VG100: INTRODUCTION TO ENGINEERING

Dimensional Analysis

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Preview

- Dimensions and units
- Buckingham Pi Theorem
- Examples

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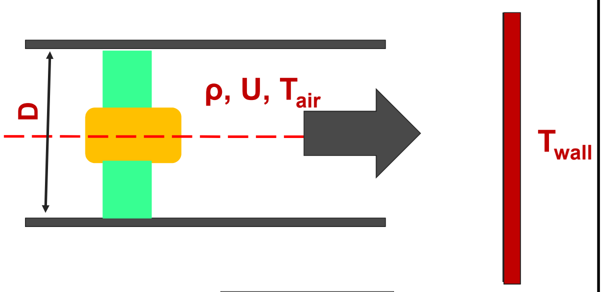
How can we scale up or scale down our project?



$$h = \frac{q}{\left(T_s - T_{ref}\right)}$$

Heating Power Q_{in}

3

Dimensions & Units

Dimension - abstract quantity (e.g. length, time, mass etc.)

Unit - a specific definition of a dimension based upon a physical reference (e.g. meter, second, gram)

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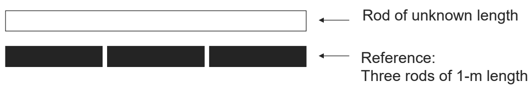
What does a "unit" mean?

How long is the rod?



The unknown rod is 3 *m* long.

unit
number

The number is **meaningless** without the unit!

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Dimensions of Some Common Physical Quantities

(absolute system)

Length – [L]
Mass – [M]
Time – [T]

Velocity – $[L][T]^{-1}$
Acceleration – $[L][T]^{-2}$
Force – $[M][L][T]^{-2}$
Density – ML^{-3}
Pressure – $ML^{-1}T^{-2}$
Energy – ML^2T^{-2}

All are powers of the fundamental dimensions:
[Any Physical Quantity] = $M^aL^bT^cQ^d$

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The International System of Units (SI)

Fundamental Dimension	Base Unit
length [*L*]	meter (m)
mass [*M*]	kilogram (kg)
time [*T*]	second (s)
electric current [*A*]	ampere (A)
absolute temperature [*θ*]	kelvin (K)
luminous intensity [*I*]	candela (cd)
amount of substance [*n*]	mole (mol)

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How do dimensions behave in mathematical formulae?

Rule 1 - All terms that are added or subtracted must have same dimensions

$$D = A + B - C$$

All have identical dimensions

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How do dimensions behave in mathematical formulae?

Rule 2 - Dimensions obey rules of multiplication and division

$$D = \frac{AB}{C} = \frac{\left(\frac{[M]}{[T^2]}\right)\left(\frac{[T^2]}{[L]}\right)}{\left(\frac{[M]}{[L^2]}\right)} = [L]$$

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Dimensionally Homogeneous Equations

An equation is said to be ***dimensionally homogeneous*** if the dimensions on both sides of the equal sign are the same.

Dimensional Homogeneity Theorem

Any physical quantity is dimensionally a power law monomial –

[Physical Quantity] = $M^a L^b T^c \ldots$

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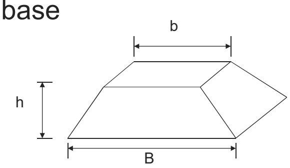
Dimensionally Homogeneous Equations

Volume of the frustrum of a right pyramid with a square base



$$V = \frac{h}{3}\left(B^2 + Bb + b^2\right)$$

$$[L]^3 = \left[\frac{L}{1}\right]\left([L]^2 + [L]^2 + [L]^2\right) = [L]^3$$

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Being a Scientist or Engineer

The steps in understanding and/or control any physical phenomena is to:

1. Identify the relevant physical variables.
2. Relate these variables using the known physical laws.
3. ~~Solve the resulting equations~~.

