

Vv156 Lecture 17

Jing Liu

UM-SJTU Joint Institute

November 11, 2018

- The idea behind integrating rational functions

$$\int \frac{P(x)}{Q(x)} dx$$

is to expand the rational function as a sum of simpler fractions, called **partial fractions**, which are easily integrated.

- For example, consider putting the following two fractions together

$$\begin{aligned} & \frac{2}{x-4} + \frac{3}{x+1} \\ &= \frac{2(x+1) + 3(x-4)}{(x-4)(x+1)} \\ &= \frac{5x-10}{x^2-3x-4} \end{aligned}$$

$$\begin{aligned} & \int \frac{5x-10}{x^2-3x-4} dx \\ &= \int \left(\frac{2}{x-4} + \frac{3}{x+1} \right) dx \\ &= \int \frac{2}{x-4} dx + \int \frac{3}{x+1} dx \\ &= 2 \ln |x-4| + 3 \ln |x+1| + C \end{aligned}$$

$$\begin{aligned} & \int \frac{1}{ax+b} dx \\ & u = g(x) = ax+b \\ & \Rightarrow g'(x) = a \\ &= \int \frac{1}{ax+b} \frac{1}{a} a dx \\ &= \int u^{-1} \frac{1}{a} du \\ &= \frac{1}{a} \ln |ax+b| + C \end{aligned}$$

Definition

This method is called **integration by partial fraction**, and the expansion is known as **partial fraction decomposition** (P.F.D.).

- Every **proper** rational function, $\deg(P) < \deg(Q)$, can be expanded as

$$\frac{P(x)}{Q(x)} = F_1(x) + F_2(x) + \cdots + F_n(x)$$

where F_1, F_2, \dots, F_n are terms of partial fractions of $\frac{P(x)}{Q(x)}$.

- The form of F_i depends on factors in $Q(x)$. For each **simple factor** in $Q(x)$,

Factor in $Q(x)$	Term in P.F.D.
$ax + b$	$\frac{A}{ax + b}$

Exercise

Find $\int \frac{x+1}{(2x+1)(x+2)} dx$ by partial fraction.

- If a particular simple factor in $Q(x)$ is repeated, which is known as a **simple repeated factor**, then

Factor in $Q(x)$	Term in P.F.D.
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^k$	$\frac{A_1}{ax + b} + \cdots + \frac{A_k}{(ax + b)^k}$

where k is a positive integer.

Exercise

Find the partial fraction decomposition for $\frac{x^2 + 1}{x(x - 1)^3}$.

- If a particular quadratic factor cannot be further factored into real terms, then it is known as **irreducible quadratic factor**,

Factor in $Q(x)$	Term in P.F.D.
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^k$	$\frac{A_1}{ax + b} + \cdots + \frac{A_k}{(ax + b)^k}$
$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$

where k is an integer.

Exercise

Expand $\frac{5x - 4}{(x + 2)(x^2 + 3)}$ using partial fraction.

Q: Is x^2 an irreducible quadratic factor or a repeated linear factor?

$$\frac{Ax + B}{x^2} \quad \text{or} \quad \frac{A}{x} + \frac{B}{x^2}$$

Q: How about $x^3, x^4, x^5 \dots$?

Factor in $Q(x)$	Term in P.F.D.
$ax + b$	$\frac{A}{ax + b}$
$(ax + b)^k$	$\frac{A_1}{ax + b} + \dots + \frac{A_k}{(ax + b)^k}$
$ax^2 + bx + c$	$\frac{Ax + B}{ax^2 + bx + c}$
$(ax^2 + bx + c)^k$	$\frac{A_1x + B_1}{ax^2 + bx + c} + \dots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}$

Exercise

Expand using partial fraction

$$\frac{4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7}{x^2 (x^2 + 1)^2}$$

You don't need to evaluate the coefficients.

- **Improper rational function** is when the degree of the top is not less than the degree of the bottom, for example,

$$\frac{x^5 - 2x^4 + x^3 + x + 5}{x^3 - 2x^2 + x - 2} = x^2 + \frac{2x^2 + x + 5}{x^3 - 2x^2 + x - 2}$$

- Long division (Polynomial division)

$$\begin{array}{r} x^2 \\ x^3 - 2x^2 + x - 2 \overline{) x^5 - 2x^4 + x^3} \\ \underline{-x^5 + 2x^4 - x^3 + 2x^2} \\ 2x^2 \end{array}$$

Exercise

Find the following integrals.

(a)

$$\int \frac{3x^2 + 2}{x^3 + 2x - 8} dx$$

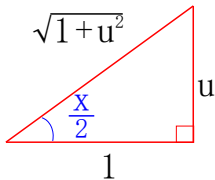
(b)

$$\int \frac{2x - 1}{x^2 + 1} dx$$

- In essence, the partial fraction decomposition helps in terms of identifying the antiderivative of a rational function by breaking down the rational function into a sum of smaller and simpler rational functions, thus PFD is just as applicable for definite integrals as it is for indefinite integral.

Q: How would you integrate a rational function of trigonometric functions?

- Consider a right angled triangle, and define x and u as indicated,



- The sine, cosine and tangent of $\frac{x}{2}$ are

$$\sin \frac{x}{2} = \frac{u}{\sqrt{1+u^2}}, \quad \cos \frac{x}{2} = \frac{1}{\sqrt{1+u^2}}, \quad \tan \frac{x}{2} = \frac{u}{1} = u$$

- Use double angle identity, we have $\sin x$ and $\cos x$ in terms of u ,

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2} = \frac{2u}{1+u^2}, \quad \cos x = 1 - 2 \sin^2 \frac{x}{2} = \frac{1-u^2}{1+u^2}$$

- Lastly, the derivative of $u = g(x)$ with respect to x in terms of u

$$u = g(x) = \tan \frac{x}{2}$$

$$\Rightarrow g'(x) = \frac{1}{2} \sec^2 \frac{x}{2} = \frac{1}{2 \cos^2 \frac{x}{2}} = \frac{1 + u^2}{2}$$

Exercise

Find the following integrals.

(a)

$$\int \frac{dx}{3 - 5 \sin x}$$

(b)

$$\int_{\pi/3}^{\pi/2} \frac{dx}{1 + \sin x - \cos x}$$