Chapter 3: Static Electric Fields

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Many slides courtesy of Sung-Liang Chen

3-10 Capacitance and Capacitors

- Deposit charges Q on a conductor ➔ V
- kQ ➔ kρ_s ➔ kV

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\rho_s}{R}\,ds' \qquad \text{(V);}}$$

- The ratio Q/V unchanged

$$\boxed{Q = CV,}$$

C: capacitance (C/V, or Farad)

Capacitor

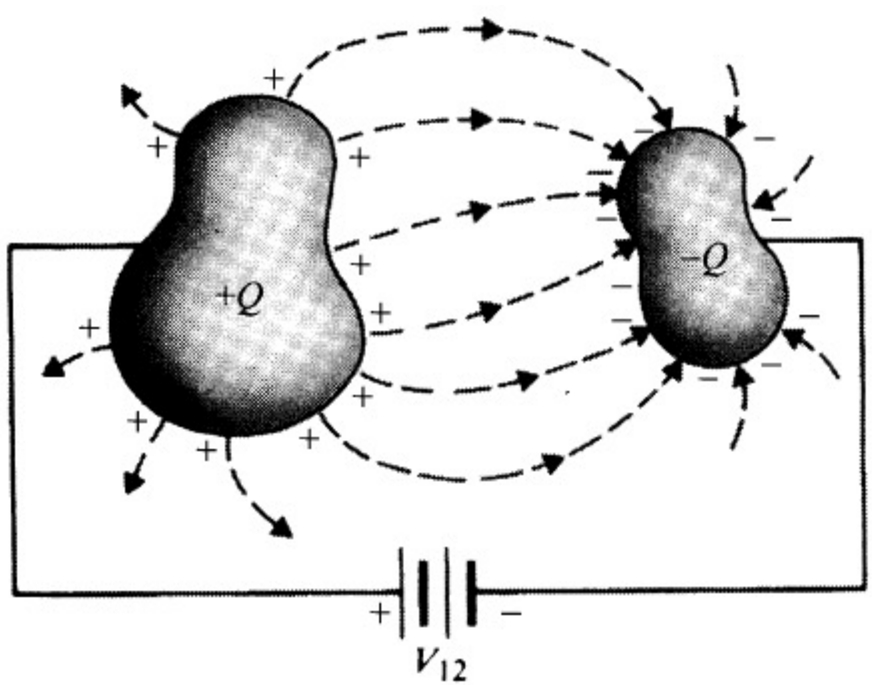


FIGURE 3–27
A two-conductor capacitor.

E $\perp$ conductor surfaces (equipotential surfaces)

$$\boxed{C = \frac{Q}{V_{12}} \qquad \text{(F)}.}$$

Capacitance

- C depends on
  - the geometry of the conductors
  - the permittivity of the medium between conductors
  - **Independent of Q and V**
- Measurement of C
  - Method 1: V_{12} known, determine Q (Chap.4)
  - Method 2: Q known, determine V_{12}

Capacitance

- Method 2: Q known, determine V_{12}
  1. Choose a proper coordinate system
  2. Assume +Q, –Q on the conductors
  3. Find **E** from Q (Gauss's law, etc.)
  4. Find V_{12} by $V_{12} = -\int_2^1 \mathbf{E} \cdot d\ell$
  5. $C=Q/V_{12}$

3-10.1 Series and Parallel Connections of Capacitors

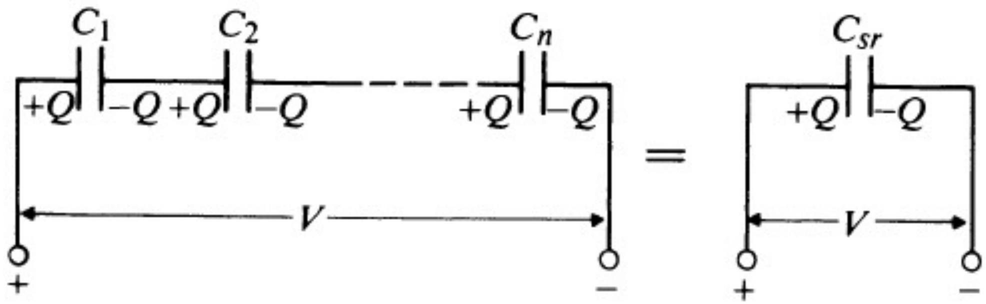


FIGURE 3–31
Series connection of capacitors.

