

## Chapter 8 – Work and Kinetic energy

UM-SJTU Joint Institute  
Physics I (Summer 2019)  
Mateusz Krzyzosiak

## 1 Motivation

## 2 Work

- How to Calculate Work in Various Situations?
- Example: Work Done on/by a Spring
- Examples: Work Along a Curved Path

## 3 Work and Motion

- Kinetic Energy and Work-Kinetic Energy Theorem
- Power

# Motivation

## Mechanics

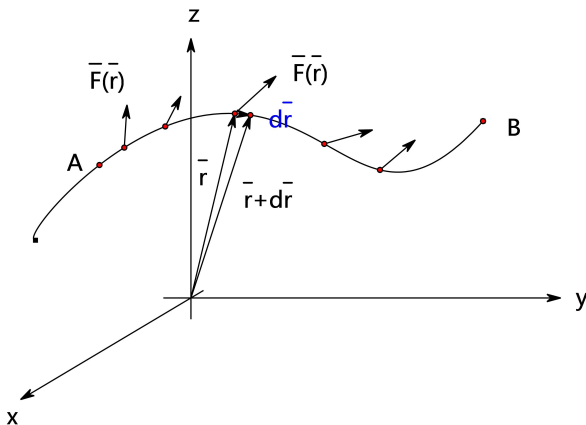
- Kinematics ( $\vec{r}, \vec{v}, \vec{a}$ ) *HOW?*
- Dynamics ( $\vec{F}, \vec{a} = \frac{\vec{F}}{m}$ ) *WHY?*

## New scalar quantities

- Concept of *work*: How to measure the "effort" put into moving the particle from one place to another?
- *Kinetic energy* and *work-kinetic energy theorem*: How does this "effort" change the state of the particle (e.g. its speed)?

# Work

# Work



In general,

$$\vec{F} = \vec{F}(\vec{r})$$

(variable force;  
vector field)

**Elementary work** done by  $\vec{F}$  when the particle moves from  $\vec{r}$  to  $\vec{r} + d\vec{r}$

$$\delta W = \vec{F} \circ d\vec{r}$$

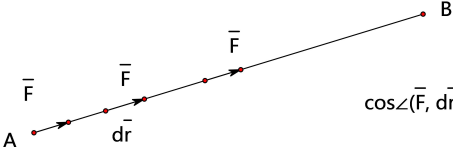
**Total work** done by  $\vec{F}$  when the particle moves from  $A$  to  $B$  along the curve  $\Gamma_{AB}$  — add all infinitesimal contributions

$$W_{AB} = \int_{\Gamma_{AB}} \vec{F} \circ d\vec{r}$$

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**How to calculate work in various situations?**

(A) Constant force in the direction of the displacement;  
displacement along a straight line

$$\delta W = \vec{F} \circ d\vec{r} = |\vec{F}| \cdot |d\vec{r}|$$


$\cos \angle(\vec{F}, d\vec{r}) = 1$

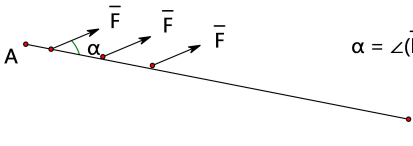
$$W_{AB} = \int_{\Gamma_{AB}} \vec{F} \circ d\vec{r} = \int_{\Gamma_{AB}} |\vec{F}| |d\vec{r}| = |\vec{F}| \underbrace{\int_{\Gamma_{AB}} |d\vec{r}|}_{S_{AB}} = |\vec{F}| S_{AB},$$

where  $S_{AB}$  is the length of  $AB$ . Hence, in this case

$$W_{AB} = |\vec{F}| S_{AB}$$



## (B) Constant force acting at an angle to the direction of straight-line displacement

$$\begin{aligned}\delta W &= \vec{F} \circ d\vec{r} \\ &= |\vec{F}| \cdot |d\vec{r}| \cos \alpha\end{aligned}$$


$\alpha = \angle(\vec{F}, d\vec{r})$  smaller angle

$$\begin{aligned}W_{AB} &= \int_{\Gamma_{AB}} \vec{F} \circ d\vec{r} = \int_{\Gamma_{AB}} |\vec{F}| \cdot |d\vec{r}| \cos \alpha = |\vec{F}| \cos \alpha \underbrace{\int_{\Gamma_{AB}} |d\vec{r}|}_{S_{AB}} \\ &= |\vec{F}| S_{AB} \cos \alpha\end{aligned}$$

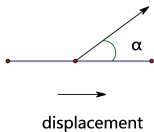
Hence, in this case,

$$W_{AB} = |\vec{F}| S_{AB} \cos \alpha$$

**Conclusion:** Work may either positive or negative or zero.

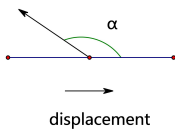
# Illustration. Positive/negative work.

(a)  $W_{AB} > 0$



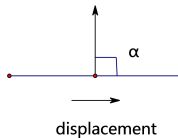
$$0 \leq \alpha < \frac{\pi}{2}$$

(b)  $W_{AB} < 0$



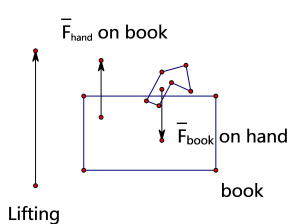
$$\frac{\pi}{2} < \alpha < \pi$$

(c)  $W_{AB} = 0$



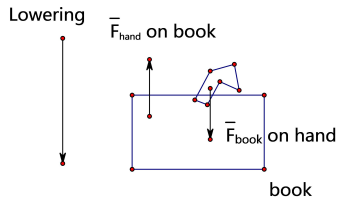
$$\alpha = \frac{\pi}{2}$$

Remember what does work on what!



$$W_{\text{book on hand}} < 0$$

$$W_{\text{hand on book}} > 0$$



$$W_{\text{book on hand}} > 0$$

$$W_{\text{hand on book}} < 0$$

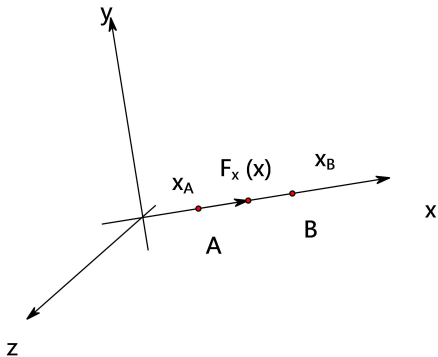
**Remark.** Elementary work done in the case when multiple forces act on a particle

$$\begin{aligned}\delta W &= (\bar{F}_1 + \bar{F}_2 + \dots + \bar{F}_N) \circ d\vec{r} = \bar{F}_1 \circ d\vec{r} + \bar{F}_2 \circ d\vec{r} + \dots + \bar{F}_N \circ d\vec{r} \\ &= \delta W_1 + \delta W_2 + \dots + \delta W_N\end{aligned}$$

or, equivalently,

$$\delta W = \bar{F}_{net} \circ d\vec{r}$$

# (C) Force of varying magnitude acting along a straight-line path



$$\vec{F} = ( \underbrace{F_x(x)}_{\text{function of } x \text{ only}}, 0, 0)$$

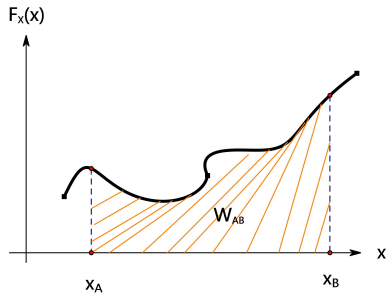
and

$$d\vec{r} = (dx, 0, 0)$$

$$\delta W = \vec{F} \circ d\vec{r} = F_x(x) dx \quad (\text{takes care of the sign})$$

