Introduction to Evolutionary Games - 3
Escuela de Bioestocástica

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Lecture 3: Evolutionary Games

Evolutionary Games and Competition

We already studied three ideas

- Reproduction
- Selection
- Mutation

however, we studied one last example at the end of Lecture 2

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$$\dot{S} = -\beta \frac{S}{N} \cdot I$$
$$\dot{I} = \beta \frac{S}{N} \cdot I - \gamma I$$
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    1. Each unit of **Infected** attacks at rate β and infects a random agent
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  1. Each unit of **Infected** attacks at rate β and infects a random agent
  2. Each unit of **Infected** attacks at rate 1 a random agent, and gains β units of mass.
- In this case (due to the interpretation we are making) everything that is loss by the susceptible is adquired by the infected, but this is not always true.

If a lion eats a zebra, the Zebras lose some mass (say M), but the Lions don't get M extra mass, and some of the mass of the Zebra will go to other animals o the enviroment

Competition and Collaboration in the Simplex

Consider two species A and B with abundance $x(t)$ and $y(t)$ with $x + y = 1$.

Competition and Collaboration in the Simplex

Consider the following tables of interactions: M_{UV} means what does the agent of type U gets when meeting an agent of type V

	A	B
A	-1	3
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1. Call x_A and x_B the abundances of species A and B respectively
2. Since at each 1 a pair of random agents meet we have that:
   - meeting between two agents of A occurs at rate $x_A^2/2$, and both get -1 of abundance
   - meeting between two agents of B occurs at rate $x_B^2/2$,
   - meeting between agents of A and B occurs at rate $X_A X_B$

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   - meeting between agents of A and B occurs at rate $X_A X_B$
3. From the point of view of A, we have

$$\dot{x}_A = \frac{x_A^2}{2} \cdot (-1) \cdot 2 + x_A x_B \cdot 3 - \phi(t) x_A$$

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and cancelling the terms, we have

$$\dot{x}_A(t) = \Big((-1) \cdot x_A(t) + 3 \cdot x_B(t) - \phi(t)\Big) x_A$$

and thus the system

$$\dot{x}_A = (-1 \cdot x_A + 3 \cdot x_B - \phi) x_A$$
$$\dot{x}_B = (0.5 \cdot x_A + x_B - \phi) x_B$$

Payoff matrices

In evolutionary game theory we will assume some sort of interaction between species, and when they meet some reward-loss occurs: this is encoded by the **Payoff matrix**

		Against	
		A	B
Play	A	M_{AA}	M_{AB}
	B	M_{BA}	M_{BB}

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This matrix means

- If an individual of type A meet another of type A, both get M_{AA}
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- if A and B meets, A gets M_{AB}, and B gets M_{BA}

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Then, we can define the fitness of both species. Let $x_A(t)$ and $x_B(t)$ be the abundance of each species, then

$$f_A(t) = M_{AA}x_A(t) + M_{AB}x_B(t)$$
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which can also be written in a matrix way as

$$f = M\overrightarrow{x}$$

Payoff matrices

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Then, we have

$$\dot{x}_A = (f_A - \phi)x_A$$
$$\dot{x}_B = (f_B - \phi)x_B$$

An interpretation is: the fitness of the species is no longer an intrinsic quantity of the species but something that depends on your interaction with other agents.

Since we are in the simplex, $x_A + x_B = 1$, then the fitness of a species is the average payoff by interacting with a random agent, assuming that all interactions are equally likely

$$f_A(t) = M_{AA}x_A(t) + M_{AB}x_B(t).$$

Games and Dynamics

With this new idea of dynamic fitness we can formulate the evolutionary dynamic in the simplex, without mutation, by

$$\dot{x}_A = x_A(f_A - \phi)$$
$$\dot{x}_B = x_B(f_B - \phi)$$

and since $\dot{x}_A + \dot{x}_B = 0$ we have

$$\phi = x_A f_A + x_B f_B = M_{AA} x_A^2 + M_{AB} x_A x_B + M_{BA} x_B x_A + M_{BB} x_B^2 = x^\intercal M x$$

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Using that $x_A = 1 - x_B$, can write

$$\begin{aligned}
\dot{x}_A = x_A(f_A - \phi) &= x_A(f_A - x_A f_A - x_B f_B) \\
&= x_A(x_B f_A - x_B f_B) \\
&= x_A x_B (f_A - f_B) \\
&= x_A(1 - x_A)(f_A - f_B)
\end{aligned}$$

Therefore, an equilibrium is found when the two fitness are the same (as expected!) or when one of the species dissapears

Games and Dynamics: Example 1

Consider a population of Zebras and Lions.

		Against	
		Z	*L*
Play	*Z*	3	1
	L	5	0

- Think about Zebras and Lions as a sort of citizienship

Games and Dynamics: Example 1

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- Think about Zebras and Lions as a sort of citiziensship
- If U is a lion, and V is a lion: U changes, being a Zebra will help him to improve the fitness
- If U is lion, and V is zebra: U is happy, changing will decrease the fitness
- if U is zebra and V is lion: U is happy
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At the end we find some sort of equilibrium where both populations are more or less the same

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At the end we find some sort of equilibrium where both populations co-exists

Games and Dynamics: Example 2

Consider a population of Zebras and Lions.

Play \ Against	Z	L
Z	5	0
L	3	1

Exercise: what happen in this case?

