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TREEPAT

① By the integral test,

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\int_2^t \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\int_2^t (\ln x)^{-p}\,d(\ln x)$$

$$= \lim_{t\to\infty} \frac{(\ln x)^{-p+1}}{-p+1}\Bigg|_2^t = \lim_{t\to\infty} \overset{0}{\cancel{\frac{(\ln t)^{-p+1}}{-p+1}}} - \frac{(\ln 2)^{-p+1}}{-p+1}$$

$$= \frac{-(\ln 2)^{1-p}}{1-p}$$

if $p>1$, $\frac{(\ln 2)^{1-p}}{p-1}$ converges, so $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges as well.

② Given $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, and by limit comparison test,

$a_n = \frac{1}{n\sqrt[n]{n}}$, $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = \frac{1}{\sqrt[n]{n}} = 1 > 0$$

$\therefore$ both Σa_n and Σb_n diverges

$\therefore \sum_{n=1}^{\infty} \frac{1}{n\sqrt[n]{n}}$ diverges.