
Question1 (3 points)

Determine whether each of the followings is true. If not, briefly explain why it is false.

(a) (1 point) If a function has the limit $\lim_{x\to\infty} f(x) = L$, then $\{f_n\}$ converges to L as well.

$$\text{where} \quad f_n = f(n), \quad \text{for} \quad n \in \mathbb{N}.$$

Solution:

1M True. This can be proved by considering the definition for each of the two limits. All we need to do is to pick an integer N that is bigger than or equal to the real number M in the definition of the limit of the function.

(b) (1 point) If a sequence $f_n = f(n)$ converges to L, then the function has the same limit

$$\lim_{x\to\infty} f(x) = L$$

Solution:

1M False. The argument used above is not valid in reverse. Consider the following

$$f(x) = \sin(\pi x)$$

(c) (1 point) If $f(x)$ is defined everywhere except at $x = a$, then the following limit exists

$$\lim_{x\to a} f(x)$$

Solution:

1M False. Being defined near $x = a$ is only a starting point, that is, it is a necessary condition, but by no means a sufficient condition. Consider the following

$$f(x) = \frac{1}{x} \qquad \text{at} \quad x = 0$$

Question2 (4 points)

Find the following limits. You may use any laws/theorems that we have covered in class. However, indicate which law/theorem you are using and justify its use at every step.

(a) (1 point) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x^2+x}\right)$

(b) (1 point) $\lim_{x\to\infty}\frac{x^2-3x+2}{2x^2+5x+1}$

(c) (1 point) $\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

(d) (1 point) $\lim_{x\to 0}\frac{\sin x}{x}$

Solution:

(a) Step (1) is a simple algebraic manipulation,

$$\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x^2+x}\right) \overset{(1)}{=} \lim_{x\to 0}\frac{x}{x(x+1)} \overset{(2)}{=} \lim_{x\to 0}\frac{1}{(x+1)} \overset{(3)}{=} 1$$

Step (2) is based on a simple cancellation or limit law 6, since

$$\frac{1}{x+1} = \frac{x}{x(x+1)} \qquad \text{for all } x \text{ except } x = 0.$$

Step (3) is based on limit law 5 since $x = 0$ is in the domain of the rational function

$$f(x) = \frac{1}{x+1}$$

(b) Step (1) is a direct application of the very last theorem in lecture 4,

$$\lim_{x\to\infty} \frac{x^2 - 3x + 2}{2x^2 + 5x + 1} \overset{(1)}{=} \lim_{x\to\infty} \frac{x^2}{2x^2} \overset{(2)}{=} \lim_{x\to\infty} \frac{1}{2} \overset{(3)}{=} \frac{1}{2}$$

Step (2) can be treated as a simple cancellation or a variation of limit law 6

$$\frac{x^2}{2x^2} = \frac{1}{2} \qquad \text{for all } x > 0.$$

Step (3) is by a variation of limit law 1 involving $x \to \infty$ instead of a finite value a.

(c) For this question, we need to consider the two one-sided limits,

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right) \overset{(1)}{=} \lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{x} \right) \overset{(2)}{=} \lim_{x\to 0^+} (0) \overset{(3)}{=} 0$$

Step (1) is due to the fact $x > 0$ when x is approaching 0 from the right.

Step (2) is an algebraic manipulation or can be taken as an application of law 6

$$\frac{1}{x} - \frac{1}{x} = 0 \qquad \text{for all } x \neq 0.$$

Step (3) is a direct application of law 1.

Now from the left, step (1) and (2) are essentially due to the same reasons as before

$$\lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right) \overset{(1)}{=} \lim_{x\to 0^-} \left(\frac{1}{x} + \frac{1}{x} \right) \overset{(2)}{=} \lim_{x\to 0^-} \left(\frac{2}{x} \right) \overset{(3)}{=} -\infty$$

Step (3) is clearly true by our understanding regarding rational functions, however, here is a formal justification behind the conclusion. For any $m < 0$, we need to show there exists $\delta > 0$ such that

$$\frac{2}{x} < m \qquad \text{whenever} \qquad -\delta < x - 0 < 0$$

since $m < 0$ and $x < 0$, the inequality is true

$$\frac{2}{x} < m \qquad \text{if and only if} \qquad \frac{2}{m} < x - 0$$

So $\delta = -\dfrac{2}{m}$ is a valid δ for any $m < 0$, that shows

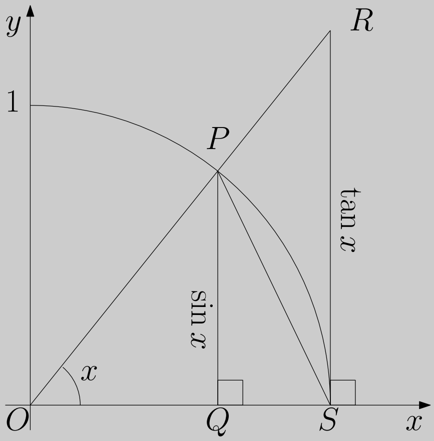
$$\lim_{x\to 0^-} \left(\frac{2}{x} \right) = -\infty$$

This leads us to conclude the left-hand limit does not equal to the right-hand limit,

$$\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{|x|} \right) \neq \lim_{x\to 0^-} \left(\frac{1}{x} - \frac{1}{|x|} \right)$$

which implies the two-sided limit does not exist.

