Vv156 Lecture 27

Jing Liu

UM-SJTU Joint Institute

December 6, 2018

Defintion

A differential equation is an equation of
an unknown function and its derivatives.

- For some unknown function

$$y = y(t)$$

the following is an example of differential equations

$$\frac{dy}{dt} = y^2 \iff \dot{y} = y^2 \iff y' = y^2$$

Q: What does the above equation say regarding the unknown function

$$y = y(t)$$

- A differential equation provides a way to specify/define a function just like an algebraic equation specifies a number

$$x^3 = 1$$

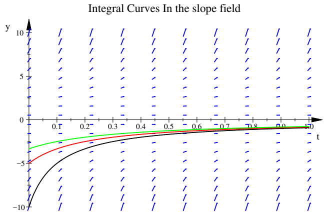
- Matlab can solve this simple equation symbolically, the output is given as

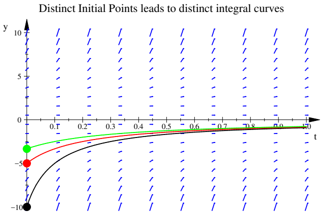
$$\left\{0,\ -\frac{1}{C+t}\right\}$$

where $y = 0$ is known as the trivial solution, and the second part

$$y = -\frac{1}{C+t} \qquad \text{where } C \text{ is an arbitrary constant.}$$

represents a family of solutions, which seem to have different initial points.





- Note a solution to a differential equation is a function instead of a number.

- If we demand the solution to have value $y = -10$ at $t = 0$ while satisfying

$$\dot{y} = y^2$$

we can narrow it down to a single solution in this case.

Definition

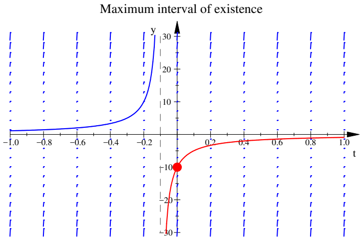
A differential equation with conditions for the value of the unknown function and possibly its derivatives at one particular point in the domain is known as an initial value problem.

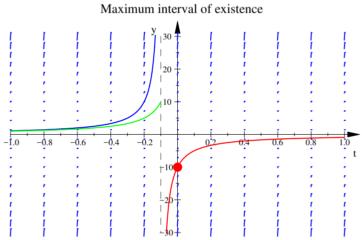
- For example, the following is an initial value problem (IVP)

$$\dot{y} = y^2, \qquad y(0) = -10$$

Q: What is the domain of the solution $y = -\dfrac{1}{0.1 + t}$ to the initial value problem

$$\dot{y} = y^2, \qquad y(0) = -10$$





- The branch in blue certainly satisfies the differential equation, but it is not relevant to the initial condition. For the green branch is just as good

$$y(t) = \begin{cases} -\frac{1}{0.1+t} & \text{for} \quad t > -0.1 \\ -\frac{1}{t} & \text{for} \quad t \leq -0.1 \end{cases}$$

  that is, a piecewise function is just as good as a function that can be written using a single formula. If we allow such functions, the solution is not unique.
- However, if we consider a smaller interval, e.g. $t > 0$, the solution is unique.
- So the domain of a solution might be smaller than the usual set in which the expression is defined.

Q: Having an essential discontinuity is problematic, and we may have to reduce the domain, but how about a solution that is everywhere continuous?

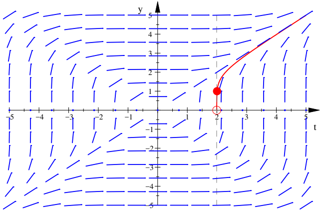
- For example, the initial value problem

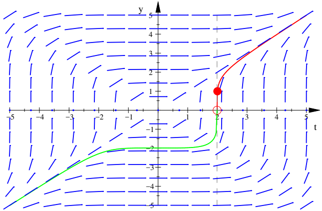
$$\dot{y} = \frac{t^4}{y^4}, \qquad y(2) = 1$$

is known to have the following solution, which is continuous everywhere,

$$y = \sqrt[5]{t^5 - 31}$$

Q: Why we have to exclude $t = \sqrt[5]{31}$.





- So we expect the following to be reasonable over an open interval.

Definition

A function

$$y = y(t)$$

defined on an interval $a < t < b$, is called a solution of the differential equation

$$\dot{y} = \Phi(t, y)$$

provided that y is a differentiable function of t on the interval

$$a < t < b$$

and the equation is defined and satisfied for every t in $a < t < b$.

- Note we say "a" solution rather than "the" solution. A differential equation, if it has a solution at all, usually has more than one solution.
- The solution to an initial value problem is a particular solution that satisfies the initial condition as well as satisfying the differential equation.

Exercise

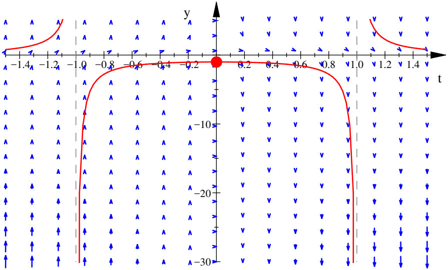
(a) Verify that $y = \dfrac{1}{t^2 + c}$, where c is an arbitrary constant, is a solution to

$$\dot{y} + 2ty^2 = 0$$

(b) Find the particular solution that satisfies $y(0) = -1$.

(c) Find the maximum interval on which the solution you found above is valid.

Maximum interval of existence



- To study how to solve differential equations, it is essential to classify them since different classes of equations often need to be solve differently.

Defintion

An ordinary differential equation (ODE) contains only derivatives of one or more dependent variables with respect to a single independent variable.

Q: Are the following equations ODEs?

$$\dot{y} + 5y = e^t \qquad \text{and} \qquad \dot{x} + \dot{y} = 2x + y$$

- ODEs are different from partial differential equations (PDEs), which involve partial derivatives of two or more independent variables.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{and} \qquad \operatorname{div}(\rho \mathbf{v}) + \frac{\partial \rho}{\partial t} = 0$$

Defintion

The order of a differential equation is the order of the highest-order derivative that occurs in the equation

Q: What is the order of the following differential equations?

$$\dddot{y} + 2e^t\ddot{y} + y\dot{y} = t^4 \qquad \text{and} \qquad \left(\dot{y}\right)^2 + t\left(\dot{y}\right)^3 + 4y = 0$$

- We will first consider first-order equations

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t), \qquad \dot{y}y = 2, \qquad \dot{y} = y^2 \qquad \text{and} \qquad \left(\dot{y}\right)^2 = 4y$$

  which might be linear or nonlinear.

- A linear first-order equation involves only $\alpha(t)y$ and $\beta(t)\dot{y}$ "by themselves".
- If any term involves a function of y or $\dot{y}$ in the equation, then it is nonlinear.

Defintion

An nth-order ordinary differential equation

$$F(t, y, y', y'', \dots, y^{(n)}) = f(t)$$

is said to be linear if F is a linear function in terms of y, y', …, $y^{(n)}$.

- A function F is said to be linear in terms of y, y', …, $y^{(n)}$ if and only if

$$F(t, ay, ay', \dots, ay^{(n)}) = aF(t, y, y', \dots, y^{(n)})$$

$$F(t, y_1 + y_2, \dots, y_1^{(n)} + y_2^{(n)}) = F(t, y_1, y_1', \dots, y_1^{(n)}) + F(t, y_2, y_2', \dots, y_2^{(n)})$$

Q: Which of the followings is a linear differential equation?

1. $\ddot{y} + \dfrac{1}{\dot{y}} = 0$
2. $\ddot{y} + \sin y = 0$
3. $t^2\ddot{y} + yt^2 \sin t = t \cos t$
4. $\dddot{y} = \exp(\ddot{y})$

- Every first-order linear equation

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t)$$

can be written in the following standard form since $\alpha(t) \neq 0$,

$$\dot{y} + P(t)y = Q(t) \qquad \text{where} \qquad P(t) = \frac{\beta(t)}{\alpha(t)} \qquad \text{and} \qquad Q(t) = \frac{\gamma(t)}{\alpha(t)}$$

- For example,

$$(4+t^2)\dot{y} + 2ty = 4t \implies \dot{y} + \frac{2t}{4+t^2}y = \frac{4t}{4+t^2}$$

- Since it is a first-order equation, we could investigate it using a slope field.

