vv256: Week 3.
Singular Solutions. Linear Spaces. Eigenvalues and Eighenvectors.

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September 24, 2019

Outline

1. Lecture 6: Implicit first-order ODEs. Singular solutions.
   - Implicit first-order ODEs
   - Singular Solutions
2. Lecture 7: Linear (vector) spaces and elements of linear algebra.
   - Structure of a linear space.
   - The Wronskian
   - Systems of linear algebraic equations
3. Lecture 8: Eigenvalues and Eigenvectors

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be general solutions of (2). What is the general solution of the equation (1)?

$$\Phi_1(t, y, C) \cdot \Phi_2(t, y, C) \cdot \ldots \cdot \Phi_k(t, y, C) = 0$$

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If a general solution of this equation has the representation $y = \Theta(p, C)$, where Θ is known and C is a constant, then

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Exercise: Find the general solution if $t = \varphi(y')$.

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What happens if $y = \psi(y')$?

Implicit first-order ODEs

What is common in cases 2 and 3?

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However, new explicit equations may not have analytical representation of the solution!!!
We are to consider two types of equations for which the approach described above works and explicit equations are solvable.

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What is the type of this equation? Linear $\Rightarrow$ Find its solution $t = \Phi(p, C)$ and obtain the general solution of the Lagrange equation in the form

$$\begin{cases} t = \Phi(p, C) \\ y = \Phi(p, C)\varphi(p) + \psi(p) \end{cases}$$

Attention! The Lagrange equation may also have special solutions of the form $y = \varphi(c)t + \psi(c)$, where c is the root of the equation $\varphi(c) = 0$. We will consider the question of special solutions later.

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Therefore, plugging C instead of y' in Clairaut's equation we immediately obtain the general solution. How we can get a singular solution from the general one? Differentiate w.r.t C.

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We have $y' = t$ and $y' = y$.Then $y = \frac{t^2}{2} + C$, $y = Ce^t$ and the general solution is

$$(y - \frac{t^2}{2} - C)(y - Ce^t) = 0.$$

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$$\begin{cases} t = \ln|p| + (p+1)e^p + C \\ y = p + p^2 e^p \end{cases}$$

We need to complement it with the obvious solution $y = 0$.

Exercises

Solve the following ODEs:

$$1.(y')^2 - 2ty' - 8t^2 = 0.$$

$$2.t^2(y')^2 + 3tyy' + 2y^2 = 0.$$

$$3.(y')^3 - y(y')^2 - t^2y' + t^2y = 0.$$

$$4.t = \ln y' + \sin y'.$$

$$5.y = \sin^{-1} y' + \ln(1 + (y')^2).$$

$$6.y = ty' + y' + \sqrt{y'}.$$

$$7.y = y' \ln y'.$$

$$8.y = 3/2ty' + e^{y'}.$$

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is called singular if the uniqueness property does not hold at any of its points, that is,

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- another non-singular solution is different form the singular one in any arbitrary small neighborhood of the point (t_0, y_0).

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Does a singular solution satisfies the equation (3)? Yes. Moreover, if $F(t, y, y')$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial y'}$ are continuous with respect to all arguments t, y, y' then any singular solution satisfies the equation

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How can we find a singular solutions from (3) and (4)? $\Rightarrow$ Eliminate y'.Elimination gives us an equation

$$\psi_p(t, y) = 0$$

which is called *p*-discriminant of the equation (3), and the integral curve corresponding *p*-discriminant is called the *p*-discriminant integral curve.

Is a *p*-discriminant curve unique? Does it define a singular solution? In general, no. $\Rightarrow$ Double-check.

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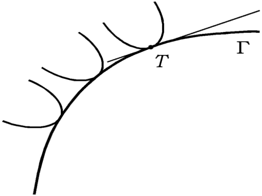
An envelope of the family of parametric curves is a smooth curve Γ that touches one curve of the family at any of its points, and any of its segments is touched by an infinite number of curves from the family. What does it mean if curves touch? A common tangent.



