Vev556 Methods of Applied Mathematics II

Sample Exercises for the Final Exam

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The following exercises are sample exercises of a difficulty comparable to those found the actual first midterm exam. The exam will usually include of 5 to 8 such exercises to be completed in 100 minutes.

Definitions and Concepts

Some questions will test your understanding of basic definitions and concepts. The answers will involve either multiple choice selections or ask you to write a sentence or two explaining the concept.

Exercise 1 Multiple Choice

In the following exercises, mark the boxes corresponding to true statements with a cross (☒). In each case, it is possible that none of the statements are true or that more than one statement is true.

i) Let (L, B_1, B_2) be a second-order ordinary differential operator with boundary operators. Suppose that the fully homogeneous problem has a non-trivial solution.

- ☐ The Green function exists but is not unique.
- ☒ The Green function does not exist.
- ☒ The modified Green's function exists but is not unique.
- ☐ The modified Green's function exists and is unique.

ii) Let (L, B_1, B_2) be a second-order ordinary differential operator with boundary operators. Which of the following always exist?

- ☒ A fundamental solution $E(x, \xi)$.
- ☐ The Green function $g(x, \xi)$.
- ☐ The adjoint Green function $g(x, \xi)$.
- ☒ The modified Green function $g_M(x, \xi)$.

iii) Let $E(x, \xi)$ denote a causal fundamental solution for an ordinary differential operator with constant coefficients and denote by E^* the causal fundamental solution for the adjoint operator. Which of the following is correct?

- ☐ $E(x, \xi) = E^*(x, \xi)$
- ☒ $E(x, \xi) = E^*(\xi, x)$
- ☐ $E(x, \xi) = E^*(-\xi, -x)$
- ☒ $E(x, \xi) = E^*(-x, -\xi)$

iv) Let $g(x, \xi)$ be the Green's function of the boundary value problem

$$[(1 + x)u']' + (\sin x)u = 0, \qquad x \in [0, 1], \qquad u(0) = u(1) = 0.$$

Then, the function $h\colon [0, 1] \to \mathbb{R}$ defined by

$$h(x) = g\left(x, \frac{1}{2}\right)$$

- ☒ is continuous.
- ☐ is discontinuous at $x = 1/2$.
- ☐ is twice differentiable on $(0, 1)$.
- ☒ is differentiable on $(0, 1)$ but not twice differentiable at $x = 1/2$.

(8 Marks)

Green's Functions for ODEs

It is important to be able to solve a Green's function problem for ordinary differential equations. Coincepts such as the conjunct, Green's formula, the adjoint boundary value problem should be familiar, as well as solvability conditions and modified Green's functions.

Exercise 2 A slightly different boundary condition
Consider the boundary value problem given by

$$Lu := -u'' = f, \quad -1 < x < 1, \qquad B_1 u := \int_{-1}^{1} x u(x)\, dx = \gamma_1, \qquad B_2 u := \int_{-1}^{1} u(x)\, dx = \gamma_2$$

for f piecewise continuous on $[-1,1]$ and $\gamma_1, \gamma_2 \in \mathbb{R}$.
Find the corresponding Green's function and write down a solution formula for the problem.
(5 Marks)

Proof. A causal fundamental solution is given by

$$E(x,\xi) = H(x-\xi)(\xi - x) + a + bx, \qquad a, b \in \mathbb{R}.$$

Adding the boundary conditions gives the Green function

$$g(x,\xi) = H(x-\xi)(\xi - x) + \frac{2 - 3\xi + \xi^3}{4} x + \frac{(\xi - 1)^2}{4}.$$

The solution formula is

$$u(x) = \int_{-1}^{1} g(x,\xi) f(\xi)\, d\xi + \frac{\gamma_2}{B_2 u_1} u_1 + \frac{\gamma_1}{B_1 u_2} u_2$$

with $u_1(x) = 1$, $u_2(x) = x$. Hence,

$$u(x) = \int_{-1}^{1} g(x,\xi) f(\xi)\, d\xi + \frac{\gamma_2}{2} + \frac{3\gamma_1}{2} x$$

