vv255: Line Integrals.

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Vector Fields

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- Vector fields can be used to visualise families of solutions to differential equations.

A vector field $F : D \longrightarrow \mathbb{R}^2$ where $D \subseteq \mathbb{R}^2$ can be visualised by selecting a finite number of (possibly uniformly distributed) points (x_1, y_1), . . . , (x_n, y_n) in D and plotting vectors corresponding to $F(x_1, y_1)$, . . . , $F(x_n, y_n)$ at the points (x_1, y_1), . . . , (x_n, y_n) on the Cartesian plane.

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Vector Fields

Example

Consider the vector field $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *defined by* $F(x, y) = y\bar{i} + (x + y)\bar{j}$

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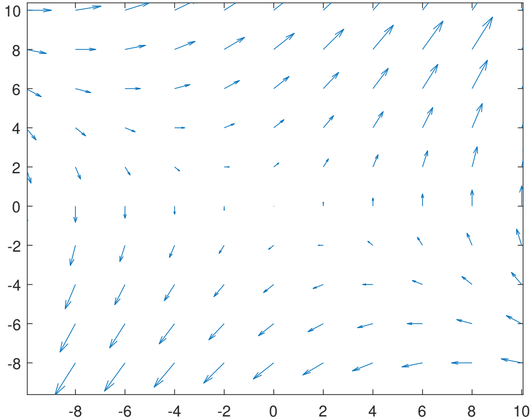

Vector Fields

Example

(Continued.) The commands that I used to produce this are...

```
>> [x,y] = meshgrid(-10:2:10);
>> dx= y;
>> dy= x+y;
>> quiver(x,y,dx,dy);
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But you can probably do better!

Vector Fields

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(Continued.) The commands that I used to produce this are...

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Example

Consider $f(x, y, z) = x \cos\left(\frac{y}{z}\right)$. $\nabla f(x, y, z)$ *is a vector field on* $\mathbb{R}^3$*:*

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

Vector Fields

Example

(Continued.)

$$\nabla f(x, y, z) = \cos\left(\frac{y}{z}\right)\bar{i} - \frac{x}{z}\sin\left(\frac{y}{z}\right)\bar{j} + \frac{xy}{z^2}\sin\left(\frac{y}{z}\right)\bar{k}$$

Vector Fields

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(Continued.)

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Definition

Let $F : D \longrightarrow \mathbb{R}^n$ where $D \subseteq \mathbb{R}^n$ be a vector field on $\mathbb{R}^n$.
We say that F is conservative if there exists a differentiable function

$$f : D \longrightarrow \mathbb{R} \text{ such that for all } \bar{x} \in D \quad F(\bar{x}) = \nabla f(\bar{x})$$

If $F : D \longrightarrow \mathbb{R}^n$ where $D \subseteq \mathbb{R}^n$ is a conservative vector field and $f : D \longrightarrow \mathbb{R}$ is such that for all $\bar{x} \in D$, $F(\bar{x}) = \nabla f(\bar{x})$, then f is called a potential function for F.

Vector Fields

Example

The vector field $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *defined by*

$$F(x, y, z) = (y^2 z^3, 2xyz^3, 3xy^2 z^2)$$

is conservative.

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is conservative. To see this, consider $f(x, y, z) = xy^2z^3$. *We have*

$$\frac{\partial f}{\partial x} = y^2z^3 \qquad \frac{\partial f}{\partial y} = 2xyz^3 \qquad \frac{\partial f}{\partial z} = 3xy^2z^2$$

so f *is a potential function for* F.

Vector Fields

The vector field $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by
$F(x, y) = (x^2 - yx)\bar{i} + (y^2 - xy)\bar{j}$ is NOT conservative.

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Proof.

Suppose, for a contradiction, that f is a potential function for F.

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$$\frac{\partial^2 f}{\partial y \partial x} = -x \text{ and } \frac{\partial^2 f}{\partial x \partial y} = -y$$

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Example

The vector field $F : D \longrightarrow \mathbb{R}^2$, where $D = \mathbb{R}^2 \backslash \{\langle 0, 0 \rangle\}$, defined by

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right)$$

is called the vortex vector field. F is NOT conservative, but we do not have the tools to show this yet!

Line Integrals

So far, we have seen how to integrate functions of more than one variable over closed rectangles and bounded regions. We now turn to defining a notion of integral that aims to simulate the integral of a function of a single variable over an arbitrary curve in 2D or 3D space.

