Rigid-body Equilibrium Problems

1. Sketch the free body diagram. (Pay attention to acting points) When consider rotation, rigid body cannot be treated as a mass point.
2. Choose an appropriately placed coordinate system. Vanish as many torques as you can. (Find the rotation axis right)
3. Write down equilibrium conditions for forces and torques.
4. Solve for unknowns

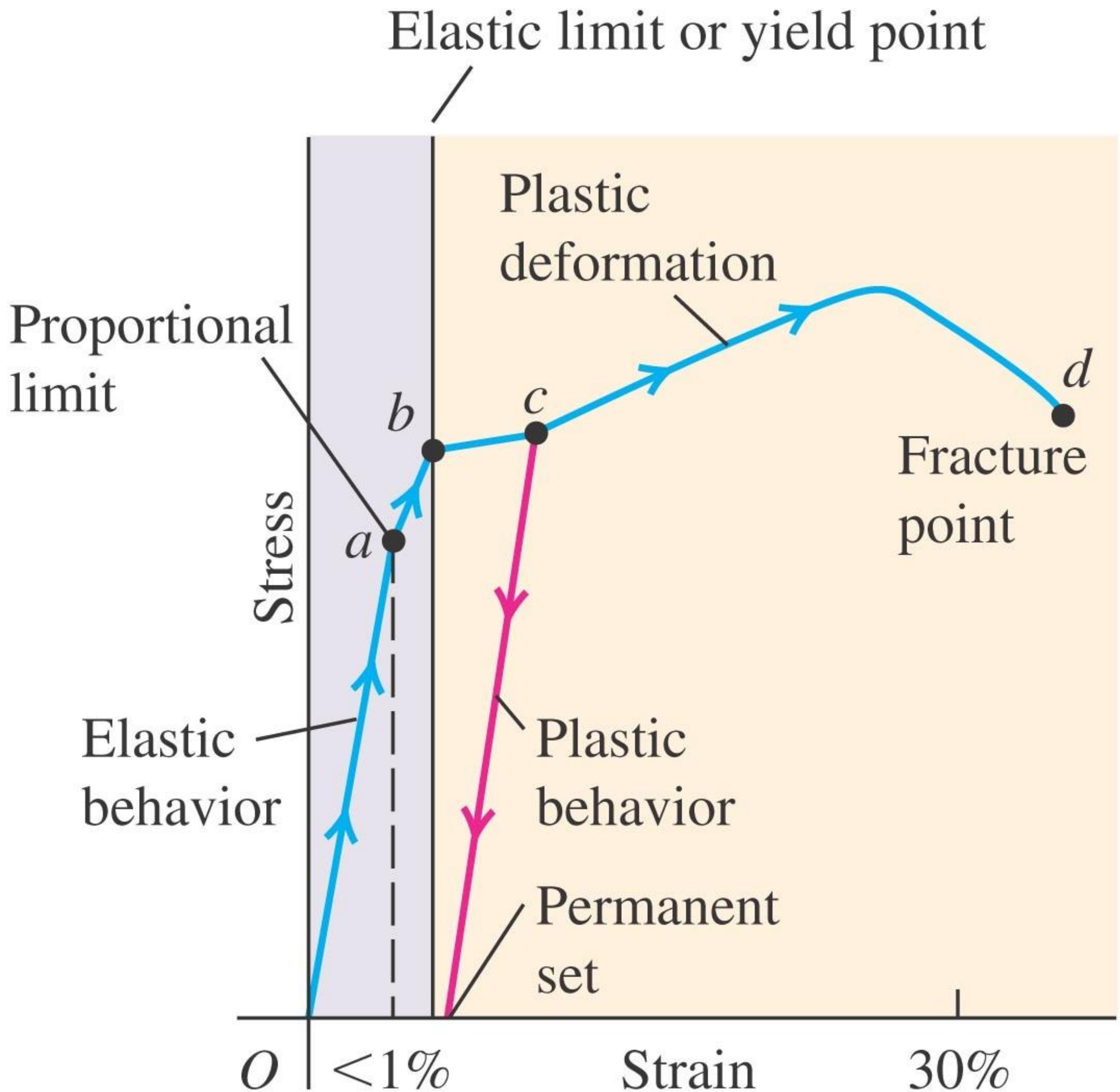
Elasticity

Stress: force per unit area
Strain: fractional deformation due to the stress
Elastic modulus: stress divided by strain