Use dimensional analysis and experiments

Sometimes solving the problem is not possible

→ lab experiments

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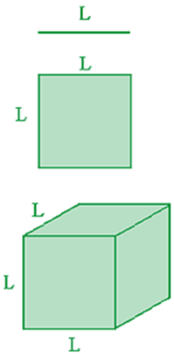
Dimensional Analysis

Physical nature of the quantity type of unit (**Dimension**) used to specify it.

- Distance has dimension [L].
- Area has dimension $[L]^2$.
- Volume has dimension $[L]^3$.
- Time has dimension [T].
- Speed has dimension [L]/[T]



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Gulliver's Lunch: Dimensional Analysis

Gulliver was 12 times taller than the Lilliputians
How much should they feed him? 12 Lunch ratios?

- A person's food needs are related to his/her body volume.

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Let $\mathbf{L_G}$ and $\mathbf{V_G}$ denote Gulliver's linear and volume dimensions.
Let $\mathbf{L_L}$ and $\mathbf{V_L}$ denote the Lilliputian's linear and volume dimensions.

Gulliver is 12 times taller than the Lilliputians $\Rightarrow$ $\mathbf{L_G} = 12\ \mathbf{L_L}$

$V_G \propto [L_G]^3$ and $V_L \propto [L_L]^3$

$$\begin{aligned} V_G / V_L &= [L_G]^3/[L_L]^3 \\ &= [12\ L_L]^3/[L_L]^3 \\ &= 12^3 \\ &= 1728 \end{aligned}$$

Gulliver needs to be fed **1728** lunch boxes!

This example has direct relevance to drug dosages in humans

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Why no small animals survive in the polar regions?

1. Identify the relevant physical variables.
2. Relate these variables using the known physical laws.
3. Use dimensional analysis

- **Heat Loss** $\propto$ Surface Area (L^2)
- Mass $\propto$ Volume (L^3)
- Heat Loss/Mass $\propto$ Area/Volume

$$= L^2 / L^3$$
$$= L^{-1}$$

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Why no small animals survive in the polar regions?

1. Identify the relevant physical variables.
2. Relate these variables using the known physical laws.
3. Use dimensional analysis

- **Heat Loss** $\propto$ Surface Area $[L]^2$
- Body heat is generated in the deep organs (e.g. liver, brain, heart, etc.)
- **Heat Generated** $\propto$ Mass $\propto$ Volume $[L]^3$
- Heat Loss/Heat Generated $\propto$ $[L]^2/[L]^3$

  $= [L]^{-1}$

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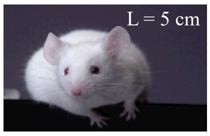
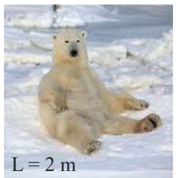
Why no small animals survive in the polar regions?

Heat Loss/Heat Generated $\propto$ $[L]^{-1}$





$1/[L] = 1/(0.05\ \text{m}) = 20\ \text{m}^{-1}$

$1/[L] = 1/(2\ \text{m}) = 0.5\ \text{m}^{-1}$

$20/0.5 = 40$ Mouse loses heat 40 times faster than the polar bear!

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Buckingham Pi Theorem

- system has ***n*** **physical variables** of relevance that depend on ***k*** **independent dimensions**
- Total of ***n-k*** independent dimensionless products

  $\pi_1, \pi_2, \ldots, \pi_{n-k}$

System behavior dimensionless equation

$$F(\pi_1, \pi_2, \ldots, \pi_{k-r}) = 0$$

- Different systems that share same description by dimensionless quantity are equivalent.

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Exponent Method

1. List all ***n*** variables involved in problem
2. Express each variable in terms of *[M] [L] [T]* dimensions (***k***)
3. # of dimensionless parameters (***n – k***)
4. Select number of repeating variables (All dimensions must be included in this set and each repeating variable must be independent of the others.)
5. Form a dimensionless parameter π by multiplying one of the non-repeating variables by the product of the repeating variables, each raised to an unknown exponent.
6. Solve for unknown exponents.
7. Repeat this process for each non-repeating variable
8. Express result as a relationship among the dimensionless parameters – $F(\pi_1, \pi_2, \pi_3, \ldots) = 0$.