(d) This limit can be found by applying the squeeze theorem, and the desired inequalities can be found by considering the following unit circle centred at the origin



Suppose $0 < x < \frac{\pi}{2}$, then

$$\begin{aligned} \text{area}\,\triangle\, OSP &< \text{area sector } OSP < \text{area}\,\triangle\, OSR \\ \frac{\sin x}{2} &< \frac{x}{2} < \frac{\tan x}{2} \\ \frac{\sin x}{2}\frac{2}{\sin x} &< \frac{x}{2}\frac{2}{\sin x} < \frac{\tan x}{2}\frac{2}{\sin x} \end{aligned}$$

Taking reciprocals, we have

$$1 > \frac{\sin x}{x} > \cos x \quad \text{for} \quad 0 < x < \frac{\pi}{2}$$

By the squeeze theorem, we have

$$\lim_{x\to 0^+} \frac{\sin x}{x} = 1$$

Since $\sin x$ and x are both odd functions, the quotient $\frac{\sin x}{x}$ is an even function, thus symmetric about y-axis, hence the left-hand limit at 0 must be equal to the right-hand limit at 0, that is,

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \implies \lim_{x\to 0} \frac{\sin x}{x} = 1$$

The value of this limit, together with the following, was used in class

$$\lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

I will leave it to you to confirm it being zero by applying an appropriate trig identity.

Question3 (2 points)

Prove the following statements using the precise definition of a limit.

(a) (1 point) $\lim_{x\to 4}(2x-1) = 7$ (b) (1 point) $\lim_{x\to 5}\sqrt{x-1} = 2$

Solution:

(a) We need to show, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|2x - 8| < \epsilon \qquad \text{if} \qquad 0 < |x - 4| < \delta$$

for every $\epsilon > 0$, the following statement is true

$$|2x - 8| < \epsilon \qquad \text{if} \qquad 2\,|x - 4| < \epsilon$$

therefore $\delta = \frac{\epsilon}{2}$ is a valid choice for any $\epsilon > 0$, and thus $\lim_{x\to 4}(2x-1) = 7$.

(b) We need to show, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\sqrt{x-1} - 2\right| < \epsilon \qquad \text{if} \qquad 0 < |x-5| < \delta$$

Since for every $\epsilon > 0$, the following is true

$$\begin{aligned}
&\left|\sqrt{x-1} - 2\right| < \epsilon \qquad \text{if} \qquad \left|\sqrt{x-1} - 2\right| \frac{\left|\sqrt{x-1}+2\right|}{\left|\sqrt{x-1}+2\right|} < \epsilon \\
\implies &\left|\sqrt{x-1} - 2\right| < \epsilon \qquad \text{if} \qquad \frac{|x-5|}{\left|\sqrt{x-1}+2\right|} < \epsilon
\end{aligned}$$

Now we use the same trick that we have used in lecture 4, let us assume

$$\begin{aligned}
\delta \le 1 \implies |x-5| < \delta \le 1 &\implies -1 < x-5 < 1 \\
&\implies \sqrt{3}+2 < \sqrt{x-1}+2 < \sqrt{5}+2
\end{aligned}$$

So the following must be true under the assumption $\delta \le 1$

$$\left|\sqrt{x-1} - 2\right| < \epsilon \qquad \text{if} \qquad |x-5| < (\sqrt{3}+2)\epsilon$$

Therefore $\delta = \min\left\{1, (\sqrt{3}+2)\epsilon\right\}$ is a valid choice for every $\epsilon > 0$, and thus

$$\lim_{x\to 5} \sqrt{x-1} = 2$$

Note it seems to suggest that for functions involving radicals, the limit of the function at a point a inside the domain of the function is the function evaluated at a, which should not be a surprise with the notion of

Question4 (1 points)

In the theory of relativity, the mass of a particle with speed v is given by

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light.

What happens as $v \to c^-$?

that is, as the speed of the particle is approaching the speed of light from below.

Solution:

1M We simply need to consider the following limit

$$\lim_{v\to c^-} m = \lim_{v\to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

If the particle has no mass, that is, $m_0 = 0$, then the mass stays as zero as $v \to c^-$.

$$\lim_{v\to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \lim_{v\to c^-} \frac{0}{\sqrt{1 - v^2/c^2}} = \lim_{v\to c^-} 0 = 0$$

However, if the particle has a positive mass at rest, then

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \infty$$

To see how we can obtain the last limit. Let

$$x = 1 - \frac{v^2}{c^2}$$

then by limit law 5

$$\lim_{v \to c^-} x = \lim_{v \to c^-} \left(1 - \frac{v^2}{c^2}\right) = 0^+$$

that is,

$$x \to 0^+ \qquad \text{as} \qquad v \to c^-$$

Let $y = \sqrt{x}$, then

$$\lim_{v \to c^-} y = \lim_{x \to 0^+} \sqrt{x} = \sqrt{0^+} = 0^+ \implies \lim_{v \to c^-} \frac{m_0}{\sqrt{1 - v^2/c^2}} = \lim_{v \to c^-} m_0 \frac{1}{y}$$
$$= m_0 \lim_{y \to 0^+} \frac{1}{y}$$
$$= \infty$$

