# Chapter 2 – Kinematics in 1D

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Physics I (Summer 2019)
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# Agenda

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3. Velocity vs Speed
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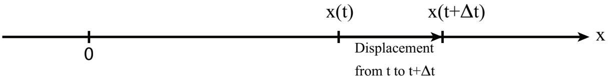
# Basic Kinematic Quantities in Motion along a Straight Line

# Position. Average Velocity

**Kinematics** – describes motion quantitatively, does not discuss the cause of motion.

In 1D all vectors have just a single component; choice of the positive direction is arbitrary. Hence, the **position** in 1D is uniquely determined by a single number $x$ (can use scalars).




In general, the position changes with time $x = x(t)$, and we can characterize these changes by, e.g., the **average velocity** over a time interval $(t, t + \Delta t)$

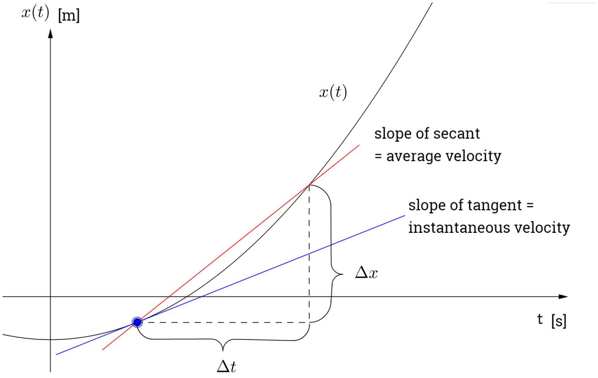
$$v_{\mathrm{av},x} \stackrel{def}{=} \frac{x(t + \Delta t) - x(t)}{\Delta t} \qquad (units\ [m/s])$$

# Instantaneous Velocity

If we let $\Delta t \to 0$, then

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{\mathrm{d}x(t)}{\mathrm{d}t} = \boxed{\dot{x}(t) \stackrel{def}{=} v_x(t)} \qquad (units\ [m/s])$$

which is called the **instantaneous velocity**.



# Average and Instantaneous Acceleration

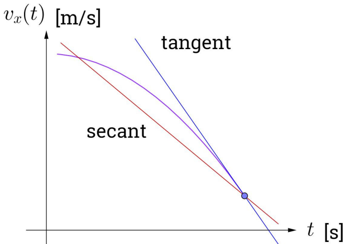
Analogously, we can define the average and the instantaneous acceleration, quantifying changes of the velocity with time.

The **average acceleration**, over a time interval $(t, t + \Delta t)$, is defined as

$$a_{\text{av,x}} \overset{def}{=} \frac{v_x(t + \Delta t) - v_x(t)}{\Delta t} \qquad (units\ [m/s^2]),$$

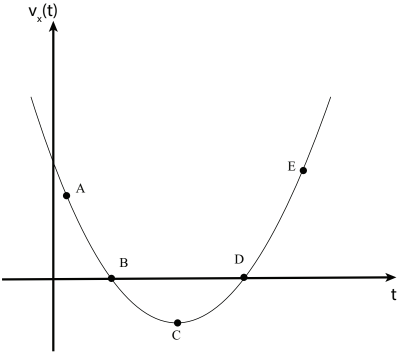
while the **instantaneous acceleration** (having the same units) is defined as

$$\boxed{a_x(t)} \overset{def}{=} \lim_{\Delta t \to 0} \frac{v_x(t + \Delta t) - v_x(t)}{\Delta t} = \frac{\mathrm{d}v_x(t)}{\mathrm{d}t} = \boxed{\dot{v}_x(t)} = \frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} = \ddot{x}(t)$$



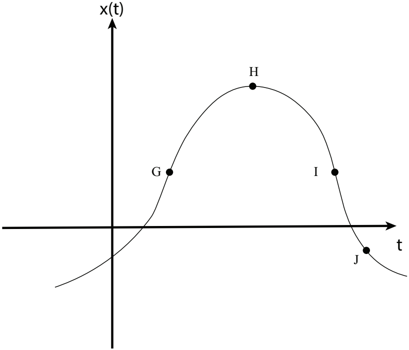
# Analysis of $v_x(t)$ and $x(t)$ Graphs

