Introduction to Evolutionary Games - 2

Escuela de Bioestocástica

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Here we have

$$\begin{aligned} \dot{x} &= ax \\ \dot{y} &= ay \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} x(t) &= x_0 e^{at} \\ y(t) &= y_0 e^{bt} \end{aligned}$$

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Therefore,

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Therefore,

$$\frac{x(t)}{y(t)} = \frac{x_0}{y_0} e^{(a-b)t}$$

- if $a = b$ then the quotient is constant: the size of the population are in the same proportion
- if $a > b$ then $\lim_{t\to\infty} \frac{x(t)}{y(t)} \to \infty$: Population A outcompete population B
- if $a < b$ then $\lim_{t\to\infty} \frac{x(t)}{y(t)} \to 0$: Population B outcompete population A

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A much more interesting case is when the total abundance is constant, i.e. we set $x(t) + y(t) = 1$. In this setting the simplest model is

$$\dot{x}(t) = (a - \phi(t))x(t)$$
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$$(a - \phi(t))x(t) + (b - \phi(t))y(t) = 0 \to ax + by = \phi(x + y) \to \phi = (ax + by)$$

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$$\dot{x} = x(1 - x)(a - b) \Rightarrow \begin{cases} x(t) = 0 & \text{if } x(0) = 0 \\ x(t) = 1 & \text{if } x(0) = 1 \\ x(t) = \frac{x_0 e^{(a-b)t}}{1 - x_0 + x_0 e^{(a-b)t}} & \text{if } x(0) \in (0, 1) \end{cases}$$

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Exercise

If $x(0) \in (0, 1)$, show that $\lim_{t\to\infty} x(t) \to 1$ if $a > b$, and $\lim_{t\to\infty} x(t) \to 0$ if $b < a$

Selection

We can have more than two species, e.g. A, B, and C

$$\dot{x}(t) = (a - \phi(t))x(t)$$
$$\dot{y}(t) = (b - \phi(t))y(t)$$
$$\dot{z}(t) = (c - \phi(t))z(t)$$

with $x(t) + y(t) + z(t) = 1$. In this case $\phi(t) = ax(t) + by(t) + cy(t)$ which is, again, the average fitness. However, solving this equation is much harder.

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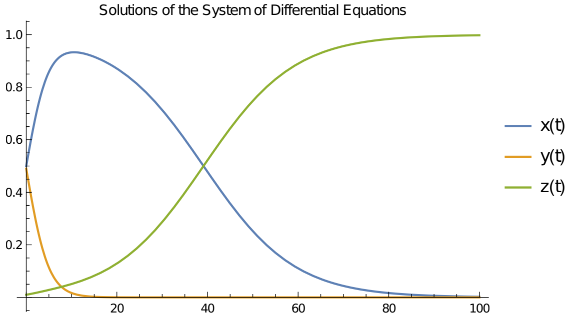
But.... Computers!

Mathematica is my to-go software for solving stuff. Other alternatives like Mathlab or several libraries in python (scypy, numpy, etc)

```
a = 1.9; b = 1.5; c = 2; (*fitness value*)
phi[t_] := a*x[t] + b*y[t] + c*z[t]; (*average fitness*)
eq1 = (x'[t] == (a - phi[t])*x[t]); (*the three equations and initial conditions*)
eq2 = (y'[t] == (b - phi[t])*y[t]);
eq3 = (z'[t] == (c - phi[t])*z[t]);
initialConditions = {x[0] == 50/100, y[0] == 49/100, z[0] == 1/100};
solution =  NDSolve[{eq1, eq2, eq3, initialConditions},
{x[t], y[t], z[t]}, {t, 0, 100}]; (*solver for ODE*)
{xSol, ySol, zSol} = {x[t], y[t], z[t]} /.
   solution[[1]]; (*extract the solutions*)
Plot[{xSol, ySol, zSol}, {t, 0, 100}, PlotLegends-> {"x(t)", "y(t)", "z(t)"},
 FrameLabel -> {"t", "Values"},  PlotLabel-> "Solutions of the System of Differential
```

Example with $a = 1.9$, $b = 1.5$, $c = 2$, $x(0) = 50/100$, $y(0) = 49/100$, $z(0) = 1/100$



DETOUR: Simplex and Equilibrium Points

Dynamics on the Simplex

In the previous examples, we had that the total abundance was constant.

The collection of points $(x_1, \ldots, x_n)$ such that $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$ are called the simplex S_n.