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Let $\mathcal{C}$ be a smooth curve in 2D space described by the parametric equations $x = f(t)$ and $y = g(t)$ where $t \in [a, b]$. Recall that the distance along $\mathcal{C}$ starting from the point a is described by the function

$$s(t) = \int_a^t \sqrt{(f'(u))^2 + (g'(u))^2}\, du$$

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Suppose that $(x_0, y_0), \ldots, (x_{n+1}, y_{n+1})$ are point on $\mathcal{C}$ and $t_0, \ldots, t_{n+1} \in [a, b]$ are such that $t_0 = a$ and $t_{n+1} = b$, and for all $0 \leq k \leq n + 1$,

$$x_k = f(t_k) \text{ and } y_k = g(t_k)$$

$(x_0, y_0), \ldots, (x_{n+1}, y_{n+1})$ look like a partition of the curve $\mathcal{C}$ and $P = \{t_0, \ldots, t_{n+1}\}$ is a partition of $[a, b]$

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$$U = \sum_{k=0}^{n} \sup\{h(f(t), g(t)) \mid t \in [t_k, t_{k+1}]\} \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$$

$$L = \sum_{k=0}^{n} \inf\{h(f(t), g(t)) \mid t \in [t_k, t_{k+1}]\} \sqrt{(x_{k+1} - x_k)^2 + (y_{k+1} - y_k)^2}$$

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Let $M_k = \sup\{h(f(t), g(t)) \mid t \in [t_k, t_{k+1}]\}$ and $m_k = \inf\{h(f(t), g(t)) \mid t \in [t_k, t_{k+1}]\}$ for all $0 \leq k \leq n$,

$$U = \sum_{k=0}^{n} M_k(t_{k+1} - t_k) \sqrt{\left(\frac{x_{k+1} - x_k}{t_{k+1} - t_k}\right)^2 + \left(\frac{y_{k+1} - y_k}{t_{k+1} - t_k}\right)^2}$$

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Line Integrals

Therefore as the partitions (x_0, y_0), ..., (x_{n+1}, y_{n+1}) and $P = \{t_0, \ldots, t_{n+1}\}$ become arbitrarily fine, U gets closer to $U(h, P)$ and L gets closer to $L(\alpha, P)$ where $U(\alpha, P)$ is the the single variable upper Darboux sum and $L(\alpha, P)$ is the single variable lower Darboux sum, and

$$\alpha(t) = h(f(t), g(t)) \sqrt{(f'(t))^2 + (g'(t))^2}$$

This motivates the following definition:

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This motivates the following definition:

Definition

Let $\mathcal{C}$ be a smooth curve in 2D space described by the parametric equations $x = f(t)$ and $y = g(t)$ where $t \in [a, b]$. Let $h(x, y)$ be a function that is defined and continuous on $\mathcal{C}$. The line integral of h along $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} h(x, y)\, ds = \int_a^b h(f(t), g(t))\sqrt{(f'(t))^2 + (g'(t))^2}\, dt$$

Line Integrals

Example

Let $\mathcal{C}$ *be the curve described by* $x = t^2$, $y = 2t$, $0 \leq t \leq 3$. *Compute*

$$\int_{\mathcal{C}} y \; ds$$

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Example

Let $\mathcal{C}$ be the curve described by the circle $x^2 + y^2 = 4$ in the left half-plane. Compute

$$\int_{\mathcal{C}} xy^4 \; ds$$

Line Integrals

Formalising the reasoning on the preceding slides would allow one to show that the line integral gives the area of the surface that lies between the graph $z = h(x, y)$ and the curve $\mathcal{C}$ on the xy-plane.

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The preceding reasoning can also be generalized to motivate the line integral of a function of three variables over a smooth curve in 3D space.

Definition

Let $\mathcal{C}$ be a smooth curve in 3D space described by the parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ where $t \in [a, b]$. Let $\gamma(x, y, z)$ be a function that is defined and continuous on $\mathcal{C}$. The line integral of γ along $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} \gamma(x, y, z)\, ds = \int_a^b \gamma(f(t), g(t), h(t)) \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, dt$$

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The assumption that the curve $\mathcal{C}$ is smooth in the above definitions is appealed to in order to get that the derivative of the curve's length is given by the function $\sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$.

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Recall that a function that is discontinuous only at a finite number of points is still integrable. Similarly, the line integral over a curve that fails to be smooth only at a finite number of points remains valid.

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Definition

A curve $\mathcal{C}$ in 2D or 3D space is said to be piecewise-smooth *if $\mathcal{C}$ is composed of a finite number of smooth curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$ such that for all $1 \leq i \leq n-1$, the initial point on $\mathcal{C}_{i+1}$ is terminal point on $\mathcal{C}_i$. If $\mathcal{C}$ is a piecewise-smooth curve in 2D or 3D space composed of the smooth curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and f is a function that is coninuous on $\mathcal{C}$, then define*

$$\int_{\mathcal{C}} f \, ds = \sum_{k=1}^{n} \int_{\mathcal{C}_k} f \, ds$$

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Example

Let $\mathcal{C}$ consist of the arc of the circle $x^2 + y^2 = 4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$. Compute

$$\int_{\mathcal{C}} (x^2 + y^2)\, ds$$

Line Integrals

We will also have cause to consider the following variants of the line integral that integrate a function with respect to x, y and z respectively.

Definition

Let $\mathcal{C}$ be a smooth curve in 3D space described by the parametric equations $x = f(t)$, $y = g(t)$ and $z = h(t)$ where $t \in [a, b]$. Let $\gamma(x, y, z)$ be a function that is defined and continuous on $\mathcal{C}$.
The line integral of γ along $\mathcal{C}$ with respect to x is defined by

$$\int_{\mathcal{C}} \gamma(x, y, z)\ dx = \int_a^b \gamma(f(t), g(t), h(t)) f'(t)\ dt$$

Line Integrals

Definition

(Continued.) The line integral of γ along $\mathcal{C}$ with respect to y is defined by

$$\int_{\mathcal{C}} \gamma(x, y, z)\ dx = \int_a^b \gamma(f(t), g(t), h(t))g'(t)\ dt$$

The line integral of γ along $\mathcal{C}$ with respect to z is defined by

$$\int_{\mathcal{C}} \gamma(x, y, z)\ dx = \int_a^b \gamma(f(t), g(t), h(t))h'(t)\ dt$$

Line Integrals

Definition

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This definitition can be restricted in the obvious way to define the line integral of a function of two variables along a curve in 2D space with respect to x and y.