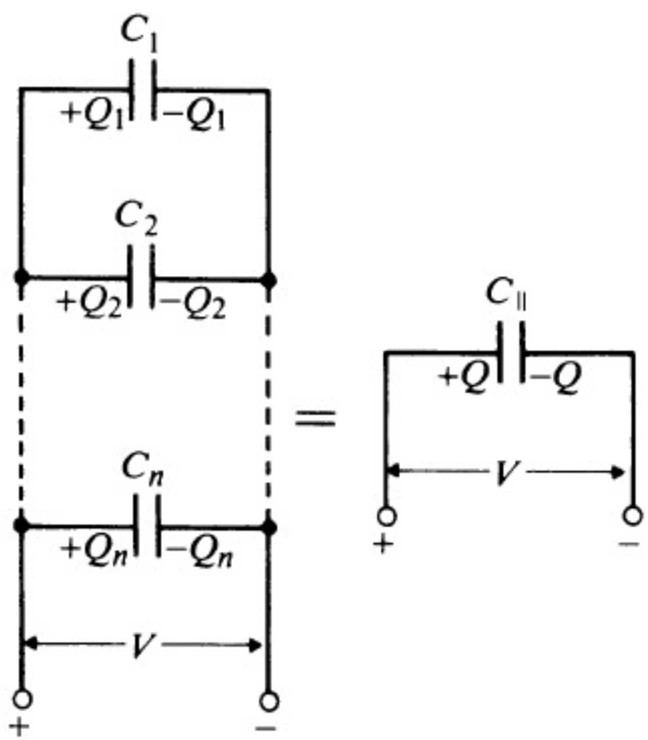


FIGURE 3–32
Parallel connection of capacitors.

Series

- V
  - ➔ +Q and –Q on two external terminals
  - ➔ +Q and –Q also induced internally

$$V=\frac{Q}{C_{sr}}=\frac{Q}{C_1}+\frac{Q}{C_2}+\cdots+\frac{Q}{C_n},$$

$$\boxed{\frac{1}{C_{sr}}=\frac{1}{C_1}+\frac{1}{C_2}+\cdots+\frac{1}{C_n}.}$$

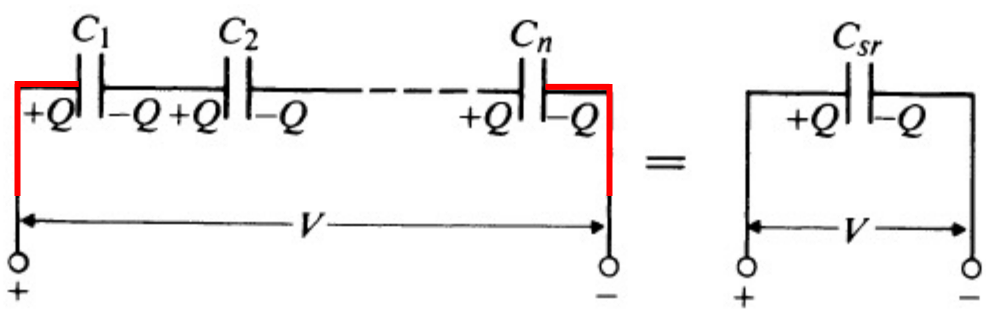
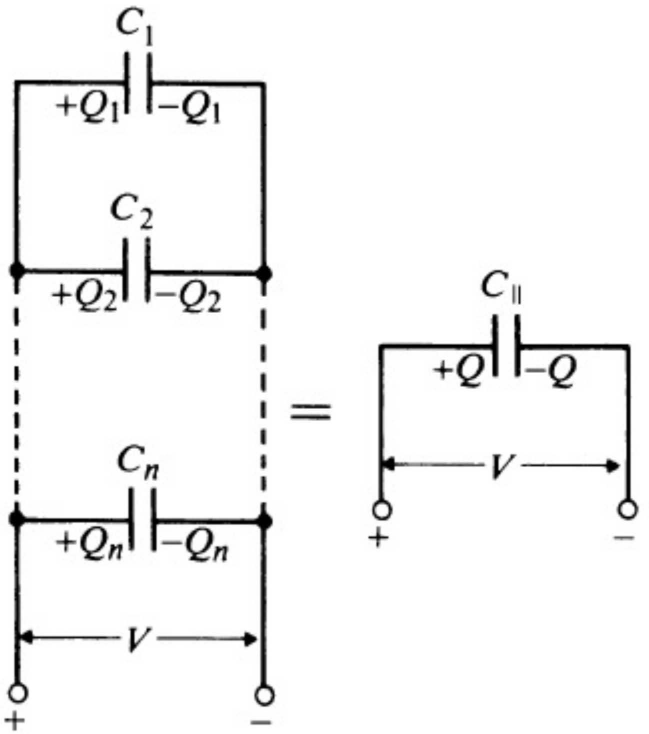


FIGURE 3–31
Series connection of capacitors.

Parallel

- V

  ➔ Q_1, Q_2, Q_3, … on each capacitor



$$Q = Q_1 + Q_2 + \cdots + Q_n$$
$$= C_1V + C_2V + \cdots + C_nV = C_{||}V$$

$$\boxed{C_{||} = C_1 + C_2 + \cdots + C_n.}$$

FIGURE 3–32
Parallel connection of capacitors.

3-10.2 Capacitances in Multiconductor Systems

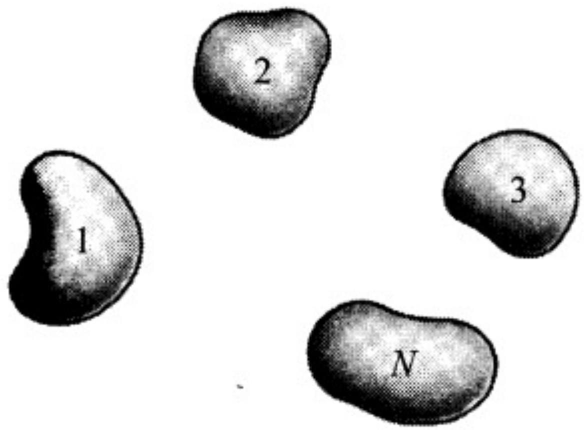


FIGURE 3–34
A multiconductor system.