Dynamics on the Simplex

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The collection of points $(x_1, \ldots, x_n)$ such that $x_i \geq 0$ and $\sum_{i=1}^{n} x_i = 1$ are called the simplex S_n.

- Each point of the simplex represents the abundance of the population.
- If we keep the total abundance fixed at 1, then any evolutionary dynamics will be a dynamic on the simplex.

A bit on equilibrium Points

When we have a system of diferential equations

$$\dot{x} = f(x, y, z)$$
$$\dot{y} = g(x, y, z)$$
$$\dot{z} = h(x, y, z)$$

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Solutions to such a system are important because they are points of "0" velocity. Those points are called **equilibria points**

A bit on equilibrium Points

No all equilibria are the same

1. **Stable Equilibrium**: the system tends to return to that point after a perturbation
2. **Unstable Equilibrium**: the system moves away after a perturbation
3. **Saddle Point**: the system returns and moves away in different directions after a perturbation
4. **Center**: the system moves around the equilibrium
5. **Others**

END OF DETOUR

Mutation

An important operator is **mutation**: this means that one species transform into others by different means (pure mutation, but also like eating).

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Let's consider again two species A and B.

- Let x and y be the abundances of A and B respectively, and assume the dynamic is on the simplex
- Suppose that A has fitness a, and mutates into B at rate m_{AB}. Similarly
- Suppose that B has fitness b and mutates into A at rate m_{BA}.
- We will assume that both $m_{AB} > 0$ and $m_{BA} > 0$

Then, the system of equations is given by

$$\dot{x} = ax - m_{AB}x + m_{BA}y - \phi x$$
$$\dot{y} = by - m_{BA}y + m_{AB}x - \phi y$$

since $x + y = 1$ and $\dot{x} + \dot{y} = 0$, we have $\phi(t) = ax + by$.

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Equilibrium Point? We shall solve $\dot{x} = 0$, $\dot{y} = 0$. Let's solve for x:

$$0 = \dot{x} = ax - m_{AB}x + m_{BA}y - (ax + by)x$$

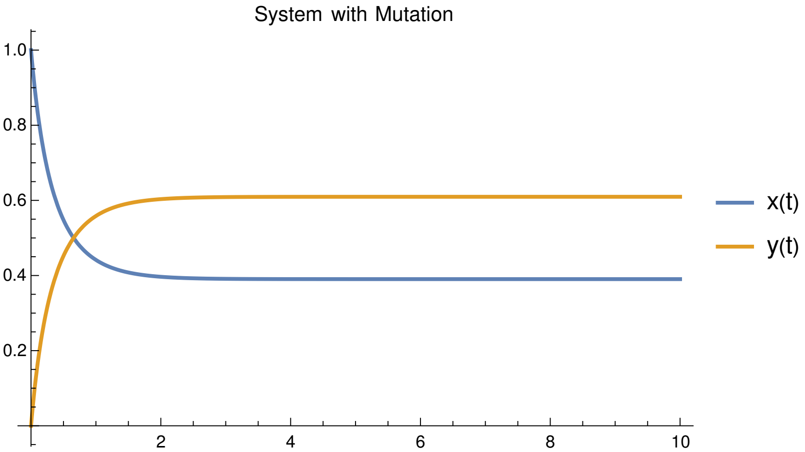
replacing $y = 1 - x$ yields

$$(b - a)x^2 + (a - b - (m_{AB} + m_{BA}))x + m_{BA}$$

- if $a = b$, then $(m_{AB} + m_{BA}))x + m_{BA}$
- if $a \neq b$, then we have a quadratic equation. The analysis of equilibrium points is harder here.

$$x \to \frac{b - a + m_{AB} + m_{BA} \pm \sqrt{(b - a + m_{AB} + m_{BA})^2 - 4(b - a)m_{BA}}}{2(b - a)}$$

A system with $a = 5$, $b = 3$, $m_A B = 2$, $m_{BA} = 0.5$



Mutation Matrix

Let's have a look again at our system of differential equations:

$$\dot{x} = ax - m_{AB}x + m_{BA}y - \phi x$$
$$\dot{y} = by - m_{BA}y + m_{AB}x - \phi y$$

Now consider another species, C, then we would have a system like

$$\dot{x} = ax - (m_{AB} + m_{AC})x + (m_{BA}y + m_{CA}z) - \phi x$$
$$\dot{y} = by - (m_{BA} + m_{BC})y + (m_{AB}x + m_{CA}z) - \phi y$$
$$\dot{z} = cz - (m_{CA} + m_{CB})z + (m_{AC}x + m_{BC}y) - \phi z$$