# Example. Analysis of a $v_x(t)$ Graph




- $A: \; v_x > 0, \; a_x < 0$ moves to the right, slowing down
- $B: \; v_x = 0, \; a_x < 0$ stops; about to start moving to the left
- $C: \; v_x < 0, \; a_x = 0$ moves to the left; no instant. acceleration
- $D: \; v_x = 0, \; a_x > 0$ stops; about to start moving to the right
- $E: \; v_x > 0, \; a_x > 0$ moves to the right; speeding up

# Example. Analysis of a $x(t)$ Graph




- $G$ : $v_x > 0$, $a_x = 0$ (inflection point)
- $H$ : $v_x = 0$, $a_x < 0$
- $I$ : $v_x < 0$, $a_x = 0$ (inflection point)
- $J$ : $v_x < 0$, $a_x > 0$

# Velocity vs Speed

# Velocity vs Speed

Note that, the instantaneous speed is just the magnitude of instantaneous velocity.

However, the average speed <u>is not</u> equal to the magnitude of the average velocity.

These two quantities differ in that

$$\begin{aligned}\text{average speed} &= \frac{\text{distance traveled}}{\text{time interval}} \\ \text{average velocity} &= \frac{\text{displacement}}{\text{time interval}}\end{aligned}$$

# Example: Free Fall

# General Problem and Solution Strategy

General remarks on mutual relations between $x,\ v_x$ and $a_x$

$$
\begin{array}{rrrr}
\text{position} & & & x(t) \\
\text{instantaneous velocity} & & v_x(t) = & \dot{x}(t) \\
\text{instantaneous acceleration} & a_x(t) = & \dot{v}_x(t) = & \ddot{x}(t)
\end{array}
$$

| | |
|---|---|
| $x \to v_x \to a_x$ | differentiate (easy) |
| $a_x \to v_x \to x$ | ? |

<u>Question</u>: Given acceleration, how to find velocity and position?

# General Problem and Strategy

...integrate!

$$a_x(t) = \frac{\mathrm{d}v_x}{\mathrm{d}t} \quad \Rightarrow \quad \mathrm{d}v_x = a_x(t)\,\mathrm{d}t \quad \Rightarrow \quad \int \mathrm{d}v_x = \int a_x(t)\,\mathrm{d}t$$

Finally we have

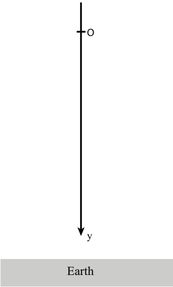
$$v_x(t) = \int a_x(t)\,\mathrm{d}t \qquad \text{(determined up to an additive constant)}$$

Similarly,

$$x(t) = \int v_x(t)\,\mathrm{d}t \qquad \text{(determined up to an additive constant)}$$

The additive constants in the two formulas above are found from *initial conditions*.

# Free Fall Close to Earth's Surface




Let us analyze Galileo's experiment, that is motion of a free falling object close to the Earth's surface, neglecting air drag.

The acceleration is constant

$$a_y(t) = g = \text{const}$$

Hence, the velocity,

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = g \quad \Rightarrow \quad \mathrm{d}v_y = g\,\mathrm{d}t \quad \Rightarrow \quad \int \mathrm{d}v_y = \int g\,\mathrm{d}t$$

Eventually

$$v_y(t) = gt + C_1 \qquad (C_1 \text{ is found from an initial condition})$$

# Free Fall Close to Earth's Surface

Combining the result with the initial condition $v_y(t_0) = v_{0y}$, we have

$$v_y(t_0) = gt_0 + C_1 \quad \Rightarrow \quad C_1 = v_{0y} - gt_0$$

and, eventually,

$$v_y(t) = g(t - t_0) + v_{0y}.$$

Similarly, for the position,

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}t} &= g(t - t_0) + v_{0y} \\
\int \mathrm{d}y &= \int [g(t - t_0) + v_{0y}]\,\mathrm{d}t \\
y(t) &= \frac{1}{2}gt^2 - gt_0 t + v_{0y} t + C_2 = \frac{1}{2}gt^2 + (v_{0y} - gt_0)t + C_2
\end{aligned}$$

