
Question1 (2 points)

(a) (1 point) Show that the square of an odd number is odd.

Solution:

1M Let $2n+1$ denote an odd number, where n is an integer.

$$(2n+1)^2 = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$$

1m The term $2n^2 + 2n$ is merely another integer, thus the square of odd number can be put in the form $2m+1$ for some integer m, hence it is odd.

(b) (1 point) Prove that $a_n = (3-2n)(-3)^n$ is the explicit formula for

$$a_0 = 3, \quad a_1 = -3, \quad a_n = -6a_{n-1} - 9a_{n-2} \qquad \text{for} \quad n \geq 2,$$

Solution:

1M This can be proved by the principle of mathematical induction.

The base step: The recursive formula involves two previous terms, we need to show that the explicit formula is true for both $n=0$ and $n=1$. When $n=0$ and $n=1$, the given explicit formula leads us to

$$a_0 = (3 - 2\cdot 0)(-3)^0 = 3 \qquad a_1 = (3 - 2\cdot 1)(-3)^1 = -3$$

which are the definitions of a_0 and a_1, so it is true for $n=0$ and $n=1$.

Now the inductive step: Assume the explicit formula

$$a_n = (3-2n)(-3)^n$$

is true for all $n \leq k$, where $n \in \mathbb{N}_0$ and $k \in \mathbb{N}_1$. Given this assumption, we need to show the explicit formula is also true for $n = k+1$. Applying the definition of a_{k+1}, we have

$$a_{k+1} = -6a_k - 9a_{k-1}$$

Using the assumption and collecting terms, we have

$$\begin{aligned} a_{k+1} &= -6(3-2k)(-3)^k - 9(3-2(k-1))(-3)^{k-1} \\ &= 2(3-2k)(-3)^{k+1} - (5-2k)(-3)^{k+1} \\ &= (6 - 4k - 5 + 2k)(-3)^{k+1} \\ &= (3 - 2(k+1))(-3)^{k+1} \end{aligned}$$

which is the same as if we use $n = k+1$ in the explicit formula, so the explicit formula is also true for $n = k+1$. Therefore the explicit formula is true for all $n \in \mathbb{N}_0$ by the principle of mathematical induction.

Question2 (1 points)
Give an example of two divergent sequences $\{a_n\}$ and $\{b_n\}$ such that $\{a_n+b_n\}$ is convergent.

Solution:

1M There are many examples, the following is just one of them,

$$\left\{n+\frac{1}{n}\right\}_{n=1}^{\infty} \qquad \left\{-n\right\}_{n=1}^{\infty} \qquad \left\{n+\frac{1}{n}-n\right\}_{n=1}^{\infty}=\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$$

Question3 (7 points)
Evaluate each of the following limits if it exits. If not, state whether it diverges ∞.

(a) (1 point)

$$\lim_{n\to\infty}\frac{1}{n^2}$$

Solution:

1M

$$\begin{aligned}
\lim_{n\to\infty}\frac{1}{n^2} &= \left(\lim_{n\to\infty}\frac{1}{n}\right)\cdot\left(\lim_{n\to\infty}\frac{1}{n}\right) && \text{Law 3}\\
&= 0\cdot 0 && \text{L2P12}\\
&= 0
\end{aligned}$$

Using Law 4 together with L2P12, we have the following

$$\lim_{n\to\infty}\frac{1}{n^k}=\left(\lim_{n\to\infty}\frac{1}{n}\right)\cdots\left(\lim_{n\to\infty}\frac{1}{n}\right)=0\cdot 0\cdots 0=0 \qquad \text{for any } k\in\mathbb{N}_1.$$

(b) (1 point)

$$\lim_{n\to\infty}\frac{n^2}{3n^2+2n+1}$$

Solution:

1M

$$\begin{aligned}
\lim_{n\to\infty}\frac{n^2}{3n^2+2n+1} &= \lim_{n\to\infty}\frac{1}{3+\frac{2}{n}+\frac{1}{n^2}}\\
&= \frac{\lim_{n\to\infty}1}{\lim_{n\to\infty}\left(3+\frac{2}{n}+\frac{1}{n^2}\right)} && \text{Law 4}\\
&= \frac{1}{3+\lim_{n\to\infty}\frac{2}{n}+\lim_{n\to\infty}\frac{1}{n^2}} && \text{Law 1, Law 2}\\
&= \frac{1}{3+2\cdot 0+0} && \text{Law 1, Law 3, part (a)}\\
&= \frac{1}{3}
\end{aligned}$$

Using Law 1, Law 3 and part (a), we have the following

$$\lim_{n\to\infty}\frac{a}{n^k}=\left(\lim_{n\to\infty}a\right)\left(\lim_{n\to\infty}\frac{1}{n^k}\right)=a\cdot 0=0\qquad\text{for any } a\in\mathbb{R},\ k\in\mathbb{N}_1.$$

(c) (1 point)

$$\lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots+n^2}{n^3+2}$$

Solution:

1M

$$\begin{aligned}
\lim_{n\to\infty}\frac{1^2+2^2+3^2+\cdots+n^2}{n^3+2}&=\lim_{n\to\infty}\frac{\frac{n(n+1)(2n+1)}{6}}{n^3+2}\\
&=\lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3+12}\\
&=\lim_{n\to\infty}\frac{2+\frac{3}{n}+\frac{1}{n^2}}{6+\frac{12}{n^3}}\\
&=\frac{\lim_{n\to\infty}\left(2+\frac{3}{n}+\frac{1}{n^2}\right)}{\lim_{n\to\infty}\left(6+\frac{12}{n^3}\right)} && \text{Law 4}\\
&=\frac{2+\lim_{n\to\infty}\frac{3}{n}+\lim_{n\to\infty}\frac{1}{n^2}}{6+\lim_{n\to\infty}\frac{12}{n^3}} && \text{Law 1, Law 2}\\
&=\frac{2+0+0}{6+0} && \text{part (b)}\\
&=\frac{1}{3}
\end{aligned}$$

(d) (1 point)

$$\lim_{n\to\infty}\frac{4^n}{n^n}$$

Solution:

1M Recall we have shown the following in class, L2P18

$$\lim_{n\to\infty}\frac{4^n}{n!}=0$$

and it is clear that

$$0\le\frac{4^n}{n^n}\le\frac{4^n}{n!}\qquad\text{for all}\quad n\in\mathbb{N}_1$$

Invoking the squeeze theorem, we have

$$\lim_{n\to\infty}\frac{4^n}{n^n}=0$$

(e) (1 point)

$$\lim_{n\to\infty}\left(\frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\cdots+\frac{(-1)^{n-1}n}{n}\right)$$

Solution:

1M Consider the sequence

$$a_n=\frac{1}{n}-\frac{2}{n}+\frac{3}{n}-\cdots+\frac{(-1)^{n-1}n}{n}$$

for even n, that is, $n=2k$, where $k\in\mathbb{N}_1$,

$$a_{2k}=\frac{1}{2k}-\frac{2}{2k}+\frac{3}{2k}-\cdots-\frac{2k}{2k}=\underbrace{-\frac{1}{2k}-\frac{1}{2k}-\cdots-\frac{1}{2k}}_{k\text{ copies}}=-\frac{1}{2}$$

Now consider a_n for odd n, that is, $n=2k-1$, where $k\in\mathbb{N}_1$,

$$\begin{aligned}a_{2k-1}&=\frac{1}{2k-1}-\frac{2}{2k-1}+\frac{3}{2k-1}-\cdots+\frac{2k-1}{2k-1}\\&=\frac{1}{2k-1}+\underbrace{\frac{1}{2k-1}+\cdots+\frac{1}{2k-1}}_{k-1\text{ copies}}\\&=\frac{k}{2k-1}\end{aligned}$$

Notice as $k\to\infty$, $n\to\infty$, and

$$a_{2k}\to-\frac{1}{2}\qquad\text{while}\qquad a_{2k-1}\to\frac{1}{2}$$

which means the sequence a_n oscillates between negative and positive values, and the gap between positive and negative values never goes aways as $n\to\infty$. Therefore, the sequence is divergent and the original limit does not exist.

