2021 Fall MAS 101
Chapter 9: First-Order Differential Equations

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General First-Order Differential Equations and Solutions

- A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y),$$

which is equivalent to

$$y' = f(x, y) \quad \text{and} \quad \frac{d}{dx}y = f(x, y).$$

- A **solution** of the equation is a differentiable function $y = y(x)$ defined on an interval I of x-values such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. The **general solution** is a solution that contains all possible solutions.

General First-Order Differential Eq. and Solutions (cont'd)

- Show that every member of the family of functions

$$y = \frac{C}{x} + 2$$

is a solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

on the interval $(0, \infty)$.

General First-Order Differential Eq. and Solutions (cont'd)

- A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

- Show that the function

$$y = (x+1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

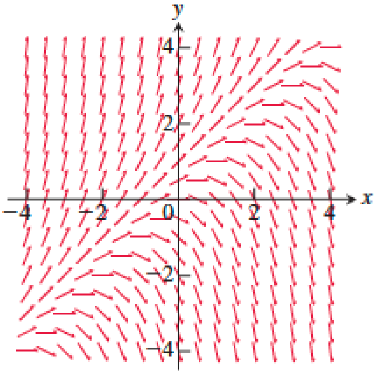
$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

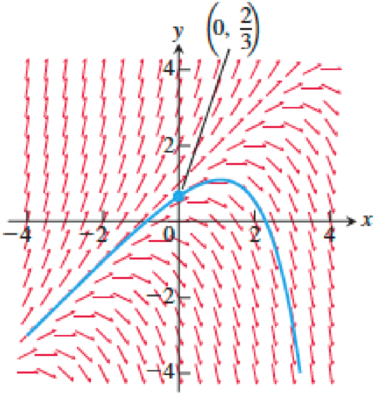
Slope Fields: Viewing Solution Curves

- Each time we specify an initial condition $y(x_0) = y_0$, the **solution curve** is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$.

Slope Fields: Viewing Solution Curves (cont'd)

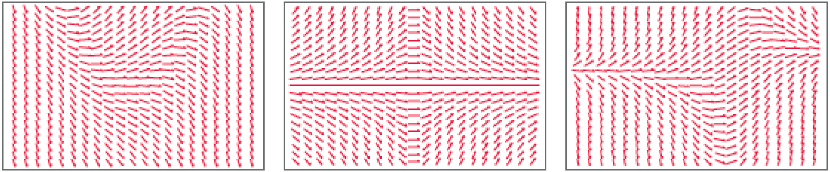
- Slope field (or direction field)



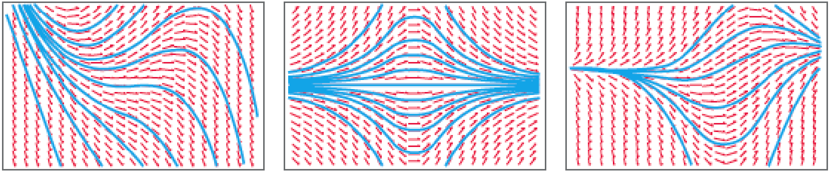


Slope Fields: Viewing Solution Curves (cont'd)

- Slope fields are useful because they display the overall behavior of the family of solution curves.



(a) $y' = y - x^2$ (b) $y' = -\dfrac{2xy}{1+x^2}$ (c) $y' = (1-x)y + \dfrac{x}{2}$



Euler's Method

- Initital value problem:

$$y' = f(x, y), \qquad y(x_0) = y_0$$

- What if we are not required or cannot immediately find an *exact* solution?

Euler's Method (cont'd)

- Initial value problem:

$$y' = f(x, y), \qquad y(x_0) = y_0$$

- Approximate the solution $y = y(x)$ by its linearization

$$L(x) =$$

Euler's Method (cont'd)

- Use Euler's method with $dx = 0.1$ to solve

$$y' = 1 + y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$, starting at $x_0 = 0$.

Euler's Method (cont'd)

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x	y (Euler)	y (exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

Solving Linear Equations

- A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

  where P and Q are continuous functions of x. The above equation is the linear equation's **standard form**.

- $\frac{dy}{dx} = ky$
- $x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0$

Solving Linear Equations (cont'd)

- We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x)$$

by multiplying both sides by a *positive* function $v(x)$, named **integrating factor**, that transforms the left-hand side into the derivative of the product $v(x) \cdot y$.

Solving Linear Equations (cont'd)

- Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solving Linear Equations (cont'd)

- Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.