VV255, Summer 2019
Final Review, Wu Zheng **Name:** ____________________

The aim of the designed final review is to go through most topics we've learned in VV255, and seek to help you with your final struggle. The answer is not yet attached here. You can pick some questions for practice, yet not necessarily all questions.

The form of the exam would be quite similarly divided into two parts. To remind, for part A you should finish all the problems and for part B you only need to choose 4 from 6 to obtain full marks.

VV255 is of significance for your further studies in either ECE of ME. For those who will devote yourselves to ME, you will uese these contexts in upper-level courses such as Hydromechanics and Dynamics, and for those ECE pursuers, you will apply those materials in upper-level courses such as Electromagnetism.

Hope that everyone of you enjoyed VV255 in this semester!

Good Luck!

Attention:

1. If you write out the solution without any intermediate steps, you may not get full marks for the question.
2. Whenever using a theorem, you shall write down the prerequisites for it to get familiar with its usage and explain why you can use it here.

I have reviewed all the contexts in the lecture slides and assignments. I know that the exercise sheet is only for practice, and that even though I may find it tedious to work it out I will still get a good grade in the coming final exam.

Signature: ____________

Part A
Part A consists 6 problems. Solve ALL problems.

Question1 (15 points)
Consider the integral

$$\iiint_E f(x,y,z)dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{y} f(x,y,z)dzdydx$$

(a) Sketch the region E.
(b) Write the other five iterated integrals which represent $\iiint_E f(x,y,z)dV$

Solution:

See the Final Review Slides.

Question2 (5 points)
Find the flux of $\overline{F}(x,y,z) = \langle e^y, ye^x, x^2y\rangle$ across the part of the paraboloid $z = x^2+y^2$ that lies above the square $0 \le x \le 1, 0 \le y \le 1$ and has upward orientation.

Solution:

Applying $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$ we have the final sol as

$$\frac{11-10e}{6}$$

Question3 (5 points)
Using spherical coordinates, parametrize the sphere $x^2+y^2+z^2 = 1$ and find the corresponding $(\overline{r}_u \times \overline{r}_v)$

Solution:

Using spherical coordinates, the sphere can be parameterized by

$$\mathbf{R}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$$

Then

$$\begin{aligned} \mathbf{R}_\phi &= \langle \cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi \rangle \\ \mathbf{R}_\theta &= \langle -\sin\phi\sin\theta, \sin\phi\cos\theta, 0 \rangle \\ \mathbf{F}(\phi, \theta) &= \langle \cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta \rangle \end{aligned} \tag{1}$$

$$\mathbf{R}_\phi \times \mathbf{R}_\theta = \left\langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi \right\rangle$$

Question4 (10 points)

(a) State the Divergence Theorem.

(b) Calculate the flux of $\overline{F}$ across S,

$$\mathbf{F}(x, y, z) = \left(\cos z + xy^2\right)\mathbf{i} + xe^{-z}\mathbf{j} + \left(\sin y + x^2 z\right)\mathbf{k}$$

S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$

Solution:

$\frac{32}{3}\pi$

Question5 (10 points)

Evaluate

$$\int_C (y + \sin x)dx + \left(z^2 + \cos y\right)dy + x^3 dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle, 0 \leqslant t \leqslant 2\pi$

(Hint: Observe that C lies on the surface z=2xy.)

Solution:

$-\pi$

Question6 (15 points)
Evaluate the line integral

$$\oint_C x^2y^2dx + xydy$$

C consists of the arc of the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$ and the line segments from $(1, 1)$ to $(0, 1)$ and from $(0, 1)$ to $(0, 0)$. And verify the Green's theorem for the case.

Solution:

$\frac{22}{105}$

Part B
Part B consists of 5 problems. Solve ANY 4(FOUR) problems.

Question7 (10 points)
For the question, choose either a or b.
(a) The temperature at a point in a ball with conductivity K is inversely proportional to the distance from the center of the ball. Find the rate of heat flow across a sphere S of radius a with center at the center of the ball.
(b) The temperature at the point (x, y, z) in a substance with con- ductivity $K = 6.5$ is $u(x, y, z) = 2y^2 + 2z^2$. Find the rate of heat flow inward across the cylindrical surface $y^2 + z^2 = 6$ $0 \leqslant x \leqslant 4$.

Solution:

(a) $4kc\pi$
(b) 1248π

Question8 (10 points)
For constants a, b, c, m, consider the vector field

$$\mathbf{F} = (ax + by + 5z)\mathbf{i} + (x + cz)\mathbf{j} + (3y + mx)\mathbf{k}$$

(a) Suppose that the flux of $\mathbf{F}$ through any closed surface is 0. What does this tell you about the value of the constants a, b, c and m?
(b) Suppose instead that the line integral of $\mathbf{F}$ around any closed curve is 0. What does this tell you about the values of the constants a, b, c and m?

Solution:

(a) If the flux of $\mathbf{F}$ through any closed surface is 0, then by the divergence theorem, the vector field must have zero divergence.

$$\nabla \cdot \mathbf{F} = a = 0$$

This tells us that $a = 0$ but it does not tell us anything about b, c or m.

(b) If the line integral of $\mathbf{F}$ around any closed curve is 0, this means that the vector field has curl equal to zero everywhere.

$$\nabla \times \mathbf{F} = (3 - c)\mathbf{i} + (5 - m)\mathbf{j} + (1 - b)\mathbf{k}$$

This tells us that $c = 3, m = 5$ and $b = 1$. It does not tell us anything about a.

Question9 (10 points)
(a) State the Stoke's Theorem.
(b) Evaluate

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where

$$\mathbf{F} = ye^x\mathbf{i} + (x + e^x)\,\mathbf{j} + z^2\mathbf{k}$$

and C is the curve

$$\mathbf{r}(t) = (1 + \cos t)\mathbf{i} + (1 + \sin t)\mathbf{j} + (1 - \sin t - \cos t)\mathbf{k}$$

Solution:

First check if F is conservative

$$\operatorname{curl}\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ ye^x & (x+e^x) & z^2 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1+e^x-e^x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \neq 0 \text{ And}$$

then by Stoke's theorem we have the final answer

$$\pi$$

Question10 (10 points)

Use geometric reasoning to find $I = \iint_S \mathbf{F}\cdot d\mathbf{S}$ by inspection for the following three situations. Explain your answers. In each case, a and b are positive constants.

(a) $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the surface consisting of three squares with one corner at the origin and positive sides facing the first octant. The squares have sides ($b\mathbf{i}$ and $b\mathbf{j}$), ($b\mathbf{j}$ and $b\mathbf{k}$), and ($b\mathbf{i}$ and $b\mathbf{k}$), respectively.

(b) $\mathbf{F}(x,y,z) = (x\mathbf{i} + y\mathbf{j})\ln(x^2+y^2)$, and S is the surface of the cylinder (including top and bottom) where $x^2+y^2 \le a^2$ and $0 \le z \le b$.

(c) $\mathbf{F}(x,y,z) = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})e^{-\left(x^2+y^2+z^2\right)}$, and S is spherical surface $x^2+y^2+z^2 = a^2$

Solution:

see the attached png.

Question11 (10 points)

Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Consider the vector field

$$\mathbf{E} = \frac{\mathbf{r}}{|\mathbf{r}|^3}$$

Find $\int_S \mathbf{E}\cdot d\mathbf{A}$ where S is the ellipsoid $x^2+2y^2+3z^2 = 6$. Give reasons for your calculation.

Solution:

$$4\pi$$