Vv156 Lecture 13

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- Differentiation can be used to solve various optimization problems.

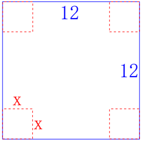
1. Maximizing or minimizing a continuous function over a finite closed interval.

Exercise

An open-top box is to be made by cutting small congruent squares from the corner of a 12cm-by-12cm sheet of tin and bending up the sides. How large should the squares cut from the corner be to make the box hold as much as possible?

Solution

- Produce a sketch. Let x denote the length of each small square.



$$
\begin{aligned}
V(x) &= (12-2x)^2 x \\
&= 144x - 48x^2 + 4x^3, \qquad 0 \le x \le 6
\end{aligned}
$$

- V is continuous in the closed and bounded interval $[0, 6]$, so EVT guarantees that there is an absolute maximum value of V in $[0, 6]$.

Solution

- Find critical points

$$
\begin{aligned}
V'(x) &= 144 - 96x + 12x^2 \\
&= 12(2-x)(6-x) \\
&\implies x = 2; \quad x = 6
\end{aligned}
$$

- Evaluate the critical points and end points

$$
\begin{aligned}
&V(x) = (12-2x)^2 x \\
\implies &V(0) = 0, \qquad V(2) = 128, \qquad V(6) = 0
\end{aligned}
$$

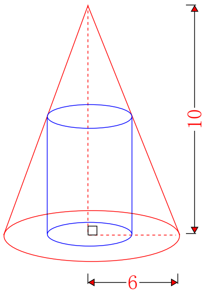
- Thus the maximum volume is 128cm^3, and the squares are 2cm-by-2cm.

Exercise

Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a right circular cone with radius 6cm and height 10cm.

Solution

- Produce a sketch. Let



$r=$radius of the cylinder
$h=$height of the cylinder
$V=$volume of the cylinder

- Find V as a function of only one variable,

$$V = \pi r^2 h$$

- Similar triangles implies

$$\frac{10-h}{r} = \frac{10}{6} \implies h = 10 - \frac{5}{3}r$$

$$\implies V = \pi r^2 (10 - \frac{5}{3}r), \qquad 0 \le r \le 6$$

Solution

- V is continuous in the closed and bounded interval $[0, 6]$, so EVT guarantees that there is an absolute maximum value of V in $[0, 6]$.
- Find critical points

$$\begin{aligned} V' &= 20\pi r - 5\pi r^2 \\ &= 5\pi r(4 - r) \end{aligned}$$

$$\implies r = 0; \quad r = 4$$

- Evaluate the critical points and end points

$$V(r) = \pi r^2 (10 - \frac{5}{3} r)$$
$$\implies V(0) = 0, \qquad V(4) = \frac{160}{3}\pi, \qquad V(6) = 0$$

- So the maximum volume is $\frac{160}{3}\pi\text{cm}^3$, and this happens when $r = 4$.

2. Maximizing or minimizing a continuous function over a noncompact interval.

Exercise

A cylindrical can is to be made to hold 1L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Solution

- Let h, r, S be the height, the radius and the surface area, respectively.
- Assume there is no waste or overlap, we need to minimise the surface area

$$S = 2\pi r^2 + 2\pi r h$$

- The volume of the can needs to be 1L $=$ 1000 cm^3, so h in terms of r is

$$1000 = \pi r^2 h \implies h = \frac{1000}{\pi r^2} \implies S = 2\pi r^2 + \frac{1000}{r}$$

- Thus we have reduced the problem to finding a value of r in the interval $[0, \infty)$ for which S is a minimum. So EVT is NOT applicable here, however

$$S' = 4\pi r - 2000r^{-2} = 2r^{-2}(2\pi r^3 - 1000)$$

Solution

- The critical points are at $r = 0$ and $r = \dfrac{10}{\sqrt[3]{2\pi}}$, by the first derivative test,

$$\begin{array}{ccc} r < 0 & S' < 0 & \text{decreasing} \\ 0 < r < \dfrac{10}{\sqrt[3]{2\pi}} & S' < 0 & \text{decreasing} \\ \dfrac{10}{\sqrt[3]{2\pi}} < r & S' > 0 & \text{increasing} \end{array}$$

- Hence

$$r = \frac{10}{\sqrt[3]{2\pi}}$$

gives a global minimum as well as a local minimum of S.

- Therefore

$$h = \frac{1000}{\pi r^2} = \frac{20}{\sqrt[3]{2\pi}} \qquad \text{and} \qquad r = \frac{10}{\sqrt[3]{2\pi}}$$

is the dimension of the can that minimises the surface area, and so the cost.