$$\dot{y} = \Phi(t,y) = -\frac{2t}{4+t^2}y + \frac{4t}{4+t^2} = \frac{2t(2-y)}{4+t^2}$$

- For "simple" first-order differential equations, like this one,

$$(4+t^2)\dot{y} + 2ty = 4t$$

we can solve it by integrating both sides of the equation directly

$$\int \left[(4+t^2)\dot{y} + 2ty\right] dt = \int 4t \, dt \qquad \text{the product/chain rule in reverse}$$

$$\int \frac{d}{dt}\left[\,(4+t^2)\,y\right] dt = 2t^2 + C_1$$

$$(4+t^2)y = 2t^2 + C_2 \implies y = \frac{2t^2 + C_2}{4+t^2}$$

- However, not every linear first-order equation can be solve in this way,

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t)$$

Q: When can we solve it in this way by directly integrating both sides?

Q: How can we solve the following equation?

$$\underbrace{(4+t^2)e^t}_{\alpha}\dot{y} + \underbrace{2te^t}_{\beta}y = 4te^t$$

- Note that β is NOT the derivative of α, however, multiplying both sides by

$$\frac{1}{e^t}$$

we obtain the previous equation without changing the underlying solutions.

$$(4+t^2)\dot{y} + 2ty = 4t$$

Q: What does the above example suggest we shall do for a general equation of

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t)$$

- Suppose there exists a function μ of t such that

$$\mu\beta = \frac{d}{dt}\Big(\mu\alpha\Big)$$

where α and β are functions of t in front of $\dot{y}$ and y, then

$$\alpha\dot{y} + \beta y = \gamma$$

$$\implies \mu\alpha\dot{y} + \mu\beta y = \mu\gamma$$

$$\Big(\mu\alpha\Big)\dot{y} + y\frac{d}{dt}\Big(\mu\alpha\Big) = \mu\gamma \qquad \text{the product/chain rule in reverse}$$

$$\frac{d}{dt}\Big[(\mu\alpha)y\Big] = \mu\gamma \implies y = \frac{\displaystyle\int \mu\gamma\, dt}{\mu\alpha}$$

Exercise

Solve the following initial value problem

$$t\dot{y} + 2y = 4t^2, \qquad y(1) = 2$$

- The function μ is known as the integrating factor, we will show it always exists and has the form given in the next theorem for any linear first-order eq.

Theorem

For a linear first-order equation

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t)$$

the integrating factor for the equation is

$$\mu = \frac{A}{\alpha} \exp\left(\int \frac{\beta}{\alpha}\, dt\right), \qquad \text{where } A \text{ is an arbitrary constant.}$$

and the general solution of the equation, which gives all possible solutions, is

$$y = \frac{\displaystyle\int \mu\gamma\, dt}{\mu\alpha}$$

Q: Why it is not a surprise to see the constant A?

Exercise

Find the solution of the initial-value problem,

$$\cos(t)\dot{y} + \sin(t)y = 2\cos^3(t)\sin(t) - 1, \qquad y\left(\frac{\pi}{4}\right) = 3\sqrt{2} \quad \text{for} \quad 0 \le t < \frac{\pi}{2}$$

- To prove the last theorem, we need to find μ for arbitrary α and β such that

$$(\mu\beta) = \frac{d}{dt}\Big(\mu\alpha\Big) = \dot{\mu}\alpha + \dot{\alpha}\mu$$

- Therefore the integrating factor is a solution to the differential equation

$$\alpha\dot{\mu} = (\beta - \dot{\alpha})\,\mu$$

- This equation is a homogeneous linear first-order equation of μ.

$$\alpha\dot{\mu} - (\beta - \dot{\alpha})\,\mu = 0$$

which is always separable and can be solved using the next theorem.

Definition

A first-order differential equation is called separable if it can be written in the form

$$\dot{y} = GF$$

where G is only a function of y and F is only a function of t.

- Note a separable equation is linear if and only if

$$G(y) = -ay + b$$

- For any α and β, the function $\mu = 0$ is always a solution to the equation

$$\alpha\dot{\mu} = (\beta - \dot{\alpha})\,\mu$$

- But we are interested in non-trivial solutions, that is, not identically zero.

