Chapter 3:
Static Electric Fields

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Many slides courtesy of Sung-Liang Chen

Everything you need to know about electrostatics in a few slides…

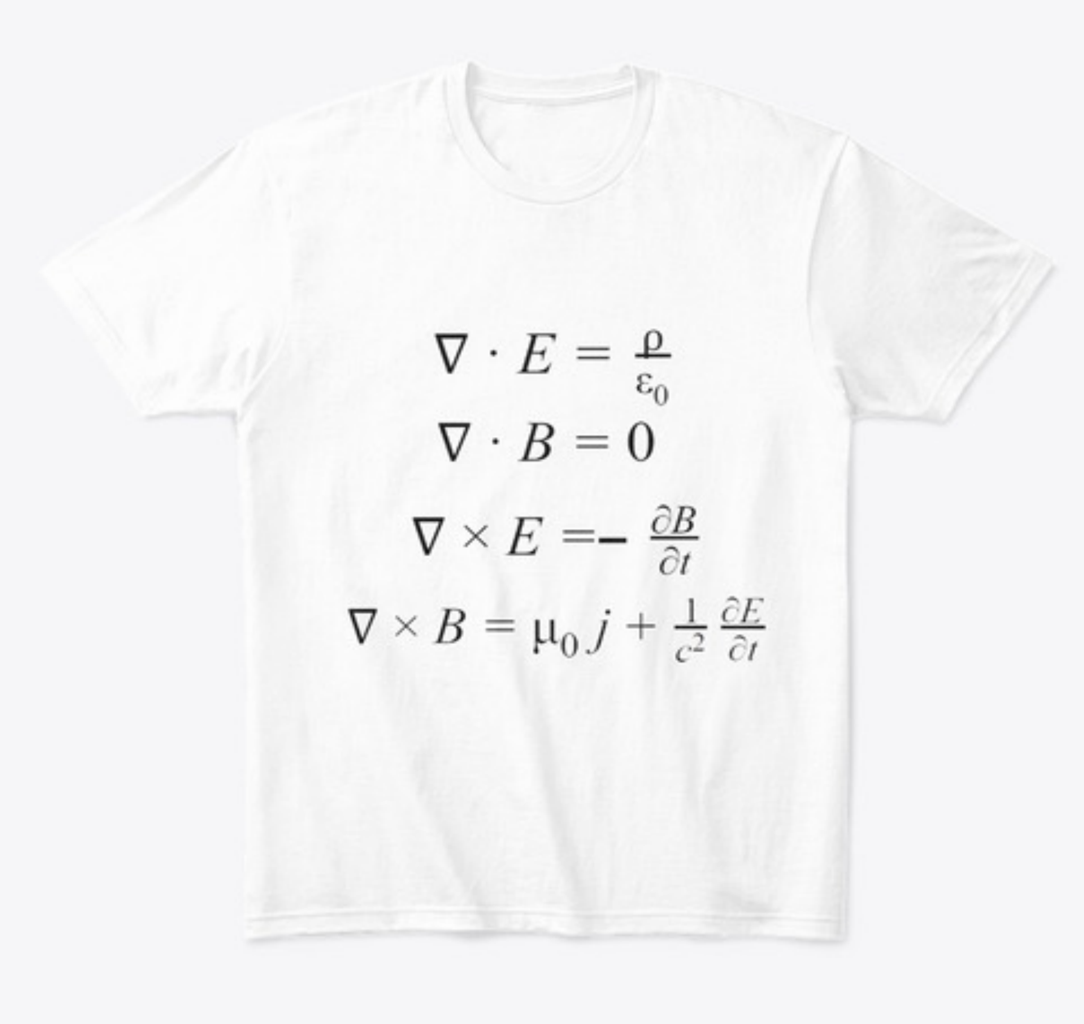

First a note on
using your imagination…

Main parts in electrostatics:

(I) Electrostatic equations
(II) Solving for E field in different scenarios
(III) Energy in E field
(IV) Conductors and insulators

Everything in electrostatics is here

Everything in electrostatics is here...



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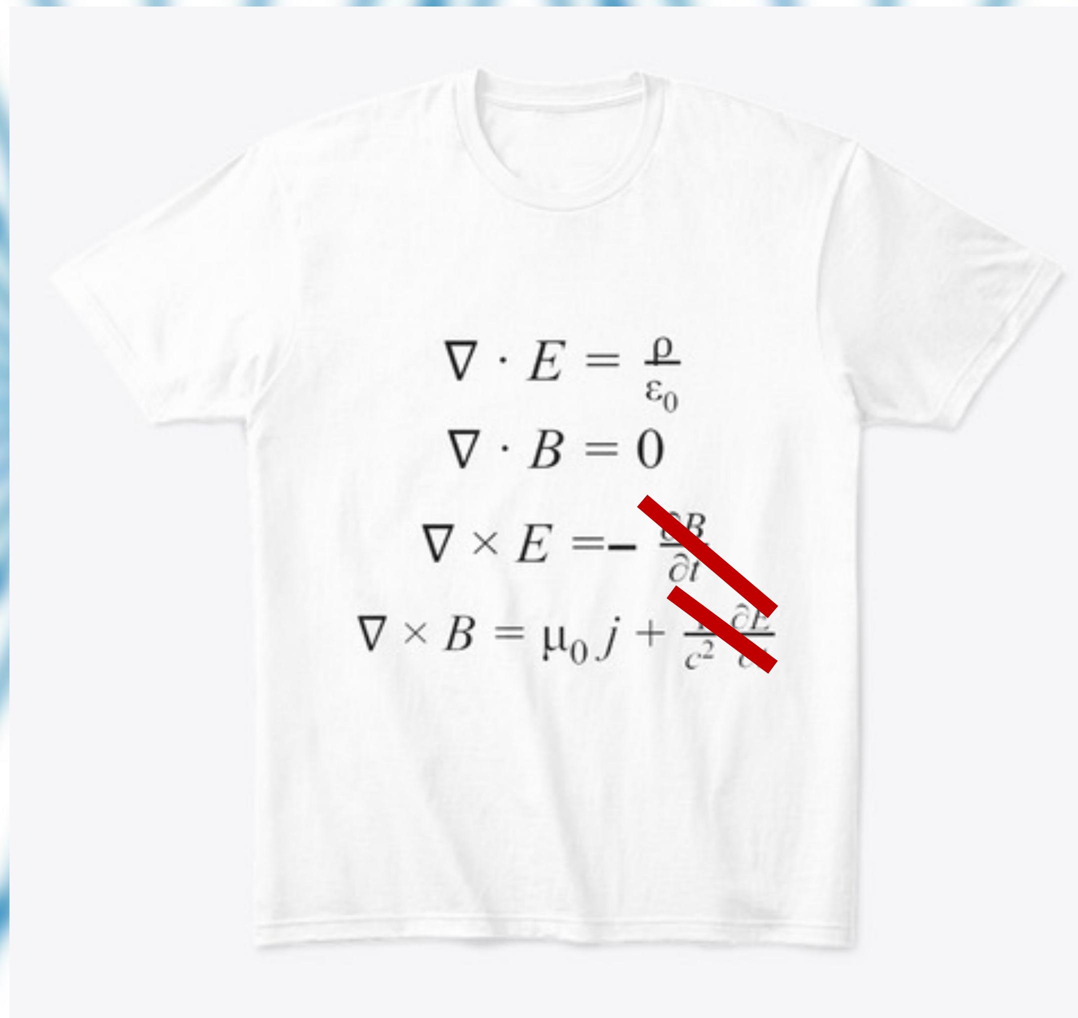
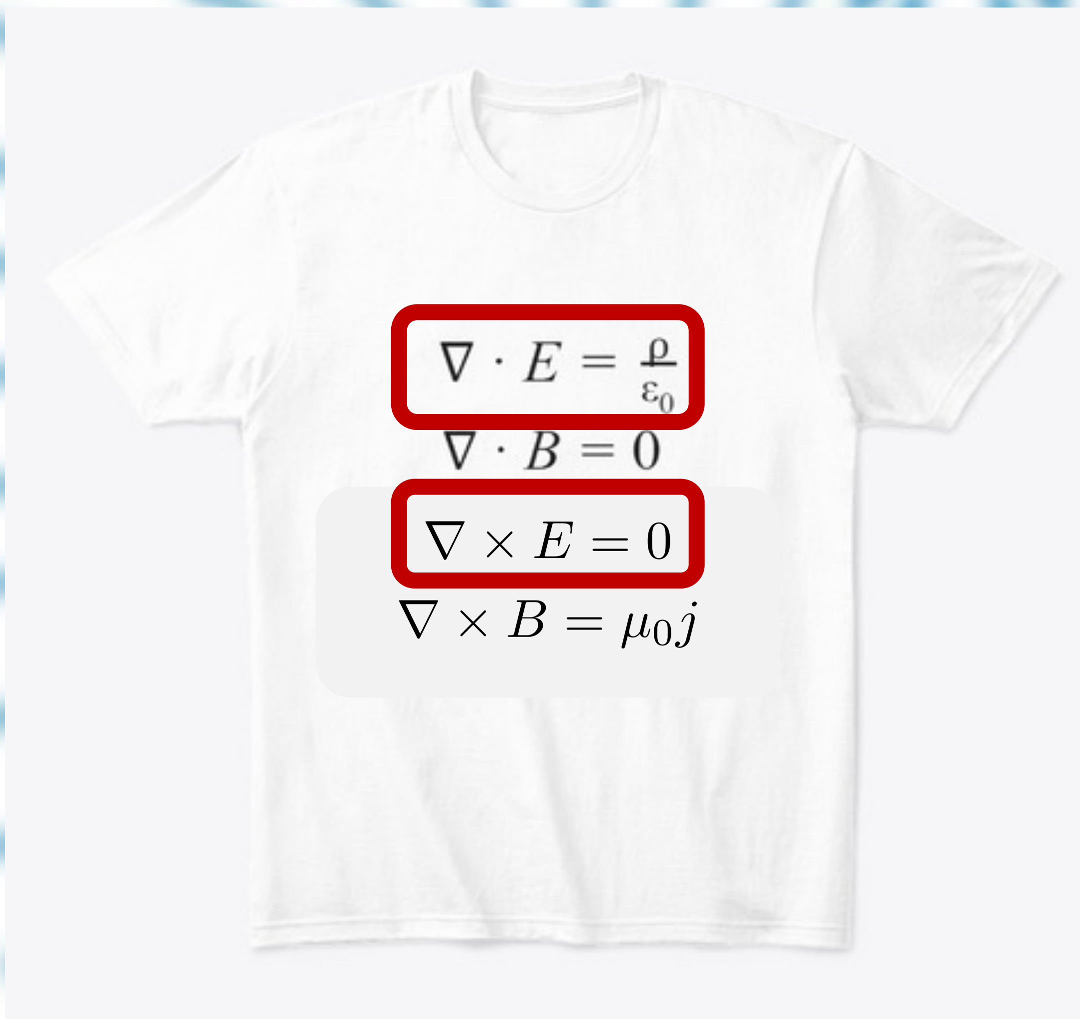
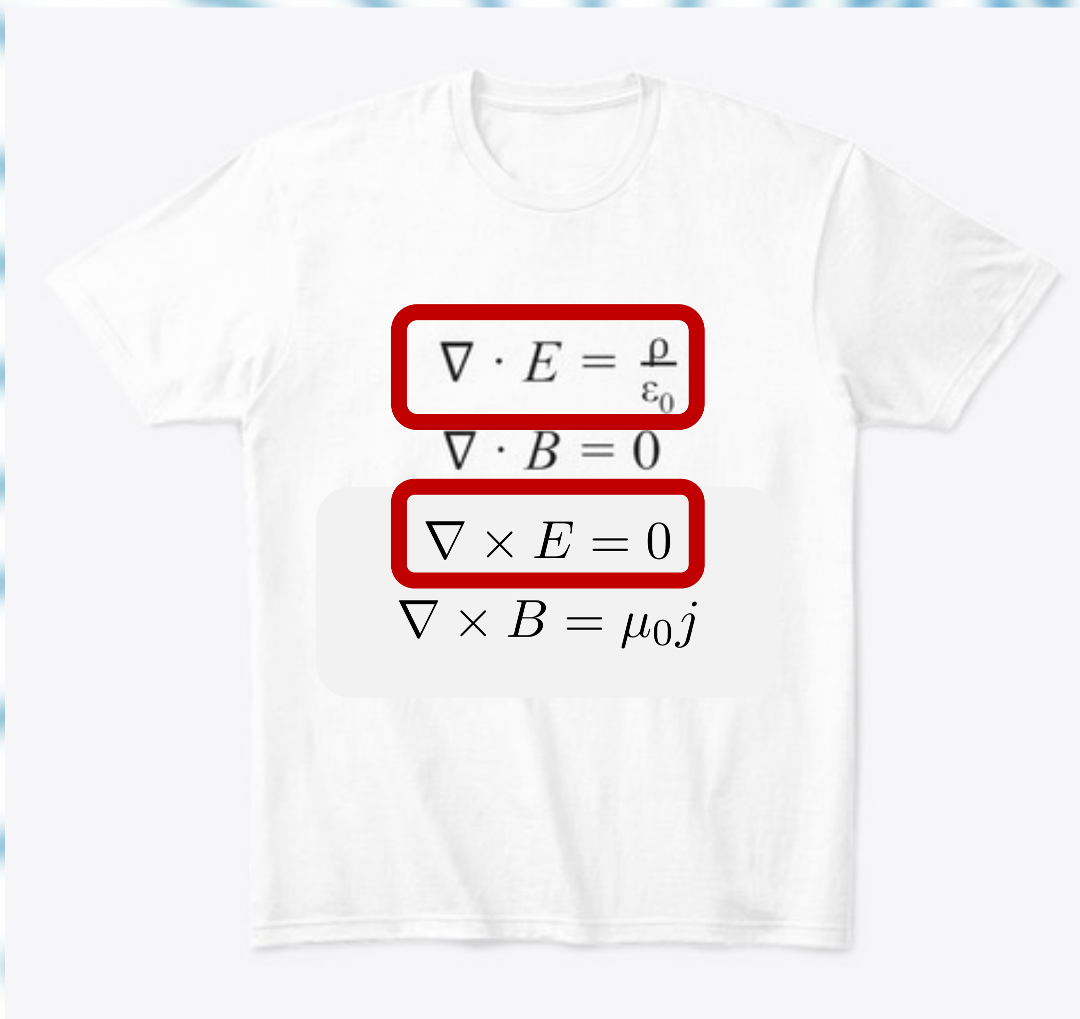


Everything in electrostatics is here…



Everything in electrostatics is here...



Electricity and magnetism become completely distinct

Electricity is curl-free
Magnetism is divergence-free

Any non-zero charge gives non-zero E field

Everything in electrostatics is here...

Everything in electrostatics is here...

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

$$\nabla \times E = 0$$

Already this tells us some important things…

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

Superposition principle

$$\nabla \cdot E_1 = \frac{\rho_1}{\epsilon_0}$$

$$\nabla \cdot E_2 = \frac{\rho_2}{\epsilon_0}$$

$$\implies \nabla \cdot E = \frac{\rho}{\epsilon_0}$$

where $E = E_1 + E_2$, $\rho = \rho_1 + \rho_2$

Very special property! Nature is simple!

Each point charge gives rise to its own E field

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

Use superposition principle to work out E field given ANY charge distribution

Already this tells us some important things…

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

Superposition principle

$$E = E_1 + E_2 + E_3 + \ldots + E_n$$
$$\rho = \rho_1 + \rho_2 + \rho_3 + \ldots + \rho_n$$

$$E = \int_V E_V dV$$
$$\rho = \int \rho_V dV$$

ANY charge distribution can be written as ρ abo

Each point charge gives rise to its own E field

Just solve for E given some ρ

$$\nabla \cdot E = \frac{\rho}{\epsilon_0}$$

Most of calculations in electrostatics is just this calculation if given some fancy ρ, nothing deeper than that

So what is knowing the E field good for?

$$\nabla \times E = 0$$

E has no curl

Means can write $E = -\nabla U$

$$\implies \nabla \times E = 0$$

What does this U mean?

So what is knowing the E field good for?

What is this U in $E = -\nabla U$?

Force $F = q(E + v \times B)$

So what is knowing the E field good for?

What is this U in $E = -\nabla U$?

Force $F = qE$

So what is knowing the E field good for?

What is this U in $E = -\nabla U$?

Force $F = qE$

$$W = -\int_a^b F \cdot dl = -q\int_a^b E \cdot dl$$

$$= q\int_a^b \nabla U dl = q(U(b) - U(a))$$

So what is knowing the E field good for?

$$W = -\int_a^b F \cdot dl = -q\int_a^b E \cdot dl$$

$$= q\int_a^b \nabla U dl = q(U(b) - U(a))$$

$$= 0 \quad \text{for closed path } a = b$$

So what is knowing the E field good for?

$$W = -\int_a^b F \cdot dl = -q\int_a^b E \cdot dl$$

$$= q\int_a^b \nabla U\, dl = q(U(b) - U(a))$$

$$= 0 \quad \text{for closed path } a = b$$

Consistent with Stokes' Theorem

So what is knowing the E field good for?

What is this U in $E = -\nabla U$?

For example, can work out
how much energy
it costs to create a certain
charge distribution or
how much energy is released
if charges repel

Main parts in electrostatics:

(I) Electrostatic equations
(II) Solving for E field in different scenarios
(III) Energy in E field
(IV) Conductors and insulators

What E fields like around conductors and insulators?

Conductors have freely moving charges on their surfaces, but inside have zero E

Atoms inside insulators can have E field induced in them by external fields

Now you know everything…

(I) Electrostatic equations

(II) Solving for E field in different scenarios

(III) Energy in E field

(IV) Conductors and insulators

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

Showing that work done in bringing a charge to its original position costs zero work.

Hint: Use Stokes' Theorem

Bonus question: Interpret 1st Maxwell's equation using divergence theorem

3-1 Introduction

- A field is a specified distribution of a scalar or vector quantity, which may or may not be a function of time.
- In this chapter, we deal with electrostatics, which means electric charges are at rest (not moving), and electric field do not change with time.
- We won't study the magnetic field in this chapter.

Development of Electrostatics

- Experimental Coulomb's law (1785)

$$\mathbf{F}_{12} = \mathbf{a}_{R_{12}} k \frac{q_1 q_2}{R_{12}^2},$$

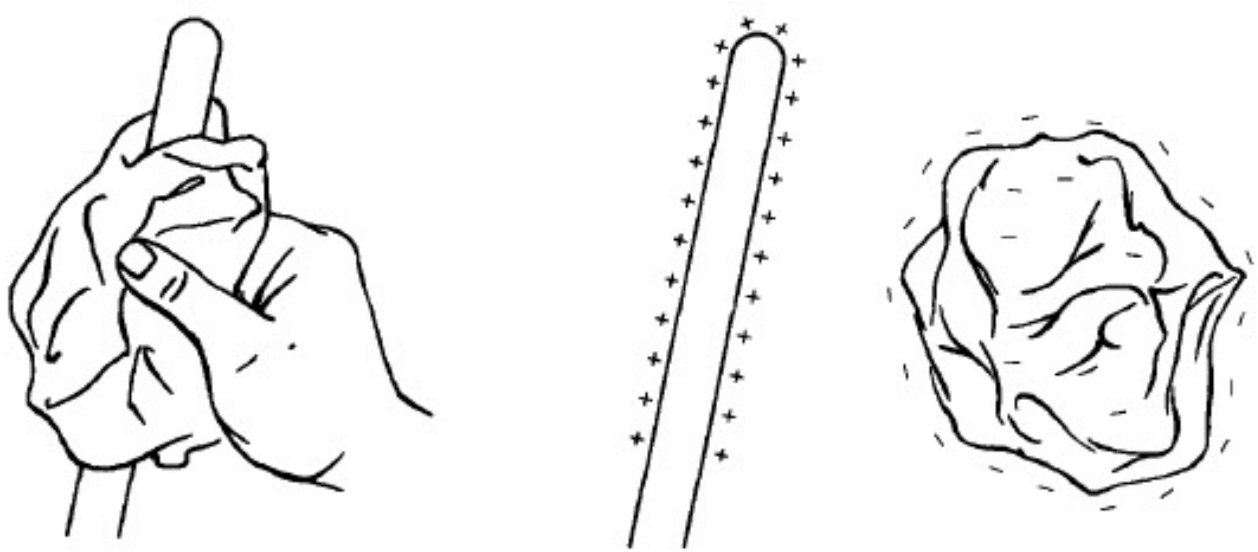
$\Rightarrow$ Gauss's law, etc.

