2021 Fall MAS 101
Chapter 5: Integrals

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Finite Sums

- Finite sum for f on the interval $[a, b]$, where $\Delta x = \frac{b-a}{n}$:

$$S_n = \sum_{k=1}^{n} f(a + k \cdot \Delta x)\Delta x$$

- You probably have learned the following relationship intuitively.

$$\int_a^b f(x)dx = \lim_{n \to \infty} S_n$$

- Is this true for any function?

Riemann Sums

- Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, where $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. This divides $[a, b]$ into n closed subintervals

$$[x_0, x_1], [x_1, x_2], \ldots, [x_{n-1}, x_n].$$

- Riemann sum for f on the interval $[a, b]$, where $\Delta x = \frac{b-a}{n}$:

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k,$$

where $c_k \in [x_{k-1}, x_k]$.

Definition of the Definite Integral

- Define the size (norm) of the partition P as

$$||P|| = \max_{k=1,\dots,n} \Delta x_k,$$

where $\Delta x_k := (x_k - x_{k-1})$.

- We say that a function f is **integrable** if the limit of S_P exists as $||P|| \to 0$.

Definition of the Definite Integral (cont'd)

Definition 1

Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral of f over** $[a, b]$ *and that J is the limit of the Riemann sums $\sum_{k=1}^{n} f(c_k)\Delta x_k$ if the following condition is satisfied:*

Given any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with $||P|| < \delta$ and any choice of c_k in $[x_{k-1}, x_k]$, we have

$$\left|\sum_{k=1}^{n} f(c_k)\Delta x_k - J\right| < \epsilon.$$

Integrable and Nonintegrable Functions

- Not every function defined over the closed interval $[a, b]$ is integrable there, even if the function is bounded.

Theorem 1 (Integrability of Continuous Functions)

If a function f is continuous over the interval $[a, b]$, of if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

- See Table 5.6 on page 284 for the properties of definite integral, most of which come from the linearity of integration.

Integrable and Nonintegrable Functions

- For integrability to fail, a function needs to be sufficiently discontinuous that the region between its graph and the x-axis cannot be approximated well by increasingly thin rectagles.

Ex. Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Mean Value Theorem for Definite Integrals

Theorem 2 (The mean value theorem for definite integrals)

If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a} \int_a^b f(x)dx.$$

Fundamental Theorem, Part 1

Theorem 3 (The fundamental theorem of calculus, part 1)

If f is continuous on $[a,b]$, then $F(x)=\int_a^x f(t)dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$F'(x)=\frac{d}{dx}\int_a^x f(t)dt=f(x).$$

Fundamental Theorem, Part 2

Theorem 4 (The fundamental theorem of calculus, part 2)

If f is continuous on $[a,b]$, and F is any antiderivative of f on $[a,b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$