vv255: Introduction: coordinate systems and vectors. Surfaces in 3D. The dot product and the cross product. Lines and planes.

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Today 05-13-2013

1. 3D space.
2. Distance.
3. Surfaces: planes, cylinders, quadric surfaces. Cross sections.
4. Vectors: binary operations, coordinates

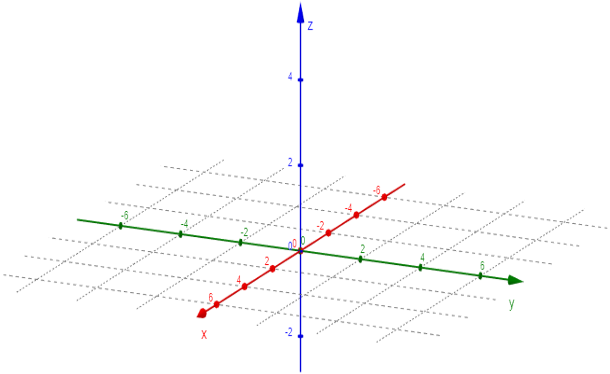
3D space

- We plot points (x, y) in an xy-plane. This is 2D space. For x, y real numbers we write $\mathbb{R}^2$ for the space.
- We plot (x, y, z) in an xyz-coordinate space. This is 3D space. For x, y, z real numbers we write $\mathbb{R}^3$ for the space.
- In 3D space: coordinate axes meet at the origin $O(0, 0, 0)$. When sketching, place axis labels at the positive end of each axis.
- Axes are right-handed. Looking down the positive z-axis gives the standard view of the xy-plane.

3D Coordinate Systems

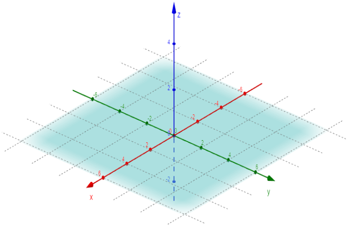
The **right-hand rule**:

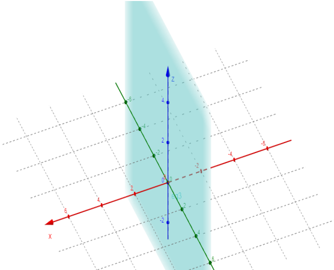
- the x-axis, y-axis and z-axis intersect at O
- the x-axis, y-axis and z-axis are pairwise perpendicular
- if you curl the fingers of your right hand around the z-axis in the direction of a 90 degree anticlockwise rotation from the positive x-axis to the positive y-axis, then your thumb points in the positive direction of the z-axis

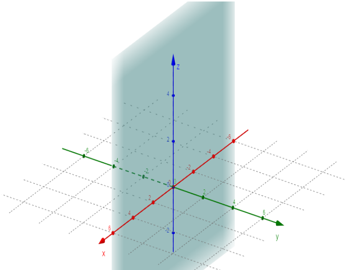


xy, yz, xz planes

The x, y and z axes determine three planes: xy-plane, yz-plane and xz-plane.

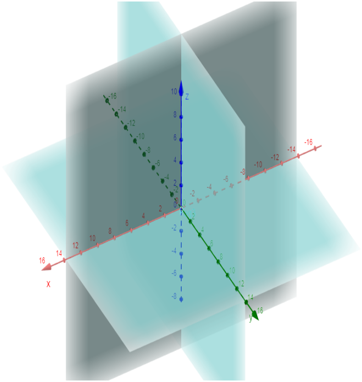






xy, yz, xz planes

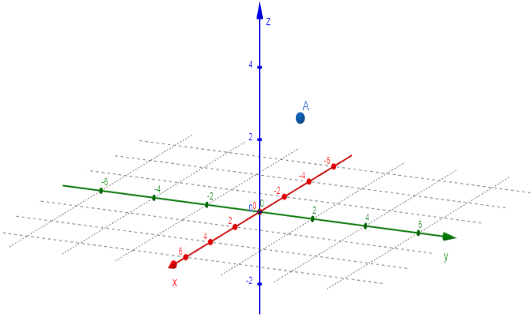
These three planes partition space into 8 regions called **octants**.



The octant bounded by the positive x-, y- and z-axis is called the **first octant**.

Coordinates

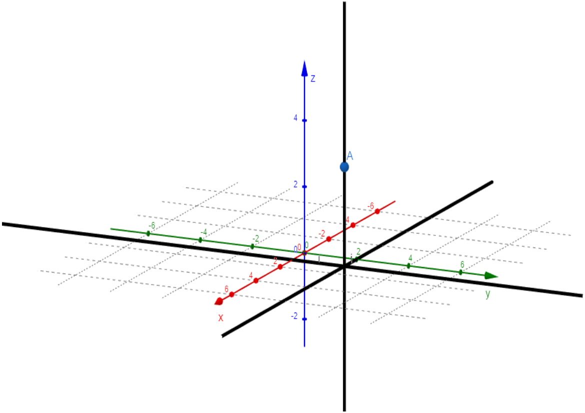
Let A be a point in 3D space.



Then A can be *uniquely* specified by the 3D rectangular coordinates (a, b, c) where a, called the x-coordinate, is the directed distance from the yz-plane to A, b, called the y-coordinate, is the directed distance from the xz-plane to A, and c, called the z-coordinate, is the directed distance from the xy-plane to A.

Coordinates

Let $A(1, 2, 3)$



This gives a one-to-one correspondence between 3D space and the set

$$\mathbb{R}^3 = \{(x, y, z) \mid x, y, z \in \mathbb{R}\}$$

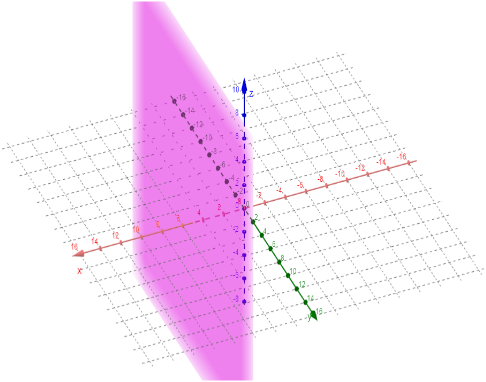
and we will often just refer to 3D space as $\mathbb{R}^3$

Surfaces

Recall, that equations in the variables x and y represent curves in 2D. Similarly, equations in the variables x, y and z represent surfaces in $\mathbb{R}^3$.

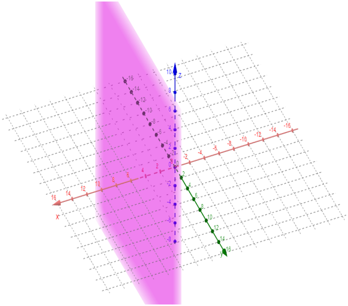
Q: What is the surface described by $x = 5$?