(f) (1 point)

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n$$

Solution:

1M Since the limit corresponds to a geometric sequence with common ratio of $1/2$, it is clearly monotonically decreasing and bounded. So MST implies the sequence is convergent and the limit exist. Thus we can let

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^n=L$$

For any $n \in \mathbb{N}_1$, we have

$$\left(\frac{1}{2}\right)^{n+1} = \frac{1}{2}\left(\frac{1}{2}\right)^n$$

Using Law 1 and Law 3, we have

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^{n+1} = \frac{1}{2}L$$

However, consider $k = n+1$, which clearly approaches ∞ as $n \to \infty$, thus

$$\lim_{n\to\infty}\left(\frac{1}{2}\right)^{n+1} = \lim_{k\to\infty}\left(\frac{1}{2}\right)^k = L$$

and

$$L = \frac{1}{2}L \implies L = 0 \implies \lim_{n\to\infty}\left(\frac{1}{2}\right)^n = 0$$

(g) (1 point)

$$\lim_{n\to\infty} 2^{1/n}$$

Solution:

1M Consider the corresponding sequence,

$$\{2^{1/n}\}$$

for $n \in \mathbb{N}_1$, we conclude it is monotonically decreasing

$$\frac{1}{n+1} < \frac{1}{n} \implies 2^{1/(n+1)} < 2^{1/n}$$

and bounded

$$0 < \frac{1}{n} \implies \log_2 1 < \frac{1}{n}\log_2 2 \implies \log_2 1 < \log_2 2^{1/n} \implies 1 < 2^{1/n}$$

Therefore, it is convergent by MST, and we can let

$$\lim_{n\to\infty} 2^{1/n} = L$$

Being monotonically decreasing and bounded below by 1 leads us to

$$1 \le L \le 2^{1/n} \implies 1 \le L^n \le 2 \implies L = 1$$

since the inequalities which would lead us to a contradiction if $L \neq 1$.

$$\lim_{n\to\infty} 2^{1/n} = 1$$

Question4 (1 points)

Use the precise definition of limit to show

$$\lim_{n\to\infty} \frac{\sqrt{n^2+4}}{n} = 1$$

Solution:

1M We need to show for every $\epsilon > 0$, there exists $N \in \mathbb{N}_1$ such that

$$\left| \frac{\sqrt{n^2+4}}{n} - 1 \right| < \epsilon \qquad \text{when} \qquad n > N$$

For an arbitrary $\epsilon > 0$, notice the inequality

$$\left| \frac{\sqrt{n^2+4}}{n} - 1 \right| < \epsilon$$

holds if and only if

$$\left| \frac{\sqrt{n^2+4}-n}{n} \right| < \epsilon$$

which in turn holds if

$$\frac{2}{n} < \epsilon$$

since

$$\begin{aligned} n^2+4 < (n+2)^2 &\implies \sqrt{n^2+4} < n+2 \\ &\implies \left| \frac{\sqrt{n^2+4}-n}{n} \right| < \left| \frac{n+2-n}{n} \right| < \left| \frac{2}{n} \right| = \frac{2}{n} \end{aligned}$$

For any arbitrary $\epsilon > 0$, it is clear that there exists N such that

$$N > \frac{2}{\epsilon} \implies \frac{2}{n} < \epsilon \qquad \text{for all} \quad n > N$$

Therefore, the limit is indeed one.

Question5 (1 points)

Consider the sequence $\{x_n\}$, where

$$x_1 = a, \quad x_2 = b, \quad x_n = \frac{x_{n-1} + x_{n-2}}{2} \qquad \text{for} \qquad n = 3, 4, \ldots$$

Determine whether $\{x_n\}$ is convergent. If so, find $\lim\limits_{n\to\infty} x_n$. If not, prove why not.