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An envelope is a part of a C-discriminant curve defined a by

$$\begin{cases} \Phi(t, y, C) = 0 \\ \frac{\partial \Phi(t,y,C)}{\partial C} = 0 \end{cases}$$

To make sure that a branch of a C-discriminant curve is an envelope, we check the following conditions.

- there exist bounded partial derivatives $\left|\dfrac{\partial \Phi}{\partial t}\right| \leq M, \ \left|\dfrac{\partial \Phi}{\partial y}\right| \leq N, \ M, N = const,$
- $\dfrac{\partial \Phi}{\partial t} \neq 0$, or, $\dfrac{\partial \Phi}{\partial y} \neq 0$

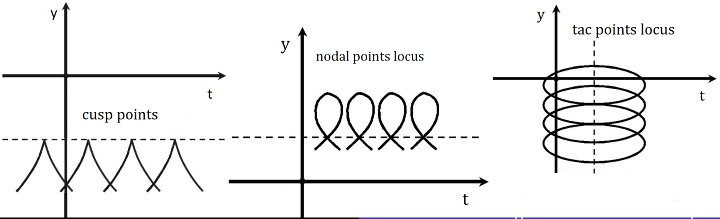
Are these condition are necessary or sufficient? Sufficient. $\Rightarrow$ if they are not satisfied on a branch of the C-discriminant curve, it can still be an envelope.

Singular Solutions

The equations of p-discriminant and C-discriminant have a certain structure

$$\psi_p(t,y) = E \cdot C \cdot T^2 = 0,$$
$$\psi_C(t,y) = E \cdot N^2 \cdot C^3 = 0,$$

where $E = 0$ is the equation of the envelope, $C = 0$ is the equation of the cusp locus, $N = 0$ is the equation of nodal locus, $T = 0$ is the equation of the tac locus. Attention! Over all locus points only the envelope is a singular solution.



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Singular Solutions: Example 1

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$$y_1(t_0) = y_2(t_0),\ y_1'(t_0) = y_2'(t_0).$$

Singular Solutions: Example 1

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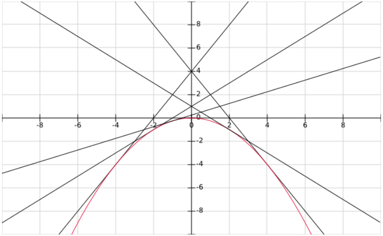

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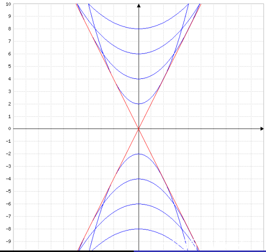

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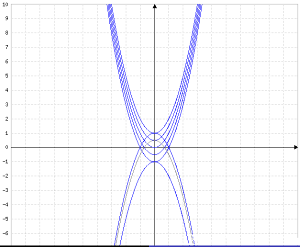

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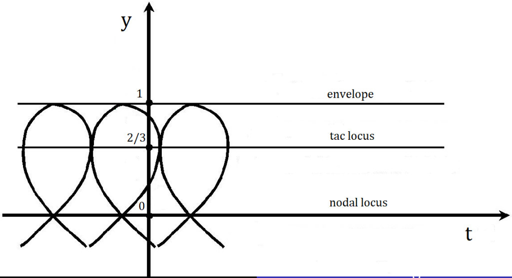

Singular Solutions: Exercises

For the following equations, find singular solutions if they exist.
1.$(1+(y')^2)y^2 - 4yy' - 4t = 0$,
2.$(y')^2 - 4y = 0$,
3.$(y')^3 - 4tyy' + 8y^2 = 0$,
4.$(y')^2 - y^2 = 0$,
5.$(ty' + y)^2 + 3t^5(ty' - 2y) = 0$.
Use C-discriminant to find singular solutions for the following equations 1.$y = (y')^2 - ty' + t^2/2$, $y = Ct + C^2 + t^2/2$,
2.$(ty' + y)^2 = y^2y'$, $y(C - t) = C^2$,
3.$y^2(y')^2 + y^2 = 1$, $(x - C)^2 + y^2 = 1$,
4.$(y')^2 - yy' + e^t = 0$, $y = Ce^t + 1/C$.

Outline

Definition

Let $\mathbb{K}$ denote a scalar field (either $\mathbb{R}$ or $\mathbb{C}$).
A set X is called a **linear (or vector) space over the scalar field** $\mathbb{K}$ if there are two binary operations of addition and scalar multiplication defined on X

$a)\,\forall x, y \in X \;\Rightarrow\; x + y \in X$

$b)\,\forall x \in X,\, \forall\alpha \in \mathbb{K} \;\Rightarrow\; \alpha x \in X$

satisfying the following properties:

$$1.\, x + y = y + x \quad \forall x, y \in X \quad \text{commutativity}$$

$$2.\, x + (y + z) = (x + y) + z \quad \forall x, y, z \in X \quad \text{associativity}$$

$$3.\, \exists\, 0 \in X :\; 0 + x = x + 0 = x \quad \forall x \in X$$

$$4.\, \forall x \in X\, \exists\, (-x) \in X :\; x + (-x) = 0$$

$$5.\, 1 \cdot x = x \quad \forall x \in X \quad 6.\, (\alpha\beta)x = \alpha(\beta x) \quad \forall x \in X\, \forall\alpha, \beta \in \mathbb{K}$$

$$7.\, \alpha(x+y) = \alpha x+\alpha y \quad 8.\, (\alpha+\beta)x = \alpha x+\beta x \quad \forall x, y \in X\, \forall\alpha, \beta \in \mathbb{K}$$

Examples of Linear Spaces

- $\mathbb{R}$, $\mathbb{C}$
- $\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n) : \ x_i \in \mathbb{R}, i = 1, \dots, n\}$, $\mathbb{C}^n$
- $l_\infty = \{x = (x_1, x_2, \dots) : x_i \in \mathbb{K}, i = \overline{1, \infty}, \ \sup_{i=\overline{1,\infty}} |x_i| < \infty\}$
- $l_1 = \{x = (x_1, x_2, \dots) : x_i \in \mathbb{K}, i = \overline{1, \infty}, \ \sum\limits_{i=1}^{\infty} |x_i| < \infty\}$
- $l_p = \{x = (x_1, x_2, \dots) : x_i \in \mathbb{K}, i = \overline{1, \infty}, \ \sum\limits_{i=1}^{\infty} |x_i|^p < \infty\}$
- $C[a, b] =$ the set of all continuous functions defined on $[a, b]$

You need to prove that for any two elements of a set their sum and a scalar product are also elements of the same set and all axioms hold.

Linear Independence

We say that elements $x_1, x_2, \ldots, x_n$ of a linear space X over $\mathbb{K}$ are **linearly independent** if the equality

$$\alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_n x_n = 0, \quad \alpha_i \in \mathbb{K}, i = \overline{1, n}$$

implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$. If there is at least one $\alpha_1 \neq 0$ then the elements $x_1, x_2, \ldots, x_n$ are called **linearly dependent**.

For example, consider two system of elements in $C[0, \pi/2]$:
$x_1(t) = 1$, $x_2(t) = \sin^2 t$, $x_3(t) = \cos 2t$ and
$x_1(t) = 1$, $x_2(t) = \sin 2t$, $x_3(t) = \cos 2t$.
Are they linearly dependent or independent? Consider $\alpha_1 x_1(t) + \alpha_2 x_2(t) + \alpha_3 x_3(t) = 0$ for all $t \in [0, \pi/2]$ and determine the values of α_i, $i = 1, 2, 3$ for the each case. What can you say about the elements of the linearly dependent system?

Exercises

Exercise 1: Prove that the system of elements of a linear space is linearly dependent if and only if one of those elements can be expressed as a linear combination of others.

Exercise 2: Prove that following systems are linearly independent

$$1, t, t^2, \dots, t^n;$$

$$e^{at}, te^{at}, t^2e^{at}, \dots, t^ne^{an},\ a \neq 0;$$

$$\cos at, \sin at, t\cos at, t\sin at, \dots, t^n\cos at, t^n\sin at,\ a \neq 0;$$

$$e^{at}\cos bt, e^{at}\sin bt, te^{at}\cos bt, te^{at}\sin bt, \dots,$$

$$t^ne^{at}\cos bt, t^ne^{at}\sin bt,\ a, b \neq 0.$$

Dimension

Dimension of a linear space X is the greatest number of linearly independent elements in X.
If $\dim X = n$ then a system $e^1, e^2, \dots, e^n$ of n linearly independent elements is said to be a **basis** in X provided any element $x \in X$ can be expressed a linear combination of basis elements

$$x = \sum_{i=1}^{n} x_i e^i.$$

Then $x = (x_1, x_2, ..., x_n)$ in the basis $\{e^i\}, i = \overline{1, n}$.
What is dimension of $\mathbb{R}$? $\dim \mathbb{R} = 1$. Why? What about $\dim \mathbb{R}^2, \dim \mathbb{R}^n, \dim C[a, b], \dim l_\infty, \dim l_1, \dim l_p$?

Dimension

- $\dim \mathbb{R}^{\mathbf{n}} = \mathbf{n}$

  Proof.

  Consider the elements $e^i = (0, \dots, 0, 1, 0, \dots, n)$, $e^i_i = 1$ of $\mathbb{R}^n$ for any $i = \overline{1, n}$ and prove that e^i are linearly independent. It will imply that $\dim \mathbb{R}^n \geq n$. Then, show that $\dim \mathbb{R}^n < n + 1$. To this aim, consider an arbitrary system of $n + 1$ elements in $\mathbb{R}^n$ and prove that it is linearly dependent. Use the property that any system of n linear equations with $n + 1$ independent variable has a non- trivial solution.

- $\dim \mathbf{C}[\mathbf{a}, \mathbf{b}] = \infty$

  Proof.

  Consider the functions $1, t, t^2, ..., t^n$ with $n \in \mathbb{N}$ and arbitrary scalars $\alpha_1, \alpha_2, \dots, \alpha_{n+1}$. If $\alpha_1 + \alpha_2 t + \alpha_3 t^2 + \dots \alpha_{n+1} t^n = 0$ then $\alpha_1 = \alpha_2 = \dots = \alpha_{n+1} = 0 \Rightarrow \{t^i\}$ is a basis but n is arbitrary $\Rightarrow$ infinite dimension.

Distance in Linear Spaces

How can we measure distance between two elements in a linear space? We need to introduce the notion of distance first!

- metric spaces
  A space with a metric $d : X \times X \to \mathbb{R}$
  $$1.\, d(x,y) \geq 0,\ d(x,y) = 0 \ \text{ iff}\, x = y,$$
  $$2.\, d(x,y) = d(y,x),\ 3.\, d(x,z) \leq d(x,y) + d(y,z)$$
- normed linear spaces
  A linear space with a norm $||\cdot|| : X \to \mathbb{R}$
  $1.\, ||x|| \geq 0,\ ||x|| = 0 \,\text{iff}\, x = 0,\ 2.\, ||\alpha x|| = |\alpha|||x||,$
  $3.\, ||x+y|| \leq ||x|| + ||y||$
- inner product spaces
  A linear space with an inner product $(\cdot,\cdot) : X \times X \to \mathbb{C}$
  $1.\, (x,x) \geq 0,\ (x,x) = 0,\ \text{iff}\, x = 0,\ 2.(\alpha x, y) = \alpha(x,y),\ (x+y,z) = (x,z) + (y,z)\, 3.(y,x) = \overline{(x,y)}$

Distance in Linear Spaces

Questions:

- Introduce metrics, norms and distances in the linear spaces $\mathbb{R}^n$, l_∞, l_p and $C[a,b]$.
- Is any metric space a linear space and vice versa? What about the connection between linear and inner product spaces?
- What do we need to know to prove that $||x|| = \sqrt{(x,x)}$?
- Are metrics, norms and inner products continuous functions? What is a continuous function?

