Chapter 10 – Momentum and Conservation of Momentum

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Physics I (Summer 2019)
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Agenda

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   - Momentum–Impulse Theorem
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5. Motion of Objects with Varying Mass.
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Momentum

Definition

Newton's idea $\rightarrow$ "quantity/ amount of motion"

2nd law of dynamics

$$\overline{F} = m\overline{a} = m\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(m\overline{v}) = \frac{\mathrm{d}\overline{p}}{\mathrm{d}t}$$

$\overline{p} = m\overline{v}$ – **momentum (linear momentum)**; units $[kg \cdot m/s]$

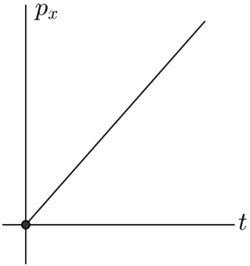
Newton's 2nd law in terms of linear momentum $\overline{p} = m\overline{v}$

$\boxed{\overline{F} = \frac{\mathrm{d}\overline{p}}{\mathrm{d}t}}$ $\rightarrow$ the net force acting on a particle is equal to the time rate of change of particle's momentum

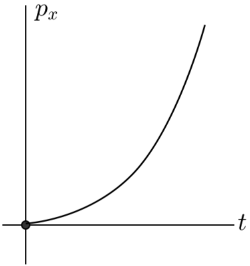
Definition

Illustration

$$\overline{F} = \frac{\mathrm{d}\overline{p}}{\mathrm{d}t}$$



constant force $F_x = \text{const} > 0$



increasing force $F_x = \dfrac{\mathrm{d}p_x}{\mathrm{d}t} > 0$

Momentum–Impulse Theorem

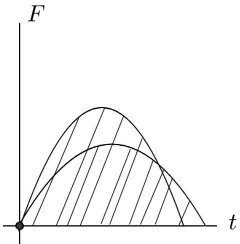
$$F_x = \frac{\mathrm{d}p_x}{\mathrm{d}t} \quad \Longrightarrow \quad F_x \mathrm{d}t = \mathrm{d}p_x \quad \Longrightarrow \quad \int_{t_1}^{t_2} F_x \mathrm{d}t = \int_{p_1}^{p_2} \mathrm{d}p_x$$

Momentum-Impulse Theorem (1D)

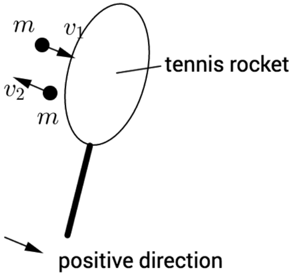
$$\boxed{p_2 - p_1 = \underbrace{\int_{t_1}^{t_2} \overset{\text{net force}}{F_x} \mathrm{d}t}_{\text{impulse of the net force}}}$$

Observation

the same net change in momentum can be achieved in various ways (different forces acting over different intervals of time), if the areas under both curves are the same, the resulting net changes of the momentum are equal.



Momentum-Impulse Theorem. Illustration



$$\frac{\Delta p}{\Delta t} = \frac{m(-v_2 - v_1)}{\Delta t} = F_{av}$$

where Δt is period of time when ball was in contact with rocket, F_{av} is average force exerted on ball.

Kinetic Energy vs. Momentum

Both are measures of the "amount of motion".

Kinetic Energy

* scalar
* changes defined by work-kinetic energy theorem

$$\mathrm{d}K = \overline{F} \cdot \mathrm{d}\overline{r}$$

force over distance

Momentum

* vector
* changes defined by momentum - impulse theorem

$$\mathrm{d}\overline{p} = \overline{F} \cdot \mathrm{d}t$$

force over period of time

Kinetic Energy vs. Momentum. Illustration

Suppose we want to bring two objects to a stop.