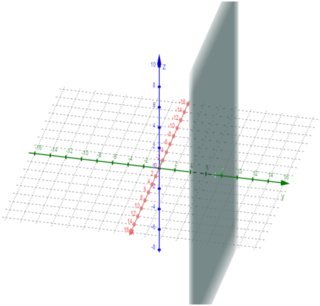
A: In $\mathbb{R}^2$, $x = 5$ is a line. In $\mathbb{R}^3$, the equation $x = 5$ describes all points $(5, y, z) \Rightarrow$ it is the plane.

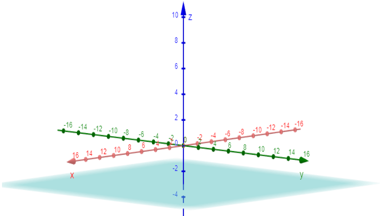


Surfaces

The surfaces described by the equations $x = k$, $y = k$ and $z = k$ are the planes parallel to the yz-plane, xz-plane and xy-plane respectively.
Example below: $x = 5$, $y = 7$, $z = -3$.

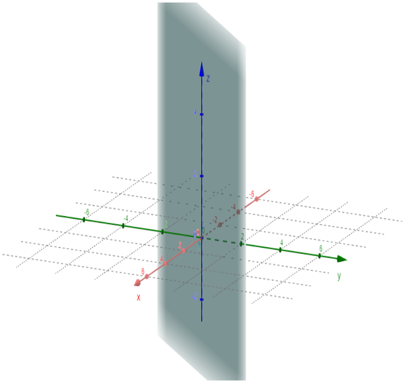






Surfaces

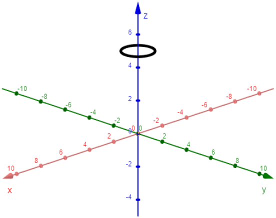
The surface $y = x$ is the plane passing through the line $y = x$.



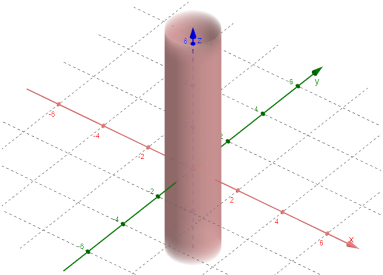
Surfaces

Q: What is the surface described by $x^2 + y^2 = 1$, $z = 5$?

A: The circle lying in the plane $z = 5$



Q: What is the surface described by $x^2 + y^2 = 1$? A: The cylinder below.



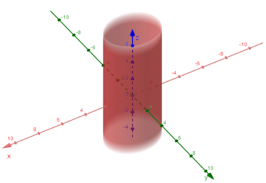
Surfaces

Q: So, what is a cylinder?

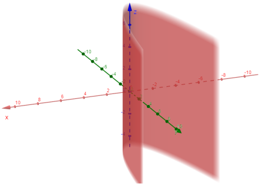
A: A cylinder is a surface that is constructed of a set of parallel lines all passing through a curve. In the example above, the curve is the circle.

Q: What happens if the curve is an ellipse, a parabola or a hyperbola?

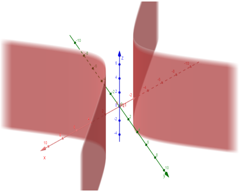
A: Passing parallel lines through an ellipse leads to an elliptical cylinder while passing parallel lines through a parabola leads to a parabolic cylinder and passing parallel lines through a hyperbola leads to a hyperbolic cylinder.



$2x^2 + 3y^2 = 10$



$y = x^2$



$2x^2 - 3y^2 = 10$

Quadric Surfaces

Definition: A quadric surface is the graph of a second-degree equation in three variables x, y, z. The most general such equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

with constant coefficients.

How to plot a quadric surface?

A: 1. Complete squares and obtain an equation of the form

$$A'x^2 + B'y^2 + C'z^2 + J' = 0.$$

2. Use cross-sections with planes parallel to $xy-$, $xz-$, $yz-$ planes to understand what curves (traces) a quadric surface make in intersection with those planes.

Quadric Surfaces: Example

Consider the quadric surface

$$x^2 + 2z^2 - 6x - y + 10 = 0$$

1. Complete the square

$$x^2 - 2 \cdot 3x + 9 - 9 + 2z^2 - y + 10 = 0 \Rightarrow (x-3)^2 + 2z^2 = y - 1$$

2. Find the shape of the intersection of the surface with each of $x = c, y = c, z = c, c = const$:

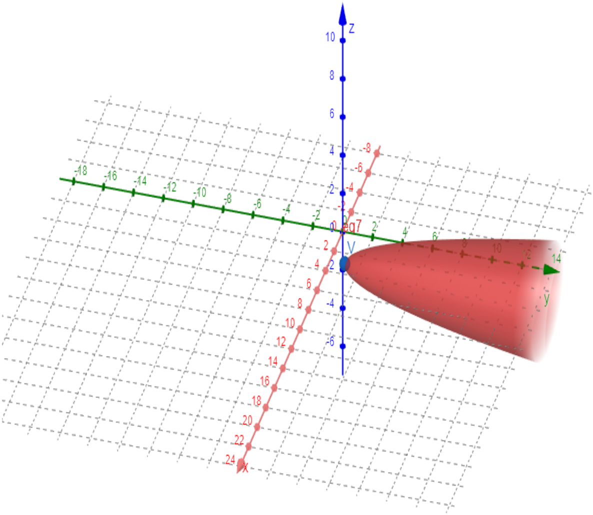
$$x = c\colon\ y = 2z^2 + ((c-3)^2 + 1) \Rightarrow \text{ a parabola in } yz-\text{plane}$$

$$y = c \neq 1\colon\ \frac{(x-3)^2}{y-1} + \frac{z^2}{(y-1)/2} = 1 \Rightarrow \text{ an ellipse in } xz-\text{plane}$$

$$y = 1 \Rightarrow \text{the point } (3, 1, 0)$$

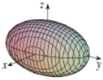
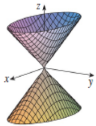
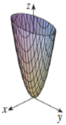
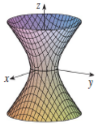
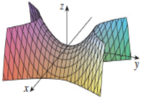
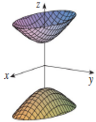
$$z = c\colon\ y = (x-3)^2 + (1 + 2c^2) \Rightarrow \text{ a parabola in } xy-\text{plane}$$

Quadric Surfaces: Example



an elliptic parabaloid

For Your Reference: Classification of Quadric Surfaces from J.Stewart

Surface	Equation	Surface	Equation
Ellipsoid 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone 	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet 	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid 	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Distance

Let $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ be two points.
Definition: The distance between P_1 and P_2, denoted $|P_1P_2|$ is the length of the line segment connecting P_1 and P_2.
Exercise: Show that

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Example

Let $P(x_0, y_0, z_0)$.Consider a sphere centered at P, i.e. the set of all points whose distance from P is r:

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

Regions

We can describe regions in 3D space using inequalities.

