Final Review Part II

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Overview

Reference: VV255 Lecture Slides by Professor Jing, Professor Olga, VV255 TA Group18SU, Demidovich, and Stewart's Textbook

Typical Questions. Part A

Description

1. All the questions shall be answered properly in part A.
2. No difficult questions presented here. Well understand those questions appeared in Slides!

1. Calculate the double integral. Especially the technique of Change of variables. Identify the boundary, and carefully conduct the calculation.
2. Using Iterated integral to represent the triple integral, in different manner (from dxdydz to dzdydx, for example). Tell the difference between the double/triple integral and the iterate integral!
3. Calculate the Type I line integral from def. Apply the Green Theorem to calculate the Type II line integral (★).
4. Tell the conservative field (★) and apply its property to calculate the corresponding line integral. Apply the FTL.
5. Apply Divergence theorem or Stoke's theorem.

Typical Questions. Part B

Description

1. Choose 4 from 6. Only counts the HIGHEST scores.
2. Integrated problems mainly involves the part after surface integral.

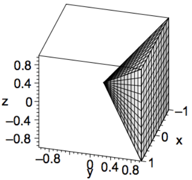
1. Properly parametrize the surface to calculate the surface integral.
2. Derive the vector field (e.g. heat flow) properly for calculating the surface integral. (Potentially physical concept and memorize the typical vector field appeared in the Text Book!)
3. Apply Stoke's to represent the surface integral of a curl as the line integral to simplify the problem.
4. Well understand the spherical coordinates and apply it to solve some integrated problems.

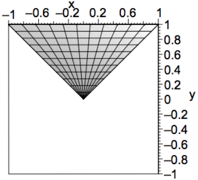
1. For each of the following regions E, express the triple integral $\iiint_E f(x,y,z)dV$ as an iterated integral in Cartesian coordinates.

   1. E is the box $[0,2] \times [-1,1] \times [3,5]$
   Solution:

   $$\iiint_E f(x,y,z)dV = \boxed{\int_0^2 \int_{-1}^1 \int_3^5 f(x,y,z)dzdydx}$$

   2. E is the pyramid with vertices
   $(0,0,0), (1,1,1), (1,1,-1), (-1,1,1),$ and $(-1,1,-1)$
   Solution:





Solution (Continued):
Top function: z = y (plane passing through (0, 0, 0), (1, 1, 1), and (1, 1, 1))
Bottom function: z = y (plane passing through (0, 0, 0), (1, 1, 1), and (1, 1, 1))

$$\iiint_E f(x,y,z)dV = \iint_D \int_{-y}^{y} f(x,y,z)dzdA$$
$$= \boxed{\int_0^1 \int_{-y}^{y} \int_{-y}^{y} f(x,y,z)dzdxdy}$$

You can see more examples in the worksheet.

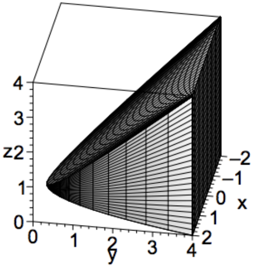
Consider the integral

$$\iiint_E f(x,y,z)dV = \int_{-2}^{2}\int_{x^2}^{4}\int_{0}^{y} f(x,y,z)dzdydx$$

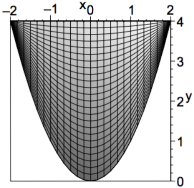
(a) Sketch the region E. (b) Write the other five iterated integrals which represent $\iiint_E f(x,y,z)dV$

Solution:

(a)



Solution(Continued):



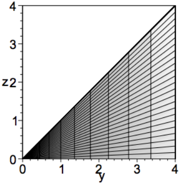


Figure: Left: The image of E on xy-plane; Right: The image of E on yz-plane

If we project E onto xy-plane, then the top function is z = y, and the bottom function is z = 0, as given in the question.
In the order of dz dx dy,

$$\iiint_E f(x,y,z)dV = \boxed{\int_0^4 \int_{-\sqrt{y}}^{\sqrt{y}} \int_0^y f(x,y,z)dzdxdy}$$