Presence of a charge on any one of the conductors
➔ Affect potential of all the other conductors

$$V_1 = p_{11}Q_1 + p_{12}Q_2 + \cdots + p_{1N}Q_N,$$
$$V_2 = p_{21}Q_1 + p_{22}Q_2 + \cdots + p_{2N}Q_N,$$
$$\vdots$$
$$V_N = p_{N1}Q_1 + p_{N2}Q_2 + \cdots + p_{NN}Q_N.$$

p_{ij}: coefficients of potential; depends on
1. Shape and position of the conductor
2. Permittivity of surroundings

For an isolated system $Q_1 + Q_2 + Q_3 + \cdots + Q_N = 0.$

$$
\begin{aligned}
Q_1 &= c_{11}V_1 + c_{12}V_2 + \cdots + c_{1N}V_N, \\
Q_2 &= c_{21}V_1 + c_{22}V_2 + \cdots + c_{2N}V_N, \\
&\vdots \\
Q_N &= c_{N1}V_1 + c_{N2}V_2 + \cdots + c_{NN}V_N,
\end{aligned}
$$

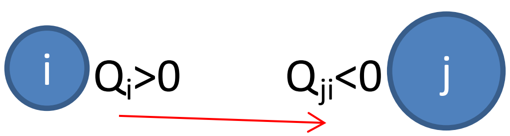
c_{ii}: coefficients of capacitance
c_{ij}: coefficients of induction ($i \neq j$)

c_{ii}: ground all other conductors, then $c_{ii}=Q_i/V_i$

c_{ji}: Induced charge $Q_i=c_{ji}V_i$

If Q_i on ith conductor, $V_i>0$ ➔ induced $Q_{ji}<0$

Thus, $c_{ii}>0$; $c_{ji}<0$



By reciprocity, $p_{ij} = p_{ji}$ and $c_{ij} = c_{ji}$

A Four-conductor System

$$
\begin{aligned}
Q_1 &= c_{11}V_1 + c_{12}V_2 + \cdots + c_{1N}V_N, \\
Q_2 &= c_{21}V_1 + c_{22}V_2 + \cdots + c_{2N}V_N, \\
&\vdots \\
Q_N &= c_{N1}V_1 + c_{N2}V_2 + \cdots + c_{NN}V_N,
\end{aligned}
$$

$$\Downarrow$$

Conductors 0,1,2,3. Let conductor 0 be grounded (i.e., $V_0=0$).

$$
\begin{aligned}
Q_1 &= c_{11}V_1 + c_{12}V_2 + c_{13}V_3, \\
Q_2 &= c_{12}V_1 + c_{22}V_2 + c_{23}V_3, \\
Q_3 &= c_{13}V_1 + c_{23}V_2 + c_{33}V_3,
\end{aligned}
$$

A Four-conductor System

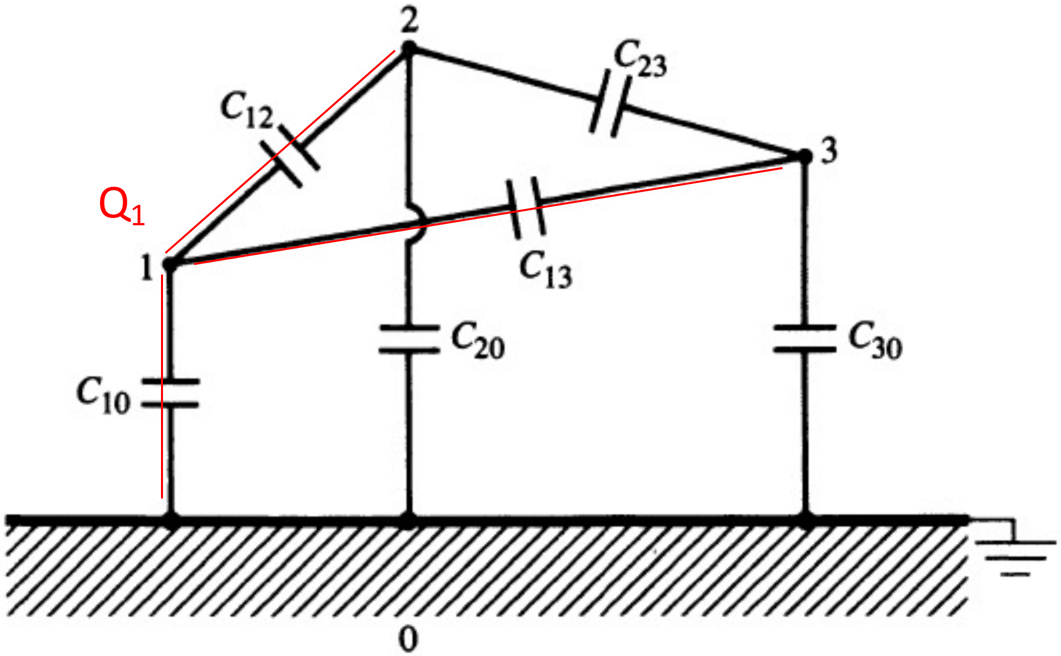


FIGURE 3–35
Schematic diagram of three conductors and the ground.

