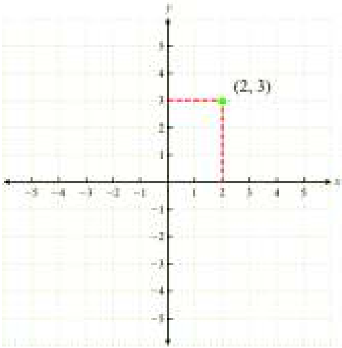
2021 Fall MAS 101
Chapter 12: Vectors and the Geometry of Space

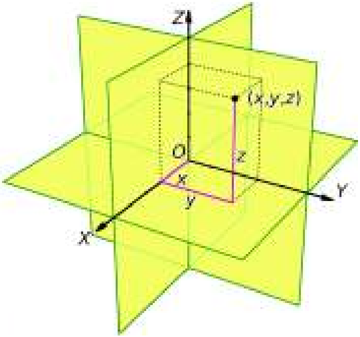
Donghwan Kim

KAIST

Nov/25,30, Dec/7,9, 2021

Three-Dimensional Coordinate Systems





Distance and Spheres in Space

- The distance between $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$

$$|P_1 P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Distance and Spheres in Space (cont'd)

- The standard equation for the sphere of radius a and center (x_0, y_0, z_0)

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = a^2$$

- Find the center and radius of the sphere

$$x^2 + y^2 + z^2 + 3x - 4z + 1 = 0$$

Component Form

Definition 1

The vector represented by the directed line segment $\overrightarrow{AB}$ *has* **initial point** A *and* **terminal point** B *and its* **length** *is denoted by* $|\overrightarrow{AB}|$. *Two vectors are* **equal** *if they have the same length and direction.*

Component Form (cont'd)

Definition 2

If $\boldsymbol{v}$ *is a* **two-dimensional** *vector in the plane equal to the vector with initial point at the origin and terminal point* (v_1, v_2), *then the* **component form** *of* $\boldsymbol{v}$ *is*

$$\boldsymbol{v} = \langle v_1, v_2 \rangle$$

If $\boldsymbol{v}$ *is a* **three-dimensional** *vector equal to the vector with initial point at the origin and terminal point* (v_1, v_2, v_3), *then the* **component form** *of* $\boldsymbol{v}$ *is*

$$\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$$

Component Form (cont'd)

- The **magnitude** or **length** of the vector $\boldsymbol{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\boldsymbol{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- Find the (a) component form and (b) length of the vector with initial point $P(-3, 4, 1)$ and terminal point $Q(-5, 2, 2)$.

Vector Algebra Operations

Definition 3

Let $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ *be vectors with* k *a scalar.*

- **Addition:** $\boldsymbol{u} + \boldsymbol{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
- **Scalar multiplication:** $k\boldsymbol{u} = \langle ku_1, ku_2, ku_3 \rangle$

- Let $\boldsymbol{u} = \langle -1, 3, 1 \rangle$ and $\boldsymbol{v} = \langle 4, 7, 0 \rangle$. Find the components of (a) $2\boldsymbol{u} - 3\boldsymbol{v}$ and (b) $|2\boldsymbol{u}|$.

Properties of Vector Operations

- Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ be vectors and a, b be scalars.

1. $\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$
2. $(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$
3. $\boldsymbol{u}+\mathbf{0}=\boldsymbol{u}$
4. $\boldsymbol{u}+(-\boldsymbol{u})=\mathbf{0}$
5. $0\boldsymbol{u}=\mathbf{0}$
6. $1\boldsymbol{u}=\boldsymbol{u}$
7. $a(b\boldsymbol{u})=(ab)\boldsymbol{u}$
8. $a(\boldsymbol{u}+\boldsymbol{v})=a\boldsymbol{u}+a\boldsymbol{v}$
9. $(a+b)\boldsymbol{u}=a\boldsymbol{u}+b\boldsymbol{u}$

Unit Vectors

- A vector $\boldsymbol{v}$ of length 1 is called a **unit vector**.
- The **standard unit vectors** are

$$\boldsymbol{i} = \langle 1, 0, 0 \rangle, \quad \boldsymbol{j} = \langle 0, 1, 0 \rangle, \quad \boldsymbol{k} = \langle 0, 0, 1 \rangle.$$

- Any vector $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ can be written as a *linear combination* of the standard unit vectors.
- Find a unit vector $\boldsymbol{u}$ in the direction of the vector from $P_1(1, 0, 1)$ to $P_2(3, 2, 0)$.

Unit Vectors (cont'd)

- If $\boldsymbol{v} = 3\boldsymbol{i} - 4\boldsymbol{j}$ is a velocity vector, express $\boldsymbol{v}$ as a product of its speed times its direction of motion.

Midpoint of a Line Segment

- The **midpoint** M of the line segment joining points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is the point

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$

Applications

- A jet airliner, flying due east at 800 km/h in still air, encounters a 110 km/h tailwind blowing in the direction 60^o north of east. The airplane holds its compass heading due east but, because of the wind, acquires a new ground speed and direction. What are they?

