vv214 SU2020 Open Book Quiz 4

Due: Thursday 30 July

Question 1

Let

$$\det\begin{pmatrix} R & S & T \\ U & V & W \\ X & Y & Z \end{pmatrix} = 11$$

Compute the following determinants, and show how you have done so.

$$a.\ \det\begin{pmatrix} X & Y & Z \\ R & S & T \\ U & V & W \end{pmatrix} \quad b.\ \det\begin{pmatrix} R & S & T \\ 2R+U & 2S+V & 2T+W \\ 3R+2U+X & 3S+2V+Y & 3T+2W+Z \end{pmatrix}$$

$$c.\ \det\begin{pmatrix} R & S & T \\ R+2U & S+2V & T+2W \\ R+2U+3X & S+2V+3Y & T+2W+3Z \end{pmatrix}$$

$$d.\ \det\begin{pmatrix} R & S & T \\ U & V & W \\ U & V & W \end{pmatrix} \quad e.\ \det\begin{pmatrix} R & S \\ U & V \\ X & Y \end{pmatrix}\begin{pmatrix} R & U & X \\ S & V & Y \end{pmatrix}$$

(20 points)

a. 2 row swaps $\Rightarrow (-1)(-1)\cdot 11 = 11$ 3

b. $\begin{vmatrix} R & S & T \\ 2R+U & 2S+V & 2T+W \\ 3R+2U+X & 3S+2V+Y & 3T+2W+Z \end{vmatrix} = \begin{vmatrix} R & S & T \\ U & V & W \\ 2U+X & 2V+Y & 2W+Z \end{vmatrix} = \begin{vmatrix} R & S & T \\ U & V & W \\ X & Y & Z \end{vmatrix} = 11$ 5

c. $\begin{vmatrix} R & S & T \\ R+2U & S+2V & T+2W \\ R+2U+3X & S+2V+3Y & T+2W+3Z \end{vmatrix} = \begin{vmatrix} R & S & T \\ 2U & 2V & 2W \\ 3X & 3Y & 3Z \end{vmatrix} = 2\cdot 3 \begin{vmatrix} R & S & T \\ U & V & W \\ X & Y & Z \end{vmatrix} = 6\cdot 11 = 66$ 5

d. 2 equal rows $\Rightarrow \det(\ ) = 0$ 3 e. 0 4

Question 2

Find $exp(At)$ for the matrix

$$A = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & -3 \end{pmatrix}$$

(30 points)

$\phi_A(\lambda)\ |A-\lambda I| = \begin{vmatrix} -\lambda & 0 & -2 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -3-\lambda \end{vmatrix} = \lambda(1-\lambda)(3+\lambda) + 2(1-\lambda) = 0$

$(1-\lambda)(3\lambda+\lambda^2+2) = 0$

$\lambda_1 = 1 \quad \lambda_2 = 1 \quad \lambda_3 = 2$ 2

$\deg \phi_A(\lambda) = 3$ 5

$g(t) = h(t)\phi_A(t) + r(t)$, $\deg r(t) \le 2 \Rightarrow r(t) = b_0 + b_1 t + b_2 t^2$ 3

$g(\lambda) = r(\lambda) \Rightarrow e^{\lambda t} = b_0 + b_1\lambda + b_2\lambda^2 \Rightarrow \left. \begin{aligned} e^{2t} &= b_0 + 2b_1 + 4b_2 \\ e^{t} &= b_0 + b_1 + b_2 \\ te^{t} &= b_1 + 2b_2 \end{aligned} \right\} \Rightarrow \begin{pmatrix} 1 & 2 & 4 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} b_0 \\ b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} e^{2t} \\ e^{t} \\ te^{t} \end{pmatrix}$ 5

$\lambda = 1$: (AM=2) $(e^{\lambda t})'_\lambda = (b_0 + b_1\lambda + b_2\lambda^2)'_\lambda$ at $\lambda = 1$

$b_2 = e^{2t} - te^{t} - e^{t}$

$b_1 = -2e^{2t} + 3te^{t} + 2e^{t}$ 10

$b_0 = e^{2t} - 2te^{t}$

$g(A) = r(A)$

$\exp(At) = b_0 I + b_1 A + b_2 A^2$ 5

Question 3

Let

$$A=\begin{pmatrix}1&3\\3&1\end{pmatrix}$$

1. Without computing anything, can it be true that A is diagonalizable? What can you say about its eigenvectors?
2. Find the eigenvalues of A and one eigenvector of A.
3. Use the spectral theorem to find another eigenvector of A.
4. Find an orthonormal eigenbasis for the transformation defined by A.
5. Find orthogonal S such that $S^{-1}AS$ is diagonal.
6. Is there S such that S^TAS is diagonal?
7. Prove or disprove that there is a unit square in $\mathbb{R}^2$ such that A takes that square to a rectangle whose sides are length 2 and 4. **(10 points)** 2

1. Yes, A is symmetric ⇒ diagonalizable; its eigenvectors are orthogonal 2

2. $|A-\lambda I| = (1-\lambda)^2-9=0 \Rightarrow \lambda^2-2\lambda-8=0 \Rightarrow \lambda_1=-2, \lambda_2=4$ 2