Experimental Basis of Electrostatics

Everything we know about electromagnetics comes from experiment. Early researchers in the 1700s discovered there are two forms of charges by rubbing different materials together.

How can you deduce from this experiment that there are 2 types (not one type) of charge?

How can you deduce from this experiment that opposite charges attract and like charges repel?

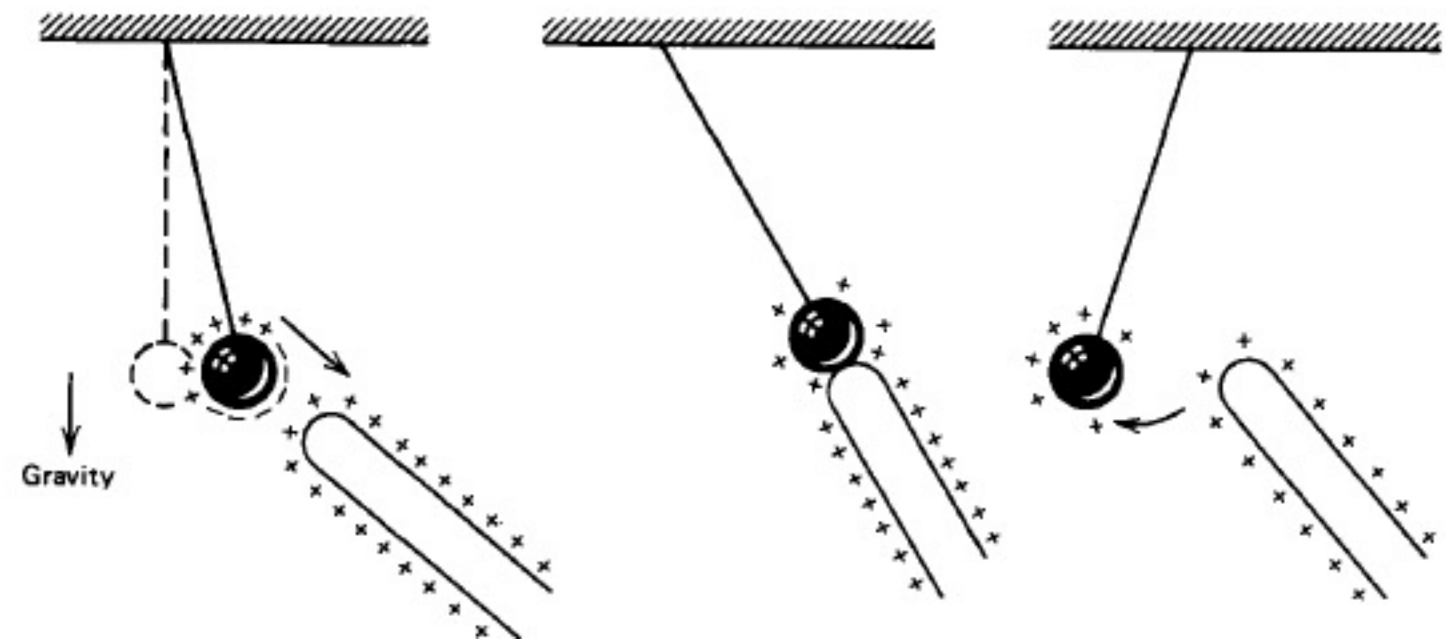


Charge Transfer Processes

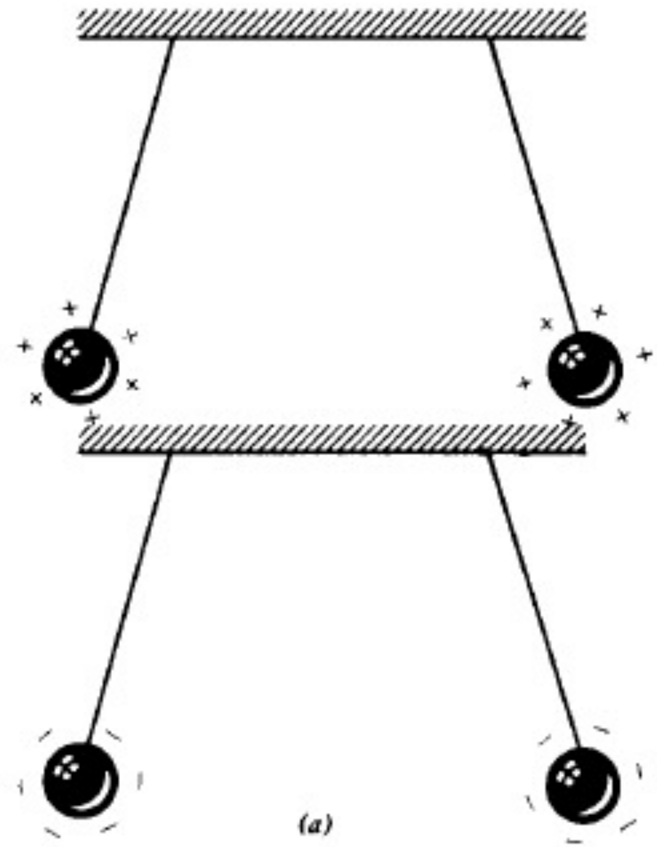
How can we know when an object is charged vs. uncharged?

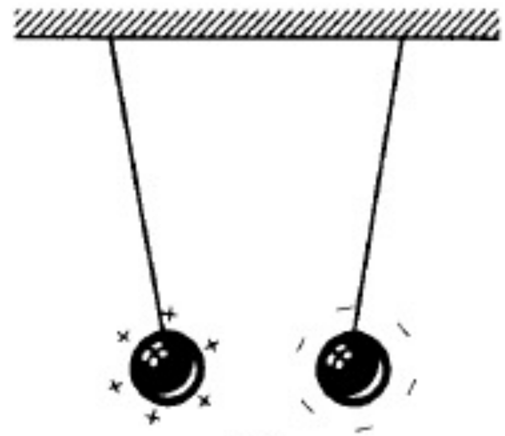
How do we know when charge is transferred from one object to another?

Does an uncharged object produce no electric field?

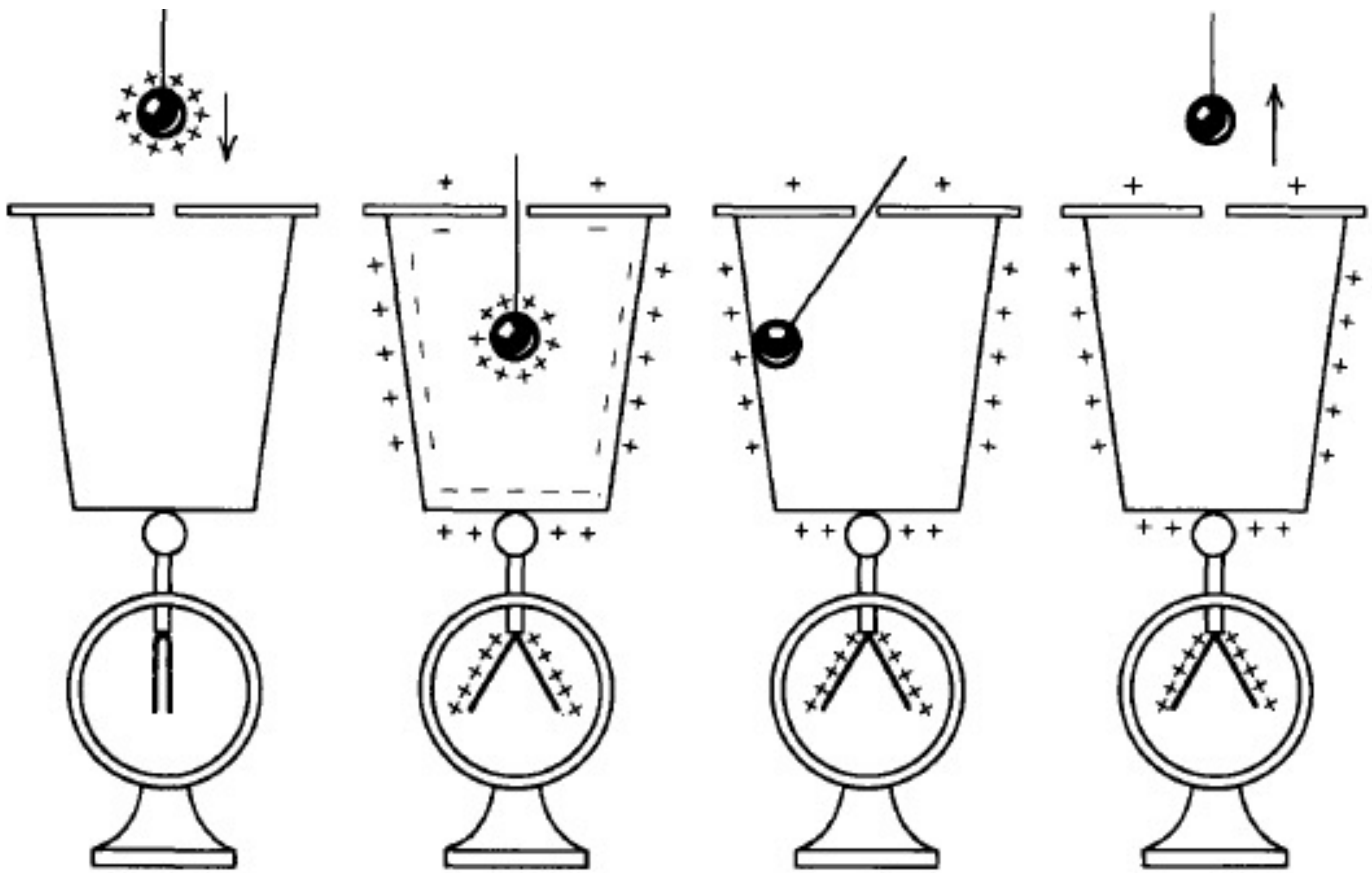


Empirical Deduction of the Laws of Electrostatic Attraction and Repulsion


(a)


(b)

Faraday Ice Pail Experiment



(a) (b) (c) (d)

3-2 Fundamental Postulates of Electrostatics in Free Space

- The simplest case:
  - Static electric charges (source) in free space ➔ electric fields
- Electric field intensity

$$\boxed{\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \qquad (\mathrm{V/m}).}$$

- If q is small enough not to disturb the charge distribution of the source, $\boxed{\mathbf{F} = q\mathbf{E} \qquad (\mathrm{N}).}$

Fundamental Postulates of Electrostatics

- In free space,

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}}$$

Not solenoidal

$$\boxed{\nabla \times \mathbf{E} = 0.}$$

Irrotational

charge if a flow source of **E** field

Integral Form

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \Rightarrow \int_V \nabla \cdot \mathbf{E}\, dv = \frac{1}{\epsilon_0} \int_V \rho\, dv. \Rightarrow \boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0},}$$

A form of Gauss's law: the total outward flux of the electric field intensity over any closed surface in free space is equal to the total charge enclosed in the surface divided by ε_0

Integral Form

$$\boxed{\nabla \times \mathbf{E} = 0.} \Rightarrow \quad \Rightarrow \boxed{\oint_C \mathbf{E} \cdot d\ell = 0.}$$

The scalar line integral of the static electric field intensity around any closed path vanishes.

KVL in circuit theory: the algebraic sum of voltage drops around any closed circuit is zero.

E is Irrotational (Conservative)

$$\boxed{\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0.}$$

$$\int_{C_1} \mathbf{E} \cdot d\boldsymbol{\ell} + \int_{C_2} \mathbf{E} \cdot d\boldsymbol{\ell} = 0$$

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell} = - \underset{\text{Along } C_2}{\int_{P_2}^{P_1}} \mathbf{E} \cdot d\boldsymbol{\ell}$$

$$\underset{\text{Along } C_1}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell} = \underset{\text{Along } C_2}{\int_{P_1}^{P_2}} \mathbf{E} \cdot d\boldsymbol{\ell}.$$

The scalar line integral of the irrotational **E** field is independent of the path; it depends only on the end points.

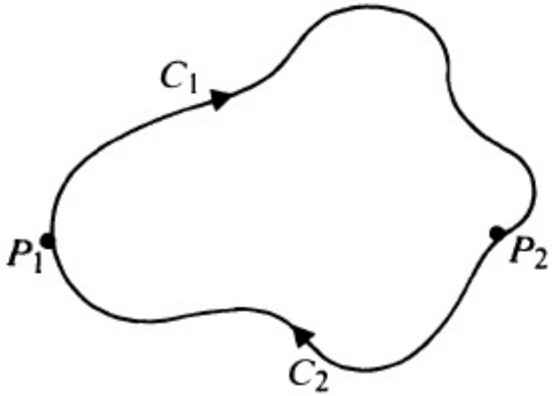


FIGURE 3–1
An arbitrary contour.

Postulates of Electrostatics in Free Space	
Differential Form	Integral Form
$\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$	$\oint_S \mathbf{E} \cdot d\mathbf{s} = \dfrac{Q}{\epsilon_0}$
$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\boldsymbol{\ell} = 0$

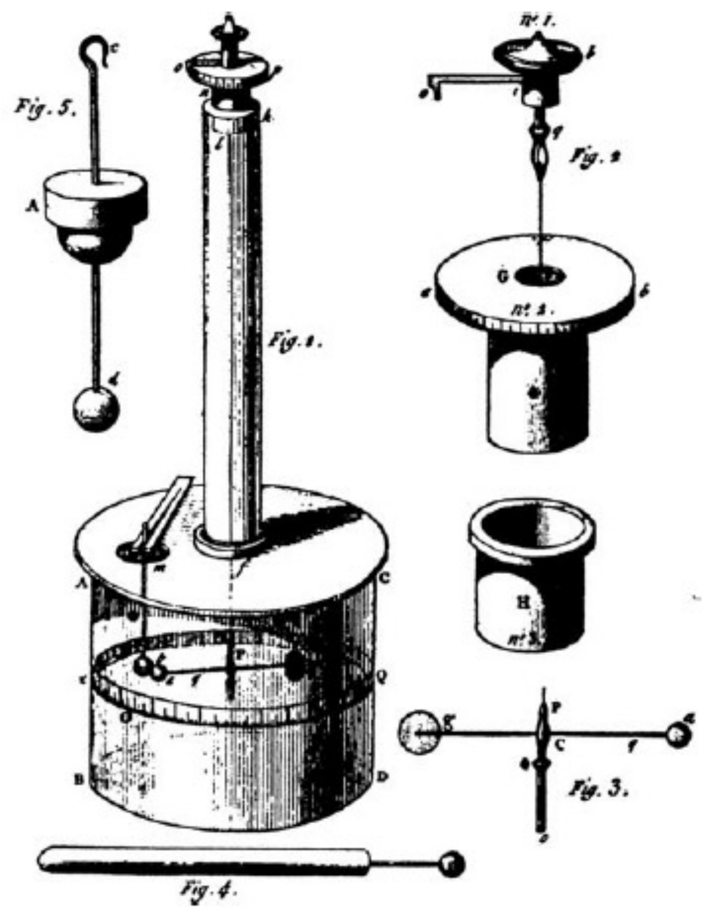
3-3 Coulomb's Law

The first person to quantify this law was Coulomb, whose experiment using torsion meters in 1785 demonstrated two things:

1) The force is proportional to the charges on each object

2) The force is inversely proportional to the square of distance

3) The force is a "central body" force (parallel to the position vector)



Coulomb's Law

- The simplest case: a single point charge

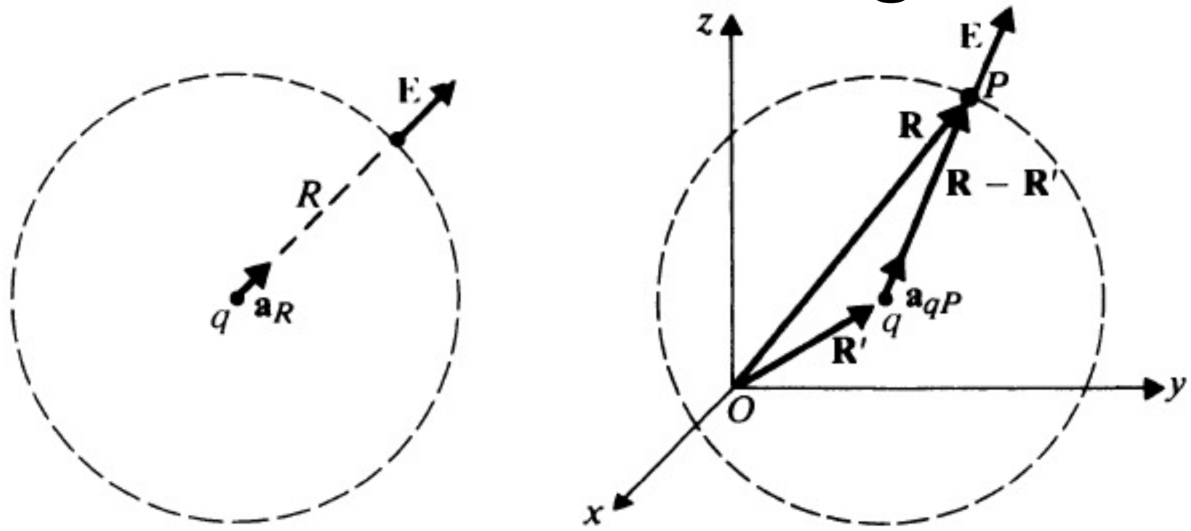
$$\oint_S \underline{\mathbf{E}} \cdot d\mathbf{s} = \oint_S \underline{(\mathbf{a}_R E_R)} \cdot \mathbf{a}_R \, ds = \frac{q}{\epsilon_0}$$

$$E_R \oint_S ds = E_R(4\pi R^2) = \frac{q}{\epsilon_0}.$$

$$\boxed{\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \qquad \text{(V/m).}}$$

The electric field intensity of a positive point charge is in the outward direction and has a magnitude proportional to the charge and inversely proportional to the square of the distance from the charge.

E of a Point Charge



(a) Point charge at the origin.   (b) Point charge not at the origin.

FIGURE 3–2
Electric field iFIGURE due to a point charge.