Then, we can write

$$Q = \begin{pmatrix} -(m_{AB} + m_{AC}) & m_{AB} & m_{AC} \\ m_{BA} & -(m_{BA} + m_{BC}) & m_{BC} \\ m_{CA} & m_{CB} & -(m_{CA} + m_{CB}) \end{pmatrix}$$

then, by using the notation $\overrightarrow{x} = (x, y, z)$, the mutation part can be written as

$$Q^{\intercal}\overrightarrow{x}$$

Mutation Matrix

In general, any mutation matrix Q is such that

1. For diagonal entries: $Q_{ii} \leq 0$
2. For odd-diagonal entries $Q_{ij} \geq 0$
3. The sum of the entries of each column is 0.

Summary

Reproduction:

- Reproduction can be model as ax with $a > 0$ being the reproduction rate, meaning that each indidivual reproduces at rate a
- Dead can be model as $-dx$ meaning that each individual dies at rate d
- the simplest model is then $\dot{x} = (a - d)x$. We can allow $a \in \mathbb{R}$ and encode reproduction and dead in the variable a

Selection and Competition:

- We allow more than one species, we can model the abundance of two species with two variables x and y
- If we fix the total abundance $x + y = 1$, then we introduce competence
- Average fitness *balance* the equations

$$\dot{x} = ax - \phi x$$
$$\dot{y} = by - \phi y$$

with $\phi(t) = ax(t) + by(t)$.

Mutation

- models the rate that one species transform into another one via a mutation

$$\dot{x} = ax - m_{AB}x + m_{BA}y - \phi x$$
$$\dot{y} = by - m_{BA}y + m_{AB}x - \phi y$$

- it can be encoded in a mutation matrix Q

Extra: Modelling Infections - SIR model

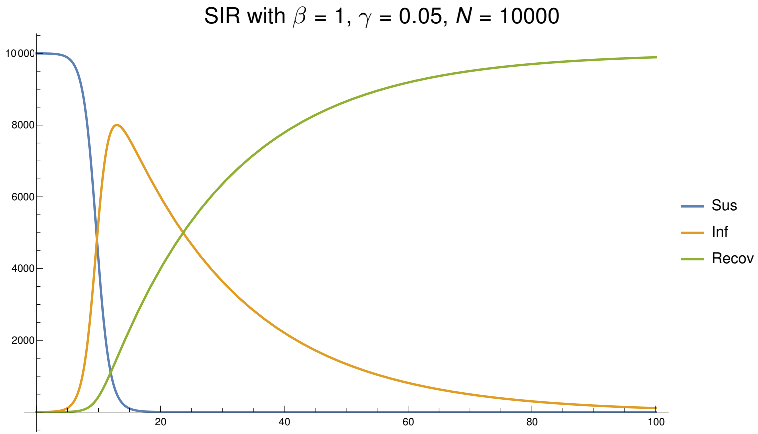
Suppose we have an infection in a population of size N.

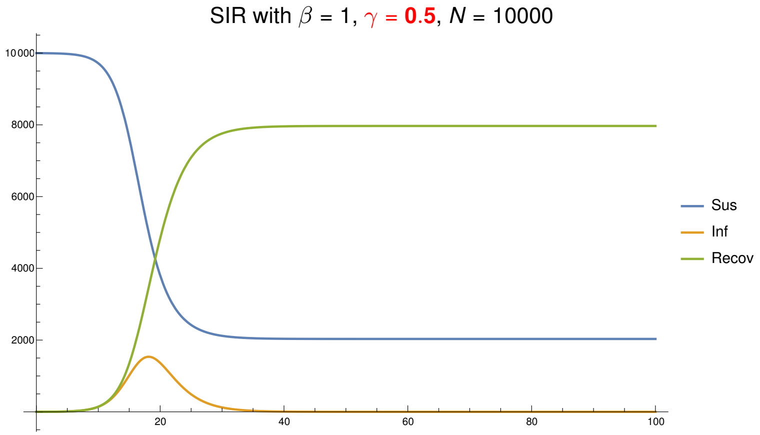
1. There are three type of subjects: infected, susceptible, and recovered
2. Susceptible agents can be infected
3. Infected can infect susceptible
4. Recovered cannot infect nor be infected (they are inmune or dead)

We have the following dynamics

1. An infected agent meets a random agent at rate β and infects her
2. Infected agents recover at rate γ

Can we write the model?