Young's modulus : $Y = \dfrac{F_{\perp}/A}{\Delta l / l_0}$

Bulk modulus : $B = \dfrac{-\Delta P}{\Delta V / V_0}$

Shear modulus : $S = \dfrac{F_{\parallel}/A}{x/h}$


Elastic limit or yield point
Plastic
deformation
Proportional
limit
b
c
d
Fracture
point
a
Stress
Elastic
behavior
Plastic
behavior
Permanent
set
O
<1%
Strain
30%

Fluids at rest

- pressure increases as depth increases

$$p = p_0 + \rho g h$$

- Pascal's Law: Pressure applied to an enclosed incompressible fluid is transmitted undiminished to every portion of the liquid and the walls of the container. $\frac{F_2}{F_1} = \frac{A_2}{A_1}$
- Pressure gauge : $p_{gauge} = p - p_0 = \rho g h$
- Buoyancy due to pressure difference

$$F_b = \rho V g$$

Fluids at motion

- Continuity equation due to mass conservation

$$A_1 v_1 = A_2 v_2$$

- Bernoulli's Equation

$$\delta W = dK + dU_{\text{grav}}$$

$$(p_1 - p_2)\ dV = \frac{1}{2}\varrho\left(v_2^2 - v_1^2\right)\ dV + \varrho g\left(y_2 - y_1\right) dV$$

$$(p_1 - p_2) = \frac{1}{2}\varrho\left(v_2^2 - v_1^2\right) + \varrho g\left(y_2 - y_1\right)$$

$$p_1 + \frac{1}{2}\varrho v_1^2 + \varrho g y_1 = p_2 + \frac{1}{2}\varrho v_2^2 + \varrho g y_2$$

$$\boxed{p + \frac{1}{2}\varrho v^2 + \varrho g y = \text{const}}$$

Chapter 18

Gravitation

Recall:

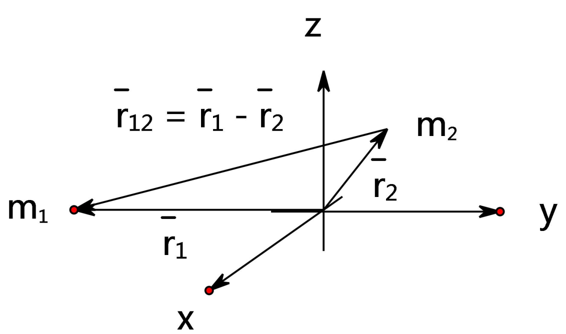
$\rightarrow$ one of the four fundermental interacions

$\rightarrow$ relative strength much smaller than that of the other three interactions

$\rightarrow$ long-range; important for the evolution of the universe

$\rightarrow$ attractive

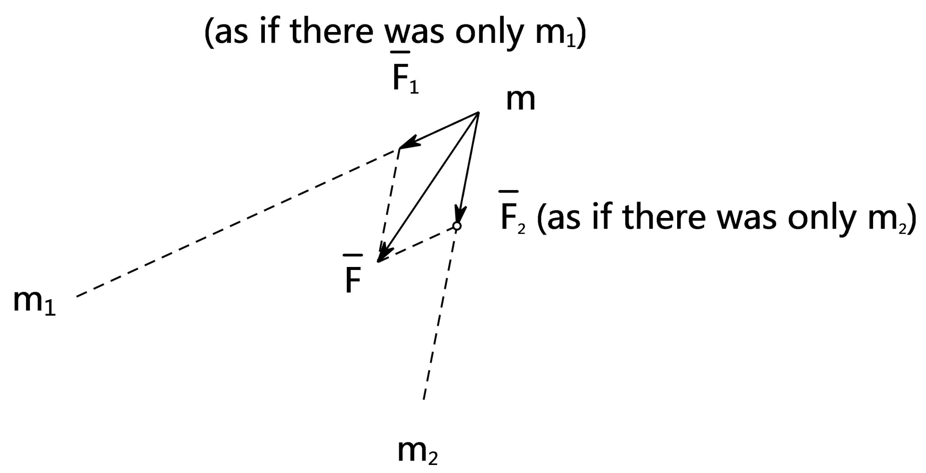
18.1 Newton's law of gravitation



$$\boxed{\underbrace{F_{12}}_{\text{on "1" due to "2"}} = -G\frac{m_1 m_2}{r_{12}^2} \cdot \frac{\bar{r}_{12}}{|\bar{r}_{12}|}} \quad ; \quad \bar{r}_{12} = \bar{r}_1 - \bar{r}_2$$

