Lecture 19

Arc length

If $f(x)$ is continuously differentiable on $[a, b]$ ($y \in [c, d]$).

$$L = \int_a^b \sqrt{1 + (\frac{dy}{dx})^2} dx$$

$$L = \int_c^d \sqrt{1 + (\frac{dx}{dy})^2} dy$$

Surface area

When $y = f(x)$ is a nonnegative smooth curve on $[a, b]$, the surface area resulted from revolution of $f(x)$ about x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + (\frac{dy}{dx})^2} dx$$

When $x = g(y)$ is a nonnegative smooth curve on $[c, d]$, the surface area resulted from revolution of $g(y)$ about y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + (\frac{dx}{dy})^2} dy$$

Lecture 20

Work of a variable force

The work done by the force $F(x)$ along x-axis is

$$W = \int_a^b F(x) dx$$

Center of mass (1D)

1. Moment of masses about the origin M

$$M = m_1 x_1 + m_2 x_2 + ... + m_n x_n$$

2. center of mass $\bar{x}$

$$\bar{x} = \frac{m_1 x_1 + m_2 x_2 + ... + m_n x_n}{m_1 + m_2 + ... + m_n}$$

$\sum(x_k - \bar{x})m_k = 0$

Center of mass (2D)

1. Moment of masses about the y-axis M

$$M_y = m_1x_1 + m_2x_2 + ... + m_nx_n$$

Moment of masses about the x-axis M

$$M_x = m_1y_1 + m_2y_2 + ... + m_ny_n$$

2. center of mass $(\bar{x}, \bar{y})$

$$\bar{x} = \frac{m_1x_1 + m_2x_2 + ... + m_nx_n}{m_1 + m_2 + ... + m_n}$$
$$\bar{y} = \frac{m_1y_1 + m_2y_2 + ... + m_ny_n}{m_1 + m_2 + ... + m_n}$$

$\sum(x_k - \bar{x})m_k = 0$
$\sum(y_k - \bar{y})m_k = 0$

Center of mass of a lamina

1. bounded by $f(x)$ and x-axis.

$$\bar{x} = \frac{\int_a^b \rho x f dx}{\int_a^b \rho f dx}$$
$$\bar{y} = \frac{\int_a^b \frac{1}{2}\rho f^2 dx}{\int_a^b \rho f dx}$$

2. bounded by $f(x), g(x)$, where $f(x) \geqslant g(x)$.

$$\bar{x} = \frac{\int_a^b \rho x(f-g)dx}{\int_a^b \rho(f-g)dx}$$
$$\bar{y} = \frac{\int_a^b \frac{1}{2}\rho(f^2-g^2)dx}{\int_a^b \rho(f-g)dx}$$

* Center of mass is called centroid if $\rho(x)$ is constant, and therefore, we can eliminate ρ in formulae above.
* When we are asked to find the centroid, and the shape of lamina is symmetry, we can directly derive $\bar{x}$ or $\bar{y}$ using symmetry.

Centroid of an arc

$f(x)$ is smooth and represents the arc.

$$\bar{x} = \frac{\int_a^b x\sqrt{1+(\frac{dy}{dx})^2}dx}{\int_a^b \sqrt{1+(\frac{dy}{dx})^2}dx}$$
$$\bar{y} = \frac{\int_a^b y\sqrt{1+(\frac{dy}{dx})^2}dx}{\int_a^b \sqrt{1+(\frac{dy}{dx})^2}dx}$$

* Centroid of a lamina equals $\frac{\rho Ax(ory)}{Area}$; centroid of an arc equals $\frac{\rho Lx(ory)}{arclength}$.

Pappus's Theorem for Volumes

Volume of the solid got from the revolution of a plane region about a line outside the region is (the density of the plane region is constant)

$$V = 2\pi DA$$

where D is the distance from the axis of revolution to the centroid.

