Part A

Problem 1

Consider a linear space $P_3(R)$ with the standard basis $S = \{1, t, t^2, t^3\}$.

a. Describe the isomorphism $P_3 \to \mathbb{R}^4$ sending $p(t) \to p_{\mathcal{S}}$.
b. Show that $\mathcal{B} = \{t-1, t+1, t^2+t, t^3\}$ is another basis for $P_3(\mathbb{R})$.
c. Let $p(t) = 3 + 2t + 4t^3$. Find $p_{\mathcal{B}}$.
d. Show that the map $P_3 \to \mathbb{R}^4$ sending $p(t) \to p_{\mathcal{B}}$ is an isomorphism.

Problem 2

Consider the basis $\mathcal{B} = \{\bar{b}_1, \bar{b}_2\} = \{\bar{e}_1 - 2\bar{e}_2, -2\bar{e}_1 - \bar{e}_2\}$ for $\mathbb{R}^2$.
a. Find $(3,-1)_{\mathcal{B}}, (3\bar{e}_1 + 7\bar{e}_2)_{\mathcal{B}}, (6\pi\bar{b}_1 + 3/2\bar{b}_2)_{\mathcal{B}}$.
b. Find the matrix that changes standard coordinates to $\mathcal{B}$-coordinates and its inverse.
c. Consider the map $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $(x_1, x_2) = (2x_1 - 3x_2, 3x_1 - 2x_2)$. Find the $\mathcal{B}$-matrix of T.
d. Find the relation between the standard matrix for T and $T_{\mathcal{B}}$.

Problem 3

1. Prove that

$$\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4\} = \{\bar{e}_1, \bar{e}_2, \bar{e}_3 + \bar{e}_2, \bar{e}_4 + \bar{e}_1\}$$

is the basis for $\mathbb{R}^4$.
2. Find $(1,1,1,1)_{\mathcal{B}}$.
3. Consider the map $T : \mathbb{R}^4 \to \mathbb{R}^4$ with the $\mathcal{B}$-matrix

$$B = \begin{pmatrix} 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \end{pmatrix}$$

Find the standard matrix of T.

Problem 4

Consider the map$T : \mathbb{C} \to \mathbb{C}$, $Tz = -2iz$.
a. Prove that T is linear and find the matrix of T in the basis $\mathcal{B}_1 = \{1, i\}$.
b. Show that $\mathcal{B}_2 = \{1, 1+i\}$ is also a basis for $\mathbb{C}$ and find the matrix of T in this basis.
c. Let a complex number has coordinates $(x,y)_{\mathcal{B}_1}$. Find $(x,y)_{\mathcal{B}_2}$ and the change of basis matrix $S_{\mathcal{B}_2 \to \mathcal{B}_1}$.

Part B (Difference Equations)

Consider the linear space of all infinite sequences of scalars

$$l = \{x = (\dots, x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n, \dots)\}$$

that represent discrete time-signals. Each signal in l is a function defined on $\mathbb{Z}$.
Sequences $\{x_k\}$, $\{y_k\}$, $\{z_k\}$ are linearly independent if $\alpha x_k + \beta y_k + \gamma z_k = 0 \quad \forall k$ implies that $\alpha = \beta = \gamma = 0$.
If α, β, γ satisfy $\alpha x_k + \beta y_k + \gamma z_k = 0$ for some k, then

$$\begin{array}{ll} \alpha x_k + \beta y_k + \gamma z_k = 0 & \\ \alpha x_{k+1} + \beta y_{k+1} + \gamma z_{k+1} = 0 & \text{for all } k. \\ \alpha x_{k+2} + \beta y_{k+2} + \gamma z_{k+2} = 0 & \end{array}$$

The matrix

$$C = \begin{pmatrix} x_k & y_k & z_k \\ x_{k+1} & y_{k+1} & z_{k+1} \\ x_{k+2} & y_{k+2} & z_{k+2} \end{pmatrix}$$

is called the Casorati matrix of the signals.
If the Casorati matrix is invertible for at least one value of k, then $\alpha = \beta = \gamma = 0$, and the three signals are linearly independent.
Problem B1: Show that $(-1)^k, (-2)^k, 3^k$ are linearly independent signals (consider $k = 0$).

