Final Exam Calculus I (MAS101), Fall 2019

Date: 2019.12.17

Time: 19:00–21:45

Name:

Student ID:

- Write your name and student ID on top of each page.
- Show your work for full credit and mark your final answer for each problem. You are encouraged to write in English but some Korean is acceptable. You can use the back of each page as scratch spaces. However **if you want parts of the back pages to be graded, you should mark this in each individual case**!
- The exam is for 2:45 hours in length. Ask permission by raising your hand if you have any question or need to go to the toilet.
- Any attempt to cheat or a failure to follow this instruction lead to serious disciplinary actions not limited to failing the exam or the course.

Problem	Score	Problem	Score
1a	/10	1b	/10
2a	/10	2b	/10
2c	/10	3	/14
4a	/8	4b	/15
5a	/5	5b	/15
5c	/6	6	/10
7a	/12	7b	/10
8a	/10	8b	/10
9	/15	10a	/10
10b	/10	**Total**	**/200**

1. Determine the set of all $x \in \mathbb{R}$ for which the following series converge:

   a) $\sum_{k=1}^{\infty} (-1)^k \dfrac{\sqrt{k+1} - \sqrt{k}}{\sqrt{k+1} + \sqrt{k}} x^k$.

   Ans.:

   b) $\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} (2x^2 - 3)^k$.

   Ans.:

2. a) Find the Taylor series for $f(x) = \frac{1}{\sqrt{x^2+1}}$ around $x = 0$. Write the terms up to at least x^5 **explicitly**.

Ans.:

b) Use a) to find the Taylor series for $g(x) = \sinh^{-1}(x)$ around $x = 0$.

Ans.:

c) Estimate the value of the integral $\int_0^1 \sinh^{-1}(x)dx$ with an error of at most $1/100$ **using Taylor series**.

(Arguments not using Taylor series will not be counted! Prove that your estimate is sufficiently close!)

Ans.:

3. Find all $a, b \in \mathbb{R}$ such that the limit $\lim_{x\to 0} \dfrac{ae^x \sin(x) + b\cos(x)\sinh(x) + \dfrac{x^2}{1+x^2}}{x^3}$ exists, and also give the value of the limit in those cases.

Ans.:

4. Let $f(x) = \sum_{n=0}^{\infty} (n^2 - 4)x^n$.

   a) Determine the interval of convergence of the power series f.

   Ans.:

   b) Find an explicit rational function $g(x) = \frac{p(x)}{q(x)}$ (with polynomials p, q) such that $f(x) = g(x)$ inside the radius of convergence of f.

   Ans.:

5. Let C be the spiral given by the polar equation $r = e^{3\theta}$, $\theta \in [0, \pi]$; and let D be the circle around the origin of radius e^{π}.

   a) Calculate the polar coordinates of the intersection point of C and D.

Ans.:

   b) Let S be the bounded segment between the spiral and the x-axis, and let T be the intersection of S with the disk around the origin of radius e^{π}. Sketch T, and calculate its area.

Ans.:

c) Calculate the length of C.

Ans.:

6. Among all lines contained in the plane $x + 2y = z$ and passing through the point $(1, 0, 1)$, find the one which has maximal distance from the point $(-1, 1, 0)$.

Ans.:

7. The equation

$$x^2 + 3x - 3y^2 - 6y + 3/2 = 0$$

describes a conic section C.

a) Identify the type of C and give its foci and directrices.

Ans.:

b) Calculate a polar equation (in the form $r = f(\theta)$) for this conic section. (For this, you should take a focus of C as the origin; furthermore, your initial ray should be chosen **parallel** to the x-axis!)

Ans.:

8. Let $a, b \in \mathbb{R}^3$. Show the following assertions:

   a) If $a \times x = b \times x$ for all $x \in \mathbb{R}^3$, then $a = b$.

Ans.:

   b) If $a \times (b \times a) = b \times (a \times b)$, then a and b are parallel.

Ans.:

9. Let $P = (x, y)$ be any point in $\mathbb{R}^2$, and let ℓ be the line parameterized by $(1, 0) + t \cdot (1, -1)$, $t \in \mathbb{R}$. Find a 2×2 matrix A and a column vector $b \in \mathbb{R}^2$ (both independent of (x, y)!) such that the closest point to (x, y) on ℓ is given as

$$A \cdot \begin{pmatrix} x \\ y \end{pmatrix} + b.$$

10. Let S be the set of all points in $\mathbb{R}^3$ bounded from above by the upper half of the unit sphere, and from below by the cone $z = \sqrt{x^2 + y^2}$. Describe S (via inequalities)

    a) in cylindrical coordinates, in the form $\{a \leq \theta \leq b, c \leq r \leq d, f(r) \leq z \leq g(r)\}$ for suitable constants a, b, c, d and functions f, g,

    b) in spherical coordinates, in the form $\{a \leq \theta \leq b, c \leq \varphi \leq d, f(\varphi) \leq \rho \leq g(\varphi)\}$ for suitable constants a, b, c, d and functions f, g.