Vv156 Lecture 2

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Dictionary

A sequence is a set of related events, movements, or things that follow each other in a particular order.

- We will be interested in sequences of a more mathematical nature;

  Mostly numbers, occasionally points, and functions in the end.

- John picks coloured marbles from a bag, first he picks a red marble, then a blue one, another blue one, a violet one, a red one and finally a blue one.
- The sequence of marbles he has chosen could be represented by

$$1, 2, 2, 3, 1, 2$$

  where 1, 2, 3 stand for red, blue and violet respectively.

- Infinite sequence, such as the Fibonacci sequence, is our primary interest

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

Notation I

- The notation for representing a general infinite sequence is:

$$a_1, a_2, a_3, \ldots$$

  where a_1 represents the 1st element in the sequence, and a_2 the 2nd, etc.

- If we wish to discuss a term in a sequence without being specific, we write

$$a_i \qquad \text{or} \qquad a_n$$

- To represent a finite sequence, we would write

$$a_1, a_2, a_3, \ldots, a_{29} \qquad \text{or} \qquad a_1, a_2, a_3, \ldots, a_N$$

  where the later one represents a finite series of some unknown length N.

Notation II

- A more compact way of representing the general infinite sequence is

$$\{a_i\}_{i=1}^{\infty}.$$

- The finite sequence a_1, a_2, …, a_N is similarly represented by:

$$\{a_i\}_{i=1}^{N}.$$

- Since we study mostly infinite sequences, we will often abbreviate further

$$\{a_i\}_{i=1}^{\infty} \quad \text{with simply} \quad \{a_i\} \quad \text{or} \quad \{a_n\}.$$

- It looks like set notation, but you should be careful not to confuse a sequence with the set whose elements are the entries of the sequence.

1. A set has no particular ordering of its elements but a sequence certainly does.
2. Each element of a set must be unique, but terms of a sequence need not be.

Specification

- There are three ways to specify a particular sequence:
  1. For a (short) finite sequence, one can simply list the terms in order.

$$1, 2, 4, 8, 16, 32, 64$$

  2. A better method is to define a sequence with an explicit formula.

$$a_n = 2^{n-1}$$

- However, an explicit formula for many sequences are hard to obtain.
  3. A third way of describing a sequence is through a recursive formula.

$$a_1 = 1, \qquad a_{n+1} = 2a_n$$

it describes the nth term of the sequence in terms of previous terms.

Mathematical Induction

- The principle of Mathematical Induction

  It is often used to prove a given statement for all natural numbers

  1. The Base Case:

     Prove the desired result for a certain n.

  2. The Inductive Step:

     Prove that if the result is true for n, then it is also true for $n+1$.

Exercise

Let $\{a_n\}$ be the sequence defined recursively by

$$a_{n+1} = a_n + (n+1), \qquad \text{and} \qquad a_1 = 1$$

Prove that in general the explicit formula for the sequence is given by

$$a_n = \frac{n(n+1)}{2}$$

Harry and Hermione playing a game

- Two people sit facing each other in a room, Harry and Hermione.
- Two consecutive natural numbers are written on their foreheads, one on each
- Harry and Hermione both know the number that's not their own.
- They also both know that the two numbers are consecutive.
- The game proceeds with one player, say Harry, asking Hermione if she knows what her number is. If she does, she says so and the game ends.
- If not, Harry's turn ends and Hermione gets her chance to ask him the same question. As before, if he does then he says so and the game ends.
- Otherwise, back to Harry asking the same question to Hermione.
- This back and forth questioning continues until someone says "Yes", if ever.
- We assume that the two players are honest, and they are "perfect reasoners", so that if there was some way for either of them at any point to deduce their own number then they would do it.

Q: Does this game ever end?

- Some sequences of real numbers get closer and closer to a single number

$$L$$

while other sequences exhibit no such behaviour.

Definition

If terms of a sequence $\{a_n\}$ can be made arbitrarily close to a real value L as we like by taking n sufficient large, then the sequence is said to be convergent and we say it has a limit of L, in which case, we use the following notation,

$$\lim_{n\to\infty} a_n = L$$

Alternatively, we denote $\{a_n\}$ converges to L using the following notation

$$a_n \to L \quad \text{as} \quad n \to \infty$$

If a sequence is not convergent, it is called divergent, or we say it diverges.

Q: What precisely does the term "arbitrarily close" to L mean?

