Lecture 16

In previous lectures, we focus on the definition and properties of definite and indefinite integrals. However, in most time, we use techniques like substitution to find antiderivatives and evaluate integrals. These techniques can simplify the original f and transform to combination of terms whose antiderivatives can be found in table.

Substitution

If $g'(x)$ is continuous on the interval $[a, b]$ and $f(u)$ is continuous on the range of

$$u = g(x),$$

then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

1. Find the relationship $u(x)$between u and x.
2. differentiate both side of $u = u(x)$. That is, $du = ...dx$. Express in $dx = ...du$. Express the correction term in u rather than x. Express the integrand in u.
3. Do the integral in terms of u and change the upper and lower limit of the integral to $g(a), g(b)$.

* Remember to change upper and lower limits w.r.t. u after substitution.
* Some substitution may switch upper and lower limits. For example, when $u = cosx$, and x is integrated from 0 to π, then u is integrated from 1 to -1.
* When the limit of integral is an expression containing x, we need to use substitution and chain rule to derive the correct answer.
* When to use substitution? The purpose of integration is simplification. In some cases, part of the correction term u' may appear in the integrand. We can set $u = ax + b$ in functions like $f(ax + b)$.
* Sometimes, substitution is obvious. eg. $\int cos(2x+3)dx$. We need not write down the relationship between u and x explicitly.

Integration by parts

Suppose f and g are continuous, and f' is continuous, then

$$\int f(x)g(x)dx = f(x)G(x) - \int f'(x)G(x)dx \quad \int_a^b f(x)g(x)dx = [f(x)G(x)]_a^b - \int_a^b f'(x)G(x)dx$$

* Integration by parts is used when
1.the integrand has just one term and its antiderivative cannot be found on table of basic integrals.
2. a substitution that can simplify the calculation is hard to find (this happens when

the integrand is the product of two different fundamental functions).
* LIATE: logarithmic, inverse trigonometric, algebraic, trigonometric, exponential.
* The difference of same indefinite integrals is a constant C rather than zero.
* Sometimes, $\int f(x)g(x)dx$ may appear to be the same as $\int f'(x)G(x)dx$.
* To evaluate an integral, we may need to use both substitution and integration by parts. Also, to evaluate $\int f'(x)G(x)dx$, we may apply integration by parts once again.

Lecture 17

Integrals evaluated by substitution and integration by parts

Reduction formula for powers of sine and cosine

For positive integers n,

$$\int sin^n x dx = -\frac{1}{n} sin^{n-1} x cos x + \frac{n-1}{n} \int sin^{n-2} x dx$$
$$\int cos^n x dx = \frac{1}{n} cos^{n-1} x sin x + \frac{n-1}{n} \int cos^{n-2} x dx$$

Evaluation of $\int sin^m x cos^n x dx$

1. either m or n is odd (containing the case where both m and n are odd).
eg. m is odd.

$$\int sin^m x cos^n x dx = \int sin^{m-1} x sin x cos^n x dx = \int -(1-cos^2 x)^{\frac{m-1}{2}} cos^n x d(cos x)$$

2. both m and n are even.
lower the power by $sin^2 x = \frac{1}{2}(1 - cos 2x), cos^n x = \frac{1}{2}(1 + cos 2x)$; apply reduction formula.

Evaluation of $sin mx cos nx$

Use equations

$$sin\alpha cos\beta = \frac{1}{2}[sin(\alpha+\beta) + sin(\alpha-\beta)]$$
$$cos\alpha sin\beta = \frac{1}{2}[sin(\alpha+\beta) - sin(\alpha-\beta)]$$
$$cos\alpha cos\beta = \frac{1}{2}[cos(\alpha+\beta) + cos(\alpha-\beta)]$$
$$sin\alpha sin\beta = -\frac{1}{2}[cos(\alpha+\beta) - cos(\alpha-\beta)]$$

Then use substitution.

