vv255_Assignment 7: Line Integrals. Green Theorem.
Due to: 2019-07-26 16:30 (use the mailbox to submit this assignment)

Problem 1:

A. Evaluate the line integrals

a. $\int_C \cos z \, ds$, $C: x(t) = \sin t, y(t) = \cos t, z(t) = t,\ t \in [0, 2\pi]$

b. $\int_C (2x + y)dx + (x^2 + y^2)dy$, C: the arc of $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$

c. $\int_C yz \ dx + xz \ dy + xy \ dz$, C: line segments joining $(1,0,0)$ to $(0,1,0)$ to $(0,0,1)$

d. $\int_C x^2 dx - xy \ dy + dz$, C: $z = x^2,\ y = 0$ from $(-1,0,1)$ to $(1,0,1)$.

B.

a. Show that the line integral of $f(x,y)$ along a path given in polar coordinates by

$$r = r(\theta), \qquad \theta_1 < \theta < \theta_2,$$

is

$$\int_{\theta_1}^{\theta_2} f(r\cos\theta, r\sin\theta)\sqrt{r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2}\, d\theta$$

b. Compute the arc length of the path $r = 1 + \cos\theta,\ 0 < \theta < 2\pi$.

Problem 2

A. Let C be a smooth path.

1. Suppose F is perpendicular to $C'(t)$ at the point $C(t)$. Show that $\int_C F \ dr = 0$

2. If F is parallel to $C'(t)$ at $C(t)$, show that $\int_C F \ dr = \int_C ||F|| \ dr$

B. Suppose C_1 and C_2 are two paths with the same endpoints and F is a vector field. Show that

$$\int_{C_1} F \ dr = \int_{C_2} F \ dr$$

is equivalent to $\int_C F \ dr = 0$, where C is the closed curve obtained by first moving along C_1 and then moving along C_2 in the opposite direction.

Problem 3

A. Evaluate the integral

$$\int_C 2xyz\ dx + x^2z\ dy + x^2y\ dz,$$

where C is an oriented simple curve from $(1,1,1)$ to $(1,2,4)$.

B. Find a function f such that $F = \nabla f$ and evaluate $\int_C F\ dr$ along the given curve C.

$$F = (\sin y, x\cos y + \cos z, -y\sin z), \qquad C:\ \bar{r}(t) = (\sin t, t, 2t), \qquad t \in [0, \frac{\pi}{2}]$$

C. Let $\nabla f(x,y,z) = 2xyze^{x^2}\bar{\imath} + ze^{x^2}\bar{\jmath} + ye^{x^2}\bar{k}$. If $f(0,0,0) = 5$, find $f(1,1,2)$.

D. A mass M at the origin in $\mathbb{R}^3$ exerts a force on a mass m located at $\bar{r}(x,y,z)$ with magnitude $\frac{GmM}{r^2}$ and directed toward the origin. Here, G is the gravitational constant, which depends on the units of measurement, and $r = ||\bar{r}|| = (x^2 + y^2 + z^2)^{\frac{1}{2}}$.

Consider the gravitational force field (with $G = m = M = 1$) defined [for $(x,y,z) \neq (0,0,0)$] by

$$F(x,y,z) = -\frac{1}{(x^2 + y^2 + z^2)^{\frac{3}{2}}}(x\bar{\imath} + y\bar{\jmath} + z\bar{k}).$$

Show that the work done by the gravitational force as a particle moves from (x_1, y_1, z_1) to (x_2, y_2, z_2) along any path depends only on the radii $R_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}$ and $R_2 = \sqrt{x_2^2 + y_2^2 + z_2^2}$.

E. Let $F = xe^y\bar{\imath} - (x\cos z)\bar{\jmath} - ze^y\bar{k}$. Find G such that $F = \nabla \times G$.

Problem 4

A. Use Green's Theorem to evaluate $\int_C F\ dr$.

1. $F(x,y) = (x^2 + y^2, x^2 - y^2)$, C: line segments joining $(0,0)$ to $(2,1)$ to $(0,1)$ to $(0,0)$.
2. $F(x,y) = (y\cos x - xy\sin x, xy + x\cos x)$, C: line segments joining $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$.

B. Under the conditions of Green's theorem, prove that

$$\int_{\partial R} PQ\ dx + PQ\ dy = \iint_R Q\left(\frac{\partial P}{\partial x} - \frac{\partial P}{\partial y}\right) + P\left(\frac{\partial Q}{\partial x} - \frac{\partial Q}{\partial y}\right)\ dxdy$$

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