Vv156 Lecture 11

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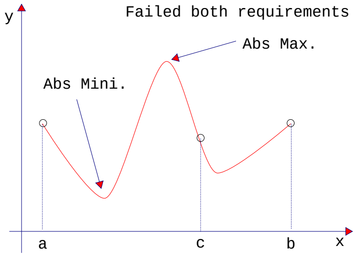
Q: Is there any curve, which is continuous on a closed and bounded interval $\mathcal{I}$, has no global maximum or has no global minimum in $\mathcal{I}$?

The Extreme-Value Theorem

If f is continuous on a closed and bounded interval $\mathcal{I}$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ where $c, d \in \mathcal{I}$.

Q: Is there any curve, which is not continuous or defined on an open/unbounded interval, has got both absolute maximum and absolute minimum?



- The extreme-value theorem (EVT) is an example of what mathematicians call an existence theorem. Such theorems state conditions under which certain objects exist, in this case absolute extrema.
- Knowing that an object exists and finding it are two separate things.
- If f is continuous on the finite closed interval $[a, b]$, the following procedures can be used to find the absolute extrema:

Procedures for finding absolute extrema

1. Find the critical point of f in (a, b)
2. Evaluate f at all the critical points and the end points
3. Compare values in step 2, the largest of them is the absolute maximum of f on $[a, b]$, the smallest is the absolute minimum.

Exercise

Find the absolute extrema of $f(x) = 6x^{4/3} - 3x^{1/3}$ on the interval $[-1, 1]$, and determine where these values occur.

- Imagine you fire a rifle straight up, assuming gravity is the only force present

Q: What can be said regarding the motion of the bullet?

- Under the usual circumstances, we expect the motion is smooth.
- And we expect the bullet goes up, and stop momentarily in the air, then comes down hitting you in the eye.

Q: Why the bullet must stop momentarily before returning?

- What seems to be a trivial truth here is the essence of a principle called

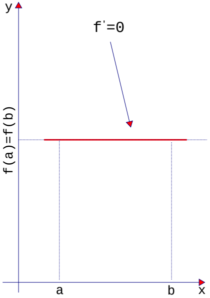
Rolle's Theorem

Let f be a function that satisfies the following three hypotheses:

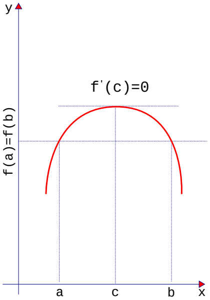
1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

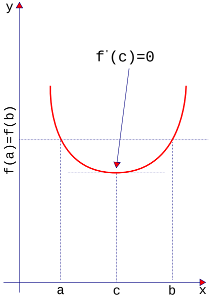
1.


y
f'=0
f(a)=f(b)
a
b
x

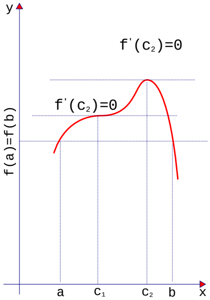
2.


y
f'(c)=0
f(a)=f(b)
a
c
b
x

3.


y
f'(c)=0
f(a)=f(b)
a
c
b
x

4.


y
f'(c₂)=0
f'(c₂)=0
f(a)=f(b)
a
c₁
c₂
b
x

Proof

- If $f(x) = k$, where k is a constant, then

$$f'(x) = 0$$

so the number c can be taken to be any number in (a, b).

- If $f(x) > f(a)$ for some x in (a, b), then by EVT,

$$\text{Hypothesis } 1 \implies f \text{ has a maximum value somewhere in } [a, b]$$

Since $f(a) = f(b)$, so $f(x) > f(b)$, and $f(a)$ and $f(b)$ cannot be the max., it must attain this maximum value at a number c in the open interval (a, b).

- Therefore f has a relative maximum at c, and

$$\text{Hypothesis } 2 \implies f'(c) = 0$$

- If $f(x) < f(a)$ for $x \in (a, b)$, then the argument is very similar, the only difference is that we have a relative minimum instead of a relative maximum.