Games and Dynamics: Example 2

Consider a population of Zebras and Lions.

		Against	
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Exercise: what happen in this case?

Answer: Two zebras are happy, two Lions are happy, but when a Zebra meets a Lion both want to change.

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Answer: Two zebras are happy, two Lions are happy, but when a Zebra meets a Lion both want to change.

In terms of dynamics:

1. Both population can survive
2. They co-exists only if $x_Z(0)$ is some very specific value
3. Otherwise, either Lions or Zebras survive

Games and Dynamics: Example 2

		Against	
		Z	L
Play	Z	5	0
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Let's find the equilibrium points: Beyond $(x_Z, x_L) = (1, 0)$ and $(0, 1)$, we shall solve $f_Z = f_L$ restricted to $x_Z + x_L = 1$.

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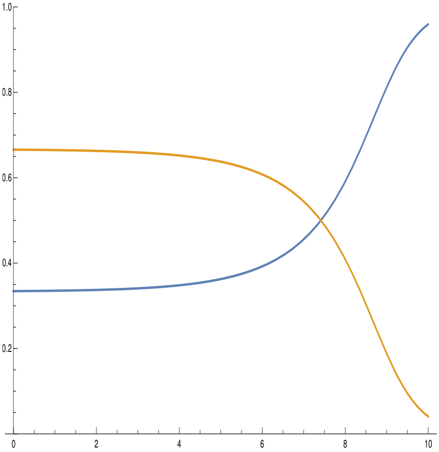
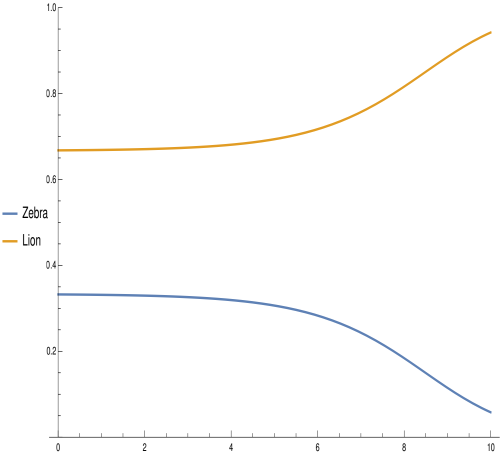
Our system of equations is

$$5x_Z = 3X_z + 1x_L$$
$$x_Z + x_L = 1$$

obtaining $(x_Z, x_L) = (1/3, 2/3)$ is a equilibrium point where both co-exist



$x_Z(0) = 1/3 + 0.001$



$x_Z(0) = 1/3 - 0.001$

Games and Dynamics: Example 3

We can make our modelling a bit more realistic by adding context to the Payoff matrix.

Example: hawks and doves

Consider a population of hawks and doves engaged in a game over a resource, like territory or food. The prize corresponds to a gain in fitness G, while an injury reduces fitness by C:

- If two doves meet, they divide the good, obtaining in average $G/2$.
- If a dove meets a hawk, it flees with 0, while that of the hawk increases by G.
- If a hawk meets a hawk, they fight. The fitness of the winner is increased by G, that of the loser reduced by C, so that the average increase in fitness is $(G - C)/2$.

This gives us the matrix

$$\begin{array}{cc} & \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{array}{c} \text{Hawk} \\[2ex] \text{Dove} \end{array} & \begin{pmatrix} \dfrac{G-C}{2} & G \\ 0 & \dfrac{G}{2} \end{pmatrix} \end{array}$$

In this case

$$\dot{x}_D = x_D((1 - x_D)f_D - (1 - x_D)f_H) = x_D(1 - x_D)\left(\frac{C - G}{2} - Cx_D/2\right).$$

Example: hawks and doves

For simplicity, assume the prize $G = 1$. Our payoff-matrix is

	H	D
H	$\frac{1-C}{2}$	1
D	0	$1/2$

We shall analyse different values for C.

1. We first should notice that $\dot{x}_D = 0$ when

$$\frac{C-1}{2} - \frac{CX_D}{2} = 0, \quad x_D = 0, \quad \text{or } x_D = 1$$

2. $\frac{C-1}{C} \in (0, 1)$ only when $C > 1$, otherwise it is negative or greater than 1

For sake of being interesting, assume that $x_D(0) \in (0, 1)$.

1. Case I: $C > 1$ hawk fights are risky
   - $DD \to H$
   - $DH \to D$
   - $HD \to H$
   - $HH \to D$

   In this case we will find a equilibrium around $x_D = (C-1)/C$.

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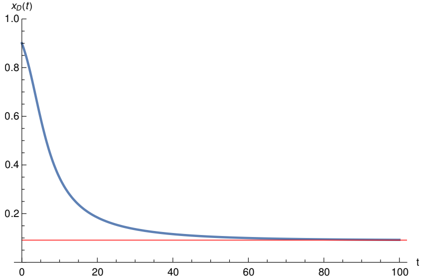
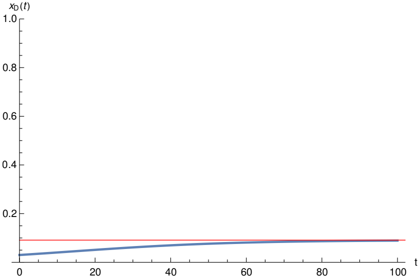
    Examples with $C = 1.1$





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