Chapter 4: Electrostatic Problems

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Summer 2021

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Many slides courtesy of Sung-Liang Chen

Everything you need to know about solutions to electrostatic problems for this course…

Main parts in solutions to electrostatic problems:

(I) Electrostatic equations to solve: Poisson and Laplace

(II) Uniqueness

(III) Special methods in special cases:

Method of images;

Boundary-value problems

What to solve for?

Charge density ρ

E field

Electric potential $E = -\nabla V$

Force $F = qE$

Work $W = -\int F \cdot dl$

Everything in electrostatics is here…

$$\nabla \cdot D = \rho$$

$$\nabla \times E = 0$$

Poisson and Laplace for linear materials

Poisson:

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

Laplace:

$$\nabla^2 V = 0$$

Poisson and Laplace for linear materials

Poisson:

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

Laplace:

$$\nabla^2 V = 0$$

Solutions to these are unique!

What to solve for?

Charge density ρ

$$\Downarrow$$

E field

Electric potential $E = -\nabla V$

What to solve for?

E field

$\downarrow$

Electric potential $E = -\nabla V$

Force $F = qE$

Work $W = -\int F \cdot dl$

Problem…

When charge density ρ
is imperfectly known…

Problem...

When charge density ρ is imperfectly known...

Method of images

Intelligent guess

Boundary-value problems

4-1 Introduction

- Electrostatic problems: **E**, V, ρ
- ρ(**r**) known exactly everywhere $\Rightarrow$ **E**(**r**), V(**r**)

- In practical problems, ρ(**r**) is not known everywhere (e.g., only partial ρ(**r**) known). We need techniques:
  - Method of images
  - Boundary-value problems

4-2 Poisson's and Laplace's Equations

Maxwell's 1st and 2nd equations

$$\nabla \cdot \mathbf{D} = \rho.$$

$$\nabla \times \mathbf{E} = 0.$$

$$\nabla \times \mathbf{E} = 0. \Rightarrow \mathbf{E} = -\nabla V.$$

$$\nabla \cdot \mathbf{D} = \rho.$$

In a linear medium

$$\mathbf{D} = \epsilon \mathbf{E},$$

$$\nabla \cdot \epsilon \mathbf{E} = \rho.$$

$$\mathbf{E} = -\nabla V.$$

$$\nabla \cdot (\epsilon \nabla V) = -\rho,$$

Poisson's Equations

$$\nabla \cdot (\epsilon \nabla V) = -\rho,$$

In a homogeneous medium
ε is a constant over space

Poisson's equation $\boxed{\nabla^2 V = -\frac{\rho}{\epsilon}.}$

Laplacian operator: $\nabla^2 = \nabla \bullet \nabla$

Poisson's equation:
ρ may be a function of space coordinates
ε must be a constant over space

Poisson's Equation in Cartesian Coordinate

$$\nabla^2 V = \nabla \cdot \nabla V = \left(\mathbf{a}_x \frac{\partial}{\partial x} + \mathbf{a}_y \frac{\partial}{\partial y} + \mathbf{a}_z \frac{\partial}{\partial z}\right) \cdot \left(\mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}\right);$$

$$\boxed{\nabla^2 V = -\frac{\rho}{\epsilon}.} \Rightarrow \boxed{\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\epsilon} \qquad (\mathrm{V/m^2}).}$$

∇^2 in Cylindrical and Spherical Coordinates

- Cylindrical:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}.$$

- Spherical:

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}.$$

Laplace's Equation

- In a simple medium where there is no free charge, $\rho=0$

Laplace's Equation $\boxed{\nabla^2 V = 0,}$

- Example to use Laplace's equation: a set of conductors at different potentials

Solve V by Laplace's equation $\Rightarrow$ $\mathbf{E}=-\nabla V$ $\Rightarrow$ $\rho_s=\varepsilon E_n$

(see example 4-1)

EXAMPLE 4–1 The two plates of a parallel-plate capacitor are separated by a distance d and maintained at potentials 0 and V_0, as shown in Fig. 4–1. Assuming negligible fringing effect at the edges, determine (a) the potential at any point between the plates, and (b) the surface charge densities on the plates.

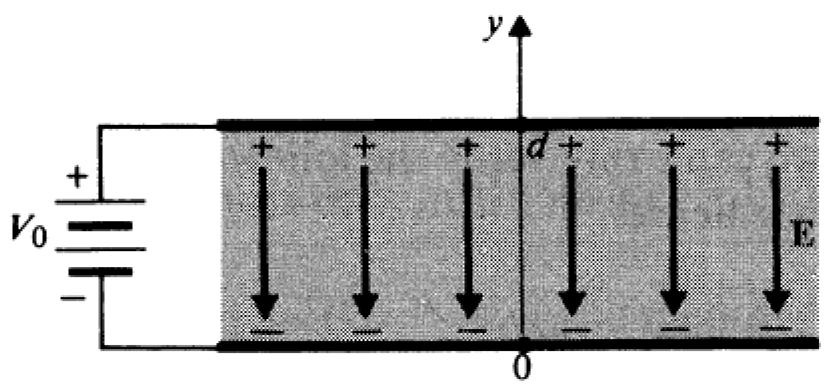


FIGURE 4–1
A parallel-plate capacitor (Example 4–1).

a) Laplace's equation is the governing equation for the potential between the plates, since $\rho = 0$ there. Ignoring the fringing effect of the electric field is tantamount to assuming that the field distribution between the plates is the same as though the plates were infinitely large and that there is no variation of V in the x and z directions. Equation (4–7) then simplifies to

$$\frac{d^2 V}{dy^2} = 0, \tag{4–11}$$

where d^2/dy^2 is used instead of $\partial^2/\partial y^2$, since y is the only space variable here.

Integration of Eq. (4–11) with respect to y gives

$$\frac{dV}{dy} = C_1,$$

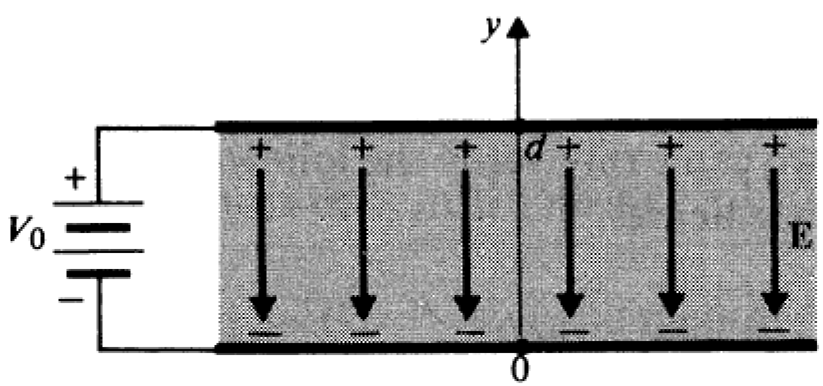


FIGURE 4–1
A parallel-plate capacitor (Example 4–1).

where the constant of integration C_1 is yet to be determined. Integrating again, we obtain

$$V = C_1 y + C_2. \tag{4–12}$$

Two boundary conditions are required for the determination of the two constants of integration, C_1 and C_2:

$$\text{At } y = 0, \qquad V = 0. \tag{4–13a}$$

$$\text{At } y = d, \qquad V = V_0. \tag{4–13b}$$

Substitution of Eqs. (4–13a) and (4–13b) in Eq. (4–12) yields immediately $C_1 = V_0/d$ and $C_2 = 0$. Hence the potential at any point y between the plates is, from Eq. (4–12),

$$V = \frac{V_0}{d} y. \tag{4–14}$$

The potential increases linearly from $y = 0$ to $y = d$.

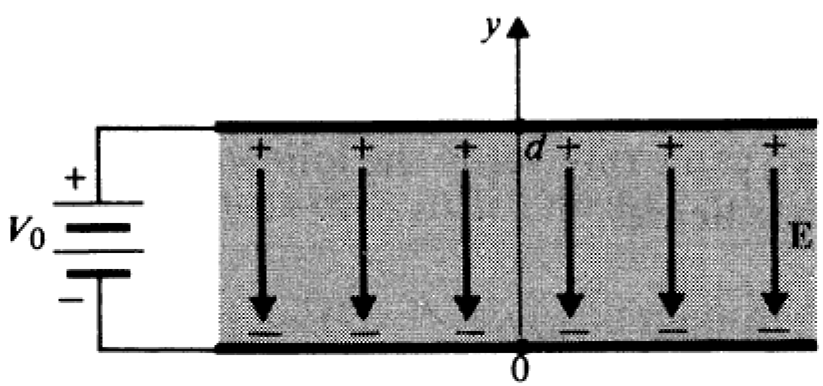


FIGURE 4–1
A parallel-plate capacitor (Example 4–1).

b) In order to find the surface charge densities, we must first find **E** at the conducting plates at $y = 0$ and $y = d$. From Eqs. (4–3) and (4–14) we have

$$\mathbf{E} = -\mathbf{a}_y \frac{dV}{dy} = -\mathbf{a}_y \frac{V_0}{d}, \tag{4–15}$$

which is a constant and is independent of y. Note that the direction of **E** is opposite to the direction of increasing V. The surface charge densities at the conducting plates are obtained by using Eq. (3–72),

$$E_n = \mathbf{a}_n \cdot \mathbf{E} = \frac{\rho_s}{\epsilon}.$$

At the lower plate,

$$\mathbf{a}_n = \mathbf{a}_y, \qquad E_{n\ell} = -\frac{V_0}{d}, \qquad \rho_{s\ell} = -\frac{\epsilon V_0}{d}.$$

At the upper plate,

$$\mathbf{a}_n = -\mathbf{a}_y, \qquad E_{nu} = \frac{V_0}{d}, \qquad \rho_{su} = \frac{\epsilon V_0}{d}.$$

Electric field lines in an electrostatic field begin from positive charges and end in negative charges. ■

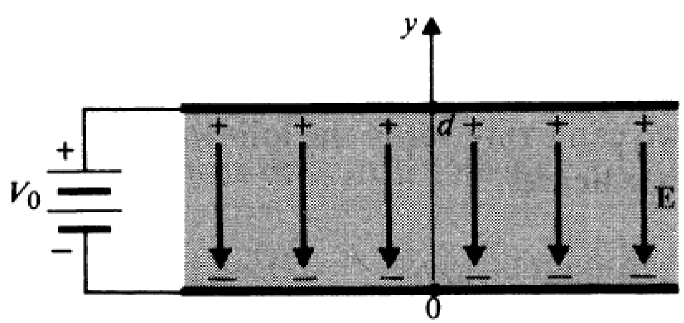
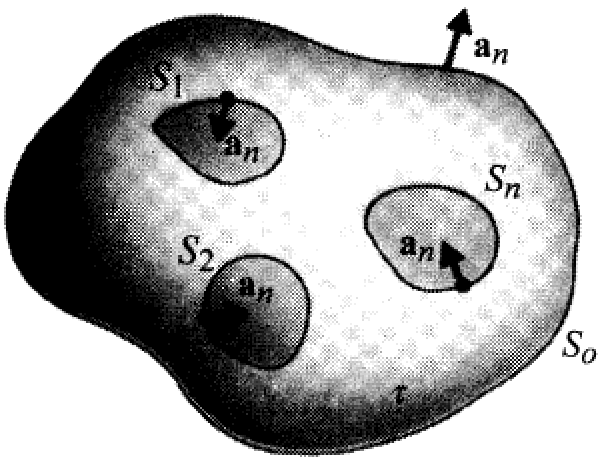


FIGURE 4–1
A parallel-plate capacitor (Example 4–1).

4-3 Uniqueness of Electrostatic Solutions

- Uniqueness theorem: a solution of Poisson's equation that satisfies the given boundary conditions is a unique solution.

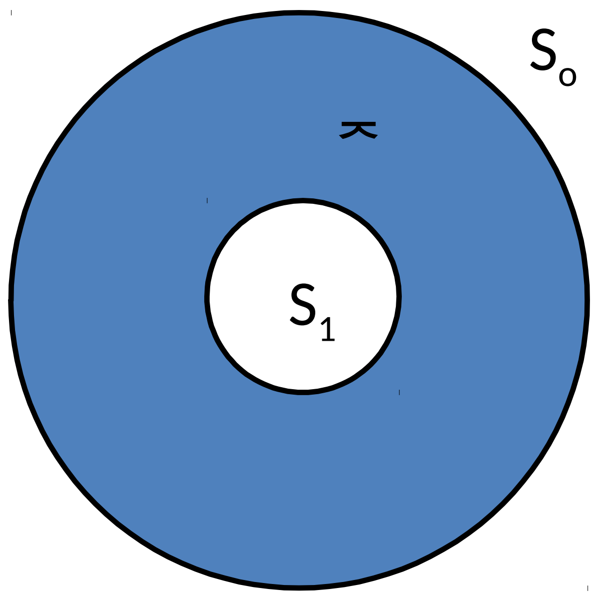
Proof of Uniqueness Theorem



S_o: outer surface enclosing the volume τ
S_1, S_2, …, S_n: surfaces of **conducting bodies**

FIGURE 4–2
Surface S_o enclosing volume τ with conducting bodies.

- Volume τ is bounded (enclosed) by a surface S_o and surfaces S_1, S_2, … S_n
- Inside S_o, many charged **conducting bodies** with surfaces S_1, S_2,…, S_n **at specified potentials**.



S_1: Surface of a conducting body

τ: Volume bounded by S_o and S_1

Proof of Uniqueness Theorem

- Uniqueness theorem: there is only one solution of potential V in τ
- To prove uniqueness theorem, we assume two solutions V_1 and V_2 in τ:

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon},$$

$$\nabla^2 V_2 = -\frac{\rho}{\epsilon}.$$

Also assume that V_1 and V_2 satisfy the same boundary conditions on S_1, S_2, ... S_n and S_o

Proof of Uniqueness Theorem

- (i) Define a potential difference V_d: $V_d = V_1 - V_2$.

$$\nabla^2 V_1 = -\frac{\rho}{\epsilon}, \quad \text{ㅆ} \quad \nabla^2 V_2 = -\frac{\rho}{\epsilon}. \qquad \nabla^2 V_d = 0. \quad \text{in } \tau$$

(ii) On conducting boundaries, **the potentials are specified ㅎ V_d=0**

Uniqueness theorem: a solution of Poisson's equation that **satisfies the given boundary conditions (= potentials are specified on surfaces of conducting bodies)** is a unique solution.

(i) In τ, $\nabla^2 V_d = 0$ (ii) On conducting boundaries, V_d=0

Proof of Uniqueness Theorem

$$\nabla \cdot (f\mathbf{A}) = f\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla f;$$

letting $f = V_d$ and $\mathbf{A} = \nabla V_d$;

$$\nabla \cdot (V_d \nabla V_d) = V_d \nabla^2 V_d + |\nabla V_d|^2,$$

Integration over τ

$\boxed{\nabla^2 V_d = 0. \quad \text{in } \tau}$

$$\oint_{\underline{S}} (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds = \int_\tau |\nabla V_d|^2 \, dv,$$

S: S_1,S_2,…S_n, and S_o

Proof of Uniqueness Theorem

$$\oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds = \int_\tau |\nabla V_d|^2 \, dv,$$

1. For S_1, S_2, ... S_n , $V_d=0$
2. For S_o

   Consider the surface of a sphere with radius $R \rightarrow \infty$

   $V_d \sim 1/R$

   $\nabla V_d \sim 1/R^2$

   $s \sim R^2$

   Thus, the integrand ~ $1/R$

   As $R \rightarrow \infty$, Left side $\rightarrow 0$

$$\Rightarrow \quad \int_\tau |\nabla V_d|^2 \, dv = 0.$$

Proof of Uniqueness Theorem

$$\int_\tau |\nabla V_d|^2\, dv = 0.$$

$$\Downarrow$$

$|\nabla V_d|^2$ is nonnegative everywhere,

$$|\nabla V_d| = 0.$$

everywhere in τ

$$\Downarrow$$

V_d is a constant everywhere in τ

(Thus, "V_d in τ" = "V_d on surfaces")

$$\Downarrow$$

We know V_d=0 on surfaces S_1, S_2,… S_n

V_d = 0 everywhere in τ

That is, $V_1 = V_2$ everywhere in τ, and there is only one possible solution!

Two Cases for the Uniqueness Theorem

- Known **potentials** of conducting bodies, which is just proved.
- Known **charge distributions** of conducting bodies:

$$(\rho_s = \epsilon E_n = -\epsilon\,\partial V/\partial n),$$

ρ_s is known on conducting bodies

$$\nabla V_d = 0.$$

on conducting body surfaces

Substitute $\nabla V_d = 0.$ on $S_1, S_2, \ldots S_n$ into $\oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds = \int_\tau |\nabla V_d|^2 \, dv,$

$\oint_S (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds$

=0 over S_1, S_2,... S_n.

Also, $\oint_{S_0} (V_d \nabla V_d) \cdot \mathbf{a}_n \, ds$ =0 as R ▭ ∞

LHS ▭ 0

$$\int_\tau |\nabla V_d|^2 \, dv = 0.$$

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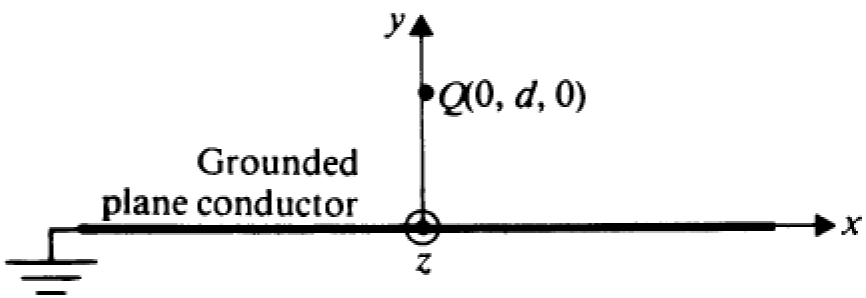
The same conclusion can be obtained!

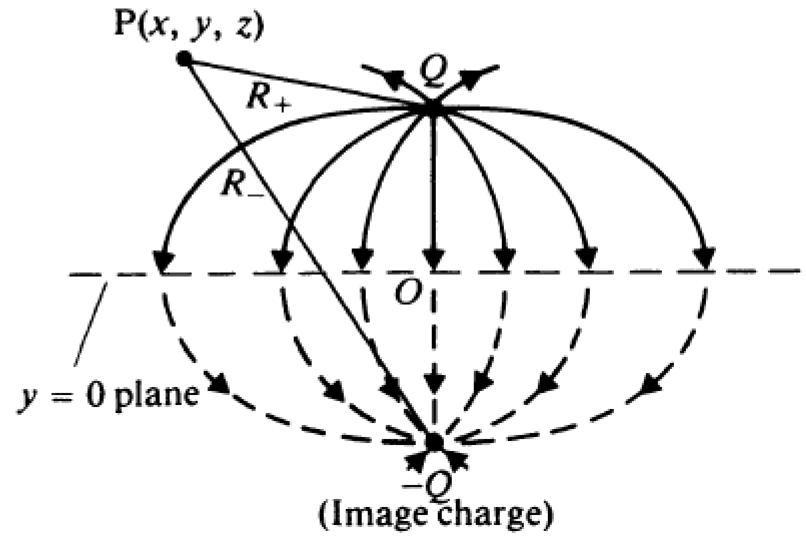
4-4 Methods of Images

- Methods of images: replacing boundaries by appropriate **image charges** in lieu of a formal solution of Poisson’s or Laplace’s equation
  – Condition on boundaries unchanged
  – V(**R**) can be determined easily

Case: Point Charge and Grounded Plane Conductor



(a) Physical arrangement.



(b) Image charge and field lines.

FIGURE 4–3
Point charge and grounded plane conductor.

Case: Point Charge and Grounded Plane Conductor

- Solved by Laplace eq.:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

Hold for y>0 except at the point charge

- **4 Conditions** should be satisfied:

1. $V(x, 0, z) = 0.$
2. for points very close to Q

$$V \to \frac{Q}{4\pi\epsilon_0 R}, \text{ as } R \to 0,$$

3. V ▭ 0 for points very far from Q

$$x \to \pm\infty, y \to +\infty, \text{ or } z \to \pm\infty$$

4. Even functions w.r.t. x and z coordinates

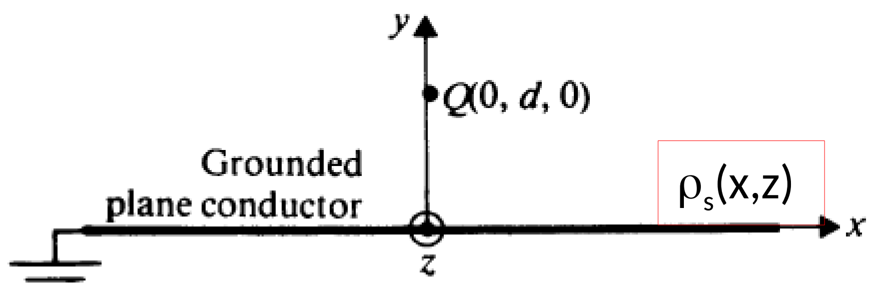
$$V(x, y, z) = V(-x, y, z) \qquad V(x, y, z) = V(x, y, -z).$$

Difficult to solve...

Case: Point Charge and Grounded Plane Conductor

- +Q □ induce ρ_s on conducting plane

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0\sqrt{x^2 + (y-d)^2 + z^2}} + \frac{1}{4\pi\epsilon_0}\int_S \frac{\rho_s}{R_1}\,ds,$$

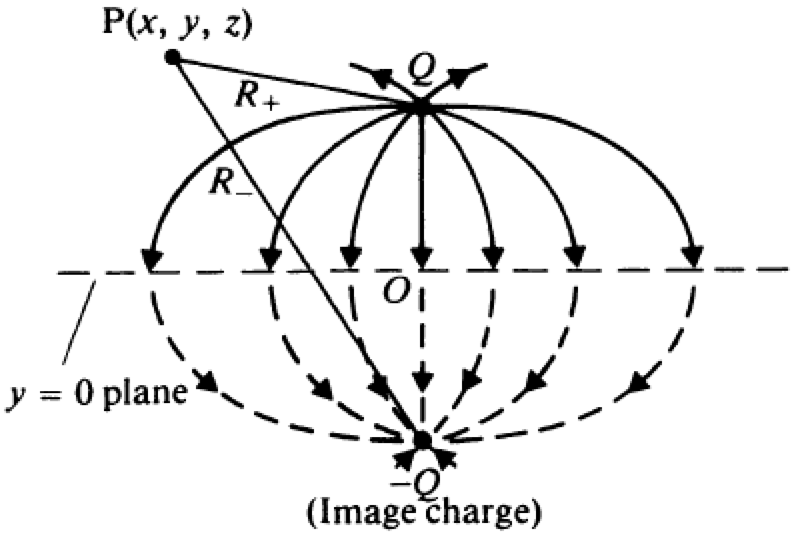


(a) Physical arrangement.

