vv214: Linear transformations II.

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May 29, 2020

Today 5/29/2020

1. Kernel and image of a linear transformation.
2. Rank-Nullity Theorem.
3. Inverse linear transformations.

Next class

Coordinates.

Review

$A = (A\bar{e}_1 \;\; A\bar{e}_2)$

$\mathbb{R}^2 \to \mathbb{R}^2$

2D: 1) Counterclockwise rotation through $\frac{\pi}{2}$

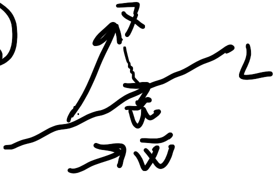
$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \bar{y} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \Rightarrow \bar{y} = \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

2) Counter. rot. through θ

$$\bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \bar{y} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

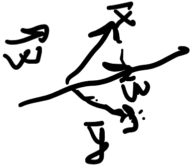
$\underline{a^2 + b^2 = 1}$

3)



$$\bar{u} = proj_L \bar{x} = \frac{1}{\|\bar{w}\|^2} \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

4)



$$\bar{y} = ref_L \bar{x} = \underbrace{\begin{pmatrix} 2w_1^2 - 1 & 2w_1 w_2 \\ 2w_1 w_2 & 2w_2^2 - 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\bar{w} \parallel L$

$\|\bar{w}\| = 1$ $\quad A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}, \; a^2 + b^2 = 1$

Examples

1. Let L be the line in $\mathbb{R}^3$ spanned by the vector $\bar{y} = (2, 1, 2)$. Find the orthogonal projection of the vector $(1, 1, 1)$ onto L.

Solution:

$$L = \operatorname{span}\left(\begin{pmatrix}2\\1\\2\end{pmatrix}\right) = \left\{\alpha\begin{pmatrix}2\\1\\2\end{pmatrix}, \alpha\in\mathbb{R}\right\}$$

$$\bar{x} = \begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\vec{w} = \begin{pmatrix}2\\1\\2\end{pmatrix}$$

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Solution:

$$proj_L \bar{x} = (\bar{x}, \bar{w})\bar{w}, \ \bar{w} || L, \ ||\bar{w}|| = 1$$

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$$\bar{y} = (2, 1, 2)||L \Rightarrow \bar{w} = \frac{\bar{y}}{||\bar{y}||} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

$\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{4}{9}}=1$

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$\bar{x} \cdot \bar{w}$

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$$proj_L \bar{x} = \left(\frac{2}{3} x_1 + \frac{1}{3} x_2 + \frac{2}{3} x_3 \right) \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3} \right)$$

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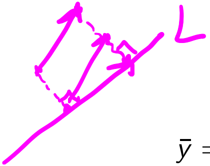
$$= \underline{\left(\frac{4}{9}x_1 + \frac{2}{9}x_2 + \frac{4}{9}x_3, \frac{2}{9}x_1 + \frac{1}{9}x_2 + \frac{2}{9}x_3, \frac{4}{9}x_1 + \frac{2}{9}x_2 + \frac{4}{9}x_3\right)} \in \mathbb{R}^3$$

Examples

1. Let L be the line in $\mathbb{R}^3$ spanned by the vector $\bar{y} = (2, 1, 2)$. Find the orthogonal projection of the vector $(1, 1, 1)$ onto L.

Solution:



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$$= \underbrace{\begin{pmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{pmatrix}}_{3\times3} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{3\times1} = \underbrace{\begin{pmatrix} \ \end{pmatrix}}_{3\times1}$$

Examples

$$= \begin{pmatrix} w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(\frac{10}{9}, \frac{5}{9}, \frac{10}{9} \right)$$

$$(x_1, x_2, x_3) = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}^T$$

Examples

$$= \begin{pmatrix} w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(\frac{10}{9}, \frac{5}{9}, \frac{10}{9}\right)$$

2. Find the matrices of the following linear transformations $\mathbb{R}^3 \to \mathbb{R}^3$:

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2. Find the matrices of the following linear transformations $\mathbb{R}^3 \to \mathbb{R}^3$:

a. the orthogonal projection onto xy plane.

$$\boxed{A = \left(A\vec{e}_1 \quad A\vec{e}_2 \quad A\vec{e}_3 \right)}$$

$$\vec{e}_1 \to \vec{e}_1 = A\vec{e}_1$$

$$\vec{e}_2 \to \vec{e}_2 = A\vec{e}_2$$

$$(xy) \perp \vec{e}_3 \to \vec{0} = A\vec{e}_3$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Examples

$$= \begin{pmatrix} w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(\frac{10}{9}, \frac{5}{9}, \frac{10}{9} \right)$$