Solution:

1M It is better to work out the explicit formula for this question,

$$\begin{aligned} x_n &= x_1 - x_1 + x_2 - x_2 + x_3 - \cdots - x_{n-1} + x_n \\ &= x_1 + (x_2 - x_1) + (x_3 - x_2) + \cdots + (x_n - x_{n-1}) \end{aligned}$$

if we consider the difference between two consecutive terms using the recurrence

$$x_n - x_{n-1} = \frac{x_{n-1} + x_{n-2}}{2} - x_{n-1} = -\frac{x_{n-1} - x_{n-2}}{2}$$

thus the difference between the difference between two consecutive terms is always a factor of $-\frac{1}{2}$. Hence the explicit formula in terms of a and b is

$$x_n = a + (b-a) - \frac{b-a}{2} + \frac{b-a}{4} - \cdots + (-1)^n \frac{b-a}{2^{n-2}} = a + \frac{2(b-a)}{3} + \frac{b-a}{3}\frac{(-1)^n}{2^{n-2}}$$

therefore the limit is

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty}\left(a + \frac{2(b-a)}{3} + \frac{b-a}{3}\frac{(-1)^n}{2^{n-2}}\right) = \frac{a+2b}{3}$$

Question6 (1 points)

If the sequence x_n is bounded and $\lim_{n\to\infty} y_n = 0$, show that $\lim_{n\to\infty} x_n y_n = 0$.

Solution:

1M Because the sequence $\{x_n\}$ is bounded, there is a number $M > 0$ so that,

$$|x_n| \le M \qquad \text{for any} \qquad n \in \mathbb{N}$$

Since $y_n \to 0$ as $n \to \infty$, for any $\epsilon_1 > 0$, there is an integer N such that

$$|y_n - 0| < \epsilon_1 \qquad \text{when} \qquad n > N$$

thus for any $\epsilon_1 > 0$, there is an integer N such that, for any $n > N$,

$$|x_n| \cdot |y_n| < M\epsilon_1 \implies |x_n y_n - 0| < M\epsilon_1$$

Hence for any $\epsilon = M\epsilon_1 > 0$, there is an integer N such that,

$$|x_n y_n - 0| < M\epsilon_1 \qquad \text{when} \qquad n > N$$

Therefore the limit by definition is

$$\lim_{n\to\infty} x_n y_n = 0$$

Question7 (1 points)

Determine whether $a_n = \cos(n\pi)$ is convergent. Justify your answer.

Solution:

1M Intuitively, the limit $\lim_{n\to\infty} \cos n\pi$ clearly doesn't exist. Let us make a more concrete argument by contradiction. Suppose that the limit exists and takes value L

$$\lim_{n\to\infty} \cos n\pi = L$$

Let us consider $m = n + 1$, then the sequence $\{\cos m\pi\}$ is essentially the same sequence without the very first term of $\{a_n\}$, that is,

$$\{a_n\}_{n=1}^{\infty} = \{a_m\}_{m=2}^{\infty}$$

Hence they must have the same limit

$$\lim_{m\to\infty} \cos(m\pi) = \lim_{n\to\infty} \cos(n\pi) = L$$

However, the following trigonometric identity

$$\cos m\pi = \cos\Big((n+1)\pi\Big) = -\cos n\pi$$

reveals the following value for the limit of $\{a_m\}$ by properties of limits,

$$\lim_{m\to\infty} \cos(m\pi) = \lim_{m\to\infty} -\cos(n\pi) = -L$$

from which we are forced to conclude that

$$L = 0$$

If there is any $\epsilon > 0$ such that NO $N \in \mathbb{N}$ exists for the following to be true

$$|a_n - 0| < \epsilon \qquad \text{when} \qquad n > N$$

then $L = 0$ is NOT the limit. Now if we consider $\epsilon = \frac{1}{2}$, then clearly the inequality

$$|\cos(n\pi)| < \frac{1}{2}$$

is never satisfied by any natural number n since cosine is either 1 or -1 for any integer copies of π. So indeed no such N exists, and we can conclude $L \neq 0$. Therefore we reach a contradiction regarding the limit of $\{a_n\}$ if we assume the limit exists, which forces us to conclude that the limit does not exist and $\{a_n\}$ is divergent.