Solution(Continued):
If we project E onto yz-plane,then the front function is $x = \sqrt{y}$, and the back function is $x = -sqrty$, as given in the question.
In the order of dx dy dz,

$$\iiint_E f(x,y,z)dV = \boxed{\int_0^4 \int_z^4 \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)dxdydz}$$

In the order of dx dz dy,

$$\iiint_E f(x,y,z)dV = \boxed{\int_0^4 \int_0^y \int_{-\sqrt{y}}^{\sqrt{y}} f(x,y,z)dxdydz}$$

Surface Integral: Concept Overview

Basic Classification

1. Parametrization of the surface; parametrized surface; Spherical coordinates; Cylindrical coordinates; Rotation surface.
2. Piecewise smooth surfaces.
3. Oriented and orientable (smooth) surfaces; positive and negative orientation.
4. Closed surface.

Surface Integral

1. Surface Integral for a Parametric Surface. Surface integral of scalar function; surface integral of vector fields.
2. Derive some typical vector fields.
3. Apply Stoke's and Divergence theorem. (covered later)

Parametric Surface

Def. Parametric surface

The vector function together with the domain D,

$$\bar{r}(u, v) = x(u, v)\mathbf{e}_x + y(u, v)\mathbf{e}_y + z(u, v)\mathbf{e}_z$$

is called a parametric surface. And $\bar{r}$ is a parametrisation of S.
Actually we have obtained the Surfaces Described by Vector Functions.

Special Parametrization: explicit z

For the surface given by the equation as

$$S = g(x, y)$$

we can find the special parametrization as

$$\bar{r}(u, v) = u\bar{i} + v\bar{j} + g(u, v)\bar{k}$$

Special Parametrization: Spherical and Cylindrical

Spherical Coordinates: example

The sphere

$$x^2 + y^2 + z^2 = a^2$$

could parametrized as

$$\bar{r}(\theta, \phi) = a \sin\phi \cos\theta \bar{i} + a \sin\phi \sin\theta \bar{j} + a \cos\phi \bar{k}$$
$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$

Cylindrical Coordinates: example

The cylinder

$$x^2 + y^2 = a^2, 0 \le z \le b$$

could be parametrized as

$$\bar{r}(z, \theta) = a \cos\theta \bar{i} + a \sin\theta \bar{j} + z \bar{k}$$
$$0 \le z \le b, \quad 0 \le \theta \le 2\pi$$

Parametric Surface

Def. Smooth Parametric surface

A parametric surface is smooth provided the following two conditions:

1. The partial derivatives, $\frac{\partial \overline{r}}{\partial u}$ and $\frac{\partial \overline{r}}{\partial v}$ are continuous
2. The cross product between partial derivatives is non-zero in the interior of the domain of $\overline{r}(u, v)$

$$\frac{\partial \overline{r}}{\partial u} \times \frac{\partial \overline{r}}{\partial v} \neq \mathbf{0}$$

Actually, it means that the surface normal vector depends continuously on the points on the surface.

Def. Piecewise Smooth Surface

A piecewise smooth surface consists of finitely many smooth surface.

Oriented and Orientable

Description. Two-Sided and One-sided

Two-sided: Across the boundary the surface normal vector $\overline{n}$ is undefined.
One-sided: Everywhere $\overline{n}$ could be well defined.

Def. Oriented Smooth Surface

A smooth surface S is said to be orientable if S is two-sided, and non-orientable if S is a one-sided surface.
An oriented surface is a surface S and a vector function $\overline{n}$ that witnesses the fact that S is orientable.

