On my honor as a KAIST student, I pledge to take this exam honestly.

You should write down all your work for full credits.

Session 2, October 19, 2021, 20:40–21:55, Four problems: 4–7

4. (15 pts) Let $w(t)$ be a differentiable function and κ and ρ be positive constants.

   (a) (10 pts) Solve

   $$\frac{d}{dt}w(t) + \kappa\, w(t) - \rho = 0 \quad \text{or} \quad w' + \kappa\, w - \rho = 0$$

   with an initial condition $w(0) = v_0$.

   (b) (5 pts) By setting $w(t) = \frac{d}{dt}y(t)$, solve

   $$\frac{d^2}{dt^2}y(t) + \kappa\frac{d}{dt}y(t) - \rho = 0 \quad \text{or} \quad y'' + \kappa\, y' - \rho = 0$$

   with $y(0) = 0$ and $\frac{d}{dt}y(t)\Big|_{t=0} = v_0$.

5. (10 pts) Evaluate

   $$\int_1^\infty \left[2\sqrt{x^2-1} - 2x + \frac{1}{x}\right] dx.$$

6. (15 pts) Let $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ for a positive integer n.

   (a) (5 pts) Use a comparison test to show that $\Gamma(n)$ converges.

   (b) (10 pts) Show that $\Gamma(n) = (n-1)!$

7. (20 pts)

   (a) (10 pts) Solve the equation $xy' + 2y = \frac{8x}{(x-1)(x+1)(x+2)}$ for $x > 1$.

   (b) (10 pts) Solve the initial value problem $y' = 2x\sqrt{1+y^2}$ with $y(0) = 0$.