2021 Fall MAS 101
Chapter 10: Infinite Sequences and Series

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- $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$
- $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \cdots$

Representing Sequences

- A sequence is a list of numbers

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

in a given order.

Convergence and Divergence

- $\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\}$
- $\{\sqrt{1}, \sqrt{2}, \sqrt{3}, \ldots, \sqrt{n}, \ldots\}$
- $\{1, -1, 1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$

Convergence and Divergence (cont'd)

Definition 1

The sequence $\{a_n\}$ **converges** *to the number* L *if for every positive number* ϵ *there corresponds to an integer* N *such that for all* n,

$$n > N \qquad \Rightarrow \qquad |a_n - L| < \epsilon.$$

If no such number L *exists, we say that* $\{a_n\}$ **diverges**. *If* $\{a_n\}$ *converges to* L, *we write* $\lim_{n\to\infty} a_n = L$, *or simply* $a_n \to L$, *and call* L *the* **limit** *of the sequence.*

- $\lim_{n\to\infty} \frac{1}{n}$
- $\{1, -1, 1, -1, \ldots, (-1)^{n+1}, \ldots\}$
- $\lim_{n\to\infty} \sqrt{n}$

Convergence and Divergence (cont'd)

Definition 2

The sequence $\{a_n\}$ **diverges to infinity** *if for every number* M *there is an integer* N *such that for all* n *larger than* N, $a_n > M$. *If this condition holds we write*

$$\lim_{n\to\infty} a_n = \infty, \quad \text{or} \quad a_n \to \infty.$$

Similarly, if for every number m *there is an integer* N *such that for all* $n > N$ *we have* $a_n < m$, *then we say* $\{a_n\}$ **diverges to negative infinity** *and write*

$$\lim_{n\to\infty} a_n = -\infty, \quad \text{or} \quad a_n \to -\infty.$$

- A sequence may diverge without diverging to infinity or negative infinity.

Calculating Limits of Sequences

Theorem 1

Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

- *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
- *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
- *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ *(any number k)*
- *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
- *Quotient Rule:* $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$ *if $B \neq 0$*

- Do sequences $\{a_n\}$ and $\{b_n\}$ have limits if their sum $\{a_n + b_n\}$ has a limit?

Calculating Limits of Sequences (cont'd)

Theorem 2 (The Sandwich Theorem for Sequences)

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

- $\frac{\cos n}{n} \to 0$

Calculating Limits of Sequences (cont'd)

Theorem 3 (The Continuous Function Theorem for Sequences)

Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

- $2^{1/n}$

Using L'Hôpital's Rule

Theorem 4

Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x \to \infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n \to \infty} a_n = L.$$

- $\lim_{n \to \infty} \frac{\ln n}{n}$

Commonly Occurring Limits

Theorem 5

The following six sequences converge to the limits listed below:

- $\lim_{n\to\infty} \frac{\ln n}{n}$
- $\lim_{n\to\infty} \sqrt[n]{n}$
- $\lim_{n\to\infty} x^{1/n}$
- $\lim_{n\to\infty} x^n$
- $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$
- $\lim_{n\to\infty} \frac{x^n}{n!}$

Recursive Definitions

- Sequences are often defined **recursively** by giving

1. The value(s) of initial term or terms, and
2. A rule, called a **recursion formula**, for calculating any later term form terms that precede it.

Bounded Monotonic Sequences

Definition 3

A sequence $\{a_n\}$ is **bounded from above** *if there exists a number M such that $a_n \leq M$ for all n. The number M is an* **upper bound** *for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the* **least upper bound** *for $\{a_n\}$.*
If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is **bounded**. *If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an* **unbounded** *sequence.*

- $1, 2, 3, \ldots, n, \ldots$
- $-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, \ldots, -\frac{n}{n+1}, \ldots$

- Is a convergent sequence bounded?
- Is a bounded sequence convergent?

