2021 Fall MAS 101
Chapter 7: Transcendental Functions

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Introduction

- Algebraic function: (analytic) function that can be defined as the root of a polynomial equation, *e.g.*, $\frac{1}{x}$, $\sqrt{x}$, $\cdots$

- Transcendental function: (analytic) function that does not satisfy a polynomial equation, *e.g.*, exponential function, logarithm, trigonometric functions

- A wide variety of phenomena of interest are best modeled by transcendental functions;
  - the growth of a biological population
  - the spread of a disease or of information throughout a human community
  - drug dosages
  - radioactive elements and their role in dating fossils
  - temperature changes
  - waves
  - electrical circuits
  - the vibrations in bridges
  - interest rates
  - probabilities

Inverse Functions

- A function that undoes (or inverts) the effect of a function f is called the inverse of f.

- The natural exponential function $y = e^x$
- The natural logarithm function $y = \ln x$

One-to-One Functions

Definition 1

A function $f(x)$ is **one-to-one** *on a domain D if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ in D.*

- $f(x) = \sqrt{x}$
- $g(x) = \sin x$

- The horizontal line test for one-to-one functions

Inverse Functions

Definition 2

Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} *(read "f inverse") is defined by*

$$f^{-1}(b) = a \quad \text{if} \quad f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

- $(f^{-1} \circ f)(x) = x$ for all x in
- $(f \circ f^{-1})(y) = y$ for all y in

- Can a function that is not one-to-one have an inverse?

Finding Inverses

1. Solve the equation $y = f(x)$ for x. This gives a formula $x = f^{-1}(y)$ where x is expressed as a function of y.
2. Interchange x and y, obtaining a formula $y = f^{-1}(x)$ where f^{-1} is expressed in the conventional format with x as the independent variable and y as the dependent variable.

- $y = \frac{1}{2}x + 1$
- $y = x^2$, $x \geq 0$

Derivatives of Inverses of Differentiable Functions

- The derivatives of $f(x) = \frac{1}{2}x + 1$ and $f^{-1}(x)$

Derivatives of Inverses of Differentiable Functions (cont'd)

Theorem 1 (The Derivative Rule for Inverses)

If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}.$$

Derivatives of Inverses of Differentiable Functions (cont'd)

- $f(x) = x^2$, $x > 0$. Find the derivative of its inverse.
- $f(x) = x^3 - 2$, $x > 0$. Find the value of $\frac{df^{-1}}{dx}$ at $x = 6 = f(2)$ without finding a formula for $f^{-1}(x)$.

Definition of the Natural Logarithm Function

Definition 3

The **natural logarithm** *is the function given by*

$$\ln x = \int_1^x \frac{1}{t} dt, \quad x > 0.$$

Definition of the Natural Logarithm Function (cont'd)

- $\ln 0.5 = -0.69$
- $\ln 1 = 0$
- $\ln 2 \approx 0.69$
- $\ln 3 \approx 1.10$
- $\ln 4 \approx 1.39$

Theorem 2 (The Intermediate Value Theorem for Continuous Functions)

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

Definition of the Natural Logarithm Function (cont'd)

Definition 4

The **number** e *is that number in the domain of the natural logarithm satisfying*

$$\ln e = \int_1^e \frac{1}{t} dt = 1.$$

The Derivative of $y = \ln x$

- The derivative of $y = \ln x$

$$\frac{d}{dx} \ln x =$$

- If u is a differentiable function of x whose values are positive,

$$\frac{d}{dx} \ln u =$$

- Find the derivative $\frac{d}{dx} \ln(x^2 + 3) =$

Properties of Logarithms

Theorem 3

For any numbers $b > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

- *Product rule:* $\ln bx =$
- *Quotient rule:* $\ln \frac{b}{x} =$
- *Reciprocal rule:* $\ln \frac{1}{x} =$
- *Power rule:* $\ln x^r =$ *(for rational r)*

4.2 The Mean Value Theorem

Theorem 4 (The Mean Value Theorem)

Suppose $y = f(x)$ *is continuous over a closed interval* $[a, b]$ *and differentiable on the interval's interior* (a, b)*. Then there is at least one point* c *in* (a, b) *at which*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

4.2 The Mean Value Theorem (cont'd)

Corollary 1

If $f'(x) = 0$ at each point x of an interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Corollary 2

If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

The Graph and Range of $\ln x$

- $\lim_{x\to\infty} \ln x =$
- $\lim_{x\to 0^+} \ln x =$
- Domain of $\ln x$:
- Range of $\ln x$:

The Integral $\int (1/u) du$

- If u is a differentiable function that is never zero,

$$\int \frac{1}{u} du = \ln |u| + C.$$

- $\int_{-\pi/2}^{\pi/2} \frac{4\cos\theta}{3+2\sin\theta} d\theta =$

The Integrals of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

- $\int \tan x dx =$
- $\int \cot x dx =$
- $\int \sec x dx =$
- $\int \csc x dx =$

Logarithmic Differentiation

- Find $\frac{dy}{dx}$, where $y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$ for $x > 1$.