  - $\rho_s(x,z)$ not easy to determine
  - 2nd term is difficult to evaluate

4-4.1 Point Charge and Conducting Planes

- Image methods:
  - Remove the conductor
  - Replace with an image point charge -Q at y=-d



(b) Image charge and field lines.

$$V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right),$$

$$R_+ = [x^2 + (y - d)^2 + z^2]^{1/2},$$
$$R_- = [x^2 + (y + d)^2 + z^2]^{1/2}.$$

Solution $$V(x, y, z) = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right),$$

- By direct substitution, we can verify:
  - Laplace's Eq. is satisfied
  - **4 conditions** are satisfied
- In view of uniqueness theorem, the solution is the only solution.

Uniqueness theorem: a solution of Poisson's equation that **satisfies the given boundary conditions (V(y=0)=0 in this case)** is a unique solution.

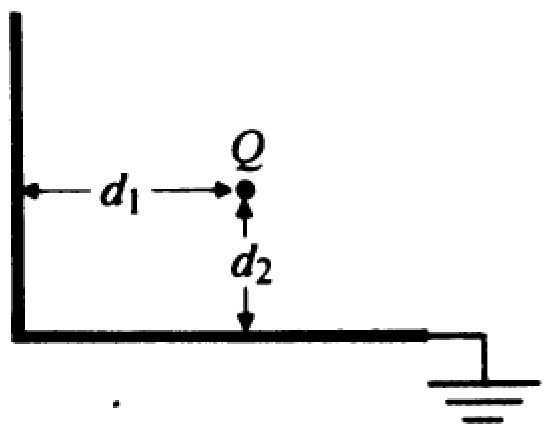
- **E**$=-\nabla$V

Extremely simple!

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

EXAMPLE 4–3 A positive point charge Q is located at distances d_1 and d_2, respectively, from two grounded perpendicular conducting half-planes, as shown in Fig. 4–4(a). Determine the force on Q caused by the charges induced on the planes.

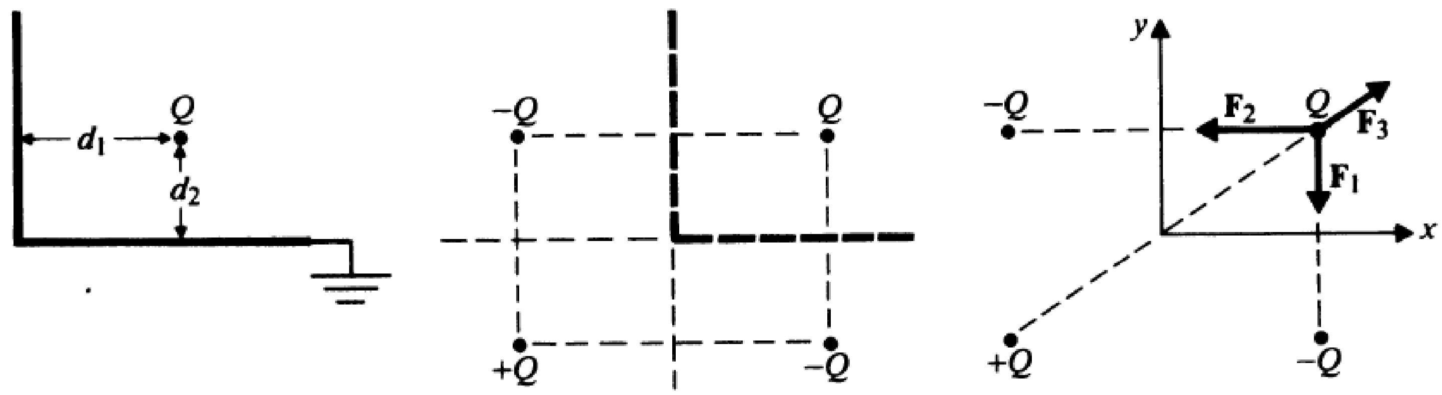


(a) Physical arrangement.

Quick classroom exercise

Solution A formal solution of Poisson's equation, subject to the zero-potential boundary condition at the conducting half-planes, would be quite difficult. Now an image charge $-Q$ in the fourth quadrant would make the potential of the horizontal half-plane (but not that of the vertical half-plane) zero. Similarly, an image charge $-Q$ in the second quadrant would make the potential of the vertical half-plane (but not that of the horizontal plane) zero. But if a third image charge $+Q$ is added in the third quadrant, we see from symmetry that the image-charge arrangement in Fig. 4–4(b) satisfies the zero-potential boundary condition on both half-planes and is electrically equivalent to the physical arrangement in Fig. 4–4(*a*).

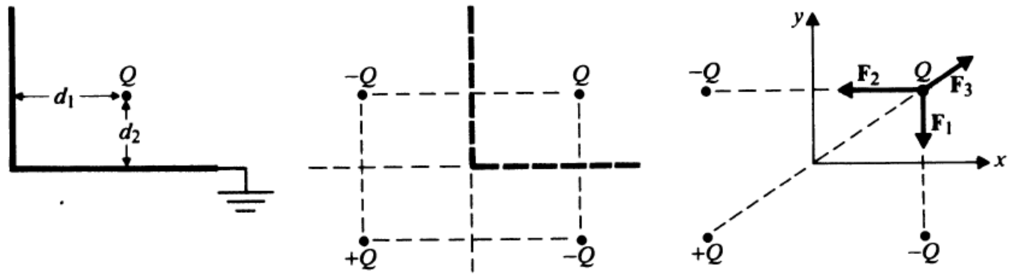
Negative surface charges will be induced on the half-planes, but their effect on Q can be determined from that of the three image charges. Referring to Fig. 4–4(c),



(a) Physical arrangement. (b) Equivalent image-charge arrangement. (c) Forces on charge Q.

FIGURE 4–4
Point charge and perpendicular conducting planes.

Quick classroom exercise



(a) Physical arrangement. (b) Equivalent image-charge arrangement. (c) Forces on charge Q.

FIGURE 4–4
Point charge and perpendicular conducting planes.

we have, for the net force on Q,

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3,$$

where

$$\mathbf{F}_1 = -\mathbf{a}_y \frac{Q^2}{4\pi\epsilon_0(2d_2)^2},$$

$$\mathbf{F}_2 = -\mathbf{a}_x \frac{Q^2}{4\pi\epsilon_0(2d_1)^2},$$

$$\mathbf{F}_3 = \frac{Q^2}{4\pi\epsilon_0[(2d_1)^2 + (2d_2)^2]^{3/2}}(\mathbf{a}_x 2d_1 + \mathbf{a}_y 2d_2).$$

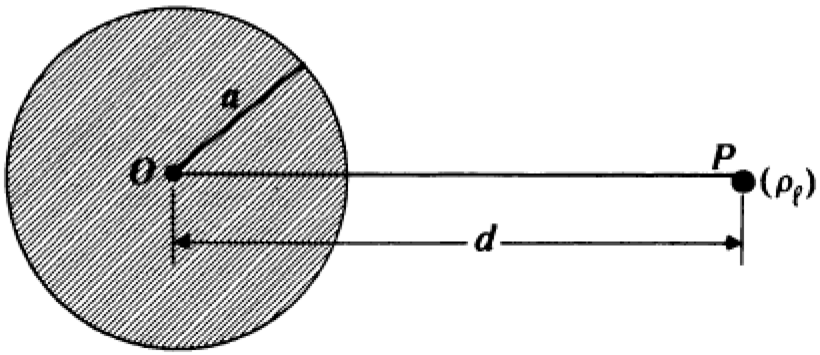
Therefore,

$$\mathbf{F} = \frac{Q^2}{16\pi\epsilon_0}\left\{\mathbf{a}_x\left[\frac{d_1}{(d_1^2 + d_2^2)^{3/2}} - \frac{1}{d_1^2}\right] + \mathbf{a}_y\left[\frac{d_2}{(d_1^2 + d_2^2)^{3/2}} - \frac{1}{d_2^2}\right]\right\}.$$

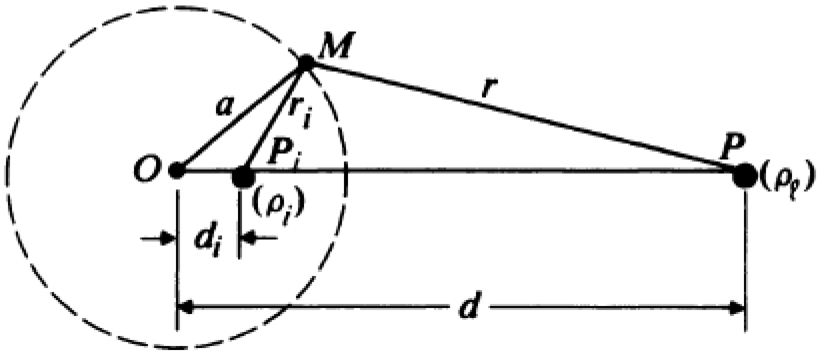
Notes of Image Methods

- The solution **cannot** be used to calculate V or **E** **in the y<0 region**.
- For y<0 region, **E**=0, V=0.

4-4.2 Line Charge and Parallel Conducting Cylinder



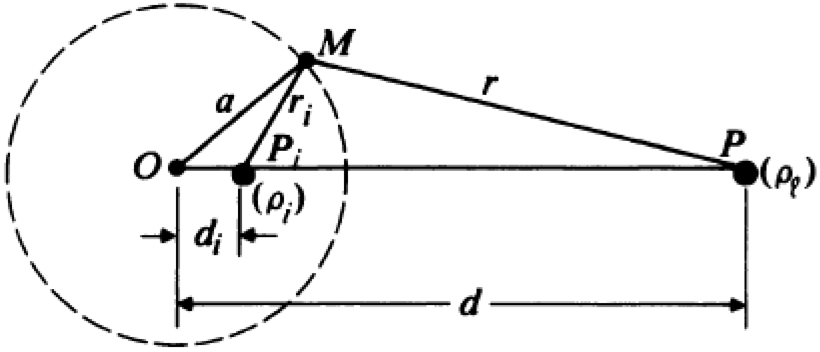
(a) Line charge and parallel conducting cylinder.