SIR with β = 1, γ = 0.05, N = 10000
10000
8000
6000
4000
2000
20
40
60
80
100
Sus
Inf
Recov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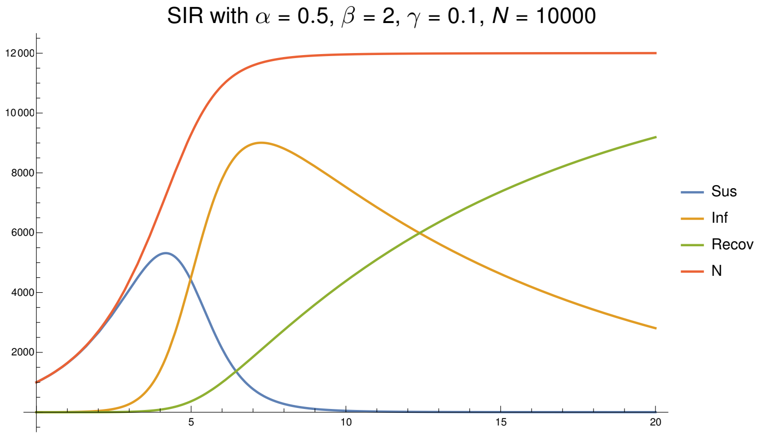

Lecture 3: Evolutionary Games

Evolutionary Games and Competition

We already studied three ideas

- Reproduction
- Selection
- Mutation

however, we studied one last example at the end of Lecture 2

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$$\dot{S} = -\beta \frac{S}{N} \cdot I$$
$$\dot{I} = \beta \frac{S}{N} \cdot I - \gamma I$$
$$\dot{R} = \gamma I$$

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  1. Each unit of **Infected** attacks at rate β and infects a random agent
  2. Each unit of **Infected** attacks at rate 1 a random agent, and gains β units of mass.
- In this case (due to the interpretation we are making) everything that is loss by the susceptible is adquired by the infected, but this is not always true.

If a lion eats a zebra, the Zebras lose some mass (say M), but the Lions don't get M extra mass, and some of the mass of the Zebra will go to other animals o the enviroment

Competition and Collaboration in the Simplex

Consider two species A and B with abundance $x(t)$ and $y(t)$ with $x + y = 1$.

We will imagine that agents are moving in such a way at rate $1/2$ a pair of agents meet

Competition and Collaboration in the Simplex Consider the following tables of interactions: M_{UV} means what does the agent of type U gets when meeting an agent of type V

	A	B
A	-1	3
B	0.5	1

So our model of competition-collaboration is the following: Each agent starts an interaction with a random agent at rate 1.

1. Call x_A and x_B the abundances of species A and B respectively
2. Since at each 1 a pair of random agents meet we have that:
   - meeting between two agents of A occurs at rate $x_A^2/2$, and both get -1 of abundance
   - meeting between two agents of B occurs at rate $x_B^2/2$,
   - meeting between agents of A and B occurs at rate $X_A X_B$
3. From the point of view of A, we have

$$\dot{x}_A = \frac{x_A^2}{2} \cdot (-1) \cdot 2 + x_A x_B \cdot 3 - \phi(t) x_A$$

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Miremos la ecuación $\dot{x}_A = \frac{x_A^2}{2} \cdot (-1) \cdot 2 + x_A x_B \cdot 3$

$$\dot{x}_A = \underbrace{\frac{x_A^2}{2}}_{\text{Rate meet AA}} \cdot \overbrace{(-1)}^{\text{outcome}} \cdot \underbrace{2}_{\text{two agents}} + \overbrace{x_A x_B}^{\text{Rate meet AB}} \cdot \underbrace{3}_{\text{outcome}} \cdot \overbrace{1}^{\text{one agent of A}} - \phi(t) x_A$$

and cancelling the terms, we have

$$\dot{x}_A(t) = (-1 \cdot x_A(t) + 3 \cdot x_B(t) - \phi(t)) x_A$$

and thus the system

$$\dot{x}_A = (-1 \cdot x_A + 3 \cdot x_B - \phi) x_A$$
$$\dot{x}_B = (0.5 \cdot x_A + x_B - \phi) x_B$$

Payoff matrices

In evolutionary game theory we will assume some sort of interaction between species, and when they meet some reward-loss occurs: this is encoded by the Payoff matrix

