/2} \Sigma ^{-1/2} e^{-\frac{1}{2}(x - \mu)^T \Sigma^{-1}(x - \mu)}$ | μ | Σ | $\exp \left\{ \mu^T s + \frac{1}{2} s^T \Sigma s \right\}$ |
| Student's t | <i>Continuous</i> | $I_x \left(\frac{\nu}{2}, \frac{\nu}{2} \right)$ | $\frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})} \left(1 + \frac{x^2}{\nu} \right)^{-(\nu+1)/2}$ | $0 \quad \nu > 1$ | $\begin{cases} \frac{\nu}{\nu - 2} & \nu > 2 \\ \infty & 1 < \nu \leq 2 \end{cases}$ | |
| Chi-square | <i>Continuous</i> | $\frac{1}{\Gamma(k/2)} \gamma \left(\frac{k}{2}, \frac{x}{2} \right)$ | $\frac{1}{2^{k/2}\Gamma(k/2)} x^{k/2-1} e^{-x/2}$ | k | $2k$ | $(1 - 2s)^{-k/2} \quad s < 1/2$ |
| Exponential | <i>Continuous</i> | $1 - e^{-x/\beta}$ | $\frac{1}{\beta} e^{-x/\beta}$ | β | β^2 | $\frac{1}{1 - \frac{s}{\beta}} \quad (s < \beta)$ |

1.3 Hypothesis testing

- Framework for filtering implausible scientific claims
- Basic steps:
 1. State relevant null hypothesis (H_0) and alternative hypothesis (H_1)
 - Two-sided: $H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$
 - One-sided: $H_0 : \theta \leq \theta_0$ vs $H_1 : \theta > \theta_0$
 2. Determine relevant test statistic (T) distribution, typically Student's t or normal distribution
 3. Select significance level (α , often 5% or 1%)
 4. Calculate rejection region (critical region), which contains all values of x for which $T(x)$ is greater than the critical value c : $R = \{x : T(x) > c\}$
 5. Determine whether to accept or reject H_0
- Alternatively, just calculate the p -value (probability given H_0 of getting a result at least as extreme as that which was observed). Reject the null hypothesis if $p \leq \alpha$.
- Common ranges for p -values are:
 - < 0.01 : very strong evidence against H_0
 - $[0.01, 0.05]$: strong evidence against H_0
 - $[0.05, 0.10]$: weak evidence against H_0
 - > 0.1 : yikes man
- Type I errors (false positives) occur when we incorrectly **reject** the null hypothesis. This is equivalent to α .
- Type II errors (false negatives) occur when we incorrectly **fail to reject** the null hypothesis.

| | Retain H_0 | Reject H_0 |
|------------|---------------------------|---------------------------|
| H_0 true | ✓ | Type I Error (α) |
| H_1 true | Type II Error (β) | ✓ (power) |

1.4 Bayesian inference

2 Linear algebra

2.1 Objects and notation

- Let scalar $s \in \mathbb{R}$

- Let vector $\mathbf{x} \in \mathbb{R}^n$. We should assume that all vectors are ‘column vectors’ (ie a matrix in $\mathbb{R}^{n \times 1}$)
- Let 2-d matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$. We’ll identify specific elements like this:

$$\begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

– We’ll denote a whole column i of a matrix as $\mathbf{A}_{:,i}$ and a row j as $\mathbf{A}_{j,:}$

- Tensors extend beyond 2d, eg: $\mathbf{A}_{i,j,k}$

2.2 Basic matrix operations review

- The **transpose** operation mirrors the matrix across the diagonal and is denoted \mathbf{A}^T .

$$\mathbf{A} = \begin{bmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \\ A_{3,1} & A_{3,2} \end{bmatrix} \Rightarrow \mathbf{A}^T = \begin{bmatrix} A_{1,1} & A_{2,1} & A_{3,1} \\ A_{1,2} & A_{2,2} & A_{3,2} \end{bmatrix}$$

- Addition of matrices is element-wise, and therefore requires them to be the same shape.

$$C_{i,j} = A_{i,j} + B_{i,j} \quad \{A, B, C\} \in \mathbb{R}^{m \times n}$$

- The **matrix product** of $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{B} \in \mathbb{R}^{n \times p}$ is $\mathbf{C} \in \mathbb{R}^{m \times p}$. Note that the number of columns in the first matrix must be equal to the number of rows in the second matrix (n). Each element in $C_{i,j}$ can be thought of as the dot product between row i of \mathbf{A} and column j of \mathbf{B} .

$$C_{i,j} = \sum_k A_{i,k} B_{k,j}$$

- Some matrix operation properties:
 - Distributive: $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
 - Associative: $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
 - **NOT** commutative: $\mathbf{AB} \neq \mathbf{BA}$
 - Transpose product: $(\mathbf{AB})^T = \mathbf{B}^T \mathbf{A}^T$

2.2.1 The identity matrix

- We’ll define the **identity matrix** \mathbf{I}_n as the matrix that does not change a vector \mathbf{x} of dimension n when they are multiplied together so that $\forall \mathbf{x} \in \mathbb{R}^n, \mathbf{I}_n \mathbf{x} = \mathbf{x}$. The identity matrix is just a square matrix with 1 on the diagonal and 0 elsewhere, so for $\mathbf{x} \in \mathbb{R}^3$:

$$\mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

2.2.2 Matrix inversion

- The **matrix inverse** of \mathbf{A} is denoted \mathbf{A}^{-1} and we define it such that:

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}_n$$

- \mathbf{A} is **invertible** if it is square ($\in \mathbb{R}^{n \times n}$) and non-singular.
 - A square matrix is **singular** \iff it has a determinant of 0
 - Singular matrices have linearly dependent columns
 - * The **determinant** of a matrix (usually denoted $\det(\mathbf{A})$ or $|\mathbf{A}|$) is a scalar factor that can be computed from the elements of a square matrix. For a 2×2 matrix:

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |\mathbf{A}| = ad - bc$$

- For other important properties of invertible matrices see [Wikipedia: Invertible matrix theorem](#)

2.3 Systems of linear equations

- We can define a **system of linear equations**, $\mathbf{Ax} = \mathbf{b}$. \mathbf{A} is a known matrix of coefficients, \mathbf{b} is a known vector, and we're trying to solve for vector \mathbf{x} . The matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ describes a system of m equations with n unknowns.
- This is really the same as writing:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

3 Differential equations

3.1 Calculus refresher

- Some useful properties with differentiable functions $f(x)$ and $g(x)$:
 - $(cf)' = c(f')$ for any constant c
 - $c' = 0$ for any constant c
 - $(f + g)' = f' + g'$
- Important rules for derivatives
 - Power rule:** $(x^n)' = nx^{n-1}$

- Product rule:** $(fg)' = f'g + g'f$
- Quotient rule:** $\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$
- Chain rule:** $f(g(x))' = f'(g)g'$

- Common derivatives:

- $\frac{d}{dx} x = 1$
- $\frac{d}{dx} cx = c$
- $\frac{d}{dx} e^x = e^x$
- $\frac{d}{dx} \ln x = \frac{1}{x}, \quad x > 0$
- $\frac{d}{dx} \ln |x| = \frac{1}{x}, \quad x \neq 0$
- $\frac{d}{dx} c^x = c^x \ln c$
- $\frac{d}{dx} \sin x = \cos x$
- $\frac{d}{dx} \cos x = -\sin x$
- $\frac{d}{dx} \tan x = \sec^2 x$
- $\frac{d}{dx} \log_c x = \frac{1}{x \ln c}, \quad x > 0$

- Common antiderivatives:

- $\int 0 \, dx = C$
- $\int 1 \, dx = x + C$
- $\int n \, dx = nx + C$
- $\int e^x \, dx = e^x + C$
- $\int \frac{1}{x} \, dx = \ln x + C$
- $\int x^n \, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$
- $\int \sin x \, dx = -\cos x + C$
- $\int \cos x \, dx = \sin x + C$

- Fundamental theorem of calculus:

$$\int_a^b \frac{dy}{dx} dx = y(b) - y(a) \iff \frac{d}{dx} \int_a^x f(s) ds = f(x)$$

- Three ways to use the fact that $\frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$
 - knowing Δx and dy/dx , we know $\Delta y \approx \Delta x \frac{dy}{dx}$ (linear approximation)
 - knowing Δy and dy/dx , we know $\Delta x \approx \frac{\Delta y}{dy/dx}$ (Newton's method)
 - approximate the derivative if we know Δy and Δx because $dy/dx \approx \frac{\Delta y}{\Delta x}$
 - note: better to take a centered difference (half step each way)*

$$\frac{dy}{dx} \approx \frac{y(x + \frac{1}{2}\Delta x) - y(x - \frac{1}{2}\Delta x)}{\Delta x}$$

- Taylor series: allows us to predict $y(x)$ from derivatives at $x = x_0$

$$y(x_0 + \Delta x) = y_0 + (\Delta x)y'_0 + \cdots + \frac{1}{n!}(\Delta x)^n y_0^{(n)}$$

$$= \sum_{n=0}^{\infty} \frac{(\Delta x)^n}{n!} y^{(n)}(x_0)$$

- e^t follows the addition rule for exponents (ie: $(e^t)(e^T) = e^{t+T}$)

3.2 1st order differential equations

A first order ordinary differential equation connects a function $y(t)$ with its derivative dy/dt . We can additionally classify first order ODEs as linear or nonlinear based on their linearity with respect to y . For example:

- linear examples: $\frac{dy}{dt} = y$, $\frac{dy}{dt} = -y$, $\frac{dy}{dt} = 2ty$
- nonlinear example: $\frac{dy}{dt} = y^2$

3.2.1 Solutions to $dy/dt = ay$

- since $\frac{d}{dt}e^t = e^t$, solutions to differential equations of the form $\frac{dy}{dt} = ay$ take the form:

$$y(t) = Ce^{at}$$

- the free constant C is the starting value, $y(0)$, so:

$$y(t) = y(0)e^{at}$$

- this solution grows exponentially when $a > 0$ and decays when $a < 0$

3.2.2 1st order DEs with source term where $dy/dt = ay + q(t)$

- This form allows a source/sink term $q(t)$. The separation of ay and $q(t)$ is critical for linear equations
- The homogeneous (null) solution when is $y_n(t) = Ce^{at}$ when $q = 0$
- The particular solution is $y_p(t)$ when $q \neq 0$
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3.3 Separable DEs

- We can solve some differential equations of two variables by separating the variables on either side of the equation using algebra and then integrating each side. Here's a full example with $\frac{dy}{dt} = ay$ (where a is some constant) showing how we arrive at Ce^{at}

1. Rearrange terms, leaving y and dy on the left side with dt on the right side

$$\frac{dy}{dt} = ay \quad \rightarrow \quad \frac{1}{y} dy = a dt$$

2. Integrate both sides of the equation to remove dy and dt terms.

$$\int \frac{1}{y} dy = \int a dt \quad \rightarrow \quad \ln|y| + C_1 = at + C_2$$

3. Solve for y . Combining the C_1 and C_2 terms just gives us another constant. Since e^C is also still just some constant, we can rename this term C to arrive at our solution.

$$|y| = e^{at+C} = e^{at}e^C = Ce^{at}$$

3.4 2nd order differential equations