□

Exercise 3 Solvability and Modified Green's Function
Consider the boundary value operator given by

$$Lu = u'', \quad 0 < x < 1, \qquad B_1 u = u(0) + u(1), \qquad B_2 u = u'(0) - u'(1)$$

i) Show that Green's function $g(x,\xi)$ doesn't exist for this problem.
(2 Marks)

ii) Give the equation that the modified Green's function $g_M(x,\xi)$ must satisfy.
(2 Marks)

iii) Construct the modified Green's function $g_M(x,\xi)$.
(3 Marks)

iv) Find the solvability condition(s) for the problem

$$Lu = f, \quad 0 < x < 1, \qquad B_1 u = \gamma_1, \qquad B_2 u = \gamma_2.$$

Give a formula for the general solution of the problem if $\gamma_1 = \gamma_2 = 0$.
(3 Marks)

Proof.

i) The completely homogeneous problem has a non-trivial solution, namely $u(x) = 1 - 2x$. To find a solvability condition, we consider the adjoint problem: Green's formula is

$$\int_0^1 (vu'' - uv'')\, dx = vu' - uv'|_0^1$$

so, setting $B_3u = u(0)$, $B_4u = u'(0)$,

$$
\begin{aligned}
J(u,v)|_0^1 &= v(1)u'(1) - u(1)v'(1) - v(0)u'(0) + u(0)v'(0) \\
&= -v(1)B_2u + v(1)B_4u - v'(1)B_1u + v'(1)B_3u - v(0)B_4u + v'(0)B_3u \\
&= -v'(1)B_1u - v(1)B_2u + (v'(1) + v'(0))B_3u + (v(1) - v(0))B_4u
\end{aligned}
$$

Hence, the adjoint boundary conditions are

$$
B_1^*v = v(1) - v(0), \qquad B_2^*v = v'(1) + v'(0), \qquad B_3^*v = -v(1), \qquad B_4^*v = -v'(1).
$$

The completely homoegenous adjoint problem has a non-trivial solution $v(x) = 1$, hence the equation $Lu = f$, $B_1u = B_2u = 0$ is only solvable if

$$
\int_0^1 f(x)\,dx = 0.
$$

This is not the case for $f(x) = \delta(x-\xi)$, so Green's function doesn't exist.

ii) $Lg_M(x,\xi) = \delta(x-\xi) - 1$

iii) We find the modified Green's function by solving $Lu = 1$, which gives $u(x) = x^2/2$. A causal fundamental solution is found as usual by solving $Lu_\xi = 0$, $u_\xi(\xi) = 0$, $u_\xi'(\xi) = 1$. This gives

$$
u_\xi(x) = x - \xi
$$

and hence

$$
g_M(x,\xi) = H(x-\xi)(x-\xi) - \frac{x^2}{2} + a + bx
$$

will satisfy $Lg_M(x,\xi) = \delta(x-\xi) - 1$. The boundary conditions yield

$$
g_M(1,\xi) = (1-\xi) - \frac{1}{2} + a + b = -g_M(0,\xi) = -a
$$

so $2a + b = 1/2 - \xi$ and

$$
g_M'(1,\xi) = 1 - 1 + b = g_M'(0,\xi) = b
$$

so b is arbitrary. Hence, we can set $b = 0$ and obtain

$$
g_M(x,\xi) = H(x-\xi)(x-\xi) - \frac{x^2-\xi}{2} - \frac{1}{4}.
$$

iv) The solvability condition is

$$
\int_0^1 f(x)(x)\,dx = \gamma_1 B_4^*v + \gamma_2 B_3^*v
$$

where $v(x) = 1$ is the non-trivial solution to the adjoint problem. Hence, the solvability condition is

$$
\int_0^1 f(x)\,dx = -\gamma_2.
$$

The general solution for $\gamma_1 = \gamma_2 = 0$ is

$$
u(x) = \int_0^1 g_M(x,\xi)f(\xi)\,dx + c(1-2x), \qquad c \in \mathbb{R}.
$$

□

Solution Formula for PDEs

You need to know solution formulas for boundary value problems involving the Laplace, heat and wave equations with different types of boundary conditions. It is important to be able to take the general formula and evaluate it in a specifc situation.

Exercise 4
Consider the boundary value problem for the heat equation on a finite interval $(0, L) \subset \mathbb{R}$:

$$\begin{aligned} u_t - c^2 u_{xx} &= F(x,t), && 0 < x < L, \\ u(0,t) &= \gamma_1, && 0 < t < T, \\ u(L,t) &= \gamma_2, && 0 < t < T, \\ u(x,0) &= f(x), && 0 < x < L. \end{aligned} \tag{$*$}$$

where $T > 0$ is some fixed time, $\gamma_1, \gamma_2 \in \mathbb{R}$, and $f\colon [0,L] \to \mathbb{R}$, $F\colon [0,L] \times \mathbb{R} \to \mathbb{R}$ suitably smooth functions.

i) Which differential equation and boundary conditions must be satisfied by the direct Green's function $g(x,t;\xi,\tau)$ for $(*)$? [No proof necessary.]
(2 Marks)

ii) Which differential equation and boundary conditions must be satisfied by the adjoint Green's function $g^*(x,t;\xi,\tau)$ for $(*)$? [No proof necessary.]
(2 Marks)

iii) Give a simple relation linking g and g^*. [No proof necessary.]
(2 Marks)

iv) The general solution formula for a parabolic problem on a domain $V = \Omega \times [0,T] \subset \mathbb{R}^n \times \mathbb{R}$ is given by

$$\begin{aligned} u(\xi,\tau) = &\int_V \varrho(x) F(x) g^*(x,t;\xi,\tau)\, d(x,t) + \int_\Omega \varrho(x) g^*(x,0;\xi,\tau) f(x)\, dx \\ &- \int_{\widetilde{S}_1} \frac{p}{\alpha}\gamma \frac{\partial g^*(\,\cdot\,;\xi,\tau)}{\partial n_x}\, d\sigma + \int_{\widetilde{S}_2 \cup \widetilde{S}_3} \frac{p}{\beta}\gamma g^*(\,\cdot\,;\xi,\tau)\, d\sigma \end{aligned}$$

Supposing that g is known, give the explicit formula for the problem $(*)$.
(3 Marks)

Proof.

i) $g_t - c^2 g_{xx} = \delta(x-\xi)\delta(t-\tau)$, $g(0,t;\xi,\tau) = g(L,t;\xi,\tau) = 0$, $g(x,0;\xi,\tau) = 0$.

ii) $-g_t - c^2 g_{xx} = \delta(x-\xi)\delta(t-\tau)$, $g(0,t;\xi,\tau) = g(L,t;\xi,\tau) = 0$, $g(x,T;\xi,\tau) = 0$.

iii) $g^*(x,t;\xi,\tau) = g(x,-t;\xi,-\tau)$ or $g^*(x,t;\xi,\tau) = g(\xi,\tau;x,t)$ are acceptable answers.

iv) We have

$$\begin{aligned} u(\xi,\tau) &= \int_V \varrho(x) F(x,t) g^*(x,t;\xi,\tau)\, d(x,t) + \int_\Omega \varrho(x) g^*(x,0;\xi,\tau) f(x)\, dx \\ &\quad - \int_{\widetilde{S}_1} \frac{p}{\alpha}\gamma \frac{\partial g^*(\,\cdot\,;\xi,\tau)}{\partial n_x}\, d\sigma + \int_{\widetilde{S}_2 \cup \widetilde{S}_3} \frac{p}{\beta}\gamma g^*(\,\cdot\,;\xi,\tau)\, d\sigma \\ &= \int_0^T \int_0^L F(x,t) g^*(x,t;\xi,\tau)\, dx\, dt + \int_0^L g^*(x,0;\xi,\tau) f(x)\, dx \\ &\quad + \gamma_1 \int_0^T \frac{\partial g^*(x,t;\xi,\tau)}{\partial x}\Big|_{x=0} dt - \gamma_2 \int_0^T \frac{\partial g^*(x,t;\xi,\tau)}{\partial x}\Big|_{x=L} dt \end{aligned}$$

$\square$

Exercise 5
Consider the problem:

$$\Delta u(x_1,x_2,x_3) = 0, \qquad x_3 > 0,\ x_1, x_2 \in \mathbb{R},$$

with boundary condition

$$u(x_1,x_2,0) = \begin{cases} u_0 & x_1^2 + x_2^2 \le a^2, \\ 0 & \text{otherwise,} \end{cases}$$

where $u_0 \in \mathbb{R}$.

i) State Green's function for this problem and give the general integral formula for u. (No calculations are required.) **(2 Marks)**

ii) Show that

$$u(0,0,x_3) = u_0\left(1 - \frac{x_3}{\sqrt{a^2+x_3^2}}\right)$$

for $x_3 > 0$.
(4 Marks)

Proof.

i) Green's function is simply

$$\begin{aligned} g(x;\xi) = E(x;\xi) - E(x;\xi^*) &= \frac{1}{4\pi}\left(\frac{1}{|x-\xi|} - \frac{1}{|x-\xi^*|}\right) \\ &= \frac{1}{4\pi}\left(\frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2}} - \frac{1}{\sqrt{(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3+\xi_3)^2}}\right) \end{aligned}$$

where $\xi^* := (\xi_1, \xi_2, -\xi_3)$. **(1 Mark)** The solution formula is

$$u(x) = -\int_{\mathbb{R}^2} u(\xi_1,\xi_2,0)\frac{\partial g}{\partial n}\, d\xi_1\, d\xi_2 = \int_{\mathbb{R}^2} u(\xi_1,\xi_2,0)\cdot \left.\frac{\partial g(x;\xi_1,\xi_2)}{\partial \xi_3}\right|_{\xi_3=0} d\xi_1\, d\xi_2.$$

(1 Mark)

ii) We first calculate

$$\begin{aligned} \left.\frac{\partial g(x;\xi_1,\xi_2)}{\partial \xi_3}\right|_{\xi_3=0} &= \frac{1}{4\pi}\left.\frac{x_3-\xi_3}{[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3-\xi_3)^2]^{3/2}}\right|_{\xi_3=0} \\ &\quad - \frac{1}{4\pi}\left.\frac{-(x_3+\xi_3)}{[(x_1-\xi_1)^2+(x_2-\xi_2)^2+(x_3+\xi_3)^2]^{3/2}}\right|_{\xi_3=0} \\ &= \frac{1}{2\pi}\frac{x_3}{[(x_1-\xi_1)^2+(x_2-\xi_2)^2+x_3^2]^{3/2}} \end{aligned}$$

(2 Marks) We then have

$$\begin{aligned} u(x_3,0,0) &= \frac{u_0}{2\pi}\int_{\xi_1^2+\xi_2^2\le a^2} \frac{x_3}{(\xi_1^2+\xi_2^2+x_3^2)^{3/2}}\, d\xi_1\, d\xi_2 \\ &= \frac{u_0}{2\pi}\int_0^a\int_0^{2\pi} \frac{x_3}{(r^2+x_3^2)^{3/2}}\, r\, d\theta\, dr \\ &= x_3u_0\int_0^a \frac{r}{(r^2+x_3^2)^{3/2}}\, dr \\ &= -x_3u_0\left.\frac{1}{(r^2+x_3^2)^{1/2}}\right|_0^a \\ &= u_0\left(1 - \frac{x_3}{\sqrt{a^2+x_3^2}}\right). \end{aligned}$$

(2 Marks)

□

Method of Images

You need to be able to calculate trigonometric Fourier series. Straightforward calculations like thse should not present any serious problems.