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2}$$

$$\mathbf{E}_P = \mathbf{a}_{qP} \frac{q}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^2}$$

R → R–R’

R: field
R’: source

$$\mathbf{a}_{qP} = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|}$$

$$\boxed{\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \qquad \text{(V/m).}}$$

Mathematical Form of Coulomb's Law

- Force on q_2 in an **E** field due to q_1

$$\boxed{\mathbf{F}_{12} = q_2\mathbf{E}_{12} = \mathbf{a}_R \frac{q_1 q_2}{4\pi\epsilon_0 R^2} \quad \text{(N).}}$$

3-3.1 Electric Field due to a System of Discrete Charges

$$\boxed{\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \qquad \text{(V/m).}}$$

$$\Downarrow$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k(\mathbf{R} - \mathbf{R}'_k)}{|\mathbf{R} - \mathbf{R}'_k|^3} \qquad \text{(V/m).}}$$

An Electric Dipole

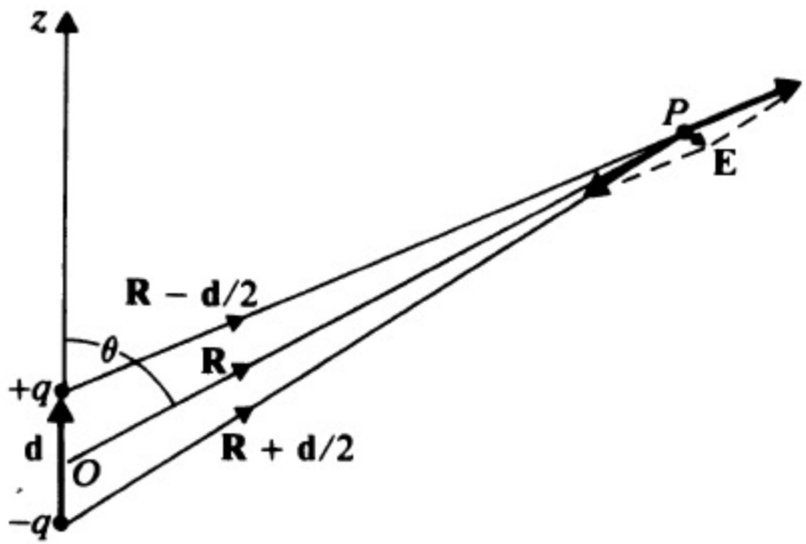


FIGURE 3–5
Electric field of a dipole.

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\dfrac{\mathbf{d}}{2}}{\left|\mathbf{R}-\dfrac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\dfrac{\mathbf{d}}{2}}{\left|\mathbf{R}+\dfrac{\mathbf{d}}{2}\right|^3}\right\}.$$

If d<<R

$$\begin{aligned}\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} &= \left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right]^{-3/2} \\ &= \left[R^2 - \mathbf{R}\cdot\mathbf{d} + \frac{d^2}{4}\right]^{-3/2} \\ &\cong R^{-3}\left[1 - \frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]^{-3/2} \\ &\cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right],\end{aligned}$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R} - \frac{\mathbf{d}}{2}}{\left|\mathbf{R} - \frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R} + \frac{\mathbf{d}}{2}}{\left|\mathbf{R} + \frac{\mathbf{d}}{2}\right|^3}\right\}.$$

$$\left|\mathbf{R} - \frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]$$

$$\left|\mathbf{R} + \frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1 - \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right].$$

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right].$$

Manipulation of vector quantities is tedious.

Any method to avoid?

Electric Dipole Moment

- Definition: The product of the charge q and the vector **d**

$$\mathbf{p} = q\mathbf{d}.$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right]$$

$$\mathbf{p} = \mathbf{a}_z p = p(\mathbf{a}_R \cos\theta - \mathbf{a}_\theta \sin\theta),$$
$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta,$$

$$\boxed{\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \quad (\text{V/m}).}$$

Why it decreases more rapidly?

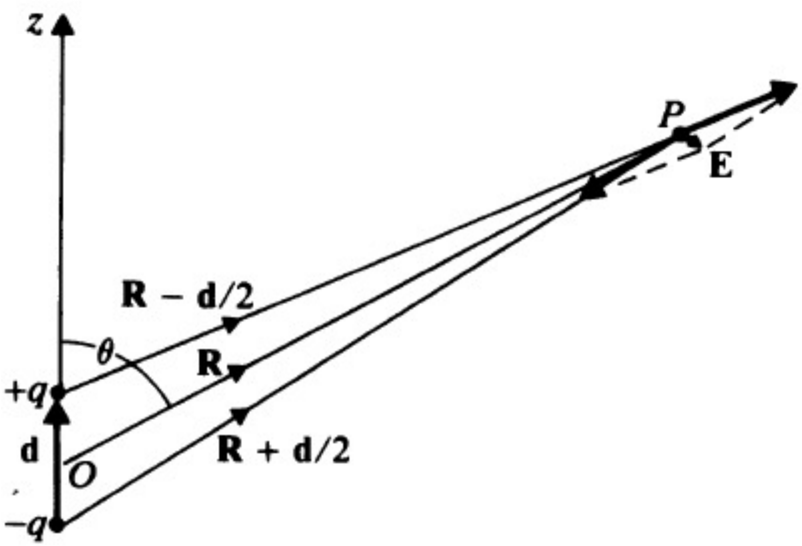


FIGURE 3–5
Electric field of a dipole.

3-3.2 Electric Field due to a Continuous Distribution of Charge

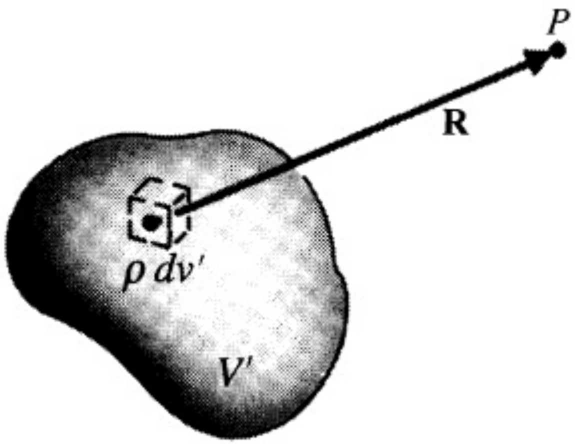


FIGURE 3–6
Electric field due to a continuous charge distribution.

$$d\mathbf{E} = \mathbf{a}_R \frac{\rho\, dv'}{4\pi\epsilon_0 R^2}.$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv' \qquad \text{(V/m)},}$$

$\mathbf{a}_R = \mathbf{R}/R$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \rho \frac{\mathbf{R}}{R^3}\, dv' \qquad \text{(V/m)}.}$$

For a Surface or Line Charge

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \qquad \text{(V/m).}}$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell' \qquad \text{(V/m),}}$$

However, from vector calculus and mechanics, we know that there is a way to describe this vector in terms of the gradient of a scalar potential:

$$\overrightarrow{F_{12}}(\vec{r}) = \frac{q_1 q_2 \widehat{r_{12}}}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|^2} = -\nabla \frac{q_1 q_2}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|} = -\nabla U$$

$$\therefore U = \frac{q_1 q_2}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|}$$

Irrotational = conservative

Thus, in a similar way to mechanics, we can define a potential energy of interaction between two charges that behaves in a corollary manner to the gravitational interaction energy between two masses.

Field Model vs. Action at a Distance

Experimental laws are based on the “action at a distance” principle, which implies that one mass acts on another **without touching it directly**.

However, another interpretation of this law is that the charge is interacting with the electric field of another charge, thus it is in fact **touching the charge through the field**.

Thus, there are two views of electrostatics:

1) All electromagnetics derived between the interaction of charges and their geometric separation. **The charge itself is the physical reality** and stores the energy of the system.

2) All electromagnetics is derived as the interaction of fields with other fields. **The field itself is the physical reality** and stores the energy of the system.

Field Model

In order to understand the field model, let us break apart the system into two parts. One part is the test charge, which we may make arbitrarily small. The other part produces the field which applied force on the test charge. Therefore, the electric field can be interpreted as the “force per unit test charge”

$$\overrightarrow{F_{12}}(\vec{r}) = \underbrace{q_2}_{\text{Test charge}} \underbrace{\frac{q_1 \widehat{r_{12}}}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|^2}}_{\text{Field}} = q_2 E_{12} \qquad \overrightarrow{E_{12}} = k q_1 \frac{\widehat{r_{12}}}{|\overrightarrow{r_{12}}|^2}$$

Likewise we may define an electric potential as the interaction energy per unit test charge.

$$U_{12}(\vec{r}) = \underbrace{q_2}_{\text{Test charge}} \underbrace{\frac{q_1}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|}}_{\text{Potential}} = q_2 V_{12} \qquad V_{12}(\vec{r}) = \frac{q_1}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|}$$

Field and Potential

$$\overrightarrow{F_{12}}(\vec{r}) = \frac{q_1 q_2 \widehat{r_{12}}}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|^2} = -\nabla \frac{q_1 q_2}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|} = -\nabla U_{12}$$

$$\therefore\ U_{12}(\vec{r}) = \frac{q_1 q_2}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|}$$

$$\overrightarrow{E_{12}}(\vec{r}) = \frac{q_1 \widehat{r_{12}}}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|^2} = -\nabla \frac{q_1}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|} = -\nabla V_{12}$$

$$\therefore\ V_{12}(\vec{r}) = \frac{q_1}{4\pi\varepsilon_0 |\overrightarrow{r_{12}}|}$$

Principle of Superposition

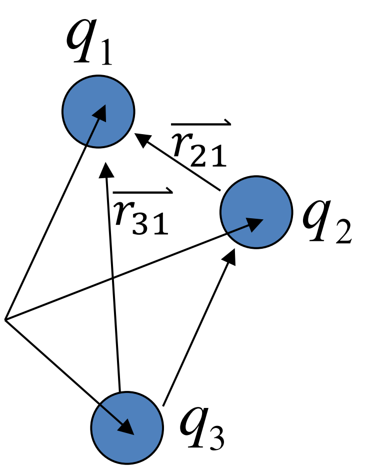
In **classical electrostatics**, it is commonly accepted that charges can be neither created nor destroyed (just like masses cannot be created nor destroyed), and that the magnitude of the fundamental unit of charge is constant.

In other words, an electron has the same charge regardless of whether it is in the presence of the field of another charge, i.e., the charge magnitude is independent of its surroundings.

This allows us to use the "*principle of superposition*":

$$\overrightarrow{E_1}(\vec{r}) = \frac{q_2 \widehat{r_{21}}}{4\pi\varepsilon_0 |\overrightarrow{r_{21}}|^2} + \frac{q_3 \widehat{r_{31}}}{4\pi\varepsilon_0 |\overrightarrow{r_{31}}|^2}$$

$$\overrightarrow{E_1}(\vec{r}) = \sum_{i=1}^{N} \frac{k q_i \widehat{r_{i1}}}{4\pi\varepsilon_0 |\overrightarrow{r_{i1}}|^2}$$ for N charges



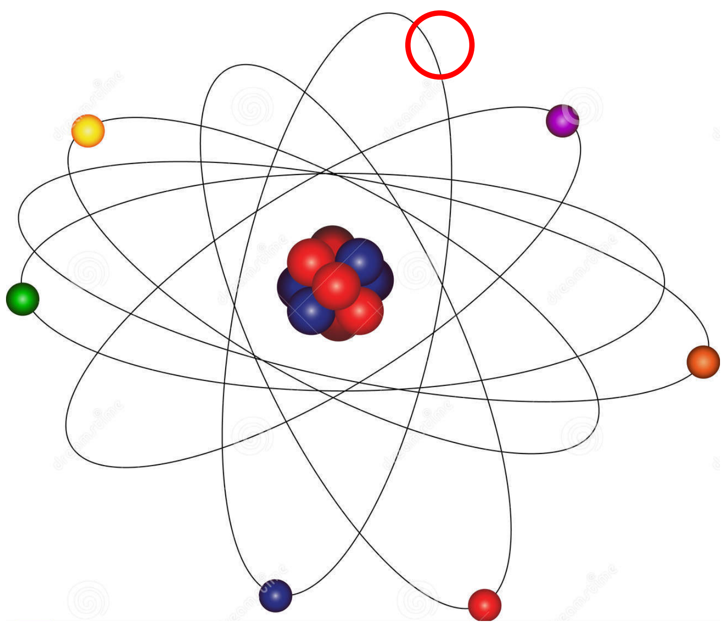
Charge Distribution

Although charges are conventionally considered to be located at a point, this would imply that the charge density is infinite, which would require infinite energy to create (i.e., a singularity).

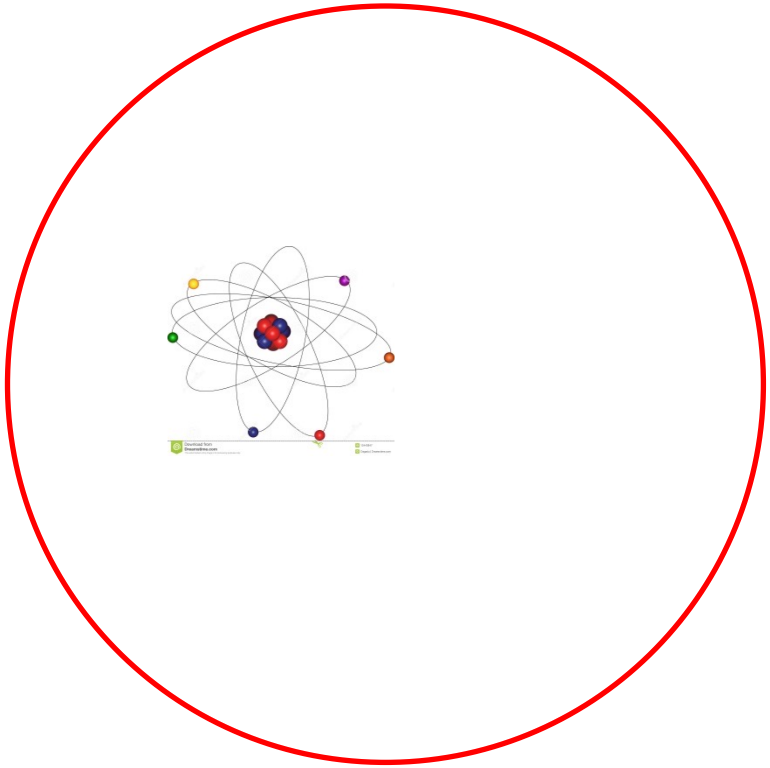
Perhaps more importantly, we can never manage to measure the electric field inside the electron. In the atom, the electron is moving around **so rapidly**, that it appears to us that the space has **a time-averaged volume charge density**.

In other words, the average charge density of the electron cloud is considered as if the individual electrons are smeared out through the volume of the electron cloud. Provided their motion is sufficiently fast, and space is isotropic (no preference for a particular direction), **the charge density within that volume is considered to be constant in time**.

Microscopic

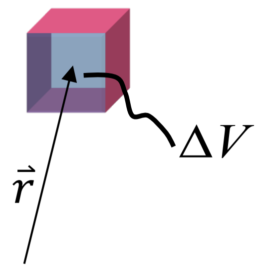


Macroscopic



The charge density within that volume is considered to be constant in time

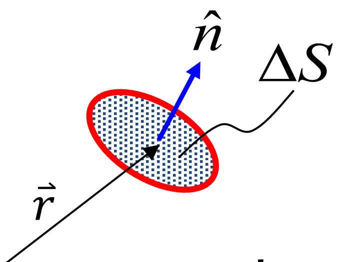
Charge Distributions



$$\rho(\vec{r}) = \lim_{\Delta V \to 0} \frac{Q_{\Delta V}}{\Delta V},$$

ΔV volume element around $\vec{r}$

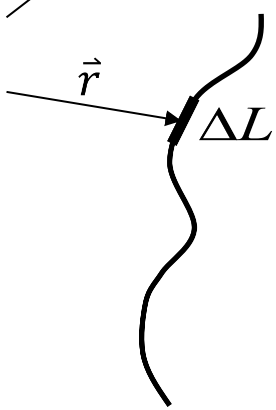
$Q_{\Delta V}$ total charge enclosed in ΔV



$$\sigma(\vec{r}) = \lim_{\Delta S \to 0} \frac{Q_{\Delta S}}{\Delta S},$$

ΔS surface element around $\vec{r}$

$Q_{\Delta S}$ total charge enclosed in ΔS



$$\lambda(\vec{r}) = \lim_{\Delta L \to 0} \frac{Q_{\Delta L}}{\Delta L},$$

ΔL line element around $\vec{r}$

$Q_{\Delta L}$ total charge enclosed in ΔL

where $\rho(\vec{r}) = q\delta(\vec{r} - \vec{r_s})$

Distribution corresponds to point charge q located at $\vec{r_s}$

Bohr Model

The electron cloud is typically considered to follow a spherically symmetric distribution around the nucleus:

$$\rho(\vec{r}) = \frac{-Q_e}{\pi a^3} e^{-2r/a}$$

where a is known as the Bohr radius.

What is the total charge in the electron cloud?

$$Q = \iiint_V \rho(\vec{r}) dV = \iiint_V \frac{-Q_e}{\pi a^3} e^{-2r/a} dV = \iiint_V \frac{-Q_e}{\pi a^3} e^{-2r/a} r^2 \sin\theta dr d\theta d\varphi$$

$$Q = \frac{-4Q_e}{a^3} \int_0^\infty e^{-2r/a} r^2 dr = -Q_e \left\{ 1 - e^{-2R/a} \left(1 + 2\left(\frac{R}{a}\right) + 2\left(\frac{R}{a}\right)^2 \right) \right\}_0^\infty = -Q_e$$

Methods for Calculating Electric Fields

If you have a distribution of charges (discrete or continuous), there are two ways to calculate the electric field:

Field:	Potential:	
$\vec{E}(\vec{r}) = \sum_{i=1} \frac{q_i(\vec{r} - \vec{r_i})}{4\pi\varepsilon_0 \lvert\vec{r} - \vec{r_i}\rvert^3}$	$V(\vec{r}) = \sum_{i=1} \frac{q_i}{4\pi\varepsilon_0 \lvert\vec{r} - \vec{r_i}\rvert}$	Point Charges
$\vec{E}(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})(\vec{r} - \vec{r_s})}{4\pi\varepsilon_0 \lvert\vec{r} - \vec{r_s}\rvert^3} dl_s$	$V(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})}{4\pi\varepsilon_0 \lvert\vec{r} - \vec{r_s}\rvert} dl_s$	Line Charges
$\vec{E}(\vec{r}) = \iint_S \frac{\sigma(\vec{r_s})(\vec{r} - \vec{r_s})}{4\pi\varepsilon_0 \lvert\vec{r} - \vec{r_s}\rvert^3} dS_s$	$V(\vec{r}) = \iint_S \frac{\sigma(\vec{r_s})}{4\pi\varepsilon_0 \lvert\vec{r} - \vec{r_s}\rvert} dS_s$	Surface Charges
$\vec{E}(\vec{r}) = \iiint_V \frac{\rho(\vec{r_s})(\vec{r} - \vec{r_s})}{4\pi\varepsilon_0 \lvert\vec{r} - \vec{r_s}\rvert^3} dV_s$	$V(\vec{r}) = \iiint_V \frac{\rho(\vec{r_s})}{4\pi\varepsilon_0 \lvert\vec{r} - \vec{r_s}\rvert} dV_s$	Volume charges

$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

Summing scalars is easier than summing vector!!!