Line Integrals

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For reasons that will be revealed shortly, line intergals with respect to x, y and z often appear together as sums.

Line Integrals

For this reason, it is convenient to write

$$\int_{\mathcal{C}} P(x, y)\ dx + Q(x, y)\ dy$$

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And

$$\int_{\mathcal{C}} P(x,y,z)\ dx + Q(x,y,z)\ dy + R(x,y,z)\ dz$$

instead of

$$\int_{\mathcal{C}} P(x,y,z)\ dx + \int_{\mathcal{C}} Q(x,y,z)\ dy + \int_{\mathcal{C}} R(x,y,z)\ dz$$

Be aware of the similarity in notation between the line integral along $\mathcal{C}$ with respect to x (or y, or z) and the notion used for the partial integral. These are two very different things!

Line Integrals

Example

Let $\mathcal{C}$ be the line running from the point $(1, 0, 0)$ to the point $(4, 1, 2)$. Compute

$$\int_{\mathcal{C}} z^2 \; dx + x^2 \; dy + y^2 \; dz$$

Example

Let $\mathcal{C}$ consist of two line segments from $(0, 0, 0)$ to $(1, 0, 1)$ and from $(1, 0, 1)$ to $(0, 1, 2)$. Evaluate

$$\int_{\mathcal{C}} (y + z) \; dx + (x + z) \; dy + (x + y) \; dz$$

Orientation of a curve

Let $\mathcal{C}$ be a smooth curve in 3D space described by the parametric equations

$$x = f(t), \quad y = g(t), \quad z = h(t),\ t \in [a, b].$$

In addition to describing $\mathcal{C}$, this parameterisation also endows $\mathcal{C}$ with a direction:
The curve starts at $(f(a), g(a), h(a))$ and finishes at $(f(b), g(b), h(b))$.

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In addition to describing $\mathcal{C}$, this parameterisation also endows $\mathcal{C}$ with a direction:
The curve starts at $(f(a), g(a), h(a))$ and finishes at $(f(b), g(b), h(b))$.
In light of this, we write $-\mathcal{C}$ for the curve that is identical to $\mathcal{C}$ except for the fact: it starts at $(f(b), g(b), h(b))$ and finishes at $(f(a), g(a), h(a))$.
$\Rightarrow -\mathcal{C}$ is the curve described by the parametric equations

$$x = f(b + a - t), \quad y = g(b + a - t), \quad z = h(b + a - t),\ t \in [a, b].$$

If $\gamma(x, y, z)$ is a function that is defined and continuous on $\mathcal{C}$, then

$$\int_{\mathcal{C}} \gamma(x, y, z)\ dx = -\int_{-\mathcal{C}} \gamma(x, y, z)\ dx$$

and the same equation holds for the line integrals of γ along $\mathcal{C}$ with respect to y and z. However,

$$\int_{\mathcal{C}} \gamma(x, y, z)\ ds = \int_{-\mathcal{C}} \gamma(x, y, z)\ ds$$

Line integrals of vector fields

Another important is notion is the line integral of a vector field.

Definition

A vector field $F : D \longrightarrow \mathbb{R}^n$, where $D \subseteq \mathbb{R}^n$, is said to be continuous on $I \subseteq D$ if each of its components is continuous on I.

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Define the line integral of F *along* $\mathcal{C}$ *by*

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Define the line integral of F along $\mathcal{C}$ by

$$\int_{\mathcal{C}} F \cdot d\bar{r} = \int_a^b F(\bar{r}(t)) \cdot \bar{r}'(t)\ dt$$

Note that by replacing F by a vector field on $\mathbb{R}^2$ and $\mathcal{C}$ by a smooth curve in $\mathbb{R}^2$ in the above definition we obtain the definition of the line integral of a vector field in $\mathbb{R}^2$ over a smooth curve in $\mathbb{R}^2$.

Line integrals of vector fields

Observe that

$$\frac{F(\bar{r}(t)) \cdot \bar{r}'(t)}{|\bar{r}'(t)|}$$

is the scalar projection of of the vector $F(\bar{r}(t))$ along the direction of $\mathcal{C}$ at the point $\bar{r}(t)$ on $\mathcal{C}$.

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$$\int_{\mathcal{C}} F \cdot d\bar{r} = \int_a^b \frac{F(\bar{r}(t)) \cdot \bar{r}'(t)}{|\bar{r}'(t)|} |\bar{r}'(t)| \; dt = \int_{\mathcal{C}} \frac{F(\bar{r}(t)) \cdot \bar{r}'(t)}{|\bar{r}'(t)|} \; ds$$

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Therefore the line integral of a vector field F along a smooth curve $\mathcal{C}$ is just the line integral of the scalar projection of F in the direction of $\mathcal{C}$ along $\mathcal{C}$.