- The speed of light depends on the medium through which it travels and tends to be slower in denser media.
- In a vacuum, it travels at the speed $c = 3 \times 10^8$m/sec, but in the earth's atmosphere it travels slightly slower than that, and even slower in glass.

Fermat's principle of least time

light travels from one point to another along a path for which the time of travel is a minimum.

Exercise

Find the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 across a straight boundary to a point B in another medium where the speed of light is c_2.

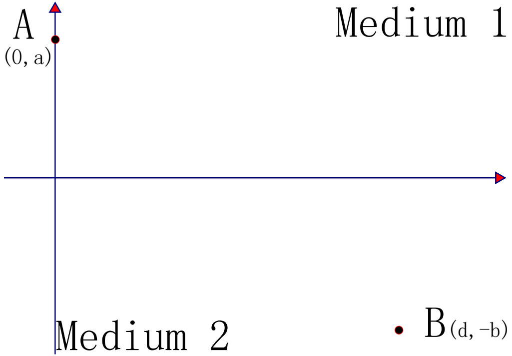
Solution

- According to Fermat's principle, we should minimise the time of travel,

$$\text{time} = \frac{\text{distance}}{\text{speed}}$$

Solution

- Let us set up the coordinate system such that point A is on the y-axis,

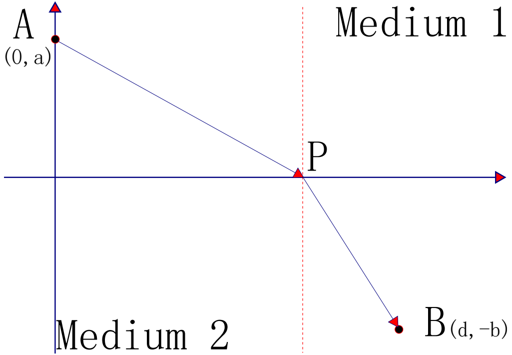


and that the line separating the two media is x-axis.

Solution

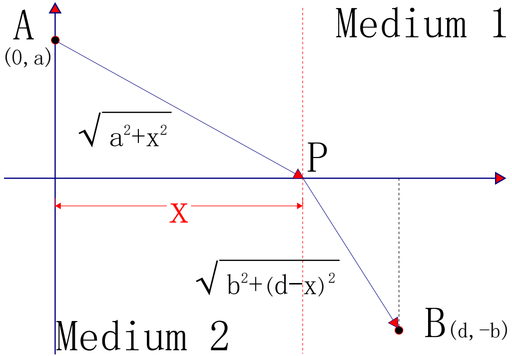
- In a uniform medium, where the speed of light remain constant, "shortest time" means "shortest path", and the ray of light will follow a straight line.

So the path from A to B will consist of a line segment from A to the boundary P, followed by another line segment from P to B.



Solution

- Let x be the x-coordinate of P, then



- The times required for light from A to P and from P to B, respectively, are

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2+x^2}}{c_1}, \qquad \text{and} \qquad t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2+(d-x)^2}}{c_2}$$

Solution

- So the total time from A and to B in terms of x is given by

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d-x)^2}}{c_2}$$

- This expresses t as a differentiable function of x for $0 \leq x \leq d$, and

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d-x}{c_2\sqrt{b^2 + (d-x)^2}}$$

  is continuous, and is negative at $x = 0$ and is positive $x = d$.

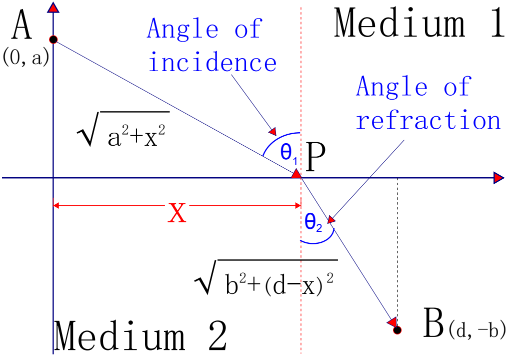
- Therefore IVT guarantees there is a point between 0 and d such that

$$0 = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d-x}{c_2\sqrt{b^2 + (d-x)^2}}$$

- There is only one such point since $\dfrac{d^2t}{dx^2} > 0$ for $0 < x < d$.

Solution

- In terms of angles, θ_1 and θ_2



- we have

$$\frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2} = 0$$

which is known as the Snell's law or the law of refraction.