- Consider the following sequence

$$a_n = \frac{1}{n}$$

- Intuitively it is clear that we mean to define zero to be the limit of $\{a_n\}$,

$$\lim_{n \to \infty} a_n = 0$$

the following statement is incorrect

$$\lim_{n \to \infty} a_n = -1$$

it cannot be made as close to -1 as we please, there always is a gap of 1.

- So when we say $\{a_n\}$ can be made arbitrarily close to L we meant it satisfies

$$|a_n - L| < \epsilon \qquad \text{for any } \epsilon > 0$$

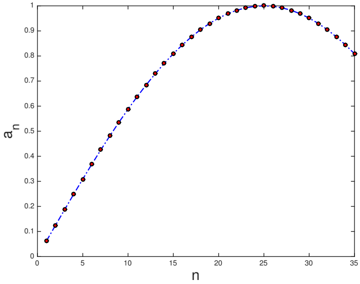
that is, it can be made as close to L as we please by having the right n.

Q: Is the following true?

$$\lim_{n\to\infty} a_n = 1$$

where

$$a_n = \sin\left(\frac{n\pi}{50}\right)$$



- In general, we don't mean to define L to be the limit of

$$\{a_n\}$$

if it is only arbitrarily close to L for some n up to a certain integer,

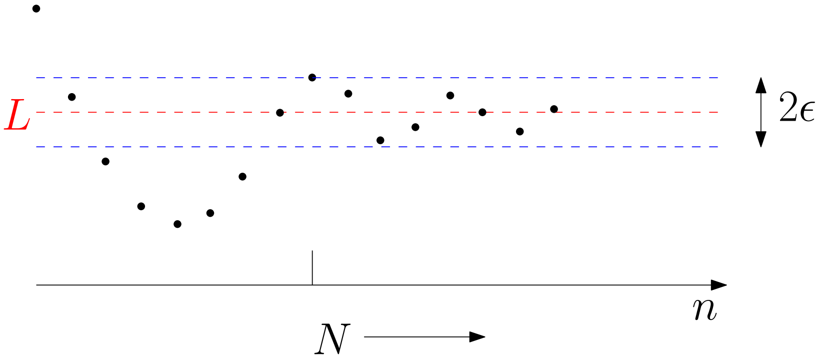
$$k$$

after which it moves away from L!

- So when we say taking n sufficiently large we meant there exists $N \in \mathbb{N}_1$ s.t.

$$|a_n - L| < \epsilon \qquad \text{whenever} \qquad n > N$$

- The number N tells you how far you have to go to get close to L up to ϵ.



Definition

A sequence $\{a_n\}$ has the limit $L \in \mathbb{R}$, and we write

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \quad \text{as} \quad n \to \infty$$

if for every $\epsilon > 0$ there is a corresponding integer N such that

$$\text{if} \quad n > N \quad \text{then} \quad |a_n - L| < \epsilon$$

- In terms of δ-neighbourhoods of L, $(L-\epsilon, L+\epsilon)$, the limit L is a real value such that $\{a_n\}$ is eventually in every δ-neighbourhood of this value.
- If such a value $L \in \mathbb{R}$ exists, we say the sequence converges or is convergent.
- Otherwise, we say the sequence diverges or is divergent.

Exercise

Show the sequence of reciprocals of natural numbers is convergent.

Definition

A sequence $\{a_n\}$ diverges to infinity, we write

$$\lim_{n\to\infty} a_n = \infty \qquad \text{and} \qquad a_n \to \infty \quad \text{as} \quad n \to \infty$$

if for each number $M \in \mathbb{R}$ there exists a number $N_M \in \mathbb{N}$ such that

$$a_n > M \qquad \text{for all} \quad n > N_M$$

Similarly, a sequence $\{a_n\}$ diverges to negative infinity, we write

$$\lim_{n\to\infty} a_n = -\infty \qquad \text{and} \qquad a_n \to -\infty \quad \text{as} \quad n \to \infty$$

if for each number $m \in \mathbb{R}$ there exists a number $N_m \in \mathbb{N}$ such that

$$a_n < m \qquad \text{for all} \quad n > N_m$$

Exercise

Show the sequence of even numbers diverges to infinity.

Limit Laws

Let $\{a_n\}$ and $\{b_n\}$ be two sequences such that $\lim\limits_{n\to\infty} = L_a$ and $\lim\limits_{n\to\infty} = L_b$.