Maxwell's 1st Equation

Let us know as the question of what is the divergence of the electric field of a point charge:

$$\nabla \cdot \vec{E} = \nabla \cdot \frac{q}{4\pi\varepsilon_0} \frac{\hat{r}}{|\vec{r}|^2} = \frac{q}{4\pi\varepsilon_0} \nabla \cdot \frac{\hat{r}}{|\vec{r}|^2} = -\frac{q}{4\pi\varepsilon_0} \nabla \cdot \nabla \frac{1}{r} = -\frac{q}{4\pi\varepsilon_0} \underbrace{\nabla^2 \frac{1}{r}}_{= -4\pi\delta(\vec{r})}$$

$$\therefore \nabla \cdot \vec{E} = \frac{q}{\varepsilon_0} \delta(\vec{r})$$

What about for a distribution of charges?

$\mathbf{r_s}$: source positions
$\mathbf{r}$: general

$$\nabla \cdot \vec{E} = \nabla \cdot \iiint_V \frac{\rho(\vec{r_s})(\hat{r} - \widehat{r_s})}{4\pi\varepsilon_0 |\vec{r} - \vec{r_s}|^2} dV_s = \iiint_V \nabla \cdot \frac{\rho(\vec{r_s})(\hat{r} - \widehat{r_s})}{4\pi\varepsilon_0 |\vec{r} - \vec{r_s}|^2} dV_s = \frac{-1}{4\pi\varepsilon_0} \iiint_V \nabla^2 \cdot \frac{\rho(\vec{r_s})}{|\vec{r} - \vec{r_s}|} dV_s$$

$$\therefore \nabla \cdot \vec{E} = \frac{-1}{4\pi\varepsilon_0} \iiint_V \rho(\vec{r_s}) \nabla^2 \cdot \frac{1}{|\vec{r} - \vec{r_s}|} dV_s = \frac{1}{4\pi\varepsilon_0} \iiint_V \rho(\vec{r_s}) 4\pi\delta(\vec{r} - \vec{r_s})\, dV_s = \frac{\rho(\vec{r})}{\varepsilon_0}$$

The result represents the first of Maxwell's equations:

$$\boxed{\nabla \cdot \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0}}$$

What happens if we now integrate the divergence of the electric field over an enclosed space?

$$\iiint_V \nabla \cdot \vec{E}\, dV = \oint\!\!\!\oint_S \vec{E} \cdot \hat{n}\, dS = \iiint_V \frac{\rho(\vec{r})}{\varepsilon_0}\, dV = \frac{Q}{\varepsilon_0}$$

This equation can also be expressed in differential form as:

$$\boxed{\oint\!\!\!\oint_S \vec{E} \cdot \hat{n}\, dS = \frac{Q}{\varepsilon_0}}$$

This equation is valid for any surface enclosing a charge Q.

Maxwell's 2nd Equation

Knowing that the electric field vector, E, is derived from the gradient of a scalar potential, φ, we may guess that E has no curl. Let us check.

$$\boxed{\nabla \times \vec{E} = -\nabla \times \nabla \phi = 0}$$

As we surmised, the electric field is irrotational. We can also derive an integral form for this statement as follows:

$$\iint_{S_L} \nabla \times \vec{E}(\vec{r}) \cdot \hat{n}\, dS = \oint_L \vec{E}(\vec{r}) \cdot \hat{t}\, dl = -\oint_L \nabla \phi(\vec{r}) \cdot \hat{t}\, dl = 0$$

$$\boxed{\oint_L \vec{E}(\vec{r}) \cdot \hat{t}\, dl = 0, \text{ for any possible } L}$$

This equation is valid for any closed line integral

In summary, the postulates we have just derived represent the first two of Maxwell's equations. These can be written in integral form or differential form (i.e. …

$$\nabla \cdot \vec{E} = \frac{\rho(\vec{r})}{\varepsilon_0} \longleftrightarrow \oiint_S \vec{E} \cdot \hat{n} dS = \frac{Q}{\varepsilon_0}$$

$$\nabla \times \vec{E} = 0 \longleftrightarrow \oint_L \vec{E} \cdot \hat{t}\, dl = 0$$

We used:

- isotropicity of space
- postulates of electrostatics to derive Coulomb's Law

$$\vec{E}(\vec{r}) = \frac{q(\vec{r} - \vec{r_s})}{4\pi\varepsilon_0 |\vec{r} - \vec{r_s}|^3} \quad , \textit{where } \vec{r_s} \textit{ is the point charge location}$$

Textbook (pp. 16-24):

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \quad \boxed{\nabla \times \mathbf{E} = 0.} \quad \xrightarrow{\text{derive}} \quad \mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2}$$

$$\mathbf{E} = -\nabla V$$

We also show (pp. 25-38):

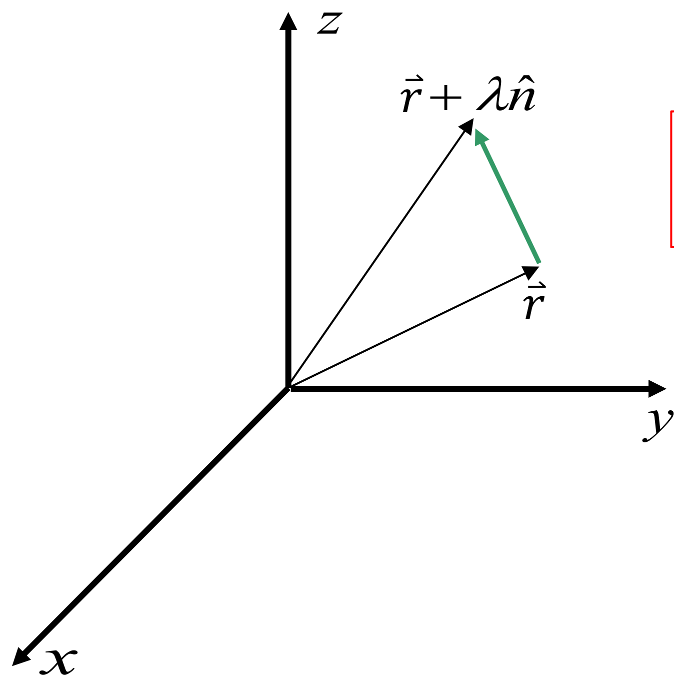
$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad \xrightarrow{\text{derive}} \quad \boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \quad \boxed{\nabla \times \mathbf{E} = 0.}$$

$$\mathbf{E} = -\nabla V$$

Vector Calculus

Directional derivative $$\frac{Du(\vec{r})}{Dn} = \lim_{\lambda\to 0}\frac{u(\vec{r}+\lambda\hat{n})-u(\vec{r})}{\lambda}$$

Gradient $$\nabla u(\vec{r})\cdot\hat{n} = \frac{Du(\vec{r})}{Dn} = \lim_{\lambda\to 0}\frac{u(\vec{r}+\lambda\hat{n})-u(\vec{r})}{\lambda}$$



Derivative along a direction **n** with a distance λ

Very Useful Formulas and Examples!

$$\left|\vec{r}+\lambda\hat{n}\right| = \sqrt{r^2 + 2\underline{\lambda\hat{n}\cdot\vec{r}} + \lambda^2} = \underline{r}\sqrt{1 + 2\underline{\frac{\lambda}{r}\hat{n}\cdot\hat{r}} + \frac{\lambda^2}{r^2}} \approx r\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)$$

when $\lambda << r$

Example $\quad \nabla r = ?$

$$\boxed{u(\vec{r}) = r}$$

$$\nabla r\cdot\hat{n} = \lim_{\lambda\to 0}\frac{\left|\vec{r}+\lambda\hat{n}\right| - r}{\lambda} \approx \lim_{\lambda\to 0}\frac{r\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right) - r}{\lambda} = \hat{r}\cdot\hat{n},\ \ \nabla r = \hat{r}$$

Example $\quad \nabla\frac{1}{r} = ?$

$$\boxed{u(\vec{r}) = \frac{1}{r}}$$

$$\nabla\frac{1}{r}\cdot\hat{n} = \lim_{\lambda\to 0}\frac{\frac{1}{\left|\vec{r}+\lambda\hat{n}\right|} - \frac{1}{r}}{\lambda} \approx \frac{1}{r}\lim_{\lambda\to 0}\frac{\frac{1}{\left(1+\frac{\lambda}{r}\hat{n}\cdot\hat{r}\right)} - 1}{\lambda} = -\frac{\hat{r}}{r^2}\cdot\hat{n},$$

or $\quad \nabla\frac{1}{r} = -\frac{\hat{r}}{r^2} = -\frac{\vec{r}}{r^3}$ $\qquad$ When r is not 0 !

Similarly When r is not 0 !

$$\nabla \frac{1}{r^n} = -n\frac{\hat{r}}{r^{n+1}} = -n\frac{\vec{r}}{r^{n+2}} \qquad \nabla \ln r = \frac{\vec{r}}{r} = \frac{\vec{r}}{r^2}$$

Example: $\nabla e^{\vec{k}\cdot\vec{r}} = ?$

Prove $\nabla e^{\vec{k}\cdot\vec{r}} = \vec{k}e^{\vec{k}\cdot\vec{r}}$

Example: $\nabla e^{\alpha r} = ?$

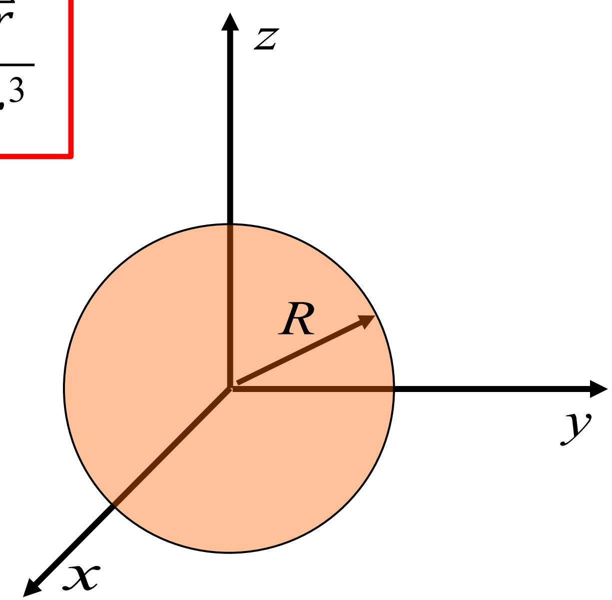
Prove $\nabla e^{\alpha r} = \alpha\hat{r}e^{\alpha r}$

Example: Determine the Laplacian of the function 1/r

$$\nabla^2 \frac{1}{r} = \nabla \cdot \nabla \frac{1}{r} = -\nabla \cdot \frac{\vec{r}}{r^3} = ?$$

$$\boxed{\nabla \frac{1}{r} = -\frac{\vec{r}}{r^3}}$$

$$\nabla \cdot \frac{\vec{r}}{r^3} = \frac{1}{r^3} \nabla \cdot \vec{r} + \vec{r} \ \nabla \frac{1}{r^3} = \frac{3}{r^3} - 3\vec{r} \cdot \frac{\vec{r}}{r^5} = 0$$



The above is valid when r ≠ 0. What happens when r = 0???

Consider volume integral of the function over all space. This integral can be found by integrating over a spherical volume shown in the figure and letting the radius increase to infinity

$$\lim_{R\to\infty} \iiint_{V_R} \nabla \cdot \frac{\vec{r}}{r^3} dV = \lim_{R\to\infty} \oiint_{S_R} \frac{\vec{r}\cdot\hat{r}}{r^3} dS = \lim_{R\to\infty} \oiint_{S_R} \frac{1}{r^2} dS = \lim_{R\to\infty} \frac{1}{R^2} 4\pi R^2 = 4\pi$$

At r≠0

$$\nabla\cdot\frac{\vec{r}}{r^3}=\frac{1}{r^3}\nabla\cdot\vec{r}+\vec{r}\cdot\nabla\frac{1}{r^3}=\frac{3}{r^3}-3\vec{r}\cdot\frac{\vec{r}}{r^5}=0$$

At r=0

$$\lim_{R\to\infty}\iiint_{V_R}\nabla\cdot\frac{\vec{r}}{r^3}dV=\lim_{R\to\infty}\oiint_{S_R}\frac{\vec{r}\cdot\hat{r}}{r^3}dS=\lim_{R\to\infty}\oiint_{S_R}\frac{1}{r^2}dS=\lim_{R\to\infty}\frac{1}{R^2}4\pi R^2=4\pi$$

$$\lim_{R\to\infty}\iiint_{V_R}\nabla\cdot\frac{\vec{r}}{r^3}dV=4\pi=\lim_{R\to\infty}\iiint_{V_R}4\pi\delta(?)dV$$

0 everywhere (known) except for r=0

$$\nabla\cdot\frac{\vec{r}}{r^3}=4\pi\delta(\vec{r})$$

What we have just shown is that the Laplacian of the function 1/r is zero everywhere except at the origin where r = 0, and yet its volume is finite and equal to 4π.

This implies that the Laplacian of 1/r is actually a Dirac Delta Function, given by:

$$\nabla^2 \frac{1}{r} = -\nabla \cdot \frac{\vec{r}}{r^3} = -4\pi\delta(\vec{r})$$

Or alternatively, we can say that the function: $\varphi = \dfrac{1}{4\pi r}$

Is a solution to the differential equation: $\nabla^2\varphi = -\delta(\vec{r})$

This result is of fundamental importance in the subject of electrostatic and magnetostatic fields !!

3-4 Gauss's Law and Applications

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \Rightarrow \boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}.}$$

Gauss's law: The total outward flux of the E-field over any closed surface in free space is equal to the total charge enclosed in the surface divided by ε_0

S: can be any hypothetical closed surface

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2}\, dv' \qquad \text{(V/m)},}$$

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2}\, ds' \qquad \text{(V/m)}.}$$

or

$$\boxed{\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}.}$$

?

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2}\, d\ell' \qquad \text{(V/m)},}$$

Gauss's law is useful when:
The normal component of the electric field intensity is constant over an enclosed surface (also called Gaussian surface)

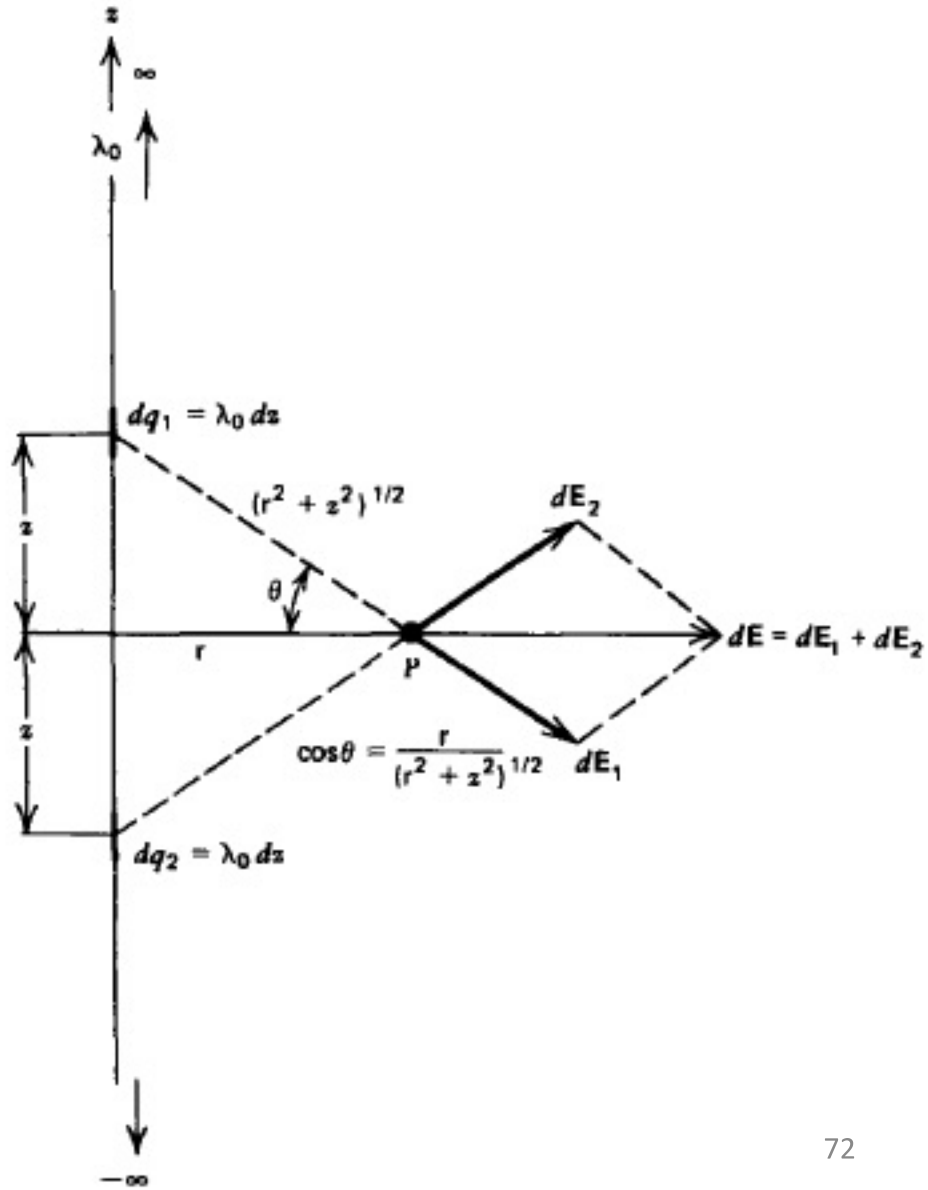
Example 1: Field of an Infinitely Long Line Charge

Let us calculate the electric field of an infinitely long line charge in two ways.

