Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 5

Date Due: 12:55 PM, Wednesday, the 31st of March 2021

Discussion Class Preparation

Please (re-)view Video files 29-31 and/or finish reading the section "Adjoint BVPs and Higher-Order Equations" in the lecture slides. You should be able to answer the following questions:

i) How are adjoint boundary conditions defined?

ii) What is the adjoint boundary value problem to a given BVP?

iii) What is the adjoint Green function?

iv) Explain the role of the conjunct in constructing the adjoint BVP.

v) Compare the solution formula using the conjunct with the solution formula obtained previously.

Exercises (15 Marks)

Exercise 5.1

We want to find a fundamental solution of the stationary equation for a simply supported beam, i.e., a function $g(x,\xi)$ satisfying

$$\frac{d^4 g}{dx^4} = \delta(x-\xi), \qquad 0 < x,\xi < 1,$$

with boundary conditions

$$g(0,\xi) = g''(0,\xi) = g(1,\xi) = g''(1,\xi) = 0.$$

i) Find a causal fundamental solution, i.e., a function E satisfying

$$\frac{d^4 E}{dx^4} = \delta(x-\xi), \qquad 0 < x,\xi < 1,$$

and $E(x) = 0$ for $x < \xi$.
(3 Marks)

ii) Add a solution of the homogeneous equation $\frac{d^4 u}{dx^4} = 0$ to E to obtain a function that satisfies the boundary conditions.
(2 Marks)

Exercise 5.2

Consider again the stationary equation for a traveling wave with wavenumber k,

$$-\frac{d^2 g}{dx^2} - k^2 g = \delta(x-\xi), \qquad 0 < x,\xi < 1,$$

this time with boundary conditions

$$g(0,\xi) = g(1,\xi) = 0.$$

Use the fundamental solution obtained in class via the Fourier transform to find a function that satisfies the boundary condition. Do this by adding a suitable solution of the homogeneous equation $-\frac{d^2u}{dx^2} - k^2 u = 0$.
(2 Marks)

Exercise 5.3
Consider the boundary value problem operator given by

$$L = \frac{d^2}{dx^2}, \quad 0 < x < 1, \qquad\qquad B_1 u = u(0).$$

Characterize M^* by three boundary functionals.
(2 Marks)

Exercise 5.4
Consider the boundary value operator given by

$$L = \frac{d^4}{dx^4}, \quad 0 < x < 1, \qquad B_1 u = u(0), \qquad B_2 u = u'''(0), \qquad B_3 = u(1), \qquad B_4 = u''(1)$$

i) Find $g(x, \xi)$.
(2 Marks)

ii) It is obvious that $L = L^*$. Find the adjoint boundary conditions and calculate $g^*(x, \xi)$.
(3 Marks)

iii) Show that $g(x, \xi) \neq g(\xi, x)$.
(1 Mark)