Example

- *The region described by*

$$1 \leq x^2 + y^2 + z^2 \leq 9$$

  is the region of points that are inside a sphere centred at O with radius 3 but not inside a phere centred at O with radius 1

- *The first octant is described by the inequalities*

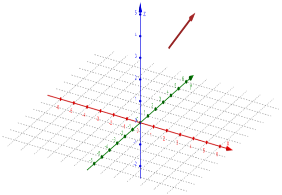
$$x \geq 0, y \geq 0 \ and \ z \geq 0$$

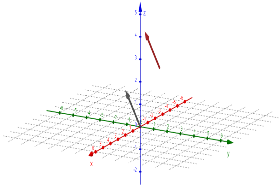
- $x^2 + y^2 \leq 4$ *and* $z = -1$

  is the region of points within a circle of radius 2 drawn on the plane $z = -1$ and centred at $(0, 0, -1)$

Vectors

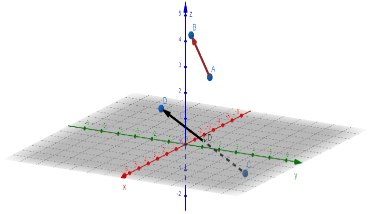
A **vector** is an object that captures a direction and a magnitude (length) in 2D/3D spaces.Geometrically, vectors are arrows in an arbitrary position in 2D/3D spaces.

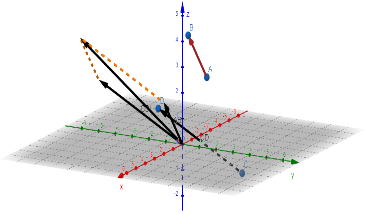


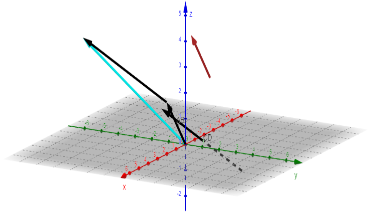


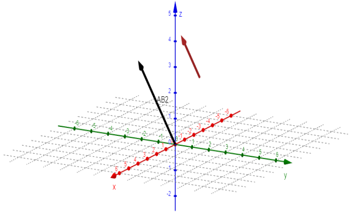
The tip of the vector is the end with the arrow, while the tail is the end without it.
A vector drawn with its tail at the origin is called a position vector.

Vector addition and scalar multiplication







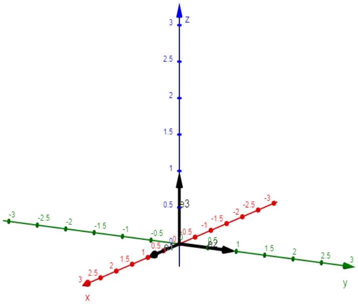


Vectors

Definition

The vectors $\bar{e}_1 = \bar{i}$, $\bar{e}_2 = \bar{j}$, $\bar{e}_3 = \bar{k}$ are vectors of length one with direction pointing along positive the x-, y-, and z-axes respectively.

$$\Rightarrow \bar{e}_1 = \bar{i} = (1,0,0),\ \bar{e}_2 = \bar{j} = (0,1,0),\ \bar{e}_3 = \bar{k} = (0,0,1)$$



Vectors

Definition

We say that we are resolving a vector into components when we write a vector in $\bar{v} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$ *form.*
In this form, we are thinking about the vector as a sum of three components (along perpendicular directions).
As an alternative, we could also provide the magnitude of the vector and indicate the direction using angles.
For a vector that is resolved into components, its magnitude is given by

$$|\bar{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

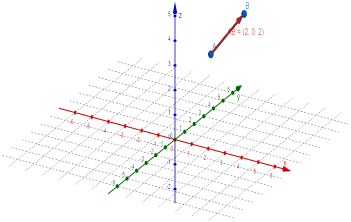
(this is coming from the distance formula).

Vectors

If $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$ are points, then the vector pointing from A to B with magnitude $|AB|$ is given by

$$\overrightarrow{AB} = (x_1 - x_0)\bar{i} + (y_1 - y_0)\bar{j} + (z_1 - z_0)\bar{k}$$

Note that $\overrightarrow{AB} = \overrightarrow{OP}$ where $P(x_1 - x_0, y_1 - y_0, z_1 - z_0)$.



Vectors

More generally, an n-dimensional vector is a direction coupled with a magnitude in n-dimensional space ($\mathbb{R}^n$). An n dimensional vector can be represented as a linear combination of n standard basis vectors. I.e.

$$\bar{v} = (v_1, \ldots, v_n) \text{ where } v_1, \ldots, v_n \in \mathbb{R}$$

The magnitude of an n-dimensional vector is given by:

$$|\bar{v}| = \sqrt{\sum_{k=1}^{n} v_k^2}$$

Properties of Vectors

Theorem

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be n-dimensional vectors, and let α and β be real numbers (scalars). Then

1. $\bar{a} + \bar{b} = \bar{b} + \bar{a}$
2. $\bar{a} + (\bar{b} + \bar{c}) = (\bar{a} + \bar{b}) + \bar{c}$
3. $\bar{a} + \bar{0} = \bar{a}$
4. $\bar{a} + (-\bar{a}) = \bar{0}$
5. $\alpha(\bar{a} + \bar{b}) = \alpha\bar{a} + \alpha\bar{b}$
6. $(\alpha\beta)\bar{a} = \alpha(\beta\bar{a})$
7. $(\alpha + \beta)\bar{a} = \alpha\bar{a} + \beta\bar{a}$
8. $1 \cdot \bar{a} = \bar{a}$

Unit vectors

Definition

We say that vectors $\bar{a}$ *and* $\bar{b}$ *are* parallel *if there exists* $c \in \mathbb{R}$ *such that* $\bar{a} = c\bar{b}$.

Definition

We say that $\bar{u}$ *is a* unit vector *if* $|\bar{u}| = 1$. *Let* $\bar{a}$ *be a vector with* $\bar{a} \neq \bar{0}$. *The unit vector of* $\bar{a}$, *written* $\hat{\bar{a}}$, *is the unit vector that points in the same direction as* $\bar{a}$, *i.e.*

$$\hat{\bar{a}} = \frac{\bar{a}}{|\bar{a}|}$$

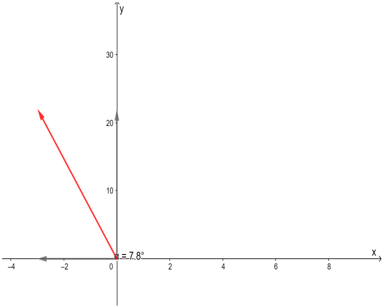
Relative Motion

Velocity, acceleration, and force are each quantities that have a magnitude and a direction$\Rightarrow$ they are well represented by vectors.For a velocity vector, we refer to its magnitude as the speed. For acceleration and force vectors we don't have special words to denote the size of the acceleration/force.
Relative motion: If an object is moving at velocity $\bar{v}$ relative to a river, and the river is moving at velocity $\bar{w}$ relative to the shore, then the object will be moving at velocity $\bar{v} + \bar{w}$ relative to the shore.

