2021 Fall MAS 101
Chapter 2: Limits and Continuity

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Rates of Change and Tangents to Curves

- Average rates of change and secant lines
- Instantaneous rates of changes and tangent lines

Limits of Function Values

- How does the function $f(x) = \frac{x^2-1}{x-1}$ behave near $x = 1$?

Limits of Function Values (cont'd)

- Suppose $f(x)$ is defined on an open interval about c, *except possibly at c itself*. If $f(x)$ is arbitrarily close to the number L for all sufficiently close to c, we say that f approaches the **limit** L as x approaches c, and write

- $f(x) = \frac{x^2-1}{x-1}$
- $g(x) = \begin{cases} \frac{x^2-1}{x-1}, & x \neq 1, \\ 1, & x = 1. \end{cases}$
- $h(x) = x + 1$

- Does a function always have a limit at a particular point?

The Limit Laws

Theorem 1 (Limit Laws)

If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \textit{and} \quad \lim_{x\to c} g(x) = M, \quad \textit{then}$$

- *Sum Rule:* $\lim_{x\to c}(f(x)+g(x)) = L+M$
- *Difference Rule:* $\lim_{x\to c}(f(x)-g(x)) = L-M$
- *Constant multiple rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$
- *Product rule:* $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$
- *Quotient rule:* $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad M\neq 0$
- *Power rule:* $\lim_{x\to c}[f(x)]^n = L^n$, $\quad n$ *a positive integer*
- *Root rule:* $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L}$, $\quad n$ *a positive integer*

  (If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

- $\lim_{x\to c}(x^3+4x^2-3)$

The Limit Laws (cont'd)

Theorem 2 (Limits of Polynomials)

If $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$, *then*

$$\lim_{x \to c} P(x) =$$

Theorem 3 (Limits of Rational Functions)

If $P(x)$ *and* $Q(x)$ *are polynomials and* $Q(c) \neq 0$, *then*

$$\lim_{x \to c} \frac{P(x)}{Q(x)} =$$

Eliminating Common Factors from Zero Denominators

- Evaluate $\lim_{x\to 1} \frac{x^2+x-2}{x^2-x}$.

Using Calculators and Computers to Estimate Limits

- Estimate the value of $\lim_{x\to 0} \frac{\sqrt{x^2+100}-10}{x^2}$.

Using Calculators and Computers to Estimate Limits

TABLE 2.3 Computed values of $f(x) = \dfrac{\sqrt{x^2+100}-10}{x^2}$ near $x = 0$

x	$f(x)$	
± 1	0.049876	approaches 0.05?
± 0.5	0.049969	
± 0.1	0.049999	
± 0.01	0.050000	
± 0.0005	0.050000	approaches 0?
± 0.0001	0.000000	
± 0.00001	0.000000	
± 0.000001	0.000000	

The Sandwich Theorem

Theorem 4 (The Sandwich Theorem)

Suppose that $g(x) \leq f(x) \leq h(x)$ *for all* x *in some open interval containing* c, *except possibly at* $x = c$ *itself. Suppose also that*

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

- Evaluate $\lim_{\theta \to 0} \sin \theta$ using $-|\theta| \leq \sin \theta \leq |\theta|$.

Informal Definition of Limit

- If $f(x)$ is arbitrarily close to the number L for all x sufficiently close to c, we say that f approaches the limit L as x approaches to c.

Informal Definition of Limit (cont'd)

- Consider the function $y = 2x - 1$ near $x = 4$. How close to $x = 4$ does x have to be so that $y = 2x - 1$ differs from 7 by less than 2?

- How should we define the limit?

Definition of Limit

Definition 1

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, *and write*

$$\lim_{x \to c} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

Examples: Testing the Definition

- Show that $\lim_{x \to 1} (5x - 3) = 2$.

Finding Deltas Algebraically for Given Epsilons

- Prove that $\lim_{x\to 2} f(x) = 4$ if $f(x) = \begin{cases} x^2, & x \neq 2, \\ 1, & x = 2. \end{cases}$

Using the Definition to Prove Theorems

- Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, prove that

$$\lim_{x\to c}(f(x) + g(x)) = L + M.$$

Using the Definition to Prove Theorems (cont'd)

- Given that $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x)) = M$, and that $f(x) \leq g(x)$ for all x in an open interval containing c (except possibly c itself), prove that $L \leq M$.

