Name and ID: ______________________________

1. Beta and Gamma function.

Question1 (0 points)
Why use Beta and Gamma (exponential form) to substitute $\sin^n x$ and $\cos^n x$?

Solution:

It could refer to the operation on the improper integral. First consider

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

which is hard to be handled in 1D case, but when we move to the double integral (you can see the proof for the transformation in the Slides).

$$\begin{aligned} I^2 &= (\int_{-\infty}^{\infty} e^{-x^2} dx) \cdot (\int_{-\infty}^{\infty} e^{-x^2} dx) \\ &= (\int_{-\infty}^{\infty} e^{-x^2} dx) \cdot (\int_{-\infty}^{\infty} e^{-y^2} dy) \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dxdy \\ &= \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} rdrd\theta \\ &= \pi \end{aligned}$$

Notice that it actually indicates there is some relationship between e (the exponential form)and the π (the trigonometric form)!

Also, think of the Euler's formula that we have all learned in high school,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Actually, there is a subject carefully considering the exponential form, namely **Complex Analysis**, which would leave deep impact on your future studies, especially for those who declare the major ECE.

Question2 (0 points)
List the definition form for Beta and Gamma function.

Solution:

Beta Function:

$$B(a,b) = \int_0^1 x^{a-1} \cdot (1-x)^{b-1}\, dx$$

Gamma Function:

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \cdot e^{-x}\, dx, \quad \alpha > 0$$

Question3 (0 points)
Specify the transformation between Beta and Gamma function.

Solution:

$$B(a,b)=\frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)},\quad \Rightarrow B(a,b)=B(b,a)$$
$$\bigstar\bigstar\bigstar\ B(a,1-a)=\Gamma(a)\cdot\Gamma(1-a)=\frac{\pi}{\sin a\pi}$$

Question4 (0 points)
Specify the other useful formula for applying beta and gamma funcrion

Solution:

The recursion expression: (a,b $\in\mathbb{R}$)

$$\begin{aligned}\Gamma(a+1)&=a\cdot\Gamma(a)\\ B(a,b)&=\frac{b-1}{a+b-1}\cdot B(a,b-1)\\ B(a,b)&=\frac{a-1}{a+b-1}\cdot B(a-1,b)\end{aligned}\tag{1}$$

Question5 (0 points)
Use Beta and Gamma function to represent

$$\int_0^{\pi/2}\sin^{a-1}\varphi\cdot\cos^{b-1}\varphi\,d\varphi,\quad (a,b>0)$$

Solution:

Consider $x=\sin\varphi$, then

$$\frac{1}{2}\cdot B(\frac{a}{2},\frac{b}{2})=\frac{1}{2}\cdot\frac{\Gamma(\frac{a}{2})\cdot\Gamma(\frac{b}{2})}{\Gamma(\frac{a+b}{2})}\tag{2}$$

Question6 (1 point)
Apply the conclusion in Question 5 and 3, calculate

$$\int_0^{\pi/2}(\sin\theta+\cos\theta)\cdot(\sin^{3/2}\theta\cdot\cos^{3/2}\theta)d\theta$$

Solution:

$$\begin{aligned}LHS&=\int_0^{\pi/2}(\sin^{5/2}\theta\cdot\cos^{3/2}\theta+\sin^{3/2}\theta\cdot\cos^{5/2}\theta)d\theta\\ &=2\times\frac{1}{2}\cdot B(\frac{1}{2}\times(\frac{5}{2}+1),\frac{1}{2}\times(\frac{3}{2}+1))\\ &=\frac{\Gamma(\frac{7}{4})\cdot\Gamma(\frac{5}{4})}{\Gamma(3)}=\frac{\pi}{\sin\frac{1}{4}\pi}\cdot\frac{1}{2!}\\ &=\frac{\sqrt{2}}{2}\pi\end{aligned}$$

2. Integration by I don't know why but anyway useful.

Question1 (1 point)
Prove: If

$$(a-c)^2+b^2\neq 0$$

then

$$\int \frac{a_1\sin x+b_1\cos x}{a\sin^2 x+2b\sin x\cos x+c\cos^2 x}dx=A\int\frac{du_1}{k_1u_1^2+\lambda_1}+B\int\frac{du_2}{k_2u_2^2+\lambda_2}$$

where λ_1,λ_2 is the root of the equation

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}=0,(\lambda_1\neq\lambda_2)$$

and

$$u_i=(a-\lambda_i)\sin x+b\cos x,\;\; k_i=\frac{1}{a-\lambda_i}\;\;(i=1,2)$$

A and B are constants to be determined.
Hint: Try first to reform the denominator according to the given determinate and then the numerator.