Bounded Monotonic Sequences (cont'd)

Definition 4

A sequence $\{a_n\}$ is **nondecreasing** *if $a_n \leq a_{n+1}$ for all n. That is, $a_1 \leq a_2 \leq a_3 \leq \dots$ The sequence is nonincreasing if $a_n \geq a_{n+1}$ for all n. The sequence $\{a_n\}$ is* **monotonic** *if it is either nondecreasing and nonincreasing.*

- $1, 2, 3, \dots, n, \dots$
- $-\frac{1}{2}, -\frac{2}{3}, -\frac{3}{4}, \dots, -\frac{n}{n+1}, \dots$
- $1, -1, 1, -1, \dots$

Bounded Monotonic Sequences (cont'd)

Theorem 6 (The Monotonic Sequence Theorem)

If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

- Is a convergent and bounded sequence monotonic?
- Does a nondecreasing and bounded sequence converge?

Infinite Series

Definition 5

Given a sequence of numbers $\{a_n\}$*, an expression of the form*

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series***. The number* a_n *is the* n**th term** *of the series. The sequence* $\{s_n\}$ *defined by*

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad \cdots,$$
$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k, \quad \cdots$$

is the **sequence of partial sums** *of the series, the number* s_n *being the* n**th partial sum***. If the sequence of partial sums converges to a limit* L*, we say that the series* **converges** *and that its* **sum** *is* L*.*

Geometric Series

- **Geometric seris** are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

  in which a and r are fixed real numbers and $a \neq 0$.

- If $|r| < 1$,
- If $|r| \geq 1$,

The nth-Term Test for a Divergent Series

Theorem 7

If $\sum_{n=1}^{\infty} a_n$ *converges, then*

Combining Series

Theorem 8

If $\sum a_n = A$ *and* $\sum b_n = B$ *are convergent series, then*

- *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
- *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
- *Constant Multiple Rule:* $\sum k a_n = k \sum a_n = kA \quad$ *(any number* k*).*

Adding or Deleting Terms

- One can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence.

Nondecreasing Partial Sums

Corollary 1

A series $\sum_{n=1}^{\infty} a_n$ of nonnegative terms converges if and only if its partial sums are

- $\sum_{n=1}^{\infty} \frac{1}{n}$

Nondecreasing Partial Sums (cont'd)

- How about $\sum_{n=1}^{\infty} \frac{1}{n^2}$?

The Integral Test

Theorem 9 (The Integral Test)

Let $\{a_n\}$ *be a sequence of positive terms. Suppose that* $a_n = f(n)$, *where* f *is a continuous, positive, decreasing function of* x *for all* $x \geq N$ *(*N *a positive integer). Then the series* $\sum_{n=N}^{\infty} a_n$ *and the integral* $\int_N^{\infty} f(x)dx$ *both converge or both diverge.*

The Integral Test (cont'd)

- $\sum_{n=1}^{\infty} \frac{1}{n^p}$

The Integral Test (cont'd)

- $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\ln n}$

The Integral Test (cont'd)

- $\sum_{n=1}^{\infty} ne^{-n^2}$

Error Estimation

- Bounds for the Remainder in the integral Test:

  Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that $\sum a_n$ converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)dx \leq R_n \leq \int_{n}^{\infty} f(x)dx.$$

- Estimate the sum of the series $\sum(1/n^2)$ using the inequalities using $n = 10$.

Comparison Tests

Theorem 10

Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with nonnegative terms. Suppose that for some integer N

$$d_n \le a_n \le c_n \quad \text{for all} \quad n > N.$$

- *If $\sum c_n$ converges, then $\sum a_n$ also converges.*
- *If $\sum d_n$ diverges, then $\sum a_n$ also diverges.*

- $\sum_{n=1}^{\infty} \frac{4}{2n-1}$
- $\sum_{n=1}^{\infty} \frac{1}{n!}$

The Limit Comparison Test

Theorem 11

Suppose that $a_n > 0$ *and* $b_n > 0$ *for all* $n \geq N$ *(*N *an integer).*

- *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, *then* $\sum a_n$ *and* $\sum b_n$ *both converge or both diverge.*
- *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
- *If* $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