Relative Motion

Example: A person walks due west on the deck of a ship at 3 mi/h. The ship is moving north at a speed of 22 mi/h. Find the speed and direction of the person relative to the surface of the water.
Hint: Let north be the positive the positive y-direction.

$$\bar{v} = (0, 22) + (-3, 0) = (-3, 22) \Rightarrow |\bar{v}| = \sqrt{493} \approx 22.2\,\text{mi/h}$$



Matlab Examples

```
>> % Magnitude of a vector:
vecv = [1,3, 2];
% Use a loop to do the addition
sumsq = 0;
for k = 1:3
sumsq = sumsq + vecv(k)^2;
end
sqrt(sumsq) %sqrt to find the magnitude
norm(vecv) %norm is a built-in command.
```

```
>>  %Find the distance between the points (0, -3, 7) and (3, 2, 4).
dist = sqrt((0-3)^2 +(-3-2)^2 + (7-4)^2)

p1=[0,-3,7], p2=[3,2,4]
dist = sqrt((p1(1)-p2(1))^2 +(p1(2)-p2(2))^2 + (p1(3)-p2(3))^2)

% The distance is exactly
sqrt(43)
```

```
>> %Plot the graph of z = x^2-y^2.
syms x y
f = @(x,y) x^2-y^2
fsurf(x, y, f(x,y))
xlabel('x'); ylabel('y'); zlabel('z')
title('surface: z = x^2 - y^2')
set(gca,'FontSize',14)
axis equal
axis([-3 3 -3 3 -5 5])
caxis([-5 5])
%Add a cross-section with x = 0
hold on
fplot3(sym(0),y,f(0,y),'LineWidth',3)
%Add a cross-section with y = x:
fplot3(x,x,f(x,x),'LineWidth',3)
```

```
>> % cross sections of z = 2x^2.
syms x y
f = @(x,y) 2*x^2;
fsurf(x,y,f(x,y),[-2 2 -2 2])
hold on
fplot3(x,sym(3),f(x,3),[-2,2],'LineWidth',3)
fplot3(sym(1),y,sym(f(1,y)),[-2 2],'LineWidth',3)
% Plot the z = 2 cross-section in the xy-plane
fimplicit(f-sym(2),[-2,2],'LineWidth',3)
xlabel('x'); ylabel('y'); zlabel('z')
set(gca,'fontsize',16)
```

Questions

1. Plot $(1, 3, 4)$ in 3D space.
2. Find the distance between$(1, 3, 4)$ and the xy-plane.
3. Find the distance between $(1, 3, 4)$ and the plane $x = 7$.
4. Find the distance from $(1, 3, 4)$ to the z-axis.
5. Write an equation for the set of points distance 2 from the point $(1, 3, 4)$.
6. Find the set of points in the intersection of the sphere of radius 3 centered around $(0, 0, 4)$ and the plane $z = 2$.
7. Sketch the surface $z = 2x^2$: find the shape of the intersections of the surface with $y = c$, $x = c$, and $z = c$
8. Sketch the surface $z = x^2 + y^2 - 6$: find the shape of the intersection of the surface with each of $x = c, y = c, z = c$.

Next

- Scalar and vector projections.
- The dot product.
- Direction angles and direction cosines.
- The cross product.
- Matrices and determinants.

Today:05-15-2019

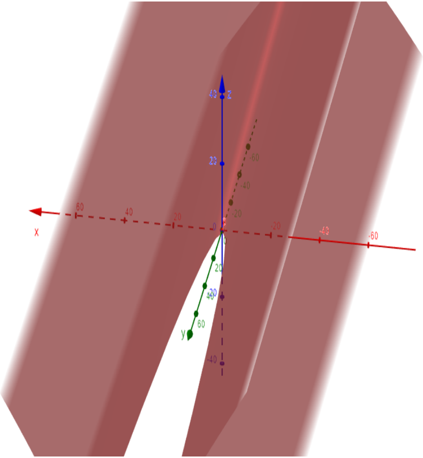
1. Review: 3D space, distance, surfaces, relative motion.
2. Scalar and vector projections.
3. The dot product.
4. Direction angles and direction cosines.
5. Matrices and determinants. Next class!
6. The cross product. Next class!
7. The triple product. Next class!

Exercises: surfaces

Match the equation with its graph:

$A. x^2 - y + 2 = 0$, $B. y = x^2 - z^2$, $C. x = z^2 - y^2$, $D. 2x^2 + y^2 + 6z^2 = 10$,

$E. x^2 = 2y^2 + z^2$, $F. 2x^2 + y^2 = z^2$, $G. 2x^2 + y^2 = z^2 + 2$, $H. y^2 + 2z^2 = x^2 - 2$



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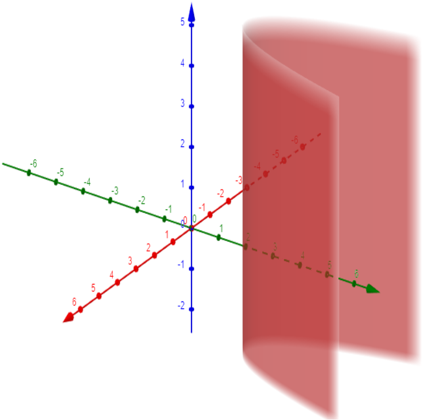

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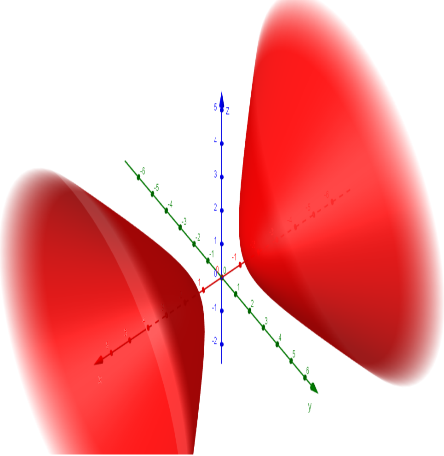

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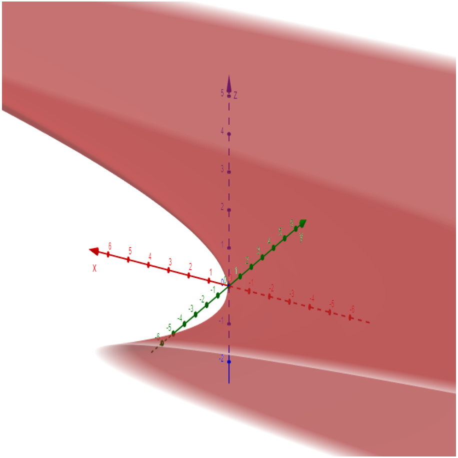

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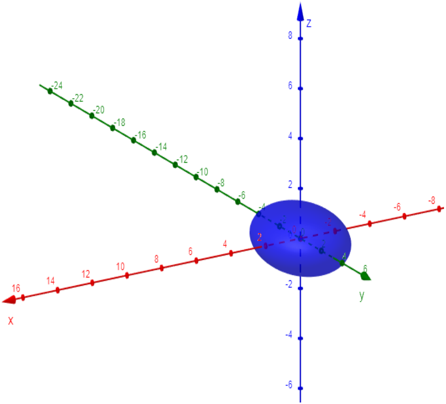