The Wronskian

The **Wronskian** of n smooth enough functions is defined by

$$W[f_1, f_2, \dots, f_n](t) = \begin{vmatrix} f_1(t) & f_2(t) & \dots & f_n(t) \\ f_1'(t) & f_2'(t) & \dots & f_n'(t) \\ \dots & \dots & \dots & \dots \\ f_1^{(n-1)}(t) & f_2^{(n-1)}(t) & \dots & f_n^{(n-1)}(t) \end{vmatrix}$$

What is a smooth enough function? A continuously differentiable function.

For example, if $f_1(t) = 2t - 1$ and $f_2(t) = t^2$ then

$$W[f_1, f_2](t) = \begin{vmatrix} 2t-1 & t^2 \\ 2 & 2t \end{vmatrix} = 4t^2 - 2t - 2t^2 = 2t^2 - 2t.$$

Exercise: Prove that the Wronskian of linearly dependent functions vanishes.

What happens if $W[f_1, \dots, f_n](t) \neq 0$? Functions are linearly independent!

The Wronskian

The **Wronskian** of n elements $x^1(t), x^2(t), \dots, x^n(t)$ of n components each is

$$W[x^1, x^2, \dots, x^n](t) = det([x^1, x^2, \dots, x^n]) = \begin{vmatrix} x_{11} & x_{21} & \dots & x_{n1} \\ x_{12} & x_{22} & \dots & x_{n2} \\ \dots & \dots & \dots & \dots \\ x_{1n} & x_{2n} & \dots & x_{nn} \end{vmatrix}.$$

If $W[x^1, x^2, \dots, x^n](t_0) \neq 0$ at some point t_0 then the system $x^1(t), x^2(t), \dots, x^n(t)$ is linearly independent. Why? How to prove that elements $x^1, x^2, ..., x^n$ are linearly dependent iff $det([x^1, x^2, \dots, x^n]) = 0$? $\Rightarrow$ Recall the theory of systems of linear algebraic equations.

Systems of Linear Algebraic Equations

Consider a system of n linear algebraic equations in n unknown written in the matrix form

$$Ax = b,$$

where $A = (a_{ij})$ is the $n \times n$ matrix, $x = (x_1, x_2, \ldots, x_n)$ is unknown and $b = (b_1, b_2, \ldots, b_n)$ is given.

- if there exists A^{-1} then $x = A^{-1}b$. What is the condition for existence of A^{-1}?
- if $\det A \neq 0$ and $b = 0$ then the only solution is the trivial one $\Rightarrow x = 0$.
- if $\det A = 0$, $b = 0$ then the system has infinitely many nonzero solutions including the trivial solution.
- if $\det A = 0$, $b \neq 0$ then the system has no solutions but if b satisfies the condition $\sum b_i y_i = 0$ for all $y = (y_1, y_2, \ldots, y_n)$ such that $\bar{A}^T y = 0$.

Systems of Linear Algebraic Equations

How can we find a solution? Cramer's rule, Gaussian elimination

Consider the procedures for the sample system

$$\begin{cases} 2x_1 - x_2 + x_3 = -3, \\ x_1 + 2x_2 + 2x_3 = 5, \\ 3x_1 - 2x_2 - x_3 = -8 \end{cases}$$

and apply Cramer's method and Gaussian elimination to solve it.

