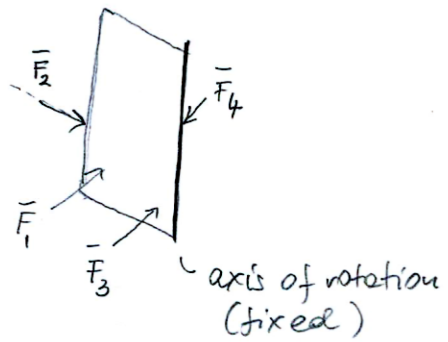
Dynamics of rotational motion: the torque and the 2nd law of dynamics for rotational motion

Example: revolving door



The force is of the same magnitude, but the effect is different

$\bar{F}_1$, $\bar{F}_3$ } door revolves, $\bar{F}_1$ "more effective" than $\bar{F}_3$

$\bar{F}_2$, $\bar{F}_4$ } door does not revolve.

Torque (of a force)

$$\bar{\tau} \overset{def}{=} \bar{r} \times \bar{F}$$

always defined with respect to a point / axis of rotation

(right-hand rule)

units: N·m (= Joule)

Note: decompose $\bar{r} = \bar{r}_{\parallel} + \bar{r}_{\perp}$

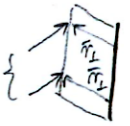
parallel/perpendicular to $\bar{F}$

then $\bar{\tau} = (\bar{r}_{\parallel} + \bar{r}_{\perp}) \times \bar{F} \overset{\bar{r}_{\parallel} \parallel \bar{F}}{=} r_{\perp} \times \bar{F}$ and $|\bar{\tau}| = |\bar{r}_{\perp}|\,|\bar{F}|$

lever arm

Magnitude of the torque (alternative ways to calculate)

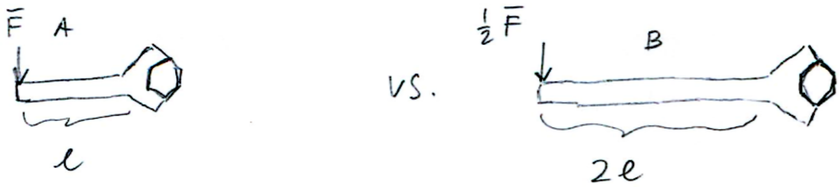
same effect {



$$|\bar{\tau}| = |\bar{r}|\,|\bar{F}| \sin\measuredangle(\bar{r},\bar{F}) = |\bar{r}_{\perp}|\,|\bar{F}| = |\bar{r}|\,|\bar{F}_{\perp}|$$

perpendicular to $\bar{r}$

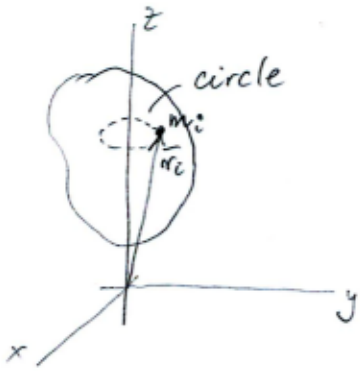
Example Practical application – wrench



smaller force needed to rotate the screw

Torque and angular acceleration for a rigid body

(rotation about a *fixed* axis; choose the z-axis to be the axis of rotation)



2^{nd} law of dynamics for the element of mass m_i

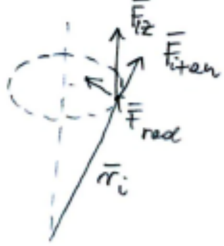
$$\bar{F}_i = m_i \bar{a}_i \qquad / \bar{r}_i \times$$

↳ net force on m_i

$$\underbrace{\bar{r}_i \times \bar{F}_i}_{} = m_i \bar{r}_i \times \bar{a}_i \qquad (*)$$

↳ net torque on mass m_i w.r.t. the origin

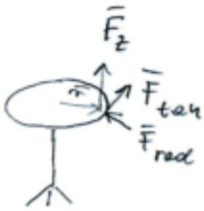
The net force can be decomposed as



$$\bar{F}_i = \bar{F}_{i\,rad} + \bar{F}_{i\,tan} + \bar{F}_{i\,z}$$

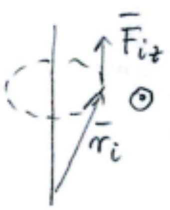
↳ radial ↳ tangential ↳ along the z-axis

Experiment: rotating chair (fixed axis)



Observation: → Only $\bar{F}_{tan}$ makes the chair rotate. The torque corresponding to $\bar{F}_{tan}$ is in the direction of the rotation axis.

→ role of $\bar{F}_z$ and $\bar{F}_{rad}$



$\odot$ $\bar{r}_i \times \bar{F}_{iz}$ tries to change the axis of rotation

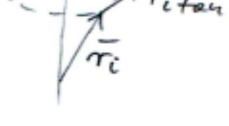
$\odot$ $\bar{r}_i \times \bar{F}_{irad}$ tries to change the axis of rotation

but the axis of rotation is *fixed*

CONCLUSION: The only component of the net force that can affect rotational motion about the (fixed) z-axis is due to the tangential component of the force $\bar{F}_{i\,tan}$

$$\bar{\tau}_{iz} = \left(\bar{r}_i \times \bar{F}_{i\,tan}\right)_z$$

↙ z-component



Consequently

$$\left(\bar{r}_i \times \bar{F}_{i\,tan}\right)_z = m_i \left(\bar{r}_i \times \bar{a}_{i\,tan}\right)_z$$

Hence, equation (*) for rotation about a fixed axis (z-axis) is of the form

$$(\bar{r}_i \times \bar{F}_{i\,tan})_z = m_i(\bar{r}_i \times \underbrace{\bar{a}_{i\,tan}}_{=\bar{\varepsilon}\times\bar{r}_{i\perp}})_z$$

- see previous lectures

↳ perpendicular to axis of rotation

But

$$(\bar{r}_i \times \bar{a}_{i\,tan})_z = (\bar{r}_i \times (\bar{\varepsilon}\times\bar{r}_{i\perp}))_z = \left[(\bar{r}_{i\parallel} + \bar{r}_{i\perp}) \times (\bar{\varepsilon}\times\bar{r}_{i\perp})\right]_z =$$

$$= \underbrace{\left[\bar{r}_{i\parallel}\times(\bar{\varepsilon}\times\bar{r}_{i\perp})\right]_z}_{=\text{zero, because has no z-component}} + \left[\bar{r}_{i\perp}\times(\bar{\varepsilon}\times\bar{r}_{i\perp})\right]_z = \left\{\begin{array}{l}\text{use } \bar{A}\times(\bar{B}\times\bar{C}) = \\ = \bar{B}(\bar{A}\cdot\bar{C}) - \bar{C}(\bar{A}\cdot\bar{B})\end{array}\right\}$$

$$= (\bar{\varepsilon}\, r_{i\perp}^2)_z - \bar{r}_{i\perp}\underbrace{(\bar{r}_{i\perp}\cdot\bar{\varepsilon})}_{=0,\text{ because } \bar{r}_{i\perp}\perp\bar{\varepsilon}} = r_{i\perp}^2\,\varepsilon_z$$

Hence

$$\tau_{iz} = m_i r_{i\perp}^2 \varepsilon_z$$

or, adding contributions from all elements m_i

$$\tau_z = \underbrace{\left(\sum_{i=1}^{N} m_i r_{i\perp}^2\right)}_{I_z}\varepsilon_z$$

I_z moment of inertia about axis z

Eventually

$$\boxed{\tau_z = I_z\, \varepsilon_z}$$

angular acceleration

torque

moment of inertia

2nd law of dynamics for a rigid body rotating about a fixed axis

Compare

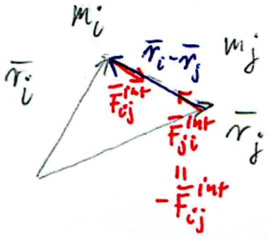
(*) particle net force ⇒ acceleration
- $\propto$ force
- $\propto \frac{1}{\text{mass}}$

(*) rigid body net torque along axis of rotation ⇒ angular acceleration
- $\propto$ torque
- $\propto \frac{1}{\text{moment of inertia}}$

Comments:

(*) the above discussion is valid for a rigid body; we assumed the same $\bar{\varepsilon}$ for all elements of mass

(*) internal forces between the elements of mass do not contribute to the torque



contribution of this pair: $\bar{r}_i \times \bar{F}_{ij}^{int} + \bar{r}_j \times \bar{F}_{ji}^{int} =$

$$\overset{3^{rd}}{\underset{\text{law}}{=}} \bar{r}_i \times \bar{F}_{ij}^{int} - \bar{r}_j \times \bar{F}_{ij}^{int} =$$

$$= (\bar{r}_i - \bar{r}_j)\times\bar{F}_{ij}^{int} = 0$$

because $(\bar{r}_i - \bar{r}_j)$ is parallel to $\bar{F}_{ij}^{int}$

valid for all pairs

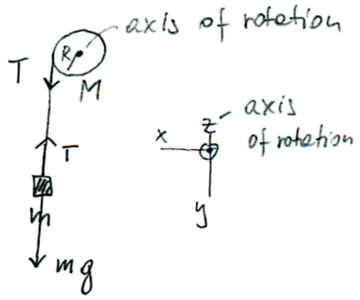
Examples

(a) unwinding string

cylinder: mass M

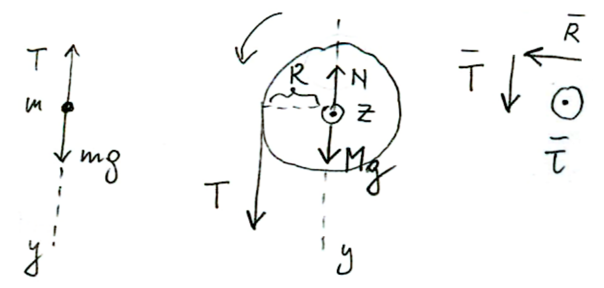
radius R

$I_z = \frac{1}{2} M R^2$



Find acceleration of the block and tension T.

Free-body diagrams



block

cylinder

Laws of dynamics

$$
\begin{cases}
(1) \quad 0 = Mg + T - N & \text{(axis of the cylinder is fixed) - statics} \\
(2) \quad m a_y = mg - T & \text{(block)} \\
(3) \quad I_z \varepsilon_z = \tau_z & \text{(cylinder)} \\
(4) \quad a_y = R \cdot \varepsilon_z &
\end{cases}
$$

$\bar{a} = \bar{\varepsilon} \times \bar{R}$

(linear acceleration of the string = tangential acceleration of the point on the rim of the cylinder; the string does not slip and is inextensible)

But $\tau_z = +RT$, substitute (4) into (2)

$$
\begin{cases}
(2') \quad m R \varepsilon_z = mg - T \\
(3') \quad RT = I_z \varepsilon_z
\end{cases}
$$

solution $\Longrightarrow$ use (3') to find T, substitute into (2')

$$\varepsilon_z = \frac{mg}{mR + \frac{I_z}{R}} = \frac{mg}{(m + \frac{1}{2}M)R}$$

$$\boxed{a_y = \frac{m}{m + \frac{1}{2}M} g \qquad T = \frac{M m g}{2m + M}}$$

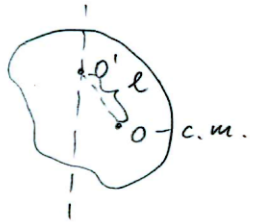
Discussion:

$\rightarrow$ if $M = 0$ $\Rightarrow$ $T = 0$; $a_y = g$; $\varepsilon_z = g/R$

$\rightarrow$ if $M \gg m$ $(\frac{m}{M} \to 0)$ $\Rightarrow$ $T \to mg$; $a_y, \varepsilon_z \to 0$

$\rightarrow$ if $M = m$ $\Rightarrow$ $T = \frac{1}{3} mg$; $a_y = \frac{2}{3} g$; $\varepsilon_z = \frac{2}{3} \frac{g}{R}$

(b) Discuss motion of a rigid body that can rotate about an axis O' parallel to axis O through the center of mass. Distance between the axes O and O' is l. g, I_o and mass of the body m are known.

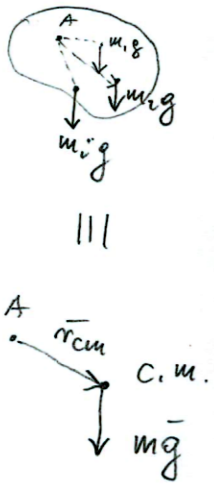


physical pendulum

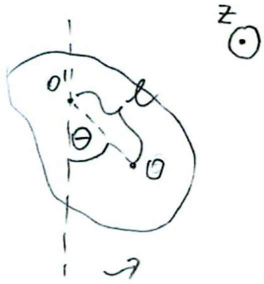
Observation: If the external force acting on a rigid body is the (constant) gravitational force, then its torque about an axis of rotation A can be found as

$$\bar{\tau} = \sum_{i=1}^{N} \bar{r}_i \times m_i \bar{g} =$$

$$= \sum_{i=1}^{N} \underbrace{(m_i \bar{r}_i)}_{m\bar{r}_{cm}} \times \bar{g} = \bar{r}_{cm} \times m\bar{g}$$



Use this observation



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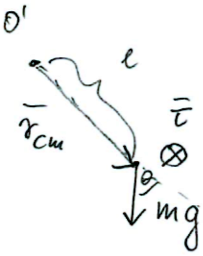


$$\bar{\tau} = \bar{r}_{cm} \times m\bar{g}$$

$$\tau_z = -mgl\sin\theta$$

Equation of motion

$$I_{o'}\varepsilon_z = \tau_z \Rightarrow I_{o'}\varepsilon_z = -mgl\sin\theta$$

But $\varepsilon_z = \ddot{\theta}$, hence

$$I_{o'}\ddot{\theta} = -mgl\sin\theta \Rightarrow \ddot{\theta} + \frac{mgl}{I_{o'}}\sin\theta = 0$$

If the angle is small $\sin\theta \approx \theta$ and $\boxed{\ddot{\theta} + \frac{mgl}{I_{o'}}\theta = 0}$

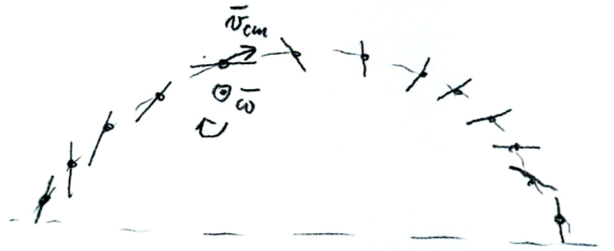
harmonic oscillator

$$\omega_o = \sqrt{\frac{mgl}{I_{o'}}} \; ; \; T = 2\pi\sqrt{\frac{I_{o'}}{mgl}}$$

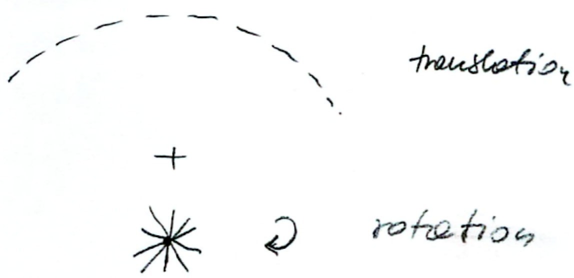
Rigid-body rotation about a moving axis

motion of a rigid-body = translational motion of the center of mass
+
rotational motion about an instanteneous axis of rotation (through the center of mass)