If the Casorati matrix is not invertible then the signals may or may not be linearly dependent.
If the signals are the solutions of the same homogeneous difference equation, then either $\exists C^{-1}\,\forall k$ and the signals are linearly independent, or C^{-1} does not exist and the signals are linearly dependent.
The equation

$$a_0 y_{k+n} + a_1 y_{k+n-1} + \ldots + a_{n-1} y_{k+1} + a_n y_k = z_k \,\forall k$$

where $a_i \in \mathbb{K}$ and a_0, $a_n \neq 0$, is called a linear difference equation of order n.
In digital signal processing a linear difference equation describes a linear filter, and $a_0, \dots, a_n$ are called the filter coefficients. $\{y_k\}$ is the input, $\{z_k\}$ is the output, and the solutions of the homogeneous equation ($z_k = 0\,\forall k$) are the signals that are filtered out.
To find a solution of a homogeneous difference equation, let $y_k = r^k$ and find the values of r for which $y_k = r^k$ satisfies the equation.
Problem B2:
1. Find all solutions of the homogeneous difference equations and a basis for the solution space:

$$1.\ y_{k+3} - 4y_{k+2} + y_{k+1} + 6y_k = 0 \qquad 2.\ y_{k+3} - 2y_{k+2} - 5y_{k+1} + 6y_k = 0$$

2. Let $\bar{x}_k = (y_k, y_{k+1}, y_{k+2})$. Find the matrix representation $A\bar{x}_{k+1} = A\bar{x}_k\,\forall k$ of the homogeneous equations from part 1.
3. Show that the signals 2^k, 3^k, $3^k \sin\frac{\pi k}{2}$, $3^k \cos\frac{\pi k}{2}$ are solutions of

$$y_{k+3} - 4y_{k+2} + 9y_{k+1} - 18y_k = 0$$

and form the basis of the solution space.
Problem B3:
When a signal is produced from a sequence of measurements made on a process (a chemical reaction, a flow of heat through a tube, a moving robot arm, etc.), the signal usually contains random noise produced by measurement errors. A standard method of reprocessing the data to reduce the noise is to smooth or filter the data. For example, a filter of a moving average replaces each y_k by its average with 2 adjacent values:

$$\frac{1}{3}y_{k+1} + \frac{1}{3}y_k + \frac{1}{3}y_{k-1} = z_k \quad k = 1, 2, \ldots$$

Let a signal y_k, $k = 0, \dots, 14$ be

$$9, 5, 7, 3, 2, 4, 6, 5, 7, 6, 8, 10, 9, 5, 7$$

Use the filter to compute z_k, $k = 1, \dots, 13$ and make broken-line graph that superimposes the original signal and the smoothed signal.

Part C (Markov Chains)

A vector with negative entries that add up to 1 is called a probability vector.
For example, a vector such as $\bar{x}_0 = (0.7, 0.3)$ could show that 70% of students start their day with tea and 30% with coffee.
A square matrix whose columns are probability vectors is called a stochastic matrix. A stochastic matrix P is regular if P^k contains strictly positive entries for some k.
A sequence of probability vectors $\bar{x}_0, \bar{x}_1, \ldots$ together with a stochastic matrix P such that

$$\bar{x}_1 = P\bar{x}_0, \ldots, \bar{x}_{n+1} = P\bar{x}_n, \ldots$$

is called a Markov chain.
For example, if $\bar{x}_0$ is a probability vector for today's morning drink, then $\bar{x}_1 = P\bar{x}_0$ is the probability vector of the morning drink tomorrow and $\bar{x}_2$ gives the drink distribution the day after tomorrow etc.
From Part B, it follows that a Markov chain is described by difference equations $\bar{x}_{k+1} = P\bar{x}_k$, $k = 0, 1, 2, \ldots$

Problem C1:

Days in Shanghai are either sunny, cloudy or rainy. If the day is sunny, there is a 75% chance it will be sunny next day, a 20% chance it will be cloudy, and a 5% chance it will be rainy.
If the day is cloudy, there is a 65% chance it will be sunny next day, and a 10% chance it will be rainy.
If the day is rainy, there is a 50% chance it will be sunny next day, and a 40% chance it will be cloudy.
Find the corresponding stochastic matrix.
If there is a 50% chance of sunny weather today, and a 50% chance of a cloudy day, what are the chances of a rainy day tomorrow and the day after tomorrow?

A vector $\bar{q}$ such that $\bar{q} = P\bar{q}$ is called a steady-state or equilibrium vector.

Problem C2:

Find the steady-state vector for the matrices

$$\begin{pmatrix} 0.4 & 0.8 \\ 0.6 & 0.2 \end{pmatrix} \qquad \begin{pmatrix} 0.4 & 0.5 & 0.8 \\ 0 & 0.5 & 0.1 \\ 0.6 & 0 & 0.1 \end{pmatrix}$$

If $P_{n\times n}$ is a regular stochastic matrix, then there exists a unique steady-state vector q, and for any initial vector $\bar{x}_0$, the sequence $\{\bar{x}_k\} : \bar{x}_{k+1} = P\bar{x}_k$ converges to q as $k \to \infty$.