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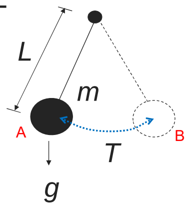
Buckingham π Theorem

Pendulum - What is the period, T (how long does it take to go from A to B and vice versa)?



1. List all ***n*** variables involved in the problem

   L, g, m, T

2. Express each variables in terms of *[M] [L] [T]* dimensions (***k***)

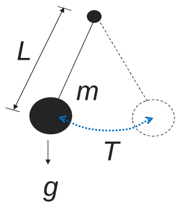
   [L], [L]/[T]^2, [M], [T] ⟶ ***p = n − k***

   1 = 4 - 3

Exponent Method

1. List all ***n*** variables involved in problem
2. Express each variable in terms of *[M] [L] [T]* dimensions (***k***)
3. # of dimensionless parameters (***n – k***)
4. Select number of repeating variables (All dimensions must be included in this set and each repeating variable must be independent of the others.)
5. **Form a dimensionless parameter π by multiplying one of the non-repeating variables by the product of the repeating variables, each raised to an unknown exponent.**
6. Solve for unknown exponents.
7. Repeat this process for each non-repeating variable
8. Express result as a relationship among the dimensionless parameters – $\boldsymbol{F(\pi_1, \pi_2, \pi_3, \ldots) = 0}$.

Pendulum - What is the period?



$$\boldsymbol{\Pi_1 = T \times m^a \times g^b \times L^c}$$

$$M^0 \times L^0 \times T^0 = [T] \quad [M]^a \quad \left[\frac{L}{T^2}\right]^b \quad [L]^c$$

$$\begin{array}{lcccccccccl}
[M] & 0 & = & a & + & 0 & + & 0 & \Rightarrow & a = 0 \\
[T] & 1 & = & 0 & - & 2b & + & 0 & \Rightarrow & b = -1/2 \\
[L] & 0 & = & 0 & + & b & + & c & \Rightarrow & c = +1/2
\end{array}$$

$$\Pi_1 = \frac{1}{T}\sqrt{\frac{L}{g}} \qquad \Rightarrow T = k\sqrt{\frac{L}{g}}$$

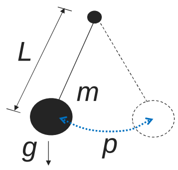
Unknown constant

Pendulum - What is the period?

Unknown constant

$$T = k\sqrt{\frac{L}{g}} \qquad T = 2\pi\sqrt{\frac{L}{g}}$$

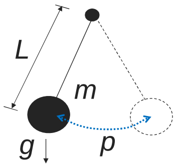
(ANALYTICAL SOLUTION)



- Dimensional analysis does not give you the exact solution but it “almost” does.
- Dimensional analysis tells you right away that the period is independent of the mass!
- The “unknown” constant, *k*, could be derived from simple laboratory tests.

Dimensionless groups are good for scaling!

$$\Pi_1 = \frac{1}{T}\sqrt{\frac{L}{g}}$$



the dimensionless group is constant.

- if I want to build a pendulum 100 times bigger than the one in the lab, T will be __? ?__ times smaller
- if I go to the moon and (g_m=g/6), my pendulum will have to be __? ?__ time shorter to have the same period

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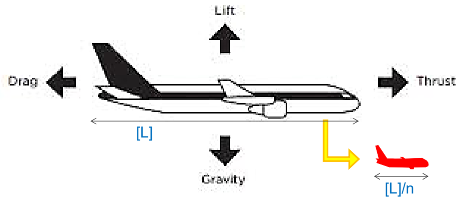
Drag Force



$$a = 0 \Leftrightarrow Drag = Thrust$$ (maximum cruising velocity)

To select the appropriate engine (thrust) required to reach the required maximum velocity you **need to know the drag force**. How do you compute it?