- $\sum_{n=1}^{\infty} \frac{2n+1}{(n+1)^2}$
- $\sum_{n=1}^{\infty} \frac{1}{2^n-1}$

Absolute Convergence

Definition 6

A series $\sum a_n$ **converges absolutely** *(is* **absolutely convergent***) if the corresponding series of absolute values, $\sum |a_n|$, converges.*

Theorem 12 (The Absolute Convergence Test)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

- $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^{1.5}}$

Tests for Absolute Convergence

- The ratio test
- The root test

The Ratio Test

Theorem 13 (The Ratio Test)

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then (a) the series converges absolutely if $\rho < 1$, (b) the series diverges if $\rho > 1$ or ρ is infinite, (c) the test is inconclusive if $\rho = 1$.

- $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^n}{n!}$

The Root Test

- $\sum a_n$, where $a_n = \begin{cases} n/2^n, & n \text{ odd}, \\ 1/2^n, & n \text{ even}. \end{cases}$

The Root Test (cont'd)

Theorem 14 (The Root Test)

Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \rho.$$

Then (a) the series converges absolutely if $\rho < 1$, (b) the series diverges if $\rho > 1$ or ρ is infinite, (c) the test is inconclusive if $\rho = 1$.

- $\sum a_n$, where $a_n = \begin{cases} n/2^n, & n \text{ odd}, \\ 1/2^n, & n \text{ even}. \end{cases}$
- $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

Alternating Series

- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Alternating Series Test

Theorem 15 (The Alternating Series Test)

The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

- *The u_n's are all positive.*
- *The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.*
- $u_n \to 0$.

- $u_n = \frac{1}{n}$
- $u_n = \frac{10n}{n^2+16}$

Alternating Series Estimation

Theorem 16 (The Alternating Series Estimation Theorem)

If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 15, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

- $\sum_{n=0}^{\infty}(-1)^n\frac{1}{2^n}$

Conditional Convergence

Definition 7

A convergent series that is not absolutely convergent is **conditionally convergent**.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$

Rearranging Series

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Theorem 17 (The Rearrangement Theorem for Absolutely Convergent Series)

If $\sum_{n=1}^{\infty} a_n$ *converges absolutely, and* $b_1, b_2, \ldots, b_n, \ldots$ *is any arrangement of the sequence* $\{a_n\}$*, then* $\sum b_n$ *converges absolutely and*

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n.$$

Summary of Tests

- **The nth-Term Test:** If it is not true that $a_n \to 0$, then the series diverges.
- **Geometric series:** $\sum ar^n$ converges if $|r| < 1$; otherwise it diverges.
- p-**series:** $\sum 1/n^p$ converges if $p > 1$; otherwise it diverges.
- **Series with nonnegative terms:** Try the Integral Test or try comparing to a known series with the Comparison Test or the Limit Comparison Test. Try the Ratio or Root Test.
- **Series with some negative terms:** Does $\sum |a_n|$ converge by the Ratio or Root Test, or by another of the tests listed above? Remember, absolute convergence implies convergence.
- **Alternating series:** $\sum a_n$ converges if the series satisfies the conditions of the Alternating Series Test.

Power Series

- Power series: infinite series of powers of some variable (*e.g.*, x)

- Section 10.8 Taylor and Maclaurin Series
- Section 10.9 Convergence of Taylor Series

Power Series and Convergence

Definition 8

A power series about $x = 0$ *is a series of the form*

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots .$$

A power series about $x = a$ *is a series of the form*

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots + c_n (x-a)^n + \cdots$$

in which the **center** a *and the* **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ *are constants.*

Power Series and Convergence (cont'd)

- Letting $c_n = 1$ yields

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

Power Series and Convergence (cont'd)

- For $-1 < x < 1$,

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots$$

- $P_n(x) = \sum_{i=0}^{n} x^i$

Power Series and Convergence (cont'd)

- Letting $c_n = \left(-\frac{1}{2}\right)^n$ and $a = 2$ yields

$$1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \cdots$$

Power Series and Convergence (cont'd)

- For what values of x do the following power series converge?