Exercise 6
Using the method of images, find Green's function for the problem

$$\Delta u(x_1, x_2, x_3) = \varrho(x_1, x_2, x_3), \qquad x_1 > 0,\ x_2 \in \mathbb{R},\ x_3 > 0,$$

with boundary conditions

$$\left.\frac{\partial u}{\partial x_3}\right|_{x_3=0} = 0, \qquad u\big|_{x_1=0} = 0.$$

(3 Marks)

Proof. There are three image charges:

$$\begin{aligned} g(x;\xi) &= E(x;\xi) - E(x;\xi^*) + E(x;\xi^{**}) - E(x;\xi^{***}) \\ &= \frac{1}{4\pi}\left(\frac{1}{|x-\xi|} - \frac{1}{|x-\xi^*|} + \frac{1}{|x-\xi^{**}|} - \frac{1}{|x-\xi^{***}|}\right) \end{aligned}$$

where $\xi^* = (-\xi_1, \xi_2, \xi_3)$, $\xi^{**} = (\xi_1, \xi_2, -\xi_3)$ and $\xi^{***} = (-\xi_1, \xi_2, -\xi_3)$. **(3 Marks)** □

Partial Eigenfunction Expansions

Exercise 7
Consider the Dirichlet problem for the half disk

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon |x|^2 \le 1,\ x_2 \ge 0\}.$$

i) Separate variables in the Dirichlet problem

$$\Delta_{(r,\theta)} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

with boundary conditions $u|_{\partial\Omega} = 0$ and find the eigenfunctions for the θ variable.
(3 Marks)

ii) Give the formal partial eigenfunction expansion for Green's function in terms of the θ eigenfunctions. Do not yet determine the coefficient functions.
(2 Marks)

iii) Determine the one-dimensional Green's function problem that the coefficients must satisfy.
(2 Marks)

iv) Solve the Green's function problem and find the coefficients, giving the partial eigenfunction expansion.
(5 Marks)

Proof.

i) We first find the eigenfucntions obtained from solving

$$\Delta_{(r,\theta)} u(r,\theta) = 0 \qquad (r,\theta) \in (0,1) \times (0,\pi).$$

Setting $u(r,\theta) = R(r) \cdot \Theta(\theta)$, we have

$$\Theta R'' + \frac{\Theta}{r} R' + \frac{R}{r^2}\Theta'' = 0$$

or

$$\frac{r^2}{R} R'' + \frac{r}{R} R' = -\frac{\Theta''}{\Theta} = \lambda.$$

Then we need to solve the two ODEs

$$\begin{aligned} \Theta'' + \lambda\Theta &= 0, \qquad & \Theta(0) = \Theta(\pi) = 0, \\ r^2 R'' + rR' - \lambda R &= 0, \qquad & R(0) = R(1) = 0 \end{aligned}$$

The first equation gives the orthonormalized eigenfunctions

$$\Theta_m(\theta) = \sqrt{\frac{2}{\pi}} \sin(m\theta), \qquad m = 1, 2, 3, \ldots$$

with eigenvalues $\lambda_m = m^2$.

ii) We have

$$g(r,\theta;\varrho,\vartheta) = \sum_{m=1}^{\infty} g_m(r;\varrho,\vartheta)\sin(m\theta)$$

where

$$g_m(r;\varrho,\vartheta) = \frac{2}{\pi}\int_0^{\pi} g(r,\theta;\varrho,\vartheta)\sin(m\theta)\,d\theta.$$

iii) Green's function for Ω, expressed in polar coordinates, satisfies

$$-\Delta_{(r,\theta)} g(r,\theta;\varrho,\vartheta) = \frac{\delta(\theta-\vartheta)\delta(r-\varrho)}{r}, \qquad (r,\theta),(\varrho,\vartheta)\in\Omega \qquad (*)$$