Approaching a Limit form One Side

- $f(x) = \frac{x}{|x|}$

Approaching a Limit from One Side (cont'd)

- If $f(x)$ is defined on an interval (c, b), where $c < b$, and approaches arbitrarily close to L as x approaches c from within that interval, then f has **right-hand limit** L at c. We write

$$\lim_{x \to c^+} f(x) = L.$$

- Consider $f(x) = \sqrt{4 - x^2}$ with a domain $[-2, 2]$

Approaching a Limit from One Side (cont'd)

Theorem 5

A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \textit{and} \quad \lim_{x\to c^+} f(x) = L.$$

Precise Definitions of One-Sided Limits

Definition 2

Let $f(x)$ be defined on an open interval about c, except possibly at c itself. We say that the **limit of $f(x)$ as x approaches c is the number L**, *and write*

$$\lim_{x\to c} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x, $\quad 0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$

Definition 3

We say that $f(x)$ has **right-hand limit L at c**, *and write*

$$\lim_{x\to c^+} f(x) = L$$

if for every number $\epsilon > 0$ there exists a corresponding number $\delta > 0$ such that for all x,

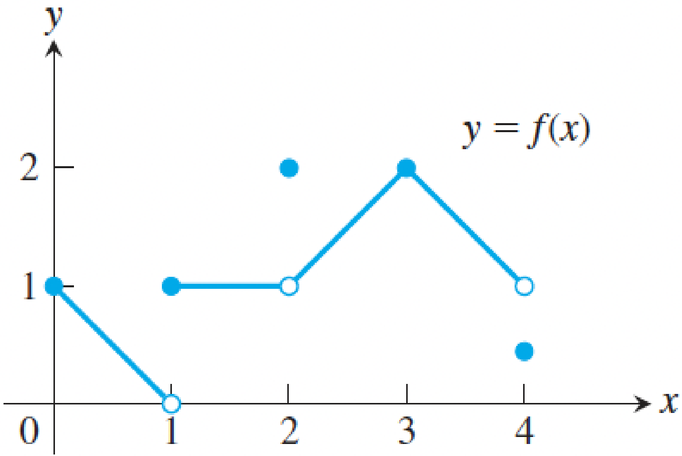
- $y = \sin(1/x)$

Limits Involving $(\sin\theta)/\theta$

Theorem 6 (Limit of the Ratio $(\sin\theta)/\theta$ as $\theta \to 0$)

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} =$$

Continuity at a Point



Continuity at a Point (cont'd)

Definition 4

Let c be a real number on the x-axis.

- *The function f is* **continuous at** c *if*

$$\lim_{x \to c} f(x) = f(c).$$

- *The function f is* **right-continuous at** c *(***or continuous from the right***) if*

$$\lim_{x \to c^+} f(x) = f(c).$$

- We say that a function is **continuous over a closed interval** $[a, b]$ if

- We say that a function is **discontinuous at** c if

Continuous Functions

- We define a **continuous function** as one that is continuous at every point in its domain.

- $y = 1/x$

Continuous Functions (cont'd)

Theorem 7 (Properties of Continuous Functions)

If the function f and g are continuous at $x = c$, then the following algebraic combinations are continuous at $x = c$.

- *Sums:* $f + g$
- *Differences:* $f - g$
- *Constant multiples:* $k \cdot f$, *for any number* k
- *Products:* $f \cdot g$
- *Quotients:* f/g, *provided* $g(c) \neq 0$
- *Powers:* f^n, n *a positive integer*
- *Roots:* $\sqrt[n]{f}$, *provided it is defined on an open interval containing* c, *where* n *is a positive integer*

- Is polynomial $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ continuous?
- How about $f(x) = |x|$?

