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TREEPAT

① Show that $\sum_{n\geq 1}^{\infty} \frac{\sin(nx)}{n^2}$ converges for all x

differentiate by term the sum, is that sum converges?

(a) By simple comparison with $\sum_{n=1}^{\infty} \frac{1}{n^2}$ which converges,

$\frac{\sin(nx)}{n^2} \leq \frac{1}{n^2}$, therefore, $\sum_{n=1}^{\infty} \frac{\sin(nx)}{n^2}$ $\boxed{\text{converges}}$ #

(b) $A = \sin(x) + \frac{\sin(2x)}{2^2} + \frac{\sin(3x)}{3^2} + \dots + \frac{\sin(nx)}{n^2}$

$\frac{dA}{dx} = \cos x + \frac{\cos(2x)}{2\cdot 2^2} + \frac{\cos(3x)}{3\cdot 3^2} + \dots + \frac{\cos(nx)}{n^3} = \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^3}$

By comparision with $\sum_{n=1}^{\infty} \frac{1}{n^3}$ which converges, $\frac{\cos(nx)}{n^3} \leq \frac{1}{n^3}$, therefore,

the ~~Suase~~ sum of the differentiation Σ. $\boxed{\text{converges}}$. #

② Find first 3 non zero term of Maclaurin series for $\frac{\ln(1+x)}{1-x}$, $|x|<1$

Given $f(x) = \frac{\ln(1+x)}{1-x}$ ~~$f(0) =$~~ ~~$f(0) = 0$~~ $f(0) = 0$

$f'(x) = \frac{\frac{1}{1+x}}{\frac{1}{(1-x)^2}} = \frac{(1-x)^2}{1+x} \longrightarrow f'(0) = 1$

$f''(x) = \frac{(1+x)(-2)(1-x) - (1-x)^2}{(1+x)^2} = \frac{-2+2x^2 - (1-x)^2}{(1+x)^2} \rightarrow f''(0) = -1$

$f'''(x) = \frac{(1+x)^2[+4x+2(1-x)] - (-2+2x^2-x^2+2x-1)(1+x)\cdot 2}{(1+x)^4} \rightarrow f'''(0) = 8$

$f(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \dots + \frac{f^n(0)x^n}{n!}$

$\boxed{\frac{\ln(1+x)}{1-x} = x - \frac{x^2}{2} + \frac{8x^3}{6}}$