Reduction formulae for tangent and secant function

When $n \geqslant 2$,

$$\int tan^n x dx = \frac{tan^{n-1}x}{n-1} - \int tan^{n-2} x dx$$
$$\int sec^n x dx = \frac{sec^{n-2}xtanx}{n-1} + \frac{n-2}{n-1}\int sec^{n-2} x dx$$

Specially,

$$\int tan^2 x dx = tanx - x + C$$
$$\int sec^2 x dx = tanx + C$$

When $n = 1$

$$\int \tan x dx = In|\sec x| + C$$
$$\int \sec x dx = In|\tan x + \sec x| + C$$

Evaluation of $\int tan^m x sec^n x dx$

1. m and n are even.
Use $\sec^2 x = \tan^2 x + 1$ or $\tan^2 x = \sec^2 x - 1$ and reduction formulae. We can also use method 2 to evaluate.
2. n is even. $u = \tan x$
3. m is odd. $u = \sec x$
* If n is even and m is odd, then u can either be $\tan x$ or $\sec x$.
4. m is even and n is odd. Use $\tan^2 x = \sec^2 x - 1$ and reduction formulae.
* Give priority to method 2 and 3 if the integrand satisfies their requirements, because we do not need to consider reduction formulae.

Trigonometric substitutions

When we encounter functions like

$$\sqrt{a^2 - x^2}, \sqrt{a^2 + x^2}, \sqrt{x^2 - a^2}$$

we use trigonometric substitution to eliminate radical.
For example, we set $x = a \sin t$ in $\sqrt{a^2 - x^2}$, where $t \in [-\pi/2, \pi/2]$. The range of t enables $\sin t$ varies from $-a$ to a and $\cos t$ is always bigger than 0. Therefore, we

do not need to add abs outside $\cos t$ when we eliminate the radical.
* The difference between substitution and trigonometric substitution:
Substitution: use u to simplify $g(x)$. To change dx to du, we'd better find some term in the integrand same to the correction term g'.
Trigonometric substitution: substitute x with a trigonometric function. To change dx to dt, we just differentiate both sides of $x = \varphi(t)$ and directly substitute dx with $\varphi' dt$.
Therefore, trigonometric substitution is actually use substitution in a reversed way.

Integration by partial fractions

* Used in rational functions.
* Separate a rational function into the sum of fractions whose integrals can be evaluated using table, substitution, etc.
* For improper fraction, we transform it to the sum of polynomials and partial fractions. Sometimes, it is hard to spot which polynomial to separate from the fraction, so we use long division. * For rational function involving trigonometric functions, we set

Factor in $Q(x)$	Term in P.F.D.
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b} + \cdots + \dfrac{A_k}{(ax+b)^k}$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c} + \cdots + \dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$

$$dx = \frac{2}{1+u^2}du$$
$$\sin x = \frac{2u}{1+u^2}$$
$$\cos x = \frac{1-u^2}{1+u^2}$$

The integrand is changed to rational function with respect to u. However, this method sometimes complicate calculations.
* Integration by partial fractions always combines with substitution.
* For fractions like $\frac{Ax+B}{ax^2+bx+c}$, we set $u = ax^2 + bx + c$. Then the fraction changes to $\frac{mu'}{u}du + \frac{const}{ax^2+bx+c}dx$. For the latter term, we use substitution and the derivative

of arctan.
* Some integrand contains radicals, and we can use substitution to change radical functions to rational functions.

Other properties of definite integral

1. If $f(x)$ is continuous on $[-a, a]$, and it is an even function, then

$$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$$

If $f(x)$ is an odd function, then

$$\int_{-a}^{a} f(x)dx = 0$$

2. If $f(x)$ is a continuous function with a period of l, then for any real number a, we have

$$\int_{a}^{a+l} f(x)dx = \int_{0}^{l} f(x)dx$$

3. If $f(x)$ is continuous on $[0, 1]$, then

$$\int_{0}^{\pi/2} f(\sin x)dx = \int_{0}^{\pi/2} f(\cos x)dx$$
$$\int_{0}^{\pi} xf(\sin x)dx = \pi\int_{0}^{\pi/2} f(\sin x)dx$$