$$\begin{aligned} v_A &= 10\ [m/s] \\ m_A &= 400\ [kg] \\ p_A &= 4000\ [kg \cdot m/s] \\ K_A &= 20\ [kJ] \end{aligned} \qquad = \qquad \begin{aligned} v_B &= 5\ [m/s] \\ m_B &= 800\ [kg] \\ p_B &= 4000\ [kg \cdot m/s] \\ K_B &= 10\ [kJ] \end{aligned}$$

What is the average stopping force if both A and B have stopped

* after travelling a distance $s = 100$ m? $\Delta K = -F \cdot s$

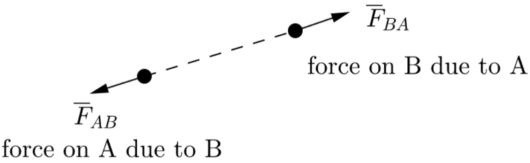
$$F_A = 200N \qquad F_B = 100N$$

* after travelling for $t = 10$ s? $\Delta p = -F \cdot t$

$$F_A = 400N \qquad F_B = 400N$$

Conservation of Momentum

Conservation of Momentum



Assume that there are no external forces acting on the two particles (or the sum of all these external force is zero), that is there are only interactions between the particles.

Newton's 3rd law

$$\overline{F}_{AB} = -\overline{F}_{BA}$$

Newton's 2nd law

$$\overline{F}_{AB} = \frac{\mathrm{d}\overline{p}_A}{\mathrm{d}t}; \qquad \overline{F}_{BA} = \frac{\mathrm{d}\overline{p}_B}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\overline{p}_A}{\mathrm{d}t} = -\frac{\mathrm{d}\overline{p}_B}{\mathrm{d}t} \quad \Longrightarrow \quad \frac{\mathrm{d}\overline{p}_A}{\mathrm{d}t} + \frac{\mathrm{d}\overline{p}_B}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\overline{p}_A + \overline{p}_B) = 0$$

Conclusion (recall: net external force is zero)

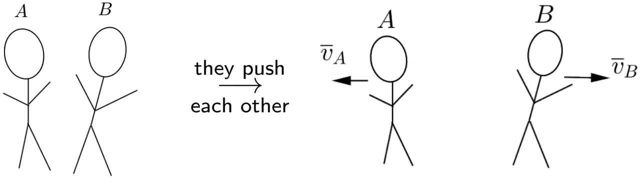
$$\boxed{\overline{p}_A + \overline{p}_B = \text{const}}$$

Law of conservation of momentum

If the sum of all external forces on the system is equal to zero, then the total momentum of the system is constant.

Illustrations

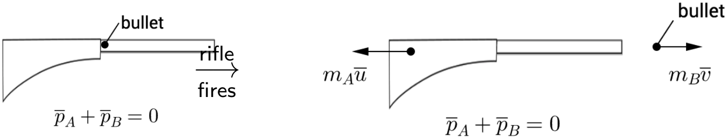
(a) Two ice-skaters



initial $\overline{p}_A + \overline{p}_B = 0$ still $\overline{p}_A + \overline{p}_B = 0$, i.e.,

$$m_A \overline{v}_A = -m_B \overline{v}_B$$

(b) recoil of a rifle



$$m_A \overline{u} = -m_B \overline{v} \quad \Longrightarrow \quad \overline{u} = -\frac{m_B}{m_A}\overline{v}$$

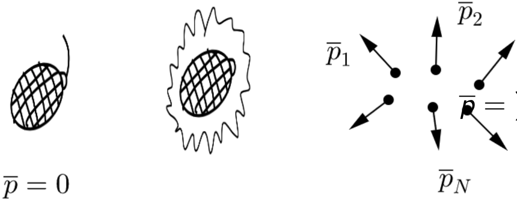
Generalisation to many-particle systems

The statement can be generalised to a system of many particles

> *If the sum of external forces on a system is equal to zero, then the total momentum of the system is constant (i.e. is conserved).*