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a. the orthogonal projection onto xy plane.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b. the reflection about xz plane.

$$A = \begin{pmatrix} A\vec{e}_1 & A\vec{e}_2 & A\vec{e}_3 \end{pmatrix}$$

$\vec{e}_1, \vec{e}_3 \subset$ the (xz)-plane

$\Rightarrow A\vec{e}_1 = \vec{e}_1$; $A\vec{e}_3 = \vec{e}_3$

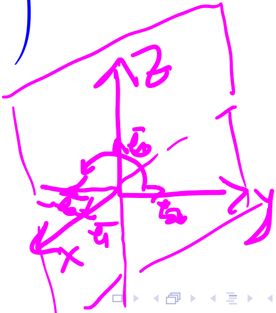
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\vec{e}_2 \perp (xz)$

$\Downarrow$

$\vec{e}_2 \to -\vec{e}_2$

$= A\vec{e}_2$



Examples

$$= \begin{pmatrix} w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(\frac{10}{9}, \frac{5}{9}, \frac{10}{9} \right)$$

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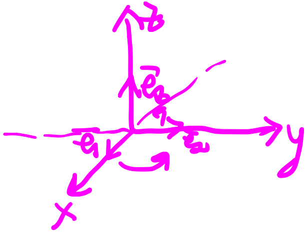
Examples

c. the rotation about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.



$\bar{e}_3 \subset Oz \Rightarrow \bar{e}_3 \to A\bar{e}_3 = \bar{e}_3$

$\bar{e}_1 \to \bar{e}_2 = A\bar{e}_1$

$\bar{e}_2 \to -\bar{e}_1 = A\bar{e}_2$

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

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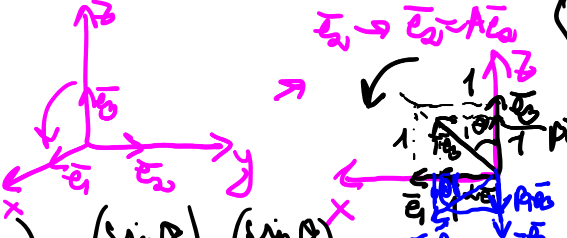
$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$A\bar{e}_1 = \cos\theta \begin{pmatrix}1\\0\\0\end{pmatrix} + \sin\theta \begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}\cos\theta\\0\\-\sin\theta\end{pmatrix}$

d. the rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.



$T\bar{e}_3 = \beta\bar{e}_3 + \alpha\bar{e}_1$

$\cos\theta = \dfrac{|\beta\bar{e}_3|}{|T\bar{e}_3|} = |\beta| = \beta$

$\sin\theta = |\alpha| = \alpha$

$A\bar{e}_3 = \begin{pmatrix}0\\0\\\cos\theta\end{pmatrix} + \begin{pmatrix}\sin\theta\\0\\0\end{pmatrix} = \begin{pmatrix}\sin\theta\\0\\\cos\theta\end{pmatrix}$

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c. the rotation about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

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d. the rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

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e. the rotation about the z-axis through $\pi/4$ turning the positive x-axis towards the positive y-axis

f. the orthogonal projection onto the line $y = x$ on the xy-plane

Composition of Linear Transformations

Any matrix defines a linear transformation $\Rightarrow$ a matrix product defines a composition of linear transformations.

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- $\star$ $D_\alpha D_\beta = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$
$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$

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$= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$
- ⋆ $D_\alpha D_\beta = D_\beta D_\alpha$

Composition of Linear Transformations

2D

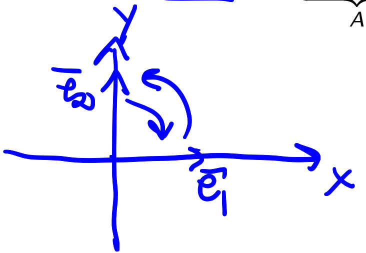
$$\bar{x}_1 = \underline{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} \to \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \bar{y}$$

$\bar{w} \parallel \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$2w_1^2 - 1 \quad 2w_1w_2$

$\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$

$a^2 + b^2 = 1$



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

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$T: \mathbb{R}^m \to \mathbb{R}^n$

$A_{n \times m}$

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The reflection about the line L with the direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

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$$\bar{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = \bar{y}$$

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$$B = \begin{pmatrix} 2w_1^2 - 1 = -1 & 2w_1w_2 \\ & 2w_2^2 - 1 \end{pmatrix}$$

The reflection about the line L with the direction vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

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The reflection about the line L with the direction vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ the rotation through } \frac{\pi}{2}$$