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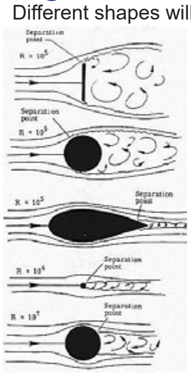
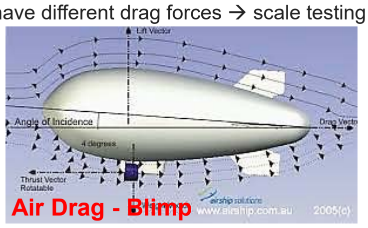
Drag Force

Different shapes will have different drag forces → scale testing





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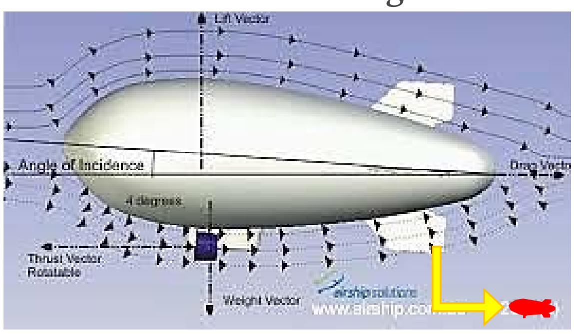
Scale Testing



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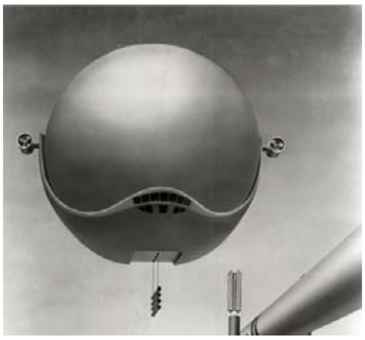
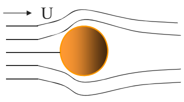
Drag Force, F



Scaled Experiment



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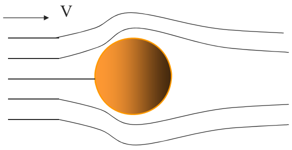
Drag Force, F



Drag force on a **smooth sphere** depends on:

a) The velocity U
b) The sphere diameter D
c) The viscosity μ of the fluid
d) The density ρ of the fluid
e) Other characteristics of secondary importance

F = f (V, D, μ, ρ)

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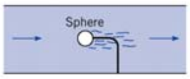
Drag Force, F



- **Find:** Drag force on **rough sphere** is function of D, ρ, μ, V and k. Express in form: $\pi_3 = f(\pi_1, \pi_2)$

F_D	D	ρ	μ	V	k
MLT^{-2}	L	ML^{-3}	$ML^{-1}T^{-1}$	LT^{-1}	L

$n = 6$ No. of dimensional parameters
$k = 3$ No. of dimensions
$p = n - k = 3$ No. of dimensionless parameters

- Select "repeating" variables: D, V, and ρ

❖ All dimensions must be included in this set and each repeating variable must be independent of the others

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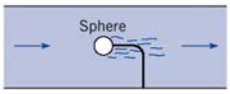
Drag Force, F



F_D	D	ρ	μ	V	k
MLT^{-2}	L	ML^{-3}	$ML^{-1}T^{-1}$	LT^{-1}	L

$$\pi_1 = \mu(D^a V^b \rho^c)$$

$$M^0 L^0 T^0 = (ML^{-1}T^{-1})(L)^a (LT^{-1})^b (ML^{-3})^c$$

$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$

$$L: \quad 0 = -1 + a + b - 3c \quad \Rightarrow \quad a = -1$$

$$T: \quad 0 = -1 - b \quad \Rightarrow \quad b = -1$$

$$\pi_1 = \frac{\mu}{DV\rho} \quad or \quad \pi_1 = \Re = \frac{\rho VD}{\mu}$$

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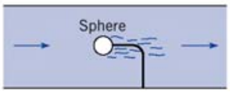
Drag Force, F