c: Coefficient of capacitance
C: Capacitance

Rewrite the Q ~ V relation

$$Q_1 = C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3),$$
$$Q_2 = C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3),$$
$$Q_3 = C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2),$$

C_{10},C_{20},C_{30}: self-partial capacitance
C_{ij} ($i \neq j$): mutual partial capacitance

$$Q_1 = (C_{10} + C_{12} + C_{13})V_1 - C_{12}V_2 - C_{13}V_3,$$
$$Q_2 = -C_{12}V_1 + (C_{20} + C_{12} + C_{23})V_2 - C_{23}V_3,$$
$$Q_3 = -C_{13}V_1 - C_{23}V_2 + (C_{30} + C_{13} + C_{23})V_3.$$

Compare with

$$\boxed{\begin{aligned} Q_1 &= c_{11}V_1 + c_{12}V_2 + c_{13}V_3, \\ Q_2 &= c_{12}V_1 + c_{22}V_2 + c_{23}V_3, \\ Q_3 &= c_{13}V_1 + c_{23}V_2 + c_{33}V_3, \end{aligned}}$$

Coefficient of capacitance:
c_{11} is the total capacitance between conductor 1 and all the other conductors connected

$$c_{11} = C_{10} + C_{12} + C_{13},$$
$$c_{22} = C_{20} + C_{12} + C_{23},$$
$$c_{33} = C_{30} + C_{13} + C_{23},$$

Coefficient of inductance:
c_{12} is negative of the C_{12} (mutual partial capacitance)

$$c_{12} = -C_{12},$$
$$c_{23} = -C_{23},$$
$$c_{13} = -C_{13}.$$

$$\Rightarrow$$

$$C_{10} = c_{11} + c_{12} + c_{13},$$
$$C_{20} = c_{22} + c_{12} + c_{23},$$
$$C_{30} = c_{33} + c_{13} + c_{23}.$$

3-10.3 Electrostatic Shielding

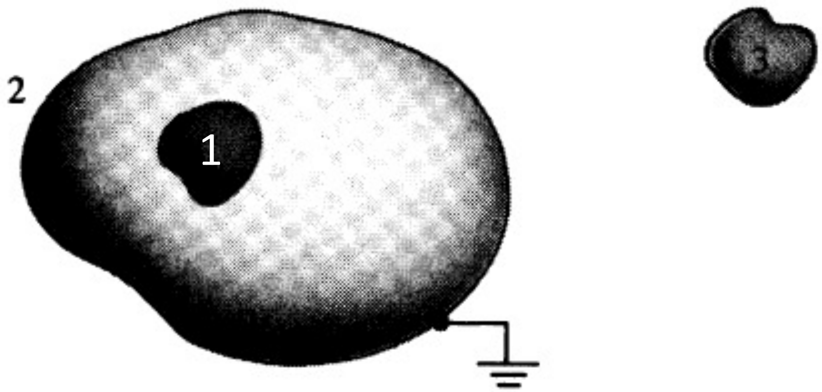


FIGURE 3–37
Illustrating electrostatic shielding.

$$\boxed{\begin{aligned} Q_1 &= C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3), \\ Q_2 &= C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3), \\ Q_3 &= C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2), \end{aligned}}$$

A three-conductor system

Setting $V_2=0$

$\Rightarrow$ $Q_1 = C_{10}V_1 + C_{12}V_1 + C_{13}(V_1 - V_3).$

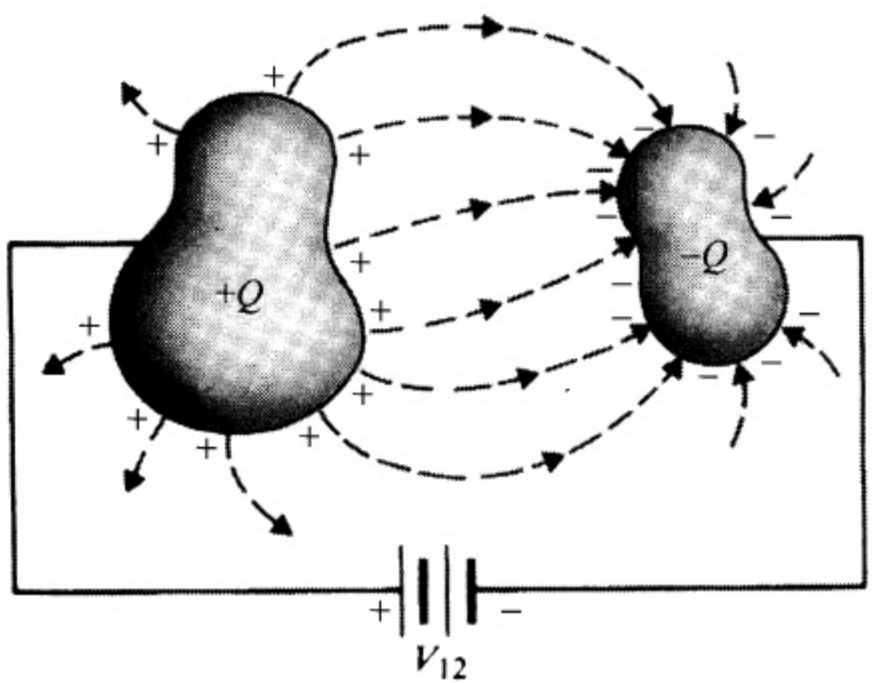
When $Q_1=0$ $\Rightarrow$ **E** inside 2=0 $\Rightarrow$ $V_1=V_2=0$ $\Rightarrow$ $0=-C_{13}V_3$ $\Rightarrow$ $C_{13}=0$

