Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 7

Date Due: 12:55 PM, Wednesday, the 14^{th} of April 2021

Discussion Class Preparation

Please (re-)view Video files 39-42 and/or finish reading the sections "Boundary Value Problems for PDEs" and "Eigenfunction Expansion" in the lecture slides. You should be able to answer the following questions:

i) State the second-order elliptic, hyperbolic and parabolic PDEs as well as the three types of boundary conditions.

ii) What are mixed boundary conditions?

iii) How do the concepts familiar from ODEs carry over to the study of PDEs

iv) Explain the following: formal adjoint, conjunct, Green's formula, adjoint boundary value problem and Green's function.

v) What role does the adjoint Green function play in the solution of the parabolic boundary value problem?

vi) How is a causal fundamental solution for a time-dependent PDE defined?

vii) Summarize the properties of the eigenvalues of the elliptic operator.

viii) Explain what full and partial eigenfunction expansions are and the difference between them.

Exercises (13 Marks)

Exercise 7.1
Suppose that $Bu = \frac{\partial u}{\partial n}|_{\partial\Omega}$ (Neumann boundary condition) and let

$$M = \{u \in C^2(\Omega) \cap C(\overline{\Omega}) : Bu = 0\}.$$

Show that if $v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\int_{\partial\Omega} J(u, v)\, d\vec{\sigma} = 0 \qquad\qquad \text{for all } u \in M$$

then

$$v \in M.$$

This proves $M^* \subset M$ for the case of Neumann boundary conditions.
(3 Marks)

Exercise 7.2
Consider the boundary value problem for the heat equation on a finite interval $(0, L) \subset \mathbb{R}$:

$$\begin{aligned} u_t - c^2 u_{xx} &= F(x,t), && 0 < x < L, \\ u(0,t) &= \gamma_1, && 0 < t < T, \\ u(L,t) &= \gamma_2, && 0 < t < T, \\ u(x,0) &= f(x), && 0 < x < L. \end{aligned} \tag{$*$}$$

where $T > 0$ is some fixed time, $\gamma_1, \gamma_2 \in \mathbb{R}$, and $f: [0, L] \to \mathbb{R}$, $F: [0, L] \times \mathbb{R} \to \mathbb{R}$ suitably smooth functions.

i) Which differential equation and boundary conditions must be satisfied by the direct Green's function $g(x, t; \xi, \tau)$ for $(*)$? [No proof necessary.]
(1 Mark)

ii) Which differential equation and boundary conditions must be satisfied by the adjoint Green's function $g^*(x,t;\xi,\tau)$ for $(*)$? [No proof necessary.]
(1 Mark)

iii) Give a simple relation linking g and g^*. [No proof necessary.]
(1 Mark)

iv) Show that

$$u(\xi,\tau) = \int_0^T \int_0^L F(x,t) g^*(x,t;\xi,\tau)\, dx\, dt + \int_0^L g^*(x,0;\xi,\tau) f(x)\, dx \\ + \gamma_1 \int_0^T \frac{\partial g^*(x,t;\xi,\tau)}{\partial x}\Big|_{x=0} dt - \gamma_2 \int_0^T \frac{\partial g^*(x,t;\xi,\tau)}{\partial x}\Big|_{x=L} dt$$

(3 Marks)

Exercise 7.3
Consider the half-disk in $\mathbb{R}^2$,

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon |x|^2 < 1,\ x_2 > 0\}.$$

Show that the solution formula for the Dirichlet problem

$$-\Delta u = 0 \quad \text{on } \Omega, \qquad u(x_1, 0) = 0 \quad \text{for } x_1 \in [-1,1], \qquad u(x_1, \sqrt{1-x_1^2}) = 1 \quad \text{for } x_1 \in (-1,1),$$

is given by

$$u(\xi) = u(\xi_1, \xi_2) = -\int_0^\pi \frac{\partial}{\partial r} g(r\cos(\theta), r\sin(\theta); \xi_1, \xi_2)\Big|_{r=1} d\theta$$

for $\xi \in \Omega$, where g is the (still undetermined) Green's function for this Dirichlet problem.
(4 Marks)