The Inverse of $\ln x$ and the Number e

- The function $\ln x$ has an inverse with domain $(-\infty, \infty)$ and range $(0, \infty)$.
- Let $\exp x$ denote the function $\ln^{-1} x$.
- Show that $\exp r = e^r$ for a rational r.
- How should we define e^x for a real number x?

The Inverse of $\ln x$ and the Number e (cont'd)

Definition 5

For every real number x, we define the **natural exponential function** *to be $e^x = \exp x$.*

- $e^{\ln x} = x$ for all $x > 0$
- $\ln(e^x) = x$ for all x

The Derivative and Integral of e^x

- The natural exponential function is differentiable. Why?

$$\frac{d}{dx}e^x =$$

- If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u \frac{du}{dx}.$$

- $\frac{d}{dx}5e^{-x} =$
- $\int_0^{\ln 2} e^{3x}dx =$

Laws of Exponents

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

- $e^{x_1}e^{x_2} = e^{x_1+x_2}$
- $e^{-x} = \frac{1}{e^x}$
- $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$
- $(e^{x_1})^r = e^{rx_1}$, if r is rational

The General Exponential Function a^x

Definition 6

For any numbers $a > 0$ *and* x, *the* **exponential function with base** a *is*

$$a^x = e^{x \ln a}$$

Proof of the Power Rule (General Version)

Definition 7

For any $x > 0$ and for any real number n,

$$x^n = e^{n \ln x}$$

Definition 8

For $x > 0$ and any real number n,

$$\frac{d}{dx} x^n = n x^{n-1}.$$

If $x \leq 0$, then the formula holds whenever the derivative, x^n, and x^{n-1} all exist.

Proof of the Power Rule (General Version) (cont'd)

- Differentiate $f(x) = x^x$, $x > 0$

The Number e Expressed as a Limit

Theorem 5

The number e can be calculated as the limit

$$e = \lim_{x \to 0} (1 + x)^{1/x}$$

The Derivative of a^u

- If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}.$$

Logarithms with Base a

Definition 9

For any positive number $a \neq 1$, $\log_a x$ is the inverse function of a^x.

Derivatives and Integrals Involving $\log_a x$

- $\frac{d}{dx}(\log_a u) =$

Exponential Change

- Exponential change: in modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t.

- Differential equation:
- Initial condition:

Exponential Change

- The solution of the initial value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0$$

is

$$y =$$

- k: rate constant of the change
- $k > 0$: exponential growth
- $k < 0$: exponential decay

Separable Differential Equations

- Differential equation

$$\frac{dy}{dx} = f(x, y)$$

- Separable differential equation

$$\frac{dy}{dx} = g(x)H(y) = \frac{g(x)}{h(y)}$$

- Solve the equation $y(x+1)\frac{dy}{dx} = x(y^2+1)$.

Unlimited Population Growth

- Unlimited population growth: $\frac{dy}{dt} = ky$ for positive k
- The number of people cured is proportional to the number y that are infected with the disease. Suppose that the number of cases of a disease is reduced by 20% for any given year. If here are 10000 cases today, how many years will it take to reduce the number to 1000.

Radioactivity

- At any given time, the rate at which a radioactive elements decays is approximately proportional to the number of radioactive nuclei present.
- Half-life:

Heat Transfer: Newton's Law of Cooling

- If H is the temperature of the object at time t, H_S is the constant surrounding temperature, then the differential equation is

$$\frac{dH}{dt} = -k(H - H_S)$$

Indeterminate Form $0/0$

- If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}?$$

- Indeterminate forms: $0/0$, ∞/∞, $\infty \cdot 0$, $\infty - \infty$, 0^0 an 1^∞.

- $\lim_{x \to 0} \frac{\sin x}{x} =$

L'Hôpital's Rule

Theorem 6 (L'Hôpital's Rule)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

assuming that the limit on the right side of this equation exists.

- Continue to differentiate f and g, until one or the other of these derivatives is nonzero at $x = a$.
- $\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} =$

L'Hôpital's Rule (cont'd)

- Consider a special case for intuition: assume that f and g have continuous derivatives and satisfy $g'(a) \neq 0$.