Illustration

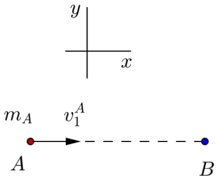
(a) grenade



$\overline{p} = \sum_{i=1}^{N} \overline{p}_i = 0$

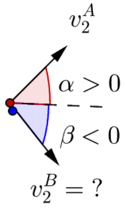
Example: Collision

Initial (1)



$$\overline{p}_1 = (m_A v_1^A, 0)$$

Final (2)



$$\begin{aligned}\overline{p}_2 =& \overline{p}_2^A + \overline{p}_2^B \\ =& (m_A v_2^A \cos\alpha + m_B v_2^B \cos\beta, \\ & m_A v_2^A \sin\alpha + m_B v_2^B \sin\beta)\end{aligned}$$

No external forces $\implies$ $\overline{p}_1 = \overline{p}_2$

$$(m_A v_1^A,\, 0) = (m_A v_2^A \cos\alpha + m_B v_2^B \cos\beta,\; m_A v_2^A \sin\alpha + m_B v_2^B \sin\beta)$$

$$\begin{cases} m_A v_1^A = m_A v_2^A \cos\alpha + m_B v_2^B \cos\beta \\ \qquad 0 = m_A v_2^A \sin\alpha + m_B v_2^B \sin\beta \end{cases}$$

$$\begin{cases} m_A(v_1^A - v_2^A \cos\alpha) = m_B v_2^B \cos\beta \\ -m_A v_2^A \sin\alpha = m_B v_2^B \sin\beta \end{cases}$$

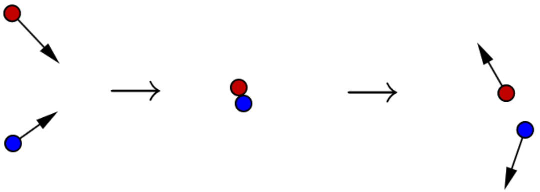
$$\Longrightarrow \boxed{\begin{aligned} \tan\beta &= -\frac{v_2^A \sin\alpha}{v_1^A - v_2^A \cos\alpha} \\ v_2^B &= -\frac{m_A v_2^A \sin\alpha}{m_B \sin\beta} \end{aligned}}$$

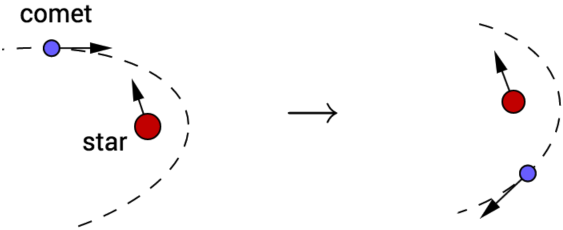
If v_1^A, v_2^A, α and m_A, m_B are known, β and v_2^B may be found.

Collisions

Collisions

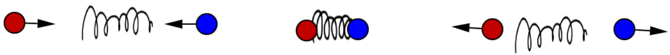
Collision – two objects interact (directly or indirectly) over a finite time-interval.





(a) elastic collision

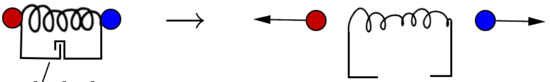
initial | intermediate | final



initial	intermediate	final
$u = \text{const}(=0)$	$u \neq 0$	$u = 0$
$k \neq 0$	$k = 0$	$k \neq 0$
	$(E = k + u = \text{const})$	

$$K_{\text{initial}} = K_{\text{final}}$$

(b) superelastic collision



locked

$u \neq 0, k = 0$

unlocked

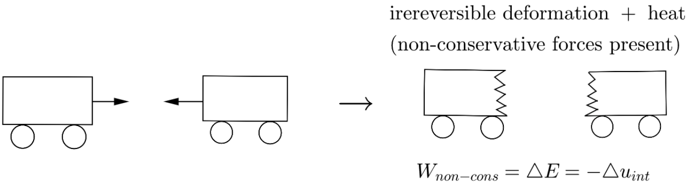
$u = 0$

$k \neq 0$

$$K_{\text{final}} > K_{\text{initial}}$$

Classification of collisions

1. inelastic collision



$$K_{\text{final}} < K_{\text{initial}}$$

In general

$$K_{\text{final}} = K_{\text{initial}} + Q \begin{cases} = 0 & \text{(elastic)} \\ > 0 & \text{(superelastic)} \\ < 0 & \text{(inelastic)} \end{cases}$$

