2021 Fall MAS 101
Chapter 8: Techniques of Integration

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Basic Integration Formulas

- $\int x^n dx = \frac{x^{n+1}}{n+1} + C$ for $n \neq -1$
- $\int \frac{dx}{x} = \ln|x| + C$
- $\int e^x dx = e^x + C$, $\int a^x dx = \frac{a^x}{\ln a} + C$ for $a > 0, a \neq 1$
- $\int \tan x dx = \ln|\sec x| + C$, $\int \sec x dx = \ln|\sec x + \tan x| + C$, $\cdots$
- $\int \frac{dx}{\sqrt{a^2 - x^2}} =$
- $\int \sinh x dx = \cosh x dx + C$
- $\int \frac{dx}{\sqrt{a^2 + x^2}} =$ for $a > 0$

Basic Integration Formulas (cont'd)

- Evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}} =$$

Basic Integration Formulas (cont'd)

- Evaluate

$$\int \frac{3x+2}{\sqrt{1-x^2}}dx =$$

Integration by Parts

- Integration by parts simplifies integrals of the form

$$\int f(x)g(x)dx$$

Product Rule in Integral Form

- If f and g are differentiable functions of x, the Product Rule says that

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Product Rule in Integral Form (cont'd)

- Integration by Parts Formula

$$\int udv = uv - \int vdu$$

- Find

$$\int e^x \cos x dx =$$

Evaluating Definite Integrals by Parts

- Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)dx$$

- Compute

$$\int_0^4 xe^{-x}dx =$$

Tabular Integration Can Simplify Repeated Integrations

- Evaluate

$$\int x^2 e^x dx =$$

- Find the integral

$$\frac{1}{\pi}\int_0^{\pi} x^3 \cos nx dx =$$

where n is a positive integer.

Products of Powers of Sines and Cosines

- Consider the form

$$\int \sin^m x \cos^n x dx,$$

where m and n are nonnegative integers.

Products of Powers of Sines and Cosines (cont'd)

1. m is odd: let $m = 2k + 1$ and use $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x$$

2. m is even and n is odd:

3. both m and n are even:

Products of Powers of Sines and Cosines (cont'd)

- Evaluate

$$\int \sin^3 x \cos^2 x dx =$$

Products of Powers of Sines and Cosines (cont'd)

- Evaluate

$$\int \cos^5 x dx =$$

Products of Powers of Sines and Cosines (cont'd)

- Evaluate

$$\int \sin^2 x \cos^4 x dx =$$

Eliminating Square Roots

- Use the identity $\cos^2\theta = (1+\cos 2\theta)/2$.
- Evaluate

$$\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx =$$

Integrals of Powers of $\tan x$ and $\sec x$

- Use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$.
- Evaluate

$$\int \tan^4 x dx =$$

Products of Sines and Cosines

- Use the identities

$$\begin{aligned}\sin mx \sin nx &= \frac{1}{2}[\cos(m-n)\,x - \cos(m+n)\,x],\\ \sin mx \cos nx &= \frac{1}{2}[\sin(m-n)\,x + \sin(m+n)\,x],\\ \cos mx \cos nx &= \frac{1}{2}[\cos(m-n)\,x + \cos(m+n)\,x].\end{aligned}$$

- Evaluate

$$\int \sin 3x \cos 5x dx =$$

Products of Powers of Sines and Cosines

- Replace the variable of integration by a trigonometric function, *e.g.*, $x = a\tan\theta$, $x = a\sin\theta$, and $x = a\sec\theta$.

- These are effective in transforming integrals involving $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ and $\sqrt{x^2-a^2}$.

Procedure for a Trigonometric Substitution

- Write down the substitution for x, calculate dx and specify the selected values of θ for the substitution.
- Substitute them into the integrand, and then simplify the results algebraically.
- Integrate the trigonometric integral (keeping in mind the restrictions on θ for reversibility).
- Draw an appropriate reference triangle to reverse the substitution in the integration result and convert it back to the original variable x.

Procedure for a Trigonometric Substitution (cont'd)

- Evaluate

$$\int \frac{dx}{\sqrt{a^2 + x^2}} =$$

Procedure for a Trigonometric Substitution (cont'd)

- Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \quad x > \frac{2}{5}$$

General Description of the Method

- The degree of $f(x)$ must be less than the degree of $g(x)$.
- We must know the factors of $g(x)$.

General Description of the Method (cont'd)

- Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$.
- Let $x^2 + px + q$ be an irreducible factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$.
- Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions.
- Equate the coefficients of corresponding powers of x and solve the resulting equations for the underdetermined coefficients.

General Description of the Method (cont'd)

- Use partial fractions to evaluate

$$\int \frac{-2x+4}{(x^2+1)(x-1)^2}dx$$

The Heaviside "Cover-up" Method for Linear Factors

- When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial functions.

The Heaviside "Cover-up" for Linear Factors (cont'd)

- Use the Heaviside Method to evaluate

$$\int \frac{x+4}{x^3+3x^2-10x}dx$$

Other Ways to Determine the Coefficients

- Differentiate
- Assign selected numerical values to x

- Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}$$

by clearing fractions, differentiating the result, and substituting $x = -1$.