Proof of L'Hôpital's Rule

Theorem 7 (Cauchy's Mean Value Theorem)

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

- L'Hôpital's rule applies to the indeterminate form ∞/∞: If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

  provided the limit on the right exists. a may be either finite or infinite.
- $\lim_{x \to \infty} \frac{\ln x}{2\sqrt{x}} =$
- $\lim_{x \to \infty} x \sin \frac{1}{x} =$
- $\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) =$

Indeterminate Powers

- If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L,$$

where a may be either finite and infinite.

- $\lim_{x\to 0^+}(1+x)^{1/x} =$

Defining the Inverses

- $y = \sin^{-1} x$ or $y = \arcsin x$
- $y = \cos^{-1} x$ or $y = \arccos x$
- $y = \tan^{-1} x$ or $y = \arctan x$
- $y = \cot^{-1} x$ or $y = \operatorname{arccot} x$
- $y = \sec^{-1} x$ or $y = \text{arcsec} x$
- $y = \csc^{-1} x$ or $y = \text{arccsc} x$

The Arcsine and Arccosine Functions

Definition 10

- *$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.*
- *$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.*

- $\sin^{-1}\left(\frac{1}{2}\right) =$
- $\cos^{-1}\left(-\frac{1}{2}\right) =$

- $\sin^{-1} x =$
- $\cos^{-1} x + \cos^{-1}(-x) =$
- $\sin^{-1} x + \cos^{-1} x =$

Inverses of $\tan x$, $\cot x$, $\sec x$, and $\csc x$

Definition 11

- $y = \tan^{-1} x$ *is the number in* $(-\pi/2, \pi/2)$ *for which* $\tan y = x$.
- $y = \cot^{-1} x$ *is the number in* $(0, \pi)$ *for which* $\cot y = x$.
- $y = \sec^{-1} x$ *is the number in* $[0, \pi/2) \cup (\pi/2, \pi]$ *for which* $\sec y = x$.
- $y = \csc^{-1} x$ *is the number in* $[-\pi/2, 0) \cup (0, \pi/2]$ *for which* $\csc y = x$.

The Derivative of $y = \sin^{-1} u$

- $y = \sin^{-1} x$ is differentiable in the interval

  and its derivative is

$$\frac{d}{dx}(\sin^{-1} x) =$$

- If u is a differentiable function of x with $|u| < 1$, we have

$$\frac{d}{dx}(\sin^{-1} u) =$$

- $\frac{d}{dx}(\sin^{-1}(2x^2+3)) =$

The Derivative of $y = \tan^{-1} u$

- The derivative of $\tan^{-1} x$ is

$$\frac{d}{dx}(\tan^{-1} x) =$$

- If u is a differentiable function of x, we have

$$\frac{d}{dx}(\tan^{-1} u) =$$

The Derivative of $y = \sec^{-1} u$

- The derivative of $\sec^{-1} x$, $|x| > 1$, is

$$\frac{d}{dx} \sec^{-1} x =$$

- If u is a differentiable function of x with $|u| > 1$, we have

$$\frac{d}{dx} (\sec^{-1} u) =$$

- How about others?

Derivatives of the Other Inverse Trigonometric Functions

- $\cos^{-1} x = \pi/2 - \sin^{-1} x$
- $\cot^{-1} x = \pi/2 - \tan^{-1} x$
- $\csc^{-1} x = \pi/2 - \sec^{-1} x$

- $\frac{d}{dx}(\cos^{-1} u) =$
- $\frac{d}{dx}(\cot^{-1} u) =$
- $\frac{d}{dx}(\csc^{-1} u) =$

Integration Formulas

- $\int \frac{dx}{\sqrt{3-4x^2}} =$

Integration Formulas (cont'd)

- $\int \frac{du}{\sqrt{a^2-u^2}} =$
- $\int \frac{du}{a^2+u^2} =$
- $\int \frac{du}{u\sqrt{u^2-a^2}} =$

- $\int \frac{dx}{\sqrt{e^{2x}-6}} =$
- $\int \frac{dx}{4x^2+4x+2} =$

Definitions and Identities

- The hyperbolic functions are formed by taking combinations of the two exponential functions e^x and e^{-x}.
- These simplify many mathematical expressions, and occur frequently in mathematical and engineering applications.