Elastic vs. Inelastic Collisions

Elastic

Internal forces involved are potential (conservative), hence the mechanical energy is conserved $\Delta E = 0$ (usually in our examples $U = \text{const}$, hence $K = \text{const}$)

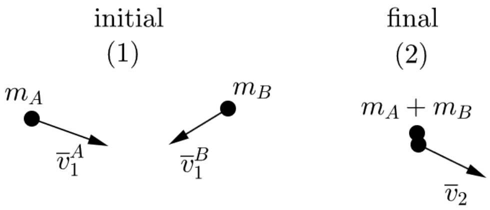
Inelastic

Internal forces are non-conservative, mechanical energy not conserved (total energy is conserved, but part of mechanical energy is transformed irreversibly into internal energy)

* **completely inelastic collisions** (colliding particles move as one object after the collision, stick to each other)

In both cases, the total momentum of the system is conserved.

Completely Inelastic Collisions. Example



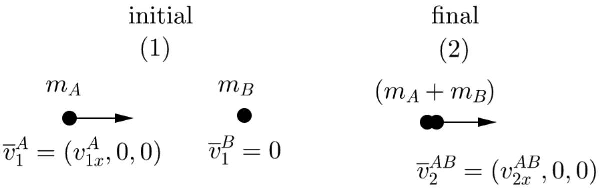
$$m_A \overline{v}_1^A + m_B \overline{v}_1^B \quad = \quad (m_A + m_B)\overline{v}_2$$

If we know: m_A, m_B and $\overline{v}_1^A$, $\overline{v}_2^B$, we can find

$$\overline{v}_2 = \frac{m_A \overline{v}_1^A + m_B \overline{v}_1^B}{m_A + m_B}$$

Completely Inelastic Collisions. Example I (head-on collision)

More specific example: head-on, completely inelastic collision.



$$\overline{p}_1 = \overline{p}_2 \qquad \Longrightarrow \qquad \boxed{v_{2x}^{AB} = \frac{m_A v_{1x}^A}{m_A + m_B}}$$

Compare the kinetic energy before and after collision (we assume U =const)

$$K_1 = \frac{1}{2} m_A (v_{1x}^A)^2$$

$$K_2 = \frac{1}{2}(m_A + m_B)(v_{2x}^{AB})^2 = \frac{1}{2}(m_A + m_B)\frac{m_A^2 (v_{1x}^A)^2}{(m_A + m_B)^2}$$

$$= \frac{1}{2}\frac{m_A^2}{m_A + m_B}(v_{1x}^A)^2 = \frac{m_A}{m_A + m_B}K_1$$

Hence

$$\Delta K = K_2 - K_1 = K_1\left(\frac{m_A}{m_A + m_B} - 1\right) < 0$$

Discussion

- limit case $\quad m_A \gg m_B$

$$\overline{v}_2^{AB} \approx (v_{1x}^A, 0, 0), \qquad \Delta K \approx 0 - \text{negligible}$$

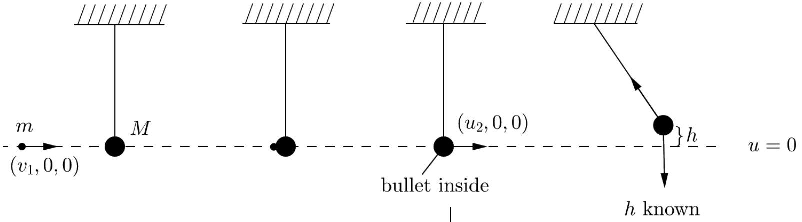
- limit case $\quad m_A \ll m_B$

$$\overline{v}_2^{AB} - \text{negligible}, \qquad \Delta K \approx -K_1$$

Almost all the initial kinetic energy is dissipated.