$(BA)(\bar{x}) = B(A\bar{x})$

Composition of Linear Transformations

$$\bar{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \bar{y}$$

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The reflection about the line L with the direction vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ the rotation through} \frac{\pi}{2}$$

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ the rotation through} -\frac{\pi}{2}$$

Image and Kernel of a Linear Transformation

over $\mathbb{K}$

Definition: The kernel and image of a linear operator $T: \underline{V} \to \underline{W}$ are defined by

$$N(T) = Ker\, T = \{v \in V: Tv = 0_W\} \subseteq V \qquad Im\, T = \{w \in W: w = Tv,\ v \in V\} \subseteq W$$

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Examples: 1. $T: \mathbb{R} \to \mathbb{R}$, $Tx = x^2$ (not linear) $T(x+y) \neq x^2+y^2$

$$\boxed{Ker\ T = \{0\},}\ Im\ T = \mathbb{R}_+ \cup \{0\}$$

$Ker\ T = \{v \in \mathbb{R} : Tv = 0 = v^2\} = \{v = 0\}$
$= \{0\}$

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$$Ker\ T = \{0\},\ Im\ T = \mathbb{R}_+ \cup \{0\}$$

2. $T\colon \mathbb{R} \to \mathbb{R}^2,\ T(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ (not linear)

$$T(t_1+t_2) \neq \begin{pmatrix} \cos t_1 \\ \sin t_1 \end{pmatrix} + \begin{pmatrix} \cos t_2 \\ \sin t_2 \end{pmatrix}$$

$$Ker\ T = \varnothing,\ Im\ T = \text{unit circle}$$

$$T(t) = 0 = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \nexists t$$

$$(\cos t)^2 + (\sin t)^2 = 1$$

$$T(t) = \begin{pmatrix} x \\ y \end{pmatrix} \qquad x^2 + y^2 = 1$$

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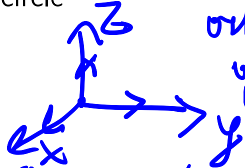
Examples: 1. $T: \mathbb{R} \to \mathbb{R},\ Tx = x^2$ (not linear)

$$Ker\ T = \{0\},\ Im\ T = \mathbb{R}_+ \cup \{0\}$$

2. $T: \mathbb{R} \to \mathbb{R}^2,\ T(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ (not linear)

$$Ker\ T = \varnothing,\ Im\ T = \text{unit circle}$$

3. $T: \mathbb{R}^3 \to \mathbb{R}^3,\ T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$



orth. proj onto the xy plane

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Image and Kernel of a Linear Transformation

Definition: The kernel and image of a linear operator $T: V \to W$ are defined by

$$Ker\ T = \{v \in V : Tv = 0\} \quad Im\ T = \{w \in W : w = Tv,\ v \in V\}$$

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$$T\bar{x} = \bar{0} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 = x_2 = 0$$

$$Ker\ T = k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ k = const,\ \boxed{Im\ T = xy \text{ plane}}$$

Image and Kernel of a Linear Transformation

4. $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T\bar{x} = A\bar{x}$, $A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$

$A \sim \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 3/2 \\ 0 & 0 \end{pmatrix}$

$\operatorname{rank}(A) = 1$

$\operatorname{Ker} T = \{ \bar{x} \in \mathbb{R}^2 ; \underline{A\bar{x} = 0} \} \neq \{ \bar{0} \}$

$A_{2\times 2} \Rightarrow n = m = 2$

$\operatorname{rank}(A) = 1 < n = m$ inf. many sol.

$A\bar{x} = \bar{0} \sim \begin{pmatrix} 1 & 3/2 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \bar{0} \Rightarrow \underline{x_1 = -3/2\, x_2}$

$\begin{pmatrix} -3/2\, x_2 \\ x_2 \end{pmatrix}$

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4. $T : \mathbb{R}^2 \to \mathbb{R}^2,\ T\bar{x} = A\bar{x},\ A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$

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$$A\bar{x} = (\bar{a}_1 \ \bar{a}_2)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_1\bar{a}_1 + x_2\bar{a}_2$$

$$Ker\ A = t\begin{pmatrix} -3/2 \\ 1 \end{pmatrix}, \quad Im\ A = span\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$A\bar{x} = x_1\begin{pmatrix} 2 \\ 6 \end{pmatrix} + x_2\begin{pmatrix} 3 \\ 9 \end{pmatrix}$$

$$= 2x_1\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 3x_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= span\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

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$A\bar{x} = (\bar{a}_1\ \bar{a}_2\ \bar{a}_3)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\bar{a}_1 + x_2\bar{a}_2 + x_3\bar{a}_3$

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$\mathbb{R}^3 \to \mathbb{R}^3 \quad A_{3\times 3}$