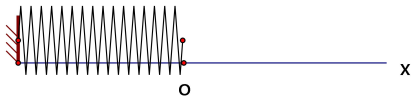
$$W_{AB} = \int_{\Gamma_{AB}} \delta W = \int_{\Gamma_{AB}} F_x(x) dx = \int_{x_A}^{x_B} F_x(x) dx$$

*Interpretation* (area under the curve)

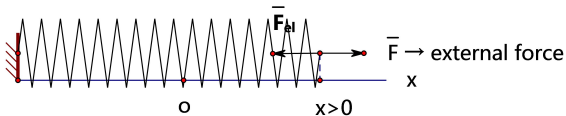


# Example: Work Done on/by a Spring

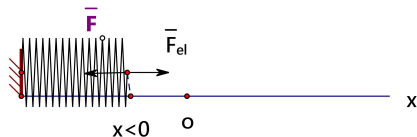
Uncompressed/  
unextended



Extending



Compressing



$$\underbrace{\vec{F}_{el}}_{\text{elastic (restoring force)}} = (-kx, 0, 0)$$

$$\vec{F} = (kx, 0, 0)$$

$$d\vec{r} = (dx, 0, 0)$$

If the position of the right (free) end of the spring changes from  $x_1$  to  $x_2$



Work done by

- the external force (e.g. hand) on the spring

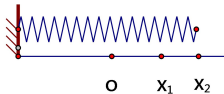
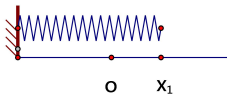
$$W_{1 \rightarrow 2}^{\text{ext}} = \int_{x_1}^{x_2} kx \, dx = \frac{1}{2} kx_2^2 - \frac{1}{2} kx_1^2$$

- the elastic force (spring) on the hand

$$W_{1 \rightarrow 2}^{\text{el}} = \int_{x_1}^{x_2} -kx \, dx = - \left( \frac{1}{2} kx_2^2 - \frac{1}{2} kx_1^2 \right) = -W_{1 \rightarrow 2}^{\text{ext}}$$

## Illustration

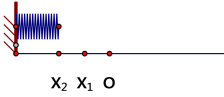
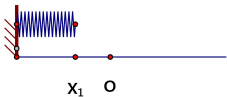
(\*)



$$W_{1 \rightarrow 2}^{\text{ext}} > 0$$

$$W_{1 \rightarrow 2}^{\text{el}} < 0$$

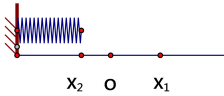
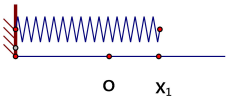
(\*\*)



$$W_{1 \rightarrow 2}^{\text{ext}} > 0$$

$$W_{1 \rightarrow 2}^{\text{el}} < 0$$

(\*\*\*)

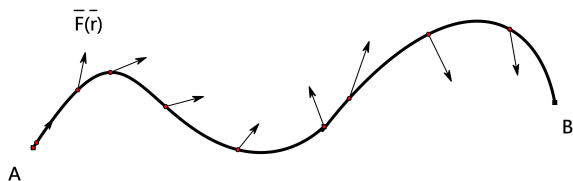


$$W_{1 \rightarrow 2}^{\text{ext}} < 0$$

$$W_{1 \rightarrow 2}^{\text{el}} > 0$$



## (D) General case – varying force, curved path



$$\vec{F} = \vec{F}(\vec{r}) \neq \text{const}$$

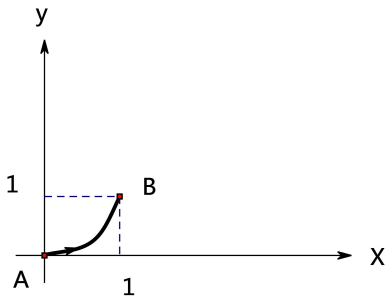
$$\delta W = \vec{F} \circ d\vec{r},$$

$$W_{AB} = \int_{\Gamma_{AB}} \vec{F} \circ d\vec{r}$$

### Example (work along a parabola)

Calculate work done by the force  $\vec{F}(\vec{r}) = (x^2 + y^2)\hat{n}_x + x\hat{n}_y$  [N], acting on a particle moving from (0,0) to (1,1) along a segment of parabola  $y = x^2$ .