Pappus's Theorem for Surface Areas

Surface area. The revolution of a smooth plane curve.

$$S = 2\pi DL$$

where D is the distance from the axis of revolution to the centroid. * Pappus's Theorem is the simplified version of formulae calculating volume and surface area.
* It is valid only when the center of mass is a centroid.

Lecture 21

* Equations can invlove more than one function.

Concepts

1. parametric curve
2. parametric equations
3. parameter
4. parameter interval
5. initial point, terminal point
6. parametrized
7. parametrization = parametric equation+parameter interval

Formulae

Suppose the parameter interval is $[a, b]$.
1. Area

$$\int_a^b f(t)\frac{dx}{dt}dt$$

2. Volume

$$\int_a^b A(t)\frac{dx}{dt}dt$$

3. arc length

$$L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

4. surface area

$$S = \int_a^b 2\pi y(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt$$

5. work

$$W = \int_a^b F(t)\frac{dx}{dt}dt$$

6. center of mass of a lamina

$$\bar{x} = \frac{\int_a^b \rho x(t)f(t)x'(t)dt}{\int_a^b \rho f(t)x'(t)dt}$$

$$\bar{y} = \frac{\int_a^b \frac{1}{2}\rho f^2(t)x'(t)dt}{\int_a^b \rho f(t)x'(t)dt}$$

7. centroid of an arc

$$\bar{x} = \frac{\int_a^b x(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt}{\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt}$$

$$\bar{y} = \frac{\int_a^b y(t)\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt}{\int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt}$$

Lecture 22

Polar coordinates

$r = r(\varphi)$
$x = \rho(\varphi)\cos\varphi; x' = \rho'(\varphi)\cos\varphi - \rho(\varphi)\sin\varphi$
$y = \rho(\varphi)\sin\varphi; y' = \rho'(\varphi)\sin\varphi + \rho(\varphi)\cos\varphi$
arc length:

$$L = \int_\alpha^\beta \sqrt{\rho^2(\varphi) + \rho'^2(\varphi)}d\varphi$$

Improper integral

- The improper integral of f over the interval $[a, \infty)$ is defined to be

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$

- The improper integral of f over the interval $(-\infty, b]$ is defined to be

$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$

- The improper integral of f over the interval $(-\infty, \infty)$ is defined to be

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$

where c is any real number.

* If the limit exists, then the improper integral converges; otherwise, it diverges.
* The improper integral in the third case exist only when both the terms on the right hand side converge.
* Although the curve can extend infinitely, the area of region it bounds is finite if the improper integral exists.

* If the limit exists, then the improper integral converges; otherwise, it diverges.
* In the third case, the numbers that approach c in two improper integrals on the right

- If f is continuous on $[a, b)$, but have an infinite discontinuity at b

$$\int_a^b f(x)\,dx = \lim_{k\to b^-} \int_a^k f(x)\,dx$$

- If f is continuous on $(a, b]$, and have an infinite discontinuity at a,

$$\int_a^b f(x)\,dx = \lim_{k\to a^+} \int_k^b f(x)\,dx$$

- If f is continuous on $[a, b]$, except for an infinite discontinuity at $c \in (a, b)$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

hand side are different. We need to evaluate the two improper integrals separately. both the terms on the right hand side converge.
* Definite and improper integrals can be interpreted as area of the regions. If the integral is convergent, then the area is defined; otherwise, the area is undefined.

Comparison test

* Comparison test is used to determine whether the improper integral is convergent or divergent without calculating the value of integral.
If f, g are continuous, $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant 0$, then

$$\int_a^\infty f(x)dx, converge \Longrightarrow \int_a^\infty g(x)dx, converge$$
$$\int_a^\infty g(x)dx, diverge \Longrightarrow \int_a^\infty f(x)dx, diverge$$

Lecture 23

Concepts of series

1. infinite series: sum of terms in an infinite sequence a_n
2. partial sum s_n: sum of first n terms of an infinite sequence
3. The series is convergent if $\lim_n \to \infty s_n$ converges; otherwise, it is divergent.