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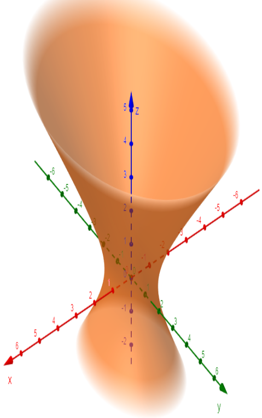

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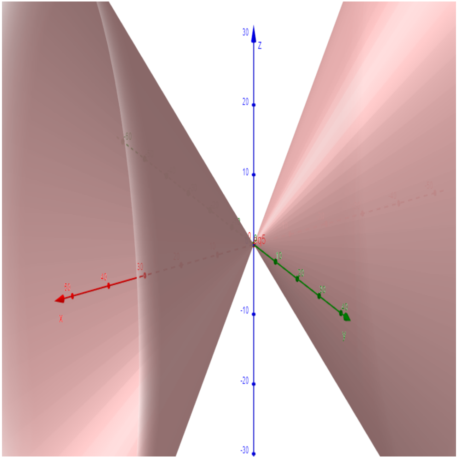

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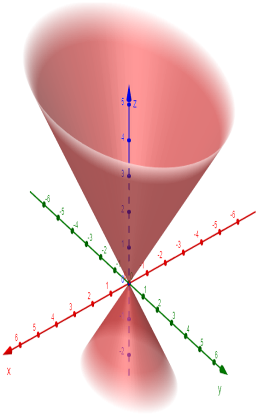


The dot product

Definition

Algebraic Definition: *Let* $\bar{a} = (a_1, a_2, a_3)$, $\bar{b} = (b_1, b_2, b_3)$.
The dot product of the vectors $\bar{a}$, $\bar{b}$ *is*

$$(\bar{a}, \bar{b}) = \bar{a} \cdot \bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Definition

Geometric Definition:

$$(\bar{a}, \bar{b}) = \bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos(\bar{a}, \bar{b})$$

Exercises

Let $\bar{v} = (3, -4, 5)$, $\bar{w} = (-2, 4, 2)$, $\bar{u} = (3, -2, 1)$.

1. Use the algebraic definition to compute $\bar{v} \cdot \bar{u}$, $\bar{w} \cdot \bar{u}$.

2. Convince yourself that $\bar{v} \cdot \bar{w} = \bar{w} \cdot \bar{v}$.

It is true in general that the dot product is commutative.

3. Show that $(\bar{v} + \bar{w}) \cdot \bar{u} = \bar{v} \cdot \bar{u} + \bar{w} \cdot \bar{u}$.

It is true in general that the dot product distributes over addition.

4. Show that $(2\bar{v}) \cdot \bar{w} = \bar{v} \cdot (2\bar{w}) = 2(\bar{v} \cdot \bar{u})$.

It is true in general that you can move scalars around this way.

5. Show that the algebraic and geometric definitions give the same answer for the dot product $\bar{i} \cdot \bar{j}$, and for the dot product $(1, 1) \cdot (0, 3)$.

Properties of the dot product

Theorem

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be 3D vectors, and let α be a scalar. Then

1. $\bar{a} \cdot \bar{a} = |\bar{a}|^2$
2. $\bar{a} \cdot \bar{b} = \bar{b} \cdot \bar{a}$
3. $\bar{a} \cdot (\bar{b} + \bar{c}) = \bar{a} \cdot \bar{b} + \bar{a} \cdot \bar{c}$
4. $(\alpha\bar{a}) \cdot \bar{b} = \alpha(\bar{a} \cdot \bar{b}) = \bar{a} \cdot (\alpha\bar{b})$
5. $\bar{0} \cdot \bar{a} = 0$

Definition

Let $\bar{a} = x_0\bar{i} + y_0\bar{j} + z_0\bar{k}$ and $\bar{b} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$ be vectors.
The angle between $\bar{a}$ and $\bar{b}$ is defined to be the angle $\angle AOB$ where $A(x_0, y_0, z_0)$ and $B(x_1, y_1, z_1)$.

Angles between vectors

Theorem

Let $\bar{a}$ and $\bar{b}$ be 3D vectors. If θ is the angle between $\bar{a}$ and $\bar{b}$, then

$$\bar{a} \cdot \bar{b} = |\bar{a}||\bar{b}| \cos \theta$$

Definition

We say that vectors $\bar{a}$ and $\bar{b}$ are perpendicular or orthogonal if the angle between $\bar{a}$ and $\bar{b}$ is $\frac{\pi}{2}$.

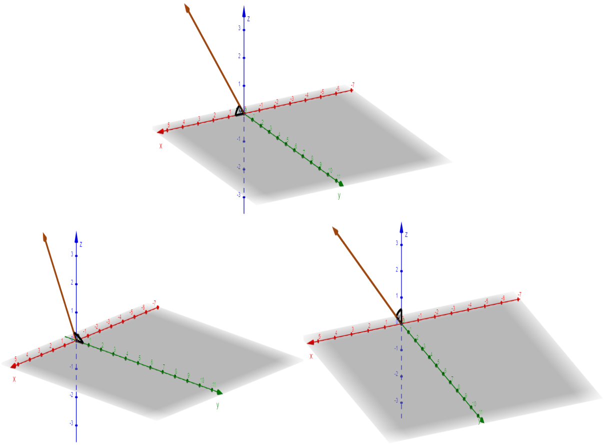
Corollary

Let $\bar{a}$ and $\bar{b}$ be 3D vectors. Then $\bar{a}$ and $\bar{b}$ are perpendicular if and only if $\bar{a} \cdot \bar{b} = 0$.

Direction angles

Definition

Let $\bar{a}$ be a 3D vector. The direction angles of $\bar{a}$ are the angles α, β and $\gamma \in [0, \pi]$ that $\bar{a}$ makes with the positive x-, y- and z-axis.



Direction angles

Definition

The *direction cosines* *of* $\bar{a} = (a_1, a_2, a_3)$ *are the cosines of the direction angles.*

$$\cos\alpha = \frac{\bar{a}\cdot\bar{i}}{|\bar{a}|\underbrace{|\bar{i}|}_{=1}} \Rightarrow \cos\alpha = \frac{a_1}{|\bar{a}|},\ \cos\beta = \frac{a_2}{|\bar{a}|},\ \cos\gamma = \frac{a_3}{|\bar{a}|}$$

Example

$\bar{a} = \bar{i} - 2\bar{j} - 3\bar{k} \Rightarrow \bar{a} = (1,\ -2,\ -3) \Rightarrow |\bar{a}| = \sqrt{1^2 + (-2)^2 + (-3)^2} = \sqrt{14}$

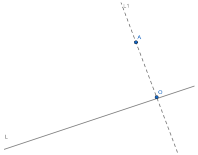
$$\cos\alpha = \frac{1}{\sqrt{14}},\ \cos\beta = \frac{-2}{\sqrt{14}},\ \cos\gamma = \frac{-3}{\sqrt{14}}$$

$\alpha = \cos^{-1}\frac{1}{\sqrt{14}} \approx 74.49^o,\ \beta = \cos^{-1}\frac{-2}{\sqrt{14}} \approx 122.3^o,\ \gamma = \cos^{-1}\frac{-3}{\sqrt{14}} \approx 143$

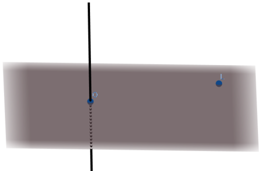
Projections

Definition

Let a line $L \subset \mathbb{R}^2$ and $A \in \mathbb{R}^2$ be a point. Draw a line L_1 passing through the point A that makes $\pi/2$ with L. The point of intersection $O = L \cap L_1$ is called the orthogonal projection of the point A onto the line L.



