Mid 1 Review Part III

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Overview

Reference: VV255 Lecture Slides by Professor Jing, Professor Olga, VV255 TA Group18SU, and Stewart's Textbook

The First Midterm Exam: Some Kind Reminds

You may feel...

1. Premise: The first three assignments was far too easy for me!
2. Premise: The content Professor Olga has covered in class was far from difficult! The concepts are quite straightforward and I am sure I have figured out all those terminologies!
3. Conclusion: Without too much effort I can also finish my mid I exam!

No! You will then get lost in those strategies!

The First Midterm Exam: Some Kind Reminds

You should ... (after RC)

1. First turn to your textbook! Remember the examples and methodology mentioned in the text! (Even for the simplest and the most straightforward one, at least taking a glance.)
2. Then follow the guidance file to check your perception of the course you have learned.
3. Close your book and guidance, try to write down all those terminologies by yourself and build up bridges and connections between them.
4. Well organize your time during the exam! Or your life will be tough and filled with struggling!

Plane Curve vs. Space Curve

Definition <Plane Curve>

A set of two parametric equations

$$x = f(t); \quad y = g(t)$$

Like the trail of an ant on the table.

Definition <Space Curve>

A set of three parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t)$$
$$\text{where} \quad a \leq t \leq b$$

Like the trail of a flying honeybee.

$$\bar{r}(t) = [f(t), g(t), h(t)]^T$$

Summary: Describe a Line

1. Vector Equation

$$\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$$

2. Parametric Equation

$$x = x_0 + v_1 t \quad y = y_0 + v_2 t \quad z = z_0 + v_3 t$$

3. Symmetric Equation

$$\frac{x - x_0}{v_1} = \frac{y - y_0}{v_2} = \frac{z - z_0}{v_3}$$

4. Line Segments from $\mathbf{r}_1$ to $\mathbf{r}_2$

$$\mathbf{r} = (1 - t)\mathbf{r}_1 + t\,\mathbf{r}_2$$
$$\text{where } 0 \leq t \leq 1$$

5. Distance D from a Point to a Line in Space

$$D = \frac{|\mathbf{P}_0\mathbf{P}_1 \times \mathbf{v}|}{|\mathbf{v}|}$$

Summary: Describe a Plane

1. Vector Equation

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

2. Scalar Equation

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

3. Linear Equation

$$ax + by + cz + d = 0 \quad \text{where} \quad d = -ax_0 - by_0 - cz_0$$

4. Line Segments from $\mathbf{r}_1$ to $\mathbf{r}_2$

$$\mathbf{r} = (1 - t)\mathbf{r}_1 + t\,\mathbf{r}_2$$
$$\text{where } 0 \le t \le 1$$

5. Distance D from a Point to a Plane

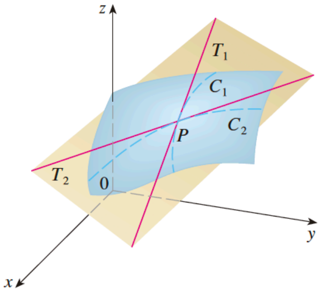
$$D = \frac{|\mathbf{P}_0\mathbf{P}_1 \cdot \mathbf{n}|}{|\mathbf{n}|}$$

Linear Approximation in 3D: Tangent Planes

Terminology: Tangent Plane

Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

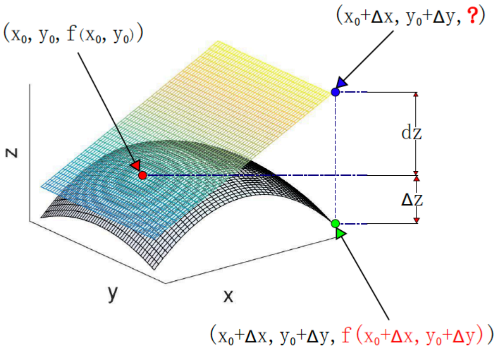
$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$



Linear Approximation in 3D: Tangent Planes

Methodology: Tangent Plane/ Linear Approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$



Practice: Planes and Lines

Guiding Target ★★★★

Find equations of lines and planes in 3D.

Analytical Skills

- Find the tangent line of the given space curve at the given point

Practice: Planes and Lines

Exercise 1

There is a line $\begin{cases} x = x_0 + t\cos\alpha \\ y = y_0 + t\sin\alpha \end{cases}$ (t as parameter). The parameters corresponding to point A, B are t_1, t_2. If point P divide the segment $\overline{AB}$ that $\overline{AP} = \lambda\overline{PB}$, find the parameter of point P.