The key step above used the following theorem:

> Suppose n is a positive integer, then the following limit is valid for all a if n is odd, and is valid for $a > 0$ if n is even.
>
> $$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

To prove the above, we need to show, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left| \sqrt[n]{x} - \sqrt[n]{a} \right| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

If $a = 0$ and n is odd, for every $\epsilon > 0$, $|\sqrt[n]{x} - 0| < \epsilon$ is true if the following is true

$$-\epsilon < \sqrt[n]{x} < \epsilon \impliedby (-\epsilon)^n < x < \epsilon^n \impliedby -\epsilon^n < x - 0 < \epsilon^n \impliedby 0 < |x - 0| < \epsilon^n$$

Thus $\delta = \epsilon^n$ is a valid choice in this case and it proves this part of the theorem

Now suppose $a > 0$, and n is either odd or even, for every $\epsilon^* > 0$ such that

$$\sqrt[n]{a} > \epsilon^*$$

the inequality $|\sqrt[n]{x} - \sqrt[n]{a}| < \epsilon^*$ is true if the following is true

$$\begin{aligned} -\epsilon^* < \sqrt[n]{x} - \sqrt[n]{a} < \epsilon^* &\Longleftarrow \sqrt[n]{a} - \epsilon^* < \sqrt[n]{x} < \sqrt[n]{a} + \epsilon^* \\ &\Longleftarrow \left(\sqrt[n]{a} - \epsilon^*\right)^n < x < \left(\sqrt[n]{a} + \epsilon^*\right)^n \\ &\Longleftarrow \left(\sqrt[n]{a} - \epsilon^*\right)^n - a < x - a < \left(\sqrt[n]{a} + \epsilon^*\right)^n - a \\ &\Longleftarrow -\left(a - \left(\sqrt[n]{a} - \epsilon^*\right)^n\right) < x - a < \left(\sqrt[n]{a} + \epsilon^*\right)^n - a \\ &\Longleftarrow 0 < |x - a| < \min\left\{a - \left(\sqrt[n]{a} - \epsilon^*\right)^n, \left(\sqrt[n]{a} + \epsilon^*\right)^n - a\right\} \end{aligned}$$

hence for small enough ϵ^*, there exists

$$\delta = \min\left\{a - \left(\sqrt[n]{a} - \epsilon^*\right)^n, \left(\sqrt[n]{a} + \epsilon^*\right)^n - a\right\}$$

such that

$$\left|\sqrt[n]{x} - \sqrt[n]{a}\right| < \epsilon^* \qquad \text{if} \qquad 0 < |x - a| < \delta$$

now for every $\epsilon > \epsilon^* > 0$, it is clear that it also works,

$$\left|\sqrt[n]{x} - \sqrt[n]{a}\right| < \epsilon^* < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

The final case is when $a < 0$ and n is an odd integer, which means the function

$$f(x) = \sqrt[n]{x}$$

is an odd function and thus is symmetric about the origin. Hence the limit

$$\lim_{x\to a} \sqrt[n]{x} = \lim_{x\to -a} -\sqrt[n]{x} = -\lim_{x\to -a} \sqrt[n]{x} = -\sqrt[n]{-a} = \sqrt[n]{a}$$

can be reduced to the previous case.

Question5 (1 points)

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$, prove the following is true

$$\lim_{x\to a} \Big(f(x) + g(x)\Big) = L + K$$

Solution:

1M We need to show for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\left|\Big(f(x) + g(x)\Big) - \Big(L + K\Big)\right| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

Since $\lim_{x\to a} f(x) = L$, for any $\epsilon_f = \dfrac{\epsilon}{2} > 0$, there exists $\delta_f > 0$ such that

$$|f(x) - L| < \epsilon_f \qquad \text{if} \qquad 0 < |x - a| < \delta_f$$

Similarly, we have, for any $\epsilon_g = \dfrac{\epsilon}{2} > 0$, there exists $\delta_g > 0$ such that

$$|g(x) - K| < \epsilon_g \qquad \text{if} \qquad 0 < |x - a| < \delta_g$$

Let $\delta = \min\{\delta_f, \delta_g\}$, then the two limit statements together imply

$$|f(x) - L| + |g(x) - K| < \epsilon_f + \epsilon_g \qquad \text{if} \qquad 0 < |x - a| < \delta$$

By the triangle inequality,

$$|f(x) - L + g(x) - K| \leq |f(x) - L| + |g(x) - K| < \epsilon_f + \epsilon_g$$

thus $\delta = \min\{\delta_f, \delta_g\}$ is a valid choice for any $\epsilon > 0$ such that

$$\Big|\Big(f(x) + g(x)\Big) - \Big(L + K\Big)\Big| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

therefore

$$\lim_{x \to a} \Big(f(x) + g(x)\Big) = L + K$$

Question6 (1 points)

Find the values of a and b such that the following function is continuous everywhere.

$$f(x) = \begin{cases} \left(x^2 - 4\right)(x - 2)^{-1} & \text{if} \quad x < 2, \\ ax^2 - bx + 3 & \text{if} \quad 2 \leq x < 3, \\ 2x - a + b & \text{if} \quad x \geq 3. \end{cases}$$

Solution:

1M We need to make sure

$$\lim_{x \to 2} f(x) = f(2) \qquad \text{and} \qquad \lim_{x \to 3} f(x) = f(3)$$

the limit at $x = 2$ from the left,

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} (x + 2) = 4$$

which needs to be equal to the limit at $x = 2$ from the right

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x) \implies 4 = 4a - 2b + 3 \implies 4a - 2b = 1 \tag{1}$$

the limit at $x = 3$ from the left,

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (ax^2 - bx + 3) = 9a - 3b + 3$$

which needs to be equal to the limit at $x = 3$ from the right

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} f(x) \implies 9a - 3b + 3 = 6 - a + b \implies 10a - 4b = 3 \tag{2}$$

Solve (1) and (2), we have

$$a = \frac{1}{2}, \qquad b = \frac{1}{2}$$

Question7 (3 points)

(a) (1 point) Prove that the absolute value function $f(x) = |x|$ is continuous everywhere.

Solution:

1M We need to show that

$$\lim_{x \to a} = f(a)$$

that is, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\Big| |x| - |a| \Big| < \epsilon \qquad \text{if} \qquad 0 < |x-a| < \delta$$

by the reverse triangle inequality

$$|x-a| \geq \Big| |x| - |a| \Big|$$

so $\Big| |x| - |a| \Big| < \epsilon$ is true if

$$0 < |x-a| < \epsilon$$

thus $\delta = \epsilon$ is a valid choice for any $\epsilon > 0$ and any a, therefore

$$\lim_{x \to a} = f(a)$$

(b) (1 point) Prove that if $g(x)$ is continuous on an open interval, then so is $|g(x)|$.

Solution:

1M If $g(x)$ is continuous on $\mathcal{I}$, then for any value $a \in \mathcal{I}$,

$$\lim_{x \to a} g(x) = g(a)$$

that is, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\Big| g(x) - g(a) \Big| < \epsilon \qquad \text{if} \qquad 0 < |x-a| < \delta$$

by the reverse triangle inequality

$$\epsilon > |g(x) - g(a)| \geq \Big| |g(x)| - |g(a)| \Big|$$

which means, for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\Big| |g(x)| - |g(a)| \Big| \qquad \text{if} \qquad 0 < |x-a| < \delta$$

therefore $|g(x)|$ is continuous on $\mathcal{I}$.

$$\lim_{x\to a} |g(x)| = |g(a)|$$

(c) (1 point) Is the converse of the statement in part (b) also true? Justify your answer.

Solution:

1M No. Consider

$$g(x) = \begin{cases} 1 & \text{if} \quad \geq 0 \\ -1 & \text{if} \quad < 0 \end{cases}$$

The absolute value of the function $|g(x)| = 1$ is clearly continuous. However,

$$\lim_{x\to 0^-} g(x) = -1 \neq 1 = \lim_{x\to 0^+} g(x)$$

which is not continuous at $x = 0$.

Question8 (2 points)
Prove two of the following LIATE functions of your choice are continuous in their domains.

(a) *L*ogarithmic:

$$\ln(x)$$

(b) *I*nverse trigonometric:

$$\arctan(x)$$

(c) *A*lgebraic:

$$\sqrt[3]{x} \quad (x > 0)$$

(d) *T*rigonometric:

$$\sin(x)$$

(e) *E*xponential:

$$\mathrm{e}^x$$

[Hint:] For $\ln(x)$ and e^x, you might use the fact

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$

Solution:

(a) We need to show, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\ln(x) - \ln(a)| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

where $a > 0$. For every $\epsilon > 0$, $|\ln(x) - \ln(a)| < \epsilon$ is true if

$$\ln(a) - \epsilon < \ln(x) < \ln(a) + \epsilon \impliedby \frac{a}{\mathrm{e}^{\epsilon}} - a < x - a < a\,\mathrm{e}^{\epsilon} - a$$

Thus $\delta = a \min\left\{1 - \frac{1}{\mathrm{e}^{\epsilon}}, \mathrm{e}^{\epsilon} - 1\right\}$ is a valid δ for every $\epsilon > 0$, therefore

$$\lim_{x \to a} \ln(x) = \ln(a), \qquad \text{where } a > 0.$$

and $f(x) = \ln(x)$ is continuous in its domain.

(b) We need to show, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\arctan(x) - \arctan(a)| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

Let $\alpha = \arctan(x)$ and $\beta = -\arctan(a)$, then by the tagent addition formula

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} \\ \tan(\arctan(x) - \arctan(a)) &= \frac{x - a}{1 + xa} \\ \arctan(x) - \arctan(a) &= \arctan\left(\frac{x - a}{1 + xa}\right) \end{aligned}$$

thus, for every $\epsilon > 0$,

$$|\arctan(x) - \arctan(a)| < \epsilon \impliedby \left|\arctan\left(\frac{x - a}{1 + xa}\right)\right| < \epsilon$$

Now let us assume $\delta \leq \dfrac{|a|}{2}$, then

$$\begin{aligned} |x - a| < \delta \leq \frac{|a|}{2} &\implies a - \frac{|a|}{2} < x < a + \frac{|a|}{2} \\ &\implies \begin{cases} 1 < 1 + a^2 - \frac{a|a|}{2} < 1 + xa < 1 + a^2 + \frac{a|a|}{2} & a > 0 \\ 1 \leq 1 + a^2 + \frac{a|a|}{2} < 1 + xa < 1 + a^2 - \frac{a|a|}{2} & a \leq 0 \end{cases} \end{aligned}$$