5. $T: P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ Tp(t) = p'(t)$ $= p'(t) = a_1 + 2a_2t + 0\cdot t^2$

$\forall p(t) \in P_2(\mathbb{R}) \qquad p(t) = a_0 + a_1t + a_2t^2$

The basis of $P_2(\mathbb{R})$ is $\{1, t, t^2\} \Rightarrow \dim P_2(\mathbb{R}) = 3$

$p(t) \rightsquigarrow \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}_{\{1,t,t^2\}} \in \mathbb{R}^3$

$P_2(\mathbb{R}) \qquad \underline{\mathbb{R}^3}$

$Tp(t) \rightsquigarrow \begin{pmatrix} a_1 \\ 2a_2 \\ 0 \end{pmatrix}_{\{1,t,t^2\}} = \overset{\frown a_0}{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} a_1 + \overset{\frown a_1 t}{\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}} a_2 + \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} a_3$

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$$\boxed{A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

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$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Ker\ T = \{p(t) :\ Tp = 0\} =$$

$a_1 = 0,\ a_2 = 0 \Rightarrow Ker\ T = \{p(t) = a_0\}$

$0 = 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2$

$a_0 = span(1)$

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$$Ker\ T = \{p(t) \colon\ Tp = 0\} = \{a_0\} = span(1),\ Im\ T = span(1, t)$$

Image and Kernel of a Linear Transformation

Lemma 1: Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be defined by the matrix $A_{n\times m}$. The columns of the matrix A are linearly independent iff

$$\boxed{Ker\, A_{n\times m} = \{\bar{0}\}_{\mathbb{R}^m}} \iff rank\, A = m \Rightarrow m \leq n$$

$A_{n\times m}\bar{x} = \bar{0}$ only $\bar{x} = \bar{0}$

inf sol. $\Rightarrow n < m$

$A = (\bar{a}_1 \;\; \bar{a}_2 \; \dots \; \bar{a}_m)$

$\bar{x} \in Ker\, A_{n\times m} \Rightarrow \bar{x} \in Ker\, T \Rightarrow A\bar{x} = \bar{0} \Rightarrow \boxed{\bar{a}_1 x_1 + \dots + \bar{a}_m x_m = \bar{0}}$

(1) $Ker\, A_{n\times m} = \{\bar{0}_{\mathbb{R}^m}\} \Rightarrow \bar{x} = \bar{0} \Rightarrow x_1 = x_2 = \dots = x_m = 0$

$\Rightarrow \bar{a}_1, \dots, \bar{a}_m$ are lin. ind.

(2) If $\bar{a}_1, \dots, \bar{a}_m$ are lin ind, then

if $c_1\bar{a}_1 + \dots + c_m\bar{a}_m = \bar{0} \Rightarrow A\begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} = \bar{0} \Rightarrow \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix} \in Ker\, A$, $= \bar{0}$

then $c_1 = \dots = c_m = 0$

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Lemma 2: Let $T\colon V \to W$ be a linear operator. $Im\, T$ and $Ker\, T$

are linear subspaces of V and W $\Rightarrow$ there exist bases of the kernel and the image of a linear transformation.

Examples:

1. A matrix A is said to be diagonally dominant if

$$|a_{ii}| > \sum_{j=1, j\neq k}^{n} |a_{ij}| \quad \forall i = \overline{1, n}$$

If $A_{n\times n}$ is diagonally dominant then $Ker\, A = \{0\}$.

Examples:

2. A matrix

$$V_{m\times n} = \begin{pmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & x_m^2 & \dots & x_m^{n-1} \end{pmatrix} \quad x_i \neq x_j \quad \forall i \neq j$$

is called a Vandermonde matrix.

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is called a Vandermonde matrix. Let $n \leq m$. Columns of $V_{m\times n}$ are linearly independent iff $Ker\, V_{m\times n} = \{0\}$

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  Let $v \in Ker\ T \Rightarrow Tv = 0 = T(0) \Rightarrow v = 0 \Rightarrow Ker\ T \subset \{0\}$.
- Let $Ker\ T = \{0\}$. If $Tu = Tv$, then
  $T(u - v) = 0 \Rightarrow u - v \in Ker\ T \Rightarrow u - v = 0 \Rightarrow u = v$

Example: $T\colon \mathbb{R}^2 \to \mathbb{R}^3,\ T(x, y) = (2x, 3y, x + 2y)$

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Image and Kernel of a Linear Transformation: Example

$$T: \mathbb{R}^6 \to \mathbb{R}^4,\ A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow Ker\ A = \{\bar{x} \in \mathbb{R}^6 :\ A\bar{x} = 0\}$$

$$\Rightarrow \begin{cases} x_2 + 2x_3 + 3x_6 = 0 & \Rightarrow x_2 = -2x_3 - 3x_6 \\ x_4 + 4x_6 = 0 & \Rightarrow x_4 = -4x_6 \\ x_5 + 5x_6 = 0 & \Rightarrow x_5 = -5x_6 \end{cases}$$

$$\bar{x} = r \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -3 \\ 0 \\ -4 \\ -5 \\ 1 \end{pmatrix} \Rightarrow \dim Ker\ A = 3$$

Rank-Nullity Theorem

$$\boxed{\dim \mathit{Ker}\ T + \dim \mathit{Im}\ T = \dim V}$$

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Rank-Nullity Theorem

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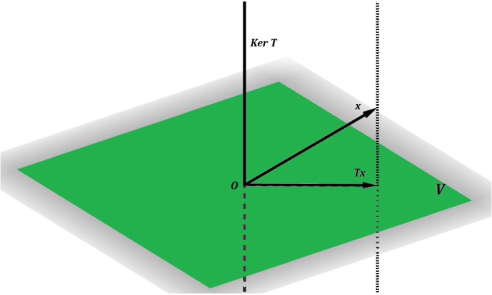
Example: $T : \mathbb{R}^3 \to \mathbb{R}^3,\ T\bar{x} = proj_V \bar{x},\ V \subset \mathbb{R}^3$

$$Ker\ T = \{\bar{x} \in \mathbb{R}^3 : proj_V \bar{x} = \bar{0}\},\ Im\ T = V$$



$$Ker\ T = \text{line orthogonal to } V$$

$$\underbrace{m}_{3} - \underbrace{\dim (Ker\ T)}_{1} = \underbrace{\dim Im\ T}_{2}$$

Rank-Nullity Theorem: Proof

Let $\dim(Ker\ T) = n$ and $\dim Ker\ T = k \Rightarrow k \leq n$.
$\Rightarrow$ there exists a basis $v_1, \dots, v_k$, of $Ker\ T$. Complete this basis up to the basis of V: $v_1, \dots, v_k,\ v_{k+1}, \dots, v_n$
We are to prove that $Tv_{k+1}, \dots,\ Tv_n$ form the basis for $Im\ T$:

1 $Tv_{k+1}, \dots,\ Tv_n$ are linearly independent:

$$\alpha_1 Tv_{k+1} + \dots + \alpha_{n-k} Tv_n = 0 \Rightarrow T(\alpha_1 v_{k+1} + \dots + \alpha_{n-k} v_n) = 0$$

$\Rightarrow \alpha_1 v_{k+1} + \dots + \alpha_{n-k} v_n \in Ker\ T$
$\Rightarrow \alpha_1 v_{k+1} + \dots + \alpha_{n-k} v_n \in span(v_1, \dots, v_k)$
But $v_1, \dots, v_k,\ v_{k+1}, \dots, v_n$ are linearly independent
$\Rightarrow \alpha_1 = \dots = \alpha_{n-k} = 0$

2 $span(Tv_{k+1}, \dots,\ Tv_n) = Im\ T$

A $w \in Im\ T \Rightarrow \exists v \in V:\ Tv = w \Rightarrow T(\beta_1 w_1 + \dots + \beta_n v_n) = w$

$$w = \beta_1 \underbrace{Tv_1}_{=0} + \dots + \beta_k \underbrace{Tv_k}_{=0} + \beta_{k+1} Tv_{k+1} + \dots + \beta_n Tv_n$$

$$w \in span(Tv_{k+1}, \dots,\ Tv_n) \Rightarrow Im\ T \subset span(Tv_{k+1}, \dots,\ Tv_n)$$

B $w \in span(Tv_{k+1}, \dots,\ Tv_n) \Rightarrow w = \alpha_{k+1} Tv_{k+1} + \dots + \alpha_{n-k} Tv_n$
$w = T(\alpha_{k+1} v_{k+1} + \dots + \alpha_{n-k} v_n) \Rightarrow w \in Im\ T$

Inverse Linear Transformations

Definition: Let V, W be linear spaces.
A linear operator $T\colon V \to W$ is called invertible if there exists a linear operator $S\colon W \to V$ such that ST equals the identity map on V and TS equals the identity map on W.

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A linear operator $S \colon W \to V$ satisfying $ST = I$ and $TS = I$ is called an inverse of T.
Here the first I is the identity map on V and the second I is the identity map on W.

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Here the first I is the identity map on V and the second I is the identity map on W. We shall denote the inverse linear operator by T^{-1}.

$$T^{-1}(Tv) = v \quad \text{and} \quad T(T^{-1}w) = w \quad \forall v \in V \, \forall w \in W$$

Inverse Linear Transformations

Lemma: A linear operator is invertible iff it is one-to-one (injective) and onto (surjective).

