
Question1 (1 points)

Consider the following set

$$\mathcal{A} = \left\{ \frac{m}{n} \mid m, n \in \mathbb{N}_1, m < n \right\}$$

Show $\mathcal{A}$ has no minimum or maximum element in spite of having $\sup(\mathcal{A}) = 1$ and $\inf(\mathcal{A}) = 0$.

Solution:

1M Note we will be using the definition of natural numbers that does not include zero in this course, so unless otherwise specified, we have

$$\mathbb{N} = \mathbb{N}_1 = \{1, 2, 3, \ldots\}$$

Consider an arbitrary pair of m and n in $\mathbb{N}_1$ such that $m < n$, and let us denote

$$a = \frac{m}{n}$$

By definition, we have $a \in \mathcal{A}$ and

$$u = \frac{m+1}{n+1} \in \mathcal{A} \qquad \text{and} \qquad \ell = \frac{m^2}{n^2} \in \mathcal{A}$$

since $m+1$, $n+1$, m^2, n^2 belong to $\mathbb{N}_1$ and

$$m+1 < n+1 \qquad \text{and} \qquad m^2 < n^2$$

It is clear, by a simple proof of contradiction,

$$u \le a \implies \frac{m+1}{n+1} \le \frac{m}{n} \implies n(m+1) \le m(n+1) \implies n+1 \le m+1$$
$$l \ge a \implies \frac{m^2}{n^2} \ge \frac{m}{n} \implies m^2 n \ge mn^2 \implies m \ge n$$

that the following holds

$$l < a < u$$

from which we can conclude for any element in $\mathcal{A}$, there is an element smaller than a, and an element bigger than a. Therefore no minimum or maximum in $\mathcal{A}$.

Question2 (1 points)

Let $\mathcal{F}$ be the collection of sets

$$\mathcal{S}_r = \{x \mid r < x \le 1 + r\}, \qquad 0 < r \le \frac{1}{2}$$

Find the union $\bigcup \{\mathcal{S}_r \mid \mathcal{S}_r \in \mathcal{F}\}$ and the intersection $\bigcap \{\mathcal{S}_r \mid \mathcal{S}_r \in \mathcal{F}\}$.

Solution:

1M The union

$$\bigcup \{\mathcal{S}_r \mid \mathcal{S}_r \in \mathcal{F}\} = \left\{x \mid 0 < x \leq \frac{3}{2}\right\}$$

1M The intersection

$$\bigcap \{\mathcal{S}_r \mid \mathcal{S}_r \in \mathcal{F}\} = \left\{x \mid \frac{1}{2} < x \leq 1\right\}$$

Question3 (3 points)

Suppose $\mathcal{A}$ is a nonempty proper subset of $\mathbb{R}$. Determine whether each of the following statements is true. If not, briefly explain why it is false.

(a) (1 point) The set $\mathcal{A}$ is either open or closed.

Solution:

1M False, the set $\mathcal{A}$ can be neither open nor closed, e.g.

$$(1, 2]$$

(b) (1 point) The interior points and boundary points of $\mathcal{A}$ are limit points.

Solution:

1M False, a boundary point of $\mathcal{A}$ might be an isolated point, e.g.

$$(1, 2) \cup \{3\}$$

(c) (1 point) If every element of $\mathcal{A}$ is an isolated point, then $\mathcal{A}$ is closed.

Solution:

1M False, the complement of $\mathcal{A}$ might not be open, e.g.

$$\left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$$

it would be true if we restrict ourselves to a finite set.

Question4 (5 points)

Let $\mathcal{D} = (-\infty, -5] \cup (3, 4) \cup \{7\}$.

(a) (1 point) Find the interior of $\mathcal{D}$, that is, the set of all interior points of $\mathcal{D}$.

Solution:

1M The interior of $\mathcal{D}$ is

$$(-\infty, -5) \cup (3, 4)$$

(b) (1 point) Find the boundary of $\mathcal{D}$, that is, the set of boundary points of $\mathcal{D}$, which is often denoted by $\partial\mathcal{D}$. The closure of $\mathcal{D}$, often denoted by $\overline{\mathcal{D}}$, is the union of $\mathcal{D}$ and $\partial\mathcal{D}$.

Solution:

1M The boundary of $\mathcal{D}$ is

$$\partial\mathcal{D} = \{-5, 3, 4, 7\}$$

(c) (1 point) Find the exterior of $\mathcal{D}$.

Solution:

1M The exterior of $\mathcal{D}$ is defined to be the interior of $\mathcal{D}^{\complement}$, thus it is

$$(-5, 3) \cup (4, 7) \cup (7, \infty)$$

(d) (1 point) Find all limit points of $\mathcal{D}$.

Solution:

1M The set of limit points of $\mathcal{D}$ is

$$(-\infty, -5] \cup [3, 4]$$

(e) (1 point) Find all isolated points of $\mathcal{D}$.

Solution:

1M The only isolated point of $\mathcal{D}$ is

$$\{7\}$$