Hence $1 + xa$ is always bigger than 1 under this assumption, which means

$$\left|\arctan\left(\frac{x - a}{1 + xa}\right)\right| < \epsilon \impliedby |\arctan(x - a)| < \epsilon \impliedby |x - a| < \tan(\epsilon)$$

Thus $\delta = \min\left\{\frac{|a|}{2}, \tan(\epsilon)\right\}$ is a valid for any $\epsilon > 0$. therefore

$$\lim_{x \to a} \arctan(x) = \arctan(a)$$

and $f(x) = \arctan(x)$ is continuous everywhere.

(c) We have proved the following theorem earlier:

Suppose n is a positive integer, then the following limit is valid for all a if n is odd, and is valid for $a > 0$ if n is even.

$$\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$$

essentially the same argument can be used for this specific case.

(d) We need to show, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\sin(x) - \sin(a)| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

For every $\epsilon > 0$,

$$|\sin(x) - \sin(a)| < \epsilon \impliedby \left|2\cos\left(\frac{x+a}{2}\right)\sin\left(\frac{x-a}{2}\right)\right| < \epsilon \impliedby \left|\sin\left(\frac{x-a}{2}\right)\right| < \frac{\epsilon}{2}$$

Let us assume $\delta \leq \frac{\pi}{2}$, then

$$\begin{aligned}
|x-a| < \delta \leq \frac{\pi}{2} &\implies -\frac{\pi}{2} < -\delta < x - a < \delta < \frac{\pi}{2} \\
&\implies -\frac{\pi}{4} \leq -\frac{\delta}{2} < \frac{x-a}{2} < \frac{\delta}{2} \leq \frac{\pi}{4} \\
&\implies -\sin\left(\frac{\delta}{2}\right) < \sin\left(\frac{x-a}{2}\right) < \sin\left(\frac{\delta}{2}\right) \\
&\implies \left|\sin\left(\frac{x-a}{2}\right)\right| < \sin\left(\frac{\delta}{2}\right)
\end{aligned}$$

Thus the valid δ is given by

$$\sin\left(\frac{\delta}{2}\right) \leq \frac{\epsilon}{2}$$

which is satisfied by $\delta = \epsilon$, if it is not clear to you, see picture in question 2 part (d). Hence $\delta = \min\left\{\frac{\pi}{2}, \epsilon\right\}$ is a valid choice for any $\epsilon > 0$, and

$$\lim_{x \to a} \sin(x) = \sin(a)$$

and $f(x) = \sin(x)$ is continuous everywhere.

(e) We need to show, for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|\exp(x) - \exp(a)| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

For every $0 < \epsilon^* < \dfrac{\exp(a)}{2}$, then $|\exp(x) - \exp(a)| < \epsilon$ is true if

$$e^a - \epsilon^* < e^x < e^a + \epsilon^* \impliedby -\Big(a - \ln(e^a - \epsilon^*)\Big) < x - a < \ln(e^a + \epsilon^*) - a$$

Thus $\delta = \min\{a - \ln(e^a - \epsilon^*), \ln(e^a + \epsilon^*) - a\}$ is a valid for every

$$0 < \epsilon^* < \frac{\exp(a)}{2}$$

Now for $\epsilon \geq \epsilon^*$, it is also valid since it represents a wilder interval

$$|\exp(x) - \exp(a)| < \epsilon^* \leq \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

Therefore

$$\lim_{x \to a} \exp(x) = \exp(a)$$

and $f(x) = \exp(x)$ is a continuous function everywhere.

Question9 (1 points)

Suppose $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, where $A > B$. Show there exits $\delta > 0$ such that

$$f(x) > g(x) \qquad \text{if} \qquad x \in \{(a - \delta, a) \cup (a, a + \delta)\}$$

that is, there is a deleted δ-neighbourhood of a such that f is bigger than g for all x in it.

Solution:

1M Let $\epsilon = \dfrac{A - B}{2} > 0$, since

$$\lim_{x \to a} f(x) = A,$$

we know by definition there exists $\delta_1 > 0$ such that

$$0 < |x - a| < \delta_1 \implies |f(x) - A| < \epsilon$$

Therefore,

$$f(x) > A - \epsilon = \frac{A + B}{2},$$

Similarly there exists $\delta_2 > 0$ such that

$$0 < |x - a| < \delta_2 \implies |f(x) - A| < \epsilon$$

Therefore,

$$g(x) < B + \epsilon = \frac{A + B}{2},$$

So if we let

$$\delta = \min\{\delta_1, \delta_2\},$$

then, when $0 < |x - a| < \delta$ we get

$$f(x) > g(x).$$

Question10 (2 points)

Suppose $f(x)$ is continuous on the interval $[0,1]$, and $f(0)=f(1)$.