- Let T^{-1} exists.
  - A Let $u, v \in V$ and $Tu = Tv$

    $$u = T^{-1}(Tu) = T^{-1}(Tv) = v \Rightarrow \; T \text{ is injective}$$

  - B Let $w \in W \Rightarrow w = T(T^{-1}w) \Rightarrow w \in Im\,T \Rightarrow W \subset Im\,T$
    As also $Im\,T \subset W$, so $W = Im\,T$
- Let T be injective and surjective. For any $w \in W$, define Sw be a unique element of V such that $T(Sw) = w$. This element exists since T is one-to-one and onto.
  - A From the definition, $TS = I$. Also

    $$T((ST)v) = (TS)(Tv) = ITv = Tv \Rightarrow STv = v \Rightarrow ST = I$$

  - B S is linear:

    $$w_1,\, w_2 \in W \Rightarrow T(Sw_1 + Sw_2) = TSw_1 + TSw_2 = w_1 + w_2$$

    Apply the definition of $S \Rightarrow S(w_1 + w_2) = Sw_1 + Sw_2$
    Similarly, $S(\alpha w) = \alpha Sw \, \forall w \in W \, \forall \alpha \in \mathbb{K}$

Inverse Linear Transformations

Remarks:

1. $(T^{-1})^{-1} = T$
2. Let $V, W = \mathbb{R}^n$. A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if the system $A\bar{x} = \bar{y}$ has a unique solution

$$\iff \textit{rank}\, A = n \iff \textit{rref}\, A = I_n$$

Definition: A square matrix A is invertible if the linear transformation $T\bar{x} = A\bar{x}$ is invertible.

Inverse Linear Transformations

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$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{array}\right)$$

$$I\bar{y} = A^{-1}\bar{x} \Rightarrow A^{-1} = \frac{1}{5}\begin{pmatrix} -2 & 3 & -1 \\ 4 & -1 & 2 \\ -7 & 3 & -1 \end{pmatrix}$$

Inverse Linear Transformations

1. Let $A_{n\times n}$. If A^{-1} exists, then the system $A\bar{x} = \bar{0}$ has a unique solution

   $$\Rightarrow rank\, A = n \Rightarrow \text{ columns of } A \text{ are linearly independent.}$$

2. If A^{-1} exists, then $A^{-1}A = AA^{-1} = I$.
3. $(AB)^{-1} = B^{-1}A^{-1}$

1. Review: basis, dimension,linear operators on finite dimensional linear spaces.
2. Coordinates of a vector in different bases of $\mathbb{R}^n$.
3. The change of basis matrix.
4. The $\mathfrak{B}$-matrix of a linear transformation.
5. Similar matrices.
6. Isomorphism. Isomorphic spaces.

Review: bases of linear subspaces.

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$$\Rightarrow \{\bar{v}_1, \; \bar{v}_2\} \text{ is the basis of } \mathbb{R}^2$$

Let $\bar{x} = \begin{pmatrix} 10 \\ 10 \end{pmatrix}$ in the standard basis $\{\bar{e}_1, \; \bar{e}_2\}$ of $\mathbb{R}^2$.

$$\bar{x} = c_1 \bar{v}_1 + c_2 \bar{v}_2 \Rightarrow \left(\begin{array}{cc|c} 3 & -1 & 10 \\ 1 & 3 & 10 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 2 \end{array}\right); \; \bar{x}_{\mathfrak{B}} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$\text{OR} \quad \bar{x} = S\bar{x}_{\mathfrak{B}} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \bar{x}_{\mathfrak{B}} \Rightarrow \bar{x}_{\mathfrak{B}} = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}^{-1} \bar{x}$$

$\mathfrak{B}$-matrix of a linear transformation

Definition: Let $T\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. A matrix $B_{n\times n}$ that transforms $\bar{x}_{\mathfrak{B}}$ into $(T\bar{x})_{\mathfrak{B}}$ is called the $\mathfrak{B}$-matrix of T:

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$$\begin{array}{ccc} \bar{x} & \xrightarrow{T} & \frac{1}{3^2+1^2} \begin{pmatrix} 3^2 & 3\cdot 1 \\ 3\cdot 1 & 1^2 \end{pmatrix} \bar{x} \\ \downarrow L \uparrow S & & \downarrow L \uparrow S \\ \bar{x}_{\mathfrak{B}} & \xrightarrow{B} & (T\bar{x})_{\mathfrak{B}} \end{array}$$

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Change of basis matrix of a linear transformation

