
Question1 (1 points)

Find the derivative function of the following function using the definition of derivative.

$$g(t) = \frac{1}{\sqrt{t}}$$

State the natural domain of $g(t)$ and the natural domain of its derivative function.

Solution:

1M By definition

$$g'(c) = \lim_{h\to 0} \frac{g(c+h) - g(c)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{c+h}} - \frac{1}{\sqrt{c}}}{h}$$

Rationalizing the numerator

$$g'(c) = \lim_{h\to 0} \frac{\frac{1}{\sqrt{c+h}} - \frac{1}{\sqrt{c}}}{h} \frac{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}}{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}} = \lim_{h\to 0} \frac{\frac{1}{c+h} - \frac{1}{c}}{h\left(\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}\right)} = \lim_{h\to 0} \frac{\frac{-h}{c(c+h)}}{h\left(\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}\right)}$$

Simplify and get rid of the factor h in the denominator

$$g'(c) = \lim_{h\to 0} \frac{\frac{-h}{c(c+h)}}{h\left(\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}\right)} = \lim_{h\to 0} \frac{\frac{-1}{c(c+h)}}{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}}$$

The natural domain of $g(t)$ is the set of values of t such that $g(t)$ takes real values

$$t > 0$$

which implies $c > 0$, and thus the function

$$\frac{\frac{-1}{c(c+h)}}{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}}$$

is a composition of continuous functions in their domain, thus continuous at $h = 0$,

$$g'(c) = \lim_{h\to 0} \frac{\frac{-1}{c(c+h)}}{\frac{1}{\sqrt{c+h}} + \frac{1}{\sqrt{c}}} = \frac{\frac{-1}{c(c+0)}}{\frac{1}{\sqrt{c+0}} + \frac{1}{\sqrt{c}}} = -\frac{1}{c^2}\frac{\sqrt{c}}{2} = -\frac{1}{2}c^{-3/2}$$

Hence and the derivative function is

$$g'(t) = -\frac{1}{2}t^{-3/2}$$

the natural domain of $g'(t)$ is also

$$t > 0$$

Question2 (1 points)

Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable but that f' is not continuous at $x = 0$.

Solution:

1M By definition,

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(h)^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \to 0} h \sin \frac{1}{h}$$

which we have shown to be 0 by the squeeze theorem in class, thus

$$f'(0) = 0$$

For $x \neq 0$, it follows from the product and chain rules that $f'(x)$ is given by

$$\underbrace{f'(x)}_{C} = \underbrace{2x \sin \frac{1}{x}}_{A} - \underbrace{\cos \frac{1}{x}}_{B}$$

We know $A \to 0$ as $x \to 0$, However, B is oscillating between -1 and 1,

$$\lim_{x \to 0} \cos \frac{1}{x}$$

This can be shown by considering the definition of the limit when $\epsilon = \frac{1}{2}$, it is clear that no value of L can satisfy the following,

$$\left| \cos \frac{1}{x} - L \right| < \frac{1}{2}$$

for all values of x in any deleted δ-neighbourhood of 0.

I don't expect a rigorous proof here. If you want one, it can be done by contradiction. See the proof for the Dirichlet function being nowhere continuous. It is similar.

Now if we assume the limit of C exists, then by the sum law the limit of

$$C - A$$

must exists, this leads to a contradiction of the limit of B does not exist, and thus force us to conclude that the limit of C also does not exists. Therefore f' is not continuous at $x = 0$.

Question3 (1 points)
Let f be defined on $(-\infty, \infty)$, and c denote a constant. Consider the following

$$f(c+h) = f(c) + Ah + \varepsilon(h)$$

where A is not a function of h. Show that $f(x)$ is continuous at c if and only if

$$\lim_{h\to 0}\varepsilon(h)=0$$

Solution:

1M Let $x=c+h$, it is clear that $x\to c$ as $h\to 0$, then

$$\lim_{h\to 0}\varepsilon(h)=\lim_{x\to c}f(x)-f(c)$$

Suppose $f(x)$ is continuous, then

$$\lim_{x\to c}f(x)=f(c)$$

and that implies

$$\lim_{h\to 0}\varepsilon(h)=0$$

Conversely, suppose

$$\lim_{h\to 0}\varepsilon(h)=0$$

then

$$\lim_{x\to c}f(x)-f(c)=0\implies\lim_{x\to c}f(x)=f(c)$$

thus continuous at any point c.

Question4 (4 points)

Find the derivative y', show all your workings.

