VE230 Final RC slides

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Review of Static Case

Fundamental Relations	Electrostatic Model	Magnetostatic Model
Governing equations	$\nabla \times \mathbf{E} = 0$ $\nabla \cdot \mathbf{D} = \rho$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
Constitutive relations	$\mathbf{D} = \epsilon \mathbf{E}$	$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$

In static case, **E**, **B** can exist together(In a conducting medium), but they won't influence each other.

Faraday's Law of Electromagnetic Induction

Content: The relationship between induced emf and the negative rate of change of flux linkage.

Expressions:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_c \mathbf{E} \cdot d\ell = -\int_s \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

Static Circuit in a Time-varying Magnetic Field

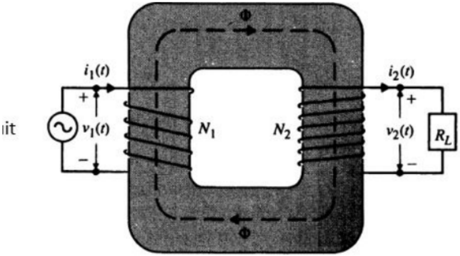
If we assign emf $V = \oint_c \mathbf{E} \cdot d\ell$ and magnetic flux $\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$, we can get the Faraday's Law

$$V = -\frac{d\Phi}{dt}$$

If there are N turns wires, the total magnetic flux is $N\Phi, V = -N\frac{d\Phi}{dt}$.
Lenz's Law: The induced emf will cause a current to flow in the closed loop in such a direction as to oppose the change in the linking magnetic flux.

Transformer

Transformer: two or more coils coupled magnetically through a common ferromagnetic core.



The general equation is

$$N_1 i_1 - N_2 i_2 = \frac{\ell}{\mu S}\Phi$$

Transformer

Ideal Transformer: $\mu \to \infty$, and then we can get

$$\frac{i_1}{i_2} = \frac{N_2}{N_1}$$

Given Faraday's Law, We get ratio of emf as

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

If we have RL in secondary circuit, we can see source Resistor $R_1 = \frac{V_1}{i_1}$ as

$$(R_1)_{eff} = \left(\frac{N_1}{N_2}\right)^2 R_L$$

Moving Conductor in Static Magnetic field

Working process: $F_m = q\mathbf{u} \times \mathbf{B} \rightarrow$ positive and negative charge move to opposite direction $\rightarrow$ built induced Electric field $E_{\text{induced}} = -\mathbf{u} \times \mathbf{B} \rightarrow$ Other charges in equilibrium (they won't move along the bar)

Motional EMF $\mathcal{V}' = \oint_c (\mathbf{u} \times \mathbf{B}) \cdot d\ell$.

Moving Circuit in a Time-varying Magnetic Field

Lorentz's force equation: $\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$.

Effective electric field E': If an observer have the same movement with q, Lorentz's force on q can be seen as effective electric field

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

General form of Faraday's law:

$$\oint_C \mathbf{E}' \cdot d\ell = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\ell$$

Moving Conductor in Static Magnetic field

Where left side talks about emf induced in a moving frame of reference, and on right side, transformer emf equals to

$$V = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

and motional emf equals to

$$V = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot dl$$

Faraday's Law also works a moving circuit. If we use $V = \oint_C \mathbf{E}' \cdot d\mathbf{l}$, then

$$V = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}$$

Maxwell's Equation

Differential Form	Integral Form	Significance
$\nabla \times \mathbf{E} = -\frac{\partial \mathrm{B}}{\partial t}$	$\oint_c \mathbf{E} \cdot d\ell = -\frac{d\Phi}{dt}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_c \mathbf{H} \cdot d\ell = I + \int_s \frac{\partial \mathrm{D}}{\partial t} \cdot d\mathbf{s}$	Ampère's circuital la
$\nabla \cdot \mathbf{D} = \rho$	$\oint_s \mathbf{D} \cdot d\mathbf{s} = Q$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_s \mathbf{B} \cdot d\mathbf{s} = 0$	No isolated magnetic c

Other useful equations:

$$\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t}$$
$$\mathbf{H} = \mathbf{B}/\mu$$
$$\mathbf{D} = \epsilon \mathbf{E}$$

Potential Function

Electric Field time-varying field:

$$E = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

where $-\nabla V$ comes from charge distribution, and $-\frac{\partial \mathbf{A}}{\partial t}$ comes from time-varying current. Strictly speaking, V and A are calculated by the Poisson's Equation in time varying field.