Multiplying $(*)$ with $\frac{2}{\pi}\sin(m\theta)$ and integrating, we obtain

$$-\int_0^{\pi}\frac{2}{\pi}\sin(m\theta)\left(\frac{\partial^2 g}{\partial r^2}+\frac{1}{r}\frac{\partial g}{\partial r}+\frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}\right)d\theta = \frac{2}{\pi r}\sin(m\vartheta)\delta(r-\varrho)$$

$$\Leftrightarrow -\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)\int_0^{\pi}\frac{2}{\pi}\sin(m\theta)g(r,\theta;\varrho,\vartheta)\,d\theta - \int_0^{\pi}\frac{2}{\pi}\sin(m\theta)\frac{1}{r^2}\frac{\partial^2 g}{\partial\theta^2}\,d\theta = \frac{2}{\pi r}\sin(m\vartheta)\delta(r-\varrho)$$

$$\Leftrightarrow -\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)g_m(r;\varrho,\vartheta) + \frac{m^2}{r^2}g_m(r;\varrho,\vartheta) = \frac{2}{\pi r}\sin(m\vartheta)\delta(r-\varrho)$$

This equation, together with boundary conitions $g_m(0;\varrho,\vartheta) = g_m(1;\varrho,\vartheta) = 0$ determines the coefficients g_m in the expansion.

iv) Writing g_m' for $\frac{\partial}{\partial r}g_m$, the general solution for the homogeneous equation

$$-r^2 g_m'' - r g_m' + m^2 g_m = 0$$

is

$$g_m(r;\varrho,\vartheta) = c_1\cdot r^m + c_2\cdot r^{-m}.$$

Solving for $r<\varrho$ with $g_m|_{r=0} = 0$, we choose the bounded solution

$$g_m(r;\varrho,\vartheta) = c_1 r^m \qquad \text{for } r<\varrho$$

In order to satisfy $g_m(1;\varrho,\vartheta) = 0$ we take

$$g_m(r;\varrho,\vartheta) = c_2\left(r^m - r^{-m}\right) \qquad \text{for } r>\varrho$$

We now require continuity of g_m,

$$c_1\varrho^m = c_2\left(\varrho^m - \varrho^{-m}\right)$$

and the jump condition

$$mc_2\left(\varrho^{m-1}+\varrho^{-m-1}\right) - mc_1\varrho^{m-1} = -\frac{2}{\pi\varrho}\sin(m\vartheta)$$

or

$$mc_2\left(\varrho^m+\varrho^{-m}\right) - mc_1\varrho^m = -\frac{2}{\pi}\sin(m\vartheta).$$

Plugging in the continuity condition,

$$mc_2\left(\varrho^m+\varrho^{-m}\right) - mc_2\left(\varrho^m-\varrho^{-m}\right) = -\frac{2}{\pi}\sin(m\vartheta)$$

so

$$c_2 = -\frac{1}{m}\frac{\varrho^m}{\pi}\sin(m\vartheta)$$

and

$$c_1 = c_2(1-\varrho^{-2m}) = -\frac{1}{m}\frac{\varrho^m-\varrho^{-m}}{\pi}\sin(m\vartheta)$$

This gives

$$g_m(r;\varrho,\vartheta) = \frac{1}{m}\frac{r^m(\varrho^{-m}-\varrho^m)}{\pi}\sin(m\vartheta) \qquad \text{for } r<\varrho$$

and

$$g_m(r; \varrho, \vartheta) = \frac{1}{m} \frac{-\varrho^m (r^m - r^{-m})}{\pi} \sin(m\vartheta) \qquad \text{for } r > \varrho.$$

Writing $r_> = \max\{r, \varrho\}$ and $r_< = \min\{r, \varrho\}$, the coefficients of the expansion can be expressed as

$$g_m(r; \varrho, \vartheta) = \frac{r_<^m (r_>^{-m} - r_>^m)}{m\pi} \sin(m\vartheta)$$

so

$$g(r, \theta; \varrho, \vartheta) = \sum_{m=1}^{\infty} g_m(r; \varrho, \vartheta) \sin(m\theta) = \sum_{m=1}^{\infty} \frac{r_<^m (r_>^{-m} - r_>^m)}{m\pi} \sin(m\vartheta) \sin(m\theta)$$

$\square$