(a) (1 point) $y=x^8-3\sqrt{x}+5x^{-3}$

(b) (1 point) $y=\sin x+2\cos^3 x$

(c) (1 point) $y=\dfrac{\sin x\cos x}{\sqrt{x}}$

(d) (1 point) $y=x^x$

Solution:

(a) Apply the sum and power rules of differentiation, we have

$$y'=8x^7-\frac{3}{2}x^{-1/2}-15x^{-4}$$

(b) Apply the sum and chain rules, we have

$$y'=\cos x-6\cos^2 x\sin x$$

(c) Apply the product and quotient rules, we have

$$y'=\frac{x^{1/2}\left(\cos^2 x-\sin^2 x\right)-\frac{1}{2}x^{-1/2}\sin x\cos x}{x}$$

(d) By the properties of logarithmic function and exponential function, we have

$$y' = (x^x)' = \left(e^{\ln x^x}\right)' = \left(e^{x\ln x}\right)'$$

by the chain rule and the product rule, we have

$$y' = e^{x\ln x} \cdot (\ln x + 1) = x^x\,(\ln x + 1)$$

We could Log both sides

$$\ln y = \ln x^x \implies \ln y = x\ln x$$

Differentiate implicitly, we have

$$\frac{1}{y}\frac{dy}{dx} = \ln x + 1 \implies y' = y(\ln x + 1) = x^x(\ln x + 1)$$

However, this needs the assumption that the function is differentiable, which is a correct assumption but I didn't give you. If you are using this assumption, you need to state it for the very least.

Question5 (1 points)
Consider the function

$$f(x) = (x^{156} - 1)g(x)$$

where $g(x)$ is continuous at $x = 1$, and $g(1) = 1$. Find the derivative of $f(x)$ at $x = 1$.

Solution:

1M By definition

$$\begin{aligned} f'(1) = \lim_{h\to 0}\frac{f(1+h) - f(1)}{h} &= \lim_{x\to 1}\frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x\to 1}\frac{(x^{156} - 1)g(x) - (1^{156} - 1)g(1)}{x - 1} \\ &= \lim_{x\to 1}\frac{(x^{156} - 1)}{x - 1}g(x) \end{aligned}$$

By the product rule of limits,

$$f'(1) = \lim_{x\to 1}\frac{(x^{156} - 1)}{x - 1}g(x) = \underbrace{\lim_{x\to 1}\frac{(x^{156} - 1)}{x - 1}}_{A}\underbrace{\lim_{x\to 1}g(x)}_{B}$$

The factor A is essentially the derivative of x^{156} at $x = 1$, thus

$$A = 156x^{155} = 156 \qquad \text{at x} = 1,$$

and B is 1 by the continuity of $g(x)$, hence

$$f'(1) = 156 \cdot 1 = 156$$

Question6 (1 points)

For a positive integer n, consider

$$f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx)$$

where $a_1, a_2, ..., a_n$ are real numbers such that $|f(x)| \leq |\sin(x)|$ for all $x \in \mathbb{R}$. Show that

$$|a_1 + 2a_2 + 3a_3 + ... + na_n| \leq 1$$

Solution:

1M Find the derivative function of f by differentiation, we have

$$f'(x) = a_1 \cos(x) + 2a_2 \cos(2x) + ... + na_n \cos(nx)$$

At $x = 0$, we have

$$f'(0) = a_1 + 2a_2 + ... + na_n$$

Now consider the definition of the derivative of f at $x = 0$, we have

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}$$

If we take the absolute value of it, and apply a basic limit law, we have

$$|f'(0)| = \left| \lim_{x \to 0} \frac{f(x)}{x} \right| = \lim_{x \to 0} \left| \frac{f(x)}{x} \right|$$

For all x values, we have

$$|f(x)| \leq |\sin(x)| \implies \left| \frac{f(x)}{x} \right| \leq \left| \frac{\sin(x)}{x} \right| \implies \lim_{x \to 0} \left| \frac{f(x)}{x} \right| \leq \lim_{x \to 0} \left| \frac{\sin(x)}{x} \right| = 1$$

Therefore

$$|f'(0)| \leq 1 \implies |a_1 + 2a_2 + ... + na_n| \leq 1$$

Question7 (1 points)

Suppose there is a bowl in the shape of a hemisphere of radius a meters. Water is pouring into the bowl with a constant rate of $5\pi a^3$ cubic meters per second. Find the rate at which the water level is rising in the bowl.

Solution:

1M The volume is given by

$$V = \frac{1}{3}\pi h^2(3a - h)$$

The instantaneous rate change of the volume with respect to h is given by

$$\frac{dV}{dh} = \pi(2ah - h^2)$$

It is clear that the volume V is a differentiable function of h, and it is 1-to-1, thus

$$\frac{dh}{dV} = \frac{1}{\pi(2ah - h^2)}$$

The rate of change V with respect to time is given, thus by the chain rule, we have

$$\frac{dV}{dt} = 5\pi a^3 \implies \frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt} = \frac{5a^3}{(2ah - h^2)}$$