Gauss's law

$$V = -\int \boldsymbol{E} \cdot dl$$

$\Rightarrow$ A change of V_3 will not affect Q_1

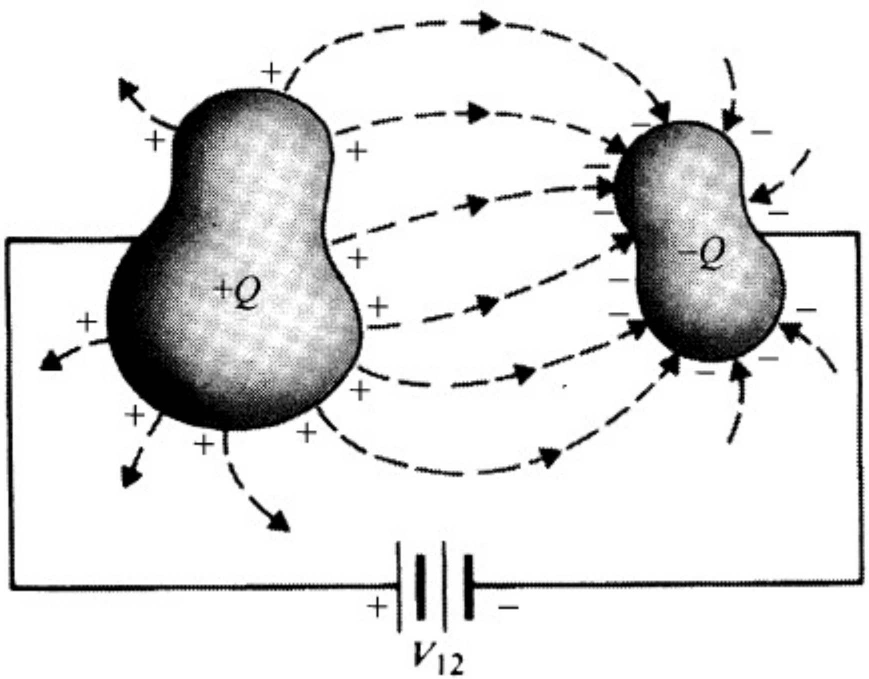
3-11 Electrostatic Energy and Forces



$$V = \frac{Q}{C}$$

FIGURE 3–27
A two-conductor capacitor.

3-11 Electrostatic Energy and Forces



$$W = -\int F \cdot dl$$

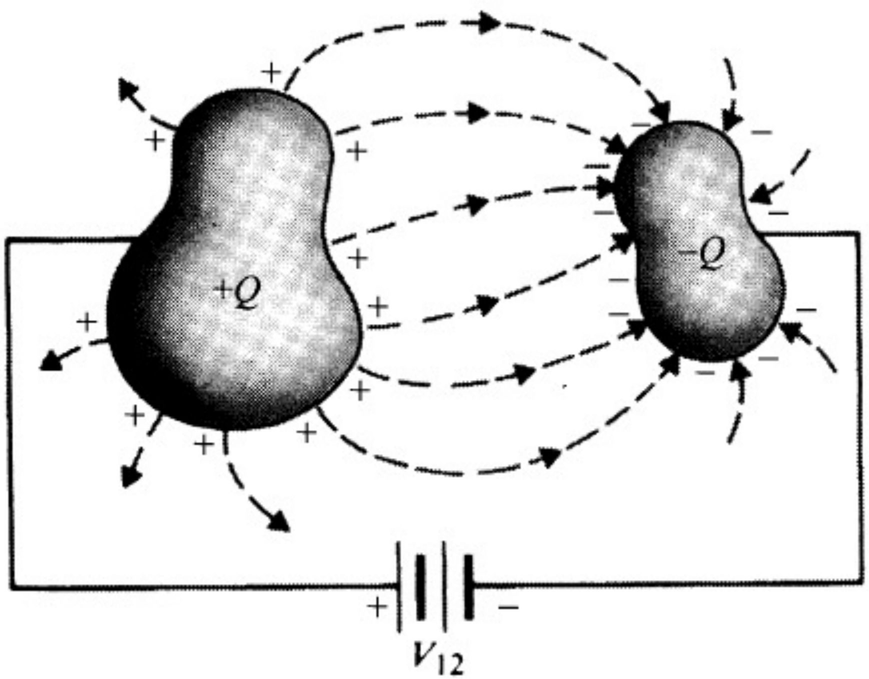
$$F = qE$$

$$E = -\nabla V$$

FIGURE 3–27
A two-conductor capacitor.

