2021 Fall MAS 101
Chapter 3: Derivatives

Donghwan Kim

KAIST

Sep/2, 2021

Finding a Tangent to the Graph of a Function

Definition 1

The **slope of the curve** $y = f(x)$ *at the point* $P(x_0, f(x_0))$ *is the number*

$$m =$$

The **tangent line** *to the curve at* P *is the line through* P *with this slope.*

- Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$.

Rate of Change: Derivative at a Point

Definition 2

The **derivative of a function** f **at a point** x_0, *denoted* $f'(x_0)$, *is*

$$f'(x_0) = \lim_{h \to 0} \frac{x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

Differentiable Functions Are Continuous

- A function $y = f(x)$ is **differentiable on an open interval** if it has a derivative at each point of the interval.

Theorem 1 (Differentiability implies continuity)

If f has a derivative at $x = c$, then f is continuous at $x = c$.

Linearization

- Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and easier to work.

Linearization (cont'd)

Definition 3

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** *of f at a. The approximation*

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** *of f at a. The point $x = a$ is the* **center** *of the approximation.*

- Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$.

Linearization (cont'd)

Approximation	True value	$\lvert \text{True value} - \text{approximation} \rvert$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$0.004555 < 10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$0.000305 < 10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$0.000003 < 10^{-5}$

Differentials

Definition 4

Let $y = f(x)$ be a differentiable function. The **differential** dx *is an independent variable. The* **differential** dy *is*

$$dy = f'(x)dx.$$

- Find dy if $y = x^5 + 37x$.

Estimating with Differentials

- Given the value of $f(x)$ at a point a, we want to estimate how much this value will change if we move to $a + dx$.

$$f(a + dx) \approx$$

- Estimate $7.97^{1/3}$.

Error in Differential Approximation

- How well does df approximate $\Delta f = f(a + \Delta x) - f(a)$?
- Approximation error $= \Delta f - df$

Error in Differential Approximation

- If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\Delta y + \epsilon \Delta x$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

Proof of the Chain Rule

Theorem 2 (The Chain Rule)

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Proof of the Chain Rule (cont'd)

Proof. Let Δx be an increment in x and let Δu and Δy be the corresponding increments in u and y.