VE230 Midterm2 Review

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Overview

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- Boundary Value Problems
- Steady Electric Currents

Boundary Value Problems

Poisson's Equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon}.$$

In Cartesian System:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}$$

In Polar System:

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}$$

Laplace Equation:

$$\nabla^2 V = 0.$$

Uniqueness Theorem

Definition

A solution of Poisson's equation that satisfies the given boundary conditions is a unique solution.

This actually means that the solution you find to solve the boundary value problem will be unique.

Solution of Laplace Equations

There are three kinds of solutions for the equations: (discuss only about 2D situation)

- $X = Ae^{-k_1x} + Be^{k_1x}, \quad Y = Ce^{-k_2x} + De^{k_2x}$
- $X = A\sin(k_1x) + B\cos(k_1x), \quad Y = C\sin(k_2x) + D\cos(k_2x)$
- $X = a + bx, \quad Y = c + dx$

You can use separation of variable to find the three forms of the solution

How to solve???

1. First, write out the expansion of the Laplace Equation, i.e.,

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

2. According to the boundary conditions given, judge whether V is independent of the variable.
3. If there is only one variable left, then the solution is the third format, i.e., $X = a + bx, \quad Y = c + dx$. Then use the boundary conditions to find the parameter of the solution.
4. If there is more than one variable involved, then tried the first format and second format to find if they can satisfy the boundary conditions. If one of the formats can be found that satisfies the boundary conditions, then stop, because of the uniqueness theorem.

Example

P.4–23 Two infinite insulated conducting planes maintained at potentials 0 and V_0 form a wedge-shaped configuration, as shown in Fig. 4–24. Determine the potential distributions for the regions: (a) $0 < \phi < \alpha$, and (b) $\alpha < \phi < 2\pi$.

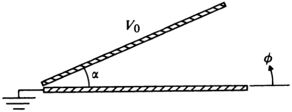

FIGURE 4–24
Two infinite insulated conducting planes maintained at constant potentials (Problem P.4–23).

Example

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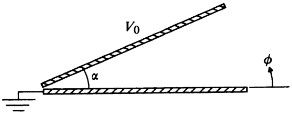


FIGURE 4–24
Two infinite insulated conducting planes maintained at constant potentials (Problem P.4–23).

P.4-23 Solution: $V(\phi) = A_0\phi + B_0$.

a) B.C. ①: $V(0) = 0 \longrightarrow B_0 = 0.$

B.C. ②: $V(\alpha) = V_0 = A_0\alpha \longrightarrow A_0 = \frac{V_0}{\alpha}.$

$\therefore V(\phi) = \frac{V_0}{\alpha}\phi, \quad 0 \le \phi \le \alpha.$

b) B.C. ①: $V(\alpha) = V_0 = A_1\alpha + B_1$

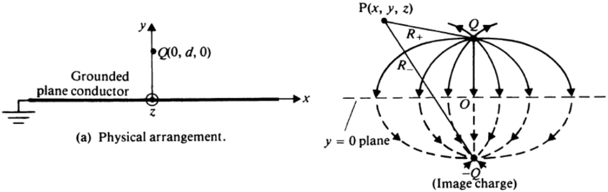
B.C. ②: $V(2\pi) = 0 = 2\pi A_1 + B$

$\longrightarrow A_1 = -\frac{V_0}{2\pi - \alpha}, \quad B_1 = \frac{2\pi V_0}{2\pi - \alpha}.$

$\therefore V(\phi) = \frac{V_0}{2\pi - \alpha}(2\pi - \phi), \quad \alpha \le \phi \le 2\pi.$

Method of Images

Method of Images is a special method to solve some special boundary value problems.



(a) Physical arrangement.

(b) Image charge and field lines.

How to use method of images

- Solved by Laplace eq.:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

Hold for y>0 except at the point charge

- **4 Conditions** should be satisfied:

1. $V(x, 0, z) = 0.$
2. for points very close to Q

$$V \to \frac{Q}{4\pi\epsilon_0 R}, \text{ as } R \to 0,$$

3. V→0 for points very far from Q

$$x \to \pm\infty, y \to +\infty, \text{ or } z \to \pm\infty$$

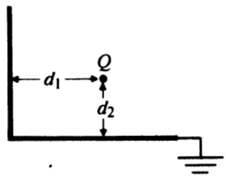
4. Even functions w.r.t. x and z coordinates

$$V(x, y, z) = V(-x, y, z) \qquad V(x, y, z) = V(x, y, -z).$$

Difficult to solve...

Quick Example

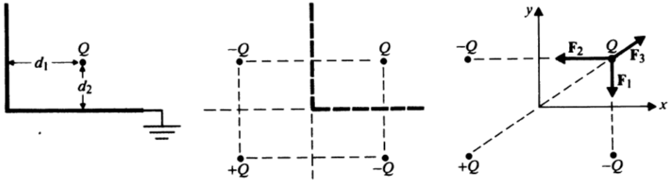
EXAMPLE 4–3 A positive point charge Q is located at distances d_1 and d_2, respectively, from two grounded perpendicular conducting half-planes, as shown in Fig. 4–4(a). Determine the force on Q caused by the charges induced on the planes.