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$$

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1}$$

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

$$\sum_{n=0}^{\infty}n!x^n$$

Power Series and Convergence (cont'd)

Theorem 18

If the power series $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots$ *converges at* $x = d \neq 0$*, then it converges absolutely for all* x *with* $|x| < |d|$*. If the series diverges at* $x = d$*, then it diverges for all* x *with* $|x| > |d|$*.*

- For series of the form $\sum c_n (x-a)^n$ we can replace $x - a$ by x'.

The Radius of Convergence of a Power Series

Corollary 2

The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

- *There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.*
- *The series converges absolutely for every x ($R = \infty$).*
- *The series converges at $x = a$ and diverges elsewhere ($R = 0$).*

The Radius of Convergence of a Power Series (cont'd)

- R is called the **radius of convergence** of the power series, and the interval of radius R centered at $x = a$ is called the **interval of convergence**.

How to Test a Power Series for Convergence

- Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely. Ordinarily, this is an open interval

$$|x - a| < R \quad \text{or} \quad a - R < x < a + R.$$

- If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint. Use a Comparison Test, the Integral Test, or the Alternating Series Test.
- If the interval of absolute convergence is $a - R < x < a + R$, the series diverges for $|x - a| > R$ (it does not even converge conditionally) because the nth term does not approach zero for those vales of x.

How to Test a Power Series for Convergence (cont'd)

Theorem 19

Suppose that $c_n \neq 0$ for all sufficiently large n and that the limit

$$R = \lim_{n \to \infty}$$

exists or diverges to inifinity. Then the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence R.

Operations on Power Series

Theorem 20

If $\sum a_n = A$ *and* $\sum b_n = B$ *are convergent series, then*

- *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
- *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
- *Constant Multiple Rule:* $\sum ka_n = k \sum a_n = kA \quad$ *(any number* k*).*

Operations on Power Series (cont'd)

Theorem 21 (The Series Multiplication Theorem for Power Series)

If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ *and* $B(x) = \sum_{n=0}^{\infty} b_n x^n$ *converge absolutely for* $|x| < R$, *and*

$$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ *converges absolutely to* $A(x)B(x)$ *for* $|x| < R$*:*

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

Operations on Power Series (cont'd)

Theorem 22

If $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} c_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

Operations on Power Series (cont'd)

Theorem 23 (The Term-by-Term Differentiation Theorem)

If $\sum c_n(x-a)^n$ has radius of convergence $R>0$, it defines a function

$$f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{on the interval} \quad a-R<x<a+R.$$

The function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x)=\sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$
$$f''(x)=\sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a-R<x<a+R$.

Operations on Power Series (cont'd)

Find series for $f'(x)$ and $f''(x)$.

- $f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $-1 < x < 1$
- $f(x) = \sum_{n=1}^{\infty} \frac{\sin(n!x)}{n^2}$ for all x

Operations on Power Series (cont'd)

Theorem 24 (The Term-by-Term Integration Theorem)

Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

converges for $a - R < x < a + R$ $(R > 0)$. *Then*

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ *and*

$$\int f(x)dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

Operations on Power Series (cont'd)

- Identify the function $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$ for $-1 < x < 1$.
- Identify the function $f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$ for $-1 < x < 1$.
- Identify the function $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Represeting Function with a Power Series

- $f(x) = \frac{1}{1-x}$

- Is a power series continuous within its interval of convergence?
- If a function has derivatives of all orders on an interval, can it be expressed as a power series?

Series Representations

- Assume that $f(x)$ is the sum of a power series about $x = a$,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n (x-a)^n \\ &= a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n + \cdots \end{aligned}$$

  with a positive radius of convergence.