(a) (1 point) Prove there exists $x \in [0,1]$ such that

$$f(x)=f\left(x+\frac{1}{2}\right)$$

Solution:

1M Let $g(x)=f(x)-f(x+\frac{1}{2})$, then we have

$$\begin{aligned} g(0)&=f(0)-f\left(\frac{1}{2}\right)\\ g\left(\frac{1}{2}\right)&=f\left(\frac{1}{2}\right)-f(1)\end{aligned} \implies g(0)+g\left(\frac{1}{2}\right)=f(0)-f(1)=0$$

If $f(0)=f\left(\frac{1}{2}\right)=f(1)$, then the result is trivial. So let us assume

$$f(0)\neq f\left(\frac{1}{2}\right)\neq f(1)$$

for the reminding of the proof. Since

$$\min\left\{g(0),g\left(\frac{1}{2}\right)\right\}<\frac{g(0)+g\left(\frac{1}{2}\right)}{2}<\max\left\{g(0),g\left(\frac{1}{2}\right)\right\}$$

there must exist a number $x\in[0,1]$ by IVT such that

$$g(x)=\frac{g(0)+g(\frac{1}{2})}{2} \implies g(x)=0 \implies f(x)=f\left(x+\frac{1}{2}\right)$$

(b) (1 point) Prove there exists $x \in [0,1]$ such that

$$f(x)=f\left(x+\frac{1}{n}\right) \qquad \text{for any positive integer } n.$$

Solution:

1M The same logic can be applied. Let $h(x)=f(x)-f(x+\frac{1}{n})$, then

$$\begin{aligned} h(0)&=f(0)-f\left(\frac{1}{n}\right)\\ h\left(\frac{1}{n}\right)&=f\left(\frac{1}{n}\right)-f\left(\frac{2}{n}\right)\\ &\vdots\\ h\left(\frac{n-1}{n}\right)&=f\left(\frac{n-1}{n}\right)-f(1)\end{aligned} \implies \sum_{i=0}^{n-1}h\left(\frac{i}{n}\right)=f(0)-f(1)=0$$

then we have

$$\min\left\{h\left(\frac{0}{n}\right),\ldots,h\left(\frac{n-1}{n}\right)\right\} < \frac{1}{n}\sum_{i=0}^{n-1} h\left(\frac{i}{n}\right) < \max\left\{h\left(\frac{0}{n}\right),\ldots,h\left(\frac{n-1}{n}\right)\right\}$$

there must exist a number $x \in [0,1]$ by IVT such that

$$h(x) = \frac{1}{n}\sum_{i=0}^{n-1} h(\frac{i}{n}) = 0 \implies h(x) = 0 \implies f(x) = f\left(x+\frac{1}{n}\right)$$

Question11 (1 points)

Suppose $f(x)$ is continuous on the interval $[a,b]$. Show that there exists $c \in [a,b]$ such that

$$f(c) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + ... + \alpha_n f(x_n)$$

where $x_i \in [a,b]$ and the coefficients are all positive and the sum is 1, that is,

$$\alpha_i > 0 \quad \text{for } n = 1,2,\ldots,n \qquad \text{and} \qquad \sum_{i=1}^{n} \alpha_i = 1$$

Solution:

1M Let

$$\mathcal{S} = \{f(x_1), f(x_2), \ldots, f(x_n)\}$$

Since the set $\mathcal{S}$ contains a finite number of elements, we must have

$$\max(\mathcal{S}) = M \qquad \text{and} \qquad \min(\mathcal{S}) = m$$

that is, there must exist a maximum value and a minimum value of $\mathcal{S}$, therefore,

$$m \le f(x_i) \le M \qquad \text{for} \qquad i = 1,2,...,n$$

If $M = m$, that is $f(x_i) = f(x_j)$ for all i, j, then c can take any x_i. So let us assume

$$M \neq m$$

for the remind of the proof. Multiply each by its coefficients,

$$\alpha_i m \le \alpha_i f(x_i) \le \alpha_i M \qquad \text{for} \qquad i = 1,2,...,n$$

Since not every i attains the maximum or the minimum, then

$$\sum_{i}^{n} \alpha_i m < \sum_{i}^{n} \alpha_i f(x_i) < \sum_{i}^{n} \alpha_i M \implies m < \sum_{i}^{n} \alpha_i f(x_i) < M$$

This shows the affine combination

$$\alpha_1 f(x_1) + \alpha_2 f(x_2) + ... + \alpha_n f(x_n)$$

is a value between m and M, by the continuity of $f(x)$, IVT guarantees there exists

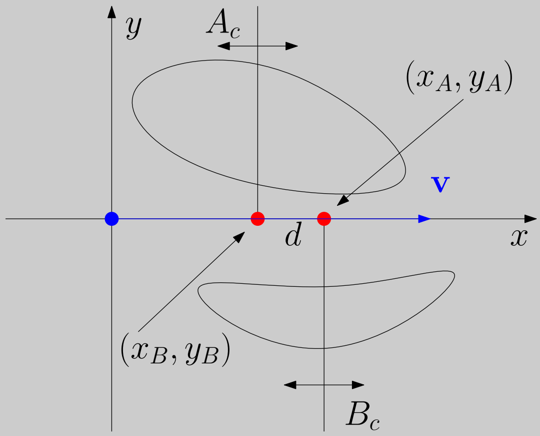
a point $c \in [a,b]$ such that $f(c) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + ... + \alpha_n f(x_n)$

Question12 (1 points)

In theory, can you always simultaneously slice two rain drops each exactly in half with a single straight-line cut, no matter the shapes of the rain drops nor their location on the windscreen.