Question8 (1 points)
Show the following limit is equal to zero.

$$\lim_{n\to\infty} \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}$$

Solution:

1M Consider the sequence corresponding to the limit

$$a_n = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}$$

and a related but distinct sequence

$$b_n = \frac{2}{3} \cdot \frac{4}{5} \cdots \frac{2n}{2n+1}$$

Suppose

$$\frac{2n-1}{2n} \geq \frac{2n}{2n+1} \implies 4n^2 - 1 \geq 4n^2$$

which means the following must be true in order to avoid a contradiction

$$\frac{2n-1}{2n} < \frac{2n}{2n+1} \implies a_n < b_n$$

It is clear both sequences are monotonically decreasing and bounded, thus MST says both must be convergent. Let us denote

$$\lim_{n\to\infty} a_n = L_a \qquad \text{and} \qquad \lim_{n\to\infty} b_n = L_b$$

Consider a third sequence

$$c_n = a_n b_n = \frac{1}{2n+1}$$

which is clearly convergent,

$$\lim_{n\to\infty} c_n = 0$$

Hence, we have

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = 0$$

which means either L_a or L_b must be zero. If L_b is zero, then

$$L_a = 0$$

since $a_n < b_n$ and a_n is bounded below by zero. If $L_b \neq 0$, then L_a must be zero. So

$$\lim_{n\to\infty} \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n} = 0$$

Question9 (1 points)

Consider consider two sequences $\{x_n\}$ and $\{y_n\}$, where $\{y_n\}$ is monotonically increasing and diverges to positive infinity. Given the following limit exits,

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}$$

show the following equality holds

$$\lim_{n\to\infty}\frac{x_n}{y_n}=\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}$$

Solution:

1M Since the following limit exists, let us denote it by L,

$$\lim_{n\to\infty}\frac{x_{n+1}-x_n}{y_{n+1}-y_n}=L$$

By definition, there must exist $N\in\mathbb{N}$ such that

$$\left|\frac{x_{n+1}-x_n}{y_{n+1}-y_n}-L\right|<\frac{\epsilon}{2}\qquad\text{where}\quad\epsilon>0$$

which means the followings are in $\left(L-\frac{\epsilon}{2},L+\frac{\epsilon}{2}\right)$ for $n>N$,

$$\frac{x_{N+2}-x_{N+1}}{y_{N+2}-y_{N+1}},\ \frac{x_{N+3}-x_{N+2}}{y_{N+3}-y_{N+2}},\cdots\ \frac{x_n-x_{n-1}}{y_n-y_{n-1}},\ \frac{x_{n+1}-x_n}{y_{n+1}-y_n}$$

Since $\{y_n\}$ is monotonically decreasing, all of the followings are positive

$$\left(y_{N+2}-y_{N+1}\right),\quad\left(y_{N+3}-y_{N+2}\right),\quad\cdots,\quad\left(y_n-y_{n-1}\right),\quad\left(y_{n+1}-y_n\right),$$

thus we have following inequalities

$$\begin{gathered}\left(L-\frac{\epsilon}{2}\right)\left(y_{N+2}-y_{N+1}\right)<x_{N+2}-x_{N+1}<\left(L+\frac{\epsilon}{2}\right)\left(y_{N+2}-y_{N+1}\right)\\ \left(L-\frac{\epsilon}{2}\right)\left(y_{N+3}-y_{N+2}\right)<x_{N+3}-x_{N+2}<\left(L+\frac{\epsilon}{2}\right)\left(y_{N+3}-y_{N+2}\right)\\ \vdots\\ \left(L-\frac{\epsilon}{2}\right)\left(y_{n+1}-y_n\right)<x_{n+1}-x_n<\left(L+\frac{\epsilon}{2}\right)\left(y_{n+1}-y_n\right)\end{gathered}$$

summing which, we have the following bound for the difference quotient

$$\begin{aligned}&\left(L-\frac{\epsilon}{2}\right)\left(y_{n+1}-y_{N+1}\right)<x_{n+1}-x_{N+1}<\left(L+\frac{\epsilon}{2}\right)\left(y_{n+1}-y_{N+1}\right)\\ \Longrightarrow\ &\left|\frac{x_{n+1}-x_{N+1}}{y_{n+1}-y_{N+1}}-L\right|<\frac{\epsilon}{2}\qquad\text{for}\quad n>N\end{aligned}$$