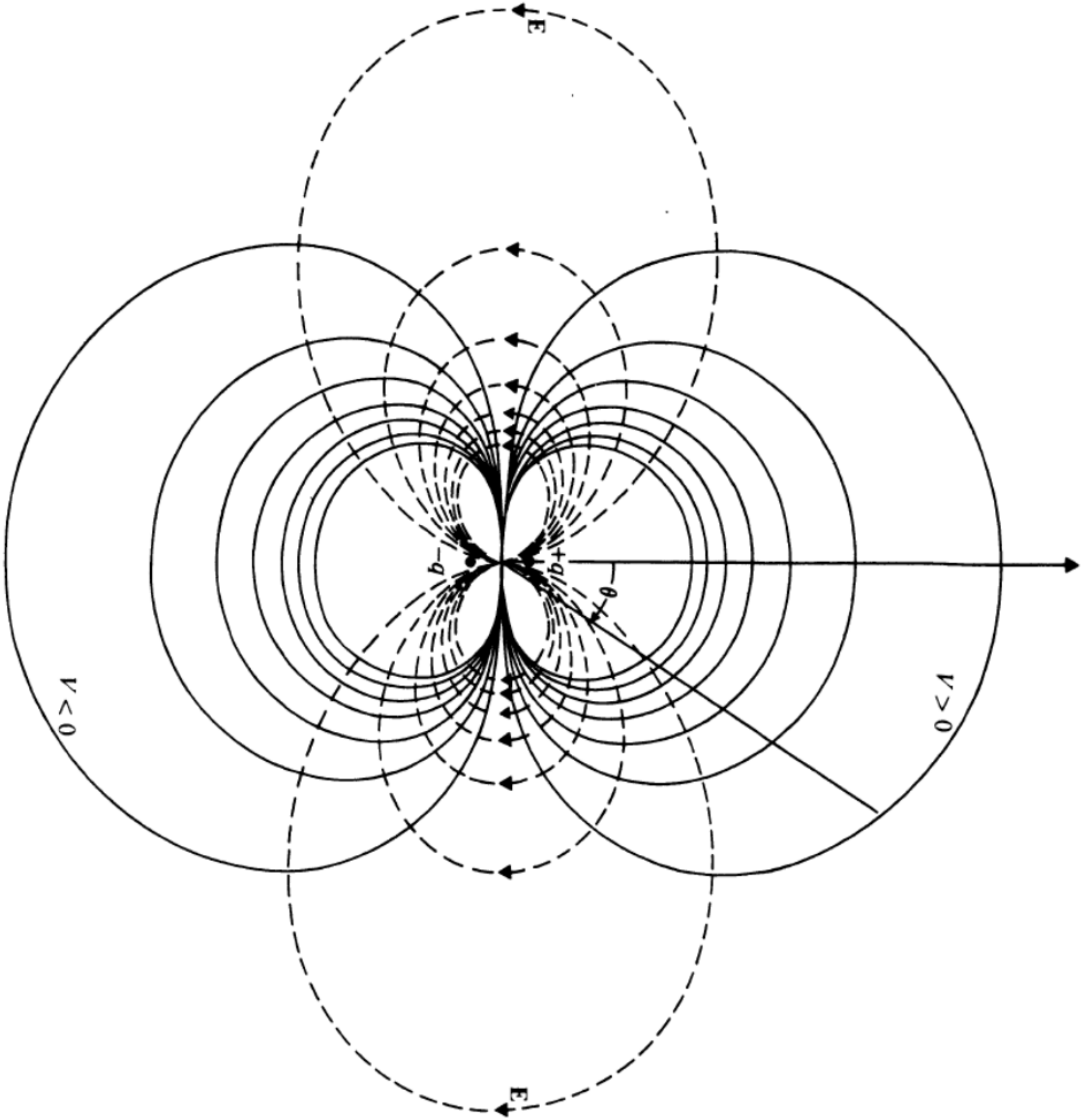
(b) Line charge and its image.

FIGURE 4–5
Cross section of line charge and its image in a parallel, conducting, circular cylinder.

1. Cylinder surface is an equi-potential surface ▭ image must be **a parallel line charge** (ρ_i) inside the cylinder
2. By symmetry of line OP, ρ_i should be on OP



(b) Line charge and its image.



Solid line: equi-
potential surface

FIGURE 3–15
Equipotential and electric field lines of an electric dipole (Example 3–8).

Assume $\boxed{\rho_i = -\rho_\ell.}$ (intelligent guess)

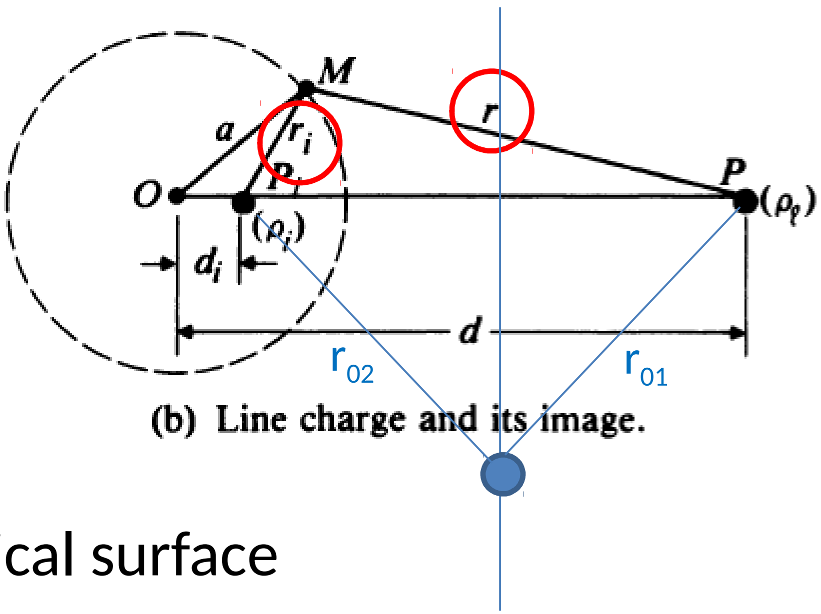
- Voltage due to ρ_l

$$V = -\int_{r_0}^{r} E_r\,dr = -\frac{\rho_\ell}{2\pi\epsilon_0}\int_{r_0}^{r}\frac{1}{r}\,dr$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_0}{r}.$$

Reference point, V=0

Point of interest



(b) Line charge and its image.

- Voltage due to ρ_l and ρ_i on cylindrical surface

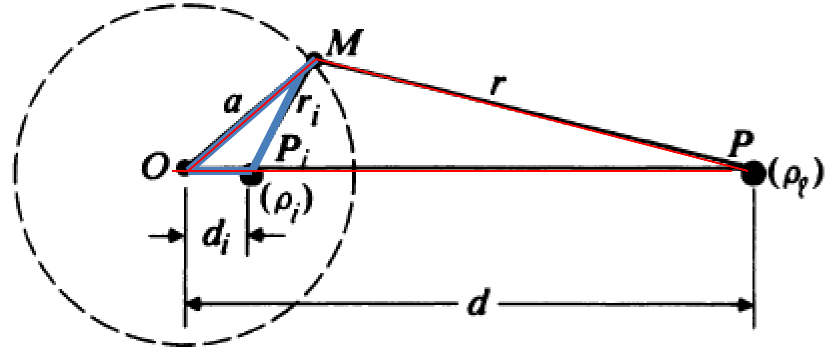
$$V_M = \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_{01}}{r} - \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_{02}}{r_i}$$

$$= \frac{\rho_\ell}{2\pi\epsilon_0}\ln\frac{r_i}{r}.$$

Choosing the same reference point with equidistance from ρ_l and ρ_i so that $\ln r_0$ terms cancel.

• To make V_M=constant

$$\frac{r_i}{r} = \text{Constant.}$$



(b) Line charge and its image.

• To make M coincide with the cylindrical surface (OM=a), P_i should be chosen to make the two triangles OMP_i and OPM similar. (Otherwise, r_i/r=constant over the cylindrical surface cannot be satisfied.)

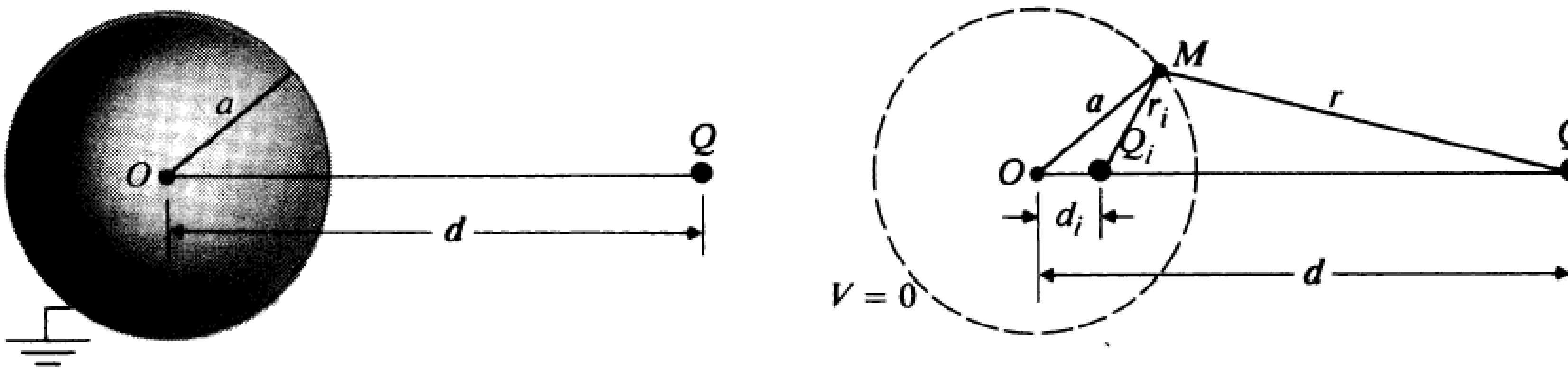
$$\frac{\overline{P_iM}}{\overline{PM}} = \frac{\overline{OP_i}}{\overline{OM}} = \frac{\overline{OM}}{\overline{OP}}$$

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant.}$$

$$\boxed{d_i = \frac{a^2}{d}}$$

P_i is called the **inverse point** of P

4-4.3 Point Charge and Conducting Sphere



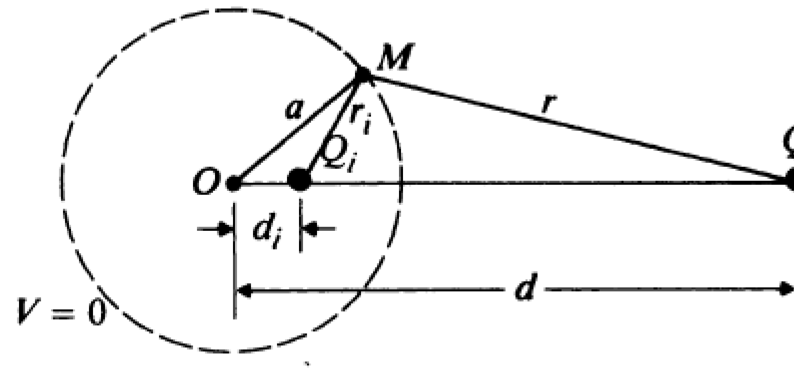
(a) Point charge and grounded conducting sphere.

(b) Point charge and its image.

FIGURE 4–11
Point charge and its image in a grounded sphere.

Intelligent Guess

- By symmetry, Q_i
  - Negative
  - Inside the sphere
  - On line OQ



(b) Point charge and its image.

- $Q_i \neq -Q$, otherwise the equi-potential surface (**V=0**) is a plane

- Thus, both Q_i and d_i should be solved

$$V_M = \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0,$$

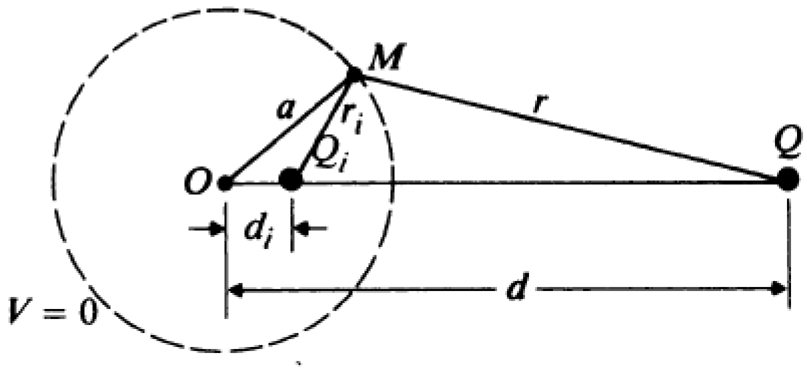
$$\frac{r_i}{r} = -\frac{Q_i}{Q} = \text{Constant.}$$

$$\frac{r_i}{r} = \frac{d_i}{a} = \frac{a}{d} = \text{Constant.}$$

$$-\frac{Q_i}{Q} = \frac{a}{d}$$

$$\boxed{Q_i = -\frac{a}{d}Q} \qquad \boxed{d_i = \frac{a^2}{d}.}$$

Q_i is called the **inverse point** of Q



(b) Point charge and its image.