Theorem

If $G(y)$ and $F(x)$ are continuous, then a separable equation has the solution,

$$\int \frac{1}{G}\,dy = \int F\,dt$$

Proof

Given a separable equation $\dot{y} = GF$, we rearrange to obtain

$$\frac{1}{G(y)}\frac{dy}{dt} = F(t)$$

Write $\dfrac{1}{G(y)}$ and $F(t)$ as the derivative of their antiderivative using FTC,

$$\frac{d}{dy}\left(\int_{y_0}^{y}\frac{1}{G(\eta)}\,d\eta\right)\frac{dy}{dt} = \frac{d}{dt}\left(\int_{t_0}^{t}F(\tau)\,d\tau\right)$$

Use the chain rule in reverse for the left-hand side

$$\frac{d}{dt}\left(\int_{y_0}^{y}\frac{1}{G(\eta)}\,d\eta\right) = \frac{d}{dt}\left(\int_{t_0}^{t}F(\tau)\,d\tau\right)$$

Two functions have the same derivative must only differ by an additive constant

$$\int_{y_0}^{y}\frac{1}{G(\eta)}\,d\eta = \int_{t_0}^{t}F(\tau)\,d\tau + C \iff \int\frac{1}{G}\,dy = \int F\,dt$$

- Let us see how this theorem leads to the formula for the integrating factor

$$\begin{aligned} \alpha\dot{\mu} = (\beta - \dot{\alpha})\,\mu \implies \int \frac{1}{\mu}\,d\mu &= \int \frac{\beta - \dot{\alpha}}{\alpha}\,dt \\ \implies \quad \ln|\mu| &= \int \frac{\beta}{\alpha}\,dt - \ln|\alpha| \end{aligned}$$

- Exponentiating both sides of the last equation

$$\begin{aligned} |\mu| = \frac{1}{|\alpha|}\exp\left(\int \frac{\beta}{\alpha}\,dt\right) &= \frac{1}{|\alpha|}\exp\left(\int \frac{\beta}{\alpha}\,dt + A_1\right) \\ &= \frac{A_2}{|\alpha|}\exp\left(\int \frac{\beta}{\alpha}\,dt\right) \\ \implies \mu &= \frac{A}{\alpha}\exp\left(\int \frac{\beta}{\alpha}\,dt\right) \end{aligned}$$

- In the standard form a linear first-order equation has the general solution of

$$y = \frac{1}{\mu}\left(\int \mu Q\,dt + C\right) \qquad \text{where} \quad \mu = A\exp\left(\int P\,dt\right)$$

Exercise

(a) Solve the following differential equation

$$\frac{dy}{dt} = -2ty$$

(b) Solve the following initial value problem

$$(1 - y^2)\dot{y} = t^2, \qquad y(1) = 3$$

(c) Find all solutions of the following differential equation.

$$\dot{y} = y(1 - y)$$

(d) Suppose an object of mass m is falling from rest near sea level. Assume the air resistance is proportional to the velocity of the object, and the drag coefficient is K. Derive a model for the motion of the object using Newton's second law, then solve it to find the velocity function.

- Recall a first-order equation is linear if it has the following form

$$\alpha(t)\dot{y} + \beta(t)y = \gamma(t)$$

- Similarly, a second-order differential equation is linear if it has the form

$$\alpha(t)\ddot{y} + \beta(t)\dot{y} + \gamma(t)y = f(t) \iff \ddot{y} + P(t)\dot{y} + Q(t)y = R(t)$$

Definition

For every second-order linear equation,

$$\alpha(t)\ddot{y} + \beta(t)\dot{y} + \gamma(t)y = f(t)$$

the following homogeneous equation is called the complementary equation

$$\alpha(t)\ddot{y} + \beta(t)\dot{y} + \gamma(t)y = 0$$

or the corresponding homogeneous equation to the original equation.

- We will see later that often the corresponding homogeneous equation has to be solved first in order to solve the original equation. So we will start with

$$a\ddot{y} + b\dot{y} + cy = 0, \qquad \text{where } a,\ b \text{ and } c \text{ are constants.}$$

Q: For example, can you think of a simple solution to the following equation?

$$\ddot{y} - y = 0$$

- To solve this equation, we are basically asking ourselves to find a function

$$y(t)$$

so that the 2nd derivative of the function is the same as the function itself.

$$\phi_1 = e^t$$

Q: Is there any other function has this property?

