Vv156 Lecture 19

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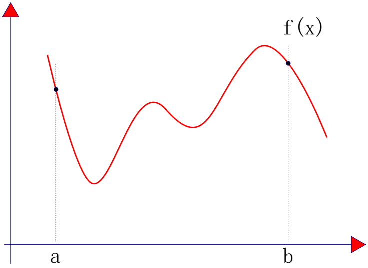
Q: What does the length of a curve represent intuitively?



Q: How can we mathematically define the length of a curve

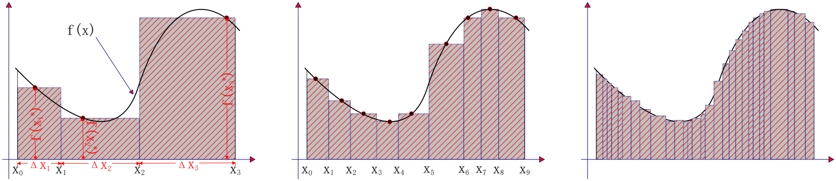
$$y = f(x)$$

over an interval $[a, b]$?

- The length of a curve is also known as the arc length.

- Recall how we mathematically define area under a continuous curve.

1. Divide the region into strips,



2. Approximate the area of each strip by the area of a rectangle,

$$\text{Strip} \approx \text{Rectangle} = \text{Height} \times \text{Width}$$

3. Add the approximations to form a Riemann sum,

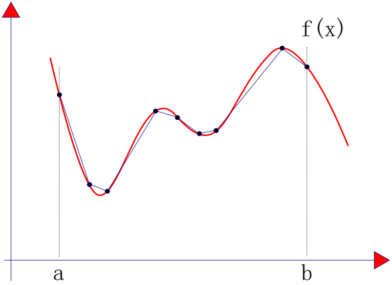
$$\sum f(x_k^*)\Delta x_k \qquad \text{or} \qquad \sum g(y_k^*)\Delta y_k$$

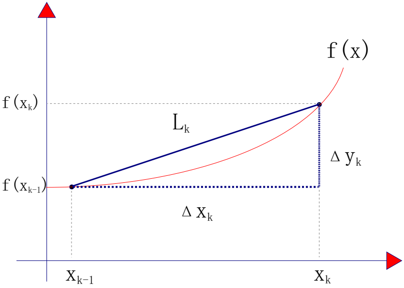
4. Take the limit of the Riemann sum to find the area.

$$\int_a^b \Big(\text{Height}\Big)\, dx \qquad \text{or} \qquad \int_c^d \Big(\text{Width}\Big)\, dy$$

- To define the arc length of a smooth curve

1. Divide the curve into small segments





2. Approximate the curve segments by line segments

$$\text{Short Curve} \approx \text{Short Line} = L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

3. Add the approximations to form a Riemann sum $L \approx \sum L_k$.

4. Take the limit of the Riemann sum to find the length, hopefully, $\sum L_k \to L$.

1. Let $y = f(x)$ be a continuously differentiable on the interval $[a, b]$,

$$L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} = \sqrt{(\Delta x_k)^2 + \Big[f(x_k) - f(x_{k-1})\Big]^2}$$

2. Apply Mean-Value Theorem

$$\frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} = f'(x_k^*) \implies f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$$

3. The length L can be approximated by the following Riemann sum

$$L \approx \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + \Big[f'(x_k^*)\Delta x_k\Big]^2} = \sum_{k=1}^{n} \sqrt{1 + \Big[f'(x_k^*)\Big]^2}\Delta x_k$$

4. In the limit, the corresponding Riemann integral gives the exact value for L

$$L = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{1 + \Big[f'(x_k^*)\Big]^2}\Delta x_k = \int_a^b \sqrt{1 + \Big[f'(x)\Big]^2}\,dx$$

Definition

If $f(x)$ is continuously differentiable on the interval $[a, b]$, then the arc length L of the curve $y = f(x)$ from $A = (a, f(a))$ to the point $B = (b, f(b))$ is

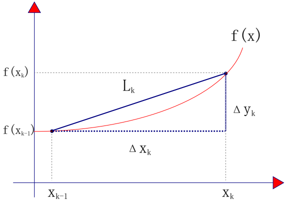
$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Exercise

Find the arc length of the curve

$$y = \frac{4\sqrt{2}}{3} x^{3/2} - 1 \qquad \text{for} \qquad 0 \le x \le 1$$

Q: Why not using Δx_k instead of L_k?



- Note that the definition is for a continuously differentiable $y = f(x)$.
- At a point on a curve where $\dfrac{dy}{dx}$ fails to exist, $\dfrac{dx}{dy}$ may exist. For example,

$$y = f(x) = \left(\frac{x}{2}\right)^{2/3} \implies \frac{dy}{dx} = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

  which is not defined at $(0, 0)$, so $y = f(x)$ is not continuously differentiable.
- However, x in terms of y, $x = g(y)$ is continuously differentiable.

$$x = 2y^{3/2} \implies \frac{dx}{dy} = 3y^{1/2}$$

- Notice that $y = f(x)$ and $x = g(y)$ represent the same curve, thus must have the same length between some points A and B.
- In those cases, we may be able to find the curve's length by expressing x as a function of y and partitioning y to have the following alternative definition of arc length for a given curve.