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Example

Let $\mathcal{C}$ be the curve described by the vector function $\bar{r} : [0, \frac{\pi}{2}] \longrightarrow \mathbb{R}^3$ where $\bar{r}(t) = \cos t\bar{i} + \sin t\bar{j}$. Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by $F(x, y) = x^2\bar{i} - xy\bar{j}$. Compute

$$\int_{\mathcal{C}} F \cdot d\bar{r}$$

Line integrals of vector fields

Let $F : D \longrightarrow \mathbb{R}^3$, where $D \subseteq \mathbb{R}^3$, be a vector field s.t. $\forall\,(x, y, z) \in D$

$$F(x, y, z) = P(x, y, z)\bar{i} + Q(x, y, z)\bar{j} + R(x, y, z)\bar{k}$$

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Let $\bar{r} : [a, b] \longrightarrow \mathbb{R}^3$ be a vector function that describes a smooth curve $\mathcal{C}$ that is contained in D and is defined by

$$\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$$

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$$\int_{\mathcal{C}} F \cdot d\bar{r} = \int_a^b F(\bar{r}(t)) \cdot \bar{r}'(t)\ dt$$

$$= \int_a^b (P(x, y, z), Q(x, y, z), R(x, y, z)) \cdot (x'(t), y'(t), z'(t))\ dt$$

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$$= \int_a^b P(x(t),y(t),z(t))x'(t)\ dt + \int_a^b Q(x(t),y(t),z(t))y'(t)\ dt$$

$$+ \int_a^b R(x(t),y(t),z(t))z'(t)\ dt = \int_{\mathcal{C}} P(x,y,z)\ dx + Q(x,y,z)\ dy + R(x,y,z)$$

Line integrals of vector fields

Example

Let $\mathcal{C}$ be the curve described by the vector function $\bar{r} : [0,1] \longrightarrow \mathbb{R}^3$ defined by

$$\bar{r}(t) = t\bar{i} + t^2\bar{j} + t^3\bar{k}$$

Let $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by

$$F(x, y, z) = xy\bar{i} + yz\bar{j} + zx\bar{k}$$

Compute

$$\int_{\mathcal{C}} F \cdot d\bar{r}$$

The Fundamental Theorem of Line Integrals

Recall that the First Fundamental Theorem of Calculus says that

$$\int_a^b F'(x)\ dx = F(b) - F(a)$$

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Theorem

(Fundamental Theorem of Line Integrals) Let $\mathcal{C}$ be a smooth curve described by the vector function $\bar{r} : [a, b] \longrightarrow \mathbb{R}^3$. Let $f : D \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^3$ and $\mathcal{C}$ is contained in D, be differentiable on D with ∇f continuous on $\mathcal{C}$. Then

$$\int_{\mathcal{C}} \nabla f \cdot d\bar{r} = f(\bar{r}(b)) - f(\bar{r}(a))$$

The Fundamental Theorem of Line Integrals

Proof.

Let $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$.

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$$\int_{\mathcal{C}} \nabla f \cdot d\bar{r} = \int_a^b \nabla f(\bar{r}(t)) \cdot \bar{r}'(t)\ dt$$

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$$\int_{\mathcal{C}} \nabla f \cdot d\bar{r} = \int_a^b \nabla f(\bar{r}(t)) \cdot \bar{r}'(t)\ dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right)\ dt$$

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$$= \int_a^b \frac{d}{dt} f(\bar{r}(t))\ dt = f(\bar{r}(b)) - f(\bar{r}(a))$$

where the last equation is obtained by applying the First Fundamental Theorem of Calculus. □

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where the last equation is obtained by applying the First Fundamental Theorem of Calculus. $\square$

The same result also holds for line integrals of gradient vector field of functions of two variables over smooth curves in 2D space.

Paths and path independence

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Let P and Q be points in 2D or 3D space. A piecewise-smooth curve $\mathcal{C}$ with initial point P and terminal point Q is called a ***path*** *between P and Q.*

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Let $F : D \longrightarrow \mathbb{R}^3$, where $D \subseteq \mathbb{R}^3$, be a continuous vector field. We say that the line integral of F is independent of path if for all P and Q in D and for all paths $\mathcal{C}_1$ and $\mathcal{C}_2$ from P to Q,

$$\int_{\mathcal{C}_1} F \cdot d\bar{r} = \int_{\mathcal{C}_2} F \cdot d\bar{r}$$

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Definition

A piecewise-smooth curve $\mathcal{C}$ is said to be a closed path if $\mathcal{C}$ starts and ends at the same point.

Paths and path independence

Theorem

Let $F : D \longrightarrow \mathbb{R}^3$, where $D \subseteq \mathbb{R}^3$, be a continuous vector field. The line integral of F is independent of path if and only if for every closed path $\mathcal{C}$,

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$\Rightarrow$: Let $\mathcal{C}$ be a closed path. Let P and Q be points on $\mathcal{C}$. So, P and Q divide $\mathcal{C}$ into two paths $\mathcal{C}_1$ and $\mathcal{C}_2$ from P to Q.

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Proof.

$\Rightarrow$: Let $\mathcal{C}$ be a closed path. Let P and Q be points on $\mathcal{C}$. So, P and Q divide $\mathcal{C}$ into two paths $\mathcal{C}_1$ and $\mathcal{C}_2$ from P to Q. We have

$$\int_{\mathcal{C}} F \cdot d\bar{r} = \int_{\mathcal{C}_1} F \cdot d\bar{r} + \int_{-\mathcal{C}_2} F \cdot d\bar{r} = \int_{\mathcal{C}_1} F \cdot d\bar{r} - \int_{\mathcal{C}_2} F \cdot d\bar{r} = 0,$$

by path independence.

