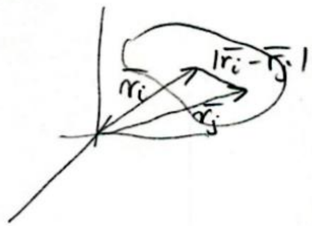
FINAL REVIEW

Part I: Rigid Body

Rigid Body

Rigid body – an object for which the distances between its parts do not change.



$|\bar{r}_i - \bar{r}_j| = \text{const.}$ for any $\bar{r}_i, \bar{r}_j$ pointing to points of the object

Examples

cannot bend, change length

non-deformable material

- 1. Rigid Body vs. Point Mass
- 2. Rotational Motion vs. Translational Motion

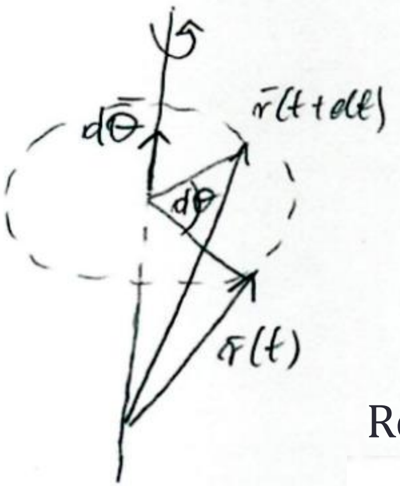
Angular velocity & acceleration; Linear velocity (fixed axis)

$$\bar\omega \overset{def}{=} \frac{d\bar\Theta}{dt}$$

[rad/s]

$$\bar v = \bar\omega \times \bar r$$

$$|\bar v| = |\bar\omega \times \bar r_\perp| = |\bar\omega|\,|\bar r_\perp|$$

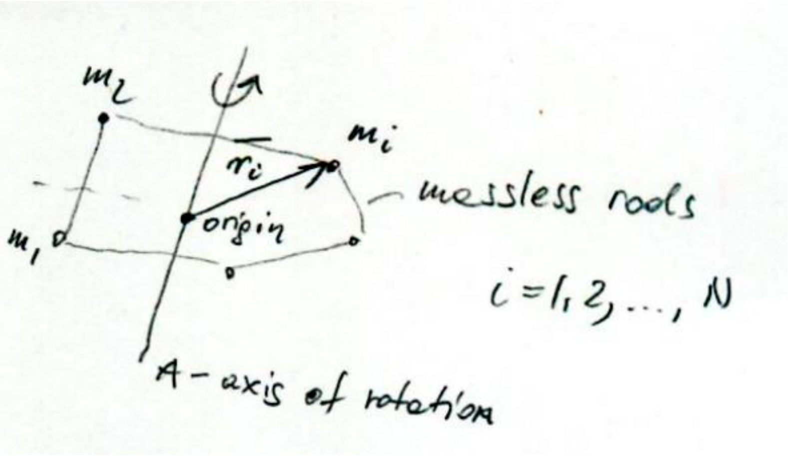


$$\bar a = \bar\varepsilon \times \bar r + \bar\omega \times (\bar\omega \times \bar r)$$

Recall: The expression for a in non-inertia FoR

$$\bar a = \bar a_{0'} + \bar a' + 2\bar\omega \times \bar v' + \frac{d\bar\omega}{dt} \times \bar r' + \bar\omega \times (\bar\omega \times \bar r')$$

Kinetic energy of a rotating rigid body (fixed axis)



Total kinetic energy

$$K = \sum_{i=1}^{N} K_i = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

But $\bar{v}_i = \bar{\omega} \times \bar{r}_i$ and $|\bar{v}_i| = |\bar{\omega}|\,|\bar{r}_{i\perp}|$

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i \omega^2 r_{i\perp}^2 = \frac{1}{2}\left(\sum_{i=1}^{N} m_i r_{i\perp}^2\right)\omega^2$$

$$\boxed{K = \frac{1}{2} I \omega^2} \quad \text{(fixed axis)}$$

I → moment of inertia about axis of rotation

Moment of inertia about an axis (General)

Total kinetic energy (add all contributions)

$$K = \int_{\text{object}} dK = \int_{\text{object}} \frac{1}{2}\omega^2 r_\perp^2 \, dm =$$

$$= \frac{1}{2}\left(\int r_\perp^2 \, dm\right)\omega^2$$

$$\boxed{I_A = \int_{\text{object}} r_\perp^2 \, dm}$$

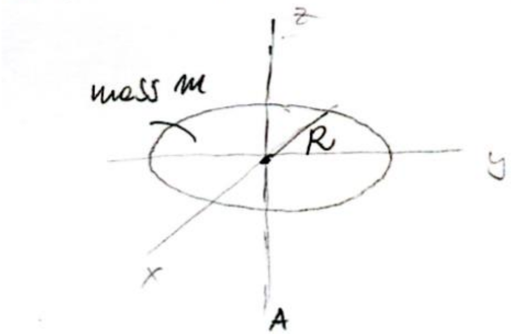
distance from axis A

Task: express dm in terms of x, y, z;
(often density is known)

Calculations of moments of inertia about an axis

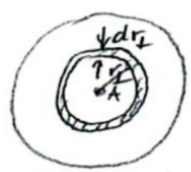
Show on blackboard (two ways)

(a) uniform disk (2D) about the axis $\perp$ to the disk through its center



x, y, z - principal axes

top view



build the disk from concentric rings

here a double integral

$$I_A = \int_{\text{object}} r_\perp^2 \, dm$$

surface density of mass $\sigma = \frac{m}{\pi R^2}$

$$dm = 2\pi r_\perp \sigma \, dr_\perp$$

$$dI_A = r_\perp^2 \, dm \Rightarrow I_A = \int_{\text{object}} r_\perp^2 \, dm$$

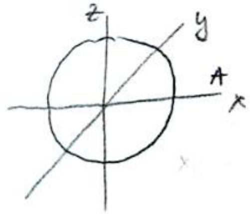
$$I_A = \int_0^R 2\pi \sigma r_\perp^3 \, dr_\perp = \sigma 2\pi \frac{R^4}{4} = \frac{1}{2} \sigma \pi R$$

$$= \frac{1}{2} m R^2$$

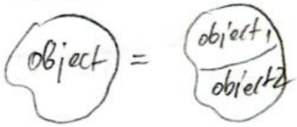
$$\boxed{I_A = \frac{1}{2} m R^2}$$

Use previous conclusion

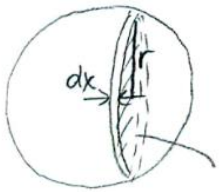
(d) ball with mass m, radius R, about any axis of symmetry

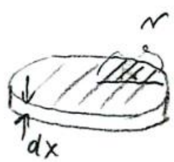


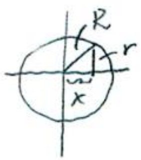
Note that $\int_{object} r_\perp^2 \, dm = \int_{object_1} r_\perp^2 \, dm + \int_{object_2} r_\perp^2 \, dm$



Idea: cut the ball into slices (cylinders of infinitesimal height), add the contributions.







$r = \sqrt{R^2 - x^2}$

$dm = \rho \pi r^2 \, dx = \rho \pi (R^2 - x^2) \, dx$

$dI_A = \frac{1}{2} r^2 \, dm$ contribution of one cylindrical slice

contribution of one half of the ball

$$I_A = \underset{\text{two halves}}{2} \cdot \int_0^R \frac{1}{2} \underbrace{(R^2 - x^2)}_{r^2} \underbrace{\rho \pi (R^2 - x^2) \, dx}_{dm} =$$

$$= \pi \rho \int_0^R \left(R^4 - 2R^2 x^2 + x^4 \right) dx =$$

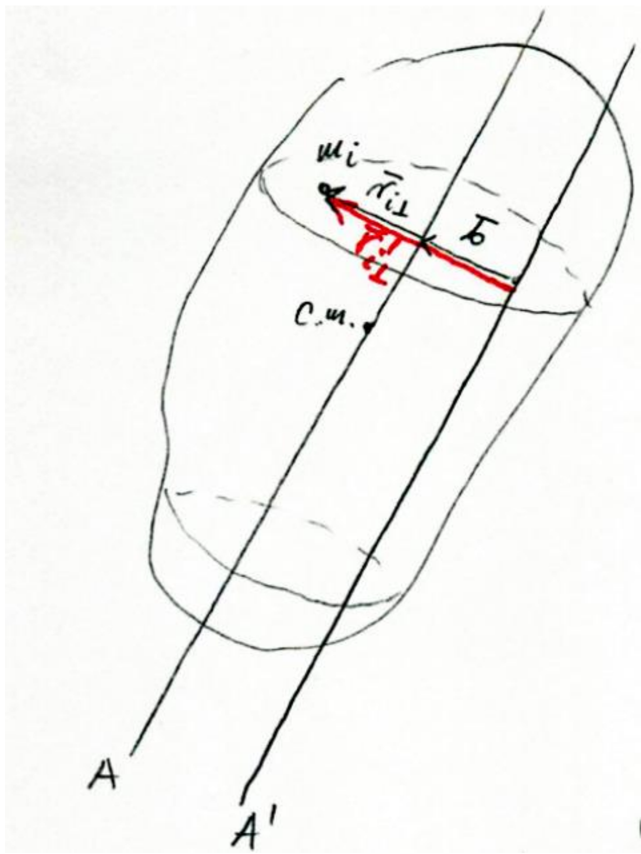
$$= \pi \rho \left(R^5 - \frac{2}{3} R^2 R^3 + \frac{1}{5} R^5 \right) = \pi \rho \frac{15 - 10 + 3}{15} R^5 =$$

$$= \frac{8}{15} R^5 \pi \rho \quad \Rightarrow \quad \boxed{I_A = \frac{2}{5} m R^2}$$

Steiner's Theorem

- When dealing with strange geometric shape:
- 1. Make use of derived result
- 2. Think of Steiner's Theorem

- E.g.: see blackboard



$$\boxed{I_{A'} = I_A + m b^2}$$

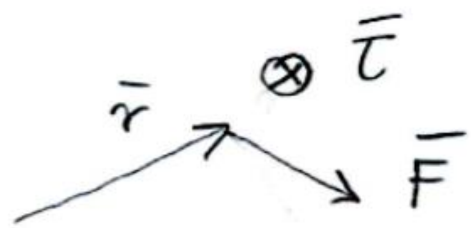
A - axis through the center of mass

distance between axes A, A'

Torque

- Always with respect to a point/ axis of rotation
- Determine the direction using right-hand rule

$$\bar{\tau} \stackrel{def}{=} \bar{r} \times \bar{F}$$



(right-hand rule)

units : $N \cdot m$ (= Joule)

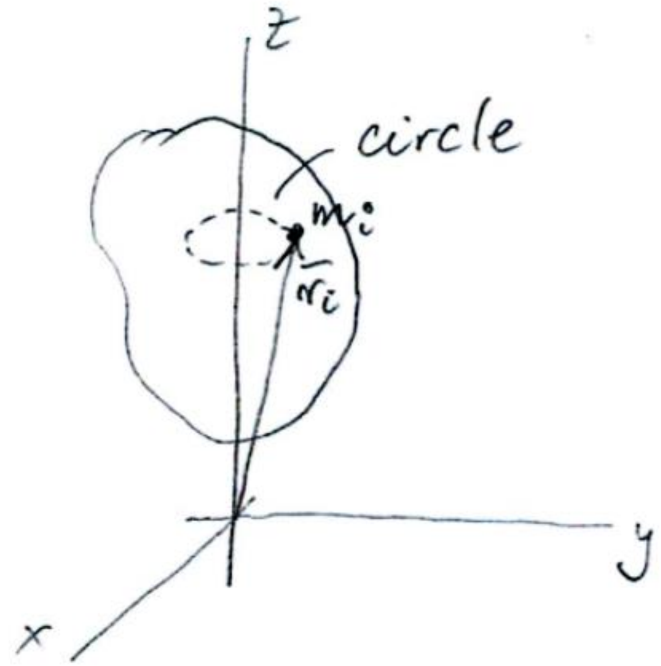
$$|\bar{\tau}| = |\bar{r}|\,|\bar{F}| \sin \measuredangle(\bar{r}, \bar{F}) = |\bar{r}_{\perp}|\,|\bar{F}| = |\bar{r}|\,|\bar{F}_{\perp}|$$

Torque and Angular Acceleration

- Rotational:

$$\tau_z = I_z \varepsilon_z$$

- Valid when axis:
- (1) through the center of mass
  (2) does not change direction



- Translational:

$$\sum_i F_i^{ext} = \frac{d\bar{P}}{dt} = M\bar{a}_{cm}$$

Rotation about a moving axis

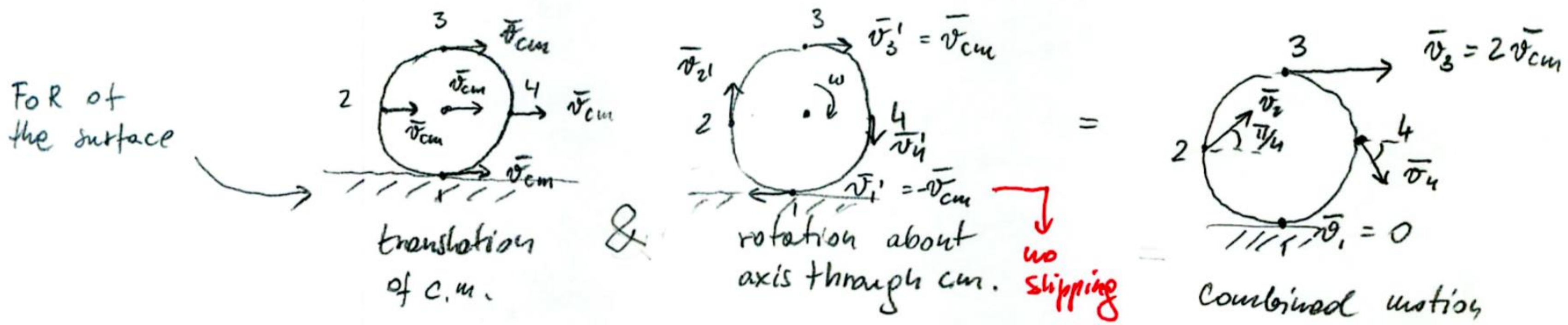
motion of a rigid-body = translational motion of the center of mass

\+

rotational motion about an instantaneous axis of rotation (through the center of mass)

Rotation about a moving axis

Rolling without slipping



No slipping $\equiv$ the point where the wheel is in contact with the surface is instantaneously at rest

Here: $v_{cm} = \omega R$

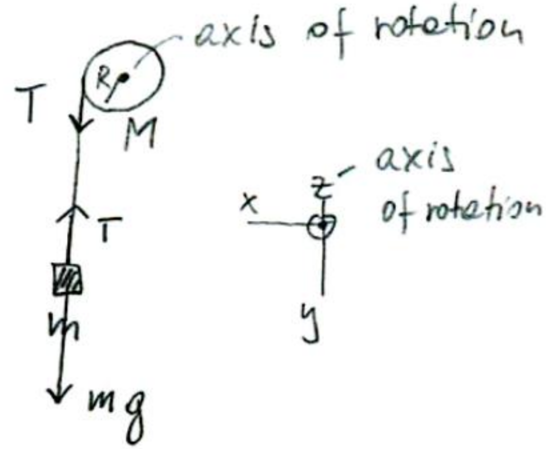
Comment: friction does no work! Provides torque in order to rotate

Examples

(a) unwinding string

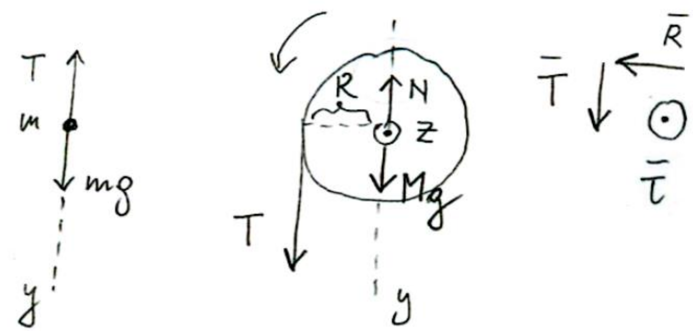
cylinder: mass M
radius R

$$I_z = \frac{1}{2} M R^2$$



Solution: force & torque

Free-body diagrams



block

cylinder

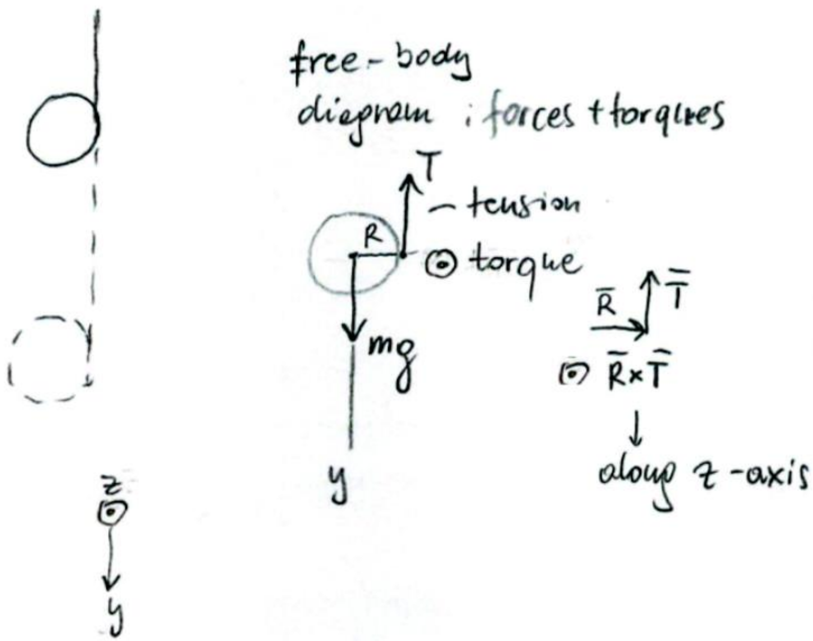
$$\begin{cases} 0 = Mg + T - N \\ m a_y = mg - T \\ I_z \varepsilon_z = \tau_z \\ a_y = R \cdot \varepsilon_z \end{cases}$$

Comment:

- 1. compare with hw2->p2
- 2. how does the force on the rope behaves? hw10->p3
- 3. Determine sign of force & torque: hw10-> p4
- 4. eqn. ; solve acceleration first

Key Point: eqn. for both forces & torque

unwinding thread



$$
\begin{cases}
mg - T = m\,a_{cm,y} & (1)\\
TR = I_{cm}\,\varepsilon_z = \frac{1}{2} m R^2 \varepsilon_z & (2)\\
a_{cm,y} = R\,\varepsilon_z & (3)
\end{cases}
$$

(no slipping; follows from $v_{cm,y} = \omega_z R$)

From (2): $T = \frac{1}{2} m R \varepsilon_z \overset{(3)}{=} \frac{1}{2} m\, a_{cm,y}$