Definitions and Identities (cont'd)

- The hyperbolic sine and hyperbolic cosine functions are defined as

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

- Similarly, define

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} \quad \text{and} \quad \coth x = \frac{\cosh x}{\sinh x} \\ \operatorname{sech} x &= \frac{1}{\cosh x} \quad \text{and} \quad \operatorname{csch} x = \frac{1}{\sinh x} \end{aligned}$$

Definitions and Identities (cont'd)

- $2\sinh x \cosh x =$
- $\cosh^2 x - \sinh^2 x = 1$

Applications of Hyperbolic Functions

- Catenary shape
- Modeling ocean waves

Derivatives and Integrals of Hyperbolic Functions

- The derivative formulas are derived from the derivative of e^u:

$$\frac{d}{dx}(\cosh u) = \frac{d}{dx}\left(\frac{e^u + e^{-u}}{2}\right) =$$

- We also have

$$\frac{d}{dx}(\operatorname{sech} u) = \frac{d}{dx}\left(\frac{1}{\cosh u}\right) =$$

Derivatives and Integrals of Hyperbolic Functions (cont'd)

- $\int \sinh u du =$
- $\int \qquad\qquad\qquad du = -\operatorname{sech} u + C$

Inverse Hyperbolic Functions

- $y = \sinh^{-1} x$
- $y = \cosh^{-1} x$
- $y = \tanh^{-1} x$
- $y = \coth^{-1} x$
- $y = \text{sech}^{-1} x$
- $y = \text{csch}^{-1} x$

- These will be useful in intergration (Chapter 8).

Useful Identities

- If $0 < x \leq 1$,

$$\operatorname{sech}\left(\cosh^{-1}\left(\frac{1}{x}\right)\right) =$$

Derivatives of Inverse Hyperbolic Functions

- If u is a differentiable function of x, then

$$\frac{d(\sinh^{-1} u)}{dx} =$$

- If u is a differentiable function of x with $|u| < 1$, then

$$\frac{d(\tanh^{-1} u)}{dx} =$$

Derivatives of Inverse Hyperbolic Functions (cont'd)

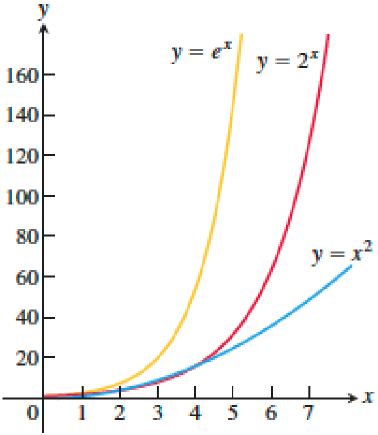
- For $a > 0$,

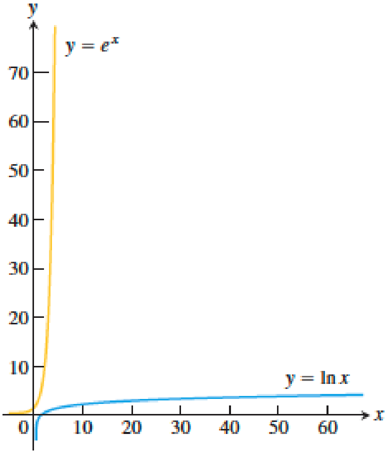
$$\int \frac{du}{\sqrt{a^2 + u^2}} =$$

- Evaluate

$$\int_0^1 \frac{2dx}{\sqrt{3 + 4x^2}} =$$

Growth Rates of Functions





Growth Rates of Functions (cont'd)

Definition 12

Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

1. f **grows faster than** g *as* $x \to \infty$ *if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

   or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

   We also say that g *grows slower than* f *as* $x \to \infty$.

2. f *and* g **grow at the same rate** *as* $x \to \infty$ *if*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L$$

   where L *is finite and positive.*

Grow Rates of Functions (cont'd)

- x vs. $2x$
- e^x vs. x^2
- x^2 vs. $\ln x$
- $\ln x$ vs. $x^{1/n}$

- If f grows at the same rate as g as $x \to \infty$, and g grows at the same rate as h as $x \to \infty$, then f grows the same rate as h as $x \to \infty$.

Order and Oh-Notation

Definition 13

A function f is **of smaller order than** g *as* $x \to \infty$ *if* $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. *We indicate this by writing* $f = o(g)$ *("f is little-oh of g").*

- $\ln x = o(?)$

Order and Oh-Notation (cont'd)

Definition 14

Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g *as $x \to \infty$ if there is a positive integer M for which*

$$\frac{f(x)}{g(x)} \leq M,$$

for x sufficiently large. We indicate this by writing $f = O(g)$ ("f is big-oh of g").

- $e^x + x^2 = O(?)$
- $f = o(g)$ implies , for functions that are positive for x sufficiently large.
- If f and g grow at the same rate, then $f = O(g)$ and .