		Against	
		A	B
Play	A	M_{AA}	M_{AB}
	B	M_{BA}	M_{BB}

This matrix means

- If an individual of type A meet another of type A, both get M_{AA}
- If B meets B, both get M_{BB}
- if A and B meets, A gets M_{AB}, and B gets M_{BA}

Then, we can define the fitness of both species. Let $x_A(t)$ and $x_B(t)$ be the abundance of each species, then

$$f_A(t) = M_{AA}x_A(t) + M_{AB}x_B(t)$$
$$f_B(t) = M_{BA}x_A(t) + M_{BB}x_B(t)$$

which can also be written in a matrix way as

$$f = M\overrightarrow{x}$$

Payoff matrices

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Then, we have

$$\dot{x}_A = (f_A - \phi)x_A$$
$$\dot{x}_B = (f_B - \phi)x_B$$

An interpretation is: the fitness of the species is no longer an intrinsic quantity of the species but something that depends on your interaction with other agents.

Since we are in the simplex, $x_A + x_B = 1$, then the fitness of a species is the average payoff by interacting with a random agent, assuming that all interactions are equally likely

$$f_A(t) = M_{AA}x_A(t) + M_{AB}x_B(t).$$

Games and Dynamics

With this new idea of dynamic fitness we can formulate the evolutionary dynamic in the simplex, without mutation, by

$$\dot{x}_A = x_A(f_A - \phi)$$
$$\dot{x}_B = x_B(f_B - \phi)$$

and since $\dot{x}_A + \dot{x}_B = 0$ we have

$$\phi = x_A f_A + x_B f_B = M_{AA} x_A^2 + M_{AB} x_A x_B + M_{BA} x_B x_A + M_{BB} x_B^2 = x^\intercal M x$$

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$$\phi = x_A f_A + x_B f_B = M_{AA}x_A^2 + M_{AB}x_A x_B + M_{BA}x_B x_A + M_{BB}x_B^2 = x^{\intercal}Mx$$

Using that $x_A = 1 - x_B$, can write

$$\begin{aligned}\dot{x}_A = x_A(f_A - \phi) &= x_A(f_A - x_A f_A - x_B f_B)\\ &= x_A(x_B f_A - x_B f_B)\\ &= x_A x_B(f_A - f_B)\\ &= {\color{red} x_A(1 - x_A)(f_A - f_B)}\end{aligned}$$

Therefore, an equilibrium is found when the two fitness are the same (as expected!) or when one of the species dissapears

Games and Dynamics: Example 1

Consider a population of Zebras and Lions.

		Against	
		Z	L
Play	Z	3	1
	L	5	0

Suppose that individuals meet in a very random fashion: choose a random individual U and then U meets a random individual V.

- Think about Zebras and Lions as a sort of citizienship
- If U is a lion, and V is a lion: U changes, being a Zebra will help him to improve the fitness
- If U is lion, and V is zebra: U is happy, changing will decrease the fitness
- if U is zebra and V is lion: U is happy
- if U is zebra and V is zebra: U changes to lion

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At the end we find some sort of equilibrium where both populations are more or less the same

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At the end we find some sort of equilibrium where both populations co-exists

Games and Dynamics: Example 2

Consider a population of Zebras and Lions.

		Against	
		Z	L
Play	Z	5	0
	L	3	1

Exercise: what happen in this case?

Games and Dynamics: Example 2

Consider a population of Zebras and Lions.

Play \ Against	Z	L
Z	5	0
L	3	1

Exercise: what happen in this case?

Answer: Two zebras are happy, two Lions are happy, but when a Zebra meets a Lion both want to change.

Games and Dynamics: Example 2

Consider a population of Zebras and Lions.