- Imagine that you are driving a Lykan for an hour, your average speed during this time is 51km/h. Suppose the speed limit is 50km/h.

Q: How can a policeman argue you have been speeding and give you a ticket?

The Mean-Value theorem

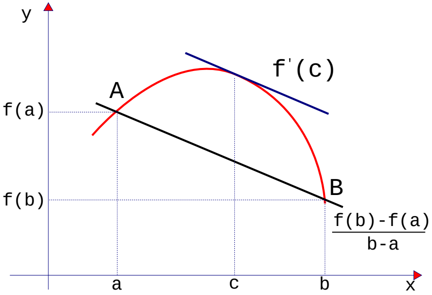
Let f be a function that satisfies the following conditions

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that

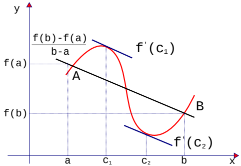
$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- The Mean-Value Theorem (MVT) states that there is a number at which



the instantaneous rate of change is equal to the average rate of change.

- Notice it did not say the number is unique.



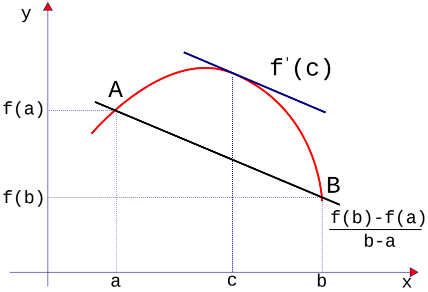
Proof

- Notice when

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a},$$

- The equation of segment is

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$



- The vertical distance between the curve and the segment is given by

$$h(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]$$

- This function $h(x)$ satisfies the three hypotheses of Rolle's Theorem.

Proof

- The function h is continuous on $[a, b]$ because it is the sum of f and a first-degree polynomial, both of which are continuous.
- The function h is differentiable on (a, b) because both f and the first-degree polynomial are differentiable, and the derivative is

$$h' = f'(x) - \frac{f(b) - f(a)}{b - a}$$

- lastly, we need to verify that $h(a) = h(b)$. This is clearly true since the vertical distance is zero at both ends.
- Therefore we can apply Rolle's theorem, which states that there is a number c in (a, b) such that $h'(c) = 0$, thus

$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a}$$
$$\implies f'(c) = \frac{f(b) - f(a)}{b - a} \quad \square$$

Theorem

- Suppose f is continuous on $[a, b]$ and differentiable on (a, b), and

1. if $f'(x) \geq 0$ for every $x \in (a, b)$ if and only if f is increasing on $[a, b]$.
2. if $f'(x) \leq 0$ for every $x \in (a, b)$ if and only if f is decreasing on $[a, b]$.

Proof

- Suppose f is increasing, then

$$\frac{f(x+h) - f(x)}{h} \geq 0$$

for all sufficiently small h, positive or negative, thus

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \geq 0$$

- Conversely suppose $f' \geq 0$ for every $x \in (a, b)$, and $a \leq x_1 < x_2 \leq b$.

Theorem

- Invoking the mean value theorem, we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \geq 0$$

for some $c \in (x_1, x_2)$, which implies that

$$f(x_2) - f(x_1) \geq 0 \qquad \text{for} \quad x_2 > x_1$$

so f is increasing.

- The statement for a decreasing function f follow in a similar fashion.

- Notice if f is strictly increasing, the derivative of f is not necessary to be strictly greater than zero for every $x \in (a, b)$.

$$f(x) = x + \sin x$$

Exercise

(a) Prove the identity

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$$

(b) Prove the following inequality

$$|\sin x - \sin y| \leq |x - y|, \qquad \text{where} \quad x, y \in \mathbb{R}$$

(c) Find all the real solutions to the equation

$$2^x + 5^x = 3^x + 4^x$$