Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 4

Date Due: 12:55 PM, Wednesday, the 24^{th} of March 2021

Discussion Class Preparation

Please (re-)view Video files 26-28 and/or finish reading the section "Second-Order Boundary Value Problems" in the lecture slides. You should be able to answer the following questions:

i) What is a fully homogeneous BVP?

ii) What are mixed and unmixed boundary conditions?

iii) How is the Green function for a BVP for a second-order ODE defined?

iv) How is a solution to an unmixed BVP constructed?

v) Explain how to find the functions u_1 and u_2 used to construct a solution for the general (mixed) BVP.

vi) Derive the solution formula for the general BVP. Explain how the function v is constructed to satisfy the boundary conditions.

Exercises (12 Marks)

Exercise 4.1 Equilibrium Diffusion

The equilibrium concentration u of a substance diffusing in a homogeneous, absorbing, infinite, one-dimensional medium (such as an infinite tube) is given by

$$Lu = -\frac{d^2u}{dx^2} + q^2u = f(x), \qquad x \in \mathbb{R}, \tag{1}$$

where f is the source density of the substance and $q > 0$ is a positive constant.

i) Let $\xi \in \mathbb{R}$ be fixed. Use the Fourier transform to find a fundamental solution $E(x;\xi)$ of L satisfying

$$LE(x;\xi) = \delta(x-\xi), \qquad \lim_{|x|\to\infty} E(x,\xi) = 0. \tag{2}$$

Is this a causal fundamental solution? Why or why not?

ii) Verify that the function found satisfies (2) by explicitly differentiating in the distributional sense.

(6 Marks)

Exercise 4.2 Traveling Wave

The goal of this exercise is to obtain a fundamental solution of the stationary equation for a traveling wave with wavenumber k, i.e., a function $g(x,\xi)$ satisfying

$$-\frac{d^2g}{dx^2} - k^2g = \delta(x-\xi), \qquad 0 < x, \xi < 1, \tag{3}$$

i) Find a causal fundamental solution, i.e., a function E satisfying (3) and $E(x;\xi) = 0$ for $x < \xi$ by setting $E(x;\xi) = u_\xi(x)H(x-\xi)$ and determining u_ξ.

ii) Verify directly, i.e., by explicitly differentiating in the distributional sense, that the causal fundamental solution satisfies (3).

(6 Marks)