Applications (cont'd)

- A 75-N weight is suspended by two wires, as shown in Figure 12.18a. Find the forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$ acting in both wires.

Angle Between Vectors

Definition 4

The **dot product** $\boldsymbol{u} \cdot \boldsymbol{v}$ *(“***u dot v***”) of vectors* $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ *is the scalar*

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

- $\langle 1, -2, -1 \rangle \cdot \langle -6, 2, -3 \rangle$
- $\left(\frac{1}{2}\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}\right) \cdot (4\boldsymbol{i} - \boldsymbol{j} + 2\boldsymbol{k})$

Angle Between Vectors (cont'd)

Theorem 1 (Angle between two vectors)

The angle θ between two nonzero vectors $\boldsymbol{u} = \langle u_1, u_2, u_3 \rangle$ and $\boldsymbol{v} = \langle v_1, v_2, v_3 \rangle$ is given by

$$\theta = \cos^{-1}\left(\frac{\boldsymbol{u} \cdot \boldsymbol{v}}{|\boldsymbol{u}||\boldsymbol{v}|}\right)$$

- Find the angle between $\boldsymbol{u} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ and $\boldsymbol{v} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$.

Orthogonal Vectors

Definition 5

Vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are orthogonal if $\boldsymbol{u} \cdot \boldsymbol{v} = 0$.

- $\boldsymbol{u} = \langle 3, -2 \rangle$ and $\boldsymbol{v} = \langle 4, 6 \rangle$

Dot Product Properties and Vector Projections

- If $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$ are any vectors and c is a scalar, then

1. $\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{u}$
2. $(c\boldsymbol{u}) \cdot \boldsymbol{v} = \boldsymbol{u} \cdot (c\boldsymbol{v}) = c(\boldsymbol{u} \cdot \boldsymbol{v})$
3. $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$
4. $\boldsymbol{u} \cdot \boldsymbol{u} = |\boldsymbol{u}|^2$
5. $\mathbf{0} \cdot \boldsymbol{u} = 0$

Dot Product Properties and Vector Projections (cont'd)

- The **vector projection** of $\boldsymbol{u}$ onto $\boldsymbol{v}$ is denoted by

$$\mathrm{proj}_{\boldsymbol{v}}\boldsymbol{u}$$

Dot Product Properties and Vector Projections (cont'd)

- Find the vector projection of $\boldsymbol{u} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ onto $\boldsymbol{v} = \boldsymbol{i} - 2\boldsymbol{j} - 2\boldsymbol{k}$ and the scalar component of $\boldsymbol{u}$ in the direction of $\boldsymbol{v}$.

Dot Product Properties and Vector Projections (cont'd)

- Verify that $\boldsymbol{u} - \mathrm{proj}_{\boldsymbol{v}}\boldsymbol{u}$ is orthogonal to $\mathrm{proj}_{\boldsymbol{v}}\boldsymbol{u}$.

Work

Definition 6

*The **work** done by a constant force $\boldsymbol{F}$ acting through a disclacement $\boldsymbol{D} = \overrightarrow{PQ}$ is*

$$W = \boldsymbol{F} \cdot \boldsymbol{D}$$

- If $|\boldsymbol{F}| = 40N$, $|D| = 3m$ and $\theta = 60^o$, compute the work done by $\boldsymbol{F}$ in acting from P to Q.

The Cross Product of Two Vectors in Space

Definition 7

The **cross product** $\boldsymbol{u} \times \boldsymbol{v}$ *("***u cross v***") is the vector*

$$\boldsymbol{u} \times \boldsymbol{v} = (|\boldsymbol{u}||\boldsymbol{v}| \sin\theta)\boldsymbol{n}.$$

- Nonzero vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are parallel if and only if $\boldsymbol{u} \times \boldsymbol{v} = \mathbf{0}$.

Properties of the Cross Product

- If $\boldsymbol{u}, \boldsymbol{v}$, and $\boldsymbol{w}$ are any vectors and r, s are scalars, then

1. $(r\boldsymbol{u}) \times (s\boldsymbol{v}) = (rs)(\boldsymbol{u} \times \boldsymbol{v})$
2. $\boldsymbol{u} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \times \boldsymbol{v} + \boldsymbol{u} \times \boldsymbol{w}$
3. $\boldsymbol{v} \times \boldsymbol{u} = -(\boldsymbol{u} \times \boldsymbol{v})$
4. $(\boldsymbol{v} + \boldsymbol{w}) \times \boldsymbol{u} = \boldsymbol{v} \times \boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{u}$
5. $\mathbf{0} \times \boldsymbol{u} = \mathbf{0}$
6. $\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}$

$|\boldsymbol{u} \times \boldsymbol{v}|$ Is the Area of a Parallelogram

- Because $\boldsymbol{n}$ is a unit vector, the magnitude of $\boldsymbol{u} \times \boldsymbol{v}$ is

$$|\boldsymbol{u} \times \boldsymbol{v}| = |\boldsymbol{u}||\boldsymbol{v}||\sin\theta||\boldsymbol{n}| = |\boldsymbol{u}||\boldsymbol{v}|\sin\theta$$

Determinant Formula for $\boldsymbol{u} \times \boldsymbol{v}$

- Calculate $\boldsymbol{u} \times \boldsymbol{v}$ from the components of $\boldsymbol{u} = u_1\boldsymbol{i} + u_2\boldsymbol{j} + u_3\boldsymbol{k}$ and $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$

- Find $\boldsymbol{u} \times \boldsymbol{v}$ and $\boldsymbol{v} \times \boldsymbol{u}$ if $\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$ and $\boldsymbol{v} = -4\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}$.