Principle of Superposition

If ϕ_1 and ϕ_2 are two solutions of a second-order homogeneous linear equation

$$\alpha(t)\ddot{y} + \beta(t)\dot{y} + \gamma(t)y = 0$$

then the function

$$y = C_1\phi_1 + C_2\phi_2,$$ where and C_1 and C_2 are two arbitrary constants.

is also a solution of the homogeneous equation.

Proof

$$\begin{aligned}
\alpha y'' + \beta y' + \gamma y &= \alpha\Big(C_1\phi_1 + C_2\phi_2\Big)'' + \beta\Big(C_1\phi_1 + C_2\phi_2\Big)' + \gamma\Big(C_1\phi_1 + C_2\phi_2\Big) \\
&= \alpha\Big(C_1\phi_1'' + C_2\phi_2''\Big) + \beta\Big(C_1\phi_1' + C_2\phi_2'\Big) + \gamma\Big(C_1\phi_1 + C_2\phi_2\Big) \\
&= C_1\underbrace{\left(\alpha\phi_1'' + \beta\phi_1' + \gamma\phi_1\right)}_{0} + C_2\underbrace{\left(\alpha\phi_2'' + \beta\phi_2' + \gamma\phi_2\right)}_{0} \\
&= 0
\end{aligned}$$

- Given the following IVP, and solutions ϕ_1 and ϕ_2

$$\alpha\ddot{y} + \beta\dot{y} + \gamma y = 0; \qquad y(t_0) = y_0, \quad \dot{y}(t_0) = y_1$$

we can try to determine the arbitrary constants c_1 and c_2 which must satisfy

$$C_1\phi_1(t_0) + C_2\phi_2(t_0) = y_0; \qquad C_1\phi_1'(t_0) + C_2\phi_2'(t_0) = y_1$$

Q: Can we always solve this linear system?

- Upon solving the above system,

$$C_1 = \frac{y_0\phi_2'(t_0) - y_1\phi_2(t_0)}{\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0)}, \quad C_2 = \frac{-y_0\phi_1'(t_0) + y_1\phi_1(t_0)}{\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0)}$$

- So C_1 and C_2 can be determined as long as the denominator is nonzero.

$$\phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0) \neq 0$$

- The denominator is actually a determinant of a 2×2 matrix.

Definition

The determinant is called the Wronskian determinant, or simply the Wronskian,

$$W(\phi_1, \phi_2)(t_0) = W(t_0) = \det \begin{bmatrix} \phi_1(t_0) & \phi_2(t_0) \\ \phi_1'(t_0) & \phi_2'(t_0) \end{bmatrix} = \phi_1(t_0)\phi_2'(t_0) - \phi_1'(t_0)\phi_2(t_0)$$

Theorem

Suppose ϕ_1 and ϕ_2 are two solutions to the following equation and $W(t_0) \neq 0$,

$$\alpha \ddot{y} + \beta \dot{y} + \gamma y = 0$$

then there is a choice of c_1 and c_2 such that the initial conditions are satisfied.

$$y(t_0) = y_0, \quad \dot{y}(t_0) = y_1$$

Exercise

Solve the initial-value problem $\quad \ddot{y} - y = 0, \quad y(0) = 1, \quad \dot{y}(0) = 1.$

- Now let us get back to the general equation with constant coefficients

$$a\ddot{y} + b\dot{y} + cy = 0$$

Q: What is the above equation actually stating?

- Given the values of a, b and c, there might be an exponential function

$$y = e^{rt}$$

may satisfy the equation for some value of r

$$\dot{y} = re^{rt} \implies \ddot{y} = r^2 e^{rt}$$

- Substitute the function and its derivatives into the equation,

$$\begin{aligned} a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0 &\implies (ar^2 + br + c)e^{rt} = 0 \\ &\implies ar^2 + br + c = 0 \end{aligned}$$

- The quadratic equation

$$ar^2 + br + c = 0$$

is known as the characteristic equation of the differential equation

$$a\ddot{y} + b\dot{y} + cy = 0$$

- The form of the solution to the original equation depends on the discriminant

$$\Delta = b^2 - 4ac$$

- If $\Delta > 0$, then the following is a solution to the original differential equation