1. The limit of a constant sequence is the constant itself.

$$\lim_{n\to\infty} a = a$$

2. The limit of a sum/difference is the sum/difference of the limits.

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = L_a + L_b$$

3. The limit of a product is the product of the limits.

$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right) = L_a L_b$$

4. The limit of a quotient is the quotient of the limits

$$\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n} = \frac{L_a}{L_b}, \qquad \text{provided} \quad L_b \neq 0 \quad \text{and} \quad b_n \neq 0$$

Exercise

Is the sequence

$$a_n = \frac{n}{10+n}$$

convergent or divergent? If it is convergent, find what it converges to.

Proof

- To prove the product law, we show that for $\epsilon > 0$, there exists N such that

$$|a_n b_n - L_a L_b| < \epsilon \qquad \text{for all} \quad n > N$$

- We take $\{a_n\}$ being convergent implies it being bounded, and let $|a_n| < M$.
- Since $\{a_n\}$ converges to L_a, for $\epsilon_1 = \dfrac{\epsilon}{2(|L_b|+1)} > 0$, there exists N_1 s.t.

$$|a_n - L_a| < \frac{\epsilon}{2(|L_b|+1)} \qquad \text{for all} \quad n > N_1$$

Proof

- Similarly, there exists N_2 such that

$$|b_n - L_b| < \frac{\epsilon}{2(M+1)} \qquad \text{for all} \quad n > N_2$$

- Let $N = \max(N_1, N_2)$. Then, for $n > N$, consider

$$\begin{aligned} |a_n b_n - L_a L_b| &= |a_n b_n - a_n L_b + a_n L_b - L_a L_b| \\ &= |a_n(b_n - L_b) + L_b(a_n - L_a)| \\ &\le |a_n(b_n - L_b)| + |L_b(a_n - L_a)| \\ &\le |a_n||b_n - L_b| + |L_b||a_n - L_a| \\ &< M\frac{\epsilon}{2(M+1)} + |L_b|\frac{\epsilon}{2(|L_b|+1)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

$$|a_n b_n - L_a L_b| < \epsilon \qquad \text{for all} \quad n > N \quad \square$$

Definition

- The sequence $\{a_n\}$ is said to be increasing if

$$a_{n+1} \geq a_n \qquad \text{for all } n.$$

  and it is said to be decreasing if

$$a_{n+1} \leq a_n \qquad \text{for all } n.$$

- A sequence $\{a_n\}$ is said to be monotonic if it is one of those cases.

Q: How can we ensure a monotonic sequence is convergent?

Monotonic Sequence Theorem

A monotonic sequence converges if and only if it is bounded.

Exercise

Suppose $a_n = \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N}$. Show the sequence is convergent.

Squeeze Theorem

Suppose $\{a_n\}$ and $\{b_n\}$ are convergent, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

If for some $N \in \mathbb{N}$,

$$a_n \le c_n \le b_n \qquad \text{for all} \quad n > N$$

then the sequence $\{c_n\}$ is convergent. Moreover,

$$\lim_{n\to\infty} c_n = L$$

Exercise

Show the sequence $\left\{\dfrac{4^n}{n!}\right\}$ converges to zero.

Proof

- We need to show that for each $\epsilon > 0$, there exists N such that

$$n > N \implies |c_n - L| < \epsilon$$

- The sequence $\{a_n\}$ converges to L, so there exists N_a, when $n > N_a$, then

$$|a_n - L| < \epsilon \iff -\epsilon < a_n - L < \epsilon$$

- Similarly, for $\{b_n\}$, there exists $n > N_b$, when $n > N_b$, then

$$|b_n - L| < \epsilon \iff -\epsilon < b_n - L < \epsilon$$

- Let $N = \max(N_a, N_b)$. For $n > N$, we have

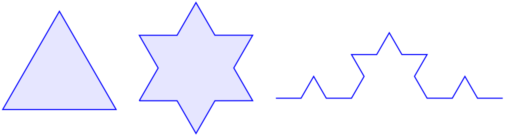
$$\begin{aligned} a_n \leq c_n \leq b_n &\iff a_n - L \leq c_n - L \leq b_n - L \\ &\iff -\epsilon < a_n - L \leq c_n - L \leq b_n - L < \epsilon \\ &\iff |c_n - L| < \epsilon \quad \square \end{aligned}$$

- The concept of sequence and limit of it is largely a stepping stone to

the limit of a function

- However, it is useful in terms of analysing anything to do with infinity.

- Consider an equilateral triangle,



1. Divide each side into three segments of equal length.
2. Create new equilateral triangles that have the middle segment in step 1. as its base and points outward.
3. Remove the line segments that are the bases of the new triangles in step 2.

- Continue the above three steps indefinitely for all sides.

Q: What is the perimeter of the object as the number of iterations $\to \infty$?