In $\mathbb{R}^3$, the orthogonal projection of the point A onto the line L is the point of intersection of the line L and a plane passing through A perpendicular to L.



Projections

Definition

The orthogonal projection of the vector $\overline{AB}$ onto the line L is the vector whose end-points are the orthogonal projections of the end-points of $\bar{AB}$ onto L.

$$proj_L \overline{AB} = \overline{O_A O_B}$$

The scalar component of the orthogonal projection of the vector $\overline{AB}$ onto the vector $\bar{l}$, $\bar{l} || L$ is

$$\pm |\overline{O_A O_B}|$$

We shall call it the scalar (component) projection *and denote* $\mathrm{comp}_{\bar{l}} \overline{AB}$

Since $\mathrm{comp}_{\bar{a}} \bar{b} = |\bar{b}| \cos(\bar{b}, \bar{a})$ for both cases $0 < \angle(\bar{b}, \bar{a}) < \pi/2$ and $\pi/2 < \angle(\bar{b}, \bar{a}) < \pi$, so

$$\bar{a} \cdot \bar{b} = |\bar{a}| \underbrace{|\bar{b}| \cos \bar{a}, \bar{b}}_{\mathrm{comp}_{\bar{a}} \bar{b}} = |\bar{a}| \mathrm{comp}_{\bar{a}} \bar{b}$$

Projections

Definition

The scalar (component) projection $\mathrm{comp}_{\bar{a}}\bar{b}$ *of* $\bar{b}$ *onto* $\bar{a}$ *is*

$$\mathrm{comp}_{\bar{a}}\bar{b} = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|}$$

The vector projection $\mathrm{proj}_{\bar{a}}\bar{b}$ *of* $\bar{b}$ *onto* $\bar{a}$*, written is defined by*

$$\mathrm{proj}_{\bar{a}}\bar{b} = \mathrm{comp}_{\bar{a}}\bar{b}\frac{\bar{a}}{|\bar{a}|} = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|^2}\bar{a}$$

Example

$$\bar{a} = (-1,4,8),\ \bar{b} = (12,1,2) \Rightarrow \mathrm{comp}_{\bar{a}}\bar{b} = \frac{-12+4+8\cdot 2}{\sqrt{1+16+64}} = \frac{8}{9},$$

$$\mathrm{proj}_{\bar{a}}\bar{b} = \frac{-12+4+16}{9^2}(-1,4,8) = \left(\frac{-8}{81},\ \frac{32}{81},\ \frac{16}{81}\right)$$

Today: 05-17-2019

1. Review: 3D space and surfaces (We shall consider rotation surfaces), projections, the dot product.
2. Matrices and determinants.
3. The cross product.
4. The triple product.
5. Applications of the dot product and the cross product in physics.
6. Equations of lines and planes in 3D.
7. Normal vectors.
8. Vector functions.

Matrices

Definition

Let n and m be whole positive number. An $n \times m$ real (complex) matrix, A, is a rectangular array of real (complex) numbers a_{ij} for $1 \leq i \leq n$ and $1 \leq j \leq m$. We write

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

Definition

Let $A = (a_{ij})$ be an $n \times m$ matrix. For all $1 \leq k \leq n$, the k^{th} row of A is the the $1 \times m$ matrix

$$\begin{pmatrix} a_{k1} & \cdots & a_{km} \end{pmatrix}$$

Matrices

Definition

Let $A = (a_{ij})$ be an $n \times m$ matrix. For all $1 \leq k \leq m$, the k^{th} column of A is the the $n \times 1$ matrix

$$\begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix}$$

Definition

The $n \times n$ identity matrix, written I_n, is defined by $I_n = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \textit{if } i = j \\ 0 & \textit{if } i \neq j \end{cases}$$

When the dimension is clear from the context or left unspecified, we just write I.

Matrices

Definition

Let $A = (a_{ij})$ be an $n \times m$ matrix. The **transpose** *of A, written A^T, is the $m \times n$ matrix entries a_{ji}. I.e. $A^T = (a_{ji})$*

Definition

(Matrix Multiplication) If $A = (a_{ij})$ is an $n \times m$ matrix and $B = (b_{ij})$ is an $m \times p$ matrix, then AB is an $n \times p$ matrix defined by $AB = (c_{ij})$ where

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj} \ for \ 1 \leq i \leq n \ and \ 1 \leq j \leq p$$

Definition

(Matrix Addition and Scalar Multiplication) If $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times m$ matrices, and α is a scalar, then $A + B$ is an $n \times m$ matrix defined by $A + B = (c_{ij})$ where $c_{ij} = a_{ij} + b_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$, and αA is an $n \times m$ matrix defined by $\alpha A = (d_{ij})$ where $d_{ij} = \alpha a_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$.

Matrices

Theorem

If A, B and C have the right dimensions to make the left-hand side make sense, then the following equations hold:

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(B + C)A = BA + CA$
4. $AI = IA = A$

Example

Note that it is NOT the case in general that if A is an $n \times n$ matrix and B is an $n \times n$ matrix, then $AB = BA$. Consider

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = BA$$

Determinants and inverses

Definition

Let $A = (a_{ij})$ be a 2×2 matrix. The **determinant** *of A, written*

$$\mathbf{det}(A) \ or \ \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

is defined by

$$\mathbf{det}(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Theorem

If $A = (a_{ij})$ is a 2×2 matrix is such that $\mathbf{det}(A) \neq 0$, then the matrix

$$A^{-1} = \frac{1}{\mathbf{det}(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix},$$

called the **inverse** *of A, is such that $AA^{-1} = A^{-1}A = I_2$.*

Determinants in general

We have defined the determinant of a 2×2 matrix. The determinant of an $n \times n$ matrix can now be defined recursively.

Definition

Let $A = (a_{ij})$ be a $n \times n$ matrix where $n > 2$. The **determinant** *of A, written*

$$\mathbf{det}(A) \; or \; \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

is defined by

$$\mathbf{det}(A) = \sum_{k=1}^{n} (-1)^{k+1} a_{1k} \mathbf{det}(A_k)$$

where A_k is the $(n-1) \times (n-1)$ matrix obtained by deleting the first row and the k^{th} column of A.