Method 1

In the first way, we will directly integrate the electrical field of a line charge as a superposition of small segments, *dl*, containing total charge *λdl*.

$$\vec{E}(\vec{r}) = \int_{-\infty}^{+\infty} \frac{\lambda\left(\vec{r} - z_s\hat{z}\right)}{4\pi\varepsilon_0\left|\vec{r} - z_s\hat{z}\right|^3} dz_s$$



Let us solve this problem in cylindrical coordinates, therefore

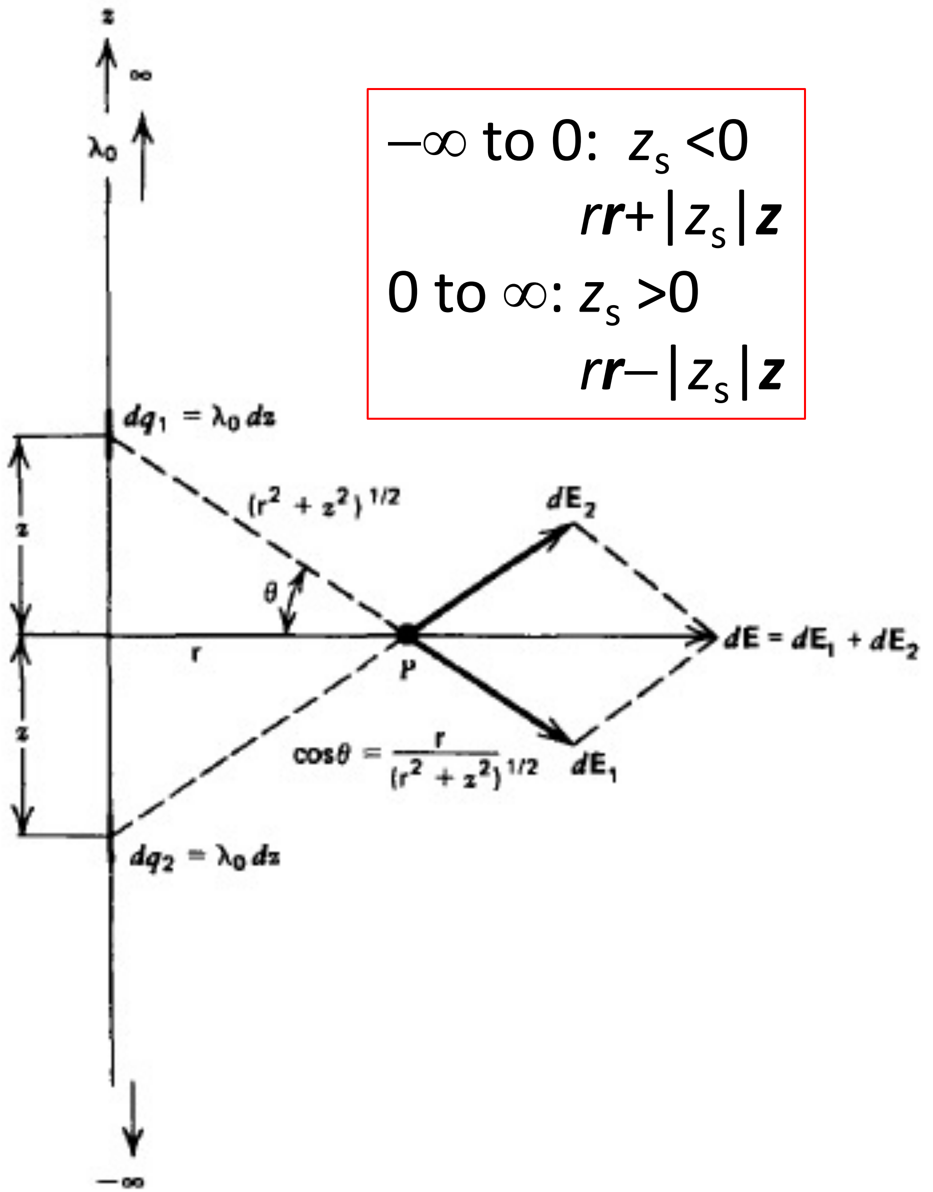
$$\vec{E}(\vec{r}) = \frac{\lambda}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty}\frac{r\hat{r} - z_s\hat{z}}{\sqrt{r^2 + z_s^2}^{\,3}}dz_s$$

$-\infty$ to 0: $z_s < 0$

$r\boldsymbol{r} + |z_s|\boldsymbol{z}$

0 to ∞: $z_s > 0$

$r\boldsymbol{r} - |z_s|\boldsymbol{z}$



Note that due to symmetry the ***z*** component of the field cancels:

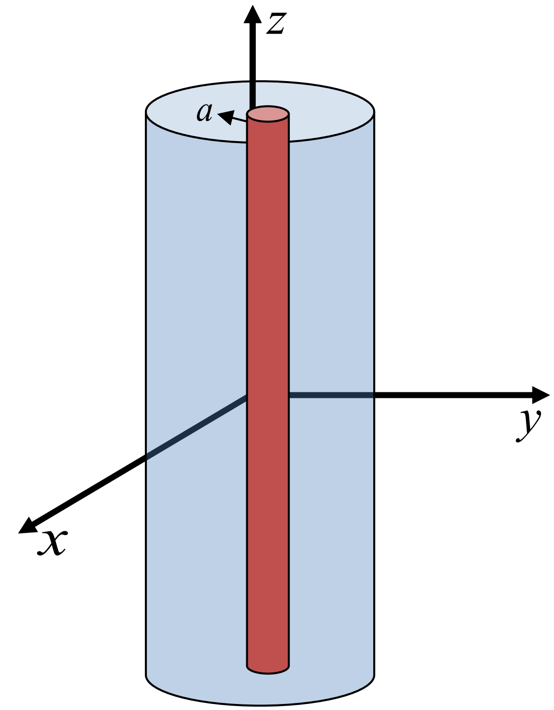
$$\vec{E}(\vec{r}) = \frac{\lambda\hat{r}}{4\pi\varepsilon_0}\int_{0}^{+\infty}\frac{2r}{\sqrt{r^2 + z_s^2}^{\,3}}dz_s$$

$$\vec{E}(\vec{r}) = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

Method 2

Now let's solve this problem using Maxwell's equations…

Let us draw a cylindrical surface around the line charge, where it is assumed to be placed at the center of the cylinder. Due to symmetry, the field evaluated anywhere on the surface must be in the ***r*** direction and must be constant.



$$\oiint_S \vec{E}\cdot\hat{n}\,dS = \frac{Q}{\varepsilon_0}$$

$$\frac{Q}{\varepsilon_0} = \int_{-\infty}^{+\infty} \frac{\lambda}{\varepsilon_0}dz \qquad \oiint_S \vec{E}\cdot\hat{n}\,dS = \oiint_S E_r\hat{r}\cdot\hat{r}r\,d\phi\,dz \;= \int_{-\infty}^{+\infty} 2\pi r E_r\,dz$$

$$\therefore \int_L 2\pi r E_r\,dz = \int_L \frac{\lambda}{\varepsilon_0}dz$$

$$\vec{E} = \frac{\lambda}{2\pi\varepsilon_0 r}\hat{r}$$

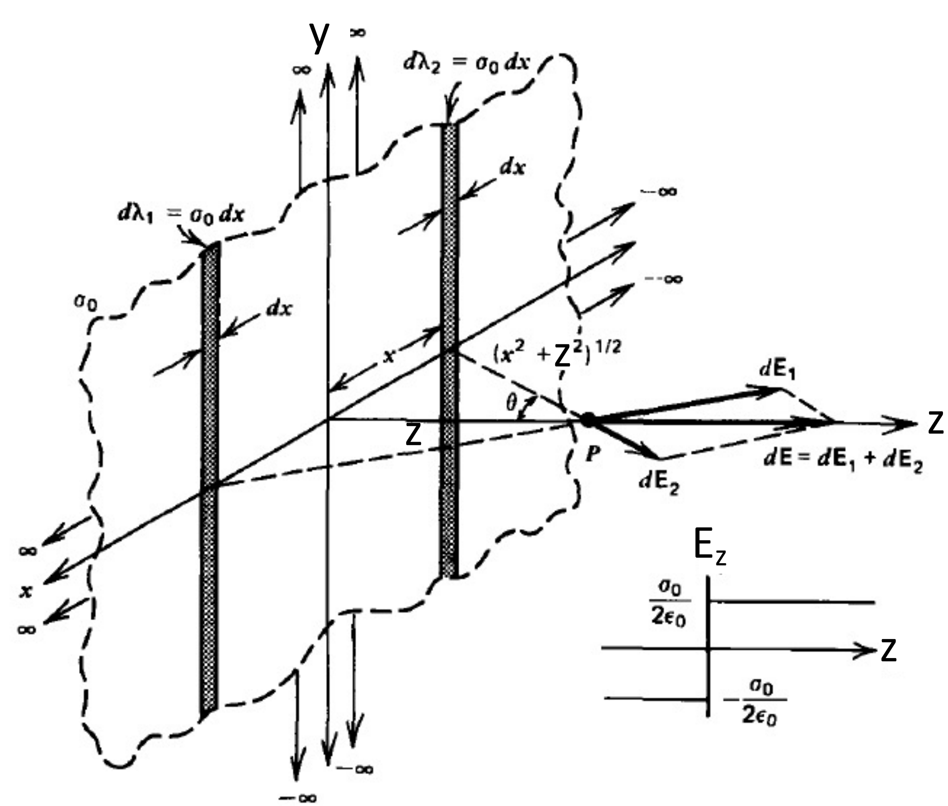
Maxwell's equations thus provide a more direct way to obtain the electric field of these highly symmetrical systems.

Example 2: Field of a Planar Sheet of Surface Charge

Let us follow the same method to calculate the field of a planar sheet of surface charge with constant charge density.

Method 1

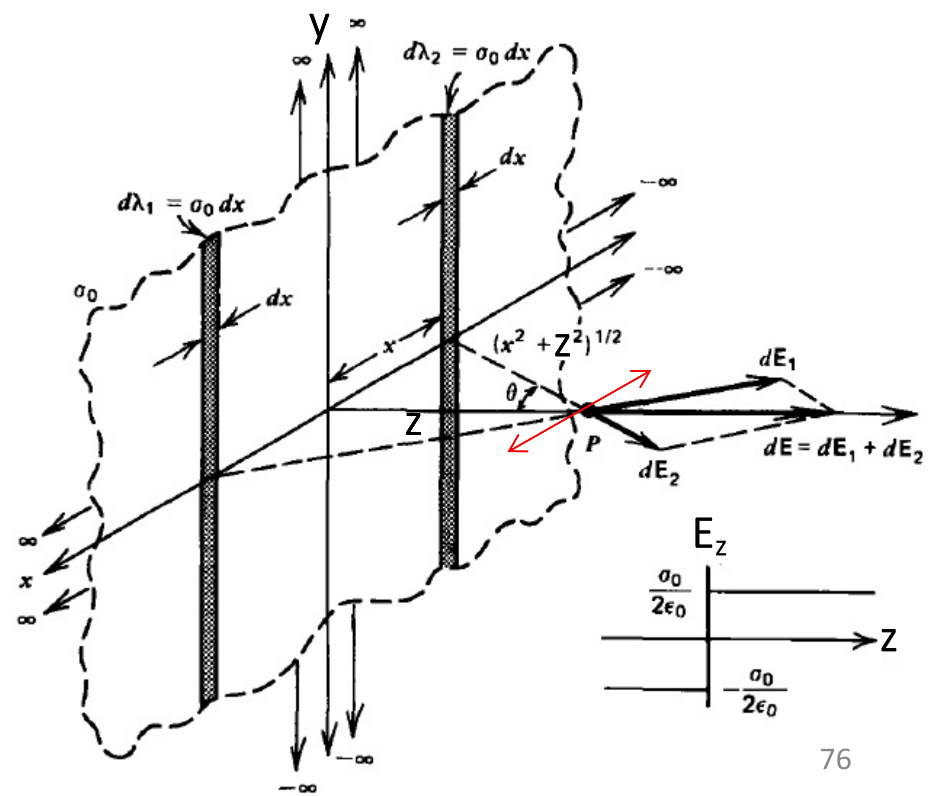
In the first way, we will directly integrate the electrical field of a patch charge as a superposition of small area segments, *dS*, containing total charge *σdS*.



$$\vec{E}(\vec{r}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sigma\left(\vec{r} - x_s\hat{x} - y_s\hat{y}\right)}{4\pi\varepsilon_0 \left|\vec{r} - x_s\hat{x} - y_s\hat{y}\right|^3} dx_s dy_s$$

$$\vec{E}(\vec{r})=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\sigma\left(\vec{r}-x_s\hat{x}-y_s\hat{y}\right)}{4\pi\varepsilon_0\left|\vec{r}-x_s\hat{x}-y_s\hat{y}\right|^3}dx_s dy_s$$

$$\vec{E}(\vec{r})=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\sigma\left(\underline{x\hat{x}}+y\hat{y}+z\hat{z}\underline{-x_s\hat{x}}-y_s\hat{y}\right)}{4\pi\varepsilon_0\sqrt{\left(x-x_s\right)^2+\left(y-y_s\right)^2+z^2}^{\,3}}dx_s dy_s$$



Field of a Planar Sheet of Surface Charge

Let us follow the same method to calculate the field of a planar sheet of surface charge with constant charge density.

In the first way, we will directly integrate the electrical field of a patch charge as a superposition of small area segments, dS, containing total charge σdS.

$$\vec{E}(\vec{r}) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\sigma\left(x\hat{x} + y\hat{y} + z\hat{z} - x_s\hat{x} - y_s\hat{y}\right)}{4\pi\varepsilon_0 \sqrt{\left(x - x_s\right)^2 + \left(y - y_s\right)^2 + z^2}^{\,3}} dx_s dy_s$$

Again, taking advantage of the symmetry properties, the fields along x and y cancel, therefore:

$$\vec{E}(\vec{r}) = \frac{\sigma z\hat{z}}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\left(x - x_s\right)^2 + \left(y - y_s\right)^2 + z^2}^{\,3}} dx_s dy_s$$

$$\vec{E}(\vec{r}) = \frac{\sigma z\hat{z}}{4\pi\varepsilon_0} \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{1}{\sqrt{\left(x - x_s\right)^2 + \left(y - y_s\right)^2 + z^2}^{\,3}} dx_s dy_s$$

$$\vec{E}(\vec{r}) = \frac{\sigma z\hat{z}}{2\pi\varepsilon_0} \int_{-\infty}^{+\infty} \frac{1}{\left(y - y_s\right)^2 + z^2} dy_s$$

$$\vec{E}(\vec{r}) = \frac{\sigma}{2\varepsilon_0}\hat{z}$$

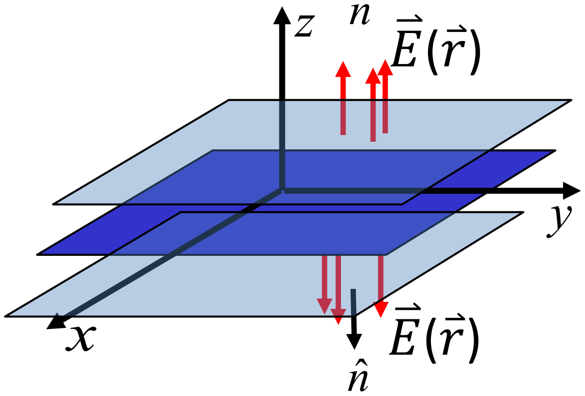
Interestingly, the electric field is constant and independent of the distance away from the sheet of plane charge.

Actually, this result is unphysical since it is not possible to have an electric field at infinity. The only way this could occur is if there were charges at infinity, which we assume is not possible. Despite this, the infinite plane charge is a good approximation for the field near a planar surface of constant charge.

Method 2

Now let's solve this problem using Maxwell's equations…

Let us draw a surface around the sheet charge, which is assumed to located at the z=0 plane. Due to symmetry, the field evaluated anywhere on the surface must be in the ***z*** direction and must be constant.



$$\oiint_S \vec{E}\cdot\hat{n}\,dS = \frac{Q}{\varepsilon_0}$$

$$\frac{Q}{\varepsilon_0} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\sigma}{\varepsilon_0}dxdy \qquad \oiint_S \vec{E}\cdot\hat{n}\,dS = \oiint_S E_z\hat{z}\cdot\hat{z}\,dxdy = \underline{2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}E_z\,dxdy$$

$$\therefore \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\sigma}{\varepsilon_0}dxdy = 2\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}E_z\,dxdy$$

$$\vec{E} = \frac{\sigma}{2\varepsilon_0}\begin{Bmatrix} +\hat{z} & z>0 \\ -\hat{z} & z<0 \end{Bmatrix}$$

Again, Maxwell's equations provide a more direct way to obtain the electric field of these highly symmetrical systems.

Example 3: Electric Field of Bohr Model

Let us consider the same two techniques for integrating the electric field near the atomic cloud. Recall that the charge density within the Bohr electron cloud is given by:

$$\rho(\vec{r}) = \frac{-Q_e}{\pi a^3} e^{-2r/a}$$

The electric field vector is thus determined by integrating over all space of the source charge distribution as follows:

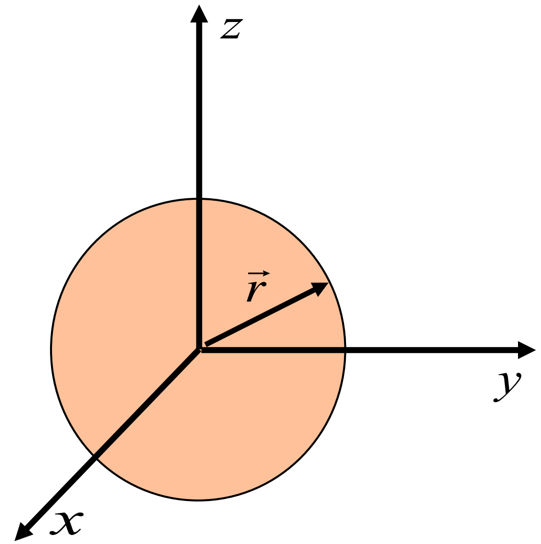
$$\vec{E}(\vec{r}) = \iiint_V \frac{\rho(\vec{r_s})(\vec{r} - \vec{r_s})}{4\pi\varepsilon_0 |\vec{r} - \vec{r_s}|^3} dV_s = \int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{-Q_e e^{-2r_s/a}}{\pi a^3} \cdot \frac{(\vec{r} - \vec{r_s})}{|\vec{r} - \vec{r_s}|^3} r_s^2 \sin(\theta_s)\, dr_s d\theta_s d\phi_s$$

Consider how challenging it is to even write the relative position vector in spherical coordinates, let alone integrate it over all space!!! Maxwell's equations are certainly much easier!

$$\frac{(\vec{r} - \vec{r_s})}{|\vec{r} - \vec{r_s}|^3}$$

Electric Field of Bohr Model

Let us now use Maxwell's Equations. First, let us use the symmetry of the problem to draw an enclosing surface that represents one of the 3 orthogonal coordinate surfaces (in this case r = const). Now integrate the charge within a sphere of radius, r.



$$\oiint_S \vec{E}\cdot\hat{n}dS = \iiint_V \frac{\rho(\vec{r})}{\varepsilon_0}dV_s = \frac{Q_{enc}}{\varepsilon_0}$$

$$Q_{enc}(r) = \int_0^R\int_0^\pi\int_0^{2\pi} \frac{-Q_e e^{-2r/a}}{\pi a^3} r^2 \sin\theta dr d\theta d\phi = -Q_e\left\{1 - e^{-2R/a}\left(1 + 2\left(\frac{R}{a}\right) + 2\left(\frac{R}{a}\right)^2\right)\right\}$$

$$\oiint_S \underline{\vec{E}}\cdot\hat{n}dS = \oiint_S \underline{E_r\hat{r}}\cdot\hat{r}r^2\sin\theta d\theta d\phi = 4\pi R^2 E_r$$

$$\therefore E_r = \frac{-Q_e}{4\pi\varepsilon_0 R^2}\left\{1 - e^{-2R/a}\left(1 + 2\left(\frac{R}{a}\right) + 2\left(\frac{R}{a}\right)^2\right)\right\}$$

Where can Maxwell's Integral Equations be Used?

- Computing electric field by the direct charge integration method will always lead to the correct field, however it can involved tedious integrals and possibly require numerical calculations.

Method 2

- **The integral form of Maxwell's equations** provides a more direct method to obtaining the electrical field of simple charge distributions, however it is only applicable when there is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.

Let's look at a few examples where the integral form of Maxwell's equations cannot be used…

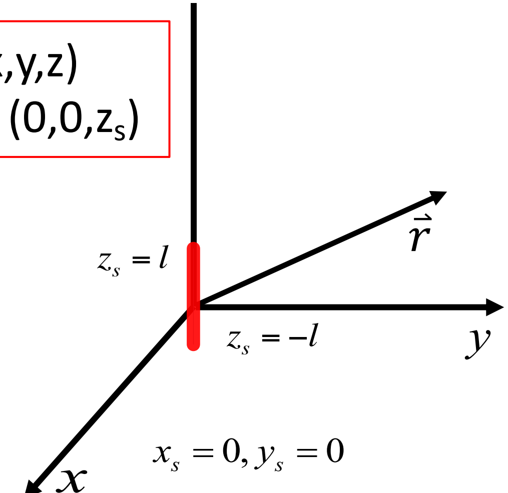
Example 4: Field of a Line charge of Finite Length

$$\varphi(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})}{4\pi\varepsilon_0 |\vec{r} - \vec{r_s}|} dl_s$$

$$\varphi(x,y,z) = \int_{-l}^{l} \frac{\lambda}{4\pi\varepsilon_0\sqrt{x^2+y^2+(z-z_s)^2}} dz_s$$

Field: (x,y,z)
Source: $(0,0,z_s)$

$$\varphi(x,y,z) = \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{z-l+\sqrt{x^2+y^2+(z-l)^2}}{z+l+\sqrt{x^2+y^2+(z+l)^2}}\right)$$



The electric field can then be computed from the negative gradient of this potential.

$$\vec{E} = -\nabla \frac{\lambda}{4\pi\varepsilon_0} \ln\left(\frac{z-l+\sqrt{x^2+y^2+(z-l)^2}}{z+l+\sqrt{x^2+y^2+(z+l)^2}}\right)$$

Why is it more difficult to obtain a simple result by directly integrating Maxwell's equations?