Solution:

Notice the target of our expression goes that

$$A\int\frac{du_1}{k_1u_1^2+\lambda_1}+B\int\frac{du_2}{k_2u_2^2+\lambda_2}$$

If we can express the numerator and the denominator, respectively as

$$(a_1\sin x+b_1\cos x)dx=Adu_1+Bdu_2$$

$$a\sin^2 x+2b\sin x\cos x+c\cos^2 x=k_iu_i^2+\lambda_i$$

For the fact that u_i is the linear combination of $\sin x$ & $\cos x$, the differential du_i is also the linear combination of them.
First we can investigate the denominator:

$$a\sin^2 x+2b\sin x\cos x+c\cos^2 x$$

which could be simplified as

$$\begin{aligned}LHS&=(a-\lambda_i)\sin^2 x+2b\sin x\cos x+(c-\lambda_i)\cos^2 x+\lambda_i\\&=\frac{1}{a-\lambda_i}((a-\lambda_i)^2\sin^2 x+2b\cdot(a-\lambda_i)\sin x\cos x\\&+(c-\lambda_i)\cdot(a-\lambda_i)\cos^2 x)+\lambda_i\\&=k_iu_i^2+\lambda_i\end{aligned}$$

We want to obtain the perfect square here, so we need the constraint

$$(c-\lambda_i)\cdot(a-\lambda_i)=b^2$$

i.e.

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix}=0$$

Notice the Δ for the equation of λ is

$$(a-c)^2+4b^2\geq(a-c)^2+b^2\neq 0\Rightarrow\lambda_1\neq\lambda_2$$

Also, notice that to ensure the existence of k_i, we here need to assume

$$b\neq 0$$

To be noticed, in the case I will leave the case when b=0 for you to fill in the blank. Then we can derive

$$u_1 = (a-\lambda_1)\sin x + b\cos x,\ \ k_1 = \frac{1}{a-\lambda_1}$$
$$u_2 = (a-\lambda_2)\sin x + b\cos x,\ \ k_2 = \frac{1}{a-\lambda_2}$$

Also the differential could be derived as

$$du_1 = ((a-\lambda_1)\cos x - b\sin x)dx$$
$$du_2 = ((a-\lambda_2)\cos x - b\sin x)dx$$

And we need to equalize the coefficient for $\sin x$ & $\cos x$ as:

$$-b(A+B) = a_1$$
$$A\,(a-\lambda_1) + B\,(a-\lambda_2) = b_1$$

Applying Cramer's rule we have

$$A = -\frac{a_1(a-\lambda_2)+bb_1}{b\,(\lambda_1-\lambda_2)}$$
$$B = \frac{a_1(a-\lambda_1)+bb_1}{b\,(\lambda_1-\lambda_2)}$$

Question2 (1 point)
Change variables and find the area of regions bounded by the following curves:

$$\left(x^3+y^3\right)^2 = x^2+y^2, \quad x \geq 0, \quad y \geq 0$$

(Hint: Apply polar transformation and utilize the conclusion in question 1)

Solution:

3. Double Integral

Question1 (1 point)
Let $R: 0 \leq x \leq t, 0 \leq y \leq t$ and

$$F(t) = \iint_R e^{-\frac{tx}{y^2}}\,dxdy$$

Compute $F'(t)$
Hint: Try to write down $F'(t)$ in terms of $F(t)$ And the transformation T:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} tu \\ tv \end{pmatrix}$$

Solution:

Question2 (1 point)
Use Generalized Polarize, compute

$$\left(\frac{x}{a}+\frac{y}{b}\right)^4=\frac{x^2}{h^2}+\frac{y^2}{k^2}\ (x>0,y>0)$$

Solution:

Question3 (1 point)
Change variables to find the area of regions bounded by the following curves:

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1,\quad \sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=2,\quad \frac{x}{a}=\frac{y}{b},\quad 4\frac{x}{a}=\frac{y}{b}\quad (a>0,b>0)$$

Hint: consider

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=u,\quad \frac{x}{y}=v$$

Solution:

Apply the transformation, and then we have

$$x=\frac{u^2v}{\left(\sqrt{\frac{v}{a}}+\frac{1}{\sqrt{b}}\right)^2},\quad y=\frac{u^3}{\left(\sqrt{\frac{v}{a}}+\frac{1}{\sqrt{b}}\right)^2}$$

$$1\leqslant u\leqslant 2,\ \frac{a}{4b}\leqslant v\leqslant\frac{a}{b}$$

and

$$|J|=\frac{2u^3}{\left(\sqrt{\frac{v}{a}}+\frac{1}{\sqrt{b}}\right)^4}$$

$$S=\int_1^2 2u^3du\int_{\frac{a}{4b}}^{\frac{a}{b}}\frac{dv}{\left(\sqrt{\frac{v}{a}}+\frac{1}{\sqrt{b}}\right)^4}$$

$$=\frac{15}{2}\cdot\int_{\frac{1}{2\sqrt{b}}}^{\frac{1}{\sqrt{b}}}\frac{2atdt}{\left(t+\frac{1}{\sqrt{b}}\right)^4}\ ,v=at^2$$

$$=15a\cdot\left(\frac{7b}{72}-\frac{37b}{648}\right)=\frac{65ab}{108}$$

Besides, you can also try the transformation

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=u,\sqrt{\frac{ay}{bx}}=v$$

the calculation of which may be of more elegance.