Paths and path independence

Proof.

(Continued.) $\Leftarrow$: Conversely, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be paths from P to Q.

Paths and path independence

Proof.

(Continued.) $\Leftarrow$: Conversely, let $\mathcal{C}_1$ and $\mathcal{C}_2$ be paths from P to Q. Let $\mathcal{C}_1$ and $-\mathcal{C}_2$ form a closed path $\mathcal{C}$ that starts and ends at P and

$$0 = \int_{\mathcal{C}} F \cdot d\bar{r} = \int_{\mathcal{C}_1} F \cdot d\bar{r} + \int_{-\mathcal{C}_2} F \cdot d\bar{r} = \int_{\mathcal{C}_1} F \cdot d\bar{r} - \int_{\mathcal{C}_2} F \cdot d\bar{r}$$

Paths and path independence

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So the line integral of F is independent of path. $\square$

Paths and path independence

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The following is an immediate consequence of the Fundamental Theorem of Line Integrals:

Theorem

Let $F : D \longrightarrow \mathbb{R}^3$, where $D \subseteq \mathbb{R}^3$, be a continuous vector field. If F is conservative, then the line integral of F is independent of path.

Paths and Path Independence: Example

Let $\mathcal{C}$ be the closed path that is composed of

$$\mathcal{C}_1 \colon x = y^2, \quad z = 0 \text{ between } (0,0,0) \text{ and } (1,1,0)$$

$$\mathcal{C}_2 \colon \text{ is the straight line joining } (1,1,0) \text{ and } (1,1,1)$$

$$\mathcal{C}_3 \colon \text{ is the straight line joining } (1,1,1) \text{ and } (0,0,0)$$

Consider the vector field $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$F(x,y,z) = yz\bar{i} + xy\bar{j} + zx\bar{k}$$

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Along $\mathcal{C}_1$, $\bar{r}(y) = y^2\bar{i} + y\bar{j}$ for $0 \leq y \leq 1$ and $F(\bar{r}(y)) = y^3\bar{j}$.

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$$\Rightarrow \frac{d\bar{r}}{dy} = 2y\bar{i} + \bar{j}$$

$$\Rightarrow \int_{\mathcal{C}_1} F \cdot d\bar{r} = \int_0^1 y^3 \, dy = \frac{1}{4}$$

Paths and Path Independence: Example (Continued.)

Along $\mathcal{C}_2$, $\bar{r}(z) = \bar{i} + \bar{j} + z\bar{k}$ for $0 \leq z \leq 1$ and $F(\bar{r}(z)) = z\bar{i} + \bar{j} + z\bar{k}$.

Paths and Path Independence: Example (Continued.)

Along $\mathcal{C}_2$, $\bar{r}(z) = \bar{i} + \bar{j} + z\bar{k}$ for $0 \leq z \leq 1$ and $F(\bar{r}(z)) = z\bar{i} + \bar{j} + z\bar{k}$.
Therefore

$$\frac{d\bar{r}}{dz} = \bar{k}$$

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$$\Rightarrow \int_{\mathcal{C}_2} F \cdot d\bar{r} = \int_0^1 z \; dz = \frac{1}{2}$$

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Therefore

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Along $\mathcal{C}_3$, $\bar{r}(x) = x\bar{i} + x\bar{j} + x\bar{k}$ for $1 \geq x \geq 0$ and
$F(\bar{r}(x)) = x^2\bar{i} + x^2\bar{j} + x^2\bar{k}$.

Paths and Path Independence: Example (Continued.)

Along $\mathcal{C}_2$, $\bar{r}(z) = \bar{i} + \bar{j} + z\bar{k}$ for $0 \leq z \leq 1$ and $F(\bar{r}(z)) = z\bar{i} + \bar{j} + z\bar{k}$. Therefore

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Along $\mathcal{C}_3$, $\bar{r}(x) = x\bar{i} + x\bar{j} + x\bar{k}$ for $1 \geq x \geq 0$ and $F(\bar{r}(x)) = x^2\bar{i} + x^2\bar{j} + x^2\bar{k}$. Therefore

$$\frac{d\bar{r}}{dx} = \bar{i} + \bar{j} + \bar{k}$$

Paths and path independence

Example

(Continued.) So,

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We now turn to developing some machinery that will allow us to show that a vector is conservative.

Definition

We say that $D \subseteq \mathbb{R}^n$ is open if for all $\bar{a} \in D$, there exists $\epsilon > 0$ such that $B(\bar{a}, \epsilon) \subseteq D$.

Conservative vector fields

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Theorem

Let F be a vector field on $\mathbb{R}^3$ (or $\mathbb{R}^2$) that is continuous on an open connected region D in $\mathbb{R}^3$ ($\mathbb{R}^2$, respectively). If the line integral of F on D is independent of path, then F is conservative.

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We will sketch a proof of this result for vector fields on $\mathbb{R}^2$. It should be reasonably clear how the argument can be modified to deal with vector fields on $\mathbb{R}^3$.

Conservative vector fields

Proof.

(Sketch.) Let $F = \alpha(x, y)\bar{i} + \beta(x, y)\bar{j}$ where α and β are continuous on the open connected region D. Let P be any point in D.