Let $\mathfrak{B}_1$ and $\mathfrak{B}_2$ be two bases of a linear space V, $\dim V = n$.

$$\mathfrak{B}_1 = \{b_1,\ b_2,\ \dots b_n\}$$

Definition: The matrix S of a linear transformation

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is called the change of basis matrix from $\mathfrak{B}_1$ to $\mathfrak{B}_2$: $S_{\mathfrak{B}_1 \to \mathfrak{B}_2}$

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Example: Let $V = span(e^x,\, e^{-x}) \in C^\infty$
The systems $\mathfrak{B}_1 = \{e^x + e^{-x},\, e^x - e^{-x}\}$ and $\mathfrak{B}_2 = \{e^x,\, e^{-x}\}$ are the bases of V.

$$S_{\mathfrak{B}_1 \to \mathfrak{B}_2} = \left((e^x + e^{-x})_{\mathfrak{B}_2} \quad (e^x - e^{-x})_{\mathfrak{B}_2}\right) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Isomorphism

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3. To generalize 2, consider a linear space with a *finite* basis $V = span(f_1, f_2 \ldots, f_n)$ and define the coordinate transformation $L_{\mathfrak{B}}\colon V \to \mathbb{R}^n$ is $L(f) = f_{\mathfrak{B}}$

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$$KerT = \{0\} \quad \text{and} \quad ImT = W$$

- If $KerT = \{0\}$, $ImT = W$, apply the rank-nullity theorem $\Rightarrow \dim V = \dim W$
- Let $\dim V = \dim W = n \Rightarrow \exists\, v_1, \ldots, v_n$ (basis of V) and $\exists\, w_1, \ldots, w_n$ (basis of W).
  Define an operator $T: V \to W$ by

$$T(c_1 v_1 + \ldots + c_n v_n) = c_1 w_1 + \ldots + c_n w_n$$

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More Examples

1. Let $V = span(\cos x, \sin x) = \{a \cos x + b \sin x,\ a, b \in \mathbb{R}\} \subset C^\infty$

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6. If S is the set of all students in your linear algebra class. Can one define operations on S that make S into a real linear space?
No.

Row-Rank=Column-Rank

Goal: to prove that the number of linearly independent columns of a matrix A is the same as the number of linearly independent rows $\Rightarrow$ the rank of a matrix is the number of linearly independent rows or columns!!!

the row-rank $=$ the column rank

How to prove:

1. Linear functionals
2. The dual space V' (of all linear functionals defined on V)
3. Dual basis
4. Dual map
5. Annihilator $U^o = \{\varphi \in V' \colon \varphi(u) = 0 \, \forall u \in U\}$ of a linear subspace
6. $U \subset V \Rightarrow \dim U + \dim U^o = \dim V$

Linear Functionals and Dual Basis

Definition: A linear operator $f\colon V \to \mathbb{R}$ is called a linear functional.

Examples:

1. $f\colon \mathbb{R}^3 \to \mathbb{R} \quad f(x,y,z) = 4x - 5y + 2z$
2. $x\colon C[a,b] \to \mathbb{R} \quad x(t) = \int_a^b x(t)\,dt$

We shall write $L(V,W)$ to denote the linear space of all linear operators from V to W.

Definition: Dual Space: $V' = L(V,\mathbb{R})$

$$\dim V' = \dim V \dim \mathbb{R} \Rightarrow \dim V' = \dim V$$

Let $v_1,\dots,v_n$ be a basis for V. The dual basis of $v_1,\dots,v_n$ is

$$\{\varphi_1,\dots,\varphi_n\} \in V'\colon \varphi_j(v_k) = \left\{\begin{array}{ll} 1 & k = j \\ 0 & k \neq j \end{array}\right.$$

Example: $e_1,\dots,e_n \in \mathbb{R}^n \Rightarrow$ let $\varphi_i(x_1,\dots,x_n) = x_j$. Then
$\varphi_j(e_k) = \left\{\begin{array}{ll} 1 & k = j \\ 0 & k \neq j \end{array}\right. \Rightarrow \{\varphi_i\}$ is the dual basis.

Dual Map

Remark: Dual basis is indeed a basis. Let $\dim V = m \Rightarrow \dim V' = m$. Then $\varphi_1, \ldots, \varphi_m$ are linearly independent:

$$\alpha_1\varphi_1 + \ldots \alpha_m\varphi_m = 0 \Rightarrow (\alpha_1\varphi_1 + \ldots \alpha_m\varphi_m)(v_k) = 0$$

$$\Rightarrow \alpha_k\varphi_k(v_k) = 0 \Rightarrow \alpha_k = 0 \quad \forall k = 1, \ldots, m$$

Definition: Dual map: $T' \colon W' \to V' \quad T'(\varphi) = \varphi \circ T \quad \forall \varphi \in W'$

Dual map is well-defined:

$$T'(\varphi)(\underbrace{v}_{\in V}) = \varphi \circ T(v) = \varphi(\underbrace{Tv}_{\in W})$$

Pick a functional φ defined on W and the $T'(\varphi)$ is a functional defined on V.