Quasi-static fields: If ρ and $\mathbf{J}$ vary slowly with time and the range of R is small in comparison with the wavelength (low frequency, long wavelength), We can use below 2 equations to find quasi-static fields.

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'}\frac{\rho}{R}dv'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{J}}{R}dv'$$

Potential Function

Non-homogeneous wave equation for vector potential: if we choose divergence and curl of **A** as

$$\nabla \cdot \mathbf{A} + \mu\epsilon\frac{\partial V}{\partial t}$$
$$B = \nabla \times \mathbf{A}$$

which is also called Lorentz condition, we can find the nonhomogeneous wave equations as

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J}$$

Non-homogeneous wave equation for Scalar Potential V:

$$\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}.$$

Electromagnetic Boundary Condition

General Boundary Condition Equations:

$$
\begin{gathered}
E_{1t} = E_{2t}(\mathrm{~V/m}) \\
\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (\mathrm{~A/m}) \\
\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad \left(\mathrm{C/m^2}\right) \\
B_{1n} = B_{2n}(\mathrm{~T})
\end{gathered}
$$

Note that (1)(4) are equivalent and (2)(3) are equivalent. Since divergence equation can be derived from curl equations with continuity equation.

Interface between two lossless Linear Media

lossless media: $\sigma = 0$, then we can get J = 0.

Usually, no free charge and no surface currents at the interface of two lossless. $(\rho_s = 0, \mathbf{J_s} = 0)$.

Boundary condition:

$$E_{1t} = E_{2t} \rightarrow \frac{D_{1t}}{D_{2t}} = \frac{\epsilon_1}{\epsilon_2}$$
$$H_{1t} = H_{2t} \rightarrow \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}$$
$$D_{1n} = D_{2n} \rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$
$$B_{1n} = B_{2n} \rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}$$

Interface between a Dielectric and a perfect conductor

perfect conductor: $\sigma \to \infty$, then we know $\mathbf{E}_{inside} = 0$, the charge only exists on the surface.

$\mathbf{D}, \mathbf{B}, \mathbf{H} = 0$ for point inside a conductor.

Boundary condition equation (2 is perfect):

On the Side of Medium 1	On the Side of Medium 2
$E_{1t} = 0$	$E_{2t} = 0$
$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$	$H_{2t} = 0$
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$

solutions for wave equations for potentials

The calculation process is on the textbook.

for given charge and current distribution ρ and **J**,in order to get the **E**, **B**, we firstly need to find solutions for **A**, **V** in nonhomogeneous wave equation.

solution for scalar potential:

$$V(R,t) = \frac{1}{4\pi\epsilon} \int_{V'} \frac{\rho(t - R/u)}{R} dv'$$

It takes time R/u for the effect of ρ to be felted at the distance R, which means there is time retardation $\Delta t = R/u$ from ρ to V.

solution for vector potential:

$$\mathbf{A}(R,t) = \frac{\mu}{4\pi} \int_{V'} \frac{\mathbf{J}(t - R/u)}{R} dv' \quad (\mathbf{W}b/\mathrm{m})$$

Source-Free Wave Equations in simple nonconducting media

Source-free; $\rho = 0, \mathbf{J} = 0$.