# Free Fall Close to Earth's Surface

To determine the constant $C_2$, another initial condition is needed: $y(t_0) = y_0$. Then

$$y(t_0) = \frac{1}{2}gt_0^2 + (v_{0y} - gt_0)t_0 + C_2 \quad \Rightarrow C_2 = \frac{1}{2}gt_0^2 - v_{0y}t_0 + y_0$$

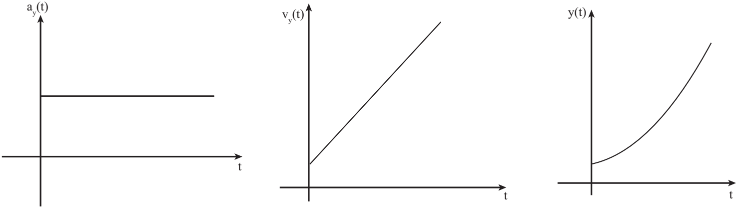
Eventually

$$y(t) = \frac{1}{2}g(t^2 + t_0^2) + v_{0y}(t - t_0) - gt_0 t + y_0$$

# Free Fall Close to Earth's Surface

If we choose $t_0 = 0$ (the usual choice)

$$v_y(t) = gt + v_{0y}, \qquad y(t) = \frac{1}{2}gt^2 + v_{0y}t + y_0$$




*Summary*

$$\begin{aligned} a_y(t) &= g = \text{const} \\ v_y(t) &= gt + v_{0y} \\ y(t) &= \frac{1}{2}gt^2 + v_{0y}t + y_0 \end{aligned}$$

# Final Remarks

(1) Using definite integrals takes care of initial conditions "automatically"

$$\int_{v_{0y}}^{v_y(t)} \mathrm{d}v_y = \int_{t_0}^{t} a_y(t)\,\mathrm{d}t \quad \Rightarrow \quad v_y(t) - v_{0y} = \int_{t_0}^{t} a_y(t)\,\mathrm{d}t$$

$$\int_{y_0}^{y(t)} \mathrm{d}y = \int_{t_0}^{t} v_y(t)\mathrm{d}t \quad \Rightarrow \quad y(t) - y_0 = \int_{t_0}^{t} v_y(t)\,\mathrm{d}t$$

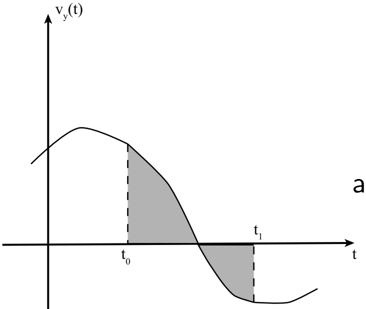
# Final Remarks

(2) The net displacement between the instants $t_0$ and $t_1$ is

$$\int_{t_0}^{t_1} v_y(t)\,\mathrm{d}t = y(t_1) - y(t_0)$$

and the corresponding distance travelled between $t_0$ and $t_1$ is the total area of the shaded region

$$\text{distance} = \int_{t_0}^{t_1} |v_y(t)|\,\mathrm{d}t$$




$$\text{average speed} = \frac{\int_{t_0}^{t_1} |v_y(t)|\,\mathrm{d}t}{t_1 - t_0}$$

$$\text{average velocity} = \frac{\int_{t_0}^{t_1} v_y(t)\,\mathrm{d}t}{t_1 - t_0}$$

# Example: Motion with Varying Acceleration

# Example. Motion with Varying Acceleration

The acceleration of an iron object moving in a straight line close to a magnet turns out to be related to its position with respect to the magnet, $x > 0$, as $a_x = -k/x$, where $k$ is a positive constant. It has been experimentally determined that the velocity of the object is $\sqrt{2}v_0 > 0$, when the object is at $x = x_0 > 0$ and it is $v_0$ when $x = 2x_0$.
What is the velocity of this particle when $x = 3x_0$?