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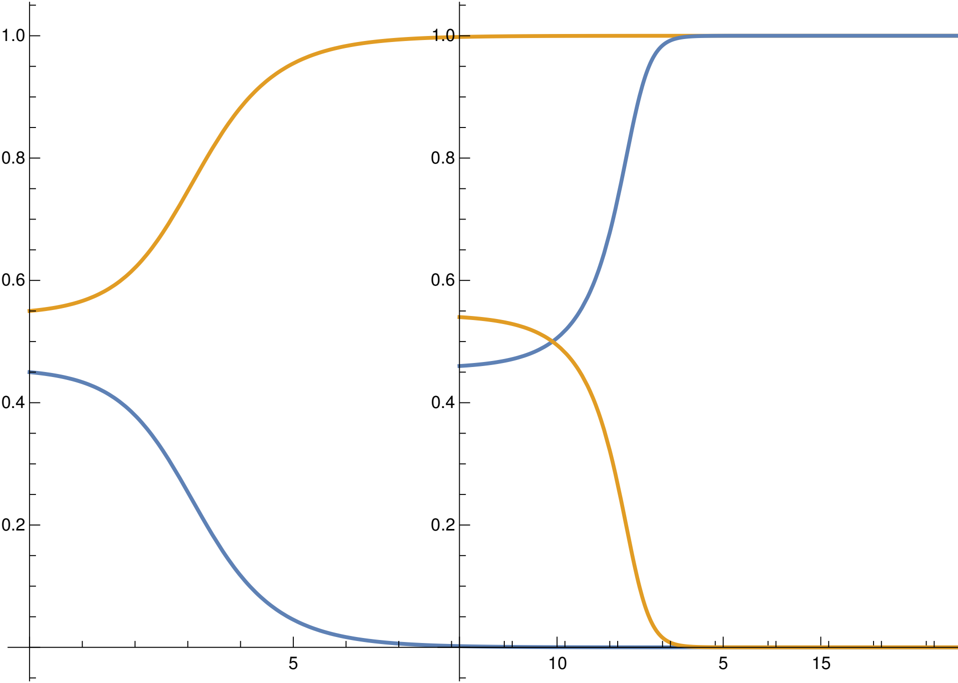
Exercise: what happen in this case?

Answer: Two zebras are happy, two Lions are happy, but when a Zebra meets a Lion both want to change.

In terms of dynamics: Both population can survive, but they cannot co-exist. This depends on the initial values


1.0
0.8
0.6
0.4
0.2
5
1.0
0.8
0.6
0.4
0.2
10
5
15

Games and Dynamics: Example 3

We can make our modelling a bit more realistic by adding context to the Payoff matrix.

Example: hawks and doves

Consider a population of hawks and doves engaged in a game over a resource, like territory or food. The prize corresponds to a gain in fitness G, while an injury reduces fitness by C:

- If two doves meet, they divide the good, obtaining in average $G/2$.
- If a dove meets a hawk, it flees with 0, while that of the hawk increases by G.
- If a hawk meets a hawk, they fight. The fitness of the winner is increased by G, that of the loser reduced by C, so that the average increase in fitness is $(G-C)/2$.

This gives us the matrix

$$\begin{array}{cc} & \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{array}{c} \text{Hawk} \\ \text{Dove} \end{array} & \begin{pmatrix} \frac{G-C}{2} & G \\ 0 & \frac{G}{2} \end{pmatrix} \end{array}$$

In this case

$$\dot{x}_D = x_D((1-x_D)f_D - (1-x_D)f_H) = x_D(1-x_D)\left(\frac{C-G}{2} - Cx_D/2\right).$$

Example: hawks and doves

For simplicity, assume the prize $G = 1$. Our payoff-matrix is

	H	D
H	$\frac{1-C}{2}$	1
D	0	$1/2$

We shall analyse different values for C.

1. We first should notice that $\dot{x}_D = 0$ when

$$\frac{C-1}{2} - \frac{CX_D}{2} = 0, \quad x_D = 0, \quad \text{or } x_D = 1$$

2. $\frac{C-1}{C} \in (0,1)$ only when $C > 1$, otherwise it is negative or greater than 1

For sake of being interesting, assume that $x_D(0) \in (0,1)$.

1. Case I: $C > 1$ hawk fights are risky
   - $DD \rightarrow H$
   - $DH \rightarrow D$
   - $HD \rightarrow H$
   - $HH \rightarrow D$

   In this case we will find a equilibrium around $x_D = (C - 1)/C$.

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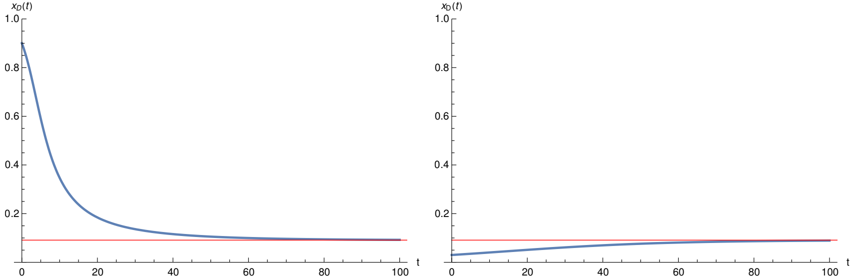
   Examples with $C = 1.1$



1. Case II: $C \leq 1$

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