Question 1
Consider $A \in \mathcal{L}(V,V)$. Prove or disprove that $\lambda = 4$ is the eigenvalue of A^2 if and only if $\lambda = 2$ or $\lambda = -2$ is the eigenvalue of A.
(20 points)

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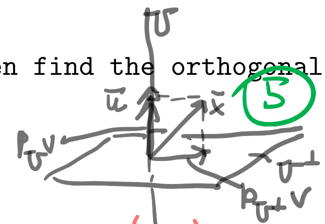
$\Rightarrow \exists\, v \neq 0, v \in V: A^2 v = 4v \Rightarrow \underline{(A^2 - 4I)v = 0} \Rightarrow (A-2I)(A+2I)v = 0$ $\quad A-2I, A+2I$ are not injective

5 $\quad \Rightarrow \exists\, v_1 \neq 0\ (A-2I)v_1 = 0 \Rightarrow Av_1 = 2v_1$, $\quad Ker(A^2-4I) \neq \{0\} \Rightarrow A^2 - 4I$ is not injective, $\exists\, v_2 \neq 0: (A+2I)v_2 = 0 \Rightarrow Av_2 = -2v_1$

$\Leftarrow$ If $\exists\, v_1 \neq 0, v_1 \in V: Av_1 = 2v_1 \Rightarrow A^2 v_1 = A(2v_1) = 2Av_1 = 2\cdot 2v_1 = 4v_1$ 5

$\exists\, v_2 \neq 0, v_2 \in V: Av_2 = -2v_2 \Rightarrow A^2 v_2 = A(-2v_2) = -2Av_2 = (-2)(-2)v_2 = 4v_2$ 5

Question 2
a. Let U be a finite-dimensional subspace of V. Define the orthogonal projection operator P_U of V onto U.
b. Show that if $U = span\{u\}$, then $P_U(v) = \frac{(v,u)}{||u||^2}u$.
c. Find the orthogonal projection of $\bar{x} = (2,0,1)$ onto $span(2,-1,3)$, and then find the orthogonal projection of $\bar{x} = (2,0,1)$ onto orthogonal complement of $span(2,-1,3)$.
(30 points)



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a. $\forall\, v \in V,\ v = u + w,\ u \in U, w \in U^\perp$. Then $\underline{P_U v = u}$

b. $U = span\, u \Rightarrow \forall\, \tilde{u} \in U\ \tilde{u} = au,\ a \in \mathbb{K}$ $\qquad \in span\, U \Rightarrow P_u v = \frac{(v,u)}{||u||^2}u$

If $\forall\, v \in V\ v = \underbrace{\boxed{\frac{(v,u)}{||u||^2}}u}_{\in U} + \left(v - \frac{(v,u)}{||u||^2}u\right)$, then $\left(v - \frac{(v,u)}{||u||^2}u, au\right)$ 10

$= \bar{a}(v,u) - \frac{(v,u)}{||u||^2}\bar{a}(u,u) = 0$

$\Rightarrow v - \frac{(v,u)}{||u||^2}u \perp u$

c. $U = span(2,-1,3)$

$P_U\bar{x} = \frac{2\cdot 2 + 0\cdot(-1) + 1\cdot 3}{2^2 + (-1)^2 + 3^2}\begin{pmatrix}2\\-1\\3\end{pmatrix} = \frac{1}{2}\begin{pmatrix}2\\-1\\3\end{pmatrix}$; $\quad P_{U^\perp}\bar{x} = \bar{x} - P_U\bar{x} = \begin{pmatrix}2\\0\\1\end{pmatrix} - \begin{pmatrix}1\\-1/2\\3/2\end{pmatrix} = \begin{pmatrix}1\\1/2\\-1/2\end{pmatrix}$

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Question 3
Let X be a subspace of $\mathbb{R}^4$ spanned by the vectors $\bar{v}_1 = (1,1,1,1)$ and $\bar{v}_2 = (1,0,3,0)$.
Find an orthonormal basis for X and for the orthogonal complement of X.
(30 points)

