Vv156 Lecture 23

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- Recall an infinite sequence is an ordered list of objects.

$$\left\{a_n\right\}_{n=1}^{\infty}$$

Definition

An infinite series, often known just a series,

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_k + \dots$$

is the sum of the terms of an infinite sequence $\left\{a_n\right\}$. For example,

$$\pi = 3 + \frac{1}{10} + \frac{4}{10^2} + \frac{1}{10^3} + \frac{5}{10^4} + \frac{9}{10^5} + \frac{2}{10^6} + \frac{6}{10^7} + \frac{5}{10^8} + \cdots$$

$$= 3.14159 \quad 26535 \quad 89793 \quad 23846 \quad 26433 \quad 83279 \quad 50288 \dots$$

Definition

A partial sum s_n is the sum of the first n terms of an infinite sequence. i.e.

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ s_4 &= a_1 + a_2 + a_3 + a_4 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i \end{aligned}$$

- Partial sums of a sequence forms a new sequence, finite or infinite,

$$\left\{s_i\right\}_{i=1}^{N} \quad \text{or} \quad \left\{s_i\right\}_{i=1}^{\infty}$$

Definition

If the new sequence $\left\{s_n\right\}$ converges to a finite value s,

$$\lim_{n\to\infty} s_n = s$$

then we say the series $\sum_{n}^{\infty} a_n$ is convergent, and the sum converges to $s = \sum_{n=1}^{\infty} a_n$.

If $\left\{s_n\right\}$ diverges, then we say $\sum_{n}^{\infty} a_n$ is divergent, and the sum is undefined.

Q: Is the series with the partial sum $s_n = \dfrac{2n}{3n+5}$ convergent?

Q: Is the series with the general formula $a_n = \dfrac{2n}{3n+5}$ convergent?

Theorem

If the series $\sum_{n}^{\infty} a_n$ is convergent, then

$$\lim_{n\to\infty} a_n = 0$$

Proof

- Since

$$a_n = s_n - s_{n-1}$$

- Consider the limit of it, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

Q Is the converse true?

- No, the converse is not true, I will give you a counterexample in the end.

Test for Divergence

If the limit

$$\lim_{n\to\infty} a_n \neq 0 \quad \text{or doesn't exist,}$$

then the series

$$\sum_{n}^{\infty} a_n \quad \text{is divergent.}$$

Exercise

Show that the following series diverges

$$\sum_{n}^{\infty} \frac{n^2}{5n^2+4}$$

Definition

A geometric sequence has the form

$$a,\, ar,\, ar^2,\, \cdots$$

where a and r are some fixed numbers.

- An explicit formula for this geometric sequence is given by

$$a_n = ar^{n-1}, \qquad n \in \mathbb{N}$$

- A recursive formula is given by

$$a_1 = a, \qquad a_n = ra_{n-1}$$

- Geometric sequences (with positive terms) are distinguished by the fact that the nth term is the geometric mean of its neighbors, i.e.

$$a_n = \sqrt{a_{n-1}a_{n+1}}$$

- The explicit formula for partial sums of Geometric sequence,

1. If $r = 1$, it is clear that

$$s_n = a + a + \ldots + a = na \to \pm\infty \qquad \text{as} \qquad n \to \infty$$

2. If $r \neq 1$, consider multiplying r to s_n

$$s_n = a + ar + \ldots + ar^{n-1} \implies rs_n = ar + ar^2 + \ldots + ar^n$$

- The difference between s_n and rs_n

$$s_n - rs_n = a - ar^n \implies s_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}$$

- It is clear that

$$\lim_{n\to\infty} s_n = \begin{cases} \dfrac{a}{1-r} & \text{if} \quad |r| < 1 \\ \text{doesn't exist} & \text{otherwise} \end{cases}$$

Exercise

(a) Is the series $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$ convergent or divergent?

(b) Write the number with recurring decimal

$$2.3\dot{1}\dot{7} = 2.3171717\ldots$$

as a fraction of integers.

(c) Determine whether the series $\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$ converges or diverges.

(d) Find the value that the following series converge to

$$\sum_{n=1}^{\infty} \left(\frac{3}{n(n+1)} + \frac{1}{2^n} \right)$$

- An ant starts to move along a rubber band 1m long at a speed of 1cm/s.
- The poor ant is actually walking on a magical rubber band which stretches by 1m suddenly and instantaneously after each second.
- So after 1 second the band is 2m long, after 2 seconds it is 3m long, etc.

Q: Will the ant ever complete his journey if the band stretches uniformly?

- It depends whether the following series is convergent as $n \to \infty$.

$$\underbrace{\underbrace{\frac{1}{100}}_{1s} + \frac{1}{100} \cdot \frac{1}{2}}_{2s} + \underbrace{\frac{1}{100} \cdot \frac{1}{3} + \cdots + \frac{1}{100} \cdot \frac{1}{n}}_{\cdots\cdots} = \frac{1}{100} \sum_{k}^{n} \frac{1}{k}$$

- The infinite series is known as Harmonic series.

$$\sum_{k}^{\infty} \frac{1}{k}$$

- If the Harmonic series converges to anything bigger than 100 or it diverges to ∞, then we have a happy ant which can complete his journey in the end.

- If we consider the following subsequence of the partial sums sequence

$$
\begin{aligned}
s_2 &= 1 + \frac{1}{2} \\
s_4 &= 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) = 1 + 1 \\
s_8 &= 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) \\
&> 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}) = 1 + \frac{3}{2}
\end{aligned}
$$

- Similarly, $s_{16} > 1 + \frac{4}{2}$, $s_{32} > 1 + \frac{5}{2}$, $s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2} \to \infty \qquad \text{as} \qquad n \to \infty$$

- Hence this shows that $\left\{ s_{2^n} \right\} \to \infty$ as $n \to \infty$ and so $\left\{ s_n \right\}$ is divergent.
- Therefore the harmonic series is divergent.
- Back to our Ant, it means that the Ant will complete his journey.

Something from nothing

- Another example of $0 = 1$,

$$
\begin{aligned}
0 &= 0 + 0 + 0 + \cdots && (1)\\
&= (1-1) + (1-1) + (1-1) + \cdots && (2)\\
&= 1 - 1 + 1 - 1 + 1 - 1 + \cdots && (3)\\
&= 1 + (-1+1) + (-1+1) + (-1+1) + \cdots && (4)\\
&= 1 + 0 + 0 + 0 + \cdots && (5)\\
&= 1 && (6)
\end{aligned}
$$

Q: What is wrong with the above manipulation?

- The series
$$\sum_{n=1}^{\infty}(-1)^{n+1} = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$
is known as Grandi's series.
- It clearly diverges since its partial sum doesn't converge to any finite value
$$\lim_{n\to\infty} s_n = \frac{1}{2}\lim_{n\to\infty}\left(1 + (-1)^{n+1}\right)$$
- However, it is sometime possible to assign a meaningful value to a series

  even if it is divergent.
- For example, Grandi's series is in a way approaching the value
$$\lim_{n\to\infty}\sigma_n = \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} s_k = \lim_{n\to\infty}\frac{s_1 + s_2 + \cdots + s_n}{n} = \frac{1}{2}$$
- Such value, when exists, is known as the Cesaro sum of the series.