Exercise 2

Find the maximum and minimum distance between the ellipse $\frac{x^2}{25} + \frac{y^2}{81} = 1$ and the line $3x + 4y - 64 = 0$

Exercise 3

If line $\begin{cases} x = t\cos\theta \\ y = t\sin\theta \end{cases}$ is tangential to a circle $\begin{cases} x = 4 + 2\cos\phi \\ y = 2\sin\phi \end{cases}$, find the slope of the line

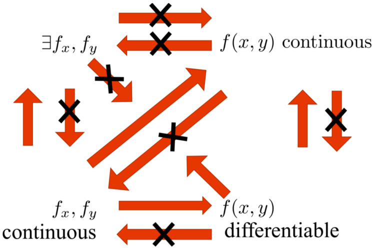
From Linear Approximation...

Definition <Differentiation>

Let $f\colon D \to \mathbb{R}$ be a function where D is an open ball of $\mathbb{R}^n$. Let $\overline{a} \in D$. The function f is differentiable at $\overline{a}$ with derivative $Df(\overline{a}) \in \mathbb{R}^n$ if

$$\frac{\|f(\overline{a}+\overline{h}) - f(\overline{a}) - Df(\overline{a})\cdot\overline{h}\|}{\|\overline{h}\|} \to 0 \text{ as } \overline{h} \to \overline{0}$$

Differentiable vs. Continuous vs. Partial Derivatives



Total Differential & The Chain Rule

Total differential: Good Approximation

Suppose the function $z = f(x, y)$ is differentiable, then

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

provides a good approximation of Δz

The Chain Rule

Suppose that u is a differentiable function of the n variables $x_1, \dots, x_n$, s.t. for all $1 \leq i \leq n$, x_i is a differentiable function of m variables $t_1, \dots, t_m$. Then u is a function of $t_1, \dots, t_m$, and for all $1 \leq j \leq m$,

$$\frac{\partial u}{\partial t_j} = \sum_{1 \leq i \leq n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial t_j}$$

Implicit Differentiation

Implicit Function Theorem

- Suppose F is a function of x, y and z, if the following conditions are satisfied

1. The partial derivatives F_x, F_y, and F_z are continuous throughout an open region R in space containing the point (x_0, y_0, z_0).
2. For some constant c, $F(x_0, y_0, z_0) = c$ and $F_z(x_0, y_0, z_0) \neq 0$, then

$$F(x, y, z) = c$$

defines z implicitly as a differentiable function of x and y near (x_0, y_0, z_0), and the partial derivatives of z are given by

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Well understand the theorem! The equation actually acts as the constraint where we shall focus on the specific function!

Implicit Differentiation: Functions and Constraints

Example

Consider

$$F(x, y, z) = x + y + z = 0$$

$$\therefore F_x \cdot \frac{\partial x}{\partial x} + F_y \cdot \frac{\partial y}{\partial x} + F_z \cdot \frac{\partial z}{\partial x} = 0$$

$$\therefore 1 \cdot \frac{\partial x}{\partial x} + 1 \cdot \frac{\partial y}{\partial x} + 1 \cdot \frac{\partial z}{\partial x} = 0$$

And we notice that

$$z = -y - x$$

$$y = -z - x$$

Contradiction occurs!

$$1 - 1 - 1 \neq 0$$

why??? Is F a function you want to investigate here?

Practice: Differential, the Chain rule, Implicit

Guiding Target ★★★

Find differentials, apply the chain rule and find tangent planes at the given points.
Probable Missing Points: Implicit Function Theorem

Analytical Skills

- Well understand the differentials and the tangent plane approximation.
- Using the differentials to perform the estimating stuff.
- Find the equation of the tangent planes to the given quadratic surface.
- Bear the actual meaning of the chain rule and implicit function theorem in mind!

Exercise 2

Simple Application of the Chain Rule

(1 point) If $z = f(t)$, where $t = x - y$, show that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

Tangent Planes

Find an equation of the tangent plane to the following ellipsoid at $(2, 1, -3)$.

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Errors

Suppose that

$$z = x(e^x + e^{-y})$$

where x and y are found to be 2 and $\ln 2$ with maximum possible errors of

$$|\Delta x| = |dx| = 0.1 \text{ and } |\Delta y| = |dy| = 0.02.$$

Exercise 2

Implicit Theorem

Suppose that the equation $F(x, y, z) = 0$ implicitly defines each of the three variables x, y, and z as functions of the other two: $z = f(x, y)$, $y = g(x, z)$, $x = h(y, z)$. If F is differentiable and F_x, F_y, and F_z are all nonzero, show that

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z} = -1$$

"Horribleeee Chain"

Let $f(x,y)$ be differentiable everywhere. Suppose $f(a,a) = a$, $f_x(a,a) = b$, $f_y(a,a) = c$, where a, b and c are constants. Find $g(a)$ and $h'(a)$ in terms of a, b and c, where

$$g(x) = f\Big(x, f\big(x, f(x,x)\big)\Big) \qquad \text{and} \qquad h(x) = [g(x)]^2$$