Example: Consider $T \colon P_n \to P_n$, $Tp(t) = p'(t)$.

Let $\varphi \colon P_n \to \mathbb{R}$, $\varphi(p) = p(3)$

$$T'(\varphi) = \varphi \circ T \Rightarrow T'(\varphi)(p) = \varphi \circ T(p) = \varphi(Tp) = \varphi(p') = p'(3)$$

Matrix of the Dual Map

Let $T\colon V \to W$, $\dim V = m$, $\dim W = n$ with the bases $v_1, \ldots, v_m$, $w_1, \ldots, w_n \Rightarrow T$ is defined by a matrix $A_{n\times m} = (a_{ij})$.
Then $T'\colon W' \to V'$, $\dim W' = n$, $\dim V' = m \Rightarrow T$ is given by a matrix $B_{m\times n} = (b_{ij})$.
Let $\psi_1, \ldots, \psi_n$ be a basis for W', and $\varphi_1, \ldots, \varphi_m$ be a basis for V'.

$$\forall j = 1, \ldots, n \quad T'(\psi_j) \in V' \Rightarrow T'(\psi_j) = \sum_{r=1}^{m} b_{rj}\varphi_r$$

But $T'(\psi_j) = \psi_j \circ T$

$$\Rightarrow \psi_j \circ T(v_k) = T'(\psi_j)(v_k) = \sum_{r=1}^{m} b_{rj}\varphi_r(v_k) = b_{kj}$$

On another hand,

$$\psi_j \circ T(v_k) = \psi_j(Tv_k) = \psi_j\left(\sum_{r=1}^{n} a_{rk}w_r\right) = a_{jk}$$

$$\Rightarrow b_{kj} = a_{jk} \quad \forall j = 1, .., n,\ k = 1..m \Rightarrow \boxed{B = A^T}$$

Annihilators

Definition: An annihilator of a linear subspace $U \subset V$ is

$$U^\circ = \{\varphi \in V' \colon \varphi(u) = 0 \,\forall u \in U\}$$

Example: Let $U = \{p(t) \in P_n(\mathbb{R}) \colon p(t) = t^2 g(t) \,\forall g(t) \in P_n(\mathbb{R})\}$

$$\Rightarrow U^\circ = \{\varphi \in P_n'(\mathbb{R}) \colon \varphi(p) = 0\}$$

If $\varphi(p) = p'(0)$, then $\varphi(p) = 0 \,\forall p \Rightarrow \varphi \in U^\circ$

Lemma: $\dim U + \dim U^\circ = \dim V$
Proof: Let $i(u) = u \quad \forall u \in U \Rightarrow i' \colon V' \to U'$

$$\dim V = \dim V' = \dim \underbrace{Ker\, i'}_{U^\circ} + \underbrace{\dim Im\, i'}_{\dim U}$$

$\dim Im\ T = \dim Im\ T'$

Lemma: $Ker\ T' = (Im\ T)^{\circ}$

Proof:

1. $\varphi \in Ker\ T' \Rightarrow 0 = T'(\varphi) = \varphi \circ T = 0$

$\Rightarrow 0 = \varphi \circ T(v) = \varphi(Tv)\, \forall v \in V \Rightarrow \varphi \in (Im\ T)^{\circ} \Rightarrow Ker\ T \subset (Im\ T)^{\circ}$

2. Let $\varphi \in (Im\ T)^{\circ}$

$\Rightarrow \varphi(Tv)\, \forall v \in V \Rightarrow 0 = T'(\varphi) \Rightarrow \varphi \in Ker\ T' \Rightarrow (Im\ T)^{\circ} \subset Ker\ T$

Theorem: $\dim Im\ T' = \dim Im\ T$

$\dim Im\ T' = \dim W' - \dim Ker\ T' = \dim W - \dim (Im\ T)^{\circ} = \dim Im\ T$

$$\boxed{rank(A^T) = rank(A)}$$

$$\boxed{\text{N of linearly independent columns} = \text{N of linearly independent rows}}$$

$$A = \begin{pmatrix} 4 & 7 & 1 & 8 \\ 3 & 5 & 2 & 9 \end{pmatrix} \Rightarrow \text{row rank} = 2 \quad \text{column rank} = 2$$