2021 Fall MAS 101
Chapter 11: Parametric Equations and Polar Coordinates

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1. 11.1 Parametrizations of Plane Curves
2. 11.2 Calculus with Parametric Curves
3. 11.3 Polar Coordinates
4. 11.4 Graphing Polar Coordinate Equations
5. 11.5 Areas and Lengths in Polar Coordinates

Curves in the Plane

- Curve as the graph of a function or equation
- Curve as the path of a moving particle whose position is chanching over time

Parametric Equations

Definition 1

If x and y are given as functions

$$x = f(t), \quad y = g(t),$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. *The equations are* **parametric equations** *for the curve.*

- The variable t is a **parameter** for the curve, and its domain I is the **parameter interval**. If I is a closed interval, $a \leq t \leq b$, the point $(f(a), g(a))$ is the **initial point** of the curve and $(f(b), g(b))$ is the **terminate point**.
- When we give parametric equations and a parameter interval for a curve, we say that we have **parameterized** the curve. The equations and interval together constitute a **parametrization** of the curve.

Parametric Equations (cont'd)

- Sketch the curve defined by the parametric equations

$$x = t^2, \quad y = t + 1, \quad -\infty < t < \infty.$$

- $x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq \pi.$

Parametric Equations (cont'd)

- Parameterize the unit circle $x^2 + y^2 = 1$.
- Parameterize the quadratic function $y = x^2$.

Parametric Equations (cont'd)

- $x = \sqrt{t}, \quad y = t, \quad t \geq 0$
- $x = t, \quad y = t^2, \quad -\infty < t < \infty$

Parametric Equations (cont'd)

- Sketch and identify the path traced by the point $P(x, y)$ if

$$x = t + \frac{1}{t}, \quad y = t - \frac{1}{t}, \quad t > 0$$

Cycloids

- A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a **cycloid**.

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Tangents and Areas

- Find slopes, lengths, and areas associated with parametrized curves

$$x = f(t) \quad \text{and} \quad y = g(t).$$

Tangents and Areas (cont'd)

- If $\frac{dy}{dt}$, $\frac{dx}{dt}$, $\frac{dy}{dx}$ exist and $\frac{dx}{dt} \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Tangents and Areas (cont'd)

- If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $\frac{dx}{dt} \neq 0$ and $y' = \frac{dy}{dx}$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

Tangents and Areas (cont'd)

- Find the tangent to the curve

$$x = \sec t, \quad y = \tan t, \quad \frac{-\pi}{2} < t < \frac{\pi}{2}$$

at the point $(\sqrt{2}, 1)$, where $t = \pi/4$.

Tangents and Areas (cont'd)

- Find the tangent and $\frac{d^2y}{dx^2}$, as a function of t, to the curve $x = t - t^2$ and $y = t - t^3$.

Length of a Parametrically Defined Curve

- Let C be a **smooth curve** given parametrically by the equations

$$x = f(t) \quad \text{and} \quad y = g(t), \quad a \leq t \leq b,$$

where we assume that f and g are **continuously differentiable** on the interval $[a, b]$, and that $f'(t)$ and $g'(t)$ are not simultaneously zero.

Length of a Parametrically Defined Curve (cont'd)

Definition 2

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of** *C is the definite integral*

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2} dt$$

- Find the length of the circle of radius r.

Length of a Parametrically Defined Curve (cont'd)

- Find the length of the astroid

$$x = \cos^3 t \quad \text{and} \quad y = \sin^3 t, \quad 0 \le t \le 2\pi$$

Length of a Parametrically Defined Curve (cont'd)

- Find the perimeter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Length of a Curve $y = f(x)$

- Consider a continuously differentiable function $y = f(x)$, $a \leq x \leq b$.

The Arc Length Differential

- The arc length function for $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$ is

$$s(t) =$$

- Then,

$$\frac{ds}{dt} =$$

The Arc Length Differential (cont'd)

- Calculate the centroid of the first-quadrant arc of the astroid

$$x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \le t \le 2\pi.$$

where the density is $\delta = 1$.

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Definition of Polar Coordinates

- A point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from an **origin** O to P, and θ gives the directed angle from the **initial ray** to ray OP.

  We then label the point P as

$$P(r, \theta).$$

- Find all the polar coordinates of the point $P(3, \pi/4)$.

Polar Equations and Graphs

- $r = 1$
- $\theta = \frac{\pi}{6}$
- $1 \leq r \leq 2$ and $0 \leq \theta \leq \frac{\pi}{2}$
- $-3 \leq r \leq 2$ and $\theta = \frac{\pi}{4}$
- $\frac{2\pi}{3} \leq \theta \leq \frac{5\pi}{6}$

Relating Polar and Cartesian Coordinates

- Equations relating polar and Cartesian coordinates

$$x = r\cos\theta, \quad y = r\sin\theta,$$
$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$$

- $x^2 + (y-3)^2 = 9$
- $r\cos\theta = -4$
- $r^2 = 4r\cos\theta$
- $r = \frac{4}{2\cos\theta - \sin\theta}$

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Symmetry

- Symmetry Tests for Polar Graphs in the Cartesian xy-Plane

1. Symmetry about the x-axis
2. Symmetry about the y-axis
3. Symmetry about the origin

Slope

- Slope of the curve $r = f(\theta)$ in the Cartesian xy-plane

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} =$$

provided $\frac{dx}{d\theta} \neq 0$ at (r, θ).

- If the curve $r = f(\theta)$ passes through the origin,

Slope (cont'd)

- Graph the curve $r = 1 - \cos\theta$ in the Cartesian xy-plane.

Converting a Graph from the $r\theta$- to xy-plane

1. Graph the function $r = f(\theta)$ in the Cartesian $r\theta$-plane
2. Use that Cartesian graph as a “table” and guide to sketch the polar coordinate graph in the xy-plane.

- Graph the curve $r^2 = \sin 2\theta$ in the Cartesian xy-plane.

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Area in the Plane

- Area of the fan-shaped region between the origin and the curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$

$$A =$$

- Area differential

$$dA =$$

Area in the Plane (cont'd)

- Area of the region $0 \leq r_1(\theta) \leq r \leq r_2(\theta)$, $\alpha \leq \theta \leq \beta$

$$A =$$

Area in the Plane (cont'd)

- Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

Area in the Plane (cont'd)

- Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = \cos\theta$.

Length of a Polar Curve

- Polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \leq \theta \leq \beta$.

Length of a Polar Curve (cont'd)

- If $r = f(\theta)$ has a continuous derivative for $\alpha \leq \theta \leq \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L =$$

Length of a Polar Curve (cont'd)

- Find the length of the cardioid $r = 1 - \cos\theta$.

Length of a Polar Curve (cont'd)

- Find the length of $r = \theta^2$ for $0 \leq \theta \leq \sqrt{5}$.