Composites

Theorem 8 (Composite of continuous functions)

If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

- $y = \sqrt{x^2 - 2x - 5}$

Composites (cont'd)

Theorem 9

If g is continuous at b and $\lim_{x\to c} f(x) = b$, then

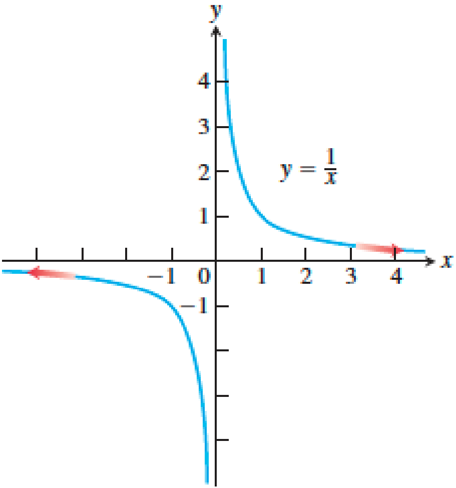
$$\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)).$$

Intermediate Value Theorem for Continuous Functions

Theorem 10 (The intermediate value theorem for continuous functions)

If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ ad $f(b)$ for some c in $[a, b]$.

Finite Limits as $x \to \pm\infty$



- What is ∞ and $-\infty$?

Finite Limits as $x \to \pm\infty$ (cont'd)

Definition 5

- *We say that $f(x)$ has the* **limit** L **as** x **approaches infinity** *and write*

$$\lim_{x\to\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

- *We say that $f(x)$ has the* **limit** L **as** x **approaches minus infinity** *and write*

$$\lim_{x\to-\infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

- Show that $\lim_{x\to\infty} \frac{1}{x} = 0$.

Finite Limits as $x \to \pm\infty$ (cont'd)

Theorem 11

All the Limit Laws are true when we replace $\lim_{x\to c}$ *by* $\lim_{x\to\infty}$ *or* $\lim_{x\to-\infty}$.

- $\lim_{x\to\infty} \left(5 + \frac{1}{x}\right) =$

Limits at Infinity of Rational Functions

- $\lim_{x\to\infty} \frac{5x^2+8x-3}{3x^2+2} =$

Horizontal Asymptotes

Definition 6

A line $y = b$ is a **horizontal asymptote** *of the graph of a function $y = f(x)$ if either*

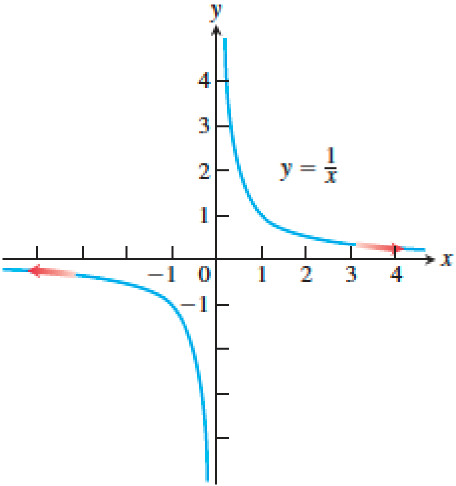
$$\lim_{x\to\infty} f(x) = b \quad \textit{or} \quad \lim_{x\to-\infty} f(x) = b.$$

- $f(x) = \frac{1}{x}$
- $f(x) = \frac{5x^2+8x-3}{3x^2+2}$
- $f(x) = 2 + \frac{\sin x}{x}$

Oblique Asymptotes

- If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique asymptote**. Why?
- $f(x) = \frac{x^2-3}{2x-4}$

Infinite Limits



Infinite Limits (cont'd)

- Discuss the behavior of $f(x) = \frac{1}{x}$ as $x \to 0$.
- Discuss the behavior of $f(x) = \frac{1}{x^2}$ as $x \to 0$.

Precise Definitions of Infinite Limits

Definition 7

- *We say that* $f(x)$ **approaches infinity as** x **approaches** c*, and write*

$$\lim_{x \to c} f(x) = \infty,$$

if for every positive real number B *there exists a corresponding* $\delta > 0$ *such that for all* x

- *We say that* $f(x)$ **approaches minus infinity as** x **approaches** c*, and write*

$$\lim_{x \to c} f(x) = -\infty,$$

if for every positive real number $-B$ *there exists a corresponding* $\delta > 0$ *such that for all* x

Vertical Asymptotes

Definition 8

A line $x = a$ is a **vertical asymptote** *of the graph of a function $y = f(x)$ if either*

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \textit{or} \quad \lim_{x \to a^-} f(x) = \pm\infty$$

- Find the horizontal and vertical asymptotes of the graph of $f(x) = -\frac{8}{x^2-4}$.

Dominant Terms

- Let $f(x) = 3x^4 - 2x^3 + 3x^2 - 5x + 6$ and $g(x) = 3x^4$. Show that they are virtually identical for $|x|$ very large.