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- If $\Delta = 0$, then the following is a solution to the original differential equation

$$y = \left(C_1 + C_2 t\right) e^{rt}$$

- If $\Delta < 0$, then the following is a solution to the original differential equation

$$y = e^{Rt}\Big(C_1 \cos\theta t + C_2 \sin\theta t\Big)$$

- Case: $b^2 - 4ac > 0$
- The solutions r_1 and r_2 of the characteristic equation are real and distinct,

$$r_1 \neq r_2$$

- Thus the two solutions to the original differential equation are simply

$$\phi_1 = e^{r_1 t} \quad \text{and} \quad \phi_2 = e^{r_2 t}$$

- The Wronskian of ϕ_1 and ϕ_2 is

$$W = \phi_1 \phi_2' - \phi_1' \phi_2 = e^{r_1 t} r_2 e^{r_2 t} - e^{r_2 t} r_1 e^{r_1 t} = (r_2 - r_1) e^{(r_1 + r_2)t}$$

$$\neq 0$$

- Hence we can determine a solution to any IVP of the given equation

$$y = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

- Case: $b^2 - 4ac = 0$
- The solutions r_1 and r_2 of the characteristic equation are real and equal.

$$r_1 = r_2 = r$$

- So we obtain only one instead of two solutions to the differential equation

$$\phi_1 = e^{rt}$$

- The method of finding ϕ_2 will covered next week, nevertheless we can verify

$$\phi_2 = te^{rt}$$

is another solution by simply substituting it back to the original equation

$$\begin{aligned} a\phi_2'' + b\phi_2' + c\phi_2 &= a(2re^{rt} + r^2te^{rt}) + b(e^{rt} + rte^{rt}) + cte^{rt} \\ &= \underbrace{(2ar + b)}_{0} e^{rt} + \underbrace{(ar^2 + br + c)}_{0} te^{rt} = 0 \end{aligned}$$

- The Wronskian is $W = e^{2rt} \neq 0$, so any IVP of the equation can be solved

$$y = C_1 e^{rt} + C_2 te^{rt} = (C_1 + C_2 t)\, e^{rt}$$

- Case: $b^2 - 4ac < 0$
- The solutions r_1 and r_2 of the characteristic equation are conjugates

$$r_1 = R + \mathrm{i}\theta, \qquad r_2 = R - \mathrm{i}\theta, \qquad \text{where} \quad \theta > 0$$

- By the definition of the complex exponential function,

$$e^{x+\mathrm{i}y} = e^x \left(\cos y + \mathrm{i} \sin y\right)$$

we can write a solution of the original differential equation

$$\begin{aligned} y = d_1 e^{r_1 t} + d_2 e^{r_2 t} &= d_1 e^{(R+\mathrm{i}\theta)t} + d_2 e^{(R-\mathrm{i}\theta)t} \\ &= d_1 e^{Rt} \left(\cos \theta t + \mathrm{i} \sin \theta t\right) + d_2 e^{Rt} \left(\cos \theta t - \mathrm{i} \sin \theta t\right) \\ &= e^{Rt} \Big[(d_1 + d_2) \cos \theta t + \mathrm{i}(d_1 - d_2) \sin \theta t \Big] \\ &= e^{Rt} \Big(C_1 \cos \theta t + C_2 \sin \theta t \Big) \end{aligned}$$

where $C_1 = d_1 + d_2$ and $C_2 = \mathrm{i}(d_1 - d_2)$.

- Therefore, we have two solutions ϕ_1 and ϕ_2 in trigonometric form,

$$\phi_1 = e^{Rt}\cos\theta t, \qquad \phi_2 = e^{Rt}\sin\theta t$$

where R is the real part, and θ is the imaginary part which is positive.

- The Wronskian is given by

$$W = \theta e^{2Rt} \neq 0$$

- Hence any IVP of the given equation can be solved

$$y = e^{Rt}\Big(C_1\cos\theta t + C_2\sin\theta t\Big)$$

Exercise

Solve the initial-value problem

$$\ddot{y} + y = 0, \qquad y(0) = 2, \qquad y'(0) = 3$$