Note: $\bar{F}_{12} = -\bar{F}_{21}$; $\qquad G = 6.67250(85) \times 10^{-11} \quad N \cdot m^2/kg^2$

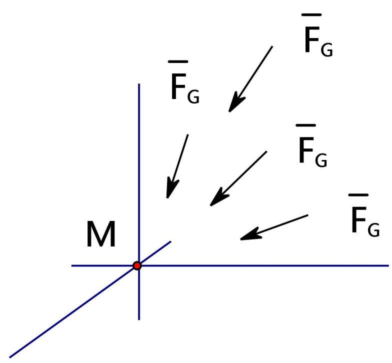
18.2 Superposition princeple



Note:

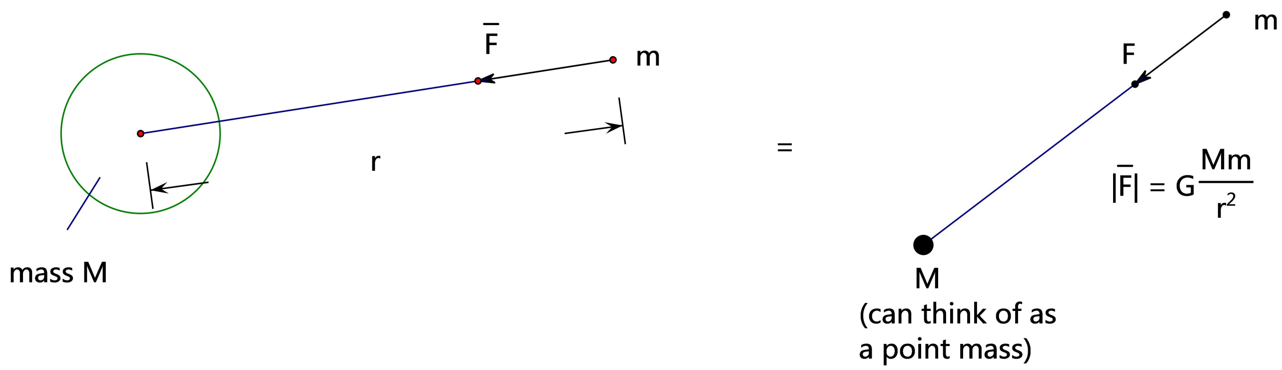
Gravitational interaction define a vector field in space

$$\bar{E}_G = -G\frac{M}{r^2}\frac{\bar{r}}{r} = \frac{\bar{F}}{m}$$

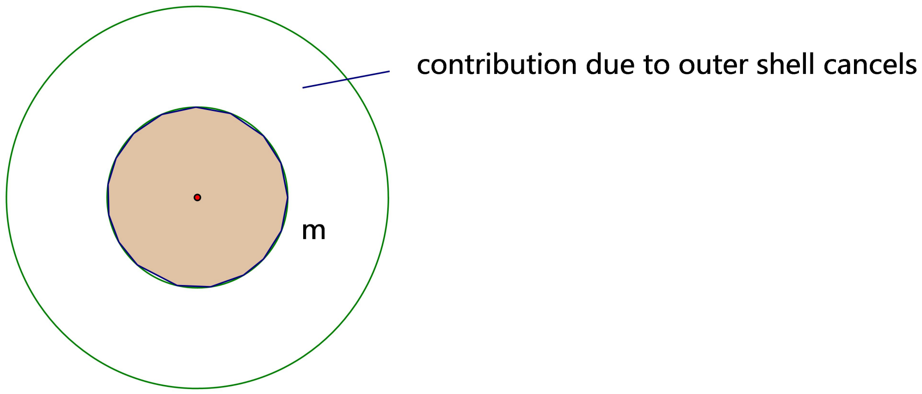


Fact

For a spherically symmetric distribution of mass in a region Ω, the gravitational field is as if the whole mass was concentrated at the center of Ω.