Completely Inelastic Collisions. Example II (ballistic pendulum)

Ballistic pendulum – used to estimate the speed of a bullet.



I here : momentum conserved, mechanical energy not conserved (dissipative internal forces present)	II here : momentum not conserved (external forces : gravity and tension) energy conserved (gravity - potential energy tension - does no work)

I use conservation of momentum (horizontal component) to find u_2

$$\begin{aligned} \overline{p}_1 &= \overline{p}_2 \\ mv_1 &= (M+m)u_2 \\ \Rightarrow u_2 &= \frac{m}{M+m} v_1 \end{aligned}$$

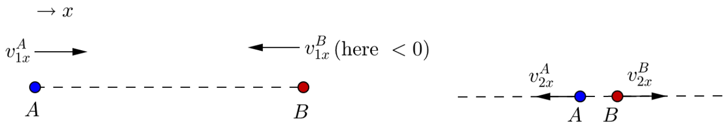
II use conservation of energy to relate h with u_2 (hence with v_1)

$$\begin{aligned} \frac{1}{2}(M+m)u_2^2 &= (M+m)gh \\ \frac{1}{2}(M+m)\frac{m^2}{(M+m)^2} v_1^2 &= (M+m)gh \end{aligned}$$

$$\boxed{v_1 = \frac{M+m}{m}\sqrt{2gh}}$$

Elastic Collisions. Example (head-on collision, general case)

General 1D case (head-on collision) $\rightarrow$ before/after the collision the velocities of both particles are colinear



1. Momentum conserved

$$m_A v_{1x}^A + m_B v_{1x}^B = m_A v_{2x}^A + m_B v_{2x}^B$$

2. Energy conserved

$$\frac{1}{2} m_A (v_{1x}^A)^2 + \frac{1}{2} m_B (v_{1x}^B)^2 = \frac{1}{2} m_A (v_{2x}^A)^2 + \frac{1}{2} m_B (v_{2x}^B)^2$$

Elastic Collisions. Example (head-on collision, specific case: one object initially at rest)

More specific example: B initially at rest ($v_{1x}^B = 0$). Then

$$\begin{cases} m_A v_{1x}^A = m_A v_{2x}^A + m_B v_{2x}^B \\ \frac{1}{2} m_A (v_{1x}^A)^2 = \frac{1}{2} m_A (v_{2x}^A)^2 + \frac{1}{2} m_B (v_{2x}^B)^2 \end{cases}$$

$$\begin{cases} m_A (v_{1x}^A - v_{2x}^A) = m_B v_{2x}^B \qquad (*) \\ m_A \underbrace{(v_{1x}^A - v_{2x}^A)(v_{1x}^A + v_{2x}^A)}_{(v_{1x}^A)^2 - (v_{2x}^A)^2} = m_B (v_{2x}^B)^2 \end{cases}$$

Divide side-by-side to get

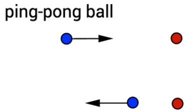
$$v_{1x}^A + v_{2x}^A = v_{2x}^B \qquad (**)$$

Substitute v_{2x}^B from (**) into (*) and solve for v_{2x}^A, substitute back into (**) to get v_{2x}^B.

$$
\boxed{\begin{aligned} v_{2x}^A &= \frac{m_A - m_B}{m_A + m_B} v_{1x}^A \\ v_{2x}^B &= \frac{2m_A}{m_A + m_B} v_{1x}^A \end{aligned}}
$$

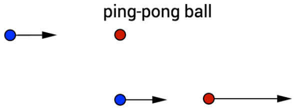
Discussion

(a) $m_A \ll m_B$



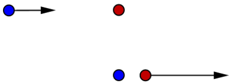
$$v_{2x}^A \approx -v_{1x}^A, \qquad v_{2x}^B \ll v_{1x}^A$$

(b) $m_A \gg m_B$



$$v_{2x}^A \approx v_{1x}^A, \qquad v_{2x}^B \approx 2v_{1x}^A$$

(c) $m_A = m_B$



$$v_{2x}^A = 0, \qquad v_{2x}^B = v_{1x}^A$$

Elastic Collisions. Example (for pool players)

A ball collides in a non-head-on elastic collision with another identical ball which is initially at rest. Show that after the collision the balls move in perpendicular directions.

