Vv156 Lecture 15

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The Fundamental Theorem of Calculus Part-I Evaluation

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f.

- This part can also be understood or interpreted as

  The total change is equal to the integral of the rate of change.

- This interpretation is wildly used in science and engineering, for example,

  Total change in displacement is equal to the definite integral of velocity.

- It is standard to denote the difference $F(b) - F(a)$ as $F(x)\Big|_a^b$ or $\Big[F(x)\Big]_a^b$

$$\int_a^b f(x)dx = F(b) - F(a) = F(x)\Big|_a^b = \Big[F(x)\Big]_a^b$$

Proof

- Let $x_1, x_2, \cdots, x_{n-1}$ be any points in $[a,b]$ such that
$$a < x_1 < x_2 < \cdots < x_{n-1} < b$$
- These values divide $[a,b]$ into n subintervals
$$[a, x_1], [x_1, x_2], \ldots, [x_{n-1}, b]$$
whose lengths, as usual, we denote by $\Delta x_1,\ \Delta x_2,\ \ldots,\ \Delta x_n$.
- Since $F'(x) = f(x)$ for all x in $[a,b]$, so F satisfies the hypotheses of MVT on each those subintervals. Hence,
$$\begin{aligned}
F(x_1) - F(a) &= F'(x_1^*)(x_1 - a) &&= f(x_1^*)\Delta x_1 \\
F(x_2) - F(x_1) &= F'(x_2^*)(x_2 - x_1) &&= f(x_2^*)\Delta x_2 \\
F(x_3) - F(x_2) &= F'(x_3^*)(x_3 - x_2) &&= f(x_3^*)\Delta x_3 \\
&\vdots \\
F(b) - F(x_{n-1}) &= F'(x_n^*)(b - x_{n-1}) &&= f(x_n^*)\Delta x_n
\end{aligned}$$

Proof

- By adding all those equations together, we have

$$F(b) - F(a) = \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

- Let us now increase n in such a way that

$$\max \Delta x_k \to 0$$

- Since f is assumed to be continuous, thus integrable, the right-hand side of the above equation approaches the definite integral. The left-hand side remains the same since it is independent of n.

$$F(b) - F(a) = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \int_a^b f(x)\,dx \quad \square$$

Exercise

(a) Evaluate $\displaystyle\int_0^{\pi/2} \frac{\sin x}{5}\,dx$.

- When applying FTC there is no need to include a constant of integration, i.e.

$$\begin{aligned}\int_a^b f(x)\,dx &= \Big[F(x) + c\Big]_a^b \\ &= [F(b) + c] - [F(a) + c] \\ &= F(b) - F(a)\end{aligned}$$

Exercise

(b) Evaluate $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx$

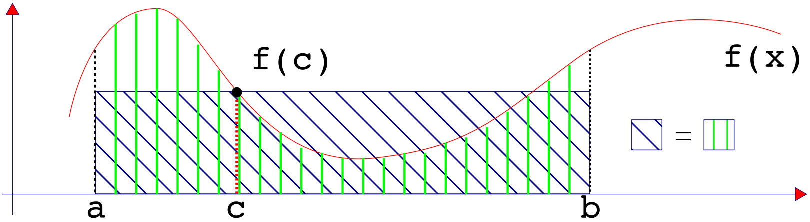
The Mean-value theorem for integrals

If f is continuous on a closed interval $[a,b]$, then there is at least one point c in $[a,b]$ such that

$$\int_a^b f(x)\,dx = (b-a)f(c)$$

The value of $f(c)$ is called the average value of f on the interval.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$$



Proof

- The function f is continuous on a closed interval, so EVT guarantees that

$$f \text{ on } [a,b] \text{ attains a maximum value } M \text{ and a minimum value } m$$

- So, for all x in $[a,b]$, we have the inequality: $m \leq f(x) \leq M$, which implies,

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a) \implies m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M$$

- The last inequality shows that

$$\frac{1}{b-a}\int_a^b f(x)\,dx, \quad \text{is a value between } m \text{ and } M.$$

- Since $f(x)$ is continuous, IVT says that there must a value of c such that

$$\frac{1}{b-a}\int_a^b f(x)\,dx = f(c) \implies \int_a^b f(x)\,dx = (b-a)f(c) \quad \square$$

- The first person to fly at a speed greater than sound was Chunk Yeager.

- On October 14, 1947, flying in an experimental X-1 rocket plane at an altitude of 12.8 kilometres, Yeager was clocked at 299.5m/sec.
- If Yeager had been flying at an altitude under 10.4 km, his speed of 299.5 m/s would not have "broken the sound barrier."

- The speed of sound in meters per second can roughly be modelled by

$$v(x) = \begin{cases} -4x + 341, & 0 \leq x < 11.5 \\ 295, & 11.5 \leq x < 22 \\ 0.75x + 278.5, & 22 \leq x < 32 \\ 1.5x + 254.5, & 32 \leq x < 50 \\ -1.5x + 404.5, & 50 \leq x < 80 \end{cases}$$

where x is the altitude in km.

Exercise

What is the average speed of sound for the interval of altitude 0-80km?