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Note that $a_x = \frac{dv_x}{dt} \overset{\text{chain rule}}{=} \frac{dv_x}{dx} \underbrace{\frac{dx}{dt}}_{v_x} = \frac{dv_x}{dx} v_x$.

Hence $-\frac{k}{x} = v_x \frac{dv_x}{dx}$. Integrating and using the information provided

$$\int_{\sqrt{2}v_0}^{v_0} v_x \, dv_x = -\int_{x_0}^{2x_0} \frac{k}{x} \, dx \qquad \Longrightarrow \qquad \frac{1}{2} v_0^2 - v_0^2 = -k \ln 2,$$

so that $\boxed{k = \frac{v_0^2}{2 \ln 2}.}$

# Example. Motion with Varying Acceleration (contd)

To find out the answer, integrate with the corresponding limits

$$\int_{\sqrt{2}v_0}^{v_x(3x_0)} v_x \, dv_x = -\int_{x_0}^{3x_0} \frac{k}{x} \, dx$$

to get

$$\frac{1}{2} v_x^2(3x_0) - v_0^2 = -k \ln 3 = -\frac{v_0^2}{2} \frac{\ln 3}{\ln 2}.$$

Solving for $v_x(3x_0)$ yields

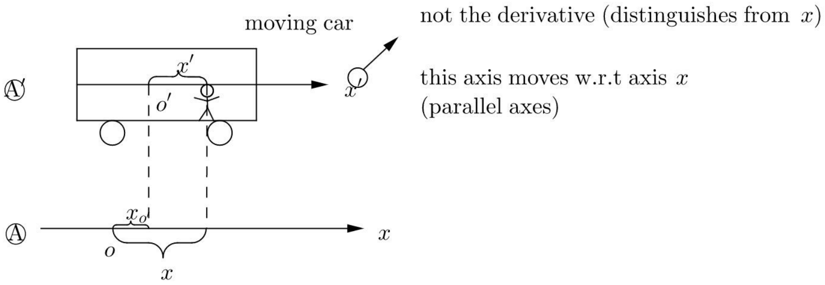
$$\boxed{v_x(3x_0) = v_0 \sqrt{2 - \frac{\ln 3}{\ln 2}}.}$$

# Galilean Transformation and Velocity Addition Rule

# Frames of Reference

In 1D kinematics, the same particle may have different positions, velocities or accelerations, depending on the choice of the coordinate axis. An axis with respect to which we describe motion is called a <u>frame of reference</u> (abbreviated as *FoR*).

*Example*. Student in a moving train car: $A$ is the *FoR* associated with the train station's platform ground; $A'$ is the FoR of the train car itself.



# Galilean Transformation

The student's position in *FoR A* is

$$x = x_{O'} + x'$$

where $x_{O'}$ is the position of the origin of $A'$ in *FoR A*, and $x'$ is the position of the student in FoR $A'$.

Differentiating the equation above with respect to time,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}x_{O'}}{\mathrm{d}t} + \frac{\mathrm{d}x'}{\mathrm{d}t} \qquad \Longrightarrow \qquad \boxed{v_x = v_{O'x} + v'_x}$$

which is the well-known velocity addition rule.

Consequently, for accelerations, $a_x = a_{O'x} + a'_x$.

# Galilean Transformation

In the special case when $O' = O$ at $t = 0$, and $v_{O'x} = \text{const}$, we have $a_x = a_x$ and the relation between student's positions in $A$ and $A'$ is

$$\boxed{x = v_{O'x}t + x'.}$$

The boxed formula is known as the **Galilean transformation**.

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There is an implicit assumption in the derivation of the Galilean transformation.

**What did we assume**?