Taylor and Maclaurin Series

Definition 9

Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** *is*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

The **Maclaurin series of f** *is the Taylor series generated by f at $x = 0$, or*

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

Taylor and Maclaurin Series (cont'd)

- Find the Taylor series generated by $f(x) = \frac{1}{x}$ at $a = 3$.

Taylor Polynomials

- The linearization of a differentiable function f at a point a:

$$P_1(x) =$$

Taylor Polynomials (cont'd)

Definition 10

Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order** n *generated by f at $x = a$ is the polynomial*

$$\begin{aligned} P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \end{aligned}$$

Taylor Polynomials (cont'd)

Find the Taylor series and the Taylor polynomials generated by $f(x)$ at $x = 0$.

- $f(x) = e^x$
- $f(x) = \cos x$
- $f(x) = \begin{cases} 0, & x = 0, \\ e^{-1/x^2}, & x \neq 0. \end{cases}$

Taylor Polynomials (cont'd)

- For what values of x can we normally expect a Taylor series to converge to its generating function?
- How accurately do a function's Taylor polynomials approximate the function on a given interval.

Taylor's Theorem

Theorem 25 (Taylor's Theorem)

If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$\begin{aligned} f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots \\ + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1} \end{aligned}$$

Taylor's Theorem

Theorem 26 (The Mean Value Theorem)

Suppose $y = f(x)$ *is continuous over a closed interval* $[a, b]$ *and differentiable on the interval's interior* (a, b)*. Then there is at least one point* c *in* (a, b) *at which*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Theorem 27 (Rolle's Theorem)

Suppose that $y = f(x)$ *is continuous over the closed interval* $[a, b]$ *and differentiable at every point of its interior* (a, b)*. If* $f(a) = f(b)$*, then there is at least one number* c *in* (a, b) *at which* $f'(c) = 0$*.*

Taylor's Formula

- If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots \\ + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \quad \text{for some } c \text{ between } a \text{ and } x.$$

Taylor's Formula (cont'd)

- If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

- Does the Taylor series generated by $f(x) = e^x$ at $x = 0$ converge to $f(x)$ for every real value of x?

Estimating the Remainder

Theorem 28 (The Remainder Estimation Theorem)

If there is a positive constant M such that $|f^{(n+1)}(t)| \leq M$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality holds for every n and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Using Taylor Series

- Find the first few terms of the Taylor series for $e^x \cos x$ using power series operations.

The Binomial Series for Powers and Roots

- The Taylor series generated by $f(x) = (1 + x)^m$, when m is constant, is

The Binomial Series

- For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!}$$

The Binomial Series

- $(1+x)^{-1}$
- $(1+x)^{1/2}$

Frequently Used Taylor Series

- $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1$
- $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n, \quad |x| < 1$
- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad |x| < \infty$
- $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \quad |x| < \infty$
- $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \quad |x| < \infty$
- $\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \quad -1 < x \leq 1$
- $\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \quad |x| \leq 1$

Evaluating Nonelementary Integrals

- Express $\sin x^2$ as a power series.
- Express $\int \sin x^2 dx$ as a power series.

Evaluating Nonelementary Integrals (cont'd)

- Estimate $\int_0^1 \sin x^2 dx$ with an error of less than 0.001.

Arctangents

- Recall: A series for $\tan^{-1} x$ was found by differentiating to get

$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

and then integrating to get

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

- This uses the term-by-term integration theorem, which we have not proved in class.

Arctangents (cont'd)

- Consider

$$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + \cdots + (-1)^n t^{2n} + \, ?$$

- Integrating both sides from $t = 0$ to $t = x$ gives

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + R_n(x)$$

Arctangents (cont'd)

- Leibniz's formula:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots + \frac{(-1)^n}{2n+1} + \cdots .$$

Evaluating Indeterminate Forms

- We can sometimes evaluate indeterminate forms by expressing the functions involved as Taylor series.
- $\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$

Evaluating Indeterminate Forms (cont'd)

- $\lim_{x\to 0} \dfrac{\cos(x^2) - e^{x^4}}{\sin(x^4)}$