# 1 Derivative

## 1.1 Definition

### 1.1.1 definition of derivative

$$\lim_{h\to 0}[\frac{f(c+h)-f(c)}{h}] = f'(c)$$

### 1.1.2 definition of one-sided derivative

right-hand

$$\lim_{h\to 0^+}[\frac{f(c+h)-f(c)}{h}] = f'(c^+)$$

left-hand

$$\lim_{h\to 0^-}[\frac{f(c+h)-f(c)}{h}] = f'(c^-)$$

### 1.1.3 differentiability, continuity, and smooth

Suppose $f(x)$ is defined for $a \le x \le b$, then $f$ is differentiable at $c \in (a,b)$ if and only if there exists a constant $A$ and a function $\varepsilon(h)$ such that

$$f(c+h) = f(c) + Ah + \epsilon(h), \qquad \text{where} \quad \lim_{h\to 0}\frac{\varepsilon(h)}{h} = 0$$

* The rate of change of $\epsilon(h)$ must be faster than the rate of change of $h$.
* $A$ is equal to $f'(c)$.
* One interpretation of the definition is: first, find $f'(c)$. If we can prove that

$$\lim_{h\to 0}\frac{\epsilon(h)}{h} = 0$$

where $\epsilon(h) = f(c+h) - f(c) - hf'(c)$, then $f(x)$ is differentiable. Otherwise, $f(x)$ is not differentiable.

* Another interpretation is : for every $A \in \mathbb{R}$, none can satisfy $lim = 0$.

Properties

1. exist derivative at $c$=differentiable at $c$
2. differentiability is a sufficient but not necessary condition of continuity
3. differentiable on an open interval I; differentiable; differentiable on a closed interval $[a, b]$
4. smooth(continuously differentiable): $f \in C^k(a, b)$, $k$ ranges from 2 to infinity

## 1.2 Laws, theorems, and techniques

### 1.2.1 laws

Suppose $f(x)$ and $g(x)$ are differentiable.

1. $\frac{d}{dx}(c) = 0$ where $c$ is a constant
2. $\frac{d}{dx}(x^r) = rx^{r-1}$ where $r$ is any real number
3. $\frac{d}{dx}(cf) = c\frac{d}{dx}f$ where $c$ is a constant
4. $\frac{d}{dx}(f \pm g) = \frac{d}{dx}(f) \pm \frac{d}{dx}(g)$

* applicable to the sum or the difference of finite number of functions

5. $\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$
6. $\frac{d}{dx}\frac{f}{g} = \frac{\frac{df}{dx} \cdot g - f \cdot \frac{dg}{dx}}{g^2}$

* If $f(x) = 1$, then $[\frac{1}{g(x)}]' = -\frac{g'}{g^2}$

$$\frac{d}{dx}c = 0 \qquad \frac{d}{dx}x^n = nx^{n-1}$$
$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = a^x \ln a$$
$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$
$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$
$$\frac{d}{dx}\csc x = -\csc x \cot x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$
$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$
$$\frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

### 1.2.2 derivative of common functions

### 1.2.3 the chain rule

$$y = f(g(x))$$
$$u = g(x)$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

### 1.2.4 implicit differentiation

eg. $y^5 - x = 0$

$$\frac{d[y(x)]^5}{dx} - \frac{dx}{dx} = \frac{d[y(x)]^5}{dy}\frac{dy}{dx} - 1 = 5y^4\frac{dy}{dx} - 1 = 0$$

* Generally, the expression of the derivative of implicit functions involve both $x$ and $y$, and we can reserve $y$ as the final result.
* The prerequisite to use implicit differentiation is that $F(x, y)$ is differentiable. In this stage, all the implicit functions required to be differentiate are differentiable.

### 1.2.5 inverse

$\frac{dx}{dy} = \frac{1}{dy/dx}$
* The equation says that the derivative of inverse function

Let $f$ be a continuous one-to-one function defined on an interval, and suppose $f$ is differentiable at $f^{-1}(b)$, then $f^{-1}$ is differentiable at $b$, and

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)} \qquad \text{provided} \quad f'\left(f^{-1}(b)\right) \neq 0$$

is equal to the reciprocal of the derivative of the original function.

* Use this theorem we can derive the formula for the derivative of anti-trigonometric functions.

### 1.2.6 parameterization

$$x = \varphi(t)$$
$$y = \phi(t)$$
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

## 1.3 Exercise

1. We have the following relationship for $t$ and $x$, where $a, b$ are constants.

$$e^t = ax + b$$

$\frac{dy}{dx}, \frac{d^2y}{dx^2}$ are known functions, please find $\frac{dy}{dt}, \frac{d^2y}{dt^2}$

2. Please find the derivative of the following functions.

$$(a) y = \arctan\left(\frac{1}{2}\tan\frac{x}{2}\right)$$
$$(b) y = e^{\arcsin\sqrt{f(x)}}$$