$\lambda_1=-2 \Rightarrow (A+2I)\bar v_1=\bar 0 \Rightarrow \begin{pmatrix}3&3\\3&3\end{pmatrix}\begin{pmatrix}v_1^1\\v_1^2\end{pmatrix}=0 \Rightarrow \bar v_1=\begin{pmatrix}1\\-1\end{pmatrix}$ 2

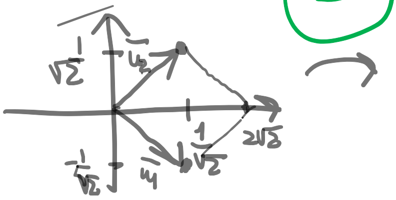
3. $\bar v_2 \perp \bar v_1 \Rightarrow \bar v_2=\begin{pmatrix}1\\1\end{pmatrix}$ 2, $A\bar v_2=\begin{pmatrix}4\\4\end{pmatrix}=4\bar v_2$

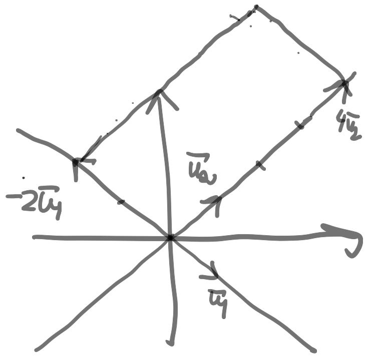
4. $\bar u_1=\frac{1}{\sqrt2}\begin{pmatrix}1\\-1\end{pmatrix}$, $\bar u_2=\frac{1}{\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}$

5. $S=\frac{1}{\sqrt2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}$ 2

6. Yes, $S^{-1}=S^T$

7.





5

$A\bar u_1=\frac{1}{\sqrt2}\cdot(-2)\bar v_1=-2\bar u_1$

$A\bar u_2=\frac{1}{\sqrt2}4\bar v_1=4\bar u_2$ 5

Question 4

Consider a SVD of a 3×4 matrix:

$\bar u_3 \perp \bar u_1, \bar u_2$; $\bar u_3=\frac{(2,4,-5)}{\|(2,4,-5)\|}$ 5

$$A=\begin{pmatrix}1/3&-2/\sqrt5&\star\\2/3&1/\sqrt5&\star\\2/3&0&\star\end{pmatrix}\begin{pmatrix}3&0&0&0\\0&2&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}\star&\star&\star&\star\\12/13&\star&\star&5/13\\5/13&\star&\star&-12/13\\\star&3/5&4/5&\star\end{pmatrix}$$

(columns of the first matrix: $\bar u_1$, $\bar u_2$, $\bar u_3$; rows of the last matrix: $\bar v_1^T$, $\bar v_2^T$, $\bar v_3^T$, $\bar v_4^T$)

$\bar v_1 \perp \bar v_2, \bar v_3, \bar v_4$; $\|\bar v_1\|=1 \Rightarrow \star=0$ 3

1. Find the eigenvalues of AA^T, A^TA and the rank of A.
2. Find a non-zero vector $\bar w$ such that $AA^T\bar w=9\bar w$, a non-zero vector $\bar w'$ such that $A^TA\bar w'=4\bar w'$ and a non-zero vector $\bar w''$ such that $A^TA\bar w''=9\bar w''$.
3. Find a basis for the kernel and the image of A and a basis for the kernel and the image of A^T.
4. Sketch the image of the unit ball in $\mathbb{R}^4$ under the map defined by A. **(10 points)** 3 9, 4, 0

1. $AA^T=U\Sigma V^TV\Sigma^TU^T=U\Sigma\Sigma^TU^T$; $\Sigma\Sigma^T=\begin{pmatrix}3&0&0&0\\0&2&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}3&0&0\\0&2&0\\0&0&0\\0&0&0\end{pmatrix}=\begin{pmatrix}9&0&0\\0&4&0\\0&0&0\end{pmatrix}$

$A^TA=V^T\Sigma^TU^TU\Sigma V=V^T\Sigma^T\Sigma V$; $\Sigma^T\Sigma=\begin{pmatrix}3&0&0\\0&2&0\\0&0&0\\0&0&0\end{pmatrix}\begin{pmatrix}3&0&0&0\\0&2&0&0\\0&0&0&0\end{pmatrix}=\begin{pmatrix}9&0&0&0\\0&4&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

2 rank $A=2$

9, 4, 0, 0 3

2. $AA^T\bar w=9\bar w \Rightarrow \bar w=\bar u_1=(1/3, 2/3, 2/3)$ 2

$A^TA\bar w'=4\bar w' \Rightarrow \bar w'=\bar v_2=(12/13, 0, 0, 5/13)$ 2

$A^TA\bar w''=9\bar w'' \Rightarrow \bar w''=\bar v_1=(0, 4/5, -3/5, 0)$ 2

$\operatorname{Im} A^T=\operatorname{span}(\bar v_1, \bar v_2)$ 2

$\operatorname{Ker} A^T=\operatorname{span}(\bar u_3)=\operatorname{span}\begin{pmatrix}2\\4\\-5\end{pmatrix}$ 3

3. $\operatorname{Ker} A=\operatorname{span}(\bar v_3, \bar v_4)$; $\operatorname{Im} A=\operatorname{span}(\bar u_1, \bar u_2)$ 3