Solution

$$\begin{array}{r} \text{momentum:} \\ \text{mechanical energy } (U = \text{const}): \end{array} \begin{cases} m\bar{v}_1 = m\bar{u}_1 + m\bar{u}_2 \\ \dfrac{1}{2}mv_1^2 = \dfrac{1}{2}mu_1^2 + \dfrac{1}{2}mu_2^2 \end{cases}$$

$$\begin{cases} \overline{v}_1 \circ \overline{v}_1 = \overline{v}_1^2 = (\overline{u}_1 + \overline{u}_2) \circ (\overline{u}_1 + \overline{u}_2) \\ \qquad v_1^2 = u_1^2 + u_2^2 \end{cases}$$

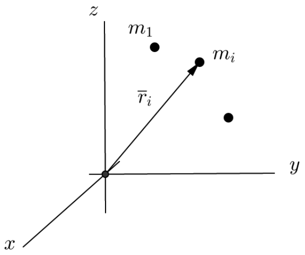
$$\begin{cases} v_1^2 = u_1^2 + u_2^2 + \underbrace{\overline{u}_1 \circ \overline{u}_2 + \overline{u}_2 \circ \overline{u}_1}_{2\overline{u}_1 \circ \overline{u}_2} \\ \\ v_1^2 = u_1^2 + u_2^2 \end{cases}$$

Subtracting side-by-side

$$\overline{u}_1 \circ \overline{u}_2 = 0 \Rightarrow \boxed{\overline{u}_1 \perp \overline{u}_2}$$

Center of Mass

Center of Mass. Definition



Position vector of the **center of mass** (system of particles)

$$\boxed{\overline{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \overline{r}_i}{\sum_{i=1}^{N} m_i}} \rightarrow \text{discrete distribution of mass}$$

Note. The sum is replaced by an integral for a continuous distribution of mass.

Center of Mass. Properties

$$\underbrace{\sum_{i=1}^{N} m_i}_{\text{total mass of the system}} \quad \overline{r}_{cm} = \sum_{i=1}^{N} m_i \overline{r}_i \qquad \Big/ \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[(\sum_{i=1}^{N} m_i)\overline{r}_{cm}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N} m_i \overline{r}_i$$

$$(\sum_{i=1}^{N} m_i)\frac{\mathrm{d}\overline{r}_{cm}}{\mathrm{d}t} = \sum_{i=1}^{N} \underbrace{m_i \underbrace{\frac{\mathrm{d}\overline{r}_i}{\mathrm{d}t}}_{\overline{v}_i}}_{\overline{p}_i}$$

Hence

$$\boxed{M\overline{v}_{cm} = \sum_{i=1}^{N} \overline{p}_i = \overline{p}} \qquad \text{where } M = \sum_{i=1}^{N} m_i$$

Center of Mass. Properties

Conclusion

The total momentum of the system is equal to the momentum of a hypothetical particle of mass M moving with velocity $\overline{v}_{cm}$.

Differentiate once again

$$\frac{\mathrm{d}\overline{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(M\overline{v}_{cm}) = M\frac{\mathrm{d}\overline{v}_{cm}}{\mathrm{d}t}$$

But $\frac{\mathrm{d}\overline{p}}{\mathrm{d}t} = \sum_{i=1}^{N} \overline{F}_i^{ext} = \overline{F}^{ext}$. *If* $\overline{F}^{ext} = 0 \Rightarrow \overline{v}_{cm} = \text{const}$

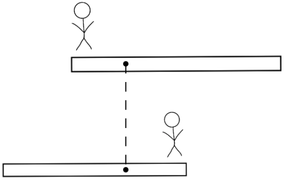
Conclusion

If the sum of all external forces acting on the system is equal to zero, the center of mass moves with a constant velocity.

