Vv156
Dr Jing Liu

Assignment 5
Due: None

Question1 (8 points)
Find the derivative y', show all your workings.

(a) (1 point) $y = x(\ln x - 1)$
(b) (1 point) $y = \sinh x$
(c) (1 point) $y = \cosh x$
(d) (1 point) $y = \tanh x$
(e) (1 point) $y = 2^{\cos x} + \ln x$
(f) (1 point) $y = \arccos(3x^2)$
(g) (1 point) $y = \sqrt{x\sqrt{1-\sin x}\ln x}$
(h) (1 point) $y = x \arcsin x + \sqrt{1-x^2}$

where $\sinh x$, $\cosh x$ and $\tanh x$ are known as the hyperbolic functions, that are defined by

$$\sinh x = \frac{1}{2}\left(e^x - e^{-x}\right), \qquad \cosh x = \frac{1}{2}\left(e^x + e^{-x}\right) \qquad \text{and} \qquad \tanh = \frac{\sinh x}{\cosh x}$$

Question2 (1 points)
Use implicit differentiation to find y' for

$$e^{x+y} + \cos(xy) = 0$$

You may assume that y' exists.

Question3 (3 points)
Suppose that f is a function with the properties:

1. f is differentiable everywhere
2. $f(x+y) = f(x)f(y)$ for all values of x and y
3. $f(0) \neq 0$
4. $f'(0) = 1$

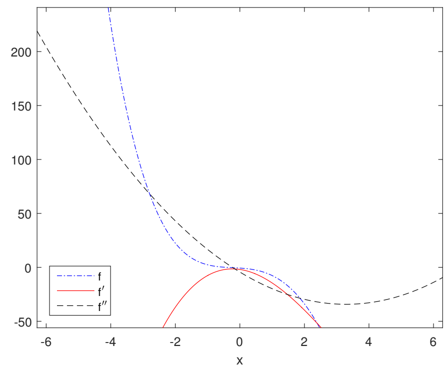
(a) (1 point) Show that $f(0) = 1$.

(b) (1 point) Show that $f(x) > 0$ for all values of x.

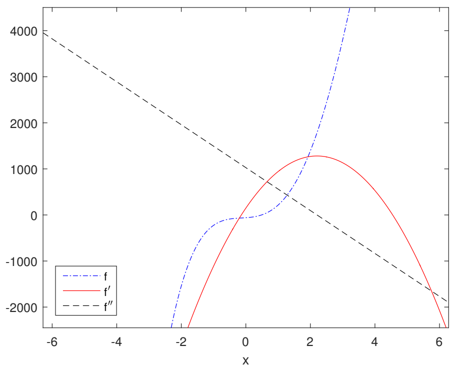
(c) (1 point) Use the definition of derivative to show that $f'(x) = f(x)$ for all x.

Question4 (2 points)
Is it possible to have $f(x)$ indicated by the following pictures? If not, explain why not.

(a) (1 point) f, f' and f'' are shown



(b) (1 point) f, f' and f'' are shown



Question5 (2 points)

(a) (1 point) Is there a differentiable function defined on $(-1, 1)$ which has no relative maximum but has got an absolute maximum? If there is one, sketch it, if not, explain.

(b) (1 point) Is there a differentiable function defined on $(-1, 1)$ which has no absolute minimum but has got a relative minimum? If there is one, sketch it, if not, explain.

Question6 (4 points)

The three cases in the first derivative test cover the situations one commonly encounters but do not exhaust all possibilities. Consider the function f, g, and h whose values at 0 are all 0 and, for $x \neq 0$,

$$f(x) = x^4 \sin \frac{1}{x}, \qquad g(x) = x^4 \left(2 + \sin \frac{1}{x}\right), \qquad h(x) = x^4 \left(-2 + \sin \frac{1}{x}\right)$$

(a) (1 point) Show that 0 is a critical number of all three functions but their derivatives change sign infinitely often on both sides of 0.

(b) (3 points) Show that f has neither a local maximum nor a local minimum at 0, g has a local minimum, and h has a local maximum.

Question7 (2 points)

Use l'Hôpital's rule to find the limits.

(a) (1 point) $\displaystyle\lim_{x \to 1^+} \left(\frac{1}{x-1} - \frac{1}{\ln x}\right)$

(b) (1 point) $\displaystyle\lim_{x \to (\pi/2)^-} (\tan x)^{(\pi/2)-x}$

Question8 (1 points)

Show the inequality $\dfrac{\arctan x_2 - \arctan x_1}{x_2 - x_1} \leq 1$ is true when $x_2 > x_1$.

Question9 (1 points)

Suppose $f(x)$ is continuous on $[0,3]$ and differentiable on $(0,3)$, and

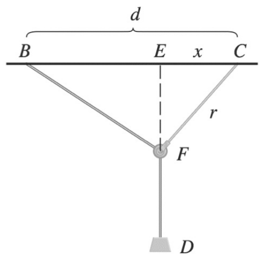
$$f(0) + f(1) + f(2) = 3 \qquad \text{and} \qquad f(3) = 1$$

Show that there must exits $c \in (0,3)$ such that $f'(c) = 0$.

Question10 (1 points)

One of the problems posed by the Marquis de L'Hospital in his calculus textbook *Analyse des Infiniment Petits* concerns a pulley that is attached to the ceiling of a room at a point C by a rope of length r. At another point B on the ceiling, at a distance d from C (where $d > r$), a rope of length l is attached and passed through the pulley F and connected to a weight W. The weight is released and comes to rest at its equilibrium position D. As L'Hospital argued, this happens when the distance $|ED|$ is maximized. Show that when the system reaches equilibrium, the value of x is

$$\frac{r}{4d}\left(r + \sqrt{r^2 + 8d^2}\right)$$



Question11 (0 points)

(a) (1 point (bonus)) A ladder is to be carried down a hallway p meters wide. Unfortunately at the end of the hallway there is a right-angled turn into a hallway q meters wide. Use optimization to find the length of the longest ladder that can be carried horizontally around the corner?

(b) (1 point (bonus)) Find the curve $y(x)$ connecting $y(a) = A$ to $y(b) = B$, assuming $a < b$ and $B < A$, along which a particle under the influence of gravity g will slide without friction in minimum time. Justify your answer.