Matrices

Example

In particular, if $A = (a_{ij})$ is a 3×3 matrix, then

$$\mathbf{det}(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

A vector $\bar{a} = x\bar{i} + y\bar{j} + z\bar{k}$ can be represented as a 3×1 matrix. I.e.

$$\bar{a} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

It should be clear from the context whether we are thinking of a vector as an ordered tuple or a matrix.

Cross product

Definition

Algebraic Definition: *Let* $\bar{a} = x_0\bar{i} + y_0\bar{j} + z_0\bar{k}$ *and* $\bar{b} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$*.*
The cross product $\bar{a} \times \bar{b}$ *is defined by*

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \end{vmatrix}$$

Example

Consider $\bar{a} = \bar{i} + 3\bar{j} - 2\bar{k}$ *and* $\bar{b} = -\bar{i} + 5\bar{k}$*.*

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 3 & -2 \\ -1 & 0 & 5 \end{vmatrix} = 15\bar{i} - 3\bar{j} + 3\bar{k}$$

Cross product

Theorem

If $\bar{a}$ is a 3D vector, then $\bar{a} \times \bar{a} = \bar{0}$

Theorem

If $\bar{a}$ and $\bar{b}$ are 3D vectors, then $\bar{a} \times \bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{b}$

Theorem

Let $\bar{a}$ and $\bar{b}$ be 3D vectors. If $\theta \in [0, \pi]$ is the angle between $\bar{a}$ and $\bar{b}$, then

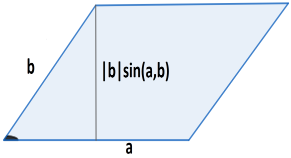
$$|\bar{a} \times \bar{b}| = |\bar{a}||\bar{b}| \sin(\theta)$$

In particular, $\bar{a}$ and $\bar{b}$ are parallel if and only if $\bar{a} \times \bar{b} = \bar{0}$

Cross product

The geometric interpretation of $\bar{a} \times \bar{b}$ is:

- (Righ-hand rule) If the fingers of your right hand curl in the direction of rotation through an angle less than π from $\bar{a}$ to $\bar{b}$, then the thumb of your right hand points in the direction of $\bar{a} \times \bar{b}$
- The magnitude of $\bar{a} \times \bar{b}$ is the area of the parallelogram with sides described by $\bar{a}$ and $\bar{b}$



Definition

Geometric Definition: $\bar{a} \times \bar{b} = \left(\begin{array}{c} \textit{the area of parallelogram} \\ \textit{with edges}\, \bar{a}, \bar{b} \end{array} \right) \bar{n}$,

where $\bar{n}$ is a unit vector perpendicular to the parallelogram with direction given by the right hand rule.

Exercises

1. Find $\bar{u} \cdot \bar{v}$, where $\bar{u} = 4\bar{i} - 6\bar{k}$ and $\bar{v} = -\bar{i} + \bar{j} + \bar{k}$.
2. Find $\bar{u} \cdot \bar{v}$ where $\bar{u} = 3\bar{i} + \bar{j} - \bar{k}$ is a vector of length 2 oriented at an angle of $\pi/4$ away from the direction of $\bar{u}$.
3. Using the geometric definition, what is $\bar{i} \times \bar{j}$ and $\bar{j} \times \bar{i}$?.
4. For $\bar{v} = 3\bar{i} - 2\bar{j} + 4\bar{k}$, $\bar{w} = \bar{i} + 2\bar{j} - \bar{k}$, find $\bar{v} \times \bar{w}$ using the algebraic and geometric definitions.

Check your results in Matlab:

Command Window

```
>> vecu = [4,0,-6]; vecv = [-1,1,1];
dot(vecu,vecv) %exercise 1
vecu = [3,1,-1];
norm(vecu)*2*cos(pi/4) %exercise 2
cross([1,0,0],[0,1,0]) %exercise 3
cross([0,1,0],[1,0,0])
vecv = [3,-2,4]; vecw = [1,2,-1];
cross(vecv,vecw) %exercise 4
```

Properties of the cross product

Theorem

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be 3D vectors, and let d be a scalar. Then

1. $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$
2. $(d\bar{a}) \times \bar{b} = d(\bar{a} \times \bar{b}) = \bar{a} \times (d\bar{b})$
3. $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$
4. $(\bar{b} + \bar{c}) \times \bar{a} = \bar{b} \times \bar{a} + \bar{c} \times \bar{a}$
5. $\bar{a} \cdot (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \cdot \bar{c}$
6. $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

Note that the cross product is NOT associative. I.e. There exists 3D vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ such that

$$\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$$

Applications of the cross product

Example

Consider the points $P(1,3,2)$, $Q(3,-1,6)$ and $R(5,2,0)$. The cross product

$$\overrightarrow{PQ} \times \overrightarrow{PR}$$

is perpendicular to the plane that passes through P, Q and R. The value

$$|\overrightarrow{PQ} \times \overrightarrow{PR}|$$

is the area of the parallelogram with adjacent sides $\overline{PQ}$ and $\overline{PR}$. Therefore the area of the triangle $\triangle PQR$ is

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}|$$

Vector triple product

Definition

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be 3D vectors. The scalar triple product *of $\bar{a}$, $\bar{b}$ and $\bar{c}$ is the value*

$$\bar{a} \cdot (\bar{b} \times \bar{c})$$

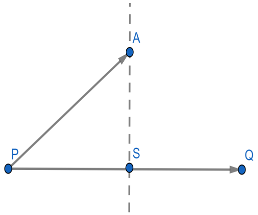
The value $|\bar{a} \cdot (\bar{b} \times \bar{c})|$ is the volume of the parallelepiped determined by the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$.

Exercise: Find the volume of the parallelepiped with sides parallel to $\bar{u} = (3, 4, 5)$, $\bar{v} = (5, 4, 3)$, $\bar{w} = (1, 1, 0)$

Examples from Physics

- The work done by the force that moves the object from P to Q pointing in the direction of the vector $\overline{PA}$ is the product of the component of the force along the displacement vector $\overline{PQ}$ and the distance moved:

$$W = \left(|\overline{PA}|\cos\left(\overline{PQ},\ \overline{PA}\right)\right)|\overline{PQ}| = \overline{PA}\cdot\overline{PQ}$$



- Exercise: Let $\bar{v} = 3\bar{i} + 4\bar{j}$ and $\bar{F} = 4\bar{i} + \bar{j}$. Find the component of the force vector $\bar{F}$ parallel to $\bar{v}$:
  a. Find the unit vector $\hat{v}$.
  b. Find $\bar{F}\cdot\hat{v}$ the length of the component of $\bar{F}$ parallel to $\bar{v}$.
  c. Construct the vector $\bar{F}_{parallel}$.