Conservative vector fields

Proof.

(Sketch.) Let $F = \alpha(x, y)\bar{i} + \beta(x, y)\bar{j}$ where α and β are continuous on the open connected region D. Let P be any point in D. Define $f : D \longrightarrow \mathbb{R}$ by: $f(x, y)$ is the line integral of F from P to (x, y) along a path from P to (x, y) in D.

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$$f(x,y) = \int_{\mathcal{C}_1} F \cdot d\bar{r} + \int_{\mathcal{C}_2} F \cdot d\bar{r}$$

Note that the line integral of F along $\mathcal{C}_1$ is constant, so

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} \int_{\mathcal{C}_2} F \cdot d\bar{r}$$

Conservative vector fields

Proof.

Now,

$$\int_{\mathcal{C}_2} F \cdot d\overline{r} = \int_{\mathcal{C}_2} \alpha(x, y)\ dx + \beta(x, y)\ dy$$

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The line integral of F along $\mathcal{C}_2$ with respect to y is 0 ($\mathcal{C}_2$ is a horizontal line).

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$$\int_{\mathcal{C}_2} F \cdot d\bar{r} = \int_{\mathcal{C}_2} \alpha(x, y)\ dx = \int_{x'}^{x} \alpha(t, y)\ dt$$

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$$\int_{\mathcal{C}_2} F \cdot d\bar{r} = \int_{\mathcal{C}_2} \alpha(x, y)\ dx = \int_{x'}^{x} \alpha(t, y)\ dt$$

And, by the Second Fundamental Theorem,

$$\frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial x} \int_{x'}^{x} \alpha(t, y)\ dt = \alpha(x, y)$$

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A similar argument where $\mathcal{C}_2$ is replaced by a vertical line shows that

$$\frac{\partial}{\partial y} f(x, y) = \beta(x, y)$$

Conservative vector fields

Proof.
Since $(x, y) \in D$ was arbitrary, it follows that $\nabla f = F$ and so F is conservative. $\square$

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Example

Recall that the vortex vector field $F : D \longrightarrow \mathbb{R}^2$*, where* $D = \mathbb{R}^2 \backslash \{(0,0)\}$*, defined by*

$$F(x, y) = \left(\frac{-y}{x^2 + y^2}\right) \bar{i} + \left(\frac{x}{x^2 + y^2}\right) \bar{j}$$

Let $\mathcal{C}$ *be the unit circle described by* $x^2 + y^2 = 1$ *(i.e.* $\mathcal{C}$ *is a closed path). Since*

$$\int_{\mathcal{C}} F \cdot d\bar{r} = 2\pi,$$

the line integral of F *is not independent of path and so* F *is NOT conservative.*

Conservative vector fields

Example

Consider $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ *defined by*

$$F(x, y, z) = yz\bar{i} + xz\bar{j} + (xy + 2z)\bar{k}$$

Find $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ *such that* $\nabla f = F$ *and compute*

$$\int_{\mathcal{C}} F \cdot d\bar{r}$$

where $\mathcal{C}$ *is the line that connects* $(1, 0, -2)$ *and* $(4, 6, 3)$*.*

Line Integrals: First Summary

1. The line integral of γ along $\mathcal{C}$ is defined by
$$\int_{\mathcal{C}} \gamma(x,y,z)\, ds = \int_a^b \gamma(f(t),g(t),h(t)) \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}\, d$$
2. The line integral of F along $\mathcal{C}$ by
$$\int_{\mathcal{C}} F \cdot d\bar{r} = \int_a^b F(\bar{r}(t)) \cdot \bar{r}'(t)\; dt$$
3. Fundamental Theorem of Line Integrals
$$\int_{\mathcal{C}} \nabla f \cdot d\bar{r} = f(\bar{r}(b)) - f(\bar{r}(a))$$
4. Theorem: The line integral of F is independent of path if and only if for every closed path $\mathcal{C}$,
$$\int_{\mathcal{C}} F \cdot d\bar{r} = 0$$
5. Theorem: If F is conservative, then the line integral of F is independent of path.
6. Theorem: If the line integral of a continuous on an open connected region D vector field F on D is independent of path, then F is conservative.

Today

1. Simple curves and simply-connected regions
2. Positively oriented simple closed curves
3. Green's Theorem

Simple Curves and Simply-connected Regions

Definition

Let $\mathcal{C}$ be a curve described by the vector function $\bar{r} : [a, b] \longrightarrow \mathbb{R}^n$. We say that $\mathcal{C}$ is simple if for all $a < t_1 < t_2 < b$, $\bar{r}(t_1) \neq \bar{r}(t_2)$. We say that $\mathcal{C}$ is a simple closed curve if $\mathcal{C}$ is simple and $\bar{r}(a) = \bar{r}(b)$.

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- A simple closed curve $\mathcal{C}$ defines the boundary of region D that we call the region enclosed by $\mathcal{C}$. Note that we will always include the points on $\mathcal{C}$ in the region enclosed by $\mathcal{C}$. So, the region enclosed by $\mathcal{C}$ is a bounded region that contains all of its boundary points.

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Definition

A region $D \subseteq \mathbb{R}^n$ (for $n = 2$ or $n = 3$) is said to be simply-connected if D is connected and for all simple closed curves $\mathcal{C}$ contained in D, the region enclosed by $\mathcal{C}$ is contained in D.