F_D	D	ρ	μ	V	k
MLT^{-2}	L	ML^{-3}	$ML^{-1}T^{-1}$	LT^{-1}	L

$$\pi_2 = k(D^a V^b \rho^c)$$
$$M^0L^0T^0 = (L)(L)^a(LT^{-1})^b(ML^{-3})^c$$
$$M: \quad 0 = c \quad \Rightarrow \quad c = 0$$
$$L: \quad 0 = 1 + a + b - 3c \quad \Rightarrow \quad a = -1$$
$$T: \quad 0 = -b \quad \Rightarrow \quad b = 0$$

$$\pi_2 = \frac{k}{D}$$

$$\pi_3 = F_D(D^a V^b \rho^c)$$
$$M^0L^0T^0 = (MLT^{-2})(L)^a(LT^{-1})^b(ML^{-3})^c$$
$$M: \quad 0 = 1 + c \quad \Rightarrow \quad c = -1$$
$$L: \quad 0 = 1 + a + b - 3c \quad \Rightarrow \quad a = -2$$
$$T: \quad 0 = -2 - b \quad \Rightarrow \quad b = -2$$

$$\pi_3 = \frac{F_D}{\rho V^2 D^2}$$

$$\boxed{\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}, \frac{k}{D}\right)}$$

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Drag Force, F

If we ignore the surface roughness (e.g., assuming the sphere is "smooth"):

$$\frac{F_D}{\rho V^2 D^2} = f\left(\frac{\rho V D}{\mu}, \cancel{\frac{k}{D}}\right)$$

$$\Downarrow$$

$$\boxed{\frac{F_D}{\rho V^2 \left(\frac{\pi}{4}\right) D^2} = \frac{F_D}{\rho V^2 A} = f\left(\frac{\rho V D}{\mu}\right)}$$

This non-dimensional group has a special name: drag coefficient, **c_d**

This non-dimensional group has a special name: Reynolds Number, **Re_D**

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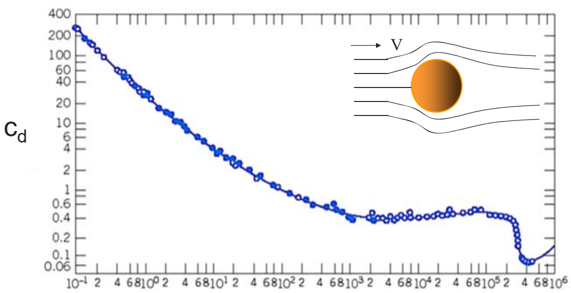
Drag Force on a Shpere



$$\frac{\rho V D}{\mu}$$

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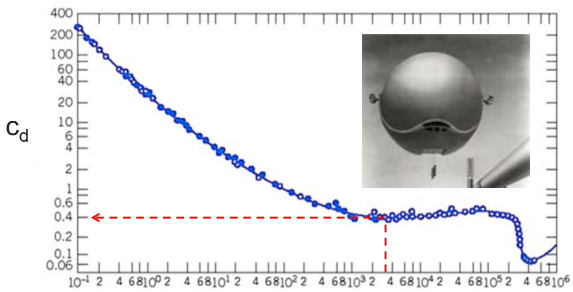
Dimensional Analysis to Design

Desire balloon velocity = 10 km/hr, balloon diameter = 10 m, then Re_D is known.
You can use the "dimension-less" plot to calculate c_d, and hence F, the drag force you need to overcome with the motors.



$$\frac{\rho V D}{\mu}$$

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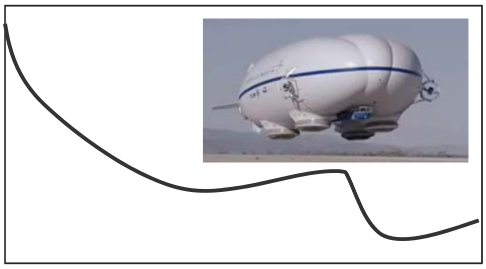
Dimensional Analysis to Design

Unique c_d vs. Re_L relationships correspond to different shapes. Testing can be carried out on scaled samples in the wind tunnel