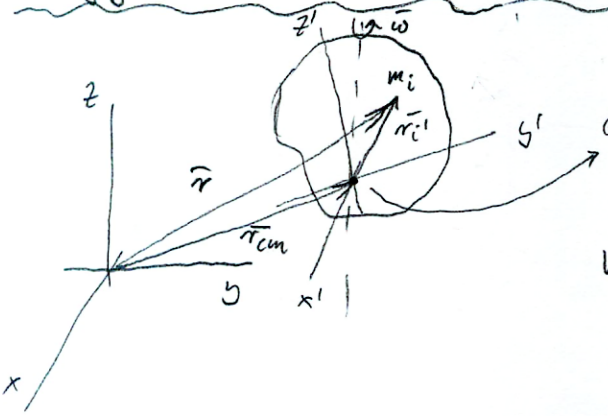
Example



(axis does not change direction)



Energy in the combined motion



origin placed at the center of mass

Velocity of element of mass m_i

$$\bar{v}_i = \bar{v}_{cm} + \bar{\omega} \times \bar{r}_i' = \text{(see previous lecture)}$$

$$= \bar{v}_{cm} + \bar{\omega} \times \bar{r}_{i\perp}'$$

↳ perpendicular to axis of rotation

Contribution to kinetic energy K_i due to m_i; Total k.e.

$$K = \sum_i K_i = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i \bar{v}_i \bar{v}_i = \frac{1}{2}\sum_i m_i \left(v_{cm}^2 + 2\bar{v}_{cm}\cdot(\bar{\omega}\times\bar{r}_{i\perp}') + (\bar{\omega}\times\bar{r}_{i\perp}')^2 \right) =$$

$$= \frac{1}{2}\sum_i m_i v_{cm}^2 + \bar{v}_{cm}\cdot\Big(\bar{\omega}\times \underbrace{\sum_i m_i \bar{r}_{i\perp}'}_{=0}\Big) + \frac{1}{2}\sum_i m_i (\bar{\omega}\times\bar{r}_{i\perp}')^2$$

But; (1) $\sum_i m_i \bar{r}_{i\perp}' = \underset{\text{mass of the Body}}{M} \cdot \bar{r}_{cm\perp}'$ and the origin of $x'y'z'$ is placed at the center of mass of the Body, so $\bar{r}_{cm}' = 0$ and $\bar{r}_{cm\perp}' = 0$

(2) $\bar{\omega} \perp \bar{r}_{i\perp}'$, hence $(\bar{\omega}\times\bar{r}_{i\perp}')^2 = \omega^2 r_{i\perp}'^2$

Eventually

$$K = \sum_i K_i = \frac{1}{2}\underbrace{\left(\sum_i m_i\right)}_{M\text{ - mass of the rigid body}} v_{cm}^2 + \frac{1}{2}\underbrace{\sum_i m_i r_{i\perp}^2}_{I_{cm}} \omega^2$$

I_{cm} ↳ about the axis of rotation through the center of mass

$$\boxed{K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm}\,\omega^2}$$

Rigid body's kinetic energy is the sum of $\frac{1}{2} M v_{cm}^2$ associated with motion of the c.m. and $\frac{1}{2} I_{cm}\omega^2$ associated with rotation about an axis through the c.m.