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The region

$$D_1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$$

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is not simply connected because the simple closed curve desrcibed by $\bar{r}(t) = \frac{1}{2}(\cos(t)\bar{i} + \sin(t)\bar{j})$ *is contained in* D_2 *and encloses the point* $(0, 0)$ *that is not in* D_2.

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Intuitively, simply connected regions are connected regions that have no holes.

Let $\mathcal{C}$ be a simple closed curve described by $\overline{r} : [a, b] \longrightarrow \mathbb{R}^2$. As was mentioned we were discussing line integral, in addition to specifying the points on $\mathcal{C}$, the parameterisation $\overline{r}$ also sepcifies a direction — $\mathcal{C}$ runs from $\overline{r}(a)$ to $\overline{r}(b)$.

Green's Theorem

In light of this, we introduce the following terminology that allows us to talk about the direction specified by the parameterisation of a curve:

Definition

Let $\mathcal{C}$ be a simple closed curve described by $\bar{r} : [a, b] \longrightarrow \mathbb{R}^2$. We say that $\mathcal{C}$ is positively oriented and call $\bar{r}$ a positive orientation of $\mathcal{C}$ if $\bar{r}(t)$ moves anticlockwise around $\mathcal{C}$ as t ranges from a to b.

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We will use the symbols

$$\oint_{\mathcal{C}} \text{ or } \ointctrclockwise_{\mathcal{C}}$$

to indicate that we are invoking a line integral around a positively oriented curve $\mathcal{C}$, or indicate that the line integral is being computed using the positive orientation of $\mathcal{C}$ if the the orientation of $\mathcal{C}$ is ambiguous.

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if $\mathcal{C}$ is *not* positively oriented.

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- One way of interpreting the First Fundamental Theorem of Calucus is by reading it as saying that the integral of a function f over an interval $[a, b]$ is completely determined by the behaviour of the antiderivative of f at the boundary points of $[a, b]$.
- Viewing the First Fundamental Theorem from this perspective, we may speculate that the double integral of a function f over a region $\mathcal{R}$ might be completely determined by the behaviour of something that looks like the antiderivative of f on the boundary of $\mathcal{R}$.

Green's Theorem

Theorem

(Green's Theorem) Let $\mathcal{C}$ be a positively oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$ and let $\mathcal{R}$ be the region enclosed by $\mathcal{C}$. Let $D \subseteq \mathbb{R}^2$ be an open region that contains $\mathcal{R}$.
If $P : D \longrightarrow \mathbb{R}$ and $Q : D \longrightarrow \mathbb{R}$ have continuous partial derivatives on D, then

$$\int_{\mathcal{C}} P\ dx + Q\ dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

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If $D \subseteq \mathbb{R}^2$ is a bounded simply-connected region that contains all of its boundary points and the boundary of D is a piecewise-smooth simple closed curve, then we wil sometimes use ∂D to denote this boundary curve. Therefore the conclusion of Green's Theorem could also be written

$$\oint_{\partial\mathcal{R}} P\ dx + Q\ dy = \iint_{\mathcal{R}} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

Green's Theorem

To give an idea of why Green's Theorem might be true and what is going on behind the scenes we will sketch the proof of a very specific case of Green's Theorem. Recall that when we were discussing double integrals we encountered regions that could be described as both type I and type II bounded regions (we used this to change the order of integration).

Green's Theorem

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(Sketch of a very specific case of Green's Theorem) We will sketch a proof of Green's Theorem in the case that $\mathcal{R}$ is a simple bounded region described by smooth functions and $\mathcal{C}$ is the boundary of $\mathcal{R}$.
Let $g_1 : [a, b] \longrightarrow \mathbb{R}$ and $g_2 : [a, b] \longrightarrow \mathbb{R}$ be smooth functions such that for all $a \leq x \leq b$, $g_1(x) \leq g_2(x)$ and

$$\mathcal{R} = \{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$$

Green's Theorem

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$$\iint_{\mathcal{R}} \frac{\partial P}{\partial y} \; dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y} \; dy \; dx$$

$$= \int_a^b \left(P(x, g_2(x)) - P(x, g_1(x)) \right) \; dx$$

Green's Theorem

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(Continued.) Now, the curves $\mathcal{C}_1$ and $\mathcal{C}_3$ are vertical lines, so

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Therefore,

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This above proof can easily be extended to regions that can be broken apart into finitely many simple regions.

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- By choosing $P(x, y) = -\frac{1}{2}y$ and $Q(x, y) = \frac{1}{2}x$, Green's Theorem yields:

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- We could also choose $P(x, y) = 0$ and $Q(x, y) = x$ to get:

$$A = \oint_{\mathcal{C}} x\ dy$$

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Example

Find the area of the ellipse described by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

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Compute

$$\oint_{\mathcal{C}} (y + e^{\sqrt{x}})\ dx + (2x + \cos(y^2))\ dy$$

where $\mathcal{C}$ *is the boundary of the region enclosed by the parabolas* $y = x^2$ *and* $x = y^2$.

Extending Green's Theorem

Green's Theorem can also be extended to regions with holes in them. This is achieved using the same idea that we indicated could be used to extend the special case of Green's Theorem proved earlier. That is, if a region has a single hole then this region can be split into two simply-connected regions.

