vv214: Markov Chains. Matrix Norms. Ranking Problem.

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1. Markov Chains.
2. Matrix Norms.
3. Ranking Problem.

Markov Chains

- We are given a set of states and initial probability distribution over states.
- A Markov chain is a memory-less stochastic process where $X_t = i$ is the state at time t and the probability of reaching the state $X_{t+1} = j$ at time $t+1$ is

$$\mathbb{P}(X_{t+1} = j | X_t = i) = p_{ij}$$

- It depends on the current state but on time or other variables.
- $T = (p)_{ij}$ is the transition matrix $p_{ij} \geq 0$

$$\sum_{i=1}^{n} p_{ij} = 1 \qquad \sum_{j=1}^{n} p_{ij} = 1$$

left stochastic matrix   right stochastic matrix

- Let $q_t \in \mathbb{R}^n$ be the distribution of the process at time t then

$$q_{t+1} = T q_t \quad \text{OR} \quad q_{t+1}^T = q_t^T T$$

Markov Chains

Let $A_{n\times n}$ be a right (left) stochastic matrix.
Definition: A non-zero vector $\bar{x} \in \mathbb{R}^n$ is called the right eigenvector of the matrix A if

$$A\bar{x} = \lambda_R \bar{x} \Rightarrow \det(A - \lambda_R I) = 0$$

Definition: A non-zero vector $\bar{x} \in \mathbb{R}^n$ is called the left eigenvector of the matrix A if

$$\bar{y}^T A = \lambda_L \bar{y}^T \Rightarrow \det(A^T - \lambda_L I) = 0$$

Are left and right eigenvalues the same?

$$0 = \det(A^T - \lambda_L I) = \det(A^T - \lambda_L I^T) = \det(A - \lambda_L I)^T = \det(A - \lambda_L I)$$

$$\lambda_L = \lambda_R$$

Operator Norms

Definition: A norm of a linear operator $T : X \to Y$ is

$$||T|| = \max_{x \in X : ||x||=1} ||Tx|| \quad \text{OR} \quad ||T|| = \max_{x \in X : x \neq 0} \frac{||Tx||}{||x||}$$

Exercise: Show that all the properties of a norm are satisfied.

Definition: A linear operator is said to be bounded if $||T|| < +\infty$

Lemma: T is bounded iff $\exists C > 0 \quad ||Tx|| \leq C \cdot ||x||$

Operator Norms

Show that

$$||A||_1 = \max_{j=1,n} \sum_{i=1}^{n} |a_{ij}| \qquad ||A||_\infty = \max_{i=1,n} \sum_{j=1}^{n} |a_{ij}|$$

Operator Norms

Operator Norms

Define

$$||A||_F = \sqrt{\sum_{i,j} a_{ij}^2} = \sqrt{trace(A^T A)}$$

Show that

$||A||_2 = \max\limits_{||\bar{x}||_2 = 1} ||A\bar{x}||_2 = \sqrt{\lambda_{max}}$, λ_{max} is the largest eigenvalue of $A^T A$

Operator Norms

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{Find} \quad ||A||_1, ||A||_\infty, \; ||A||_F, \; ||A|_2$$

Exercise: Show that $\forall \bar{x} \in \mathbb{R}^n \quad ||\bar{x}||_\infty \leq ||\bar{x}||_2 \leq ||\bar{x}||_1$

- $\forall \bar{x} \in \mathbb{R}^n \quad ||\bar{x}||_i \leq \alpha ||\bar{x}||_j$ where α is the (i, j) of the matrix

$$\begin{pmatrix} \star & \sqrt{n} & n \\ 1 & \star & \sqrt{n} \\ 1 & 1 & \star \end{pmatrix}$$

- $\forall A_{n \times n} \quad ||A||_i \leq \alpha ||A||_j$ where α is the (i, j) of the matrix

$$\begin{pmatrix} \star & \sqrt{n} & n & \sqrt{n} \\ \sqrt{n} & \star & \sqrt{n} & 1 \\ n & \sqrt{n} & \star & \sqrt{n} \\ \sqrt{n} & \sqrt{n} & \sqrt{n} & \star \end{pmatrix}$$

Operator Norms

MATLAB

- `norm(A,2)` returns $||A||_2$
- `norm(A)` is the same as `norm(X,2)`
- `norm(A,'fro')` returns $||A||_F$
- `norm(V,P)` returns the p-norm of V
- `norm(V,Inf)` returns $||V||_\infty$

Eigenvalues of a Stochastic matrix

Ranking Problem

- Consider the results of a tournament

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[dl] \\
C \arrow[r] & D \arrow[ul]
\end{tikzcd}
$$

where "$A \to B$" means A defeated B.

How to rank the players fairly?

- The Page-Brin idea:it should *be worth more* to defeat a *better player* $\Leftarrow$ based on Perron (1907)-Frobenius (1912) Theorem

How do we know who is better before ranking them?

- Define recursion!

Ranking Problem

* Give everyone the initial score of 1

$$\bar{x}_0 = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \begin{array}{c} A \\ B \\ C \\ D \end{array}$$

* Define for all $n \geq 0$

$$\bar{x}_{n+1} = A\bar{x}_n,$$

where

$$A = \begin{array}{c} \\ A \\ B \\ C \\ D \end{array} \begin{array}{c} \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

Ranking Problem

$$\bar{x}_1 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

$$\bar{x}_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

$$\bar{x}_3 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 2 \\ 3 \end{pmatrix} \begin{matrix} A \\ B \\ C \\ D \end{matrix}$$

The (n+1)th score of a player A is the sum of the nth scores of the players that the player A defeated.

Ranking Problem

$$\bar{x}_5 = \begin{pmatrix} 8 \\ 6 \\ 3 \\ 5 \end{pmatrix}, \bar{x}_{10} = \begin{pmatrix} 35 \\ 34 \\ 21 \\ 26 \end{pmatrix}, \bar{x}_{100} = \begin{pmatrix} 1037 \\ 933 \\ 547 \\ 731 \end{pmatrix}$$

Is $A > B > D > C$? $\Rightarrow$ analyze $\bar{x}_n = A^n \bar{x}_0$ as $n \to \infty$

Theorem (Perron-Frobenius): There exists a *largest positive* eigenvalue λ_{PF} for a nonnegative matrix A such that the rescaled system

$$\bar{x}_n = \left(\frac{1}{\lambda_{PF}} A \right)^n \bar{x}_0$$

converges to an equilibrium state $\bar{x}_\infty$.

$$\bar{x}_\infty = \bar{x}_{\infty+1} = \frac{1}{\lambda_{PF}} A \bar{x}_\infty \Rightarrow A\bar{x}_\infty = \lambda_{PF} \bar{x}_\infty$$

The equilibrium state is the eigenvector associated with λ_{PF}!!!

Ranking Problem

The largest positive eigenvalue is

$$\lambda_{PF} = 1.3953369...$$

and

$$\bar{x}_\infty = \begin{array}{cc} \begin{pmatrix} 0.321... \\ 0.288... \\ 0.165... \\ 0.230... \end{pmatrix} & \begin{array}{c} A \\ B \\ C \\ D \end{array} \end{array}$$

$$A > B > D > C$$