Introduction to Evolutionary Games
Escuela de Bioestocástica

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Evolutionary Operators

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  - **mutation**: species 'transform' into other.
- Combination of them allows us to create different evolutionary dynamics

Reproduction

Let's imagine a single bacterial cell in a perfect enviroment containing all the nutrients required for it to survive, and reproduce.

- Suppose the bacteria divides into two every 20 minutes
- Then after 20 minutes we will have 2 bacteria
- after 40 we will have 4
- after 60 we weill have 8
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The solution for this difference equation is given by

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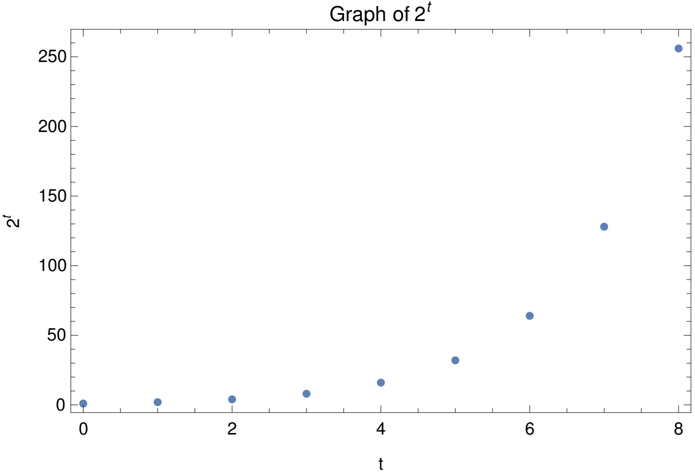
In our case $x_0 = 1$ since we have one bacterium



This is the famous exponential growth.

Reproduction

So far so good...

However, instead of thinking about reproduction over epochs, it is more convenient to think about reproduction rates.

There are many reasons

1. Solving difference equations is hard
2. Modelling with difference equations is hard
3. understanding difference equations, without solving them, is hard

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On the other hand, working with rates lead to differential equations. These has been used for many years in phisical modelling, and we understand them much better than difference equations as they tend to be easier to solve and study.

In some sense, we want to do classical mechanics in biology, but instead of spheres and cars with mass, we have biological entities

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- $x(t)$ will also be continuous, meaning that it can take value 1.2 or π,
- It is ok to think that $x(t)$ is the number of cells, but probably it is more convenient to use a continuous measure, such as
  1. weight (biomass)
  2. volume

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Differential Equation for reproduction only

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The solution of this differential equation is

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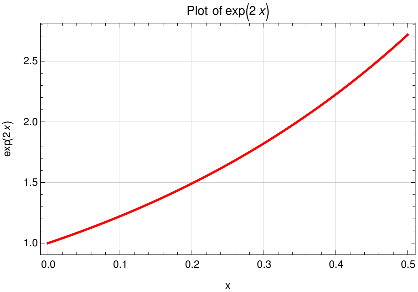

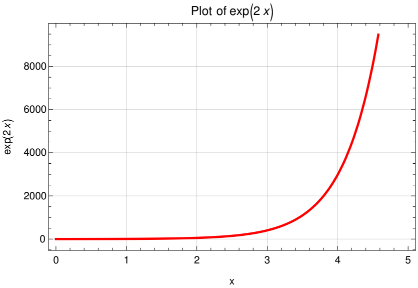

In the pictures we have $x(0) = 1$ and $r = 2$.

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A continuous-time model

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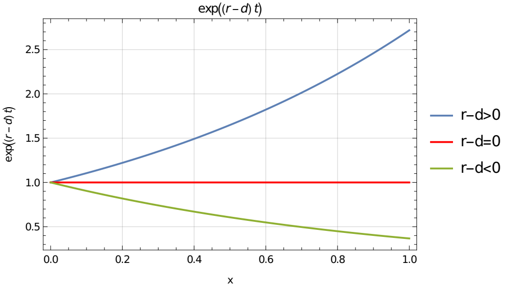
Here we have three behaviours

1. $r > d$, then the population size grows to infinity
2. $r < d$, then the population size tends to 0
3. $r = d$, the population size remains fixed. Note, however, that a minimal change in r and d will change the behaviour of the dynamic.



A continuous-time model: size-dependent effects

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We can make our model a bit more complex

- **Birth:** cells divides at rate r
- **Death:** cells die at rate $x \cdot d$ when the abundance is x
- In this model the more cells we have, the faster they die: e.g. they are competing for nutrients, space, or just fight
- In this case, the dynamics satisfies the equation

$$\dot{x}(t) = (r - d \cdot x(t))x(t) \quad t > 0,$$

which is commonly written as

Logistic equation

$$\dot{x}(t) = rx(t)(1 - x(t)/K)$$

where $K = r/d$.

whose solution is

Logistic function

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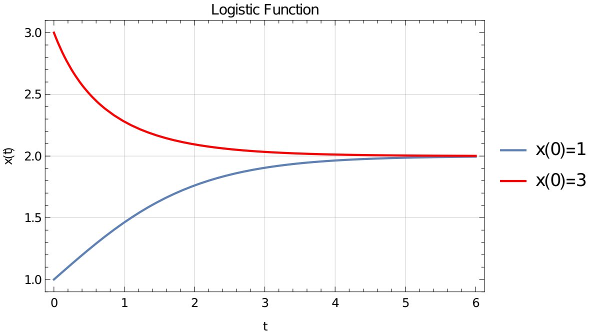

Example with $K = 2$ and $r = 1$

Writing models

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To write our own model, we need to be willing to

1. write equations that roughly explain the situation
2. make concessions

Writing models

Exercise: Imagine a bird population on an isolated island. Birds reproduce at rate 3 per year, and they die at rate 1 every 10 years. Unfortunately, these Birds attack each other: in particular, each pairs of birds fight at rate 1 every three months. The outcome of the fight is that 0.1% of the time both die, 0.1% only one dies, and the rest of the time both survive.

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8. Finally

$$\begin{aligned}\dot{x} = 2.9x - \frac{6}{1000}x^2 &= x\left(2.9 - 0.006x\right)\\ &= rx\left(1 - x/K\right)\end{aligned}$$

   with $r = 2.9$ and $K = \frac{2.9}{0.006} \approx 483.333$.
9. We recognise a logistic function, so for large times t we have a population of 773 birds, approximately.

Writing models

Our logistic function with $r = 2.9$ and $K \approx 483.333$ looks like