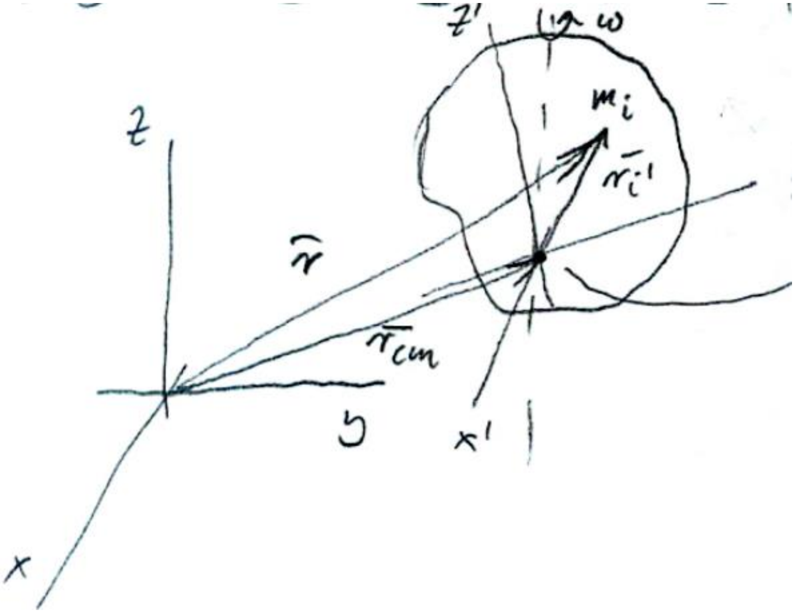
From (1):

$$mg = \frac{3}{2} m\, a_{cm,y}$$

$$\boxed{a_{cm,y} = \frac{2}{3} g}$$

$$\boxed{T = \frac{1}{3} mg}$$

Energy in the combined motion



$$K = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{cm} \omega^2$$

Comment: 1. motion of the c.m.

2. rotation about an axis through the c.m.

Find the speed of c.m. at the bottom

Static friction is conservative

$$\boxed{\delta W_{non\text{-}cons} = d(K+U) = dE}$$

(c)

different objects $I = c \cdot mR^2$

ball, ring, cylinder, ...

find v_{cm} at the bottom

static friction

(no slipping)

ignore rolling friction effects

no slipping

$$K_1 + U_1 = K_2 + U_2$$

$$mgh = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} I \left(\frac{v_{cm}}{R}\right)^2$$

$$gh = \frac{1}{2}(1+c) v_{cm}^2$$

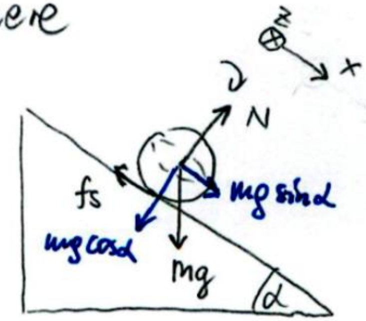
$$\Downarrow$$

$$\boxed{v_{cm} = \sqrt{\frac{2gh}{1+c}}}$$

Same situation: dynamics

(b) rolling sphere

static friction
(no slipping)



$\otimes$ torque $\vec{R} \times \vec{f_s}$

$\vec{f_s} \swarrow \vec{R}$

$$\begin{cases} mg\sin\alpha - f_s = m a_{cm,x} \\ f_s R = I_{cm}\varepsilon_z = \frac{2}{5} mR^2 \varepsilon_z \\ a_{cm,x} = \varepsilon_z R \end{cases}$$

$$a_{cm,x} = \frac{5}{7} g \sin\alpha$$

$$f_s = \frac{2}{7} mg \sin\alpha$$

Comments:

× friction still directed uphill for a ball rolling uphill

Hint: 1. differentiate from sliding friction
2. consider the direction of rotation

See hw10->Problem2