In a simple nonconducting media: ϵ, μ are constant, $\sigma = 0$

Rewrite the Maxwell Equations:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

Wave Equations for **E**, **H** can be found directly.

$$\nabla^2 \mathbf{E} - \frac{1}{u^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\nabla^2 \mathbf{H} - \frac{1}{u^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Time Harmonic Electromagnetic

Time harmonic Vector Phasor:

$$\mathbf{E}(x,y,z,t) = \mathrm{Re}\left[\mathbf{E}(x,y,z)e^{jwt}\right]$$

Different or integrate it, we can find

$$\partial\mathbf{E}(x,y,z,t)/\partial t = j\omega\mathbf{E}(x,y,z)$$
$$\int \mathbf{E}(x,y,z,t)dt = \mathbf{E}(x,y,z)/j\omega$$

Revised Maxwell Equation

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$
$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E}$$
$$\nabla \cdot \mathbf{E} = \rho/\epsilon$$
$$\nabla \cdot \mathbf{H} = 0$$

Time Harmonic Electromagnetic

Time harmonic wave equations:

$$\nabla^2 V + k^2 V = -\frac{\rho}{\epsilon}$$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu \mathbf{J}$$

$$\text{where} \quad k = \omega\sqrt{\mu\epsilon} = \frac{\omega}{u} = 2\pi/\lambda$$

Phasor solutions:

$$V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho e^{-jkR}}{R} dv' \quad (V)$$

$$\mathbf{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J} e^{-jkR}}{R} dv' \quad (\mathbf{Wb}/\mathrm{m})$$

Note that $kR = 2\pi\frac{R}{\lambda} << 1$ when $R << \lambda$.

Procedure for determining E and H

Find V and A by

$$V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho e^{-jkR}}{R} dv' \quad \mathbf{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J} e^{-jkR}}{R} dv'$$

Find B and E by

$$\mathbf{E}(R) = -\nabla V - j\omega \mathbf{A} \quad \mathbf{B}(R) = \nabla \times \mathbf{A}$$

Find Instantaneous E(t) and B(t) by

$$\mathbf{E}(R,t) = \mathcal{R}e\left[\mathbf{E}(R)e^{j\omega t}\right] \quad \mathbf{B}(R,t) = \mathcal{R}e\left[\mathbf{B}(R)e^{j\omega t}\right]$$

Source-Free Fields in non-conducting Simple Media

Find Wave function given previous section

$$\nabla^2\mathbf{E} - \frac{1}{u^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \quad \nabla^2\mathbf{H} - \frac{1}{u^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = 0$$

Find homogeneous equations in phasor form:

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \quad \nabla^2\mathbf{H} + k^2\mathbf{H} = 0$$

If medium is conducting, We use $\nabla \times \mathbf{H} = j\omega\epsilon_c\mathbf{E}$ where complex permitivity $\epsilon_c = \epsilon - j\frac{\sigma}{\omega}(F/m)$.

Flow of Electromagnetics Power and the Poynting Vector

Power flow per unit area, Poynting vector, $\vec{\mathcal{P}}$ is defined as:

$$\mathcal{P} = \vec{E} \times \vec{H}$$

Poynting vector is a power density vector associated with an electromagnetics field.
Poynting's theorem: the surface integral of $\mathcal{P}$ over a closed surface, equals the power leaving the enclosed volume, that is,

$$\begin{aligned}
-\oint_S \mathcal{P} \cdot d\vec{s} &= -\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{s} \\
&= \frac{\partial}{\partial t} \int_V \left(\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right) dv + \int_V \sigma E^2 dv \\
&= \frac{\partial}{\partial t} \int_V (w_e + w_m) dv + \int_V p_\sigma dv
\end{aligned}$$

Flow of Electromagnetics Power and the Poynting Vector

In the previous slides,

$$w_e = \frac{1}{2}\epsilon E^2 = \frac{1}{2}\vec{E}\cdot\vec{E}^* = \text{Electric energy density}$$

$$w_m = \frac{1}{2}\mu H^2 = \frac{1}{2}\mu\vec{H}\cdot\vec{H}^* = \text{Magnetic energy density}$$

$$p_\sigma = \sigma E^2 = J^2/\sigma = \sigma\vec{E}\cdot\vec{E}^* = \vec{J}\cdot\vec{J}^*/\sigma = \text{Ohmic power density}$$

If the region is loseless ($\sigma = 0$),

total power flow in $=$ rate of increase of the stored electric and magnetic

,

moreover, for static situation, w_e and w_m vanish, the total power flowing into the closed surface equals to the ohmic power (usually heat) dissipated in the enclosed volume.