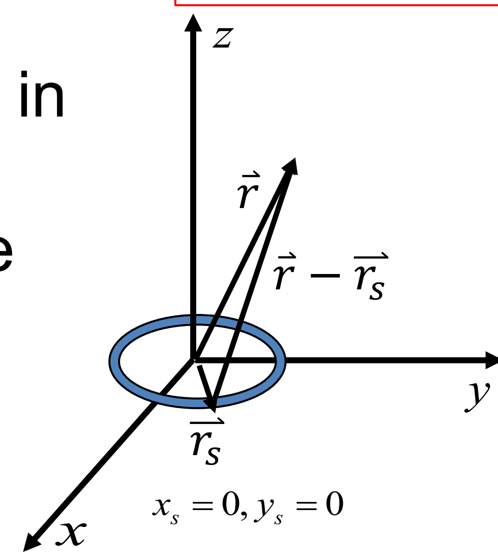
Example 5: Field of a Charged Ring

Consider a ring with radius a and uniform charge density with a center at the origin and located in the xy- plane.

$$\varphi(\vec{r}) = \int_L \frac{\lambda(\vec{r_s})}{4\pi\varepsilon_0|\vec{r}-\vec{r_s}|} dl_s$$

$$\varphi(x,y,z) = \frac{\lambda a}{4\pi\varepsilon_0} \oint \frac{d\theta}{\sqrt{(x-a\cos\theta)^2 + (y-a\sin\theta)^2 + z^2}}$$

Field: (x,y,z)
Source: (acosθ,asinθ,0)



This integral is more difficult to solve, and in many cases they don't have an analytical solution. In these and other examples, the potential and field must be computed numerically on a computer:

$$\vec{E} = -\nabla \frac{\lambda a}{4\pi\varepsilon_0} \oint \frac{d\theta}{\sqrt{(x-a\cos\theta)^2 + (y-a\sin\theta)^2 + z^2}}$$

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ (Fig. 2.23
Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to th
right of both.

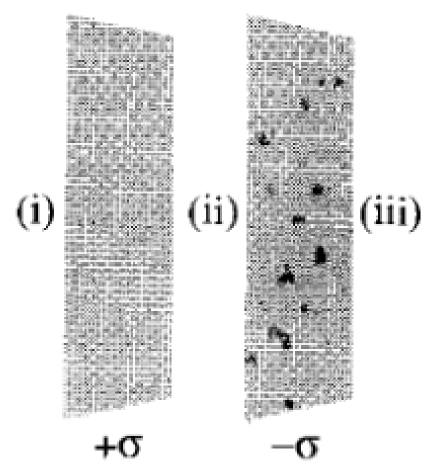


Figure 2.23

Quick classroom exercise

Two infinite parallel planes carry equal but opposite uniform charge densities $\pm\sigma$ (Fig. 2.23). Find the field in each of the three regions: (i) to the left of both, (ii) between them, (iii) to the right of both.

Solution: The left plate produces a field $(1/2\epsilon_0)\sigma$ which points away from it (Fig. 2.24)—to the left in region (i) and to the right in regions (ii) and (iii). The right plate, being negatively charged, produces a field $(1/2\epsilon_0)\sigma$, which points *toward* it—to the right in regions (i) and (ii) and to the left in region (iii). The two fields cancel in regions (i) and (iii); they conspire in region (ii). *Conclusion:* The field is $(1/\epsilon_0)\sigma$, and points to the right, between the planes; elsewhere it is zero.

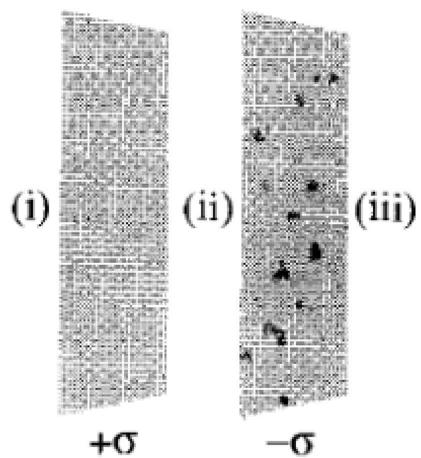


Figure 2.23

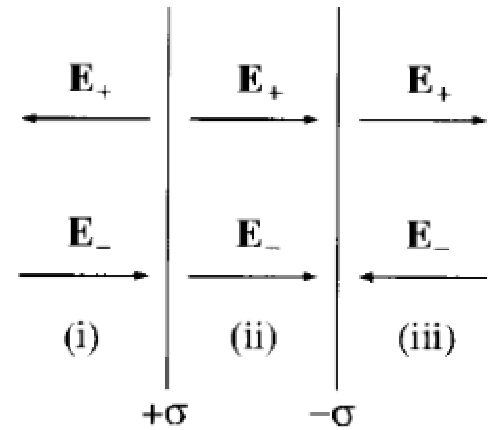


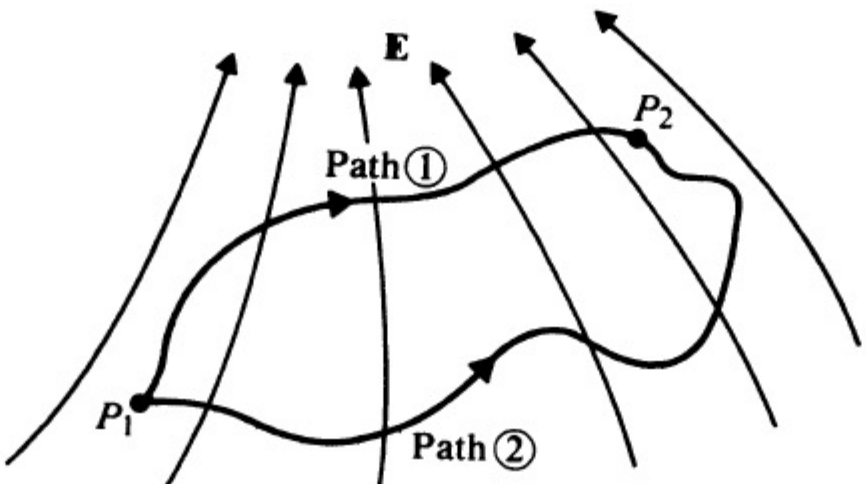
Figure 2.24

3-5 Electric Potential

$$\nabla \times \mathbf{E} = 0 \quad \Rightarrow \quad \boxed{\mathbf{E} = -\nabla V}$$

Scalar quantities are easier to handle than vector quantities

Physical meaning: Work done in carrying a charge from one point to another



$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\boldsymbol{\ell} \qquad \text{(J/C or V).}$$

E=F/q

Work is done against the field

FIGURE 3–11
Two paths leading from P_1 to P_2 in an electric field.

Electrical Potential Difference

$$\boxed{\mathbf{E} = -\nabla V}$$

$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \qquad \text{(J/C or V)}.$$

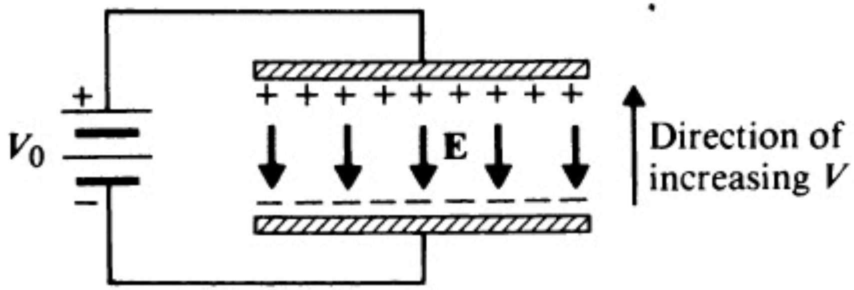
$$-\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell = \int_{P_1}^{P_2} (\nabla V) \cdot (\mathbf{a}_\ell \, d\ell)$$
$$= \int_{P_1}^{P_2} dV = V_2 - V_1.$$

See Eq. 2-88:
$(\nabla V) \cdot \mathbf{a}_l = dV/dl$
The component of ∇V in $\mathbf{a}_l$ direction

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \qquad \text{(V)}.}$$

Usually, the zero-potential point is taken at infinity (P_1).

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \qquad \text{(V).}}$$



E and increasing V are in opposite direction

FIGURE 3–12
Relative directions of **E** and increasing V.

In going against the E field the electric potential V **increases**

∇V

Its direction is ⊥ constant-V surfaces

$$\boxed{\mathbf{E} = -\nabla V}$$

E ⊥ constant-V surfaces

Field lines
Streamlines

Equipotential lines
Equipotential surfaces

3-5.1 Electric Potential due to a Charge Distribution

- V(*R*) of **a point charge at origin**

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \qquad \text{(V)}.}$$

With reference point at infinity

$$V = -\int_{\infty}^{R} \left(\mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \right) \cdot (\mathbf{a}_R \, dR),$$

$$\boxed{V = \frac{q}{4\pi\epsilon_0 R} \qquad \text{(V)}.}$$

- Potential difference between any two points

$$V_{21} = V_{P_2} - V_{P_1} = \frac{q}{4\pi\epsilon_0} \left(\frac{1}{R_2} - \frac{1}{R_1} \right).$$

A charge is moved from P_1 to P_2 (against the E field if $V_{21}>0$)

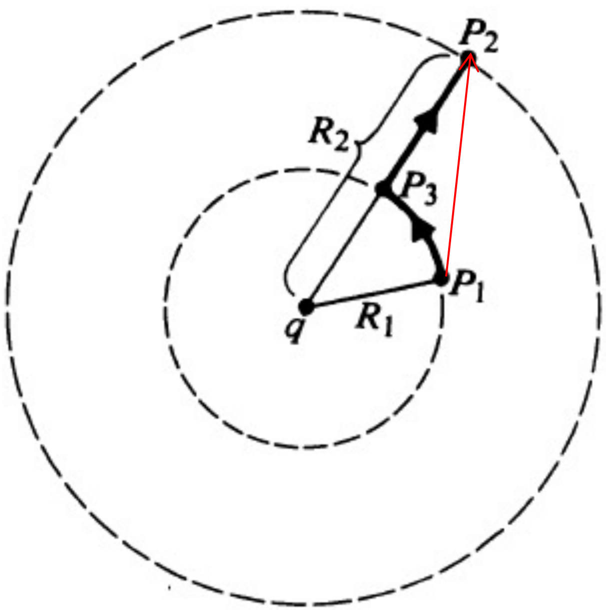


FIGURE 3–13
Path of integration about a point charge.

$V_{21}=V_{31}+V_{23}$

V due to n Discrete Point Charges

$$V = \frac{q}{4\pi\epsilon_0 R} \quad \text{(V).}$$

Reference point at infinity

$\boldsymbol{R} \rightarrow \boldsymbol{R} - \boldsymbol{R'}$ (Charges located at $\boldsymbol{R'}$)

$\sum$

$$V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k}{|\mathbf{R} - \mathbf{R}'_k|} \quad \text{(V).}$$

V of an Electric Dipole

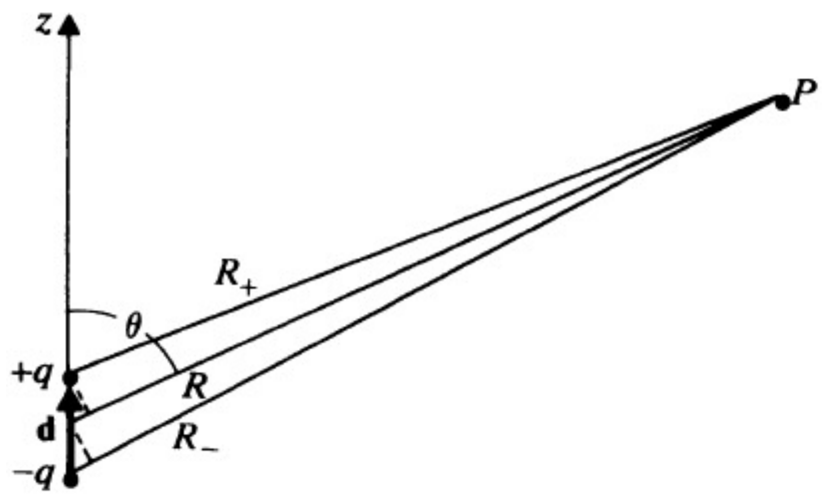


FIGURE 3–14
An electric dipole.

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right).$$

If $d \ll R$

$$\frac{1}{R_+} \cong \left(R - \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 + \frac{d}{2R}\cos\theta\right)$$

$$\frac{1}{R_-} \cong \left(R + \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 - \frac{d}{2R}\cos\theta\right).$$

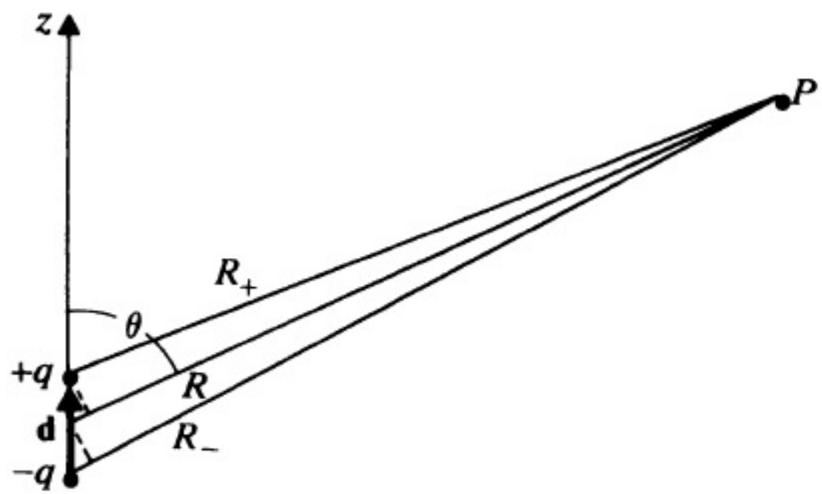


FIGURE 3–14
An electric dipole.

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2}$$

$$\boxed{V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},}$$

where $\mathbf{p} = q\mathbf{d}$.

$\boxed{\mathbf{d}\cdot\mathbf{a}_R = d\cos\theta}$

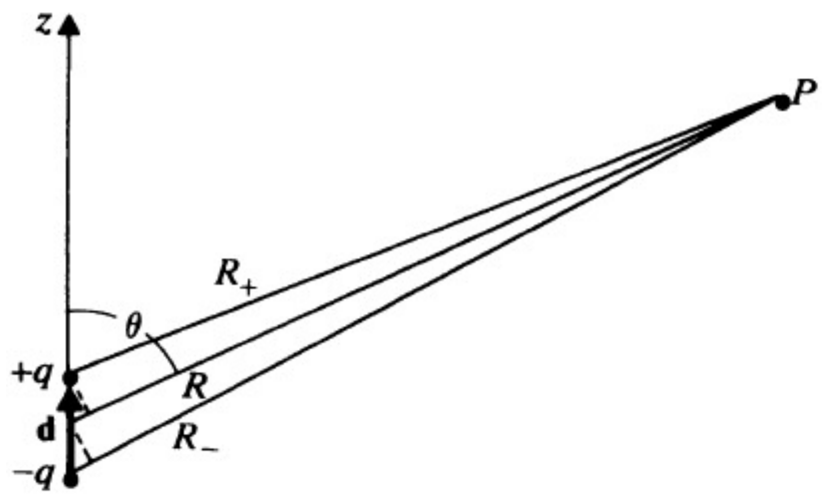


FIGURE 3–14
An electric dipole.

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2}$$

$$\mathbf{E} = -\nabla V = -\mathbf{a}_R\frac{\partial V}{\partial R} - \mathbf{a}_\theta\frac{\partial V}{R\,\partial\theta}$$

$$= \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta).$$

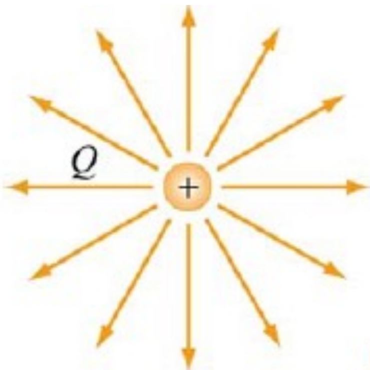
A much simpler approach

Supplementary Material

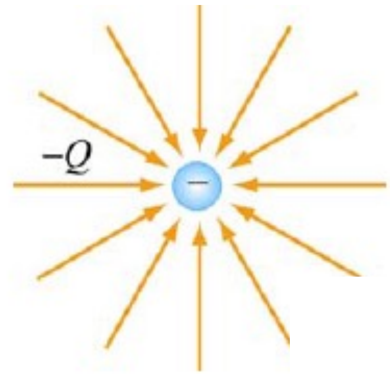
- Dipole
- Multipole
- Force and Torque

Dipole Field

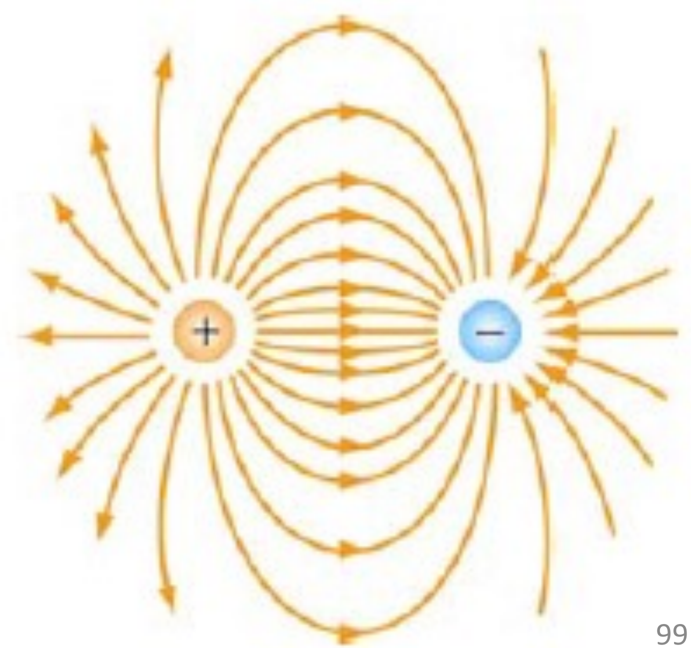
The field of two isolated charges are spherically symmetric and always pointing away (or towards) the origin for positive (or negative) charges.