Solution:

1M Let us introduce a coordinate system so that two rain drops are entirely in the first quadrant and the fourth quadrant respectively.



In class, we have discussed that it is always possible to cut one rain drop in half and there are infinitely many ways to do so under the assumption of continuity. Let A_c and B_c to be such cuts, one for each rain drop such that both cuts are perpendicular to a vector $\mathbf{v}$, which is parallel to the x-axis initially.

Let the directed distance between the two cuts A_c and B_c be d, that is, d can be negative as well as being positive. Specifically, let $\mathbf{v}$ be the reference. It is clear if d takes the value zero, then we have a single straight-line cut that divides both drops in half.

Suppose d is positive initially. If we rotate the vector $\mathbf{v}$ counter-clockwise by an angle of θ, then we expect d will change, so d is a function of θ

$$d(\theta)$$

Now if we rotate $\theta = \pi$, then $\mathbf{v}$ is pointing in the opposite direction to the initial direction. Thus the directed distance

$$d(\pi) = -d(0)$$

So if d is continuous, then IVP implies there must exist an angle θ such that $d = 0$. This can be derived from the assumption that whatever we assumed to be divided in half is a continuous function of where to cut. Since a composition of continuous functions is continuous. I will omit the details here.

Question13 (1 points)

Find the largest interval, if it exists, on which the following function is continuous.

$$f(x) = \sum_{n=1}^{\infty} \frac{B(nx)}{n^2}, \qquad \text{with} \quad B(x) = \begin{cases} x - [x] & \text{if } x \neq k/2, \\ 0 & \text{if } x = k/2, \end{cases}$$

where k is an integer and $[x]$ denotes the nearest integer to x. Justify your answer.

Solution:

1M The function is actually nowhere continuous, so it is not possible to find such interval.

I don't expect you to show this conclusion rigorously. You only need to explain how you have got the conclusion. However, you have to demonstrate that you have done more than guessing the correct answer or listening to others. For the very least, you have to demonstrate you understand the definition of $f(x)$ and intuitively know why $f(x)$ cannot be continuous.

First, notice the function $A(x) = x - [x]$ is zero only and if only

$$x \text{ takes integer values,}$$

and $A(x)$ has discontinuities only at

$$x = \frac{i}{2} \qquad \text{where } i \text{ takes odd integers.}$$

so the horizontal distance d_1 between two consecutive discontinuities of $A(x)$ is always

$$d_1 = 1$$

Now let us consider the case $x = \dfrac{k}{2}$ for a fixed k value, if k is even,

$$B(x) = A(x) \qquad \text{for all } x.$$

and if k is odd, then

$$B(x) = \begin{cases} A(x) + \dfrac{1}{2} & \text{if } x = \dfrac{k}{2} \\ A(x) & \text{Otherwise} \end{cases}$$

Note going from $A(x)$ to $B(x)$ does not introduce any new discontinuity or remove any discontinuity of $A(x)$. The change, if any, merely is on the value that the dependent variable takes at $x = k/2$. Thus the distance between two consecutive discontinuities is still

$$d_1 = 1$$

So we don't really need to worry about the point $x = k/2$ for it will not affect the continuity of $B(x)$. Note $B(x)$ thus is an odd periodic function whose period is $d_1 = 1$ excluding the point $x = k/2$.

Now consider,

$$B(2x)$$

the effect of multiplying 2 to x is compressing in the x-axis by a factor of 2. So $B(2x)$, which is merely a compressed version of $B(x)$, has the same features but a different period of

$$d_2 = \frac{1}{2}$$

Now consider

$$\frac{B(2x)}{2^2}$$

the effect of dividing $2^2 = 4$ is compressing in the y-axis by a factor of 4. It will affect the value that the dependent variable takes, but again it will not affect where the discontinuities are.

Now let us continue building towards $f(x)$, thus consider the function

$$B(x) + \frac{B(2x)}{2^2}$$

it is an odd periodic function whose period is 1, but the distance between two consecutive discontinuities is now

$$d_2 = \frac{1}{2}$$

In general, consider

$$\frac{B(nx)}{n^2} \qquad \text{for} \quad n = 2, 3, \ldots$$

the effect will be compressing in both x-axis and y-axis, and thus changing the distance between two consecutive discontinuities of $B(x)$ from d_1 to

$$d_n = \frac{1}{n}$$

Thus we can conclude that

$$f(x) = \sum_{n=1}^{\infty} \frac{B(nx)}{n^2}, \qquad \text{with} \quad B(x) = \begin{cases} x - [x] & \text{if } x \neq k/2, \\ 0 & \text{if } x = k/2, \end{cases}$$

is an odd periodic function whose period is 1. So we only need to focus on

$$\mathcal{I} = [0, 1]$$

Since the period of $B(nx)$ is strictly decreasing as $n \to \infty$, and

$$f(x) = \lim_{m \to \infty} \sum_{n=1}^{m} \frac{B(nx)}{n^2}$$

the number of discontinuities of $f(x)$ in $\mathcal{I}$ is strictly increasing as $m \to \infty$. Therefore $d_n \to 0$ as $n \to \infty$ implies $f(x)$ is nowhere continuous.