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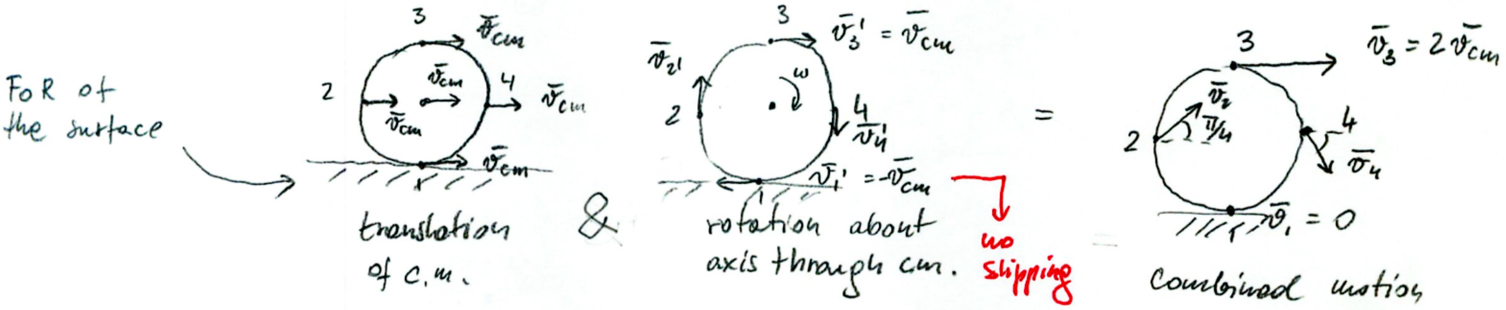
Rolling without slipping



No slipping ≡ the point where the wheel is in contact with the surface is instantaneously at rest

Hence the condition for no slipping: $v_1' = v_{cm}$

but $v_1' = \omega R$

$$\boxed{v_{cm} = \omega R}$$

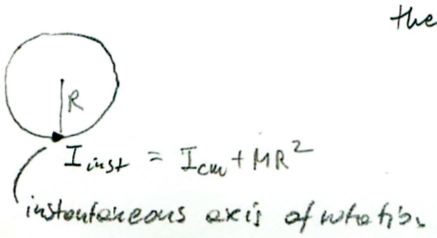
Comment

We can also treat this motion as rotational motion about an instanteneous axis of rotation through the point of contact with the surface with the same angular velocity

then $K = \frac{1}{2} I_{inst}\,\omega^2 \overset{\text{Steiner's thm}}{=} \frac{1}{2}\left(I_{cm} + MR^2\right)\omega^2 =$

$= \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M (\omega R)^2$ rolling without slipping

$= \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2$ → same result



Examples (a) 'fast start'

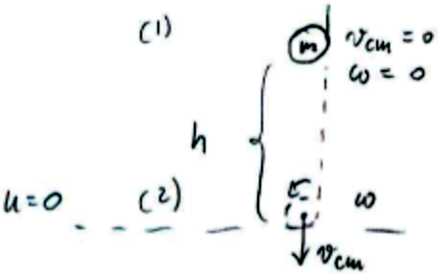
$\omega R > v_{cm}$

braking without ABS

car skids

$\omega R < v_{cm}$

(b) unwinding thread; no slipping



Conservation of energy

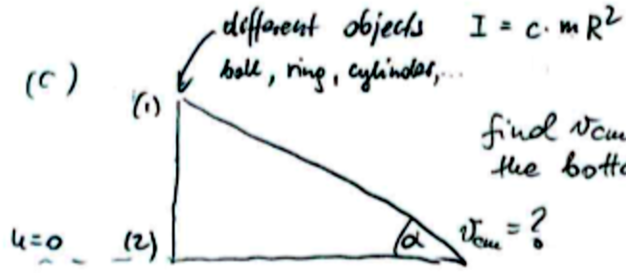
$$K_1 + U_1 = K_2 + U_2$$

$$mgh = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \overset{\frac{1}{2}mR^2}{\overset{\shortparallel}{I_{cm}}} \omega^2 \qquad \underline{\underline{v_{cm} = \omega R}}$$

$$= \frac{1}{2} m v_{cm}^2 + \frac{1}{4} m v_{cm}^2 = \frac{3}{4} m v_{cm}^2$$

$$\boxed{v_{cm} = \sqrt{\frac{4}{3} gh}}$$

(c)