Determinant Formula for $\boldsymbol{u} \times \boldsymbol{v}$ (cont'd)

- Find a unit vector perpendicular to the plane of $P(1,-1,0)$, $Q(2,1,-1)$ and $R(-1,1,2)$.
- Find the area of triangle with vertices P, Q and R.

Triple Scalar or Box Product

- Find the volume of the paralleleppiled (parallelogram-sided box) defined by three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$.

Triple Scalar or Box Product (cont'd)

- The product $(\boldsymbol{u} \times \boldsymbol{v}) \cdot \boldsymbol{w}$ is called the **triple scalar product** of $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$.

- Find the volumn of the parallelepiped determined by $\boldsymbol{u} = \boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k}$, $\boldsymbol{v} = -2\boldsymbol{i} + 3\boldsymbol{k}$, and $\boldsymbol{w} = 7\boldsymbol{j} - 4\boldsymbol{k}$.

Lines and Line Segments in Space

- **A vector equation for the Line** L **through** $P_0(x_0, y_0, z_0)$ **parallel to** $\boldsymbol{v}$ is

$$\boldsymbol{r}(t) = \qquad\qquad , \quad -\infty < t < \infty,$$

where $\boldsymbol{r}$ is the position vector of a point $P(x, y, z)$ on L and $\boldsymbol{r}_0$ is the position vector of $P_0(x_0, y_0, z_0)$.

Lines and Line Segments in Space (cont'd)

- **The standard parametrization of the line through** $P_0(x_0, y_0, z_0)$ **parallel to** $\boldsymbol{v} = v_1\boldsymbol{i} + v_2\boldsymbol{j} + v_3\boldsymbol{k}$ is

$$x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3, \quad -\infty < t < \infty.$$

- Find parametric equations for the line through $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

Lines and Line Segments in Space (cont'd)

- Parametrize the line segment joining the points $P(-3, 2, -3)$ and $Q(1, -1, 4)$.

- A helicopter is to fly directly from a helipad at the origin in the direction of the point $(1, 1, 1)$ at a speed of $60m/s$. What is the position of the helicopter after $10s$.

The Distance from a Point to a Line in Space

- Distance from a point S to a line through P parallel to $\boldsymbol{v}$

$$d =$$

- Find the distance from the point $S(1, 1, 5)$ to the line

$$L: \quad x = 1 + t, \quad y = 3 - t, \quad z = 2t.$$

An Equation for a Plane in Space

- Suppose that plane M passes through a point $P_0(x_0, y_0, z_0)$ and is normal to the nonzero vector $\boldsymbol{n} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$.

An Equation for a Plane in Space

- The plane through $P_0(x_0, y_0, z_0)$ normal to $\boldsymbol{n} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$ has
- **Vector equation:**
- **Component equation:**
- **Component equation simplified:**

- Find an equation for the plane through $A(0, 0, 1)$, $B(2, 0, 0)$ and $C(0, 3, 0)$.

Lines of Intersection

- The planes are **parallel** if and only if their normals are parallel, or $\boldsymbol{n}_1 = k\boldsymbol{n}_2$ for some scalar k.
- Two planes that are not parallel intersect in a line.

- Find a vector parallel to the line of intersection of the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.
- Find its parametric equations.

The Distance from a Point to a Plane

- If P is a point on a plane with normal $\boldsymbol{n}$, the distance from S to the plane is

$$d =$$

where $\boldsymbol{n} = A\boldsymbol{i} + B\boldsymbol{j} + C\boldsymbol{k}$ is normal to the plane.

- Find the distance from $S(1, 1, 3)$ to the plane $3x + 2y + 6z = 6$.

Angles Between Planes

- Find the angle between the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Cylinders

- A **cylinder** is a surface that is generated by moving a straight line along a given planar curve while holding the line parallel to a given fixed line. The curve is called a **generating curve** for the cylinder.

- Find an equation for the cylinder made by the lines parallel to the z-axis that pass through the parabola $y = x^2$, $z = 0$.

Quadric Surfaces

- A **quadric surface** is the graph in space of a second-degree equation x, y, and z, *e.g.*,

$$Ax^2 + By^2 + Cz^2 + Dz = E.$$

  The basic quadric surfaces are **ellipsoids**, **paraboloids**, **elliptical cones**, and **hyperboloids**.

Quadric Surfaces

- The **ellipsoid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Quadric Surfaces (cont'd)

- The **hyperbolic paraboloid**

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = \frac{z}{c}, \quad c > 0$$