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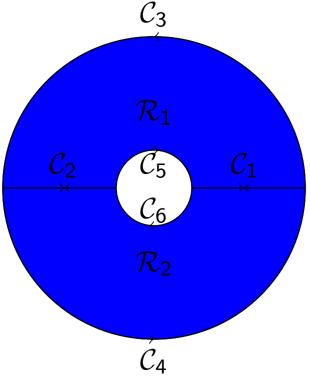

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Let $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$. Suppose that D is open and $\mathcal{R}$ is contained in D, and $P : D \longrightarrow \mathbb{R}$ and $Q : D \longrightarrow \mathbb{R}$ have continuous partial derivatives on D.

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Example

Consider again the vortex vector field $F : D \longrightarrow \mathbb{R}^2$, *where* $D = \mathbb{R}^2 \backslash \{(0,0)\}$, *defined by*

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When we first starting talking about conservative vector field we observed the following consequence of Clairaut's Theorem:

Theorem

If $F(x,y) = \alpha(x,y)\bar{i} + \beta(x,y)\bar{j}$ is a conservative vector field where α and β have continuous first-order partial derivatives on $D \subseteq \mathbb{R}^2$, then for all $(u,v) \in D$, $\alpha_y(u,v) = \beta_x(u,v)$

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Proof.

(Sketchy Outline.)

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- The general result can then be obtained by observing that a general closed curve can be viewed as a collection of simple curves that meet at a point.

$\square$

Curl and divergence

Definition

The vector differential operator*, pronouced "del" or "nabla", is the operator defined by*

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}$$

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This is consistent with the notation that we have been using for the gradient of a function f with three independent variables:

$$\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$$

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The vector differential operator*, pronouced "del" or "nabla", is the operator defined by*

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Let $F : D \longrightarrow \mathbb{R}^3$, where $D \subseteq \mathbb{R}^3$, be a vector field such that the components of F are differentiable on D. The curl of F is defined by $\mathrm{curl}(F) = \nabla \times F$. The divergence of F is defined by $\mathrm{div}(F) = \nabla \cdot F$.

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Let $F(x, y, z) = P(x, y, z)\bar{i} + Q(x, y, z)\bar{j} + R(x, y, z)\bar{k}$ be a vector field. Then

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Note that $\mathrm{div}(F)$ is a scalar. Physical interpretations of these operations can be obtained if one considers a vector field F that describes the flow of a fluid through space.

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Theorem

Let F be a vector field on $\mathbb{R}^3$ whose component functions have continuous second-order partial derivatives. Then $\operatorname{div}(\operatorname{curl}(F)) = 0$.

Curl and divergence

Example

Consider $F(x, y, z) = (y^2 \cos(x) + z^3)\bar{i} + (2y \sin(x) - 4)\bar{j} + (3xz^2 + 2)\bar{k}$.
Find $\mathrm{div}(F)$ *and* $\mathrm{curl}(F)$.

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This looks strange! The first ∇ operation on F (∇F) is not a dot product.

Identities involving divergence and curl

Theorem

Let F and G be a vector fields in $\mathbb{R}^3$ and let f and g be functions of three variables. If the partial derivatives of F, G, g and f that appear in the following equations exist and are continuous, then the following equations hold:

1. $\mathrm{div}(F+G) = \mathrm{div}(F) + \mathrm{div}(G)$
2. $\mathrm{curl}(F+G) = \mathrm{curl}(F) + \mathrm{curl}(G)$
3. $\mathrm{div}(fF) = f\,\mathrm{div}(F) + F \cdot \nabla f$
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5. $\mathrm{div}(F \times G) = G \cdot \mathrm{curl}(F) - F \cdot \mathrm{curl}(G)$
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Consider $F(x,y,z) = ye^x\bar{i} + (x^2+z)\bar{j} + y^3\cos(zx)\bar{k}$. Compute $\operatorname{curl}(\operatorname{curl}(F))$, $\nabla\operatorname{div}(F)$ and $\nabla^2 F$.

Vector formulation of Green's Theorem

Let $\mathcal{C}$ be a positively oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$ and let $\mathcal{R}$ be the region enclosed by $\mathcal{C}$.
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So, Green's Theorem yields:

$$\boxed{\int_{\mathcal{C}} F \cdot d\bar{r} = \iint_{\mathcal{R}} (\mathrm{curl}(F)) \cdot \bar{k}\ dA}$$

2D Divergence Theorem

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The unit normal vectors of the tangent lines to the curve $\mathcal{C}$ are described by the vector function

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$\Rightarrow$ for all $t \in [a, b]$, $\bar{n}(t) \cdot \hat{\bar{r}}'(t) = 0$ and $|\bar{n}(t)| = 1$.

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$$\int_{\mathcal{C}} F \cdot \overline{n}\ ds = \int_a^b (F \cdot \overline{n})(t)|\overline{r}'(t)|\ dt$$

2D Divergence Theorem

Now,

$$(F \cdot \bar{n})(t) = \frac{P(x(t), y(t))y'(t)}{|\bar{r}'(t)|} - \frac{Q(x(t), y(t))x'(t)}{|\bar{r}'(t)|}$$

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So

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by Green's Theorem.

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$$\int_{\mathcal{C}} F \cdot \bar{n}\ ds = \iint_{\mathcal{R}} \mathrm{div}(F)\ dA$$