
Question1 (3 points)

Determine whether each of the followings is true. If not, briefly explain why it is false.

(a) (1 point) If a function has the limit $\lim_{x\to\infty} f(x) = L$, then $\{f_n\}$ converges to L as well.

$$\text{where} \quad f_n = f(n), \quad \text{for} \quad n \in \mathbb{N}.$$

(b) (1 point) If a sequence $f_n = f(n)$ converges to L, then the function has the same limit

$$\lim_{x\to\infty} f(x) = L$$

(c) (1 point) If $f(x)$ is defined everywhere except at $x = a$, then the following limit exists

$$\lim_{x\to a} f(x)$$

Question2 (4 points)

Find the following limits. You may use any laws/theorems that we have covered in class. However, indicate which law/theorem you are using and justify its use at every step.

(a) (1 point) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{x^2+x}\right)$

(b) (1 point) $\displaystyle\lim_{x\to\infty}\frac{x^2-3x+2}{2x^2+5x+1}$

(c) (1 point) $\displaystyle\lim_{x\to 0}\left(\frac{1}{x} - \frac{1}{|x|}\right)$

(d) (1 point) $\displaystyle\lim_{x\to 0}\frac{\sin x}{x}$

Question3 (2 points)

Prove the following statements using the precise definition of a limit.

(a) (1 point) $\displaystyle\lim_{x\to 4}(2x-1) = 7$

(b) (1 point) $\displaystyle\lim_{x\to 5}\sqrt{x-1} = 2$

Question4 (1 points)

In the theory of relativity, the mass of a particle with speed v is given by

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$

where m_0 is the mass of the particle at rest and c is the speed of light.

$$\text{What happens as } v \to c^-?$$

that is, as the speed of the particle is approaching the speed of light from below.

Question5 (1 points)

Suppose $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = K$, prove the following is true

$$\lim_{x\to a}\Big(f(x) + g(x)\Big) = L + K$$

Question6 (1 points)

Find the values of a and b such that the following function is continuous everywhere.

$$f(x) = \begin{cases} \left(x^2-4\right)(x-2)^{-1} & \text{if} \quad x < 2, \\ ax^2 - bx + 3 & \text{if} \quad 2 \le x < 3, \\ 2x - a + b & \text{if} \quad x \ge 3. \end{cases}$$

Question7 (3 points)

(a) (1 point) Prove that the absolute value function $f(x) = |x|$ is continuous everywhere.

(b) (1 point) Prove that if $g(x)$ is continuous on an open interval, then so is $|g(x)|$.

(c) (1 point) Is the converse of the statement in part (b) also true? Justify your answer.

Question8 (2 points)

Prove two of the following LIATE functions of your choice are continuous in their domains.

(a) *L*ogarithmic:

$$\ln(x)$$

(b) *I*nverse trigonometric:

$$\arctan(x)$$

(c) *A*lgebraic:

$$\sqrt[3]{x} \quad (x > 0)$$

(d) *T*rigonometric:

$$\sin(x)$$

(e) *E*xponential:

$$\mathrm{e}^x$$

[Hint:] For $\ln(x)$ and e^x, you might use the fact

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

Question9 (1 points)

Suppose $\lim_{x \to a} f(x) = A$ and $\lim_{x \to a} g(x) = B$, where $A > B$. Show there exits $\delta > 0$ such that

$$f(x) > g(x) \qquad \text{if} \qquad x \in \{(a - \delta, a) \cup (a, a + \delta)\}$$

that is, there is a deleted δ-neighbourhood of a such that f is bigger than g for all x in it.

Question10 (2 points)

Suppose $f(x)$ is continuous on the interval $[0, 1]$, and $f(0) = f(1)$.

(a) (1 point) Prove there exists $x \in [0, 1]$ such that

$$f(x) = f\left(x + \frac{1}{2}\right)$$

(b) (1 point) Prove there exists $x \in [0, 1]$ such that

$$f(x) = f\left(x + \frac{1}{n}\right) \qquad \text{for any positive integer } n.$$

Question11 (1 points)

Suppose $f(x)$ is continuous on the interval $[a, b]$. Show that there exists $c \in [a, b]$ such that

$$f(c) = \alpha_1 f(x_1) + \alpha_2 f(x_2) + ... + \alpha_n f(x_n)$$

where $x_i \in [a, b]$ and the coefficients are all positive and the sum is 1, that is,

$$\alpha_i > 0 \quad \text{for } n = 1, 2, \ldots, n \qquad \text{and} \qquad \sum_{i=1}^{n} \alpha_i = 1$$

Question12 (1 points)

In theory, can you always simultaneously slice two rain drops each exactly in half with a single straight-line cut, no matter the shapes of the rain drops nor their location on the windscreen.

Question13 (1 points)

Find the largest interval, if it exists, on which the following function is continuous.

$$f(x) = \sum_{n=1}^{\infty} \frac{B(nx)}{n^2}, \qquad \text{with} \quad B(x) = \begin{cases} x - [x] & \text{if } x \neq k/2, \\ 0 & \text{if } x = k/2, \end{cases}$$

where k is an integer and $[x]$ denotes the nearest integer to x. Justify your answer.

Question14 (2 points)

Suppose $f(x)$ is continuous on the interval $\mathcal{I} = [a, b]$.

(a) (1 point) Show why $f(x)$ is bounded on this interval.

(b) (1 point) Show why $f(x)$ attains its supremum and infimum on this interval.

Question15 (0 points)

A function $f(x)$ defined on an interval $\mathcal{I}$ is said to have the *intermediate value property* if for all a and b in $\mathcal{I}$ with $a < b$ and for any number y between $f(a)$ and $f(b)$, there exists a number x in $[a, b]$ such that $f(x) = y$.

A basic property of continuous functions defined on an interval is that they have the intermediate value property, that is basically what the intermediate value theorem is about. It was widely believed by many mathematicians in the nineteenth century that the intermediate value property is equivalent to continuity. However, the French mathematician Jean Gaston Darboux (1842–1917) proved in 1875 that this not the case. He proved that, for real-valued functions defined on intervals,

$$\text{Continuity} \implies \text{Intermediate Value Property},$$

but the converse is not true.

(a) (1 point (bonus)) Prove that $f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$ has the Intermediate Value Property on the interval $[0, B]$, for any $B > 0$, despite not being continuous at $x = 0$.

(b) (1 point (bonus)) Prove that if f is monotonic on the interval $\mathcal{I}$ and has the Intermediate Value Property on $\mathcal{I}$, then f is continuous on $\mathcal{I}$.