## Example 1 (work along a parabola)



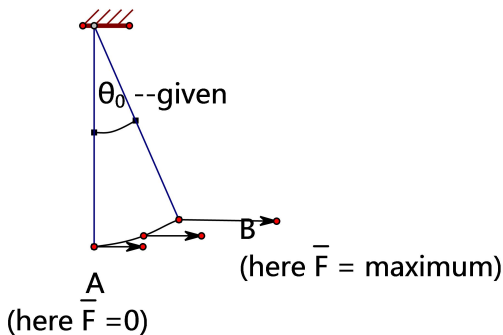
$$\Gamma_{AB} : \begin{cases} 0 \leq x \leq 1 \\ y = x^2 \end{cases}$$

$$\begin{aligned} W_{AB} &= \int_{\Gamma_{AB}} \vec{F} \circ d\vec{r} = \int_{\Gamma_{AB}} F_x dx + F_y dy = \int_{\Gamma_{AB}} (x^2 + y^2) dx + x dy \\ &= \int_0^1 [(x^2 + x^4) + (x \cdot 2x)] dx = \frac{1}{3} + \frac{1}{5} + \frac{2}{3} = \frac{18}{15} \quad [J] \end{aligned}$$

Note that for all the points on the parabola  $dy = 2x dx$ .

## Example 2 (less abstract one; work along a curve)

Find the minimum work needed to move an object of weight  $Q$ , suspended on a light rod with length  $R$ , from  $A$  to  $B$ , acting with a horizontal force.



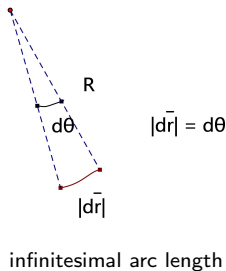
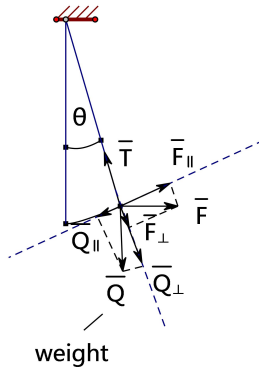
$$W_{A \rightarrow B} = \int_{\Gamma_{AB}} \vec{F} \circ d\vec{r}$$

## First Method

Idea: Decompose the force

$$\delta W = \vec{F} \circ d\vec{r} = (\vec{F}_{\parallel} + \vec{F}_{\perp}) \circ d\vec{r} = \underbrace{\vec{F}_{\parallel}}_{\text{varies}} \circ d\vec{r}$$

and express everything in terms of  $\theta$



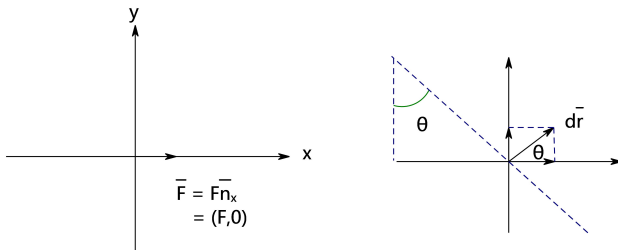
$$\delta W = F_{\parallel} |d\vec{r}| = Q \sin \theta R d\theta$$

Work done by  $\vec{F}$  on the path  $A \rightarrow B$  (arc of a circle with

$$\begin{aligned}\boxed{W_{A \rightarrow B}} &= \int_{\Gamma_{AB}} \vec{F} \cdot d\vec{r} = \int_0^{\theta_0} \underbrace{Q \sin \theta R}_{F_{\parallel}} d\theta = QR \int_0^{\theta_0} \sin \theta d\theta \\ &= \boxed{QR [1 - \cos \theta_0]}\end{aligned}$$