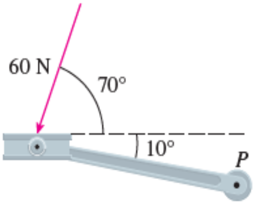
Examples from Physics

- Consider a force F acting on a rigid body at a point given by a position vector r. The torque $\bar{\tau}$ measures the tendency of the body to rotate about the origin. It is defined as the cross product of the position and force vectors

$$\bar{\tau} = \bar{r} \times \bar{F}$$

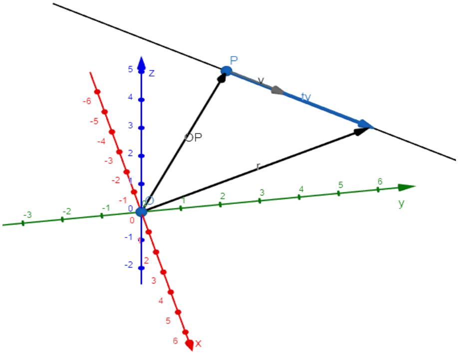
  The direction of the torque vector indicates the axis of rotation.

- **Example (Stewart):** A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.



Lines

Let L be a line in 3D space. Let P be a point on L and let $\bar{v}$ be a vector that is parallel to L.



For all $t \in \mathbb{R}$,

$$\bar{r}(t) = \overrightarrow{OP} + t\bar{v} \tag{1}$$

is a vector that points from the origin (O) to a point on L. Equation (1) is called the vector equation of L.

Lines

Therefore if $P(x_0, y_0, z_0)$ and $\bar{v} = a\bar{i} + b\bar{j} + c\bar{k}$, then for all $t \in \mathbb{R}$, the point $Q(x, y, z)$ where

$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = z_0 + tc \tag{2}$$

lies on L. (2) are called the parametric equations of L. Rearranging (2) we get

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{3}$$

These are called the **symmetric equations** of L.

Definition

We say two lines L_1 and L_2 is 3D space are skew if L_1 and L_2 are not parallel and don't intersect.

Planes

We want to find the equation of a plane perpendicular to the vector $\bar{n} = \bar{i} + \bar{j} - \bar{k}$ and passing through the point $(0, 0, -1)$.

- We are looking for points (x, y, z) that sit in the plane. Create a displacement vector, $\bar{v}$ between a point (x, y, z) and the point $(0, 0, -1)$.
- We want this displacement vector to be perpendicular to $\bar{n}$, so we want $\bar{v} \cdot \bar{n} = 0$: Plug your displacement vector and the information for $\bar{n}$ into this dot product. Expand and simplify.
  You should get $z = x + y - 1$ for the displacement vector to be perpendicular to $\bar{n}$.
- You have found an equation for a plane. Show that it passes through $(0, 0, -1)$.
- Is any vector parallel to this plane perpendicular to $\bar{n}$? Choose two points on the plane and convince yourself that the vector between those points is perpendicular to $\bar{n}$. This can be shown to hold in general, but just choose enough pairs of points to convince yourself.

Planes: now we are to generalize the previous example

A plane $\mathcal{P}$ in $\mathbb{R}^3$ is completely determined by a point P that lies on the plane and a vector $\bar{n}$, called a/the normal vector, that points in a direction which is perpendicular to $\mathcal{P}$. To see this, observe that for any point Q with $Q \neq P$ that lies on $\mathcal{P}$, the vector $\overrightarrow{PQ}$ is perpendicular to $\bar{n}$. Therefore $\bar{n} \cdot \overrightarrow{PQ} = 0$. In other words, if $\bar{r}$ is a vector that points from the origin (O) to a point on $\mathcal{P}$, then $\bar{r}$ satisfies

$$\bar{n} \cdot (\bar{r} - \overrightarrow{OP}) = 0 \tag{4}$$

(4) is called the vector equation of $\mathcal{P}$. If $P(x_0, y_0, z_0)$ and $\bar{n} = a\bar{i} + b\bar{j} + c\bar{k}$, then this yields

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{5}$$

which is called the scalar equation of $\mathcal{P}$. Therefore a plane $\mathcal{P}$ with normal vector $\bar{n} = a\bar{i} + b\bar{j} + c\bar{k}$ is described by the equation

$$ax + by + cz = d$$

where d can be determined by any point P on $\mathcal{P}$

Exercises

1. Let points $(0, 1, 2), (2, -1, 3)$ and $(0, 0, 1)$ form a triangle that lies in a plane.
a. Find a normal vector to the plane and construct an equation for the plane.
b. Find the area of the triangle.
2. Find the point at which the line $x = t - 1$, $y = 1 - 2t, z = 3 - t$ intersects the plane $3x - y + 2z = 5$

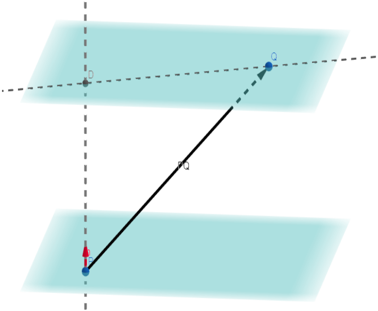
Planes

Definition

Two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel if their normal vectors are parallel.

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel planes, then the normal vector, $\bar{n}$, of either of these planes describes the direction of the shortest path between $\mathcal{P}_1$ and $\mathcal{P}_2$. Therefore, if P lies on $\mathcal{P}_1$ and Q lies on $\mathcal{P}_2$, then the shortest distance between $\mathcal{P}_1$ and $\mathcal{P}_2$ is given by

$$D = |\mathrm{comp}_{\bar{n}}(\overrightarrow{PQ})| = \frac{|\overrightarrow{PQ} \cdot \bar{n}|}{|\bar{n}|}$$



Planes

Similarly, if $\mathcal{P}$ is a plane with normal vector $\bar{n}$, P is a point on $\mathcal{P}$ and Q is a point that does not lie on $\mathcal{P}$, then the shortest distance between $\mathcal{P}$ and Q is given by

$$D = |\mathrm{comp}_{\bar{n}}(\overrightarrow{PQ})|$$

Vector functions

Definition

A vector-valued function or vector function is a function whose domain is a subset of the reals and range is a set of vectors, i.e we say that $\bar{r}$ is a vector function if $\bar{r} : A \longrightarrow \mathbb{R}^3$ where $A \subseteq \mathbb{R}$.

By interpreting vectors as arrows that point from the origin to a point in $\mathbb{R}^3$, we can interpret vector functions as describing a curve in $\mathbb{R}^3$. That is, if $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$, then $\bar{r}(t)$ describes the curve in $\mathbb{R}^3$ with parametric equations

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

Example

We have already seen how to compute vector-valued functions that describe lines in $\mathbb{R}^3$.

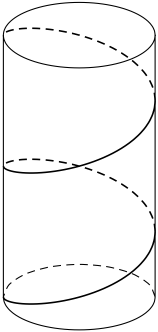
Vector functions

Example

The vector function

$$\bar{r}(t) = \cos(t)\bar{i} + \sin(t)\bar{j} + t\bar{k}$$

describes a spiral around the surface of an infinitely long cylinder of radius 1 *centred around the z-axis. This curve is called a* **helix**.



Next Week

- Vector functions: derivatives and integrals.
- Arc length and curvature.
- Motion in space. Kepler's laws of planetary motion.
- Functions of several variables.