Definition

If $g(y)$ is continuously differentiable on the interval $[c, d]$, then the arc length L of this curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Exercise

Find the length of the curve

$$y = \left(\frac{x}{2}\right)^{2/3}$$

from $x = 0$ to $x = 2$.

- The arc length formulae often lead to an integrand for which we do not know an antiderivative and so cannot apply the Fundamental Theorem of Calculus.
- In those situations, the definition is still valid, but the evaluation of the definite integral must be done using some numerical methods.

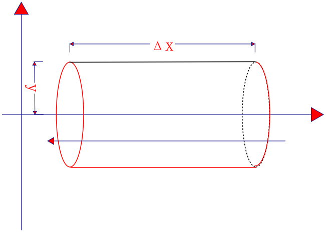
- If you revolve a region in the plane that is bounded by the graph of a function over an interval, it sweeps out a solid of revolution.

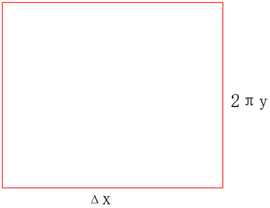
Q: What will you create if you revolve only the bounding curve itself?

Surface that surrounds the solid

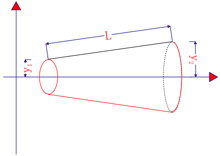
- Before considering general curves, recall if we rotate the horizontal segment AB of length Δx about the x-axis, then the surface generated has an area of

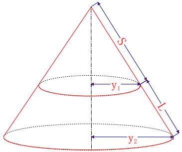
$$2\pi y \Delta x$$

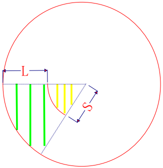




- Suppose the segment AB has length L and is slanted rather than horizontal,







$$\frac{L+S}{y_2} = \frac{L}{y_2 - y_1} \implies L+S = \frac{Ly_2}{y_2 - y_1} \implies S = \frac{Ly_2}{y_2 - y_1} - L = \frac{Ly_1}{y_2 - y_1}$$

$$\text{Recall the area of a sector of circle} = \frac{\text{Arc length}}{2\pi r}\pi r^2 = \frac{1}{2}\text{Arc length} r$$

$$\implies A = \frac{1}{2}2\pi y_2 \frac{Ly_2}{y_2 - y_1} - \frac{1}{2}2\pi y_1 \frac{Ly_1}{y_2 - y_1} = 2\pi \frac{y_1 + y_2}{2} L$$

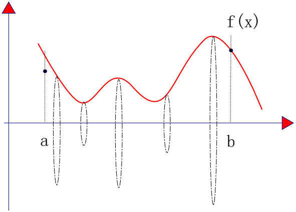
Q: What is A representing?

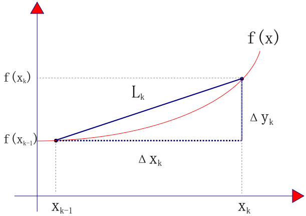
- Suppose we want to find the area of the surface, A.K.A surface area, created by revolving about the x-axis the graph of a nonnegative smooth function

$$y = f(x), \qquad a \le x \le b,$$

- We approach this as usual,

1. Divide the curve into small curve segments.





2. Approximate the area using a line segment instead of the curve segment.
3. Add the approximations to form a Riemann sum.
4. Take the limit of the Riemann sum in the hope that it exists.

1. Suppose that $y = f(x)$ is a smooth curve on the interval $[a, b]$.

$$S_k = 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \sqrt{(\Delta x_k)^2 + [f(x_k) - f(x_{k-1})]^2}$$

2. Apply the Mean-Value Theorem

$$f(x_k) - f(x_{k-1}) = f'(x_k^*)\Delta x_k$$

3. The area S can be approximated by the following sum

$$S \approx \sum_{k=1}^{n} \pi[f(x_{k-1}) + f(x_k)]\sqrt{(\Delta x_k)^2 + [f'(x_k^*)\Delta x_k]^2}$$

4. Apply the Intermediate-Value Theorem

$$\frac{1}{2}[f(x_k) + f(x_{k-1})] = f(x_k^{**})$$

5. Hence the corresponding Riemann integral gives the exact value for S

$$S = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 2\pi f(x_k^{**})\sqrt{1 + [f'(x_k^*)]^2}\Delta x_k$$

Definition

Suppose that $y = f(x)$ is a nonnegative smooth curve on the interval $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve about the x-axis is defined as,

$$S = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\,dx = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

Moreover, if $x = g(y)$ is a nonnegative smooth curve on the interval $[c, d]$, then the surface area S of the surface of revolution that is generated by revolving the portion of a curve about the y-axis can be expressed as

$$S = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\,dy = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

Exercise

Find the surface area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.