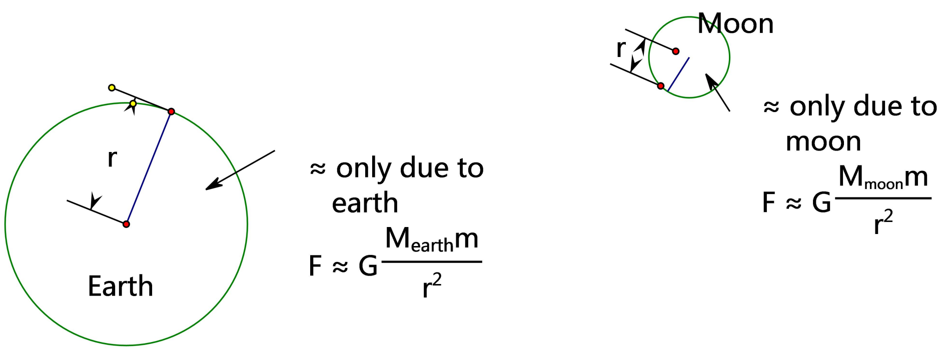
Inside of Ω



18.3 Weight

Weight: total gravitational force exerted on a body by all other objects in the universe

Practically,



At earth's surface,

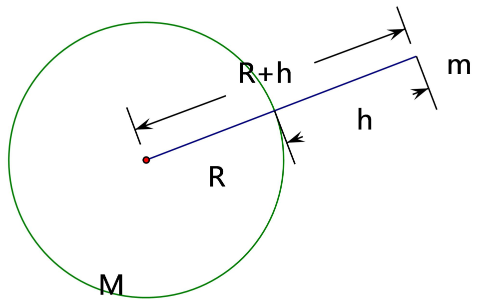
$$F = G\frac{M_{earth}m}{\underbrace{R^2}_{\text{radius of the earth}}} = mg$$

Comment: By measuring g, the mass of the earth may be estimated

$$M_{earth} = g\frac{R^2}{G} \approx 5.98 \times 10^{24} kg$$

$$\text{average} \quad \text{density :} \quad g_{av} = \frac{M_{earth}}{\frac{4}{3}\pi R^3} \approx 5.5 \times 10^3 kg/m^3$$

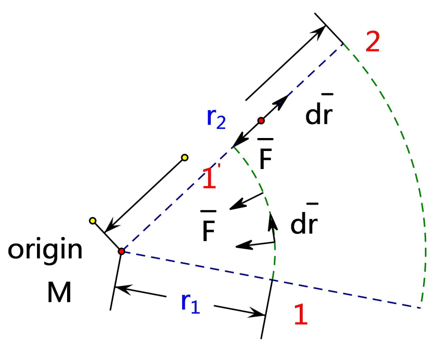
18.4 Change of weight (due to earth) with altitude



$$\begin{aligned} \text{Weight} &= G\frac{Mm}{(R+h)^2} = G\frac{Mm}{R^2(1+\frac{h}{R})^2} = G\frac{M}{R^2}m(1+\frac{h}{R})^{-2} \\ &= G\frac{M}{R^2}m(1-2\frac{h}{R}+3(\frac{h}{R})^2-...) \quad [h \ll R] \\ &\approx \underbrace{G\frac{M}{R^2}}_{=g} m = mg \ \ \text{(close the earth's surface)} \end{aligned}$$

18.5 Potential energy (gravitational)

Recall: central forces are conservative $\implies$ work does not depend on path.



$$\bar{F} = -G\frac{Mm}{r^2}\frac{\bar{r}}{|r|} = \underbrace{f(r)}_{\text{central force}} \quad \bar{r}$$

Choose path $1 \to circle \to 1' \to radius \to 2'$

$$W_{1\to 2} = \int_{1\to 2} \bar{F}\cdot d\bar{r} = \underbrace{\int_{1\to 1'} \bar{F}\cdot d\bar{r}}_{=0 \quad (\bar{F}\perp d\bar{r})} + \int_{1'\to 2'} \bar{F}\cdot d\bar{r} = -\int_{r_1}^{r_2} |\bar{F}|\cdot|d\bar{r}|$$

$$= \underbrace{-\int_{r_1}^{r_2} G\frac{Mm}{r^2}dr}_{\angle(\bar{F},d\bar{r})=\pi} = G\frac{Mm}{r}\Big|_{r_1}^{r_2} = G\frac{Mm}{r_2} - G\frac{Mm}{r_1}$$

On the other hand, for potentiaL forces, $W_{1\to 2} = u_1 - u_2$

Hence,

$$\underbrace{u(r)}_{\text{gravitational potential energy}} = -G\frac{Mm}{r} + C$$

