Mid 1 Review

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Overview

1. Extreme Value Theorem
2. Mean Value Theorem
3. L'Hospital's Rule

Extremum: Definition

Definition <Monotonicity>

Suppose f is defined on an interval $\mathbb{I}$, and x_1 and x_2 denote points in $\mathbb{I}$,$x_1 < x_2$, then

1. f is increasing if $f(x_1) \leqslant f(x_2)$.
2. f is increasing if $f(x_1) \geqslant f(x_2)$.

Definition <Concavity>

1. Concave up $\Leftrightarrow$ f'(x) $\uparrow$
2. Concave down $\Leftrightarrow$ f'(x) $\downarrow$

To see the definition of the global/local extremum, please refer to the slides.

Extremum: Theorem

Theorem <Monotonicity> (Sufficient and Necessary)

1. f' $\geqslant 0 \Leftrightarrow$ f is increasing. (can possibly be strictly increasing)
2. f' $\leqslant 0 \Leftrightarrow$ f is decreasing. (can possibly be strictly decreasing)

Theorem <Concavity> (Sufficient)

1. f" $> 0 \Rightarrow$ f concaves up.
2. f"$<0 \Rightarrow$ f concaves down.

1^{st} and 2^{nd} Derivative Test: Choice

- 1^{st} derivative Test: When f" doesn't exist or f" is too complex.
- 2^{nd}derivative Test: When f" is easy to acquire.

Critical and Inflection Point

Critical Point

<Case I> $f'(c)=0 \Rightarrow$ Stationary Point.
<Case II> $f'(c)$ doesn't exist.

Inflection Point

The point where concavity changes.
<Necessary Condition> $f''=0$

Extremum: Property

Fermat Theorem

If the f is differentiable at the extemum $\Rightarrow$ f'=0 <Stationary Point>.
Yet the stationary point is not necessarily the extremum.

Extreme Value Theorem

1. Existence Theorem

The Extreme-Value Theorem

If f is continuous on a closed and bounded interval I, then f attains an absolute maximum value f(c) and an absolute minimum value f(d) where c,d $\in \mathbb{I}$.

Procedure to find Absolute Extremum

1. Find the critical point
2. Evaluate critical points/end points
3. Compare

Mean Value Theorem

If:

1. f is continuous on [a, b].
2. f is differentiable on (a, b).

Then

$$\text{there } \exists\ c \in (a,b), \text{ s.t. } f'(c) = \frac{f(b)-f(a)}{b-a}$$

Rolle's expression

$f(a)=f(b)$, $f'(c)=0$: always related to the proof for the existence of a point at which the derivative is zero.

Example 1

Example

If $e < a < b < e^2$, show that $ln^2 b - ln^2 a > \frac{4}{e^2}(b - a)$

Solution

We need to prove that

$$\frac{ln^2 b - ln^2 a}{b - a} > \frac{4}{e^2}$$

Let $f(x) = ln^2 x$, so $f'(x) = \dfrac{2lnx}{x}$

Let $\exists e < a < c < b < e^2$

By applying MVT, we can get

$$\frac{f(b) - f(a)}{b - a} = f'(c) = \frac{2lnx}{x} > \frac{4}{e^2}$$

Exercise 2

Example

If $f(x)$ is a function continuous on $[a, b]$ and differentiable on (a, b), show that $\exists c \in [a, b]$ so that

$$cf'(c) + f(c) = \frac{bf(b) - af(a)}{b - a}$$

Hint

By using your Mathematical Intuition, you can guess

$$g(x) = xf(x)$$

Since $g'(x) = xf'(x) + f(x)$

Exercise 3

Example

If $f(x), g(x)$ is sufficiently differentiable on $[a, b]$, and $f(a) = f(b) = g(a) = g(b) = 0$. Show that $\exists \varepsilon \in [a, b]$ so that

$$f(\varepsilon)g''(\varepsilon) - f''(\varepsilon)g(\varepsilon) = 0$$

Solution

You can guess the function

$$F(x) = f(x) \cdot g'(x) - f'(x) \cdot g(x)$$

Summary: Three Existence Theorem

Here follows the main strategy to apply those theorem.

IVT

Guarantee the existence of the intermediate value.

EVT

Guarantee the existence of maximum and minimum $\Rightarrow$
Replace f(x), $f_1(x) + f_2(x)$, ... by a simple maximum M or minimum m.
e.g.

$$m \leq f(x) \leq M$$

MVT

Relate the value of the function with the value of its derivatives.

$$f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Exercise

See Assignment 5, Question 9 to get familiar with the applicable strategy.

Exercise

Example

If $f(x)$ is continuous on $[0, 3]$ and differentiable on $(0, 3)$. If $f(0) + f(1) + f(2) = 3$ and $f(3) = 1$. Show that $\exists c \in (0, 3)$ so that $f'(c) = 0$.

Solution

Please show the sol to the Assignment 5.

L'Hospital's Rule: Conditions

Applied to the indeterminate form.

Condition I

$\frac{f(x)}{g(x)}$ is in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ (or after conversion).

Condition II

$\lim\limits_{x\to a/x\to\infty} \frac{f'}{g'}$exists or diverges to infty.

Condition III

In some neighborhood of c, possibly except at c: f', g' exist and f' & g' is not zero simultaneously.

Notice that all the three conditions need to be satisfied.

Exercise

Example

Find

1. $$\lim_{x\to 0} \frac{\sqrt{1+\sin x}-\sqrt{1+x}}{x^3}$$

2. $$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^{100}}$$

Hint

Hint

1. Rationalize.

$$\frac{\sqrt{1+\sin x}-\sqrt{1+x}}{x^3} = \frac{\sin x - x}{x^3(\sqrt{1+\sin x}+\sqrt{1+x})}$$
$$= \frac{\sin x - x}{x^3}\cdot\frac{1}{\sqrt{1+\sin x}+\sqrt{1+x}}$$

2. Substitution.

$$\lim_{x\to 0}\frac{e^{-\frac{1}{x^2}}}{x^{100}} = \lim_{\frac{1}{x^2}\to\infty}\frac{\left(\frac{1}{x^2}\right)^{50}}{e^{\frac{1}{x^2}}} \overset{t=\frac{1}{x^2}}{=\!=\!=} \lim_{t\to\infty}\frac{t^{50}}{e^t} = 0$$

Notice we here repeatedly apply the L'Hospital's Law to acquire the final answer.

Thanks!