3-11 Electrostatic Energy and Forces

$$dW = \frac{Q}{C} dQ$$



$$W = \frac{1}{2}\frac{Q^2}{C}$$

FIGURE 3–27
A two-conductor capacitor.

3-11 Electrostatic Energy and Forces

$$W = -\int F \cdot dl$$

$$F = qE$$

$$E = -\nabla V$$

3-11 Electrostatic Energy and Forces

Work required to bring a charge *q* from P_1 to P_2

$W = qV_{21}$

$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell = V_{21}$$

- A charge Q_1 in free space. Work required to bring a **second** charge Q_2 from infinity to a distance R_{12} (position 2): $W = Q_2 V_{2\infty} = Q_2 V_2$

$$W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}}$$

Against E field of charge Q_1
(V_2 is due to charge Q_1)

Rewrite $$W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}}$$

$$W_2 = Q_1 \frac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1 V_1.$$

$Q_1V_1=Q_2V_2$

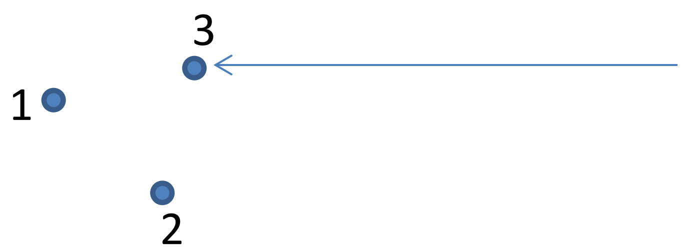
$\Rightarrow$ $Q_1V_1+Q_2V_2= 2Q_1V_1=2W_2$

$$W_2 = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2).$$

- Another charge Q_3. Work required to bring a **third** charge Q_3 from infinity to a distance R_{13} from Q_1 and R_{23} from Q_2: $\Delta W = Q_3 V_{3\infty}$

Against E field of charge Q_1 and E field of charge Q_2
V_3 is due to charges Q_1 and Q_2

$$\Delta W = Q_3 V_3 = Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right).$$



- Total work to assemble the 3 charges Q_1, Q_2, and Q_3: $W_3=W_2+\Delta W$

$$W_3 = W_2 + \Delta W = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1 Q_2}{R_{12}} + \frac{Q_1 Q_3}{R_{13}} + \frac{Q_2 Q_3}{R_{23}}\right).$$

Rewrite: 3 terms divided into 6 terms

$$\begin{aligned} W_3 &= \frac{1}{2}\left[Q_1\underline{\left(\frac{Q_2}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{13}}\right)} + Q_2\left(\frac{Q_1}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{23}}\right)\right. \\ &\quad \left. + Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right)\right] \\ &= \tfrac{1}{2}(Q_1 \underline{V_1} + Q_2 V_2 + Q_3 V_3). \end{aligned}$$

Potential V_1 is caused by charges Q_2 and Q_3

Different from the previous V_1 due to Q_2 only $W_2 = Q_1 \dfrac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1 V_1.$

General expression

Self energy: Work required to assemble the individual point charges

Mutual energy: the interacting energy

Initially, Q_1 in space

Introduce Q_2 $\Delta W = Q_2 V_{2\infty}$

Introduce Q_3 $\Delta W = Q_3 V_{3\infty}$

+

$$W_2 = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2).$$

$$W_3 = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3).$$

.

.

.

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \quad \text{(J)},}$$

Potential V_k is caused by all the other charges

$$V_k = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1\\(j\neq k)}}^{N}\frac{Q_j}{R_{jk}}.$$

3-11.1 Electrostatic Energy in terms of Field Quantities

- For a continuous charge distribution of density ρ

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \qquad \text{(J)},}$$

$$\boxed{W_e = \tfrac{1}{2}\int_{V'} \rho V \, dv \qquad \text{(J).}}$$

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

Find the energy required to assemble a uniform sphere of charge of radius b and volume charge density ρ.