4-4.4 Charged Sphere and Grounded Plane

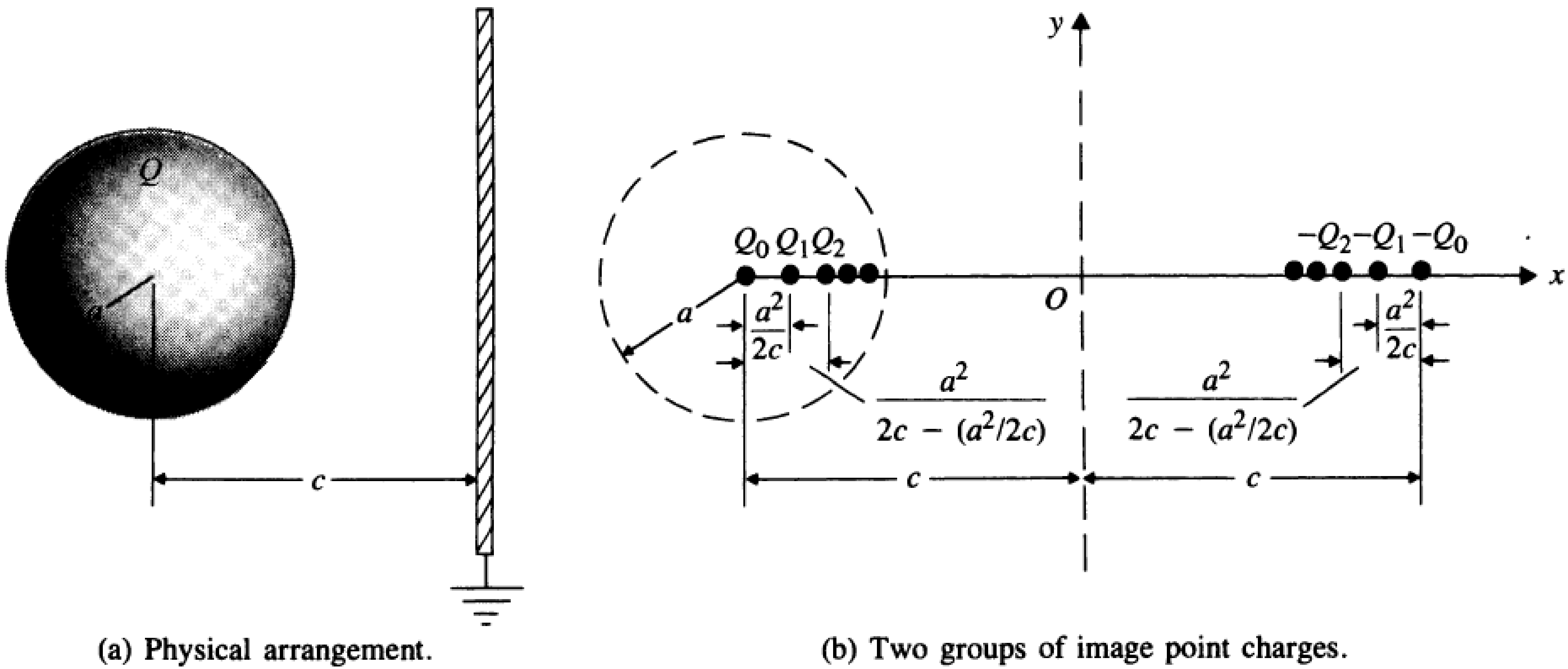
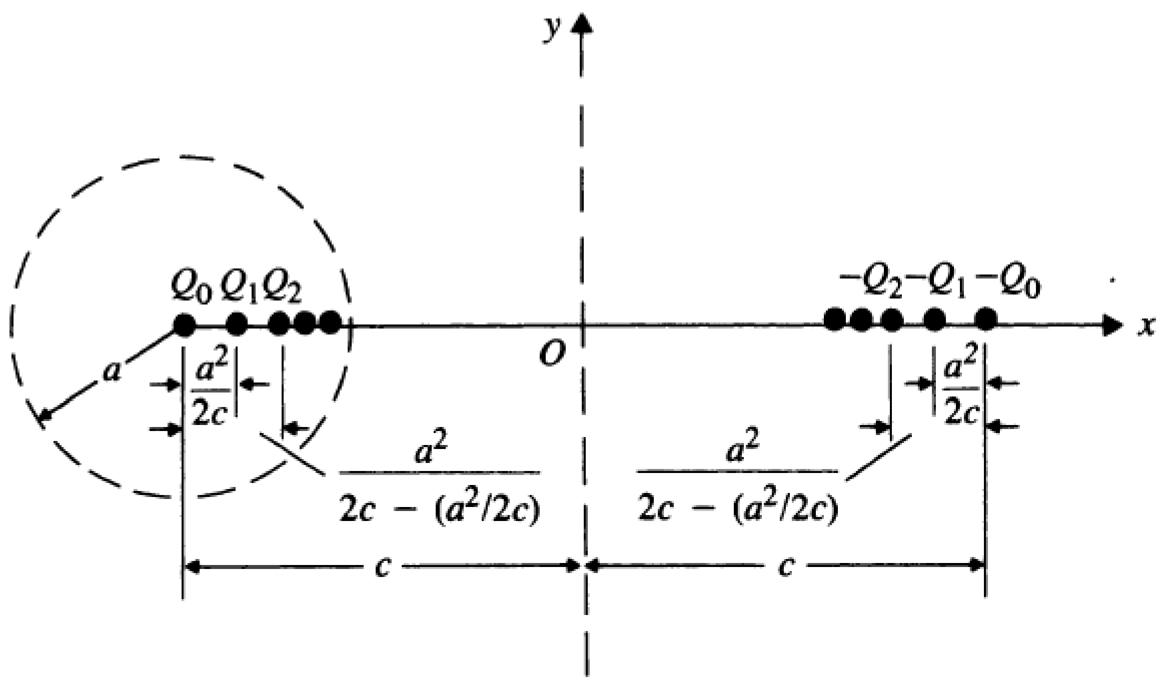


FIGURE 4–13
Charged sphere and grounded conducting plane.

- Assume that a charge Q_0 is at the center.
- The sphere and the plane must be equi-potential surfaces.
- Method of images: The charged sphere and grounded plane can be replaced by charges



(b) Two groups of image point charges.

Q_0 at (-c,0) ⇒ the sphere is an equi-potential plane

To make yz plane equi-potential (V=0)

$-Q_0$ at (c,0)

Destroy equi-potential of the sphere

To make the sphere surface equi-potential

Q_1 inside the sphere

Destroy equi-potential of yz plane (V=0)

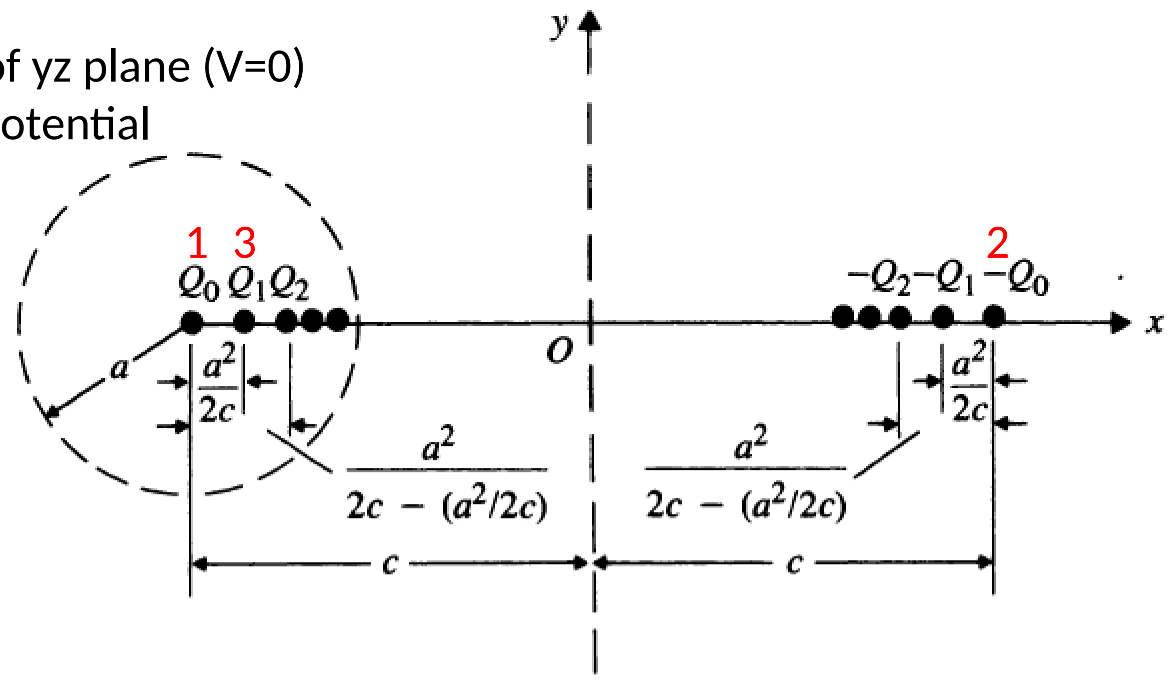
To make yz plane equi-potential

$-Q_1$

.

.

.



(b) Two groups of image point charges.

$$Q_1 = \left(\frac{a}{2c}\right) Q_0 = \alpha Q_0,$$

$$Q_2 = \frac{a}{\left(2c - \dfrac{a^2}{2c}\right)} Q_1 = \frac{\alpha^2}{1 - \alpha^2} Q_0, \qquad \alpha = \frac{a}{2c}.$$

$$Q_3 = \frac{a}{2c - \dfrac{a^2}{\left(2c - \dfrac{a^2}{2c}\right)}} Q_2 = \frac{\alpha^3}{(1-\alpha^2)\left(1 - \dfrac{\alpha^3}{1-\alpha^3}\right)} Q_0,$$

$$\vdots$$

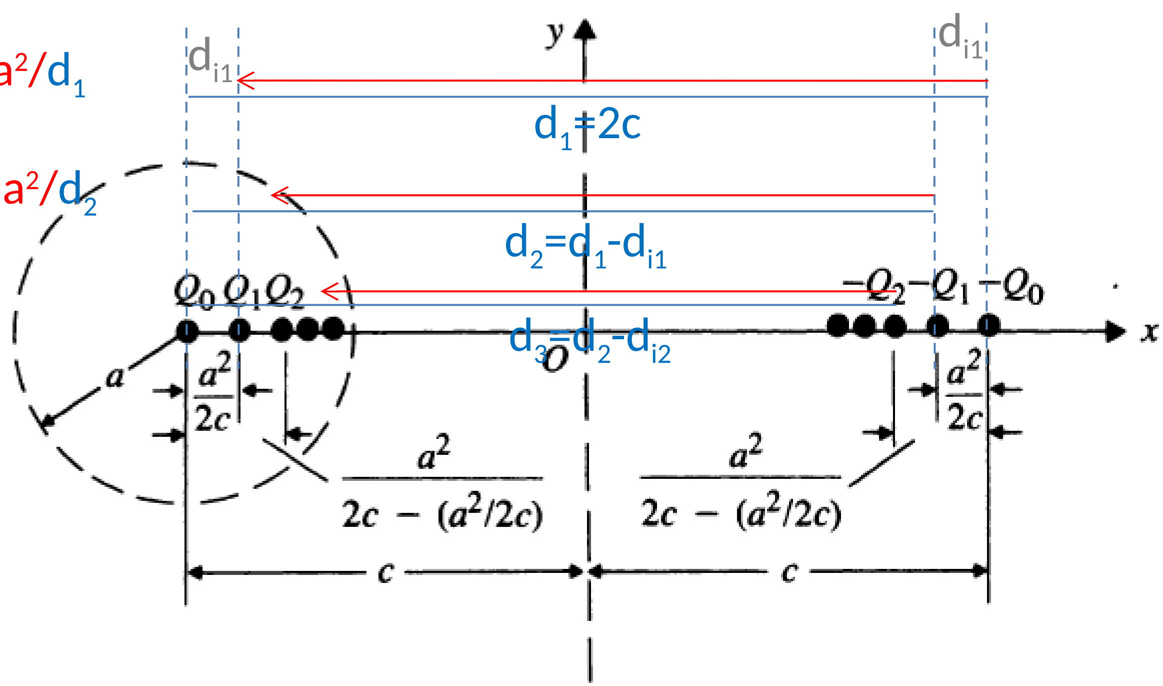
$$\boxed{Q_i = -\frac{a}{d} Q} \qquad \boxed{d_i = \frac{a^2}{d}.}$$

d: from the charge to the sphere center

Q_0 to Q_1 ⇒ $Q_1=(a/d_1)Q_0$; $d_{i1}=a^2/d_1$

Q_1 to Q_2 ⇒ $Q_2=(a/d_2)Q_1$; $d_{i2}=a^2/d_2$

Q_2 to Q_3 …



(b) Two groups of image point charges.