$X = span(\underbrace{(1,1,1,1)}_{\bar{v}_1}, \underbrace{(1,0,3,0)}_{\bar{v}_2}) \Rightarrow ||\bar{v}_1|| = \sqrt{1^2+1^2+1^2+1^2} = 2 \Rightarrow \bar{u}_1 = \frac{1}{2}(1,1,1,1)$

$\bar{u}_2 = \frac{\bar{v}_2 - (\bar{v}_2,\bar{u}_1)\bar{u}_1}{||\bar{v}_2 - (\bar{v}_2,\bar{u}_1)\bar{u}_1||} = \frac{(1,0,3,0) - \frac{1}{2}(1\cdot1+0\cdot1+3\cdot1+0\cdot1)\frac{1}{2}(1,1,1,1)}{||\bar{v}_2 - (\bar{v}_2,\bar{u}_1)\bar{u}_1||}$

10 $\quad = \frac{(1,0,3,0)-(1,1,1,1)}{||(1,0,3,0)-(1,1,1,1)||} = \frac{(0,-1,2,-1)}{\sqrt{0^2+(-1)^2+2^2+(-1)^2}}$

$= \frac{1}{\sqrt{6}}(0,-1,2,-1)$

X: $\boxed{\bar{u}_1 = \frac{1}{2}(1,1,1,1),\ \bar{u}_2 = \frac{1}{\sqrt{6}}(0,-1,2,-1)}$

$X^\perp$: $\boxed{\bar{w}_1 = \frac{1}{\sqrt{2}}(0,-1,0,1),\ \bar{w}_2 = \frac{1}{\sqrt{12}}(3,-1,1,1)}$

Let $\bar{x} = (x_1,x_2,x_3,x_4) \perp \bar{v}_1, \bar{v}_2 \Rightarrow \begin{cases} x_1 + x_2 + x_3 + x_4 = 0 \\ x_1 + 3x_3 = 0 \end{cases}$
$\bar{x} \in X^\perp$

$\begin{pmatrix}1&1&1&1\\1&0&3&0\end{pmatrix} \sim \begin{pmatrix}1&0&3&0\\1&1&1&1\end{pmatrix} \sim \begin{pmatrix}1&0&3&0\\0&1&-2&1\end{pmatrix}$ 10

$x_1 = -3x_3$
$x_2 = 2x_3 - x_4$ $\Rightarrow \bar{x} = \begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = \begin{pmatrix}-3\\2\\1\\0\end{pmatrix}x_3 + \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}x_4 \Rightarrow X^\perp = span\left(\begin{pmatrix}-3\\2\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\0\\1\end{pmatrix}\right) \Rightarrow$

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$\bar{w}_1 = \frac{(0,-1,0,1)}{||(0,-1,0,1)||} = \frac{(0,-1,0,1)}{\sqrt{2}}$

$\bar{w}_2 = \frac{(-3,2,1,0) - \frac{1}{\sqrt{2}}(3\cdot0+2\cdot(-1)+1\cdot0+0\cdot1)\frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\0\\1\end{pmatrix}}{||\cdot||}$

$= \frac{(-3,1,1,1)}{||\cdot||} = \frac{(-3,1,1,1)}{\sqrt{9+1+1+1}}$

Question 4 Linear approximation which is the best least squares to fit:

x	-1	0	0	1	2
$f(x)$	0	1	1	2	4

(20 points)

$$\phi(x) = \beta_0 + \beta_1 x \Rightarrow \begin{cases} \beta_0 - \beta_1 = 0 \\ \beta_0 + 0 = 1 \\ \beta_0 + \beta_1 = 2 \\ \beta_0 + 2\beta_1 = 4 \end{cases} \Rightarrow \underbrace{\begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}}_{\overline{\beta}} = \underbrace{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}}_{\overline{\phi}} \Rightarrow A\overline{\beta} = \overline{\phi}$$

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The least squares solution is a solution of $A^T A \overline{\beta}^* = A^T \overline{\phi}$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \overline{\beta}^* = \begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \\ 4 \end{pmatrix}$$

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$$\begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix} \overline{\beta}^* = \begin{pmatrix} 7 \\ 10 \end{pmatrix} \Rightarrow \overline{\beta}^* = \frac{1}{20} \begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} 7 \\ 10 \end{pmatrix}$$

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