Integral Tables

- Find

$$\int \frac{dx}{x\sqrt{2x-4}}$$

Reduction Formulas

- Repeated integrations by parts can sometimes be shortened by applying reduction formulas like

$$\int \tan^n x dx =$$

$$\int (\ln x)^n dx =$$

Integration with a CAS

- Computer algebra system (CAS) can integrate symbolically.
  e.g., **int** in Maple, **Integrate** in Mathematica, and **int** in Matlab.

Integration with a CAS (cont'd)

- Evaluate using Maple

$$f(x) = x^2\sqrt{a^2 + x^2}$$

- `> f:=x^2*sqrt(a^2+x^2);`
- `> int(f,x);`
- `> simplify(%);`
- `> int(f,x=0..Pi/2);`
- `> a:=1;`
- `> int(f,x=0..1);`

Integration with a CAS (cont'd)

- Evaluate using Matlab (with Symbolic Math Toolbox)

$$f(x) = x^2\sqrt{a^2 + x^2}$$

- `syms x a`
- `f(x,a) = x^2*sqrt(a^2+x^2);`
- `int(f,x)`
- `int(f,x,[0 pi/2])`

Nonelementary Integrals

- Nonelementary integrals: integral of functions that cannot be expressed as finite combinations of elementary functions (the functions we have been studying).

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \quad \int \sin x^2 dx, \quad \int \sqrt{1+x^4} dx, \quad \cdots$$

- ~~Numerical integration: Section 8.7~~
- Express with infinite series: Chapter 10

Motivating Examples

- $\int_1^\infty \frac{\ln x}{x^2} dx$
- $\int_0^1 \frac{1}{\sqrt{x}} dx$

Motivating Examples (cont'd)

Definition 1

A **probability density function** *for a continuous random variable is a function f defined over $(-\infty, \infty)$ and having the following properties:*

- *f is continuous, except possibly at a finite number of points.*
- *f is nonnegative, so $f \geq 0$.*
- *$\int_{-\infty}^{\infty} f(x)dx = 1$.*

If X is a continuous random variable with probability density function f, the probability that X assumes a value in the interval between $X = c$ and $X = d$ is the area integral

$$P(c \leq X \leq D) = \int_c^d f(X)dx.$$

Infinite Limits of Integration

- $\int_0^\infty e^{-x/2} dx$

Infinite Limits of Integration (cont'd)

Definition 2

Integrals with infinite limits of integration are **improper integrals of Type I**.

- *If* $f(x)$ *is continuous on* $[a, \infty)$, *then* $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx.$
- *If* $f(x)$ *is continuous on* $(-\infty, b]$, *then*

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx.$$

- *If* $f(x)$ *is continuous on* $(-\infty, \infty)$, *then*

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^\infty f(x)dx,$$

  where c *is any real number.*

In each case, if the limit is finite we say that the improper integral **converges** *and that the limit is the* **value** *of the improper integral. If the limit fails to exist, the improper integral* **diverges**.

Infinite Limits of Integration (cont'd)

- Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?
- Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

The Integral $\int_1^\infty \frac{dx}{x^p}$

- For what values of p does the integral $\int_1^\infty dx/x^p$ converge? When the integral does converge, what is its value?

Integrands with Vertical Asymptotes

- $\int_0^1 \frac{dx}{\sqrt{x}}$

Integrands with Vertical Asymptotes (cont'd)

Definition 3

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

- *If* $f(x)$ *is continuous on* $(a, b]$ *and discontinuous at* a*, then*

- *If* $f(x)$ *is continuous on* $[a, b)$ *and discontinuous at* b*, then*

- *If* $f(x)$ *is discontinuous at* c*, where* $a < c < b$*, and continuous on* $[a, c) \cup (c, b]$*, then*

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

In each case, if the limit is finite we say the improper integral **converges** *and that the limit is the* **value** *of the improper integral. If the limit does not exist, the integral* **diverges**.

Integrands with Vertical Asymptotes (cont'd)

- $\int_0^1 \frac{1}{1-x}dx$
- $\int_0^3 \frac{dx}{(x-1)^{2/3}}$

Improper Integrals with a CAS

- Evaluate using Maple

$$\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)} dx$$

- `> f:=(x+3)/((x-1)*(x^2+1));`
- `> int(f,x=2..infinity);`

Tests for Convergence and Divergence

- Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

Tests for Convergence and Divergence (cont'd)

Theorem 1 (Direct Comparison Test)

Let f *and* g *be continuous on* $[a, \infty)$ *with* $0 \le f(x) \le g(x)$ *for all* $x \ge a$. *Then*

- $\int_a^\infty f(x)dx$ *converges if* $\int_a^\infty g(x)dx$ *converges.*
- $\int_a^\infty g(x)dx$ *diverges if* $\int_a^\infty f(x)dx$ *diverges.*

Tests for Convergence and Divergence (cont'd)

- Does $\int_1^\infty \frac{1}{\sqrt{x^2-0.1}} dx$ converge?
- Does $\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}} dx$ converge?

Tests for Convergence and Divergence (cont'd)

Theorem 2 (Limit Comparison Test)

If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)dx \quad \text{and} \quad \int_a^\infty g(x)dx$$

both converge or both diverge.

Tests for Convergence and Divergence (cont'd)

- Show that

$$\int_1^\infty \frac{dx}{1+x^2}$$

converges by comparison with $\int_1^\infty (1/x^2)dx$. Find and compare the two integral values.