- Total charge on the sphere

$$Q = Q_0 + Q_1 + Q_2 + \cdots$$
$$= Q_0\left(1 + \alpha + \frac{\alpha^2}{1-\alpha^2} + \cdots\right).$$

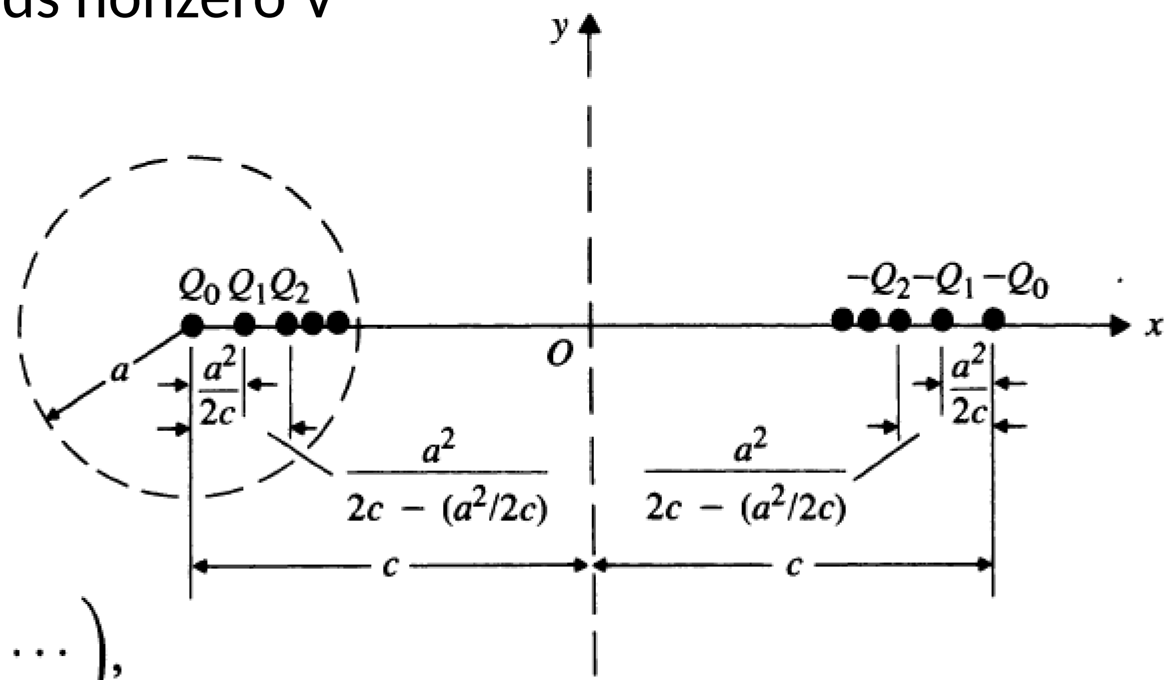
- The V on the sphere

Pairs $(-Q_0, Q_1), (-Q_1, Q_2), \ldots$ yield zero potential on the sphere
Only Q_0 at the center yields nonzero V

$$V_0 = \frac{Q_0}{4\pi\epsilon_0 a}.$$

- The C between the sphere and the conducting plane

$$C = \frac{Q}{V_0} = 4\pi\epsilon_0 a\left(1 + \alpha + \frac{\alpha^2}{1-\alpha^2} + \cdots\right),$$



(b) Two groups of image point charges.

4-5 Boundary-Value Problems in Cartesian Coordinates

- Method of images: useful for the case with **isolated free charges**
- Laplace's equation: can be used to solve the case **w/o isolated free charges** (Example 4-1: charges on conductors)
  - Known boundary values (potential or its normal derivative specified), so called boundary-value problems.

Three Types of Boundary Conditions

- Dirichlet: V is specified on boundaries
- Neumann: dV/dn is specified on boundaries
- Mixed: V specified on some boundaries; dV/dn specified over the remaining boundaries.

Separation of Variables

Laplace's equation $$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

By separation of variables

$$V(x, y, z) = X(x)Y(y)Z(z),$$

$$Y(y)Z(z)\frac{d^2X(x)}{dx^2} + X(x)Z(z)\frac{d^2Y(y)}{dy^2} + X(x)Y(y)\frac{d^2Z(z)}{dz^2} = 0,$$

$$\times 1/X(x)Y(y)Z(z)$$

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0.$$

f(x) + f(y) + f(z) =0

f(x)+f(y)+f(z)=0 to be satisfied for all values of x, y, z

f(x): function of x only

f(y): function of y only

f(z): function of z only

$\Downarrow$

f(x), f(y), f(z) must be a constant

$\Downarrow$

df(x)/dx=0, df(y)/dy=0, df(z)/dz=0

df(x)/dx=0

$$\frac{d}{dx}\left[\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}\right] = 0,$$

Integration on both sides

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = \underline{-k_x^2},$$

Integration constant

$$\frac{d^2X(x)}{dx^2} + k_x^2X(x) = 0.$$

Possible Solutions of $X''(x) + k_x^2X(x) = 0$

k_x^2	k_x	$X(x)$	Exponential forms† of $X(x)$
0	0	$A_0x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1e^{jkx} + D_1e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2e^{kx} + D_2e^{-kx}$

k is real

$$\frac{d^2X(x)}{dx^2} + k_x^2 X(x) = 0.$$

$$\frac{d^2Y(y)}{dy^2} + k_y^2 Y(y) = 0$$

$$\frac{d^2Z(z)}{dz^2} + k_z^2 Z(z) = 0,$$

$$\boxed{\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0.}$$

f(x) + f(y) + f(z) =0

$-k_x^2$ $-k_y^2$ $-k_z^2$ =0

$k_x^2 + k_y^2 + k_z^2 = 0.$, which should be satisfied.

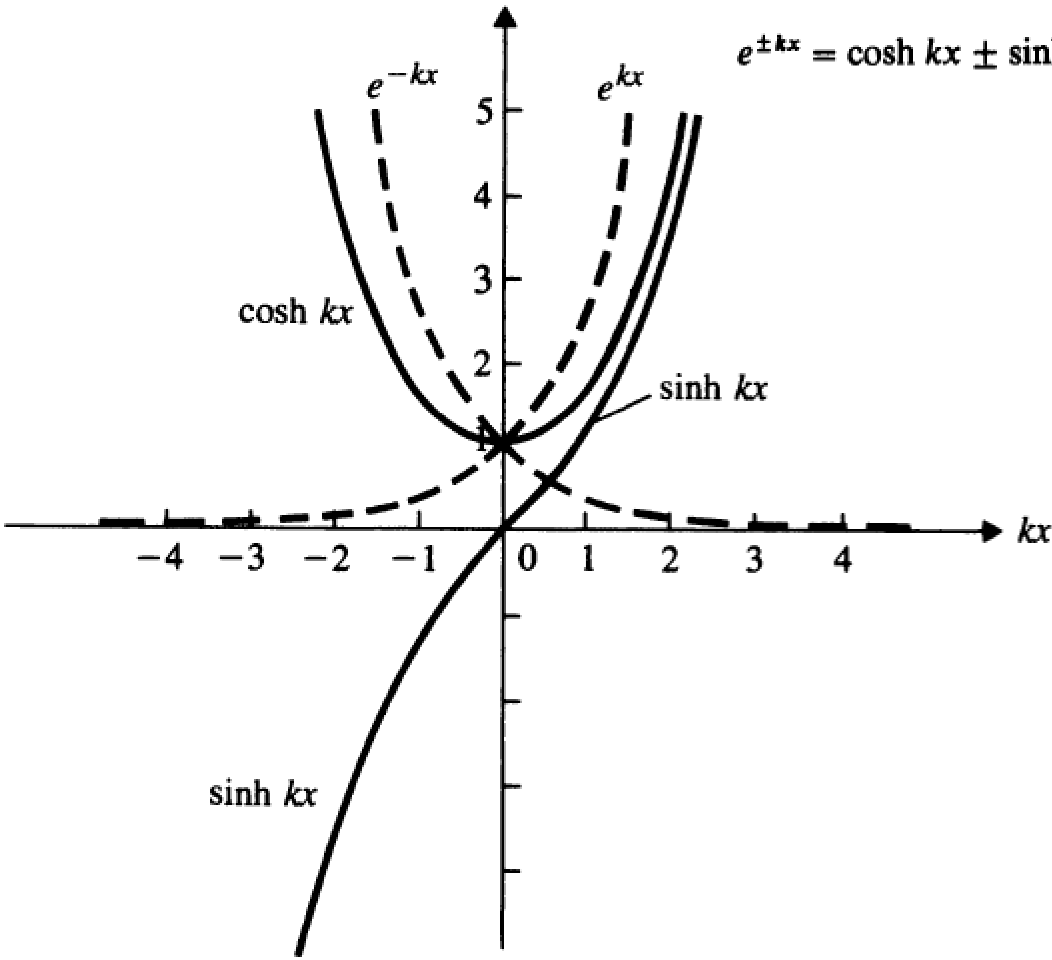
Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

k_x^2	k_x	$X(x)$	Exponential forms† of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

k is real

† $e^{\pm jkx} = \cos kx \pm j \sin kx, \quad \cos kx = \frac{1}{2}(e^{jkx} + e^{-jkx}), \quad \sin kx = \frac{1}{2j}(e^{jkx} - e^{-jkx});$

$e^{\pm kx} = \cosh kx \pm \sinh kx, \quad \cosh kx = \frac{1}{2}(e^{kx} + e^{-kx}), \quad \sinh kx = \frac{1}{2}(e^{kx} - e^{-kx}).$



A, B or C, D will be determined by boundary conditions

FIGURE 4–14
Hyperbolic and exponential functions.

4-6 Boundary-Value Problems in Cylindrical Coordinates

Laplace's equation $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0.$$

General solution: Bessel functions

Assuming z independent

$\partial^2 V/\partial z^2 = 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}=0.$$

By separation of variables

$V(r, \phi) = R(r)\Phi(\phi),$

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]+\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=0.$$

To hold for all values of r and ϕ

$$\boxed{\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]+\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=0.}$$

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]=k^2$$

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}=-k^2, \xrightarrow[\text{rewrite}]{} \frac{d^2\Phi(\phi)}{d\phi^2}+k^2\Phi(\phi)=0.$$

$$\boxed{V(r, \phi) = R(r)\underline{\Phi(\phi)},}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0.$$

For circular cylindrical configurations, if ϕ is unrestricted, $\Phi(\phi)$ is periodic (same values at a certain ϕ)

- k must be an integer n
- sinh, cosh are not periodic!