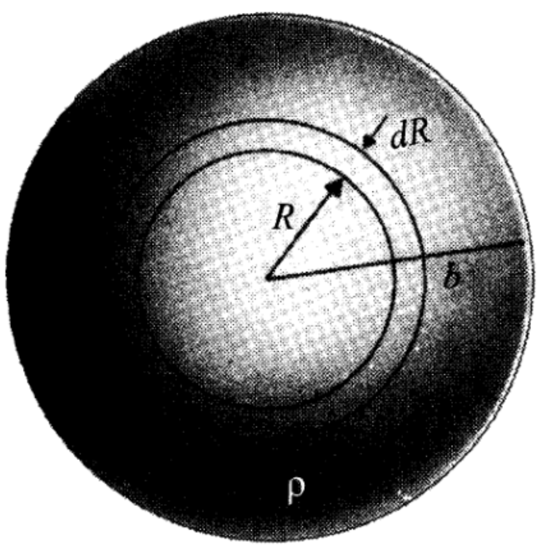


FIGURE 3–38
Assembling a uniform sphere of charge

Quick classroom exercise

assume that the sphere of charge is already in place. Since ρ is a constant, it can be taken out of the integral sign. For a spherically symmetrical problem,

$$W_e = \frac{\rho}{2}\int_{V'} V\,dv = \frac{\rho}{2}\int_0^b V\,4\pi R^2\,dR, \tag{3-171}$$

where V is the potential at a point R from the center. To find V at R, we must find the negative of the line integral of $\mathbf{E}$ in two regions: (1) $\mathbf{E}_1 = \mathbf{a}_R E_{R1}$ from $R = \infty$ to $R = b$, and (2) $\mathbf{E}_2 = \mathbf{a}_R E_{R2}$ from $R = b$ to $R = R$. We have

$$\mathbf{E}_{R1} = \mathbf{a}_R \frac{Q}{4\pi\epsilon_0 R^2} = \mathbf{a}_R \frac{\rho b^3}{3\epsilon_0 R^2}, \qquad R \ge b,$$

and

$$\mathbf{E}_{R2} = \mathbf{a}_R \frac{Q_R}{4\pi\epsilon_0 R^2} = \mathbf{a}_R \frac{\rho R}{3\epsilon_0}, \qquad 0 < R \le b.$$

Consequently, we obtain

$$\begin{aligned} V &= -\int_\infty^R \mathbf{E}\cdot d\mathbf{R} = -\left[\int_\infty^b E_{R1}\,dR + \int_b^R E_{R2}\,dR\right] \\ &= -\left[\int_\infty^b \frac{\rho b^3}{3\epsilon_0 R^2}\,dR + \int_b^R \frac{\rho R}{3\epsilon_0}\,dR\right] \\ &= \frac{\rho}{3\epsilon_0}\left(b^2 + \frac{b^2}{2} - \frac{R^2}{2}\right) = \frac{\rho}{3\epsilon_0}\left(\frac{3}{2}b^2 - \frac{R^2}{2}\right). \end{aligned} \tag{3-172}$$

Substituting Eq. (3–172) in Eq. (3–171), we get

$$W_e = \frac{\rho}{2}\int_0^b \frac{\rho}{3\epsilon_0}\left(\frac{3}{2}b^2 - \frac{R^2}{2}\right)4\pi R^2\,dR = \frac{4\pi\rho^2 b^5}{15\epsilon_0}$$

3-11.1 Electrostatic Energy in terms of Field Quantities

- For a continuous charge distribution of density ρ

$$\boxed{W_e = \frac{1}{2}\int_{\underline{V'}} \rho \underline{V}\, dv \qquad \text{(J).}}$$

Volume

Electrical potential

$$W_e = \frac{1}{2}\int_{V'} (\nabla \cdot \mathbf{D}) V\, dv.$$

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \qquad \text{(J),}}$$

$$\nabla \cdot (V\mathbf{D}) = V\nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla V,$$

$$W_e = \frac{1}{2}\int_{V'} \nabla \cdot (V\mathbf{D})\,dv - \frac{1}{2}\int_{V'} \mathbf{D} \cdot \nabla V\,dv$$
$$= \frac{1}{2}\oint_{S'} V\mathbf{D} \cdot \mathbf{a}_n\,ds + \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E}\,dv,$$

- *V'* can be any volume
- Choose its radius R→∞ ➔ 1st term disappears

$$\boxed{W_e = \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E}\,dv \qquad \text{(J).}}$$

$\mathbf{D} = \epsilon\mathbf{E}$ For a linear medium

$$\boxed{W_e = \frac{1}{2}\int_{V'} \epsilon E^2\,dv \qquad \text{(J)}}$$

$$\boxed{W_e = \frac{1}{2}\int_{V'} \frac{D^2}{\epsilon}\,dv \qquad \text{(J).}}$$

Electrostatic Energy Density w_e

$$\boxed{W_e = \frac{1}{2}\int_{V'} \mathbf{D}\cdot\mathbf{E}\,dv \qquad \text{(J)}.}$$

$$\boxed{W_e = \frac{1}{2}\int_{V'} \epsilon E^2\,dv \qquad \text{(J)}}$$

$$\boxed{W_e = \frac{1}{2}\int_{V'} \frac{D^2}{\epsilon}\,dv \qquad \text{(J)}.}$$

$$W_e = \int_{V'} w_e\,dv.$$

$$w_e = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E} \qquad (\text{J/m}^3)$$

$$w_e = \tfrac{1}{2}\epsilon E^2 \qquad (\text{J/m}^3)$$

$$w_e = \frac{D^2}{2\epsilon} \qquad (\text{J/m}^3).$$

Definition of density form is artificial;
Volume integral form can be verified.