Orientation

Def. Orientation

If $\overline{n}$ witnesses the fact that S is orientable, then we say that $\overline{n}$ is an orientation of S, where there are two orientations in total,
positive orientation:

$$\overline{\mathbf{n}_1} = \frac{\overline{\mathbf{r}_u} \times \overline{\mathbf{r}_v}}{|\overline{\mathbf{r}_u} \times \overline{\mathbf{r}_v}|}$$

negative orientation: $\overline{\mathbf{n}_2} = -\overline{\mathbf{n}_1}$

Orientation for Surface explicitly defined by equation

For the surface defined by the equation, we have the orientation as

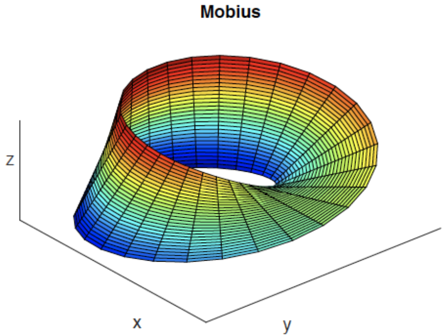
$$\Phi(x, y, z) = 0$$

$$\mathbf{n}_1 = \frac{\nabla\Phi}{|\nabla\Phi|} \Longrightarrow \nabla\Phi = \mathbf{r}_u \times \mathbf{r}_v$$

Orientation: for fun

Illustrate some non-orientable surfaces.

Solution: Mobius strip or Klein bottle.



Surface Integral

Def. of a scalar-valued function

Suppose S is a smooth parametric surface defined by $\overline{r}(u, v)$ over D and $f(x, y, z)$ is a continuous function, then the surface integral of f over S is defined to be

$$\iint_S f(x, y, z) dS = \iint_{\mathcal{R}} f(\overline{r}(u, v)) \left| \overline{r}_u \times \overline{r}_v \right| dA$$

Def. of a vector field

Suppose $\overline{F}$ is a continuous vector field and S is an oriented smooth surface, then

$$\iint_{\mathcal{S}} \overline{F} \cdot \overline{n} \, dS = \iint_{\mathcal{S}} \overline{F} \cdot d\overline{S} = \iint_D \overline{F}(\overline{r}(u, v)) \cdot (\overline{r}_u \times \overline{r}_v) \, dA$$

is known also as the flux integral of $\overline{F}$ across S.

Surface Integral: Type I

Special case: explicit z parametrization

If the surface is $z = g(x, y)$

$$|\overline{r}_x \times \overline{r}_y| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

Application: Surface Area

Let S be a smooth surface that is parametrized by

$$\mathbf{r}(u, v), \quad \text{where} \quad (u, v) \in \mathcal{D}$$

where D is a region in the $uv-$ plane, then the surface area of S is

$$A(\mathcal{S}) = \iint_{\mathcal{R}} |\overline{r}_u \times \overline{r}_v| \, dA = \iint_{\mathcal{S}} dS$$

Surface Integral: Type II

For Piecewise-smooth S

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \cdots + \iint_{S_n} \vec{F}\cdot d\vec{S}$$

Orientation

1. Default: just given parametrization of the surface $\Rightarrow$ Assume positive oriented. (where the normal is outward)
2. Given Description: "inward" for negative and "outward" for positive.

Heat Flow

Provided $u(x, y, z)$ be the temperature at a point (x, y, z), then the temperature gradient, a.k.a. heat flow $\overline{F}$ can be represented as,

$$\overline{F} = -K\nabla u$$

Heat Flow

Provided $u(x, y, z)$ be the temperature at a point (x, y, z), then the temperature gradient, a.k.a. heat flow $\overline{F}$ can be represented as,

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where K is an experimentally determined constant called the conductivity of the substance.
And

$$\iint_S F \cdot d\overline{S}$$

is the total rate of heat flow or flux across the surface S.

Exercises

Surface Integral. Type I

Example: Evaluate the surface integral $\iint_S yz dS$, where S is the helicoid with vector equation $\vec{R}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi$.

Surface Integral. Type II and the spherical coordinates

Find the outward flux of $\vec{F}(x,y,z) = \langle z, y, x\rangle$ across the sphere $x^2 + y^2 + z^2 = 1$.

Surface Integral. Heat Flow

Example: The temperature at the point (x,y,z) in a substance with conductivity $K = 6.5$ is $T(x,y,z) = 2(x^2+y^2)$. Find the rate of heat flow inward across the cylindrical surface $x^2 + y^2 = 6$ for $0 \le z \le 4$.