find v_{cm} at the bottom

Static friction
(no slipping)

ignore rolling friction effects

$$K_1 + U_1 = K_2 + U_2$$

no slipping

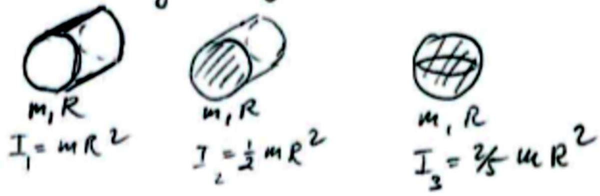
$$mgh = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \left(\frac{v_{cm}}{R}\right)^2$$

$$gh = \frac{1}{2}(1+c) v_{cm}^2$$

$$\Downarrow$$

$$\boxed{v_{cm} = \sqrt{\frac{2gh}{1+c}}}$$

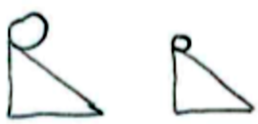
Conclusions: (1) small c's give greater v_{cm}



$v_{1cm} < v_{2cm} < v_{3cm} \longrightarrow$ less energy in rotational motion; more available for translational motion

(2) v_{cm} does not depend on R, m

E.g. all balls irrespective of size and mass have the same speed at the bottom



Combined motion: dynamics

Discussed before $\quad \sum_i \bar{F}_i^{ext} = \frac{d\bar{P}}{dt} = M \bar{a}_{cm}$

For rotational motion

$$\tau_z = I_z \varepsilon_z$$

net torque

→ derived under assumption that axis of rotation is fixed

! true also if the axis is moving

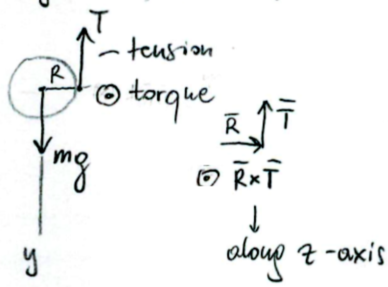
(1) axis is a symmetry axis (through the center of mass)

and

(2) axis does not change direction

Examples

(a) unwinding thread



$$\begin{cases} (1) \quad mg - T = m\, a_{cm,y} \\ (2) \quad TR = I_{cm}\, \varepsilon_z = \frac{1}{2} m R^2 \varepsilon_z \\ (3) \quad a_{cm,y} = R\, \varepsilon_z \quad \text{(no slipping; follows from } v_{cm,y} = \omega_z R\text{)} \end{cases}$$

From (2): $T = \frac{1}{2} m R \varepsilon_z \overset{(3)}{=} \frac{1}{2} m\, a_{cm,y}$

From (1):

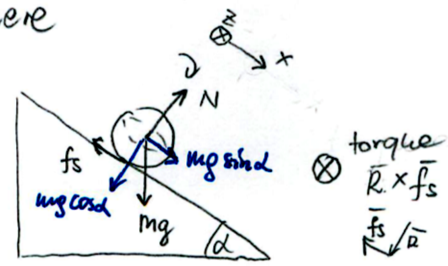
$$mg = \frac{3}{2} m\, a_{cm,y}$$

$$\boxed{a_{cm,y} = \frac{2}{3} g}$$

$$\boxed{T = \frac{1}{3} mg}$$

(b) rolling sphere

static friction (no slipping)



$$\begin{cases} mg \sin\alpha - f_s = m\, a_{cm,x} \\ f_s R = I_{cm}\, \varepsilon_z = \frac{2}{5} m R^2 \varepsilon_z \\ a_{cm,x} = \varepsilon_z R \end{cases}$$

Solution: $\quad a_{cm,x} = \frac{5}{7} g \sin\alpha$

$$f_s = \frac{2}{7} mg \sin\alpha$$

Comments:

* friction still directed uphill for a ball rolling uphill
* comment on rolling friction (perfect rigidity): example: car tire