JOINT INSTITUTE 交大密西根学院

vv214_SU2020_Assignment 2 **due to June 5, 2020**

Problem 1

1. The classical form of Hölder's inequality states that if $p > 1$ and $q > 1$ are real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ and $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \mathbb{R}^n$, then

$$\sum_{i=1}^{n} |x_i y_i| \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} |y_i|^q\right)^{\frac{1}{q}}$$

Derive this inequality by executing the following steps:

a. By considering the function $f(t) = (1-\lambda) + \lambda t - t^{\lambda}$ for $0 < \lambda < 1$, establish the inequality

$$\alpha^{\lambda}\beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta, \qquad \alpha, \beta \in \mathbb{R}_+ \cup \{0\} \qquad \textit{Young's inequality}$$

Hint: find the intervals where the function increases and decreases, and set t.

b. Let $\hat{x} = \frac{x}{\|x\|}, \hat{y} = \frac{y}{\|y\|}$. Apply the inequality of part **a.** to obtain

$$\sum_{i=1}^{n} |\hat{x}_i \hat{y}_i| \le \frac{1}{p}\sum_{i=1}^{n} |\hat{x}_i|^p + \frac{1}{q}\sum_{i=1}^{n} |\hat{y}_i|^q$$

c. Deduce the classical form of Hölder's inequality by the suitable choice of $\hat{x}, \hat{y}$.

2. The triangle inequality

$$\|x+y\|_p \le \|x\|_p + \|y\|_p$$

for a general p −norm is the classical Minkowski inequality, which states that for $p \ge 1$

$$\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \le \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}$$

Derive Minkowski's inequality.

Hint: For $p > 1$, let q be the number such that $\frac{1}{q} = 1 - \frac{1}{p}$. Verify that for scalars α and β,

$$|\alpha+\beta|^p = |\alpha+\beta||\alpha+\beta|^{\frac{p}{q}} \le |\alpha||\alpha+\beta|^{\frac{p}{q}} + |\beta||\alpha+\beta|^{\frac{p}{q}}$$

and make use of Hölder's inequality.

3. Find $\|x\|_p, p = 1,2$ and $\|x\|_\infty$ for $x = (2, 1, -4, 2)$ and $y = (1+i,\ 1-i,\ 1,\ 4i)$

4. Sketch the unit circle $\|x - 0\| = 1,\ x = (x_1, x_2)$ with $\|\cdot\|_1, \|\cdot\|_2, \|\cdot\|_\infty$

Problem 2

For each of the following subsets of $\mathbb{R}^3$, determine whether or not it is a linear subspace of $\mathbb{R}^3$ and if yes, find its basis.

a. $\{(x,y,z) \in \mathbb{R}^3:\ x + 2y + 3z = 0\}$
b. $\{(x,y,z) \in \mathbb{R}^3:\ x + y - z = 1\}$
c. $\{(x,y,z) \in \mathbb{R}^3:\ xyz = 0\}$
d. $\{(x,y,z) \in \mathbb{R}^3:\ x = 2y\}$

Problem 3

Which of the following vectors do form bases of $\mathbb{R}^3$? Justify your answer.

- **a.** $(1,2,0)$ and $(0,1,-1)$
- **b.** $(1,1,-1)$, $(2,3,4,)$, $(4,1,-1)$, $(0,1,-,1)$
- **c.** $(1,2\ ,2)$, $(-1,2,1)$, $(0,8,0)$
- **d.** $(1,2\ ,2)$, $(-1,2,1)$, $(0,8,6)$

Problem 4

- **a.** Let U be the subspace of $\mathbb{R}^5$ defined by
$$U = \{(x_1,x_2,x_3,x_4,x_5):\ x_1 = 3x_2, x_3 = 7x_4\}.$$
- **b.** Find a basis of U.
- **c.** Extend the basis in part a. to a basis of $\mathbb{R}^5$.
- **d.** Find a subspace W of $\mathbb{R}^5$ such that $\mathbb{R}^5 = U \oplus W$.

Problem 5

Prove or give a counterexample:

If $v_1, v_2, \dots, v_m$ are linearly independent vectors in V, then $5v_1 - 4v_2, v_2, \dots, v_m$ are linearly independent.

Problem 6

Prove or give a counterexample:

Let v_1, v_2, v_3, v_4 form a basis of V. Is $v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$ another basis of V?

Problem 7

Let $U = \{p(t) \in P_4(\mathbb{R}):\ p(2) = p(5)\}$.

- **a.** Find a basis of U.
- **b.** Extend the basis in part a. to a basis of $P_4(\mathbb{R})$.
- **c.** Find a subspace W of $P_4(\mathbb{R})$ such that $P_4(\mathbb{R}) = U \oplus W$.

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