Directional Derivatives & Gradients

Definition <Directional Derivatives>

The rate of change of f at (x_0, y_0) in the direction of $\mathbf{v}$, known as a directional derivative, is defined and denoted as

$$\mathrm{D}_{\mathbf{v}} f(x_0, y_0) = f'_{\mathbf{v}}(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + hu_1, y_0 + hu_2) - f(x_0, y_0)}{h}$$

where $\mathbf{u} = u_1 \mathbf{e}_x + u_2 \mathbf{e}_y$ is the unit vector in the direction of $\mathbf{v}$, i.e. $\mathbf{u} = \hat{\mathbf{v}}$

Definition <Gradient: Maximizing the d.d.>

The gradient vector (or simply gradient) of a function of several variables $x_1, \ldots, x_n$ is,

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} = \frac{\partial f}{\partial x_1} \mathbf{e}_1 + \cdots + \frac{\partial f}{\partial x_n} \mathbf{e}_n$$

Directional Derivatives & Gradients

A Useful Theorem:

If f is differentiable in an open region, then the derivative f in the direction of v, where ∇f is the gradient of f.

$$f'_{\mathrm{v}} = \hat{\mathbf{v}} \cdot \nabla f$$

A Useful Theorem: by dot product

$$f'_{\mathrm{v}} = \hat{\mathbf{v}} \cdot \nabla f = \|\nabla f\| \|\hat{\mathbf{v}}\| \cos\theta = \|\nabla f\| \cos\theta$$

Then f increases most rapidly at any point in the domain

$$f'_{\nabla f} = \|\nabla f\|$$

And decreases most rapidly in the direction of $-\nabla f$

$$f'_{-\nabla f} = -\|\nabla f\|$$

Gradients: Other Properties

Relationship to TNB frame

Consider a differentiable function $f(x, y)$ takes a constant value c along a smooth curve

$$\mathcal{C} : \mathbf{r} = x(t)\mathbf{e}_x + y(t)\mathbf{e}_y$$

then the graph of $\mathbf{r}$ is a level curve of the function f, and,

$$f(x(t), y(t)) = c$$

then

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{e}_1 + \frac{\partial f}{\partial y}\mathbf{e}_2\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dx}{dt}\mathbf{e}_x + \frac{dy}{dt}\mathbf{e}_y\right)}_{\mathbf{r}'} = 0$$

$\Rightarrow \nabla f$ & $\overline{N}$ is always parallel but not necessarily in the same direction.

Gradients: Other Properties

Theorem. find the tangent to the level curve

Core: Every vector $\overline{P_0P}$ on the tangent line through a point P_0 is orthogonal to the gradient ∇f evaluated at P_0

$$\mathbf{P}_0\mathbf{P} \cdot (\nabla f)_{P_0} = 0 \quad \text{where} \quad \mathbf{P}_0\mathbf{P} = \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \quad \text{is on the line.}$$

The scalar form,

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

Practice: Direction Derivatives and Gradients

Guiding Target ★★★★★

Overall, the directional derivatives and gradients serve as the best tool to understand the partial and our previous frame of work.

1. Compute the directional derivative of a function of two or three variables, determine the sign
2. Estimate the value of the directional derivative of a function of two variables at a point (based on a contour diagram, a graph, or a table)
3. Identify a relationship between the values of $f_x(a, b)$ and $f_y(a, b)$ and the value of the directional derivative at (a,b) in a specified direction.
4. Compute the gradient of a function of two or three variables.
5. Geometric definition of the dot product of the gradient.
6. Construct tangent lines or planes based on the gradient vector and simplify problems using directional derivative or the gradient.

Exercise 3

Basic Evaluation of directional derivatives

Suppose f(x)= $\begin{cases} \dfrac{xy^2}{x^2+y^4}, & x^2+y^2 \neq 0 \\ 0, & x^2+y^2 = 0 \end{cases}$

Find the directional derivative at (0,0) along the direction of $\mathbf{e}(\cos\theta, \sin\theta)$.

The Gradient of the function and the geometric meaning

(1 point) Show that the curve

$$\mathbf{r}(t) = (\ln t)\mathbf{e}_x + (t \ln t)\mathbf{e}_y + t\mathbf{e}_z$$

is tangent to the surface

$$xz^2 - yz + \cos xy = 1 \qquad \text{at} \qquad (0,0,1)$$

Exercise 3

Directional Derivatives and the gradient

Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two non-parallel directions given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find ∇f at this point? If so, how would you do it?

The implication of the gradient

(1 point) Is there a direction $\mathbf{v}$ in which the rate of change of

$$F(x, y) = x^2 - 3xy + 4y^2$$

at $(1, 2)$ equals 14? Justify your answer.

Total derivatives and the gradient

(1 point) Express $\dfrac{dw}{dt}$ as a dot product, where

$$w = Q(x, y, z),\ x = f(t),\ y = g(t), \text{ and } z = h(t)$$

Exercise 3

Application Problem!

Try to solve some application problem by yourself. For example James Stewart's 8th, section 14.6 exe34.

Organize the logic of those concepts!
Get prepared for the potentially tedious calculation!