For $\epsilon>0$, we need to show, there exists $N^*\in\mathbb{N}_1$ such that

$$\left|\frac{x_n}{y_n}-L\right|<\epsilon\qquad\text{when}\quad n>N^*$$

Direction computation reveals that the following decomposition holds,

$$\frac{x_n}{y_n} - L = \frac{x_{N+1} - Ly_{N+1}}{y_n} + \left(1 - \frac{y_{N+1}}{y_n}\right)\left(\frac{x_{n+1} - x_{N+1}}{y_{n+1} - y_{N+1}} - L\right)$$

Taking absolute values and applying the triangle inequality, we have

$$\left|\frac{x_n}{y_n} - L\right| = \left|\frac{x_{N+1} - Ly_{N+1}}{y_n} + \left(1 - \frac{y_{N+1}}{y_n}\right)\left(\frac{x_{n+1} - x_{N+1}}{y_{n+1} - y_{N+1}} - L\right)\right|$$
$$\left|\frac{x_n}{y_n} - L\right| \leq \left|\frac{x_{N+1} - Ly_{N+1}}{y_n}\right| + \left|\left(1 - \frac{y_{N+1}}{y_n}\right)\left(\frac{x_{n+1} - x_{N+1}}{y_{n+1} - y_{N+1}} - L\right)\right|$$

Using the bound we derived for the difference quotient, we have

$$\left|\frac{x_n}{y_n} - L\right| < \left|\frac{x_{N+1} - Ly_{N+1}}{y_n}\right| + \left|1 - \frac{y_{N+1}}{y_n}\right|\frac{\epsilon}{2} \qquad \text{for} \quad n > N$$

Since $\{y_n\}$ is monotonically increasing and diverges to positive infinity, we have

$$0 \leq \left|1 - \frac{y_{N+1}}{y_n}\right| < 1 \qquad \text{for sufficiently big } N.$$

and there exists $N^* > N$ such that y_n, for $n > N^*$, is sufficiently big such that

$$\left|\frac{x_{N+1} - Ly_{N+1}}{y_n}\right| < \frac{\epsilon}{2}$$

combining those results, we have

$$\left|\frac{x_n}{y_n} - L\right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \qquad \text{for} \quad n > N^*$$

which lead us to conclude

$$\lim_{n\to\infty}\frac{x_n}{y_n} = L = \lim_{n\to\infty}\frac{x_{n+1} - x_n}{y_{n+1} - y_n}$$

Question10 (3 points)
Consider the sequence

$$\begin{aligned} a_1 &= \sqrt{6} \\ a_2 &= \sqrt{6 + \sqrt{6}} \\ a_3 &= \sqrt{6 + \sqrt{6 + \sqrt{6}}} \\ a_4 &= \sqrt{6 + \sqrt{6 + \sqrt{6 + \sqrt{6}}}} \\ &\vdots \end{aligned}$$

(a) (1 point) Find a recursion formula for the sequence.

Solution:

1M It is that
$$a_1 = \sqrt{6}, \qquad a_n = \sqrt{6 + a_{n-1}}$$

(b) (1 point) Show this sequence converges.

Solution:

1M By the monotonic sequence theorem, all need to be shown is the sequence is monotonic and bounded. Use induction to show it is increasing, firstly the base step
$$a_2 - a_1 = \sqrt{6 + \sqrt{6}} - \sqrt{6} > 0$$
then the inductive step, let us assume that $a_n > a_{n-1}$ for $n = k$, then
$$a_{k+1} - a_k = \sqrt{6 + a_k} - \sqrt{6 + a_{k-1}} \implies a_{k+1} - a_k > 0$$
Thus it is monotonically increasing by the principle of mathematical induction. And $\sqrt{6}$ follows immediately being a lower bound for this sequence. Now use induction to show it is bounded above, firstly the base step,
$$\sqrt{6} < \sqrt{9} = 3$$
then the inductive step, let us assume that $a_n < 3$ for $n = k$, then
$$a_k + 6 < 9 \implies \sqrt{a_k + 6} < \sqrt{9} \implies a_{k+1} < 3$$
So it is bounded above by the principle of mathematical induction. Thus it is bounded as well as being monotonic, and is convergent by the monotonic sequence theorem.