- There is a close relationship between the definite and indefinite integrals, but they differ in some important ways,

An indefinite integral is a set of functions

while

a definite integral is a real number

- They also differ in the role played by the variable of integration.

$$\int x^2\,dx = \frac{x^3}{3} + C; \qquad \int t^2\,dt = \frac{t^3}{3} + C;$$

$$\int_1^3 x^2\,dx = \left[\frac{x^3}{3}\right]_1^3 = \frac{26}{3}; \qquad \int_1^3 t^2\,dt = \left[\frac{t^3}{3}\right]_1^3 = \frac{26}{3};$$

- Because the variable of integration in a definite integral plays no role in the end result, it is often referred to as a dummy variable.

The Fundamental Theorem of Calculus Part-II Differentiation

If $f(x)$ is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by

$$F(x) = \int_a^x f(t)\,dt$$

is an antiderivative of f; that is,

$$F'(x) = f(x), \qquad \text{for each } x \text{ in the interval.}$$

In an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

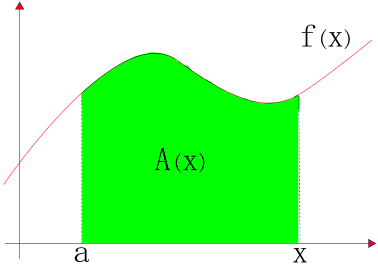
- This part of FTC essentially is saying that the effect of an indefinite integration can be reversed by a differentiation.
- It guarantees the existence of antiderivatives for continuous functions.

- A special case of this theorem:

  Suppose f is continuous and nonnegative on $[a, b]$, and let $A(x)$ be the area under the graph of $y = f(x)$ and x-axis over the interval $[a, x]$, then

$$A(x) = \int_a^x f(t)\,dt \implies A'(x) = f(x)$$

- Notice this is true for all x in $[a, b]$, and A' actually doesn't depend on a.
- Graphically,



Proof

- We need to show that $F'(x) = f(x)$, where $F(x)$ is the function

$$F(x) = \int_a^x f(t)\,dt$$

- If x is not an endpoint, then it follows from the definition

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + \int_x^a f(t)\,dt\right] \\ &= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt \end{aligned}$$

Proof

- Applying MVT for integrals,

$$\frac{1}{h}\int_x^{x+h} f(t)\,dt = \frac{1}{h}\left[f(c)\cdot h\right] = f(c), \qquad \text{where } c \in [x, x+h].$$

- Because c is trapped between x and $x+h$, it follows that

$$c \to x \qquad \text{as} \qquad h \to 0$$

- Thus the continuity of f implies $f(c) \to f(x)$ as $h \to 0$. Therefore

$$F'(x) = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x)$$

- If x is an endpoint of the interval, then the two-sided limits in the proof must be replaced by the appropriate one-sided limits. $\square$

Exercise

(a) Is the following function increasing or decreasing?

$$f(x) = \int_1^x \frac{1}{t}\,dt, \qquad \text{where } x > 1.$$

(b) Find

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \sin\frac{k}{n}$$

(c) Find

$$\int_{-1}^{1} 3x^2 \sqrt{x^3+1}\,dx$$

- Because FTC links the concept of the derivative of a function with the concept of the integral, so for definite integration as well as indefinite integration it is essential to find the antiderivative. However, a table of derivatives doesn't tell us how to evaluate every integrals, for example

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

- The following technique, can often be used to transform complicated integrals into simpler ones.

Substitution

If $g'(x)$ is continuous on the interval $[a,b]$ and $f(u)$ is continuous on the range of

$$u = g(x),$$

then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Proof

- Let F denote any antiderivative of f, then

$$\frac{d}{dx}F(g(x)) = F'(g(x)) \cdot g'(x) = f(g(x)) \cdot g'(x)$$

- Using integral notation, the above is equivalent to

$$F(g(x)) = \int f(g(x)) \cdot g'(x)\,dx + C$$

- Now if we consider the definite integral over $[a, b]$, FTC states

$$\begin{aligned} \int_a^b f(g(x)) \cdot g'(x)\,dx &= \Big[F(g(x))\Big]_{x=a}^{x=b} \\ &= F(g(b)) - F(g(a)) \\ &= \Big[F(u)\Big]_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f(u)\,du \quad \square \end{aligned}$$

Exercise

(a) Evaluate

$$\int_{-1}^{1} 3x^2 \sqrt{x^3 + 1}\, dx$$

(b) Find

$$\int x\sqrt{2x + 1}\, dx$$

(c) Evaluate

$$\int_{0}^{9} \sqrt{4 - \sqrt{x}}\, dx$$

(d) Find

$$\int \frac{dx}{e^x + e^{-x}}$$

Q: Can we apply FTC directly to the following

$$\frac{d}{dx}\int_0^{x^2} \cos(t)\, dt$$

In other words, is the following true?

$$\frac{d}{dx}\int_0^{x^2} \cos(t)\, dt = \cos(x^2)$$

Exercise

Find the following derivative.

$$\frac{d}{dx}\int_{\sqrt{x}}^{x^2-3x} \tan(t)\, dt$$