(a)



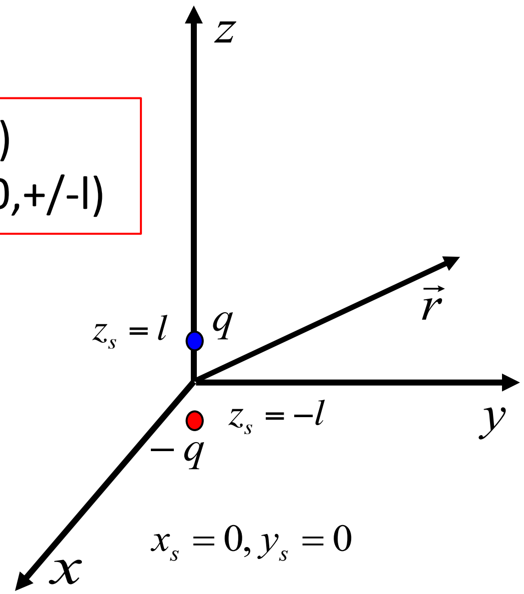
When two opposite charges are brought into close vicinity, the field distribution **changes substantially**. This resembles the electric field produced by a point dipole, which we will discuss shortly…



Dipole Field

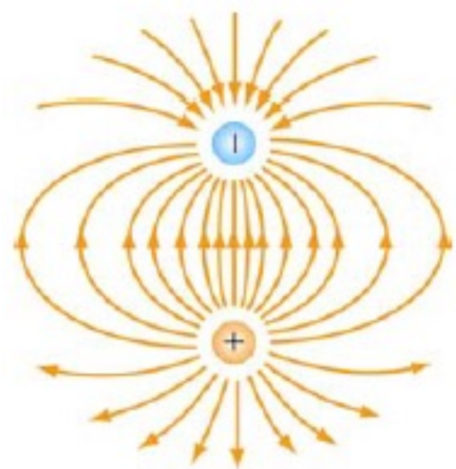
Let us first consider the field of the two charges in the **z = 0 plane**.

Field: (x,y,0)
Source: (0,0,+/-l)



$$\vec{E} = \overrightarrow{E_+} + \overrightarrow{E_-} = \frac{q\left(x\hat{x} + y\hat{y} - l\hat{z}\right)}{4\pi\varepsilon_0\sqrt{x^2 + y^2 + l^2}^{\,3}} + \frac{-q\left(x\hat{x} + y\hat{y} + l\hat{z}\right)}{4\pi\varepsilon_0\sqrt{x^2 + y^2 + l^2}^{\,3}}$$

$$\vec{E} = \overrightarrow{E_+} + \overrightarrow{E_-} = \frac{-2ql}{4\pi\varepsilon_0\sqrt{x^2 + y^2 + l^2}^{\,3}}\hat{z}$$



As expected, the x- and y- fields cancel due to symmetry and only the z- field remains for the z = 0 plane. We also notice that when the observation point is much larger than the charge separation (i.e., r >> *l*) the electric field decays with the cube of distance instead of the square of distance.

$$\vec{E}\left(r >> l\right) = \frac{-2ql}{4\pi\varepsilon_0 r^3}\hat{z} = \frac{-p}{4\pi\varepsilon_0 r^3}\hat{z}$$

Dipole Field

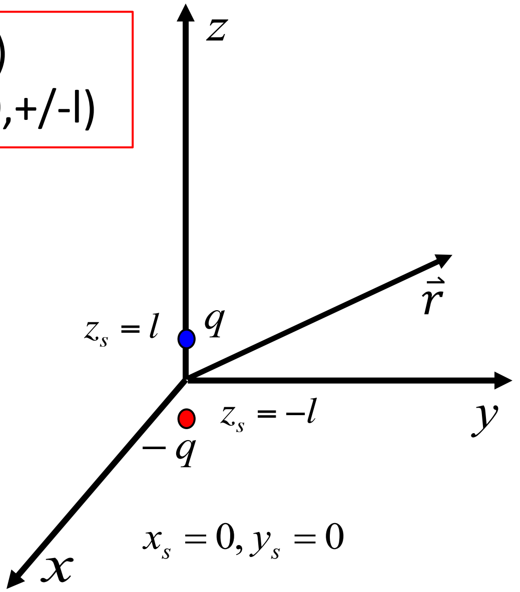
Let us first consider the field of the two charges **along the z-axis**.

$$\vec{E} = \overrightarrow{E_+} + \overrightarrow{E_-} = \frac{q\left(z\hat{z} - l\hat{z}\right)}{4\pi\varepsilon_0\left(z-l\right)^3} + \frac{-q\left(z\hat{z} + l\hat{z}\right)}{4\pi\varepsilon_0\left(z+l\right)^3}$$

Field: (0,0,z)
Source: (0,0,+/-l)

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{\left(z-l\right)^2} - \frac{1}{\left(z+l\right)^2}\right]\hat{z} = \frac{ql}{4\pi\varepsilon_0}\left[\frac{4z}{\left(z-l\right)^2\left(z+l\right)^2}\right]\hat{z}$$

$$\vec{E}\left(z >> l\right) \approx \frac{4ql}{4\pi\varepsilon_0 z^3}\hat{z} = \frac{2p_z}{4\pi\varepsilon_0 z^3}\hat{z}$$



The result again indicates that the field decays with the cube (not square) of distance. Notice also that the field along the z-axis is twice as strong as the field along the z=0 plane for large distances (r >> l) field. What is the origin of this inverse cubic distance dependence?

To understand this, let us consider the **Taylor series expansion** of the field of the charge Q that is moved to the origin. Recall the Taylor series expansion is given by:

$$f(x) = f(x_0) + \frac{df(x_0)}{dx}(x - x_0) + \frac{1}{2}\frac{d^2 f(x_0)}{dx^2}(x - x_0)^2 + \ldots + \frac{1}{n!}\frac{d^n f(x_0)}{dx^n}(x - x_0)^n$$

Consider now the sum of the two charge expanded **about the origin**:

$$\overrightarrow{E_+} \cdot \hat{z} = \frac{q}{4\pi\varepsilon_0}\frac{1}{(z - \underline{l})^2} \approx \frac{q}{4\pi\varepsilon_0 z^2} - \frac{2ql}{4\pi\varepsilon_0 z^3} + \frac{1}{2}\frac{6ql^2}{4\pi\varepsilon_0 z^4} + \ldots$$

Along the z-axis
Field: (0,0,z)
Source: (0,0,+/-l)

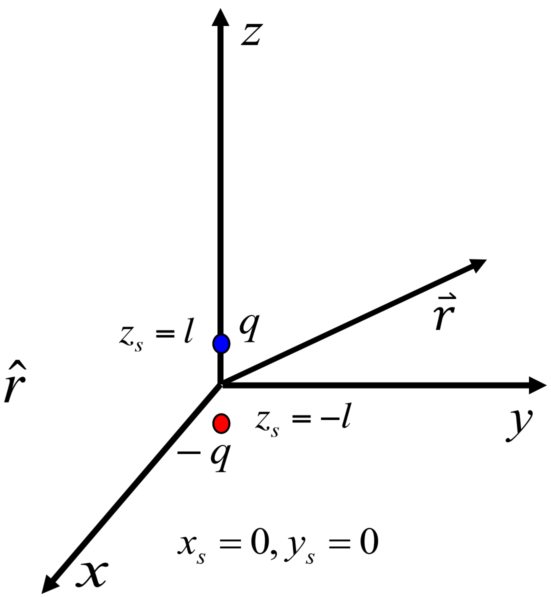
$$\overrightarrow{E_-} \cdot \hat{z} = \frac{(-q)}{4\pi\varepsilon_0}\frac{1}{(z \underline{+ l})^2} \approx \frac{(-q)}{4\pi\varepsilon_0 z^2} - \frac{2(-q)(-l)}{4\pi\varepsilon_0 z^3} + \frac{1}{2}\frac{6(-q)(-l)^2}{4\pi\varepsilon_0 z^4} + \ldots$$

$$\vec{E} \cdot \hat{z} = \overrightarrow{E_+} \cdot \hat{z} + \overrightarrow{E_-} \cdot \hat{z} \approx \frac{4ql}{4\pi\varepsilon_0 z^3} + \ldots \approx \frac{2p}{4\pi\varepsilon_0 z^3}$$

Thus, the dipole field ($1/r^3$) is just the first non-cancelling term in the Taylor series expansion of a point charge field ($1/r^2$)!

Dipole Potential

Let us first now do the same thing for the potential, but a little more exact:



$$\varphi(\vec{r}) = \varphi_+(\vec{r}) + \varphi_-(\vec{r}) = \frac{q}{4\pi\varepsilon_0 \left|\vec{r} - l\hat{z}\right|} + \frac{-q}{4\pi\varepsilon_0 \left|\vec{r} + l\hat{z}\right|}$$

$$\varphi(\vec{r}) = \lim_{r>>l} \frac{q}{4\pi\varepsilon_0 r}\left[\frac{1}{\left(1 - \frac{l}{r}\hat{z}\cdot\hat{r}\right)} - \frac{1}{\left(1 + \frac{l}{r}\hat{z}\cdot\hat{r}\right)}\right] = \frac{(q)(2l)}{4\pi\varepsilon_0 r^2}\hat{z}\cdot\hat{r}$$

$$\varphi(\vec{r}) = \frac{p_z \hat{z}\cdot\hat{r}}{4\pi\varepsilon_0 r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3}$$

This expression becomes exact in the limit $\vec{p} = \lim_{\substack{2l\to 0\\ q\to\infty \\ 2ql=const}} (2l)(q)\hat{l} = (2l)(q)\hat{l}$

From this expression, it is much easier to compute the field of a dipole by taking the negative gradient of this potential

Derivation of Dipole Field by E= $-\nabla V$

By vector relations, the dipole field can be expressed as:

$$\vec{E} = -\nabla\varphi = -\nabla\frac{\vec{p}\cdot\vec{r}}{4\pi\varepsilon_0 r^3} = \frac{1}{4\pi\varepsilon_0}\nabla\left(\vec{p}\cdot\nabla\frac{1}{r}\right)$$

$$\boxed{\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})}$$

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left\{\vec{p}\times\underbrace{\left(\nabla\times\nabla\frac{1}{r}\right)}_{=0} + \nabla\frac{1}{r}\times\underbrace{\left(\nabla\times\vec{p}\right)}_{=0} + \left(\vec{p}\cdot\nabla\right)\nabla\frac{1}{r} + \underbrace{\left(\nabla\frac{1}{r}\cdot\nabla\right)\vec{p}}_{=0}\right\}$$

independen dent of r

$\partial/\,\partial r$ only

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\vec{p}\cdot\nabla\right)\nabla\frac{1}{r}$$

$$\boxed{\vec{p} = (2l)(q)\hat{l}}$$

This leads to the most commonly accepted expression for the field of a point dipole as:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\left(\vec{p}\cdot\hat{r}\frac{\partial}{\partial r} + \vec{p}\cdot\hat{\theta}\frac{\partial}{\partial\theta}\right)\frac{-\vec{r}}{r^3} = \frac{-\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0}\frac{\partial}{\partial r}\frac{\vec{r}}{r^3} = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

Dipole Field in Different Coordinates

In spherical coordinates, the field of a dipole oriented along the z- axis (polar axis) can be expressed as:

$$\vec{E}(r,\theta) = -\nabla\varphi = -\hat{r}\frac{\partial}{\partial r}\frac{p\cos\theta}{4\pi\varepsilon_0 r^2} - \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta}\frac{p\cos\theta}{4\pi\varepsilon_0 r^2} - \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$\boxed{\phi(\vec{r}) = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}}$$

$$\vec{E}(r,\theta) = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}\hat{r} + \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}\hat{\theta}$$

In Cartesian coordinates the dipole field can be expressed as:

p=p$_x$**x**+p$_y$**y**+p$_z$**z**
r=x**x**+y**y**+z**z**

r̂=**r**/|**r**|

$$\vec{E}(x,y,z) = \frac{1}{4\pi\varepsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right]$$

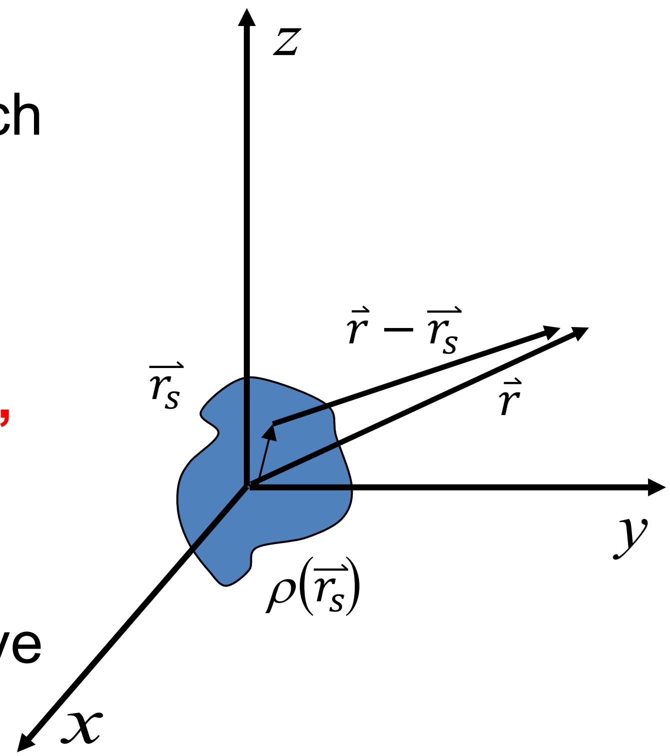
$$\vec{E}(x,y,z) = \frac{1}{4\pi\varepsilon_0\sqrt{x^2+y^2+z^2}^{\,5}}\begin{bmatrix} 3x^2-r^2 & 3xy & 3xz \\ 3xy & 3x^2-r^2 & 3yz \\ 3xz & 3yz & 3z^2-r^2 \end{bmatrix}\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

At θ=0 (Field/Potential along z axis)

	CHARGE	DIPOLE
POTENTIAL	$V \approx \dfrac{q}{4\pi\varepsilon_0 r}$	$V \approx \dfrac{p}{4\pi\varepsilon_0 r^2}$
FIELD	$E \approx \dfrac{q}{4\pi\varepsilon_0 r^2}$	$E \approx \dfrac{p}{4\pi\varepsilon_0 r^3}$

Multipole Expansion of Potential

So far, we have shown that the dipole terms comes from the Taylor series expansion of two equal and opposite point charges (which we may call the **monopole** term). Higher order terms of the Taylor series expansion leads to other field quantities known as multipoles. These include the **quadrupole, octupole, hexadecapole**, etc., which correspond to higher derivatives of 1/r.



Let us now formalize this process and derive the multipole expansion of the potential by taking the Taylor series of 1/r.

$$\frac{1}{|\vec{r} - \vec{r_s}|} = ???$$

Taylor Series of 1/r

$$\frac{1}{|\vec{r}-\vec{r_s}|} = \frac{1}{r} + (x_s)\frac{\partial}{\partial x}\frac{1}{r} + (y_s)\frac{\partial}{\partial y}\frac{1}{r} + (z_s)\frac{\partial}{\partial z}\frac{1}{r}$$ 3 terms

$$+\left(\frac{x_s^2}{2}\right)\frac{\partial^2}{\partial x^2}\frac{1}{r} + \left(\frac{x_s y_s}{2}\right)\frac{\partial^2}{\partial x \partial y}\frac{1}{r} + \left(\frac{x_s z_s}{2}\right)\frac{\partial^2}{\partial x \partial z}\frac{1}{r} + \left(\frac{y_s^2}{2}\right)\frac{\partial^2}{\partial y^2}\frac{1}{r} + \ldots$$ 9 terms

$$+\left(\frac{x_s^3}{6}\right)\frac{\partial^3}{\partial x^3}\frac{1}{r} + \left(\frac{x_s^2 y_s}{6}\right)\frac{\partial^3}{\partial x^2 \partial y}\frac{1}{r} + \left(\frac{x_s^2 z_s}{6}\right)\frac{\partial^3}{\partial x^2 \partial z}\frac{1}{r} + \cdots$$ 27 terms

The electrostatic potential is then expressed as:

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{all\,space}\rho(\vec{r_s})\frac{1}{|\vec{r}-\vec{r_s}|}dV_s$$

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0 r}\iiint\limits_{all\,space} \rho(\vec{r_S})\,dV_s$$ Monopole

Dipole

$$+\frac{1}{1!}\frac{1}{4\pi\varepsilon_0 r^2}\left\{\frac{x}{r}\iiint\limits_{all\,space}\rho(\vec{r_S})\,x_s\,dV_s + \frac{y}{r}\iiint\limits_{all\,space}\rho(\vec{r_S})\,y_s\,dV_s + \frac{z}{r}\iiint\limits_{all\,space}\rho(\vec{r_S})\,z_s\,dV_s\right\}$$

$$+\frac{1}{2!}\frac{1}{4\pi\varepsilon_0 r^3}\left\{\frac{3x^2-r^2}{r^2}\iiint\limits_{all\,space}\rho(\vec{r_S})\,x_s^2\,dV_s + \frac{3xy}{r^2}\iiint\limits_{all\,space}\rho(\vec{r_S})\,x_s y_s\,dV_s + \ldots\right\}$$

Quadrupole

$$+\frac{1}{3!}\frac{1}{4\pi\varepsilon_0 r^4}\left\{\begin{aligned}&\frac{15x^3-9xr^2}{r^3}\iiint\limits_{all\,space}\rho(\vec{r_S})\,x_s^3\,dV_s\\ &+\frac{15x^2y-3yr^2}{r^2}\iiint\limits_{all\,space}\rho(\vec{r_S})\,x_s^2 y_s\,dV_s\\ &+\frac{15xyz}{r^2}\iiint\limits_{all\,space}\rho(\vec{r_S})\,x_s y_s z_s\,dV_s + \ldots\end{aligned}\right\}$$ Octupole

$+\ldots$

Dipole: due to 1st order terms in Taylor series expansion

The expansion can also be written as **Multipoles**

For x, y, z

$$\varphi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\left[\begin{array}{l} \frac{1}{r}\Omega + \\ \frac{1}{r^2}\sum_{i=1}^{3}\frac{x_i}{r}p_i + \\ \frac{1}{r^3}\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2!}\left[\frac{3x_i^2 - r^2}{r^2}Q_{ii} + \frac{3x_i x_j}{r^2}Q_{ij}\right] + \\ \frac{1}{r^4}\sum_{i=1}^{3}\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{1}{3!}\left[\begin{array}{l} \frac{\left(15x_i^2 - 9r^2\right)x_i}{r^3}O_{iii} \\ + \frac{\left(15x_i^2 - 3r^2\right)x_j}{r^3}O_{iij} \\ + \frac{15x_i x_j x_k}{r^3}O_{ijk} \end{array}\right] + \ldots \end{array}\right]$$

$$\Omega = \iiint_{all\ space} \rho(\vec{r_s})dV_s$$

$$p_i = \iiint_{all space} x_{is}\rho(\vec{r_s})dV_s$$

p=qd

$$Q_{ij} = \iiint_{all\ space} x_{is}x_{js}\rho(\vec{r_s})dV_s$$

$Q=qd_id_j$

$$O_{ijk} = \iiint_{all\ space} x_{is}x_{js}x_{ks}\rho(\vec{r_s})dV_s$$

$O=qd_id_jd_k$

Some Idealized Charge Distributions

Monopole – a charge distribution that has non-zero monopole moment, while all higher order moments are zero.