Outline

Eigenvalues and Eigenvectors

Consider the equation $Ax = y$, where $A = (a_{ij})_{n \times n}$ is a matrix of scalars. Recall that any matrix defines a mapping on finite dimensional linear spaces and vice versa. How can we prove it? Therefore, the equation shows that a vector x is mapped to another vector y through the linear mapping defined by the matrix A. Denote $y = \lambda x$ where $\lambda \in \mathbb{K}$ and obtain

$$Ax = \lambda x.$$

Definition. The value of λ for which there are nonzero vectors x satisfying the eq. is called the eigenvalue of A, and those nonzero vectors x are called the eigenvectors of A associated with λ. How can we find x? $(A - \lambda I)x = 0 \Rightarrow det(A - \lambda I) = 0$ for nonzero solutions. This is a polynomial equation of degree n whose n roots are the eigenvalues of A. The roots can be real or complex, single or repeated, or any combination of these cases.

Eigenvalues and Eigenvectors

$$A = \begin{pmatrix} 5 & -4 \\ 8 & -7 \end{pmatrix} \quad \Rightarrow det(A-\lambda I) = \begin{vmatrix} 5-\lambda & -4 \\ 8 & -7-\lambda \end{vmatrix} = \lambda^2+2\lambda-3$$

There are two eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -3$. For the eigenvalue $\lambda = 1$, $(A-\lambda)x = 0$, and

$$\begin{pmatrix} 4 & -4 \\ 8 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -4x_1 \quad -4x_2 &= 0 \\ 8x_1 \quad -8x_2 &= 0 \end{aligned}$$

Thus, $x_1 - x_2 = 0$. $\Rightarrow$ if $x_1 = a$ then

Eigenvalues and Eigenvectors

$$x = \begin{pmatrix} a \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; a \neq 0$$

Similarly, for the eigenvalue $\lambda = -3$: $(A + 3I)x = 0$ and hence, $2x_1 - x_2 = 0$

$$x = \begin{pmatrix} b \\ 2b \end{pmatrix} = b \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \; b \neq 0$$

Consider another case

$$A = \begin{pmatrix} 0 & 0 & -1 \\ -1 & 1 & -1 \\ 2 & 0 & 3 \end{pmatrix}$$

Eigenvalues and Eigenvectors

Then $\det(A-\lambda I) = \begin{vmatrix} -\lambda & 0 & -1 \\ -1 & 1-\lambda & -1 \\ 2 & 0 & 3-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = \lambda_3 = 1$.

The eigenvector corresponding to $\lambda_1 = 2$ are nonzero solutions of the system $(A-2I)x = 0$

$$-2x_1 - x_3 = 0,$$

$$-x_1 - x_2 - x_3 = 0,$$

$$2x_1 + x_3 = 0.$$

Therefore,

$$x = \begin{pmatrix} a \\ a \\ -2a \end{pmatrix}$$

Eigenvalues and Eigenvectors

What is the eigenvector corresponding $\lambda_2 = \lambda_3 = 1$?

$$x = \begin{pmatrix} -c \\ b \\ c \end{pmatrix} = b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \; b, c \neq 0$$

Exercise: Compute eigenvalues and corresponding eigenvectors for the following matrices

$$\begin{pmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 2 \\ -2 & 3 & -4 \\ 1 & -1 & 3 \end{pmatrix}, \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 2 \\ 2 & -2 & 0 \end{pmatrix}$$

Eigenvalues and Eigenvectors

- If a real matrix A has a complex eigenvalue λ and x is a corresponding eigenvector, then the complex conjugate $\bar{\lambda}$ is also an eigenvalue with y, the conjugate vector of x, as a corresponding eigenvector. Prove it.
- A matrix A has an inverse matrix A^{-1} if and only if it does not have zero as an eigenvalue. If $\lambda_1, \ldots, \lambda_n \neq 0$ are the eigenvalues of A, then the eigenvalues of A^{-1} are $1/\lambda_1, \ldots, 1/\lambda_n$. Prove it.
- The eigenvalues of A are the same as the eigenvalues of A^T.
- If λ is an eigenvalue of A with an eigenvector x, then λ^k is an eigenvalue of A^k with a corresponding eigenvector x, $c\lambda$ is an eigenvalue of cA with a corresponding eigenvector x, and $C_m\lambda^m + C_{m-1}\lambda^{m-1} + \ldots + C_1\lambda + C_0$ is an eigenvalue of $C_mA^m + C_{m-1}A^{m-1} + \ldots + C_1A + c_0I$ with a corresponding eigenvector x. Prove it.