$e^{jk\phi} \neq e^{jk(\phi+2\pi)}$

$e^{jn\phi} = e^{jn(\phi+2\pi)}$

$$\Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi,$$

$$\boxed{V(r, \phi) = \underline{R(r)}\Phi(\phi),}$$

$$\frac{r}{R(r)} \frac{d}{dr}\left[r \frac{dR(r)}{dr}\right] = k^2$$

k ▭ n
Product rule

$$r^2 \frac{d^2R(r)}{dr^2} + r \frac{dR(r)}{dr} - n^2R(r) = 0,$$

Solution: $R(r) = A_r r^n + B_r r^{-n}.$

(Verified by direct substitution)

$$\Phi(\phi) = A_\phi \sin n\phi + B_\phi \cos n\phi,$$

$$R(r) = A_r r^n + B_r r^{-n}.$$

Combine the two solutions

$$\boxed{V(r, \phi) = R(r)\Phi(\phi),}$$

$$V_n(r, \phi) = r^n(A_n \sin n\phi + B_n \cos n\phi) + r^{-n}(A_n' \sin n\phi + B_n' \cos n\phi), \qquad n \neq 0.$$

where $A_n=A_rA_\phi$; $B_n=A_rB_\phi$; $A'_n=B_rA_\phi$; $B'_n=B_rB_\phi$;

If region of interest (ROI) includes r=0, 2nd term cannot exist.
If region of interest (ROI) includes r=∞, 1st term cannot exist

A Special Case: k=0

$$\boxed{\frac{d^2\Phi(\phi)}{d\phi^2} + k^2\Phi(\phi) = 0.}$$

$$\boxed{\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right] = k^2}$$

$$\frac{d^2\Phi(\phi)}{d\phi^2} = 0.$$

$$\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right] = 0,$$

for k=0

$$\Phi(\phi) = A_0\phi + B_0$$

$$R(r) = C_0 \ln r + D_0,$$

4-7 Boundary-Value Problems in Spherical Coordinates

Laplace's equation

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{R^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

Assuming ϕ independent

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial V}{\partial R}\right) + \frac{1}{R^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) = 0.$$

By separation of variables

$$V(R, \theta) = \Gamma(R)\Theta(\theta).$$

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2 \frac{d\Gamma(R)}{dR}\right] + \frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta \frac{d\Theta(\theta)}{d\theta}\right] = 0.$$

To hold for all values of R and θ

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right]=k^2$$

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right]=-k^2,$$

$$\boxed{V(R,\theta) = \underline{\Gamma(R)}\Theta(\theta).}$$

$$\boxed{\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right] = k^2}$$

$$\Downarrow$$

$$R^2\frac{d^2\Gamma(R)}{dR^2} + 2R\frac{d\Gamma(R)}{dR} - k^2\Gamma(R) = 0,$$

Solution: $\Gamma_n(R) = A_nR^n + B_nR^{-(n+1)}.$

where $n(n+1) = k^2,$

n=0,1,2… is a positive integer

(Verified by direct substitution)

$$\boxed{V(R,\theta) = \Gamma(R)\underline{\Theta(\theta)}.}$$

$$\boxed{\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] = -k^2,}$$

$$\boxed{n(n+1) = k^2,}$$

$$\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] + n(n+1)\Theta(\theta)\sin\theta = 0,$$

Legendre's equation

Solution: $\Theta_n(\theta) = P_n(\cos\theta).$

Legendre's functions
if involving full range of θ=[0,π]

For integer values of n,

Several Legendre Polynomials

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta - 1)$
3	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

The Legendre polynomials are **orthogonal**

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

EXAMPLE 4–6 Two grounded, semi-infinite, parallel-plane electrodes are separated by a distance b. A third electrode perpendicular to and insulated from both is maintained at a constant potential V_0 (see Fig. 4–15). Determine the potential distribution in the region enclosed by the electrodes.

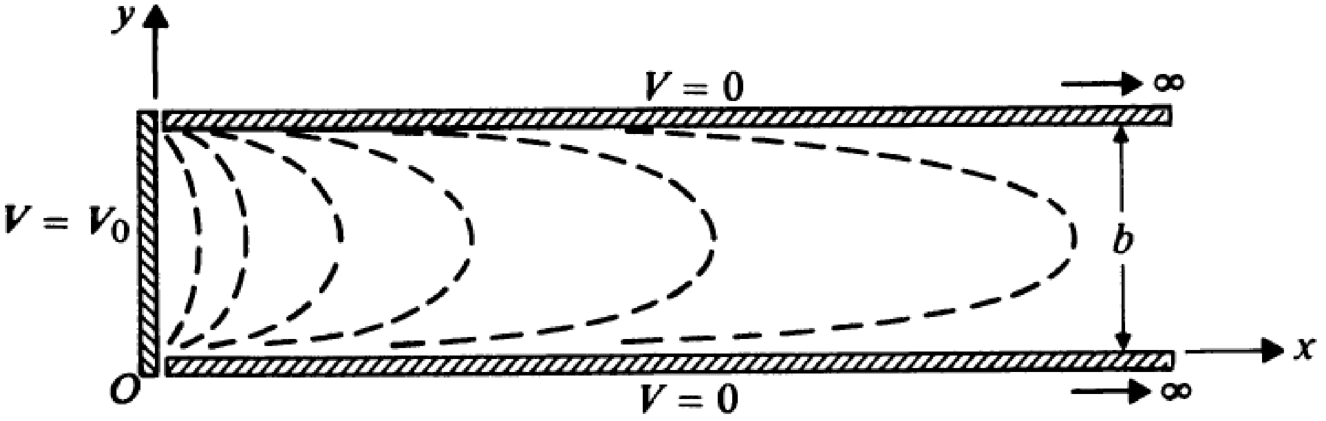


FIGURE 4–15
Cross-sectional figure for Example 4–6. The plane electrodes are infinite in z-direction.

Quick classroom exercise

TABLE 4–1
Possible Solutions of $X''(x) + k_x^2 X(x) = 0$

k_x^2	k_x	$X(x)$	Exponential forms† of $X(x)$
0	0	$A_0 x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1 e^{jkx} + D_1 e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2 e^{kx} + D_2 e^{-kx}$

Quick classroom exercise

Solution Referring to the coordinates in Fig. 4–15, we write down the boundary conditions for the potential function $V(x, y, z)$ as follows.

With V independent of z:

$$V(x, y, z) = V(x, y). \tag{4–90a}$$

In the x-direction:

$$V(0, y) = V_0 \tag{4–90b}$$

$$V(\infty, y) = 0. \tag{4–90c}$$

In the y-direction:

$$V(x, 0) = 0 \tag{4–90d}$$

$$V(x, b) = 0. \tag{4–90e}$$

Condition (4–90a) implies $k_z = 0$, and from Table 4–1,

$$Z(z) = B_0. \tag{4–91}$$

The constant A_0 vanishes because Z is independent of z. From Eq. (4–89) we have

$$k_y^2 = -k_x^2 = k^2, \tag{4–92}$$

where k is a real number. This choice of k implies that k_x is imaginary and that k_y is real. The use of $k_x = jk$, together with the condition of Eq. (4–90c), requires us to choose the exponentially decreasing form for $X(x)$, which is

$$X(x) = D_2 e^{-kx}. \tag{4–93}$$

In the y-direction, $k_y = k$. Condition (4–90d) indicates that the proper choice for $Y(y)$ from Table 4–1 is

$$Y(y) = A_1 \sin ky. \tag{4–94}$$

Combining the solutions given by Eqs. (4–91), (4–93), and (4–94) in Eq. (4–82), we obtain an appropriate solution of the following form:

$$\begin{aligned} V_n(x, y) &= (B_0 D_2 A_1) e^{-kx} \sin ky \\ &= C_n e^{-kx} \sin ky, \end{aligned} \tag{4–95}$$

where the arbitrary constant C_n has been written for the product $B_0 D_2 A_1$.

Now, of the five boundary conditions listed in Eqs. (4–90a) through (4–90e) we have used conditions (4–90a), (4–90c), and (4–90d). To meet condition (4–90e), we require

$$V_n(x, b) = C_n e^{-kx} \sin kb = 0, \tag{4–96}$$

which can be satisfied, for all values of x, only if

$$\sin kb = 0$$

or

$$kb = n\pi$$

or

$$k = \frac{n\pi}{b}, \qquad n = 1, 2, 3, \ldots. \tag{4–97}$$

Therefore, Eq. (4–95) becomes

$$V_n(x\ y) = C_n e^{-n\pi x/b} \sin \frac{n\pi}{b} y. \tag{4–98}$$

Electrostatic Analogs!!

Why physics analogies are powerful:

The same mathematical equations have the same solutions…

Many different physical systems happen to obey the same form of equations…

Not just in physics…

(i) Economics
(ii) Biology
(iii) Networks…

Imagination developed in one area can often help with another

Electrostatic analogs:

(i) Gravity
(ii) Heat flow
(iii) Stretched membrane
(iv) Irrotational fluid

Electrostatics

$$\nabla \cdot (\epsilon E) = \frac{\rho}{\epsilon_0}$$

$$\nabla \times E = 0$$

Electrostatics

$$\nabla \cdot (\epsilon E) = \frac{\rho}{\epsilon_0}$$

$$\nabla \times E = 0$$

$$E = -\nabla V$$

Poisson equation

$$\nabla \cdot (\epsilon \nabla V) = -\frac{\rho}{\epsilon_0}$$

Newtonian Gravity

$$F = -\nabla U_g$$

$$F = \frac{Gm'}{|x'-x|^2}$$

$$U_g = -\frac{Gm'}{|x'-x|}$$

Newtonian Gravity

$$F = -\nabla U_g$$

$$F = \frac{Gm'}{|x' - x|^2}$$

$$U_g = -\frac{Gm'}{|x' - x|}$$

$$N \text{ masses: } U_g = -G \sum_{i=1}^{N} \frac{m'_i}{|x'_i - x|}$$

$$\text{Smooth: } U_g = -G \int_{V'} \frac{\rho(x')}{|x' - x|} dV'$$

Newtonian Gravity

$$U_g = -G \int_{V'} \frac{\rho(x')}{|x' - x|} dV'$$

Newtonian Gravity

$$U_g = -G \int_{V'} \frac{\rho(x')}{|x' - x|} dV'$$

$$\nabla^2 \left(\frac{1}{|x' - x|} \right) = -4\pi\delta(x' - x)$$

Newtonian Gravity

$$U_g = -G \int_{V'} \frac{\rho(x')}{|x' - x|} dV'$$

$$\nabla^2 \left(\frac{1}{|x' - x|} \right) = -4\pi\delta(x' - x)$$

Poisson equation

$$\nabla^2 U_g = 4\pi G \rho(x)$$

More complicated as we go beyond Newtonian gravity…

Heat flow

$$\nabla \cdot h = \text{rate of heat out/unit time}$$

Heat flow

$$\nabla \cdot h = \text{rate of heat out/unit time}$$

Steady heat flow

$$\nabla \cdot h = s$$

Heat flow

$$\nabla \cdot h = \text{rate of heat out/unit time}$$

Steady heat flow

$$\nabla \cdot h = s$$

$$h = -K\nabla T$$

$$K = \text{thermal conductivity}$$

$$T = \text{temperature}$$

Heat flow

$$\nabla \cdot h = s$$

$$h = -K \nabla T$$

Poisson equation

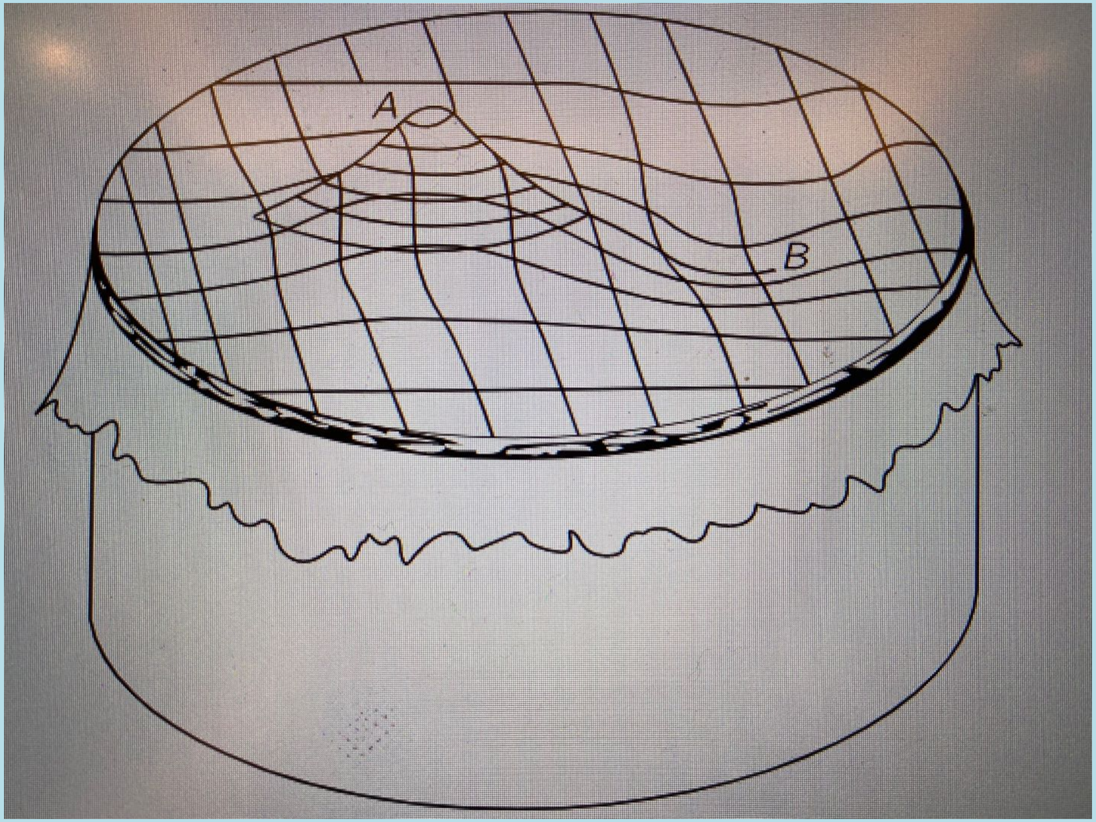
$$\nabla \cdot (K \nabla T) = -s$$

Stretched membrane

Thin rubber sheet stretched over a membrane over cylindrical drum.

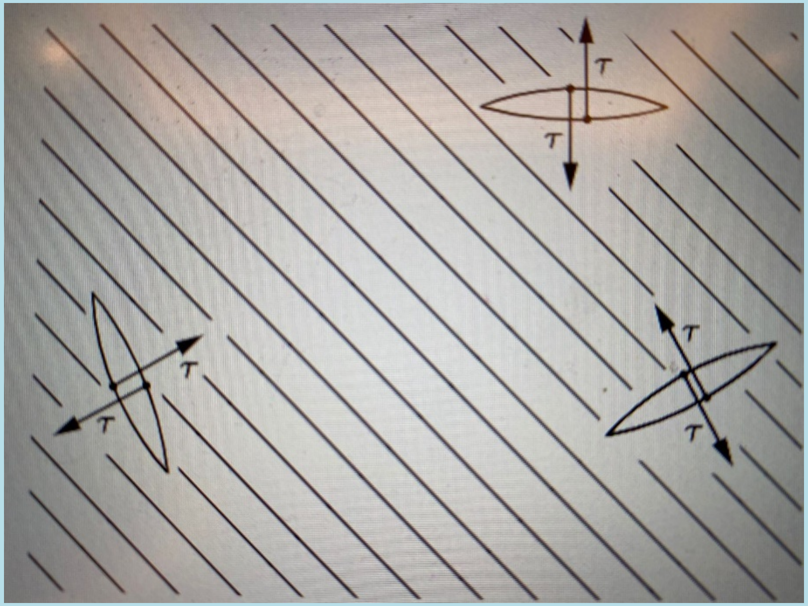
If sheet pushed up at A and down at B, what is shape of surface?

Stretched membrane

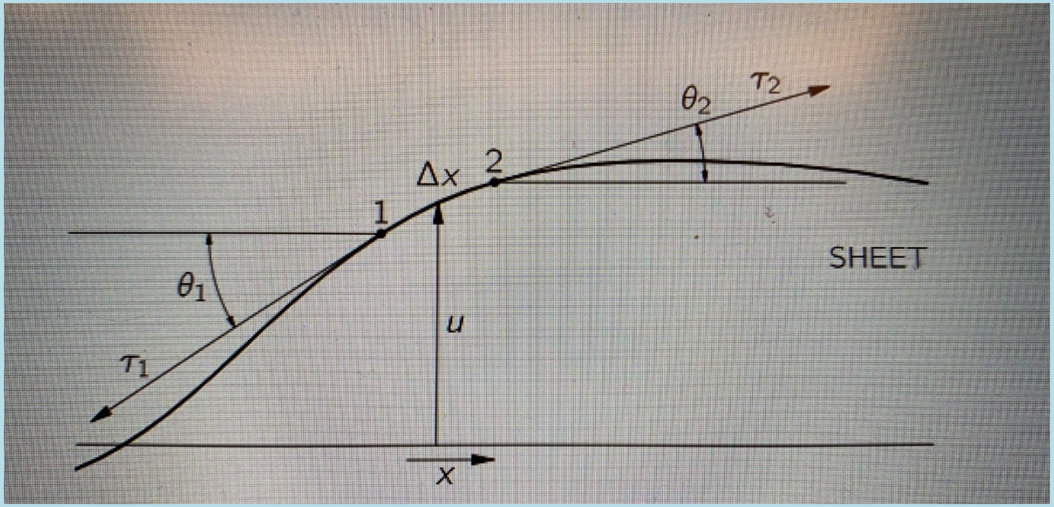


Stretched membrane

Surface tension τ = force/unit length across line



Stretched membrane



$$\Delta F = \tau_2 \Delta y \sin\theta_2 - \tau_1 \Delta y \sin\theta_1$$

Small slopes...

Stretched membrane

Small slopes...

$$\Delta F \sim \tau_2 \Delta y \sin\theta_2 - \tau_1 \Delta y \sin\theta_1$$

$$\sim \left(\tau_2 \left(\frac{\partial u}{\partial x} \right)_2 - \tau_1 \left(\frac{\partial u}{\partial x} \right)_2 \right) \Delta y$$

$$\sim \frac{\partial}{\partial x} \left(\tau \frac{\partial u}{\partial x} \right) \Delta x \Delta y$$

Stretched membrane

Define $$f \equiv -\frac{\Delta F}{\Delta x \Delta y}$$

$$f = -\nabla \cdot (\tau \nabla u)$$

Poisson equation $$\nabla^2 u = -\frac{f}{\tau}$$

Stretched membrane

Poisson equation

$$\nabla^2 u = -\frac{f}{\tau}$$

Since u is the height profile of membrane, does it make it easier to believe in the uniqueness solution of the electrostatic potential?

Irrotational fluid

Conservation of matter

$$\nabla \cdot (\rho v) + \frac{\partial \rho}{\partial t} = 0$$

v = velocity

Irrotational fluid

Incompressible fluid

$$\nabla \cdot (\rho v) + \frac{\partial \rho}{\partial t} = 0 \implies$$

$$\nabla \cdot v = 0$$

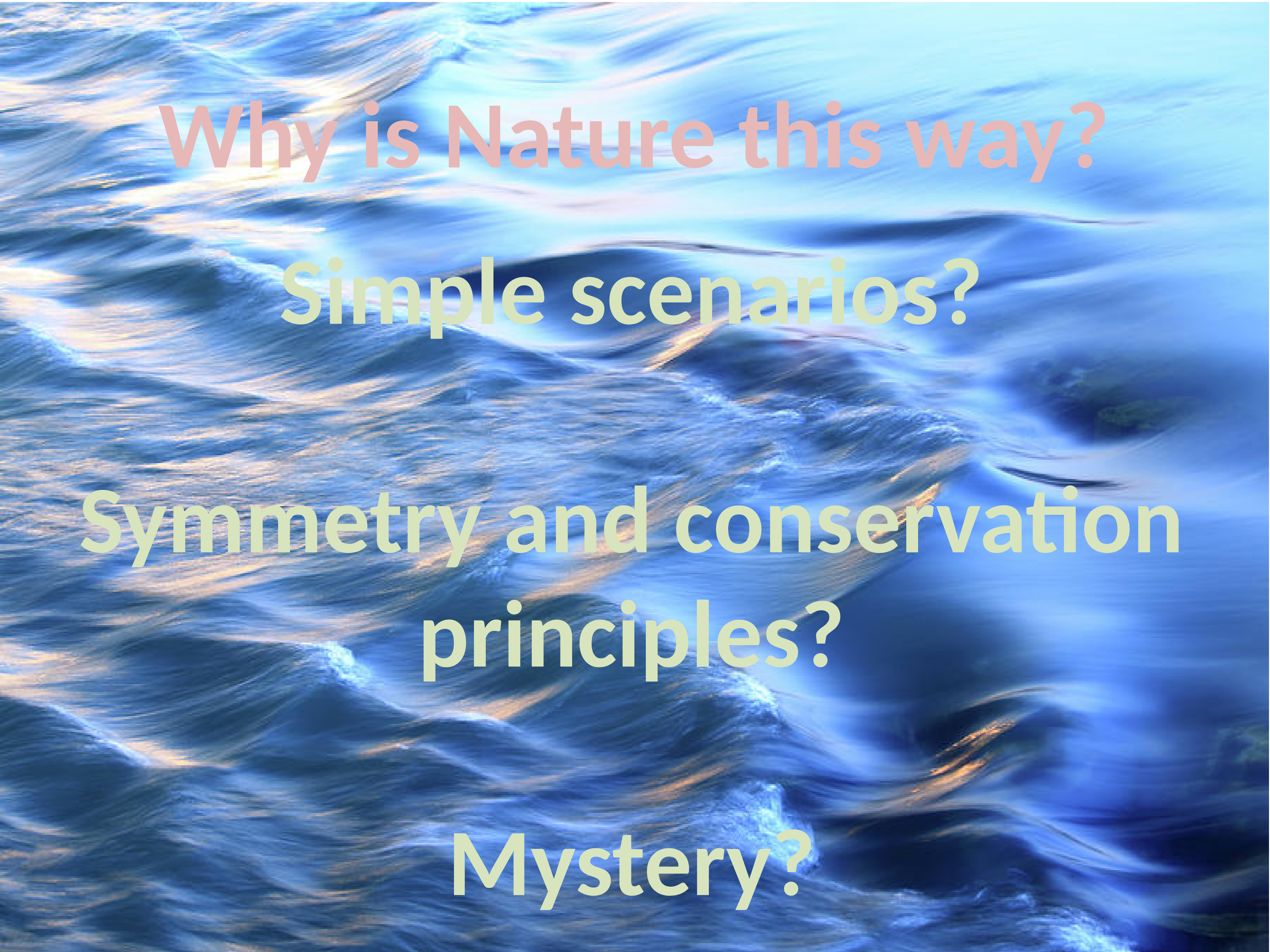
Irrotational fluid

Irrotational fluid

$$\nabla \times v = 0 \implies v = -\nabla\phi$$

Incompressible fluid $\nabla \cdot v = 0$

Laplace equation $\nabla^2\phi = 0$


Why is Nature this way?
Simple scenarios?
Symmetry and conservation principles?
Mystery?