Question14 (2 points)
Suppose $f(x)$ is continuous on the interval $\mathcal{I} = [a, b]$.

(a) (1 point) Show why $f(x)$ is bounded on this interval.

Solution:

1M Intuitively, this seems to be very obvious. However, checking intuition with rigorous reasoning is what this question is about. Firstly let me explain why this question is not as trivial as it seems. A function $f(x)$ is said to be bounded on some set $\mathcal{U}$ if there exists a real number M such that

$$|f(x)| \leq M \qquad \text{for all } x \in \mathcal{U}.$$

Loosely and incorrectly, you might argue such M must exist for a function $f(x)$ that is continuous on the interval $[a, b]$ because the continuity of $f(x)$ implies $f(x)$ is finite for every x, thus M is the absolute value of either the maximum or the minimum of the set of $f(x)$ values for all possible values of $x \in [a, b]$. This is a valid argument if the existence of the maximum and the minimum is guaranteed or given. If the set of $f(x)$ values were finite, then I would not protest against you jumping to the conclusion. However, the set of $f(x)$ values is not finite, it is not so clear why the maximum and the minimum must exist. Simply consider the open interval $(0, 1)$, this set is continuous, but it has not maximum or minimum. Thus the argument is not good enough.

The following argument is trying to make it slightly more rigorous. Suppose that f is not bounded on the interval $[a, b]$. Let c be the mid point of $[a, b]$, then f is unbounded in at least one of the two interval

$$[a, c] \qquad \text{and} \qquad [c, b]$$

Pick the interval on which f is unbounded, in case it is unbounded on both, pick the left one, and denote it as

$$[a_1, b_1]$$

Carry this bisection indefinitely and let

$$[a_{n+1}, b_{n+1}]$$

denote the half of

$$[a_n, b_n]$$

in which f is unbounded, if f is unbounded on both halves, the left half is always selected. The length of the nth interval is

$$\frac{(b-a)}{2^n}$$

Let $\mathcal{A}$ denote the set of leftmost endpoints $a, a_1, a_2, \ldots$ obtained from this process of bisection, and let α denote the supremum of $\mathcal{A}$. It is clear that $\alpha \in [a, b]$.

$$\alpha = \sup(\mathcal{A})$$

Since f is continuous at α, there exists $\delta > 0$ such that

$$|f(x) - f(\alpha)| < 1 \qquad \text{if} \qquad 0 < |x - \alpha| < \delta$$

By the reverse triangle inequality

$$|f(x)| - |f(\alpha)| \leq |f(x) - f(\alpha)| < 1 \implies |f(x)| < 1 + |f(\alpha)|$$

Hence $f(x)$ is bounded in the interval

$$(\alpha - \delta, \alpha + \delta)$$

Since $\alpha \geq a_n$, if the length of $[a_n, b_n]$, $\dfrac{(b-a)}{2^n}$, is less than δ, then

$$[a_n, b_n] \text{ lies entirely in the interval } (\alpha - \delta, \alpha + \delta).$$

which will lead us to a contradiction and force us to conclude that

$$f \text{ is bounded on } [a, b].$$

Now to see why there exists some n such that

$$\frac{(b-a)}{2^n} < \delta$$

Note the sequence $\{a_n\}$ is monotonic and bounded. Thus α is approaching a specific value in $[a, b]$ and the change in α becomes smaller and smaller, thus we expect the change in δ becomes smaller and smaller. Since at the limiting α, $\delta > 0$, we expect δ to approach a non-zero value. Thus we can guarantee

$$\frac{(b-a)}{2^n} < \delta \qquad \text{for some large enough integer } n.$$

since

$$\lim_{n\to\infty} \frac{(b-a)}{2^n} = 0$$

Similar things can be said if α happens to be the boundary points, that is, if

$$\alpha = a$$

then the interval should be

$$[a, a + \delta)$$

if

$$\alpha = b$$

then the interval should be

$$(b - \delta, b]$$

Thus it will not affect our argument. So f is bounded if it is continuous on $\mathcal{I}$.

(b) (1 point) Show why $f(x)$ attains its supremum and infimum on this interval.

Solution:

1M Suppose the value

$$M = \sup_{x \in \mathcal{I}} f(x)$$

is not assumed by the function $f(x)$ for $x \in \mathcal{I}$, then

$$f(x) < M \qquad \text{for all } x \in \mathcal{I}.$$

Let

$$g(x) = \frac{1}{M - f(x)}$$

It is clear that

$$g(x) > 0 \qquad \text{for all } x \in \mathcal{I}.$$

and $g(x)$ is continuous on $\mathcal{I}$, thus $g(x)$ is bounded on $\mathcal{I}$. Hence there is $K > 0$

$$g(x) < K \qquad \text{for all } x \in \mathcal{I}.$$

Thus for every $x \in \mathcal{I}$,

$$K > \frac{1}{M - f(x)} \implies M - \frac{1}{K} > f(x)$$

This contradicts the fact M being the least upper bound of $f(x)$ on $\mathcal{I}$. So M must be assumed, that is, there exists $c \in \mathcal{I}$ such that

$$f(c) = M$$

The argument for infimum is very similar and thus omitted.