MAS 101 Calculus I : Final Exam

June 11, 2019 7:00 PM ~ 10:00 PM

Spring 2019 MAS 101

Full Name : ______________________________

Student Number: ____________________ **Class:** ____________

Do not write in this box.

Instruction

- No items other than pen, pencil, eraser and your student ID card are allowed. You must leave all of your belongings that are not allowed at the designated area in the front of the classroom.
- Before you start, fill out the identification section on the title page and on the header of each page with an inerasable pen.
- Show your work for full credit and write the final answer to each problem in the box provided. You are encouraged to write in English but some Korean is acceptable. You can use the back of each page as scratch spaces but anything written there will not be graded.
- The exam is for three hours in length. Ask permission by raising your hand if you have any question or need to go to toilet. You are not allowed to go to toilet for the first 30 minutes and the last 30 minutes.
- Any attempt to cheat or a failure to follow this instruction lead to serious disciplinary actions not limited to failing the exam or the course.

Do not write in this table.

Problem	Score	Problem	Score	Problem	Score
1(a)	/ 10	1(b)	/ 10	2(a)	/ 10
2(b)	/ 10	3(a)	/ 10	3(b)	/ 10
4(a)	/ 10	4(b)	/ 10	5(a)	/ 10
5(b)	/ 10	6(a)	/ 10	6(b)	/ 10
7(a)	/ 10	7(b)	/ 10	8(a)	/ 10
8(b)	/ 10	9(a)	/ 10	9(b)	/ 10
10(a)	/ 10	10(b)	/ 10	**Total**	**/ 200**

1. Find all values of x for which the following series converge:

(a) $\displaystyle\sum_{n=1}^{\infty}(\tanh n)x^n$

Ans:

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{(x+1)^n\ln(n+1)}{2nx^n}$

Ans:

2. Let $f(x) = \dfrac{\ln(x+1)}{x}$.

   (a) Find the Maclaurin series of $f(x)$ and its interval of convergence.

   Ans:

   (b) Estimate the definite integral $\displaystyle\int_0^{1/2} f(x)\,dx$ with an error less than $\dfrac{1}{200}$.

   Ans:

3. Find the sum of the following series:

(a) $1 - \dfrac{\pi^2}{16 \cdot 2!} + \dfrac{\pi^4}{256 \cdot 4!} - \cdots + (-1)^n \dfrac{\pi^{2n}}{2^{4n}(2n)!} + \cdots.$

Ans:

(b) $\dfrac{1}{3} + \dfrac{4}{9} + \dfrac{9}{27} + \cdots + \dfrac{n^2}{3^n} + \cdots.$

Ans:

4. Let C be the parametric curve given by $x = t\cos t, y = t\sin t$ for $-2\pi \le t \le 2\pi$.

   (a) Sketch the curve C.

   (b) Find the area of the innermost region, that is, the region enclosed by the smallest loop.

Ans:

5. Let P be the intersection of the polar curve $r = r(\theta)$ with the line $y = (\tan\theta)x$ in $\mathbb{R}^2$. And let α be the angle between the line tangent to $r = r(\theta)$ at P and the line $y = (\tan\theta)x$.

   (a) Show that $\tan\alpha = \pm\dfrac{r(\theta)}{r'(\theta)}$.

   (b) For $r(\theta) = \cos^2\dfrac{\theta}{2}$ and $\alpha = \dfrac{\pi}{3}$, find the polar coordinates of such P's.

Ans:

6. Let C be the conic in $\mathbb{R}^2$ given by the equation $x^2 + y^2 = (2 - x + y)^2$.

   (a) Find the polar equation $r = f(\theta)$ for C and its eccentricity.

Ans:

   (a) Find the polar coordinates of the center, the vertices and the foci, and the polar equations of directrices for C.

Ans:

7. For each of the following statements, prove if it is true or give an example to disprove.

   (a) For non-zero vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^n$, $\|\boldsymbol{b}\|\boldsymbol{a} + \|\boldsymbol{a}\|\boldsymbol{b}$ bisects the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$.

   (b) For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$, $\boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \times (\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{b}$.

8. Consider the line ℓ containing two points $(-1, 1, 1)$ and $(0, 2, 1)$ and the line m containing two points $(2, 1, 1)$ and $(2, -1, -1)$.

   (a) Find the distance between ℓ and m.

   Ans:

   (b) Find a vector parametrization of the line that intersects perpendicularly both ℓ and m.

   Ans:

9. Consider the region in $\mathbb{R}^3$ defined by the inequalities

$$x^2 + y^2 + z^2 \le 4 \quad \text{and} \quad -1 \le z \le 1$$

in Cartesian coordinates.

(a) Express the region as a union of the following forms (called elementary regions) in cylindrical coordinates:

$$f_1(\theta, z) \le r \le f_2(\theta, z), \quad g_1(z) \le \theta \le g_2(z), \quad a \le z \le b$$

where f_1, f_2, g_1, and g_2 are appropriate functions that can be constants and a, b are constants.

Ans:

(b) Express the region as a union of the following forms (called elementary regions) in spherical coordinates:

$$f_1(\phi,\theta) \le \rho \le f_2(\phi,\theta), \quad g_1(\theta) \le \phi \le g_2(\theta), \quad a \le \theta \le b$$

where f_1, f_2, g_1, and g_2 are appropriate functions that can be constants and a, b are constants.

Ans:

10. Let $\boldsymbol{x}$ be a point in $\mathbb{R}^3$ given by $(\rho, \varphi, \theta) = (1, \frac{\pi}{6}, \frac{\pi}{6})$ in the spherical coordinates.

    (a) Find the standard bases at $\boldsymbol{x}$ for cylindrical coordinates.

    Ans:

    (b) Find the standard bases at $\boldsymbol{x}$ for spherical coordinates.

    Ans: