Vv156 Lecture 25

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- Until now we have considered series of numbers, e.g.

$$1 + 2 + 3 + 4 + \cdots + n + \cdots = \sum_{n=1}^{\infty} n$$

- Now we start turning our attention to series of real-valued functions

$$f_0 + f_1 + f_2 + \cdots + f_n + \cdots = \sum_{n=0}^{\infty} f_n$$

where f_n are functions of $x \in \mathbb{R}$.

Definition

A series of power functions of x is known as a power series

$$\sum_{n=0}^{\infty} c_n x^n = \underbrace{c_0}_{f_0} + \underbrace{c_1 x}_{f_1} + \underbrace{c_2 x^2}_{f_2} + \cdots + \underbrace{c_n x^n}_{f_n} + \cdots$$

where c_n's are constants called the coefficients of the series.

Power series about a

A power series is said to be about $x = a$ if the series has the following form,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + \cdots$$

where a is a constant called the center, and c_n's are coefficients.

- A power series may converge for some values of x and diverge for other values of x. Notice that it is c_n's and a define a power series.
- Taking all the coefficients to be 1, a power series about $x = 0$ gives,

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots$$

which is a geometric series that converges to

$$\frac{1}{1-x} \quad \text{for} \quad -1 < x < 1$$

Exercise

(a) Is the following power series convergent when $x = 1$?

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

(b) Is it convergent when

$$x = -1$$

(c) Use the ratio test to check whether the series is converge when

$$x = 0.5$$

(d) For what values of x do the power series converge?

The convergence theorem for power series

If the power series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$$

converges at $x = \delta \neq 0$, then it converges absolutely for all x with

$$|x| < |\delta|.$$

If the series diverges at $x = d$, then it diverges for all x with

$$|x| > |d|.$$

Proof

- The proof for the first half uses the comparison test. We compare the given series to a geometric series, $\displaystyle\sum_{n=0}^{\infty} \left|\frac{x}{\delta}\right|^n$, which is convergent for $|x| < |\delta|$.

Proof

- Now if the series $\sum_{n=0}^{\infty} c_n \delta^n$ converges, then

$$\lim_{n\to\infty} c_n \delta^n = 0$$

- Hence, there is an integer N such that

$$\begin{aligned} |c_n \delta^n| &< 1, \qquad \text{for all} \qquad n > N \\ |c_n| &< \frac{1}{|\delta|^n} \\ |c_n||x|^n &< \frac{|x|^n}{|\delta|^n} = \left|\frac{x}{\delta}\right|^n \qquad \text{for all} \qquad n > N \end{aligned}$$

- By the comparison test, the series $\sum_{n=0}^{\infty} |c_n x^n|$ converges, so the original power series converges absolutely for $-|\delta| < x < |\delta|$.

Proof

- The second half of the theorem can be proved by contradiction.
- Suppose that the series $\sum_{n=0}^{\infty} c_n x^n$ diverges at $x = d$, but convergent for

$$|x| > |d|$$

- Suppose $|x| > |d|$ and the series converges at x, then the series converges at d by the first half of the theorem we have proved.
- This is a contradiction to our assumption, therefore

$$\text{the series diverges for all } x \text{ with } |x| > |d| \quad \square$$

- For power series with nonzero center, $\sum c_n(x-a)^n$, we can substitute

$$(x-a) \qquad \text{by} \qquad x^*$$

and apply the above theorem to the series $\sum c_n(x^*)^n$.

Corollary

For a given power series $\sum c_n(x-a)^n$ there are only three possibilities:

1. There is a positive number R such that the series diverges for x with
$$|x-a| > R$$
but converges absolutely for x with
$$|x-a| < R.$$
The series may or may not converge at either of the endpoints
$$x = a - R \quad \text{and} \quad x = a + R.$$
2. The series converges absolutely for every x ($R = \infty$).
3. The series converges at $x = a$ and diverges elsewhere ($R = 0$).

Radius and interval of convergence

Definition

1. R is called the radius of convergence of the power series, and the interval of radius R centered at $x = a$ is called the interval of convergence.

$$(a-R, a+R), \quad [a-R, a+R], \quad (a-R, a+R], \quad [a-R, a+R).$$

2. If the series converges for all values of x, we say

   its radius of convergence is infinite.

3. If it converges only at $x = a$, we say

   its radius of convergence is zero.

Testing a power series for convergence

Procedure

1. Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely. Usually, this is an open interval
$$|x-a|<R \qquad \text{or} \qquad a-R<x<a+R.$$
2. If the interval of absolute convergence is $a-R<x<a+R$, then
   the series diverges for $|x-a|>R$.
3. If the interval of absolute convergence is finite, then we need to
   test for convergence or divergence at each endpoint.

Exercise

(a) For what values of x do the following power series converge?

$$\sum_{n=0}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

(b) For what values of x do the following power series converge?

$$\sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

(c) For what values of x do the following power series converge?

$$\sum_{n=0}^{\infty} n!x^n = 1 + x + 2!x^2 + 3!x^3 + \cdots$$

Theorem

If the power series $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely for $|x| < R$, then

$$\sum_{n=0}^{\infty} c_n \Big(f(x)\Big)^n$$

converges absolutely for any continuous function f on $|f(x)| < R$.

Exercise

For what values of x does the following power series converge absolutely?

$$\sum_{n=0}^{\infty} \left(4x^2\right)^n$$

The term-by-term differentiation and integration

Theorem

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ has radius of convergence $R > 0$, then function

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable and integrable on the interval $(a-R, a+R)$, and

1. $\dfrac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \dfrac{d}{dx}\left[c_n(x-a)^n\right] = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$
2. $\displaystyle\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int c_n(x-a)^n\, dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

The radii of convergence of the two series above are both R.

Proof

- Consider the case $a = 0$, that is,

$$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n x^n\right] = \sum_{n=1}^{\infty} n c_n x^{n-1}$$

and suppose $|x| < \rho < R$, and let

$$r = \frac{|x|}{\rho} \implies r \in (0,1)$$

- Notice this allows us to rewrite the terms in the derivative

$$\left|n c_n x^{n-1}\right| = \frac{n}{\rho}\left(\frac{|x|}{\rho}\right)^{n-1} \rho^n \left|c_n\right|$$

$$= n r^{n-1} \frac{\left|c_n \rho^n\right|}{\rho}$$

Proof

- The ratio test shows the following series is convergent

$$\sum_{n=1}^{\infty} nr^{n-1}$$

because $r \in (0,1)$,

$$\lim_{n\to\infty} \left| \frac{(n+1)r^n}{nr^{n-1}} \right| = \lim_{n\to\infty} \left| \left(1+\frac{1}{n}\right) r \right| = r < 1$$

- So the sequence $\{nr^{n-1}\}$ must be bounded, that is, for $M \in \mathbb{R}$,

$$\begin{aligned} nr^{n-1} \le M \implies \left| nc_n x^{n-1} \right| &= nr^{n-1} \frac{|c_n \rho^n|}{\rho} \\ &\le \frac{M}{\rho} \left| c_n \rho^n \right| \end{aligned}$$

Proof

- Since the series $\sum_{n=0}^{\infty} c_n \rho^n$ is convergent by construction since $|x| < \rho < R$.

$$\left| n c_n x^{n-1} \right| \leq \frac{M}{\rho} \left| c_n \rho^n \right|$$

  Then the series $\sum_{n=1}^{\infty} n c_n x^{n-1}$ must be convergent by the comparison test.

- Now suppose $|x| > R$, then the series $\sum_{n=0}^{\infty} c_n x^n$ diverges and $\sum_{n=1}^{\infty} n c_n x^{n-1}$ is also divergent by the comparison test

$$\left| n c_n x^{n-1} \right| = \frac{1}{|x|} \left| n c_n x^n \right| \geq \frac{1}{|x|} \left| c_n x^n \right| \qquad \text{for} \quad n \geq 1$$

- So the radii of convergence are the same, and the integral follows directly.

Exercise

(a) Express the function $f(x) = \dfrac{1}{(1-x)^2}$ as a power series by differentiating

$$\frac{1}{1-x}$$

(b) Find an approximation

$$\int_0^{0.5} \frac{dx}{1+x^7}$$

correct to within 10^{-7} .