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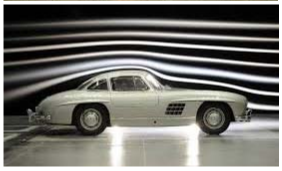
Wind Tunnel Testing

Unique c_d vs. Re_L relationships correspond to different shapes. Testing can be carried out on scaled samples in the wind tunnel



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Automobile experiment

- Drag force is $F = f\,(V, \rho, \mu, L)$
- Through dimensional analysis, we can reduce the problem to

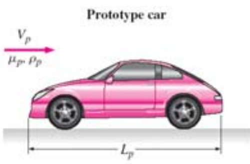
$$\Pi_1 = f\,(\Pi_2)$$

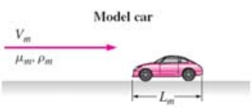
- where

$$\Pi_1 = \frac{F_D}{\rho V^2 L^2} = C_d$$

and

$$\Pi_2 = \frac{\rho V L}{\mu} = Re$$





The Reynolds number is the most well known and useful dimensionless parameter in all of fluid mechanics.

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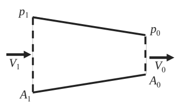
Pressure drop in a pipe flow

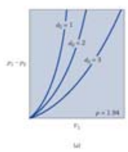
- Study pressure drop as function of velocity (V_1) and diameter (d_o)
- Carry out numerous experiments with different values of V_1 and d_o and plot the data

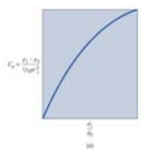


ΔP	ρ	V_1	d_1	d_2
$ML^{-1}T^{-2}$	ML^{-3}	LT^{-1}	L	L

5 parameters:
$\Delta p, \rho, V_1, d_1, d_o$

2 dimensionless parameter groups:
$\Delta P/(\rho V^2/2)$, (d_1/d_o)





- **Much easier to establish functional relations with 2 parameters, than 5**

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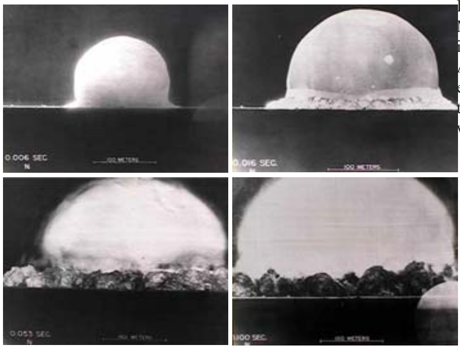
G. I. Taylor's 1947 Analysis



In 1947 the "Times Magazine" published images from an Atomic Bomb explosion (Power of the blast, 18 kiloton, was **classified**)

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Nuclear Explosion Shock Wave

The propagation of a nuclear explosion shock wave depends on: E, r, ρ, and t. $r = f(E, \rho, t)$

$n = 4$ No. of variables
$k = 3$ No. of dimensions
$n - k = 1$ No. of dimensionless parameters

E	ρ	r	t
ML^2T^{-2}	ML^{-3}	L	T

Select "repeating" variables: E, t, and ρ

Combine these with the rest of the variables: r

$$\pi_1 = r \times (E^a t^b \rho^c)$$

$$M^0 L^0 T^0 = (L)(ML^2T^{-2})^a (T)^b (ML^{-3})^c$$

$$M: \quad 0 = a + c \quad \Rightarrow \quad c = -a$$

$$L: \quad 0 = 1 + 2a - 3c \Rightarrow \quad a = -\frac{1}{5}$$

$$T: \quad 0 = -2a + b \quad \Rightarrow \quad b = -\frac{2}{5}$$

$$\pi_1 = RE^{-1/5} t^{-2/5} \rho^{1/5} = \frac{R}{E^{1/5} t^{2/5} \rho^{-1/5}}$$

$$F(\pi_1) = 0 \quad \Rightarrow \quad \pi_1 = C$$

$$\frac{R}{E^{1/5} t^{2/5} \rho^{1/5}} = C \quad \Rightarrow \quad R = CE^{1/5} t^{2/5} \rho^{-1/5}$$

$$\Rightarrow \quad R = C\left(\frac{Et^2}{\rho}\right)^{1/5}$$

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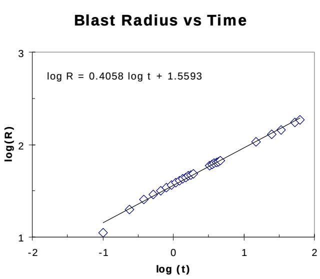
$R = (E/\rho)^{1/5}\, t^{2/5}$