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

In Fig. 3–39 a parallel-plate capacitor of area S and separation d is charged to a voltage V. The permittivity of the dielectric is ϵ. Find the stored electrostatic energy.

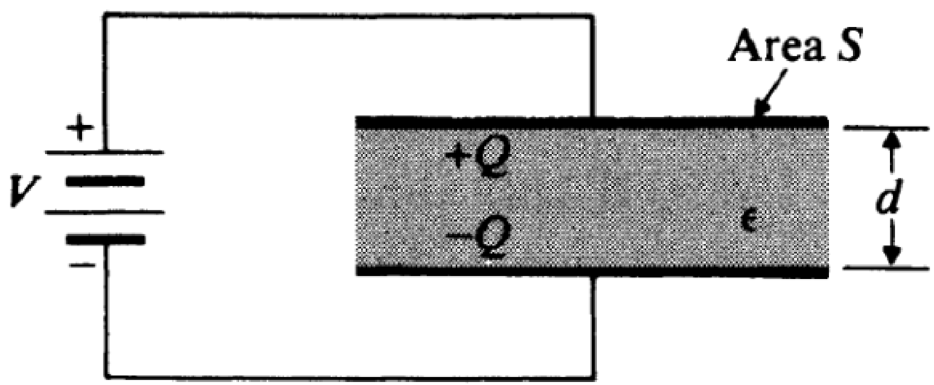


FIGURE 3–39
A charged parallel-plate capacitor

Quick classroom exercise

Solution With the d-c source (batteries) connected as shown, the upper and lower plates are charged positive and negative, respectively. If the fringing of the field at the edges is neglected, the electric field in the dielectric is uniform (over the plate) and constant (across the dielectric) and has a magnitude

$$E = \frac{V}{d}.$$

Using Eq. (3–176b), we have

$$W_e = \frac{1}{2}\int_{V'} \epsilon \left(\frac{V}{d}\right)^2 dv = \frac{1}{2}\epsilon \left(\frac{V}{d}\right)^2 (Sd) = \frac{1}{2}\left(\epsilon \frac{S}{d}\right) V^2. \qquad (3\text{–}179)$$

$$\boxed{W_e = \tfrac{1}{2}\int_{V'} \epsilon E^2\, dv \qquad \text{(J)}} \qquad (3\text{–}176\text{b})$$

3-11.2 Electrostatic Forces

- Using Coulomb's law to determine the force on one body that is caused by the charges on other bodies would be very tedious…can we make use of the known electrostatic energy?
- Thus, a simple method of **principle of virtual displacement** is introduced.
  - System of bodies with fixed charges
  - System of conducting bodies with fixed potentials

System of Bodies with Fixed Charges

- An isolated system consisting of charged conductor and dielectric bodies.
- Condition: Charges are constant.
- Electric force displaces one of the bodies by dl (a virtual displacement)
  - Mechanical **work** done **by the system**:

$$dW = \mathbf{F}_Q \cdot d\ell,$$

$\mathbf{F}_Q$: total electric force acting on the body

– In other words, **reduced stored electrostatic energy** produces the mechanical **work**

$$dW = \underline{-dW_e} = \mathbf{F}_Q \cdot d\ell.$$

Reduced stored electrostatic energy

$$dW_e = (\nabla W_e) \cdot d\ell$$

$$\boxed{\mathbf{F}_Q = -\nabla W_e \qquad \text{(N)}.}$$

A very simple formula for the calculation of $\mathbf{F}_Q$ from the electrostatic energy of the system

System of Conducting Bodies with Fixed Potentials

- Condition: potentials are fixed.
- System connected to **external sources** to maintain fixed potentials
- A displacement dl ➔ dW_e, dQ_k to maintain fixed potentials V_k
  - 1. Work done by the external sources:

$$dW_s = \sum_k V_k dQ_k$$

  - 2. Produced mechanical work:

$$dW = \mathbf{F}_V \cdot d\ell$$

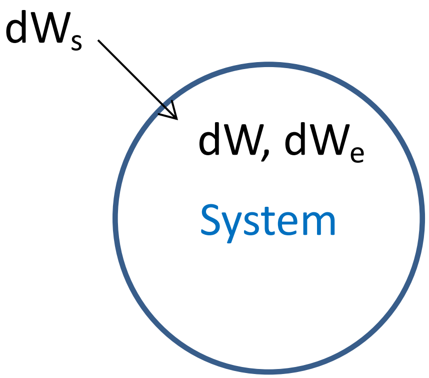
$\mathbf{F}_V$: electric force acting on the body

– 3. Change of electrostatic energy due to dQ_k:

$$dW_e = \frac{1}{2}\sum_k V_k \, dQ_k = \frac{1}{2} dW_s$$

- Thus, $dW + dW_e = dW_s.$

$$\boxed{dW_e = \frac{1}{2}\sum_k V_k \, dQ_k = \frac{1}{2} dW_s}$$



$dW = dW_e$

$$\mathbf{F}_V \cdot d\ell = dW_e = (\nabla W_e) \cdot d\ell$$

$$\boxed{\mathbf{F}_V = \nabla W_e \qquad (\mathrm{N}).}$$