[In particular, if it was at rest initially, it remains at rest.]

Center of Mass. Examples

a) boat



$$\overline{F}^{ext} = 0$$
$$\overline{v}_{cm} = 0$$

b) grenade

$$\overline{F}^{ext} \neq 0$$
$$\frac{\mathrm{d}\overline{v}_{cm}}{\mathrm{d}t} = \frac{\overline{F}^{ext}}{m}$$
$$\Downarrow$$
$$\overline{r}_{cm} = \overline{r}_{cm}(t) \rightarrow \text{parabola}$$

Motion of Objects with Varying Mass. Rocket Propulsion

Rocket Propulsion. Idea

propeller

↓

needs air to work

rocket

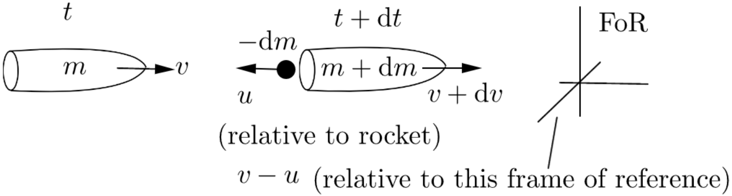
↓

works in vacuum

Idea: burn fuel $\Rightarrow$ eject burned fuel products $\Rightarrow$

$\Rightarrow$ rocket moves forward

Rocket Propulsion. Quantitative Analysis



Momentum conservation

[all momenta in the same (fixed, inertial) frame of reference!]

$$mv = \overbrace{(m+\mathrm{d}m)(v+\mathrm{d}v)}^{\text{rocket}}\ \overbrace{\underbrace{-dm}_{>0}\underbrace{(v-u)}_{<0}}^{\text{burned fuel}}$$

$$mv = mv + m\mathrm{d}v + v\mathrm{d}m + \underbrace{(\mathrm{d}m)(\mathrm{d}v)}_{\substack{\text{much smaller than} \\ \text{the other terms} \\ \text{(product of 2 infinitesimally} \\ \text{small quantities)}}} - v\mathrm{d}m + u\mathrm{d}m$$

Eventually,

$$m\,\mathrm{d}v = -u\,\mathrm{d}m \quad (\#)$$

and

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -u\frac{\mathrm{d}m}{\mathrm{d}t}$$

$$\boxed{a} = \frac{\mathrm{d}v}{\mathrm{d}t} = \boxed{-\frac{u}{m(t)}\frac{\mathrm{d}m}{\mathrm{d}t}}$$

Conclusion. An effective rocket

- burns fuel at a fast rate ($|\mathrm{d}m/\mathrm{d}t|$ is large),
- ejects burned fuel products with a great speed (u is large).

To find the velocity as a function of time, look at Eq. (#) again

$$\mathrm{d}v = -u\frac{\mathrm{d}m}{m}$$

$$\int_{v_0}^{v(t)} \mathrm{d}v = -u \int_{m_0}^{m(t)} \frac{\mathrm{d}m}{m}$$

$$\boxed{v(t)} = v_0 - u \ln \frac{m(t)}{m_0} = \boxed{v_0 + u \ln \frac{m_0}{m(t)}}$$

Conclusions

* The ratio of the initial mass of the rocket to the mass of the rocket after all fuel is burned should be as large as possible.
* The final velocity is greater in magnitude than u if $m_0/m(t_{final}) > e$ (for $v_0 = 0$).

Numerical example: If $m(t_{final}) = m_0/4, t_{final} = 30$ s, and $u = 2400$ m/s, $v_0 = 0$, then $v_{final} = 3327$ m/s [''final'' means after all fuel has been used up].