(c) (1 point) Find the limit of this sequence.

Solution:

1M To work out the value of limit we need the following theorem,

If $\{a_n\}$ converges to L with $a_n \geq 0$, then $\{\sqrt{a_n}\}$ converges to $\sqrt{L}$.

Let us see why the theorem is true. There are two possibilities since $a_n \geq 0$.
$$L = 0 \qquad \text{and} \qquad L > 0$$
First suppose $L = 0$, then for every $\epsilon^* > 0$, there exists one N such that
$$|a_n - 0| < \epsilon^* \qquad \text{when} \qquad n > N$$
which implies for every $\epsilon^* > 0$, there exists one N such that
$$\sqrt{a_n} < \sqrt{\epsilon^*} \qquad \text{when} \qquad n > N$$

thus for every $\epsilon = \sqrt{\epsilon^*} > 0$, there exist one N such that

$$\left|\sqrt{a_n} - \sqrt{0}\right| < \epsilon \qquad \text{when} \qquad n > N$$

This shows the result is true for

$$L = 0$$

Now suppose $L > 0$, then for every $\epsilon^* > 0$, there exists one N such that

$$|a_n - L| < \epsilon^* \qquad \text{when} \qquad n > N$$

which implies every $\epsilon^* > 0$, there exists one N such that

$$\frac{|a_n - L|}{\sqrt{L}} < \frac{\epsilon^*}{\sqrt{L}} \qquad \text{when} \qquad n > N$$

Therefore the following inequality

$$\left|\sqrt{a_n} - \sqrt{L}\right| = \frac{|a_n - L|}{\sqrt{a_n} + \sqrt{L}} < \frac{|a_n - L|}{\sqrt{L}}$$

implies for every $\epsilon = \frac{\epsilon^*}{\sqrt{L}} > 0$, there exists one N such that

$$\left|\sqrt{a_n} - \sqrt{L}\right| < \epsilon \qquad \text{when} \qquad n > N$$

This shows the theorem is true. Applying it with some basic properties of limits,

$$\lim_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + L}$$

However, the sequence $\left\{\sqrt{6 + a_n}\right\}$ is essentially $\{a_n\}$ without the first term, hence surely converges to the same L, thus we have an equation of L

$$\sqrt{6 + L} = L \implies L^2 = 6 + L \implies L = 3$$

since L is positive.

Question11 (1 points)
Suppose $\{a_n\}$ is monotonic and bounded. Use the definition of the supremum and the infimum to argue $\{a_n\}$ must be convergent by the definition of convergence.

Solution:

1M Since $\{a_n\}$ is bounded, then both the supremum and the infimum of the set $\mathcal{S}$ of all terms of $\{a_n\}$ must exist. First let us suppose $\{a_n\}$ is increasing, and let $L = \sup(\mathcal{S})$. If we can show this L is the limit of the $\{a_n\}$, then $\{a_n\}$ is convergent by the definition of convergence.

Since we have assumed $\{a_n\}$ is increasing and L is the supremum. For any $\epsilon > 0$, there exists a natural number N such that

$$L - \epsilon < a_n \qquad \text{for all } n > N.$$

which means the following must be true for any $\epsilon > 0$,

$$L - a_n < \epsilon \qquad \text{for all } n > N. \tag{1}$$

By the definition of L,

$$a_n \leq L \qquad \text{for all } n.$$

Hence, for any $\epsilon > 0$,

$$a_n < L + \epsilon \qquad \text{for all } n.$$

which means the following must be true for any $\epsilon > 0$,

$$-\epsilon < L - a_n \qquad \text{for all } n > N. \tag{2}$$

If we combine (1) and (2), we conclude the following statement is true for any $\epsilon > 0$,

$$|a_n - L| < \epsilon \qquad \text{when} \qquad n > N$$