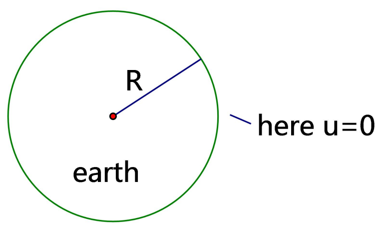
C can be chosen arbitrarily (no physical meaning), only Δu is measurable/-physical

Choice of C (choice of gauge) - infinitely many posibilities

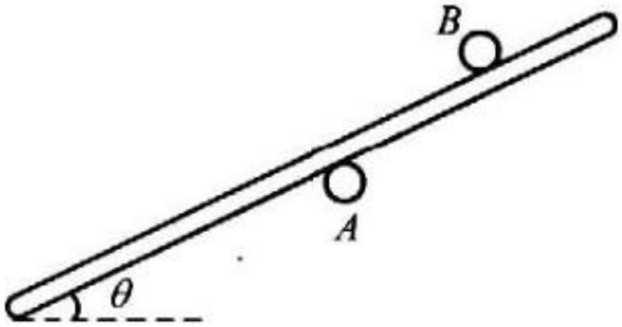
Eg.

$$\to u(\infty) = 0 \implies C = 0 \quad and \quad u(r) = -G\frac{Mm}{r}$$

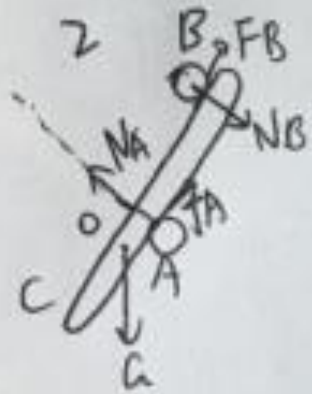
$$\to u(R) = 0 \implies 0 = u(R) = -G\frac{Mm}{R} + C \implies C = G\frac{Mm}{R} \quad and \quad u(r) = GMm($$



Two stick A and B with distance 2a between them is fixed on the wall parallel to the ground, a stick C is put above A and under B with angle θ to the ground. (the stick does not touch the ground) The friction coefficient between sticks is μ. To make C at equilibrium, which requirements does the distance between center mass of C and A need satisfy?



2



if center of mass at section 2 of C,

C will rotate about A

so center of mass at section 1

suppose N_A, N_B, G, f_A, f_B the distance of OA is x

① net force = 0

$$\begin{cases} G\sin\theta - f_A - f_B = 0 \\ G\cos\theta + N_B - N_A = 0 \end{cases}$$

② net torque = 0

can only rotate about A

so $G\cos\theta \cdot x = N_B \cdot 2a$

$$N_B = \frac{G\cos\theta x}{2a},$$

$$N_A = \frac{G\cos\theta(x+2a)}{2a},$$

$$f_A + f_B = G\sin\theta \le \mu(N_A + N_B)$$

so $G\sin\theta \le \frac{\mu G\cos\theta(2x+2a)}{2a}$,

$$\frac{1}{\mu}\tan\theta \le \frac{x+a}{a}$$

$$x \geq \frac{a(\tan\theta - \mu)}{\mu}.$$

① if $\mu > \tan\theta$,

then $\frac{a(\tan\theta-\mu)}{\mu}$ is negative.

x can't be negative,

so $x \geq 0$

② if $\mu \leq \tan\theta$.

then $x \geq \frac{a(\tan\theta-\mu)}{\mu}$

Final Review Exercise - Gravitation

1. Suppose Earth is a uniform ball with radius R, for a well with depth d, find the ratio of gravitational acceleration at the bottom of the well and on the ground.

   Solution:

   Suppose the density of the earth is ρ, we have

   $$mg = G\frac{Mm}{R^2} = G \cdot m \cdot \frac{\rho\frac{4}{3}\pi R^3}{R^2} = \frac{4}{3}\pi G m \rho R$$

   Recall that only the inner part contribute to gravitational force inside a ball.

   (because inside a uniform sphere, the gravitation force is 0. For a mass inside a uniform ball with distance a from center, outer part can be seen as many sheers of spheres, thus only inner part($r < a$) have force on m)

   Therefore

   $$mg' = G\frac{M'm}{(R-d)^2} = \frac{4}{3}\pi G m \rho (R-d)$$

   So

   $$\frac{g'}{g} = \frac{R-d}{R}$$

GOOD LUCK!

& Welcome to ME!