## Second Method

*Idea:* Rewrite both the force and the elementary displacement in terms of their Cartesian coordinates.



$$d\vec{r} = |d\vec{r}| \cos \theta \hat{n}_x + |d\vec{r}| \sin \theta \hat{n}_y = (|d\vec{r}| \cos \theta, |d\vec{r}| \sin \theta)$$

$$\delta W = \vec{F} \circ d\vec{r} = F \cos \theta \underbrace{|d\vec{r}|}_{R d\theta} = F \cos \theta R d\theta$$

and  $F \cos \theta = Q \sin \theta$ , so that  $\delta W = Q \sin \theta R d\theta$ . Eventually,

$$W_{A \rightarrow B} = \int^{\theta_0} Q \sin \theta R d\theta = QR [1 - \cos \theta_0]$$

# Work and Motion

## *Observations*

- $W > 0 \quad \implies \quad$  particle speeds up
- $W < 0 \quad \implies \quad$  particle slows down
- $W = 0 \quad \implies \quad$  no change in speed

**Conclusion.** Work done by the net force implies a change in the particle's speed (or another quantity that is a function of speed).



# Kinetic Energy and Work-Kinetic Energy Theorem

Recall that  $\delta W = \vec{F} \circ d\vec{r}$  so, the rate of work being done by the net force on a particle

$$\underbrace{\frac{\delta W}{dt}} = \vec{F} \circ \frac{d\vec{r}}{dt} = \vec{F} \circ \dot{\vec{r}} = m \vec{a} \circ \vec{v} = m \dot{\vec{v}} \circ \vec{v}$$

But  $v^2 = \vec{v} \circ \vec{v}$ , so  $\frac{d}{dt}(v^2) = \frac{d}{dt}(\vec{v} \circ \vec{v}) = \dot{\vec{v}} \circ \vec{v} + \vec{v} \circ \dot{\vec{v}} = 2\vec{v} \circ \dot{\vec{v}}$

$$\frac{\delta W}{dt} = \frac{d}{dt} \underbrace{\left( \frac{1}{2} m v^2 \right)}_K$$

where  $\boxed{K = \frac{1}{2} m v^2}$  is the **kinetic energy**. Hence,

$$\frac{\delta W}{dt} = \frac{dK}{dt} \quad \text{or} \quad \boxed{\delta W = dK}$$

# Work–Kinetic Energy Theorem

*Work done by the net force on a particle is equal to the change in the particle's kinetic energy.*

For finite changes

$$W = \Delta K$$

$$W > 0 \quad \implies \Delta K > 0 \quad \implies \Delta v > 0$$

$$W < 0 \quad \implies \Delta K < 0 \quad \implies \Delta v < 0$$

$$W = 0 \quad \implies K = \text{const} \quad \implies v = \text{const}$$

- 1 Derived in a general case; no assumptions about the nature of the force or the particle's trajectory made
- 2 Used Newton's second law; can use only in inertial FoRs.
- 3 Kinetic energy has the units of work [ $J = N \cdot M = kg \frac{m^2}{s^2}$ ]

**Power** – characterizes "how fast" work is being done

$$\underbrace{\frac{\delta W}{dt}}_{\text{rate of work done}} = \vec{F} \circ \vec{v} = \underbrace{P}_{\text{instantaneous power}}$$

$$\frac{W}{\Delta t} = P_{av}$$

Units: [W] (Watt)

Alternative units:  $\underbrace{\text{hp}}_{\text{horse power}} = 746W = 0.746kW$

**Example** (*Power of engines in vehicles*): Abrams battle tank 1500 hp, Volvo C30 108hp, Harley-Davidson 1600ccm 63 hp, Porsche 911 GT2 523 hp, Boeing 747-300 (@ 0.84 Mach, 35k feet) 330 000 hp, Diesel locomotive 5000 hp