$\log R = 0.4 \log t + 0.2 \log(E/\rho)$



$0.2 \log(E/\rho) = 1.56$

$\rho = 1\ \text{kg/m}^3$

$\Downarrow$

$E = 7.9\text{x}10^{13}$ J
$= 19.8$ kilotons TNT

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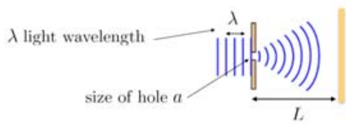
Some common dimensionless groups

Reynolds number $\quad Re = \dfrac{\rho U L}{\mu} = \dfrac{UL}{\nu}$

Euler number $\quad Eu = \dfrac{\Delta p}{\rho V^2}$

Mach number $\quad M = \dfrac{V}{c}$

Fresnel number



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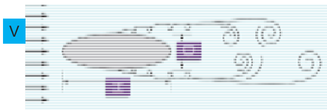
Practice

A long structural component of a bridge has an elliptical cross section (as shown in figure). It is known that when a steady wind blows past this type of bluff body, vortices may develop on the downwind side that are shed in a regular fashion at some definite frequency. Since these vortices can create harmful periodic forces acting on the structure, it is important to determine the shedding frequency. For the specific structure of interest, $D=0.1\ m$, $H=0.3\ m$ and a representative wind velocity is $50\ km/hr$. Standard air can be assumed. The shedding frequency is to be determined through the use of a small-scale model that is to be tested in a water tunnel.

For the model $D_m=20\ mm$ and the water temperature is 20^o. Determine the model dimension, and the velocity at which the test should be performed. If the shedding frequency for the model is found to be $49.9\ Hz$, what is the corresponding frequency for the prototype? $\omega = \mathcal{F}(D, H, V, \rho, \mu)$

$$\omega \doteq T^{-1},\ D \doteq L,\ H \doteq L,\ V \doteq L\,T^{-1},\ \rho \doteq M\,L^{-3},\ \mu \doteq M\,L^{-1}T^{-1}$$



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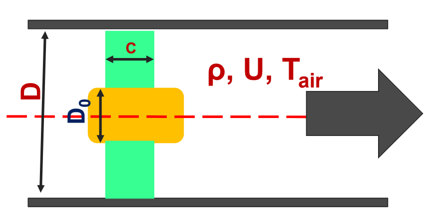
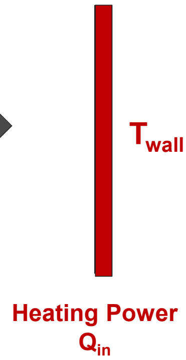
Important dimensionless groups in our project





$$Re = \frac{\rho U L}{\mu} = \frac{UL}{\nu}$$

$$M = \frac{V}{c}$$

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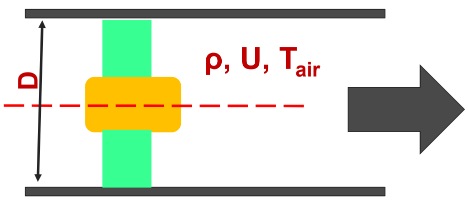
Important dimensionless groups in our project



$$W = f(\rho, \mu, \omega, \Delta P, Q, D)$$

$$\frac{\rho\omega D^2}{\mu} \quad \frac{Q}{\omega D^3} \quad \frac{\Delta P}{\rho\omega^2 D^2} \quad \frac{\dot{W}}{\rho\omega^3 D^5}$$

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Review

- Dimensions and units
- Buckingham Pi Theorem
- Examples

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