Eigenvalues and Eigenvectors

Theorem. Eigenvectors associated with distinct eigenvalues are linearly independent.

Proof. Consider the case $n = 2$, you can prove that it will hold for any n by induction.
Let A be a 2×2 matrix with eigenvalues $\lambda_1 \neq \lambda_2$ and the corresponding eigenvectors x^1, x^2. If

$$C_1 x^1 + C^2 x^2 = 0$$

then

$$0 = (A - \lambda_1 I)(C_1 x^1 + C_2 x^2) = C_1 (A - \lambda_1 I) x^1 +$$

$$+ C_2 (A - \lambda_1 I) x^2 = C_2 (A x^2 - \lambda_1 x^2) = C_2 (\lambda_2 - \lambda_1) x^2,$$

and hence, $C_2 = 0$. Prove that $C_1 = 0$ as well. $\Rightarrow$ x^1, x^2 are linearly independent.

Diagonalizable matrices

Two $n \times n$ matrices A and B are said to be **similar** if there exists a matrix T (similarity transformation matrix), such that

$$B = T^{-1}AT.$$

In this case

- $\det(B-\lambda I) = \det(T^{-1}AT - T^{-1}\lambda IT) = \det(T^{-1}(A-\lambda I)T) = \det(T^{-1}) \cdot \det(A-\lambda I) \cdot \det(T) = \det(A-\lambda I)$.
- if λ is an eigenvalue of B with the corresponding eigenvector x, then $A(Tx) = T(T^{-1}AT)x = T(Bx) = T(\lambda x) = \lambda(Tx)$.

Therefore, if A and B are similar matrices them they have the same characteristic polynomial, the same eigenvalues, and if x is an eigenvector of B then Tx is the eigenvector of A.
Prove that $A^k = TB^kT^{-1}$, $k = 1, 2, \ldots$.

Diagonalizable matrices

Prove that the eigenvalues of an upper triangular matrix and a lower triangular matrix are the diagonal elements.

A matrix A is said to be **diagonalizable** if it is similar to a diagonal matrix B.

Theorem. An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Proof.

1. If A is diagonalizable, i.e. $D = T^{-1}AT$, $D = diag(d_1, \ldots, d_n)$, then A has the same eigenvalues as D, i.e. $d_1, \ldots, d_n$. The corresponding eigenvectors $e^1 = (1, 0, \ldots, 0), e^2 = (0, 1, \ldots, 0), \ldots, e^n = (0, \ldots, 0, 1)$ are linearly independent. $\Rightarrow$ the eigenvectors $Te^1, Te^2, \ldots, Te^n$ of A are linearly independent as well.

Diagonalizable matrices

2. If A has n linearly independent eigenvectors $x^1, x^2, \dots, x^n$, $Ax^i = \lambda_i x^i$. Denote by T the matrix whose columns are vectors $(x^1, \dots, x^n)$. Then the rank of T is n, and T^{-1} exists.

$$(x^1 \dots x^n) \cdot diag(\lambda_1, \dots, \lambda_n) = (\lambda_1 x^1 \dots \lambda_n x^n) =$$

$$= (Ax^1 \dots Ax^n) = A(x^1 \dots x^n) = AT \Rightarrow$$

$$diag(\lambda_1, \dots, \lambda_n) = T^{-1}AT$$

Exercise: Prove that the matrix $A = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{pmatrix}$ can be diagonalized as follows

$$A = \begin{pmatrix} 2 & -2 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1/5 & 3/5 & 1/5 \\ -2/5 & 4/5 & 3/5 \\ 1/5 & -2/5 & 1/5 \end{pmatrix}$$