Quick Example

Solution A formal solution of Poisson's equation, subject to the zero-potential boundary condition at the conducting half-planes, would be quite difficult. Now an image charge $-Q$ in the fourth quadrant would make the potential of the horizontal half-plane (but not that of the vertical half-plane) zero. Similarly, an image charge $-Q$ in the second quadrant would make the potential of the vertical half-plane (but not that of the horizontal plane) zero. But if a third image charge $+Q$ is added in the third quadrant, we see from symmetry that the image-charge arrangement in Fig. 4–4(b) satisfies the zero-potential boundary condition on both half-planes and is electrically equivalent to the physical arrangement in Fig. 4–4(*a*).

Negative surface charges will be induced on the half-planes, but their effect on Q can be determined from that of the three image charges. Referring to Fig. 4–4(c),



(a) Physical arrangement.

(b) Equivalent image-charge arrangement.

(c) Forces on charge Q.

Types of electric currents caused by the motion of free charges:

1. **conduction currents**: drift motion of conduction electrons and/or holes in conductors/semiconductors.
2. electrolytic currents: migration of positive and negative ions.
3. convenction currents: motion of electrons and/or ions in a vacuum.

Current Density and Ohm's Law

$$I = \int_S \vec{J} \cdot d\vec{s} \quad (A)$$

where $\vec{J}$ is the volume current density or current density, defined by

$$\vec{J} = Nq\vec{u} \quad (A/m^2)$$

,
where N is the number of charge carriers per unit volume, each of charges q moves with a velocity $\vec{u}$.
Since Nq is the free charge per unit volume, by $\rho = Nq$, we have:

$$\vec{J} = \rho\vec{u} \quad (A/m^2)$$

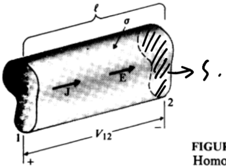
Current Density and Ohm's Law

For conduction currents,

$$\vec{J} = \sigma \vec{E} \quad (A/m^2)$$

where $\sigma = \rho_e \mu_e$ is conductivity, a macroscopic constitutive parameter of the medium. $\rho_e = -Ne$ is the charge density of the drifting electrons and is negative. $\vec{u} = -\mu_e \vec{E} \quad (m/s)$ where μ_e is the electron mobility measured in $(m^2/V \cdot s)$.
Materials where $\vec{J} = \sigma \vec{E} \quad (A/m^2)$ holds are called ohmic media.
The form can be referred as the point form of Ohm's law.
Derivation of voltage-current relationship of a piece of homogeneous material by the point form of Ohm's law.

Current Density and Ohm's Law



σ, S

FIGURE 5–3
Homogeneous conductor with a constant cross section.

Thus, the resistance is defined as

$$R = \frac{l}{\sigma S} \quad (\Omega)$$

where l is the length of the homogeneous conductor, S is the area of the uniform cross section.

Current Density and Ohm's Law

The conductance G (reciprocal of resistance), is defined by

$$G = \frac{1}{R} = \sigma \frac{S}{l} \quad (S)$$

1. Resistance in series:

$$R_{sr} = R_1 + R_2$$

2. Resistance in parallel:

$$\frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2}$$

, or

$$G_{||} = G_1 + G_2$$

Electromotive Force and Kirchhoff's Voltage Law

A steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field, which is:

$$\oint_C \frac{1}{\sigma}\vec{J}\cdot d\vec{l} = 0$$

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistance, which is:

$$\sum_j V_j = \sum_k R_k I_k \quad (V)$$

Equation of Continuity and Kirchhoff's Current Law

Equation of continuity:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \quad (A/m^3)$$

,
where ρ is the volume charge density.
For steady currents, as $\partial \rho / \partial t = 0$, $\nabla \cdot \vec{J} = 0$. By integral, we have Kirchhoff's current law, stating that the algebraic sum of all the currents flowing out of a junction in an electric circuit is zero:

$$\sum_j I_j = 0$$

Equation of Continuity and Kirchhoff's Current Law

For a simple medium conductor, the volume charge density ρ can be expressed as:

$$\rho = \rho_0 e^{-(\rho/\epsilon)t} \quad (C/m^3)$$

where ρ_0 is the initial charge density at $t = 0$. The equation implies that the charge density at a given location will decrease with time exponentially.
Relaxation time: an initial charge density ρ_0 will decay to $1/e$ or 36.8% of its original value:

$$\tau = \frac{\epsilon}{\sigma} \quad (s)$$

Power Dissipation and Joule's Law

For a given volume V that the total electric power converted to heat is:

$$P = \int_V \mathbf{E} \cdot \mathbf{J} dv.$$

$$P = \int_L E dl \int_S J ds = VI = I^2 R$$