Dipole – a charge distribution that has non-zero dipole moment, while all other moments are zero

$\boxed{\text{Dipole: due to } 1^{st} \text{ order terms in Taylor series expansion}}$

Quadrupole – a charge distribution that has non-zero quadrupole moment, while all other moments are zero

Octupole – a charge distribution that has non-zero octupole moment, while all other moments are zero

…

What is the Multipole Content of This Charge Distribution?

$$\boxed{\Omega = \iiint\limits_{all\ space} \rho(\vec{r}_s)\, dV_s}$$

$$\left\{\Omega = q + q = 2q\right.$$

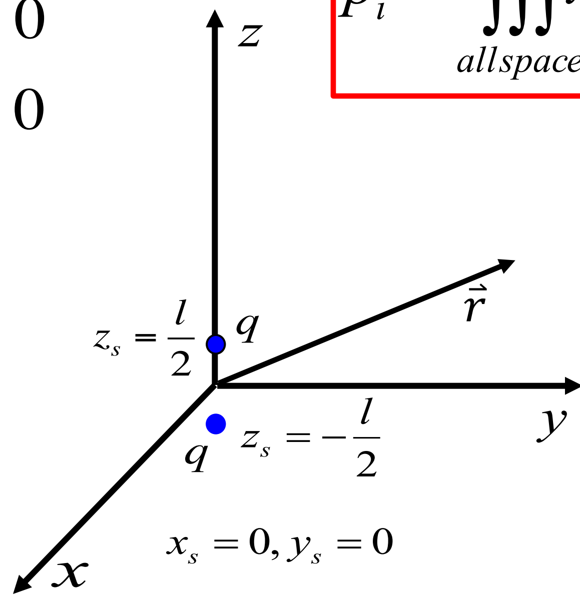
$$\begin{cases} Q_{zz} = \left(\frac{l}{2}\right)^2 q + \left(-\frac{l}{2}\right)^2 q = \underline{\frac{ql^2}{2}} \\ Q_{ij} = 0 \end{cases}$$

$\boxed{\text{0 when } l \to 0}$

$$\begin{cases} O_{zzz} = \left(\frac{l}{2}\right)^3 q + \left(-\frac{l}{2}\right)^3 q = 0 \\ O_{iij} = 0 \\ O_{ijk} = 0 \end{cases}$$

$$\begin{cases} p_z = \frac{l}{2} q + \left(-\frac{l}{2}\right) q = 0, \\ p_x = 0 \\ p_y = 0 \end{cases}$$

$$\boxed{p_i = \iiint\limits_{allspace} x_{is} \rho(\vec{r}_s)\, dV_s}$$



In the limit that the charge separation distance, *l*, we have the definition of a **POINT MONOPOLE**. All other multipole terms are zero for this charge distribution.

What is the Multipole Content of This Charge Distribution?

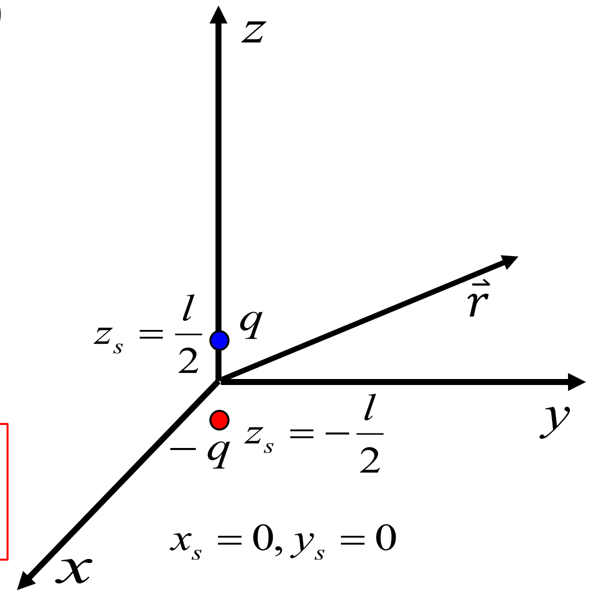
$$\left\{ \Omega = q + (-q) = 0 \right.$$

$$\begin{cases} Q_{zz} = \left(\frac{l}{2}\right)^2 q + \left(-\frac{l}{2}\right)^2 (-q) = 0 \\ Q_{ij} = 0 \end{cases}$$

$$\begin{cases} O_{zzz} = \left(\frac{l}{2}\right)^3 q + \left(-\frac{l}{2}\right)^3 (-q) = \underline{\frac{l^3}{4} q} \\ O_{iij} = 0 \\ O_{ijk} = 0 \end{cases}$$

0 when
$l \rightarrow 0$ and ql=const.

$$\begin{cases} p_z = \frac{l}{2} q + \left(-\frac{l}{2}\right)(-q) = lq, \\ p_x = 0 \\ p_y = 0 \end{cases}$$



In the limit that the charge separation distance, *l*, shrinks to zero while maintaining the product of **ql=const**, we have the definition of a **POINT DIPOLE**. All other multipole terms are zero for this charge distribution.

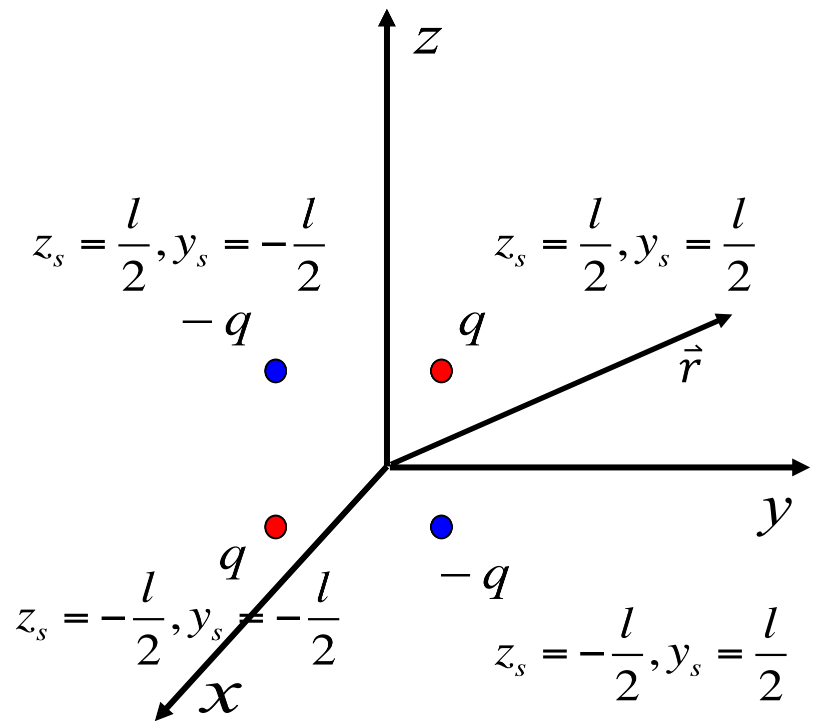
What is the Multipole Content of This Charge Distribution?

$$\left\{\Omega = 2q + 2(-q) = 0\right.$$

$$\begin{cases} Q_{zz} = \left(\frac{l}{2}\right)^2 q + \left(\frac{l}{2}\right)\left(\frac{-l}{2}\right)(-q) + \left(\frac{l}{2}\right)\left(\frac{-l}{2}\right)(-q) + \left(-\frac{l}{2}\right)^2 q = ql^2 \\ Q_{ij} = 0 \end{cases}$$

$$\begin{cases} p_z = \frac{l}{2}(q-q) + \left(-\frac{l}{2}\right)(q-q) = 0 \\ p_y = \frac{l}{2}(q-q) + \left(-\frac{l}{2}\right)(q-q) = 0 \\ p_x = 0 \end{cases}$$

$$\begin{cases} O_{zzz} = 0 \\ O_{iij} = 0 \\ O_{ijk} = 0 \end{cases}$$

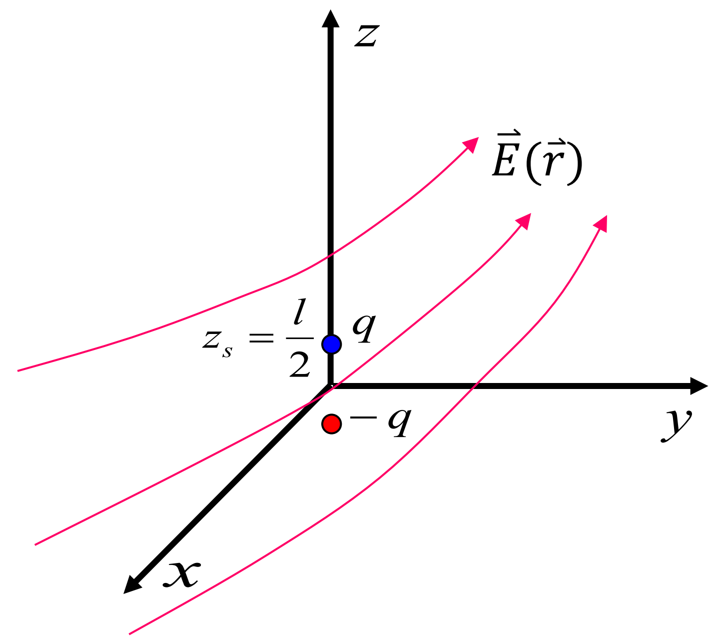


In the limit that the charge separation distance, l, shrinks to zero while maintaining the product of **ql^2=constant**, we have the definition of a **POINT QUADRUPOLE**. All other multipole terms are zero for this charge distribution.

Force and Torque on Multipoles

Now that we have defined certain multipole distributions, we can begin to discuss the forces and torque on these multipoles. Let us begin with the dipole interaction.

Let us assume that there is an external field that is non-uniform like that shown on the right. A dipole oriented along the z-direction is present in this field. The potential energy is given by:



$$U = q_+\varphi_+ + q_-\varphi_- = q\varphi\left(\frac{l}{2}\hat{z}\right) + (-q)\varphi\left(-\frac{l}{2}\hat{z}\right)$$

$$U = ql\,\frac{\varphi\left(\frac{l}{2}\hat{z}\right) - \varphi\left(-\frac{l}{2}\hat{z}\right)}{l} = -p_z E_z = -\vec{p}\cdot\vec{E}$$

$\boxed{\begin{array}{l} ql = p_z \\ dV/dl = -E \end{array}}$

$\boxed{\text{E and increasing } \varphi \text{ are in opposite direction}}$

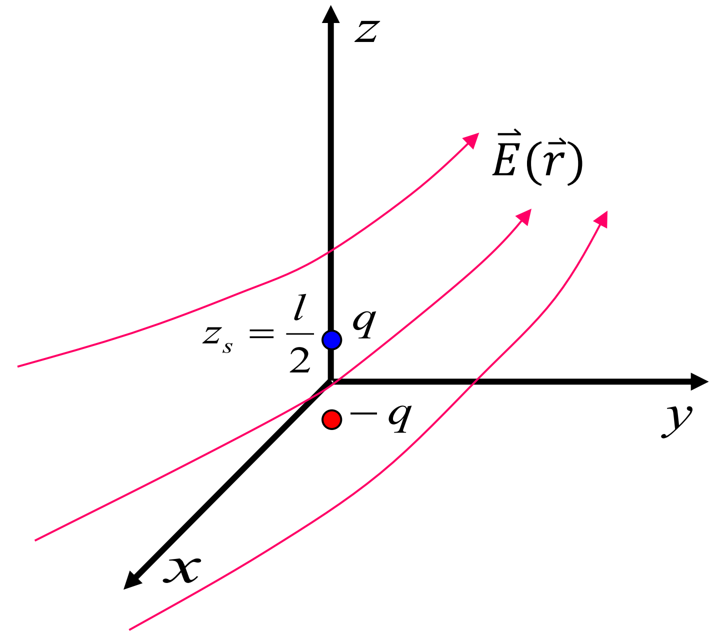
Force on a Point Dipole

From this we can determine the force:

$$F = -\nabla U = \nabla\left(\vec{p}\cdot\vec{E}\right) = \left\{ \vec{p}\times\Big(\underbrace{\nabla\times\vec{E}}_{=0}\Big) + \vec{E}\times\Big(\underbrace{\nabla\times\vec{p}}_{=0}\Big) + \left(\vec{p}\cdot\nabla\right)\vec{E} + \underbrace{(\vec{E}\cdot\nabla)\vec{p}}_{=0} \right\}$$

$$F = \left(\vec{p}\ \nabla\right)\vec{E}$$

We see that it is the **gradient of the field** that leads to a force on a point dipole. In other words, it is not the strength of the field, but how quickly **the field changes in space that leads to a force on a point dipole**. A dipole in a uniform electric field feels no linear force.

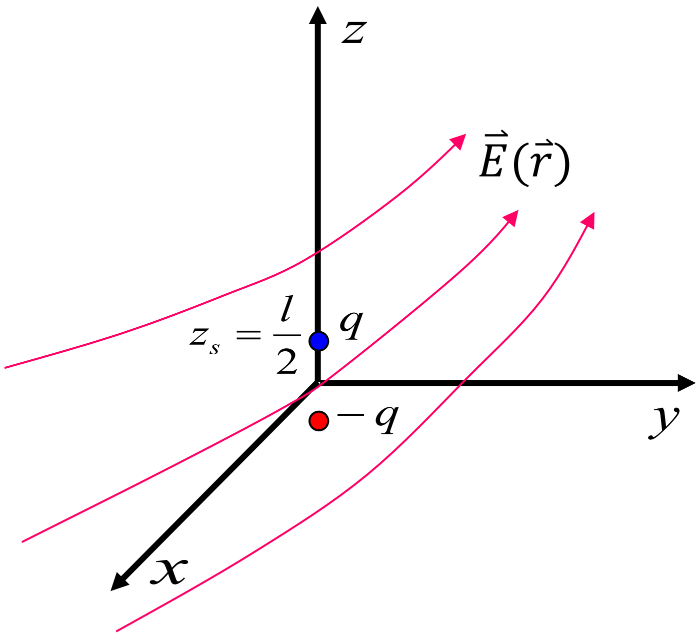


Torque on a Point Dipole

From the potential energy expression we can also determine the torque:

$$\vec{T} = -\hat{z}\frac{\partial}{\partial\theta}U = \hat{z}\frac{\partial}{\partial\theta}\left(pE\cos\theta\right) = \hat{z}pE\sin\theta = \vec{p}\times\vec{E}$$

We see that an external field acts to **align the dipole along the field direction**. When the dipole is parallel to the external field, the torque is zero. The torque is maximal when the dipole is perpendicular to the field.



Interaction between a Charge and Dipole

We can calculate this interaction in two ways:

1) The force is equal to the field of the **dipole** at the location of the **charge**.

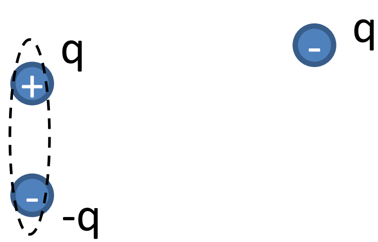
2) The force is equal to the field gradient of the **charge** at the location of the **dipole**.

$$\boxed{F = \left(\vec{p}\cdot\nabla\right)\vec{E}}$$

Let us show that these produce the same result.

$$\overrightarrow{F_q} = q\overrightarrow{E_{dip}} = \frac{q}{4\pi\varepsilon_0 r^3}\left[3\left(\vec{p}\cdot\hat{r}\right)\hat{r} - \vec{p}\right]$$

$$\overrightarrow{F_{dip}} = \left(\vec{p}\cdot\nabla\right)\overrightarrow{E_q} = -\nabla\left(\vec{p}\cdot\overrightarrow{E_q}\right) = -\nabla\left(\frac{q}{4\pi\varepsilon_0 r^2}\vec{p}\cdot\hat{r}\right) = \frac{q}{4\pi\varepsilon_0 r^3}\left[3\left(\vec{p}\cdot\hat{r}\right)\hat{r} - \vec{p}\right]$$



Torque on a Dipole due to a Point Charge

Assume a dipole is located at a position *r* away from the origin and pointing along the polar axis (z-direction), and a charge is located at the origin:

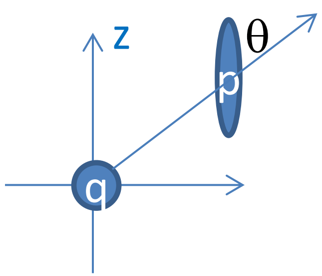
$$\vec{T} = \vec{p}\times\vec{E} = \vec{p}\times\frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{qp}{4\pi\varepsilon_0 r^2}\hat{z}\times\hat{r} = \frac{qp\sin(\theta)}{4\pi\varepsilon_0 r^2}\hat{\phi}$$

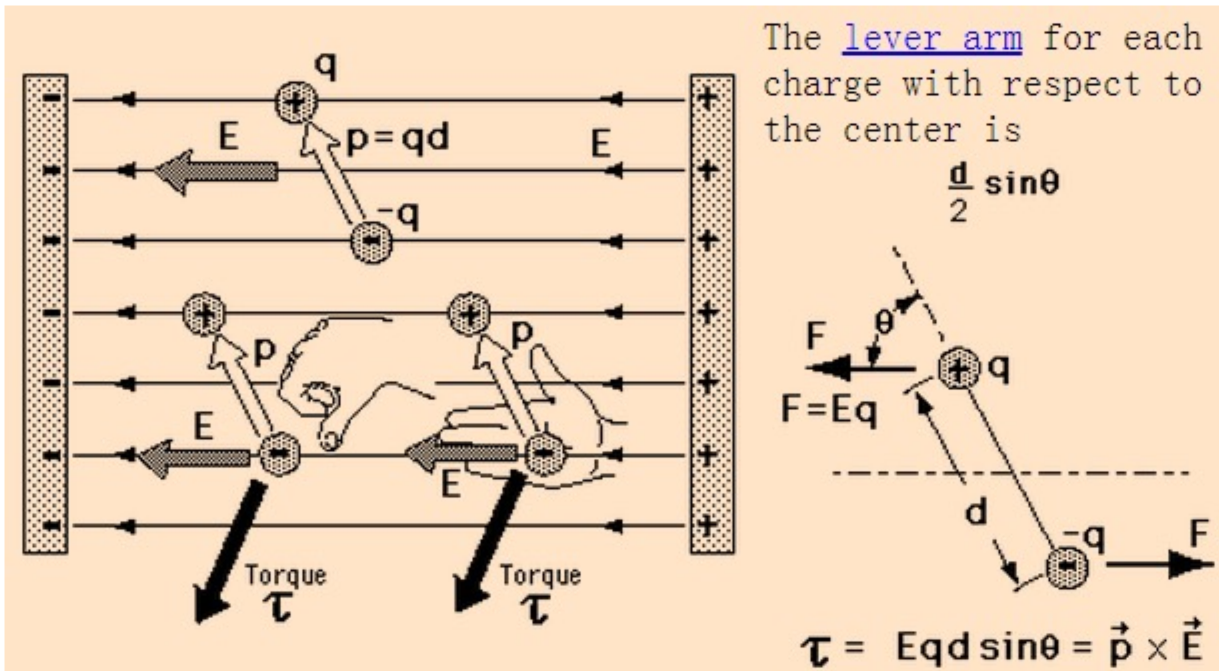
Field due to the charge

Thus, the dipole rotates until it is pointing in the direction of the position vector connecting the dipole and charge.

θ=0

These forces and torques between dipoles and charges play an important role in solid materials and in fluids.




The lever arm for each charge with respect to the center is
$\frac{d}{2}\sin\theta$
q
p=qd
E
E
-q
p
p
E
E
Torque
τ
Torque
τ
F
θ
q
F=Eq
d
-q
F
$\tau = Eqd\sin\theta = \vec{p} \times \vec{E}$