Midterm Exam Calculus I (MAS101), Fall 2019

Date: 2019.10.22
Time: 19:00–21:45

Name:

Student ID:

- Write your name and student ID on top of each page.
- Show your work for full credit and mark your final answer for each problem. You are encouraged to write in English but some Korean is acceptable. You can use the back of each page as scratch spaces but anything written there will not be graded.
- The exam is for 2:45 hours in length. Ask permission by raising your hand if you have any question or need to go to the toilet.
- Any attempt to cheat or a failure to follow this instruction lead to serious disciplinary actions not limited to failing the exam or the course.

Problem	Score	Problem	Score
1	/15	2	/12
3a	/10	3b	/10
3c	/12	4a	/10
4b	/10	4c	/10
5a	/12	5b	/12
5c	/12	6a	/10
6b	/10	7a	/10
7b	/10	8	/10
9	/10	10	/15
		Total	**/200**

1. Determine explicitly a solution $y = y(x)$ of the following initial value problem:

$$y \cdot y' = \sec^2(x) e^{-y^2}, \qquad y(0) = 1$$

Ans.:

2. Use derivatives to show the following identity for all $0 < x < 1$:

$$sech^{-1}(x) = \ln(\frac{1+\sqrt{1-x^2}}{x}).$$

3. Evaluate the following integrals:

a) $\int \frac{x^3+1}{x^2(x^2+1)} dx.$

Ans.:

b) $\int \frac{\tan^3(\sinh^{-1}(x))}{\sqrt{1+x^2}}dx.$

Ans.:

c) $\int \frac{\sqrt{1-(\ln(x))^2}}{x\ln(x)}dx.$

Ans.:

4. Determine whether the following improper integrals converge or diverge:

a) $\int_1^\infty \frac{\sinh(1/x)}{x} dx.$

Ans.:

b) $\int_0^1 \frac{e^{-1/x}}{x^2 - 1} dx.$

Ans.:

c) $\int_1^\infty \frac{\sin^2(x)}{x} dx$.

Ans.:

5. Decide which of the following series are absolutely convergent, conditionally convergent or divergent.

a) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{\ln(\ln(n))}{\ln(n)^2}$.

Ans.:

b) $\sum_{n=1}^{\infty}(-1)^n \frac{\sinh^{-1}(n)\tanh^{-1}(n)}{n^2-n+4}$

Ans.:

c) $\sum_{n=1}^{\infty}(-1)^n \frac{\cosh^{-1}(n)}{\ln(n+1)}$

Ans.:

6. The following statements are **false**. Show this by providing a **counterexample**.

   a) If $(a_n)_{n\in\mathbb{N}}$ is a sequence of positive terms with $a_n \in o(1/n)$, then $\sum_{n=1}^{\infty} a_n$ converges.

   b) Let $(a_n)_{n\in\mathbb{N}}$ be a sequence, and set $b_n := a_{n+1} - a_n$. If $\lim_{n\to\infty} b_n = 0$, then $(a_n)_{n\in\mathbb{N}}$ converges.

7. a) Estimate the integral $\int_0^\pi \sin^3(x)dx$ using Simpson's rule with $n = 4$ subintervals.

Ans.:

b) Now choose $n = 30$ subintervals, Show that the trapezoid estimate for $\int_0^\pi \sin^3(x)dx$ then differs from the true value by at most $\frac{1}{100}$.
You may use the - in fact, rather good - estimate $\pi^3 = 31$.

8. Find all $a \in \mathbb{R}$ such that the sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$a_n = \frac{\sin^2(\frac{1}{n}) + 4\sinh(n) - a\cosh(n)}{(\tan^{-1}(\frac{1}{n}))^2}$$

converges, and also determine the limit in those cases.

Ans.:

9. Determine the exact sum of the series

$$\sum_{n=2}^{\infty} \frac{n-1}{n!}$$

Ans.:

10. Consider the series

$$\sum_{k=1}^{\infty} \frac{\ln(k)}{k^3}.$$

Find the smallest integer n such that the partial sum s_n approximates the series with an error of at most $1/10$. Give a proof that your value n is correct.

(You may use any of the following estimates:
$\ln(2) = 0.7$, $\ln(3) = 1.1$, $\ln(5) = 1.6$).

Ans.: