# Chapter 3:
# Static Electric Fields

Lecturer: Nana Liu

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JOINT INSTITUTE
交大密西根学院



Many slides courtesy of Sung-Liang Chen

Main parts in electrostatics:

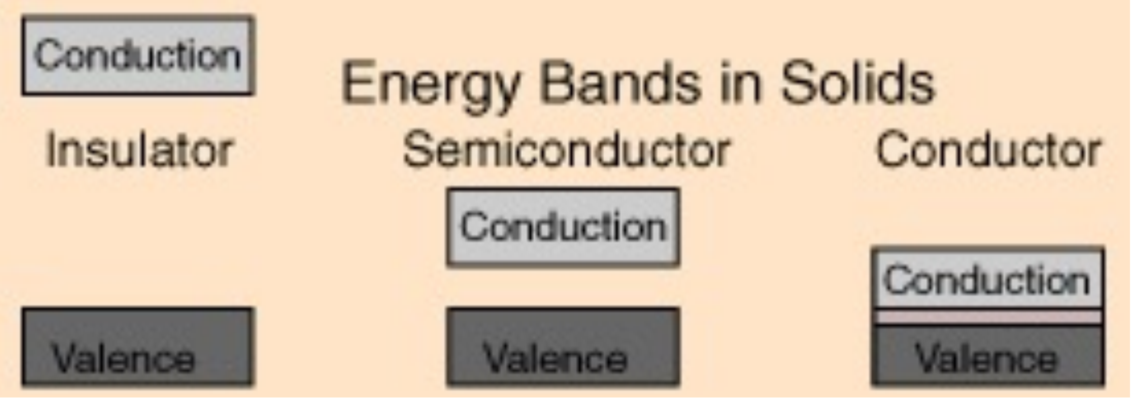
(I) Electrostatic equations
(II) Solving for E field in different scenarios
(III) Energy in E field
(IV) Conductors and insulators

# 3-6 Conductors in Static Electric Field

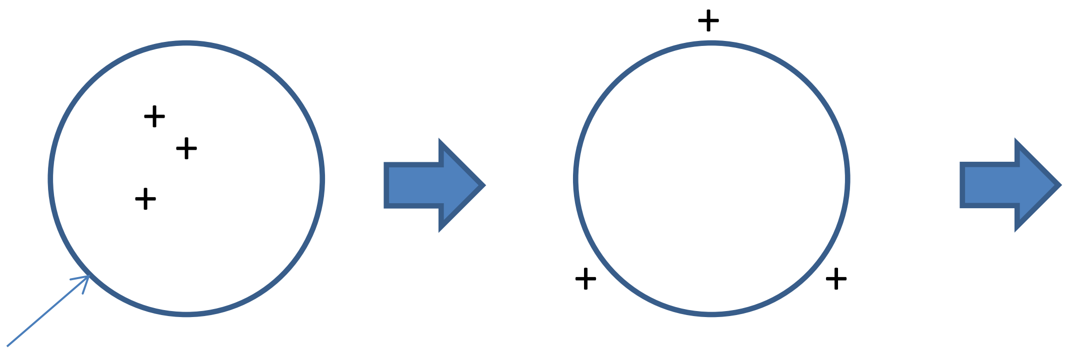
- 3 types: conductors, semiconductors, insulators (or dielectrics)
- Conductors: Orbiting electrons are loosely held by an atom and migrate easily from one atom to another.
- Insulators: Electrons confined to their orbits
- Semiconductors: A small number of freely moveable charges (between conductors and insulators)

# Band theory

- Crucial to the conduction process is whether or not there are electrons in the conduction band



# $\mathbf{E}$ and $\rho$ inside a Conductor



Conductor surface

Charge distribution due to electrostatic force

$\rho_{inside}=0$

**Inside a Conductor**
*(Under Static Conditions)*

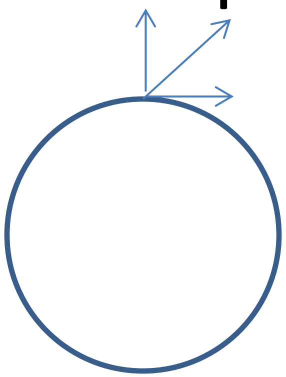
$\rho = 0$

$\mathbf{E} = 0$

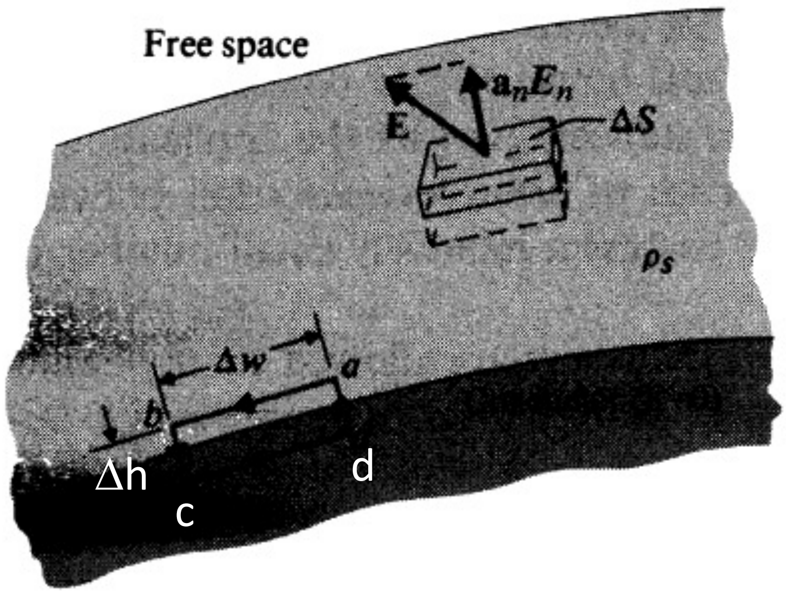
By Gauss's law

# Equilibrium

- At a state of equilibrium (static charges), tangential **E**=0. Otherwise, charges move… ➔ only normal **E** components



- Under static conditions the **E** field on a conductor surface is everywhere normal to the surface. The surface of a conductor is an **equipotential surface** under static conditions.




Conductor surface

Inside conductor

**FIGURE 3–18**
A conductor–free space interface.

$$\oint_{abcda} \mathbf{E} \cdot d\boldsymbol{\ell} = E_t \Delta w = 0$$

(1) To check E on surface, let $\Delta h \rightarrow 0$ (integrals along bc, da = 0)
(2) By Gauss's Law, E inside=0 ➔ $E_t$ inside = 0 (integral along cd = 0)
(3) The integral along ab = $E_t \Delta w$

$$E_t = 0,$$

The tangential component of the E field on a conductor surface is zero.

# Normal Component of **E**

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = E_n \Delta S = \frac{\rho_s \Delta S}{\epsilon_0}$$

6 faces:

(1) Let $\Delta h \rightarrow 0$ ➔ integrals over 4 side surfaces = 0
(2) $E_{inside}=0$ ➔ Integral over the inside surface = 0
(3) The integral over the top surface = $E_n \Delta S$

$$E_n = \frac{\rho_s}{\epsilon_0}.$$

The normal component of the E field at a conductor/free space boundary is equal to the surface charge density on the conductor divided by the permittivity of free space.

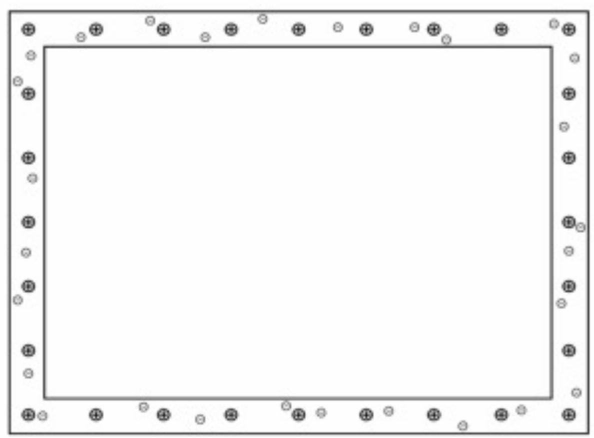
| Boundary Conditions at a Conductor/Free Space Interface |
|---|
| $E_t = 0$ |
| $E_n = \dfrac{\rho_s}{\epsilon_0}$ |

# An Uncharged Conductor in a Static **E** Field

- $E_{external}$ ➔ Electrons moving ➔ $E_{induced}$
- $E_{induced}$ cancels $E_{external}$ both inside the conductor and tangent to its surface

$E_{external}$

Faraday Cage



$E_{induced}$

# Faraday cage demonstration

https://www.youtube.com/watch?v=QU0fLnucE6A

# Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

An uncharged spherical conductor centered at the origin has a cavity of some weird shape carved out of it (Fig. 2.46). Somewhere within the cavity is a charge $q$. *Question:* What is the field outside the sphere?

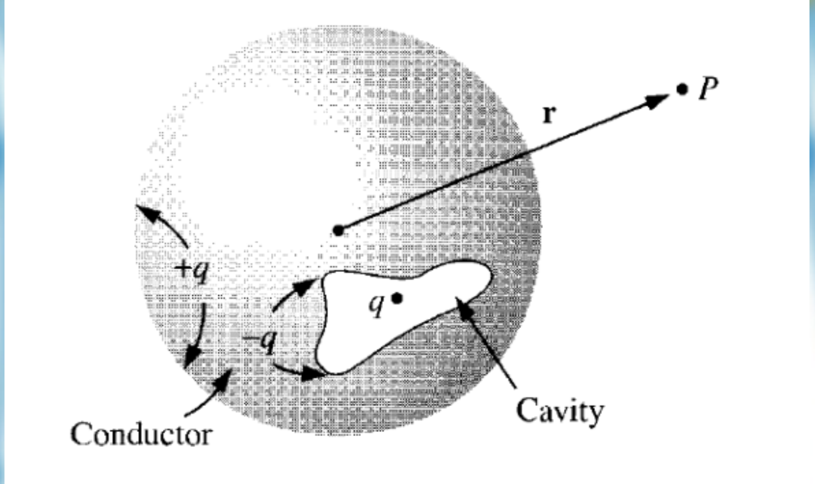



Figure 2.46

# Quick classroom exercise

Answer:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

# 3-7 Dielectrics in Static Electric Field

- Dielectrics: bound charges
- $\mathbf{E}_{external}$ ➔ polarize a dielectric material and create electric dipoles, which is $\mathbf{E}_{induced}$ ➔ modify **E** both inside and outside the dielectric material

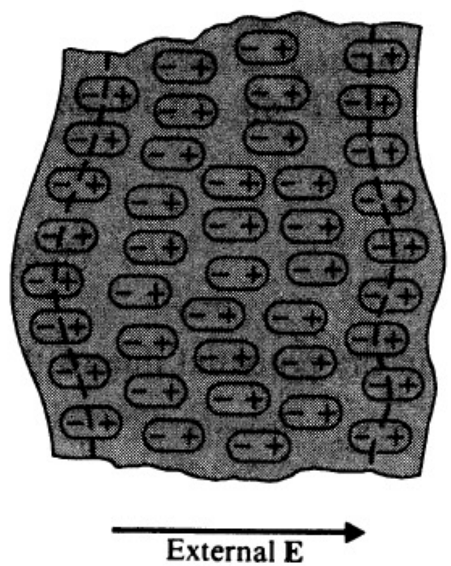



**FIGURE 3–20**
A cross section of a polarized dielectric medium.

# Permanent Dipole Moments

- Some materials have non-zero dipole moments in the absence of external **E** field
  - E.g., $H_2O$ (polar molecule)

- Macroscopic viewpoint
  - Without external **E**: no net dipole moment
  - With **E**: molecules aligned due to external **E** ➔ nonzero net dipole moment

# An Electric Dipole: revision

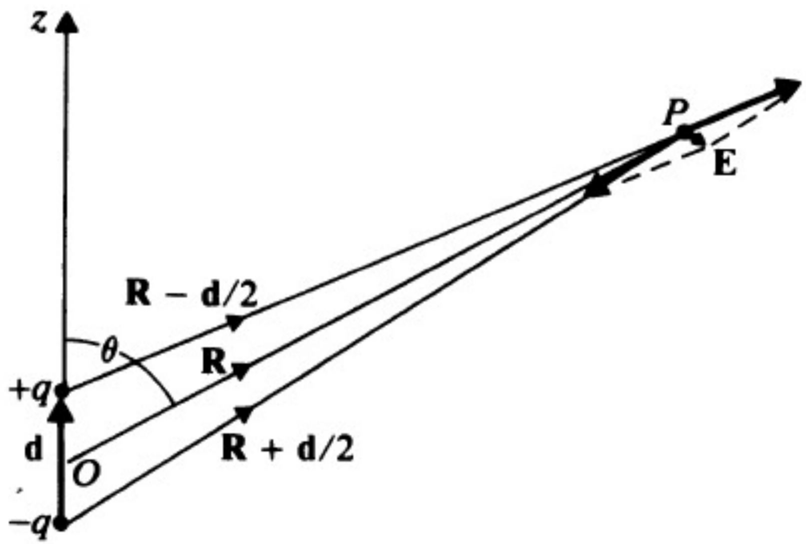



**FIGURE 3–5**
Electric field of a dipole.

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}.$$

If d<<R

$$\begin{aligned}\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} &= \left[\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\cdot\left(\mathbf{R}-\frac{\mathbf{d}}{2}\right)\right]^{-3/2} \\ &= \left[R^2 - \mathbf{R}\cdot\mathbf{d} + \frac{d^2}{4}\right]^{-3/2} \\ &\cong R^{-3}\left[1 - \frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]^{-3/2} \\ &\cong R^{-3}\left[1 + \frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right],\end{aligned}$$

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}.$$

$$\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1+\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right]$$

$$\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^{-3} \cong R^{-3}\left[1-\frac{3}{2}\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\right].$$

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right].$$

Manipulation of vector quantities is tedious.

Any method to avoid?

# Electric Dipole Moment: revision

- Definition: The product of the charge q and the vector **d**

$$\mathbf{p} = q\mathbf{d}.$$

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{p}}{R^2}\mathbf{R} - \mathbf{p}\right]$$

$$\mathbf{p} = \mathbf{a}_z p = p(\mathbf{a}_R \cos\theta - \mathbf{a}_\theta \sin\theta),$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta,$$

$$\boxed{\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta \sin\theta) \quad \text{(V/m)}.}$$

Why it decreases more rapidly?

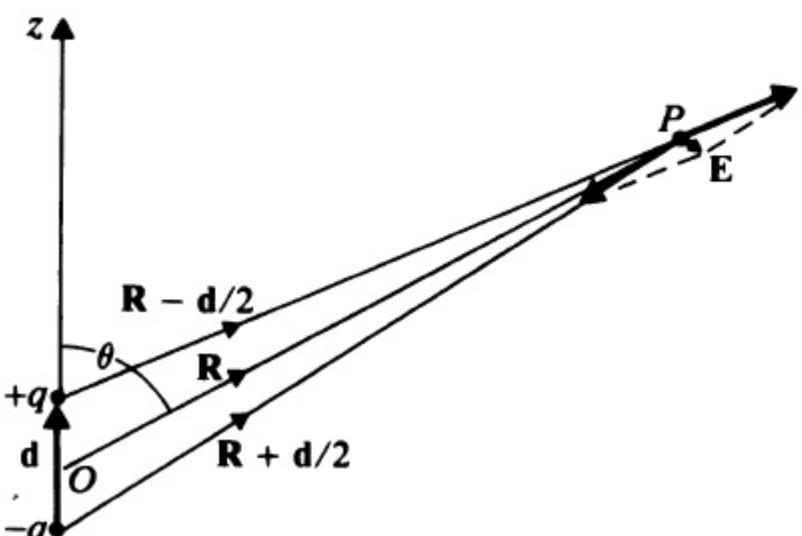



FIGURE 3–5
Electric field of a dipole.

# Electrical Potential Difference

$$\boxed{\mathbf{E} = -\nabla V}$$

$$\frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \qquad \text{(J/C or V).}$$

$$-\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell = \int_{P_1}^{P_2} (\nabla V) \cdot (\mathbf{a}_\ell \, d\ell)$$

$$= \int_{P_1}^{P_2} dV = V_2 - V_1.$$

$$\boxed{V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell \qquad \text{(V).}}$$

Usually, the zero-potential point is taken at infinity ($P_1$).

# $V$ of an Electric Dipole

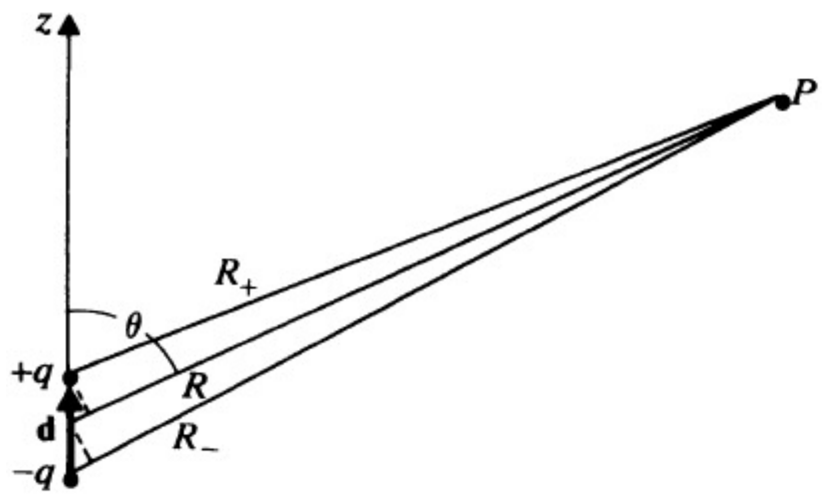



**FIGURE 3–14**
An electric dipole.

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_+} - \frac{1}{R_-}\right).$$

If $d \ll R$

$$\frac{1}{R_+} \cong \left(R - \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 + \frac{d}{2R}\cos\theta\right)$$

$$\frac{1}{R_-} \cong \left(R + \frac{d}{2}\cos\theta\right)^{-1} \cong R^{-1}\left(1 - \frac{d}{2R}\cos\theta\right).$$

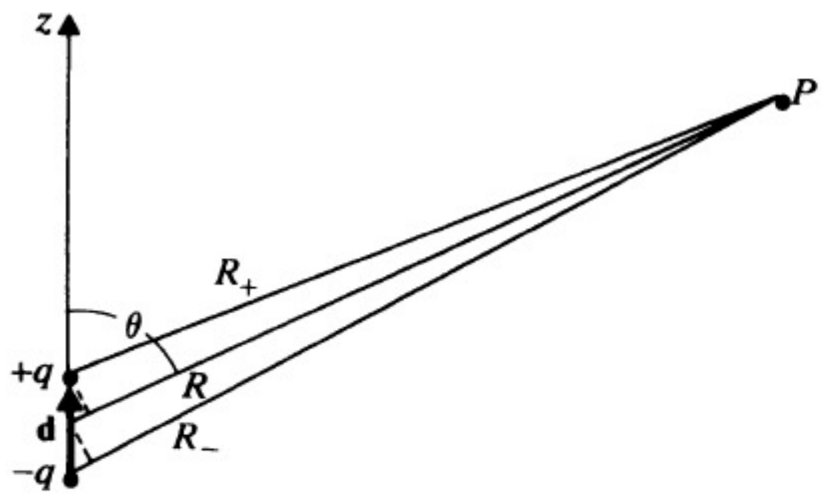



**FIGURE 3–14**
An electric dipole.

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2}$$

$$\boxed{V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},}$$

where $\mathbf{p} = q\mathbf{d}$.

$\boxed{\mathbf{d}\cdot\mathbf{a}_R = d\cos\theta}$

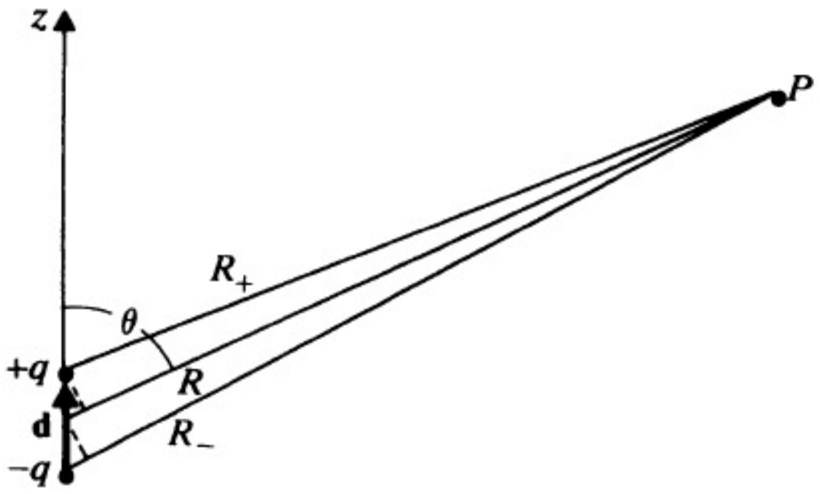



**FIGURE 3–14**
An electric dipole.

$$V = \frac{qd\cos\theta}{4\pi\epsilon_0 R^2}$$

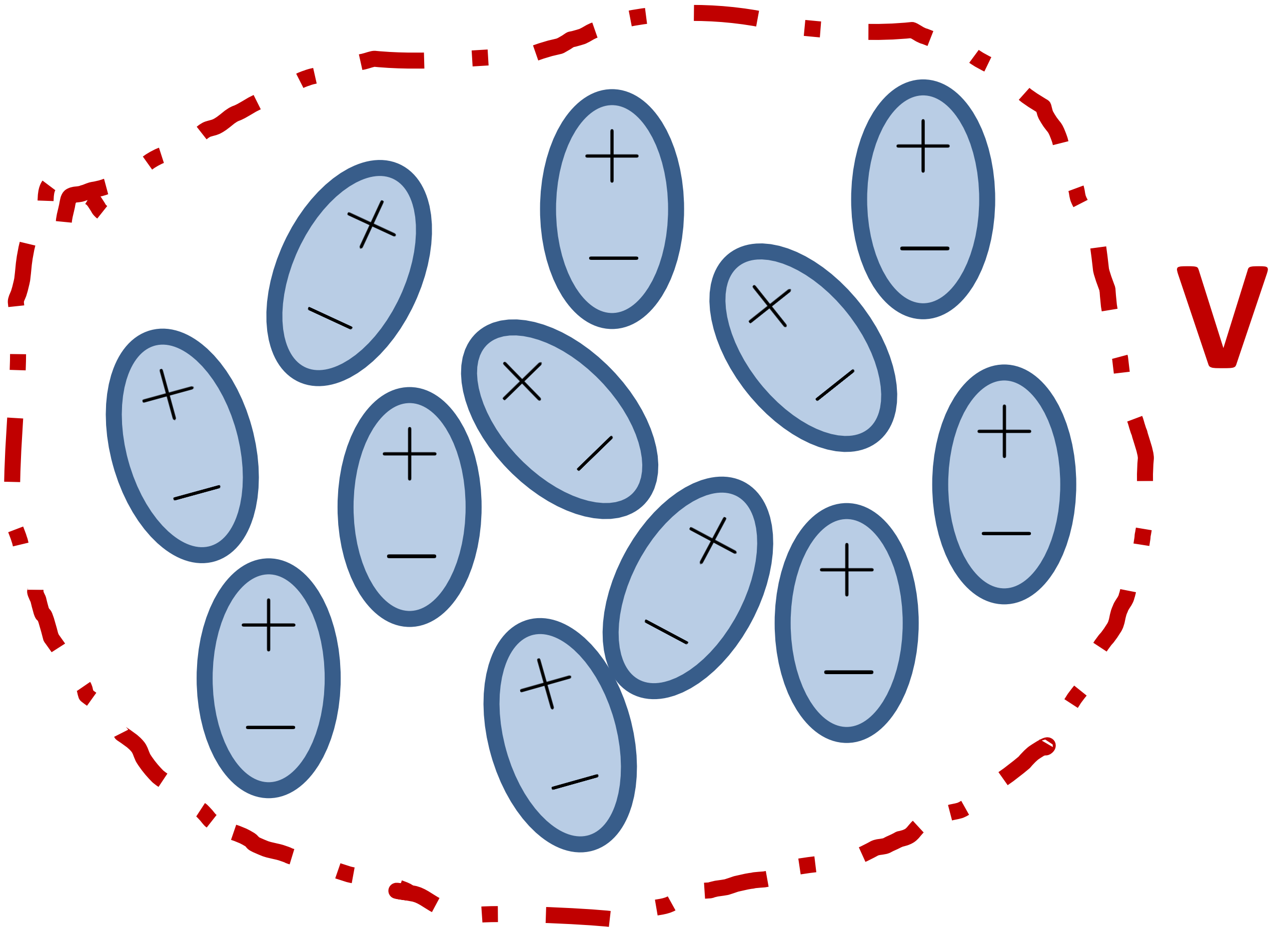
## Polarisation vector

$$\boxed{V = \frac{\mathbf{p}\cdot\mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad \text{(V)},}$$

where $\mathbf{p} = q\mathbf{d}$.

$\boxed{\mathbf{d}\cdot\mathbf{a}_R = d\cos\theta}$

# Charge Distributions of Polarized Dielectrics



# 3-7.1 Equivalent Charge Distributions of Polarized Dielectrics

- Macroscopic effect
- Polarization vector:

N = n$\Delta$v,
where N is the Total # in a volume ($\Delta$v);
n is the number density

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v} \qquad (\mathrm{C/m^2}),$$

**P**: volume density of electric dipole moment **p=qd**

$$d\mathbf{p} = \mathbf{P}\,dv'$$

- Derivation for dielectrics

$$\boxed{V = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} \qquad (\mathrm{V}),} \Rightarrow dV = \frac{\mathbf{P} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2}\,dv'.$$

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\mathbf{P} \cdot \mathbf{a}_R}{R^2}\, dv',$$

Primed is the coordinate of source

$$R^2 = (x - x')^2 + (y - y')^2 + (z - z')^2,$$

$$\nabla'\left(\frac{1}{R}\right) = \frac{\mathbf{a}_R}{R^2}.$$

Taking the integral over the primed coordinates and the observer location at the unprimed coordinates

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{P} \cdot \nabla'\left(\frac{1}{R}\right) dv'.$$

$$\nabla' \cdot (f\mathbf{A}) = f\nabla' \cdot \mathbf{A} + \mathbf{A} \cdot \nabla' f,$$ letting $\mathbf{A} = \mathbf{P}$ and $f = 1/R$,

$$V = \frac{1}{4\pi\epsilon_0}\left[\int_{V'} \nabla' \cdot \left(\frac{\mathbf{P}}{R}\right) dv' - \int_{V'} \frac{\nabla' \cdot \mathbf{P}}{R}\, dv'\right].$$

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\mathbf{P} \cdot \mathbf{a}'_n}{R} \, ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{(-\nabla' \cdot \mathbf{P})}{R} \, dv',$$

By divergence theorem

V = contribution of surface charge distribution

+

contribution of volume charge distribution

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

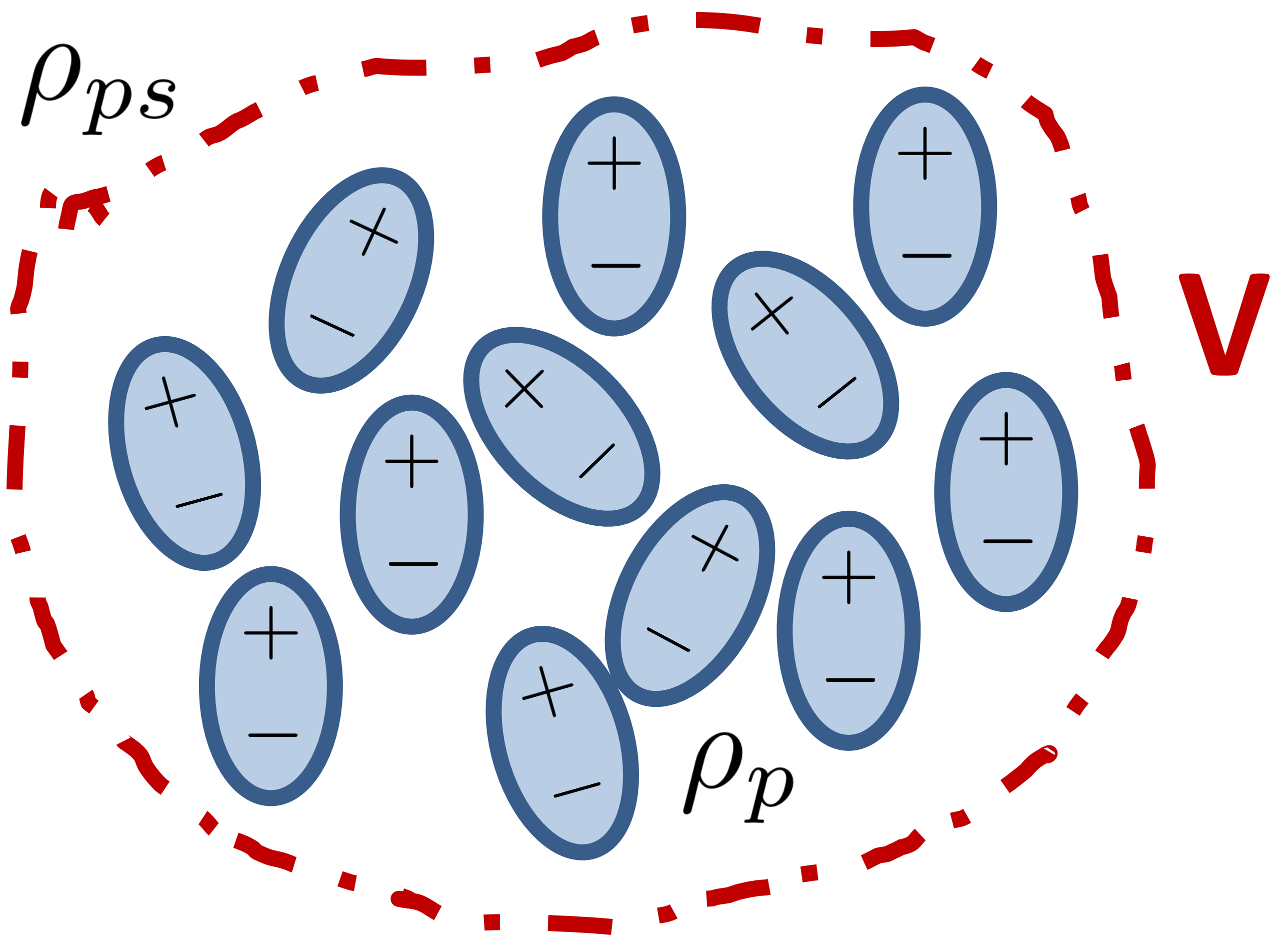
’ has been dropped for simplicity

$$V = \frac{1}{4\pi\epsilon_0} \oint_{S'} \frac{\rho_{ps}}{R} \, ds' + \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho_p}{R} \, dv'.$$

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

Polarization surface charge densities, and polarisation bound-charge densities



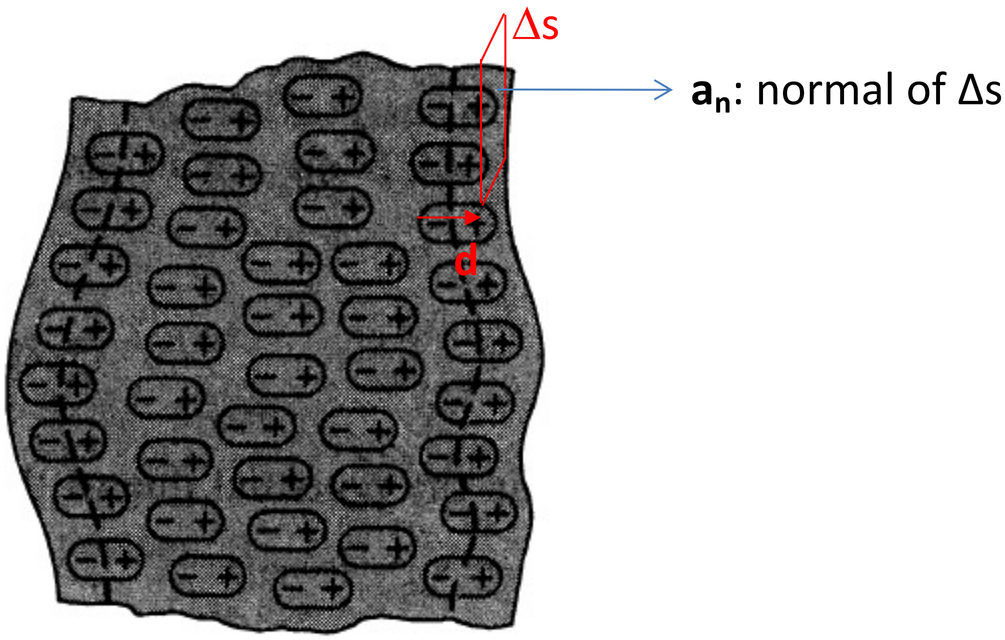



**FIGURE 3–20**
A cross section of a polarized dielectric medium.

External **E**

➔ causes a separation d of bound charges:

+q to d/2 along **E**,

-q to d/2 against **E**

Total charge crossing the surface Δs: $nq\,\underline{d}(\Delta s)$,

$$\Delta Q = nq\underline{(\mathbf{d}\cdot\mathbf{a}_n)}(\Delta s).$$ for $\mathbf{d} \nparallel \mathbf{a}_n$

$$\Delta Q = nq(\mathbf{d} \cdot \mathbf{a}_n)(\Delta s).$$

**p**=q**d**
**P**=n**p**=nq**d**

$$\Delta Q = \mathbf{P} \cdot \mathbf{a}_n(\Delta s)$$

$$\rho_{ps} = \frac{\Delta Q}{\Delta s} = \mathbf{P} \cdot \mathbf{a}_n,$$

**P** is along **d**

- Right side surface: $\mathbf{P}\cdot\mathbf{a}_n > 0 \Rightarrow \rho_{ps} > 0$
- Left side surface: $\mathbf{P}\cdot\mathbf{a}_n < 0 \Rightarrow \rho_{ps} < 0$

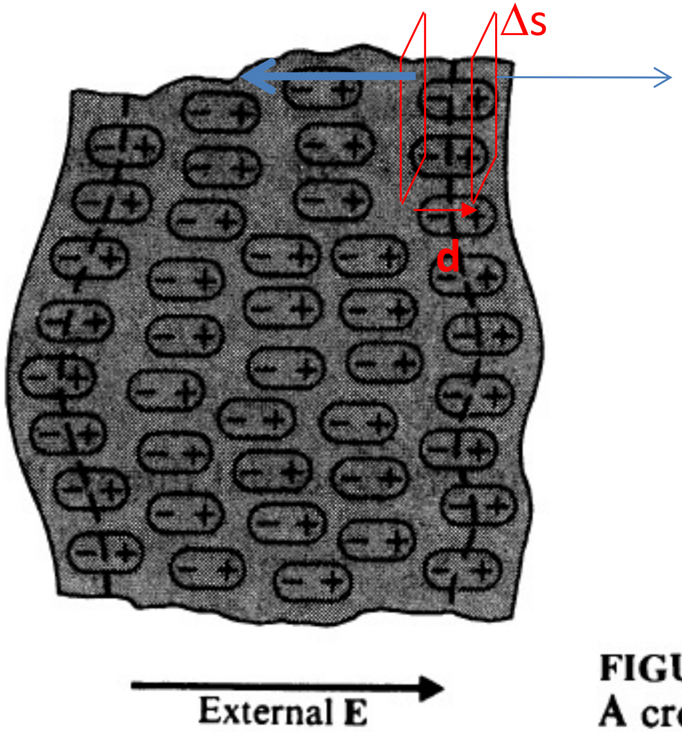



**FIGURE 3–20**
A cross section of a polarized dielectric medium.

$$\boxed{\Delta Q = \mathbf{P} \cdot \mathbf{a}_n(\Delta s)}$$

The net charge remaining within the volume V is the negative of the integral

$$Q = -\oint_S \mathbf{P} \cdot \mathbf{a}_n \, ds$$
$$= \int_V (-\nabla \cdot \mathbf{P})\, dv = \int_V \rho_p \, dv,$$

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

Since starting with an electrically neutral dielectric body, the total charge of the body after polarization must remain zero

$$\text{Total charge} = \oint_S \rho_{ps} \, ds + \int_V \rho_p \, dv$$
$$= \oint_S \mathbf{P} \cdot \mathbf{a}_n \, ds - \int_V \nabla \cdot \mathbf{P} \, dv = 0,$$

Verified.

$$\boxed{\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n}$$

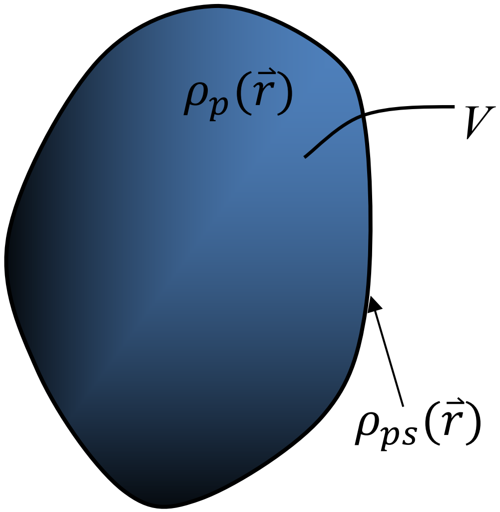
$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

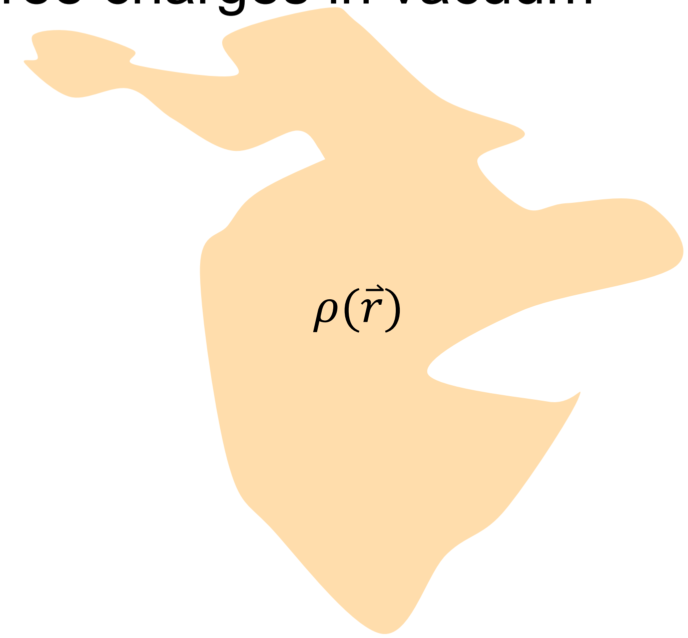
- I. Field expression in terms of charges (bound charge distribution)
- II. Expression in terms of polarizations (bound dipole distribution)

# I.

## Fields in the Presence of Material Media

Material (bound charges $\rho_b$) Free charges in vacuum




The effect of material media on the electrostatic field everywhere is described phenomenologically by the presence of ***bound charges***. Bound charges are **fictitious**. Electromagnetics itself does not tell us how to find appropriate values of those fictitious charges. We have to use other models of physical reality to find these charges.

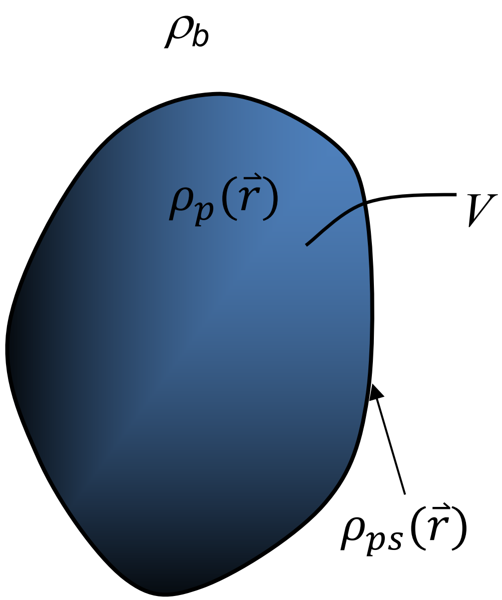
In fact, the field due to bound charges (electrons and protons inside the atom, ions, etc.) cannot be distinguished from the field due to free charges. They **behave identically**. They apply forces on other charges in the same exact way. **Thus, we must mathematically treat these bound charges in the same way that we treat the free charges.**

Subscript (b) will now refer to the bound charges. Maxwell's equations for bound charges are therefore:

$$\nabla \cdot \boldsymbol{E_b}\,(\boldsymbol{r}) = \frac{\rho_b(\boldsymbol{r})}{\varepsilon_0}$$ and $$\nabla \cdot \boldsymbol{E_b}\,(\boldsymbol{r}) = 0$$ when $\boldsymbol{r} \in V\ surface$

We also define an electrostatic potential for bound charges in the same way that we do for free charges:

$$\boldsymbol{E_b}\,(\boldsymbol{r}) = -\nabla \varphi_b\,(\boldsymbol{r})$$

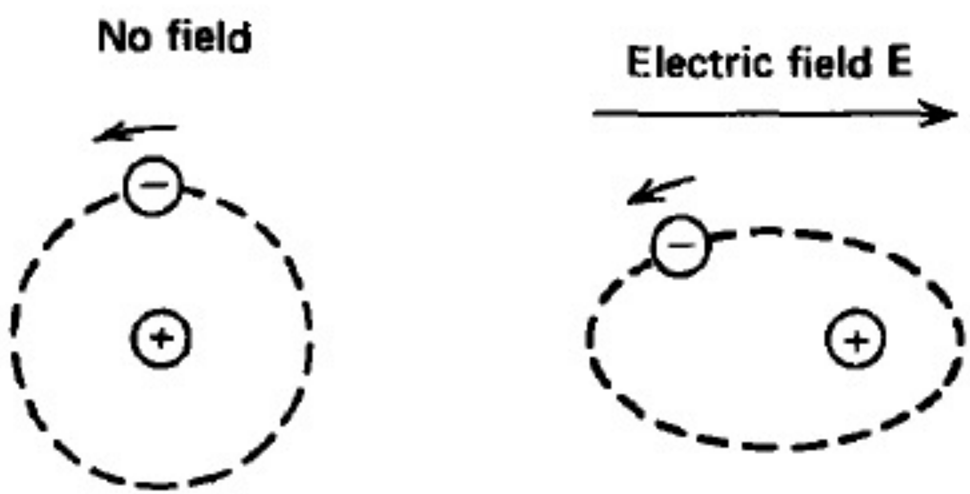



Bound charges may be found **on the surface ($\rho_{ps}$)** of a material, and they may also be found **within the interior ($\rho_p$)** of the material. We know the potential must therefore look like:

$$\varphi_b\,(\boldsymbol{r}) = \oiint_{S_V} \frac{\rho_{ps}(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}dS_S + \iiint_V \frac{\rho_p(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}dV_S$$

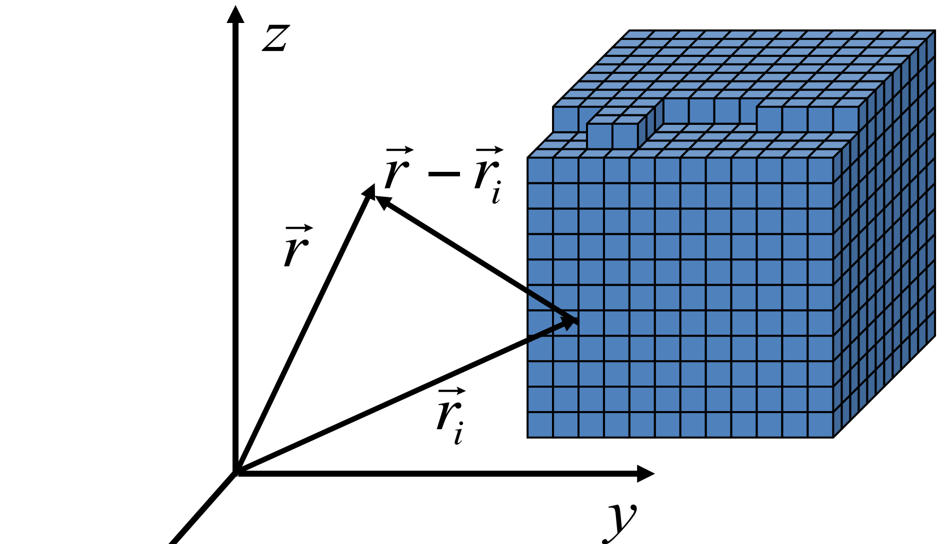
# II.
## Materials Viewed as a Collection of Dipoles

However, another equivalent method of **viewing neutrally charged materials is as a distribution of electric dipoles**. These dipoles are commonly found in atoms, whose electron cloud is displaced by an external electric field.




Thus, it is natural to ask whether we can derive equations for materials in terms of a bound dipole distribution, on the surface and within the interior.

# View of Materials as a Collection of Point Dipoles




$$\varphi(\vec{r}) = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{\vec{p}_i \cdot (\vec{r} - \vec{r}_i)}{\left|\vec{r} - \vec{r}_i\right|^3}$$

However, we know that there is no such thing as a *true point dipole*, like there is no such thing as a *true point mass*. All materials occupy some discrete volume, and thus it is more relevant to consider a material as having a mass defined by a mass density and the volume it occupies. The same is done for dipoles, which are considered to have a "***dipole density***" or "***polarization***".

# View of Materials as an Integral of Polarization

**p** = ∫**P**dV$_s$

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_V \frac{\boldsymbol{P}(\boldsymbol{r_s})\cdot(\boldsymbol{r}-\boldsymbol{r}_s)}{|\boldsymbol{r}-\boldsymbol{r}_s|^3}\,dV_S = \frac{1}{4\pi\varepsilon_0}\iiint_V \boldsymbol{P}(\boldsymbol{r_s})\cdot\nabla\frac{-1}{|\boldsymbol{r}-\boldsymbol{r}_s|}\,dV_S$$

$$\varphi(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_V \underline{\boldsymbol{P}(\boldsymbol{r_s})\cdot\nabla_S\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_s|}}\,dV_S$$

Switching the Gradient with respect to $\boldsymbol{r}_s$

Now using the product rule it can be rewritten as:

$$\boxed{\nabla'\cdot(f\mathbf{A}) = f\nabla'\cdot\mathbf{A} + \underline{\mathbf{A}\cdot\nabla' f},}$$

$$\varphi(\boldsymbol{r}) = \iiint_V \nabla_S\cdot\left(\frac{\boldsymbol{P}(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}\right)dV_S - \iiint_V \frac{\nabla_S\cdot\boldsymbol{P}(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}\,dV_S$$

$$\varphi(\boldsymbol{r}) = \oiint_{S_V} \frac{\boldsymbol{P}(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}\cdot\hat{\boldsymbol{n}}\,dS_s + \iiint_V \frac{-\nabla_S\cdot\boldsymbol{P}(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}\,dV_S$$

# A duality: Two Views of the Field Due to Materials

I. View of materials in terms of **bound charge distribution**:

$$\varphi_b(\boldsymbol{r}) = \oiint_{S_V} \frac{\rho_{ps}(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}dS_s + \iiint_V \frac{\rho_p(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}dV_S$$

II. View of materials in terms of **bound dipole distribution**:

$$\varphi_b(\boldsymbol{r}) = \oiint_{S_V} \frac{\boldsymbol{P}(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}\cdot\hat{\boldsymbol{n}}dS_s + \iiint_V \frac{-\nabla_s\cdot\boldsymbol{P}\,(\boldsymbol{r_s})}{4\pi\varepsilon_0|\boldsymbol{r}-\boldsymbol{r}_s|}dV_S$$

Conclusion: Materials can be viewed either as a collection of bound charges or as a collection of bound dipoles. They are related by:

$$\rho_{ps} = \boldsymbol{P}\cdot\hat{\boldsymbol{n}} \qquad\qquad \rho_p = -\nabla\cdot\boldsymbol{P}$$

In other words, we can treat a material as having a **surface charge distribution**, or we can treat the same material as having a discontinuity in the normal component of the Polarization at an interface.

$$\boxed{\rho_{ps} = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}}$$

Likewise, we can treat a material as having a **volume charged distribution**, or we can treat the same material as having a divergence in the polarization within the interior of the material.

$$\boxed{\rho_p = -\nabla \cdot \boldsymbol{P}}$$

These two views lead to identical results.

Where these relations become useful is in the modification of the description of Maxwell's Equations (Section 3.8).

# 3-8 Electric Flux Density and Dielectric Constant

$$\boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

- In 3-7, polarization **P** or bound volume charge density $\rho_p$ ➔ produces **E** field due to $\rho_p$

- Modification of divergence postulates:

$$\boxed{\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}} \Rightarrow \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho + \underline{\rho_p}).$$

$$\Downarrow \quad \boxed{\rho_p = -\nabla \cdot \mathbf{P}.}$$

$$\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho.$$

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\mathrm{C/m^2}).}$$

Where **D**: electric flux density, electric displacement

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\mathrm{C/m^3}),}$$

$$\int_V \nabla \cdot \mathbf{D}\, dv = \int_V \rho\, dv$$

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \qquad (\mathrm{C}).}$$

Integral form

Another from of Gauss's law: The total outward flux of the **electric displacement** (or, simply, the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.

# Modified Maxwell’s Equations

Equations of Electrostatics in Any Medium

$$\boxed{\nabla \cdot \mathbf{D} = \rho \qquad (\mathrm{C/m^3}),}$$

$$\boxed{\nabla \times \mathbf{E} = 0.}$$

# $\chi_e$ and $\varepsilon_r$

- Electric susceptibility
  - For linear and isotropic medium,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E},$$

$\chi_e$ dimensionless quantity called ***electric susceptibility***

- Relative permittivity (dielectric constant)

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \qquad (\text{C/m}^2).}$$

$$\Rightarrow$$

$$\boxed{\begin{aligned} \mathbf{D} &= \epsilon_0 (1 + \chi_e) \mathbf{E} \\ &= \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E} \qquad (\text{C/m}^2), \end{aligned}}$$

$$\epsilon_r = 1 + \chi_e = \frac{\epsilon}{\epsilon_0}$$

$\varepsilon$: absolute permittivity (or simply permittivity)

# A Simple Medium

- Linear: $\chi_e$ is dependent of **E** only (not $|\mathbf{E}|^2$, $|\mathbf{E}|^3$…)
- Homogeneous: $\chi_e$ is independent of space, $\chi_e(\cancel{r})$
- Isotropic: $\chi_e$ is a scalar, not a tensor ➔ **P//E**

- A simple medium: linear, homogeneous, and isotropic

- $\varepsilon_r$ in a simple medium is a constant

# Anisotropic Medium

- The $\varepsilon_r$ is different for different directions of the electric field
  - **D** and **E** vectors generally have different directions
  - $\overline{\overline{\varepsilon}}$ is a tensor

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

- For crystals, choosing a proper coordinate system, $\overline{\overline{\varepsilon}}$ can be simplified

# Anisotropic: Biaxial and Uniaxial

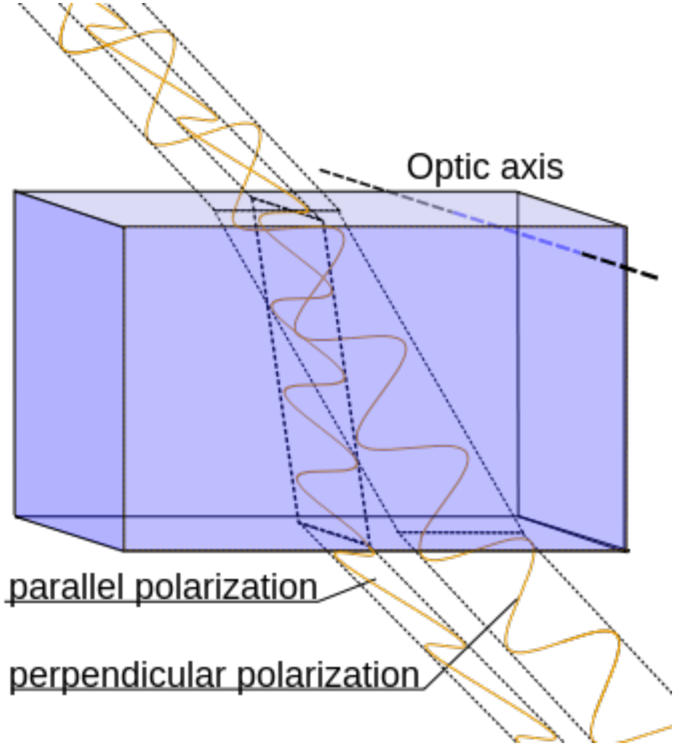
- Biaxial: $\varepsilon_1 \neq \varepsilon_2 \neq \varepsilon_3$

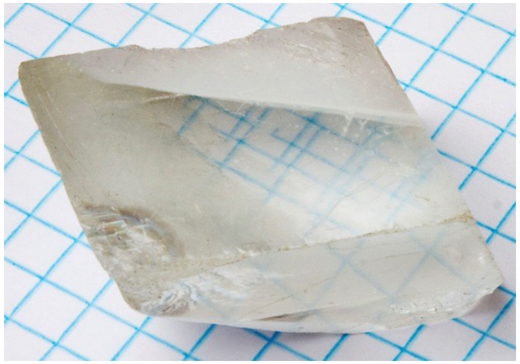
$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}.$$

$$D_x = \epsilon_1 E_x,$$
$$D_y = \epsilon_2 E_y,$$
$$D_z = \epsilon_3 E_z.$$

- Uniaxial: $\varepsilon_1 = \varepsilon_2 \neq \varepsilon_3$

# Optical Birefringence



$\chi_e$ depends on directions of $\mathbf{E}$

Different polarizations (**E** in different directions) ➔ see different $\chi_e$ or $\varepsilon_r$ ➔ see different refractive index n ➔ different refraction

# 3-8.1 Dielectric Strength

- External **E** → Displacement of bound charges → Polarization
- Dielectric breakdown: If very strong external E causes **permanent dislocation of electrons** and damage in the material, avalanche effect of ionization due to collisions may occur. **The material becomes conducting** and may result in large currents.

- Dielectric strength: the maximum **E** intensity that a dielectric material can withstand without breakdown

**TABLE 3–1**
**Dielectric Constants and Dielectric Strengths of Some Common Materials**

| Material | Dielectric Constant | Dielectric Strength (V/m) |
|---|---|---|
| Air (atmospheric pressure) | 1.0 | $3 \times 10^6$ |
| Mineral oil | 2.3 | $15 \times 10^6$ |
| Paper | 2–4 | $15 \times 10^6$ |
| Polystyrene | 2.6 | $20 \times 10^6$ |
| Rubber | 2.3–4.0 | $25 \times 10^6$ |
| Glass | 4–10 | $30 \times 10^6$ |
| Mica | 6.0 | $200 \times 10^6$ |

# Supplementary Material

- Relationships between **D** and **E**
- Linear materials
- Microscopic pictures of polarizations
- Example: Point Charge Embedded in a Dielectric Sphere
- Nonlinear materials

Electric Fields inside material are not able to distinguish whether the field is produced by true charges in vacuum or material induced charges. Thus, the Maxwell's equations must be modified as follows:

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = \frac{\rho(\boldsymbol{r}) + \rho_b(\boldsymbol{r})}{\varepsilon_0} = \frac{\rho(\boldsymbol{r}) - \nabla \cdot \boldsymbol{P}(\boldsymbol{r})}{\varepsilon_0}$$

$$\nabla \cdot (\varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r})) = \rho(\boldsymbol{r})$$

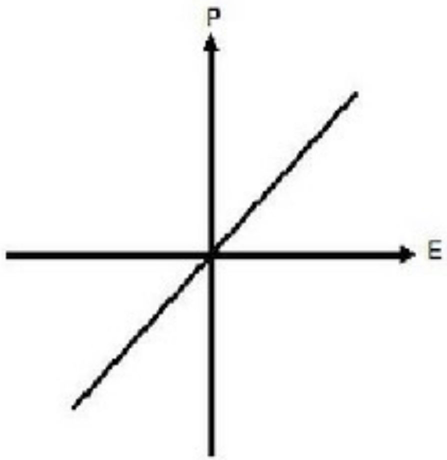
$$\nabla \cdot \boldsymbol{D}(\boldsymbol{r}) = \rho(\boldsymbol{r}) \qquad \boldsymbol{D}(\boldsymbol{r}) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r}))$$

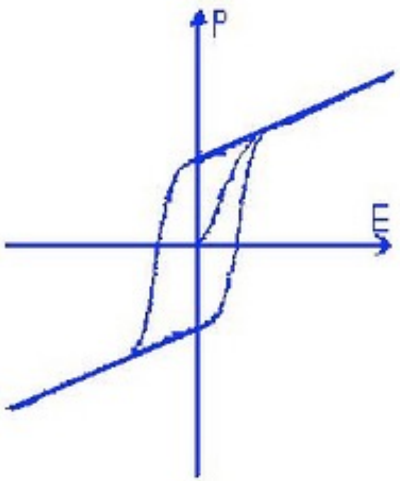
- Here, it is convenient to introduce a new field, called the **electric displacement field, D**, which is **the field due only to true vacuum charges** and does not incorporate the field due to materials.
- **E**, on the other hand, is the total field produced by both true charges in vacuum as well as the field from materials.

Unfortunately, these relationships by themselves are not sufficient for determining the fields everywhere in space. We still do not know the relationship between **D** and **E** fields. These relationships are different for each materials and can only be **experimentally determined**. These are known as "*material constitutive relationships*".

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r}))$$

In many materials, the polarization follows a linear relationship with the electric field. In other materials (e.g., ferroelectric materials), there is "*hysteresis*"



# Linear Materials: Dielectric Constant

For the vast majority of materials, the polarization follows a linear relationship as follows:

$$\boldsymbol{P}(\boldsymbol{r}) = \chi_e \varepsilon_0 \boldsymbol{E}(\boldsymbol{r})$$

Where $\chi_e$ is the **slope** in the **P-E** graph, and is called the "dielectric susceptibility". This term tells us **how susceptible a material is to polarize in an external field**. For this class of materials, we thus have the relationship"

$$\boldsymbol{D}(\boldsymbol{r}) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{P}(\boldsymbol{r}) = \varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) + \chi_e \varepsilon_0 \boldsymbol{E}(\boldsymbol{r})$$

$$\boldsymbol{D}(\boldsymbol{r}) = (1 + \chi_e) \varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) = \varepsilon_r \varepsilon_0 \boldsymbol{E}(\boldsymbol{r}) = \varepsilon \boldsymbol{E}(\boldsymbol{r})$$

$$\varepsilon_r = (1 + \chi_e)$$

This term is known as the relative permittivity of a material

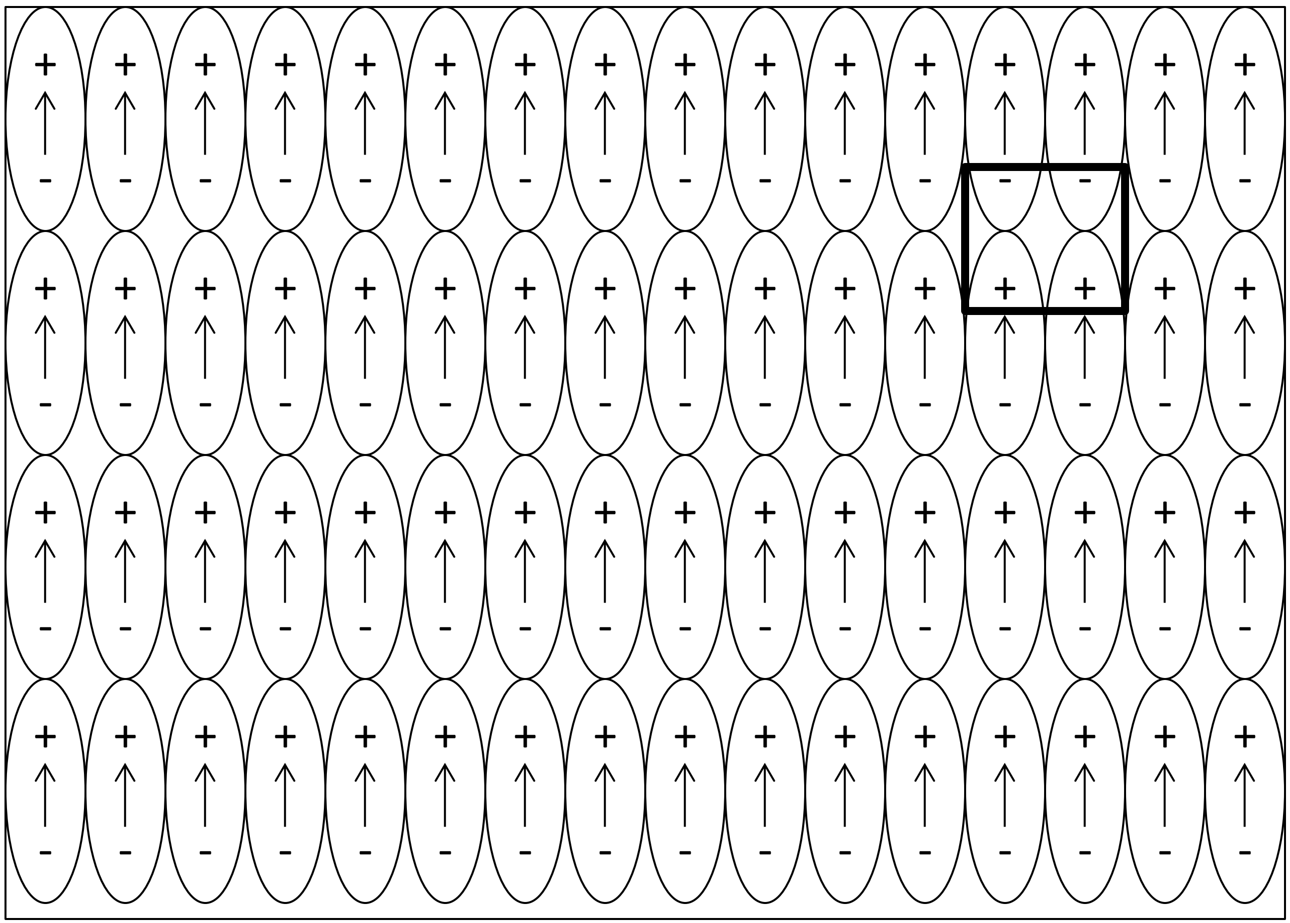
| | $\varepsilon_r = \varepsilon/\varepsilon_0$ |
|---|---|
| Carbon Tetrachloride[a] | 2.2 |
| Ethanol[a] | 24 |
| Methanol[a] | 33 |
| *n*-Hexane[a] | 1.9 |
| Nitrobenzene[a] | 35 |
| Pure Water[a] | 80 |
| Barium Titanate[b] (with 20% Strontium Titanate) | >2100 |
| Borosilicate Glass[b] | 4.0 |
| Ruby Mica (Muscovite)[b] | 5.4 |
| Polyethylene[b] | 2.2 |
| Polyvinyl Chloride[b] | 6.1 |
| Teflon[b] (Polytetrafluorethylene) | 2.1 |
| Plexiglas[b] | 3.4 |
| Paraffin Wax[b] | 2.2 |

[a] *From* Lange's Handbook of Chemistry, *10th ed., McGraw-Hill, New York, 1961, pp. 1234–37.*

[b] *From A. R. von Hippel (Ed.) Dielectric Materials and Applications, M.I.T., Cambridge, Mass., 1966, pp. 301–370*

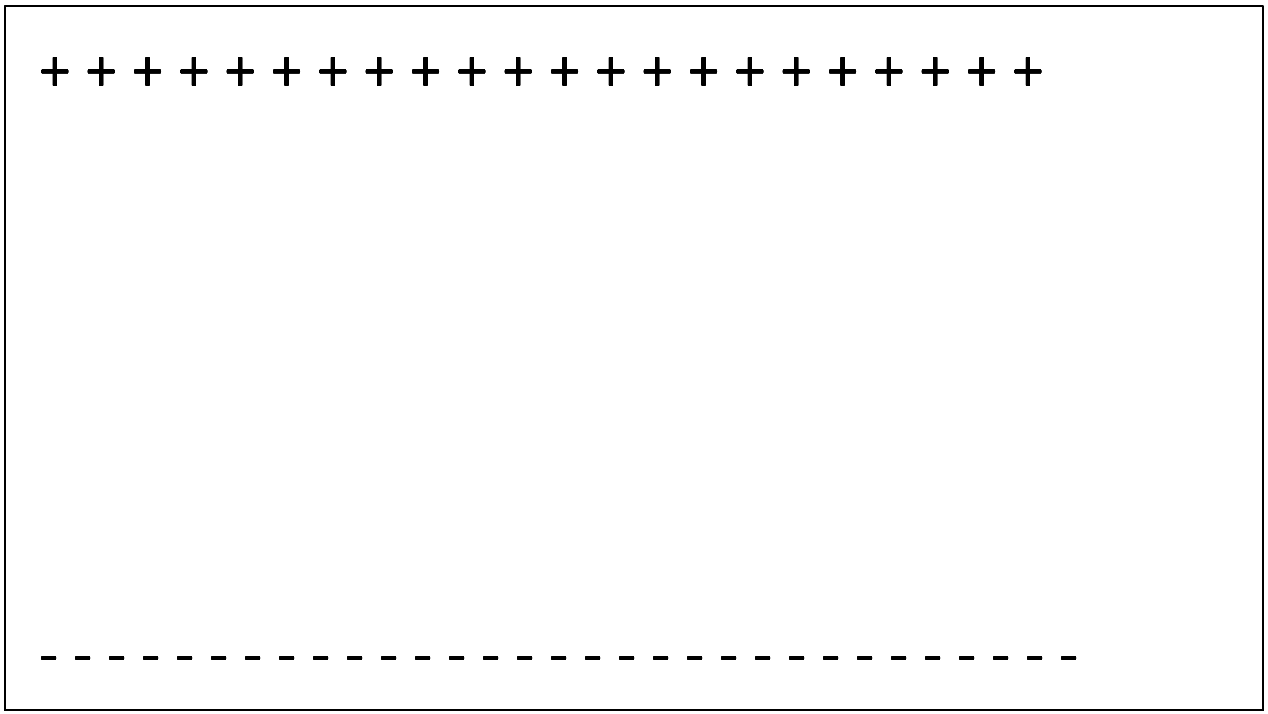
# Microscopic Picture of Uniform Polarization

Each molecule in a material can be visualized as a dipole. And the overall charge distribution of an array of these dipoles looks like:



# Illustration of the Equivalent Charge Distribution

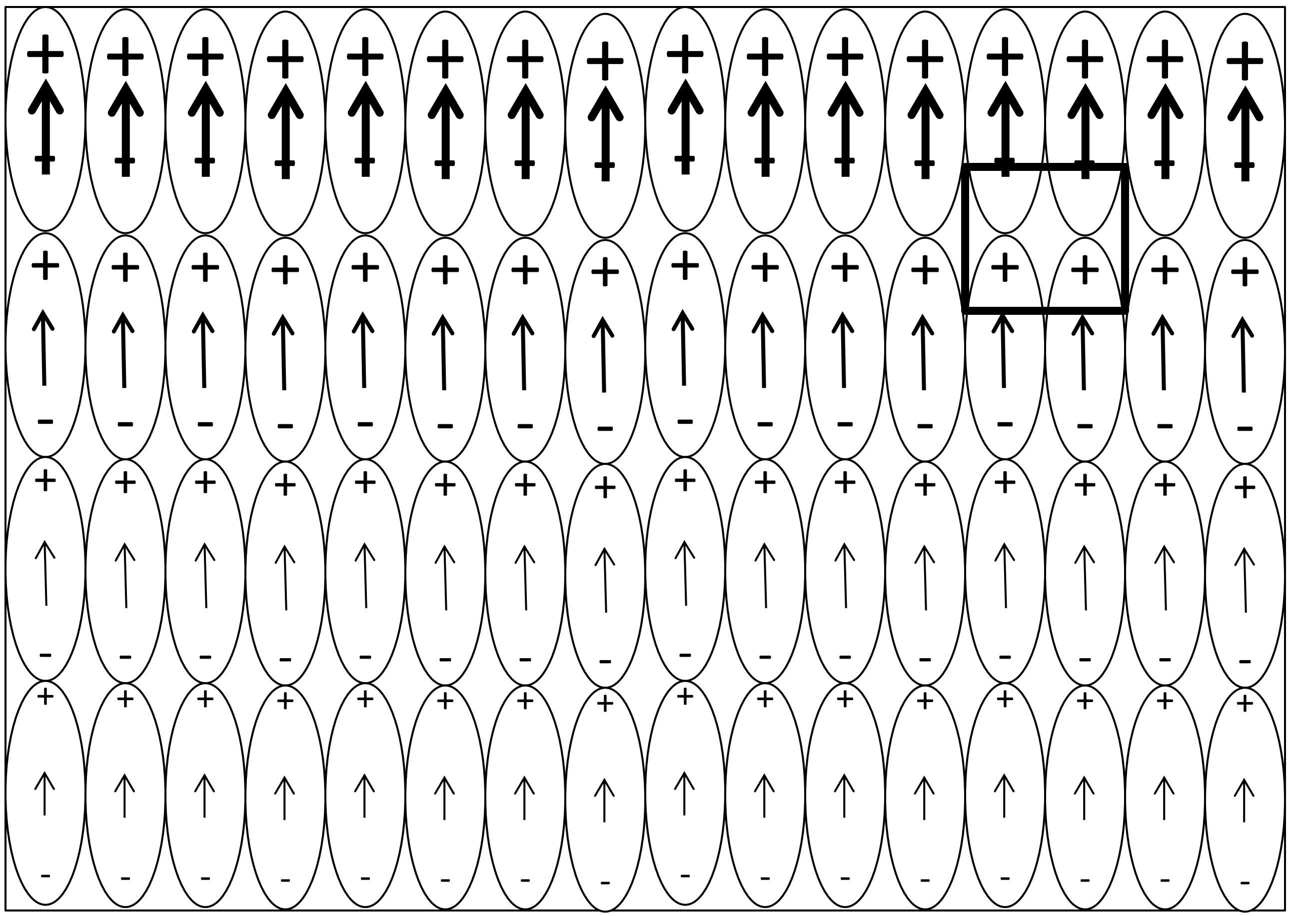
If each molecule in the material has the same dipole moment and is polarized in the same direction, then **all the internal charges cancel**, and the overall field resembles a material with two charge surfaces at the boundary of the material. This is what happens in capacitors.



Provided you know the polarization strength, it becomes a simple matter to determine the field of a material.

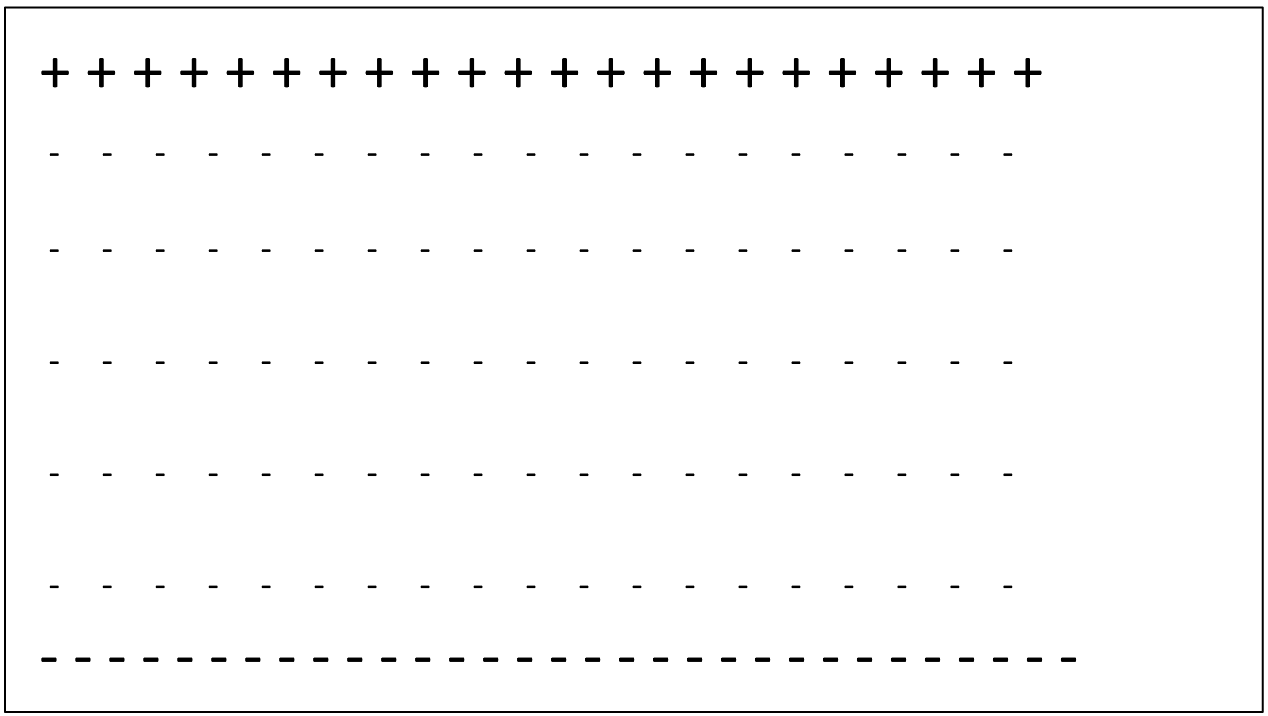
# Microscopic Picture of Non-Uniform Polarization

Each molecule in a material can be visualized as a dipole. And the overall charge distribution of an array of these dipoles looks like:



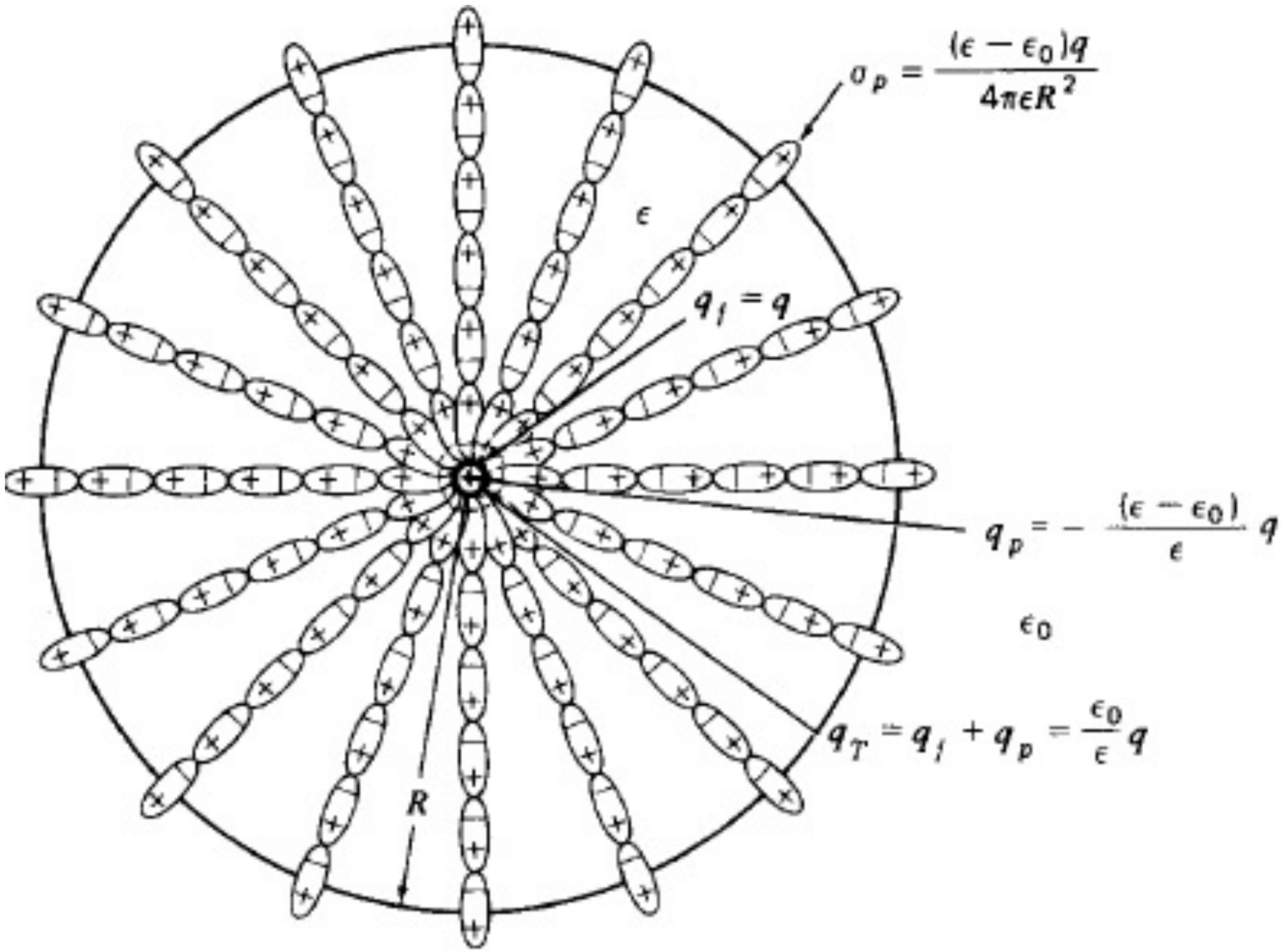
# Illustration of the Equivalent Charge Distribution

If there is a polarization gradient, then **the internal charges do not cancel**, and the overall field must be treated as a combination of a **surface and volume charge distribution**.



# Example: Point Charge Embedded in a Dielectric Sphere

Let's use Maxwell's equations to solve the fields inside materials.



# Point Charge Embedded in a Dielectric Sphere

$$\oiint \vec{\boldsymbol{D}} \cdot \hat{n}\, dS = \iiint_V \rho\, dV_s$$

$$D_r 4\pi r^2 = Q$$

$$\therefore \vec{\boldsymbol{D}} = \frac{Q}{4\pi r^2}\hat{r}$$

$$\vec{\boldsymbol{E}} = \frac{Q}{4\pi}\begin{cases} \dfrac{1}{\varepsilon r^2}\hat{r} & r < R \\ \dfrac{1}{\varepsilon_0 r^2}\hat{r} & r > R \end{cases}$$

$$\boldsymbol{P} = \boldsymbol{D} - \varepsilon_0 \boldsymbol{E} =$$

$$\vec{\boldsymbol{P}} = \frac{Q}{4\pi r^2}\begin{cases} \left(\dfrac{\varepsilon - \varepsilon_0}{\varepsilon}\right)\hat{r} & r < R \\ 0 & r > R \end{cases}$$

$$\boldsymbol{E} = -\nabla\varphi$$

$$\varphi(\vec{r}) = \frac{Q}{4\pi}\begin{cases} \dfrac{1}{\varepsilon r} + \dfrac{1}{\varepsilon R}\left(\dfrac{\varepsilon - \varepsilon_0}{\varepsilon_0}\right) & r < R \\ \dfrac{1}{\varepsilon_0 r} & r > R \end{cases}$$

# Polarization Induced Charge on Sphere Surface

At the outer surface of the sphere $\boldsymbol{P}(\boldsymbol{R}) \cdot \hat{\boldsymbol{n}} = P_r(\boldsymbol{R}) = \rho_{ps} = \dfrac{Q(\varepsilon - \varepsilon_0)}{4\pi\varepsilon \ R^2}$

In order to maintain charge neutrality, we must have an equal and opposite polarization induced point charge at the center of the sphere. The total polarization charge must sum to zero.

At the center: $$q_p = -Q\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon}\right)$$

$\boxed{q_p + q_{ps} = 0}$

Thus the sphere can be modeled as the combination of the true and polarization charge at the center along with the polarization charge on the outer surface.

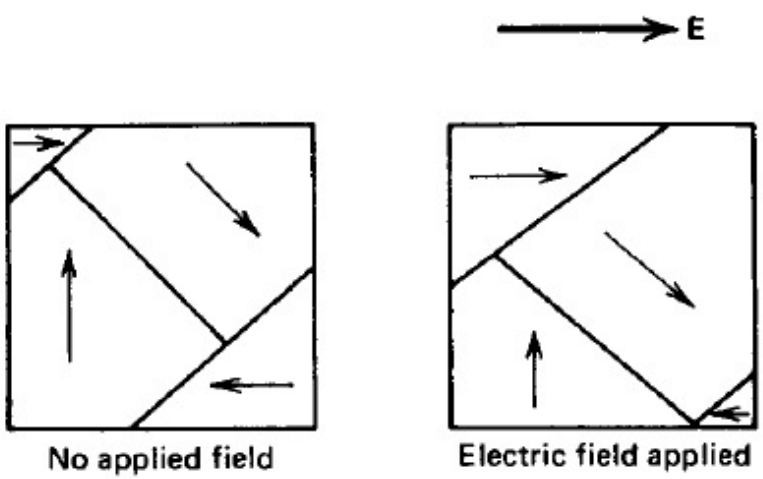
At the center: $$q_T = q + q_p = Q\frac{\varepsilon_0}{\varepsilon} = \frac{Q}{\varepsilon_r}$$

$\boxed{<Q}$

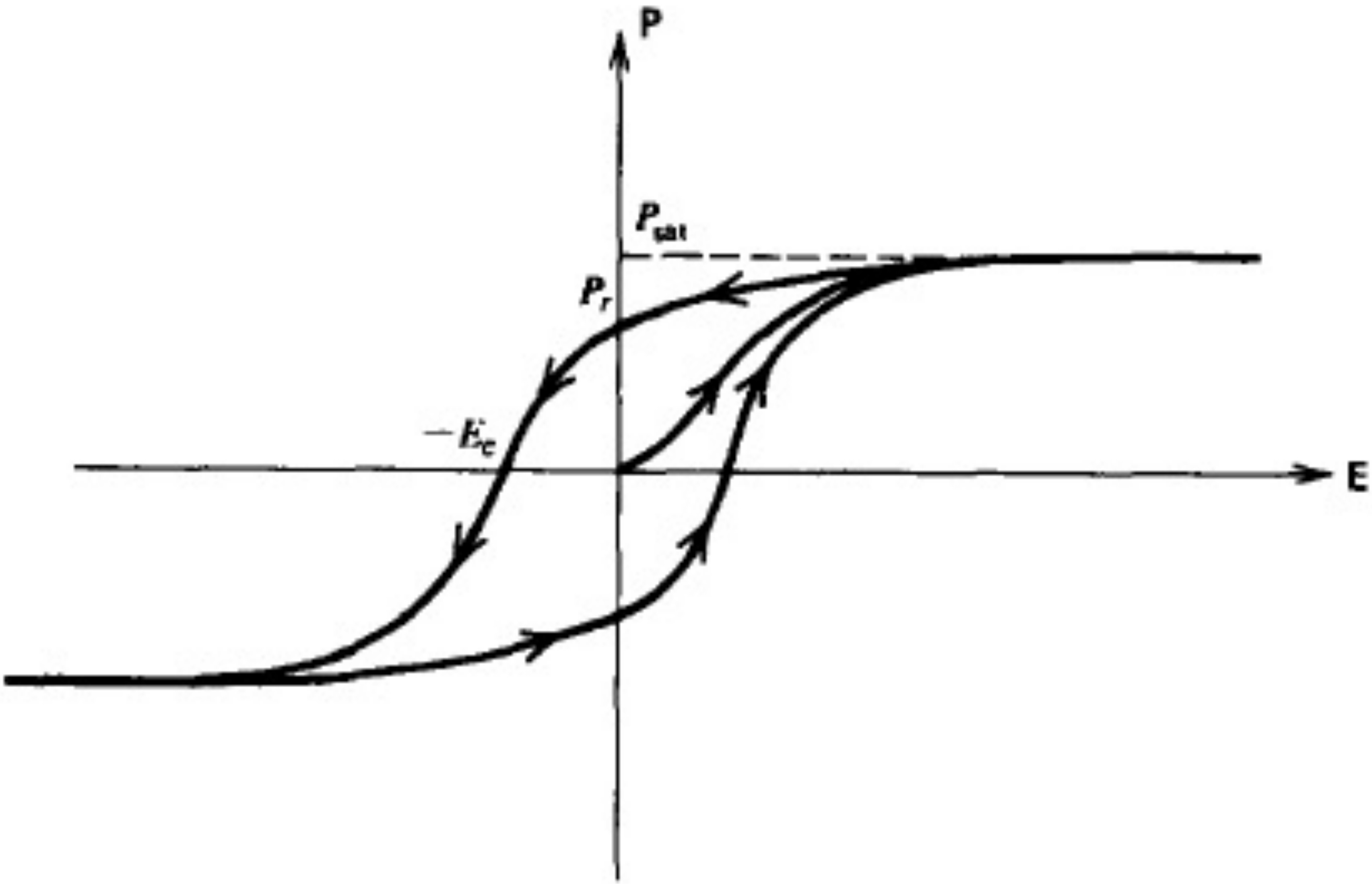
The effect of the dielectric is therefore **the displace some of the original charge to its outer surface**.

# Non-Linear Materials: Hysteresis

- Ferroelectric materials: such materials exists **spontaneous polarization** even when there is no external field. Ferromagnetic materials (permanent magnets) also display similar behavior.
- Domains: There is a whole field for describing how the material changes its polarization when an external field is applied. It is also well know that there are “domains” inside material that have constant polarization and well defined boundaries. The application of an electric field amounts to the **shrinking or growing** of these domains.



# Ferroelectric Hysteresis Curve



# Other Interesting Polarization Behavior

**Electrets**: Usually waxes or other polymers that can accept an trap an injected charge (Flash memory is based on this effect). Other types of electrets can **permanently store electric polarization** by applying an electric field while it cools from a liquid state. The dipoles in the material become permanently aligned.

**Electrostriction:** This class of materials experience a change in the dielectric constant with strain. Often forms the basis of strain gauges as well as acoustic actuators and impedance measurement systems.

Strain ➔ change $\varepsilon_r$

**Piezoelectricity:** This class of materials experiences a shape deformation in the presence of an electric field. These materials are used in ultrasound actuators and in step motors of various kinds.

Electric signal ➔ Deformation ➔ Ultrasound

# 3-9 Boundary Conditions for Electrostatic Fields

- Knowledge of the relations of the **field quantities at an interface** between two media is of importance for electromagnetic problems.

# B.C.: Tangential Component

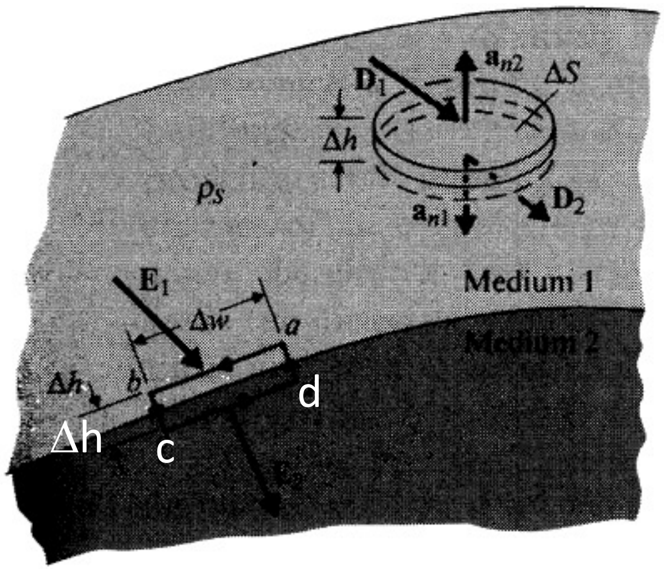



**FIGURE 3–23**
An interface between two media.

$\boxed{\text{let sides } bc = da = \Delta h \text{ approach zero}}$

$$\oint_{abcda} \mathbf{E} \cdot d\ell = \mathbf{E}_1 \cdot \Delta\mathbf{w} + \mathbf{E}_2 \cdot (-\Delta\mathbf{w}) = E_{1t}\Delta w - E_{2t}\Delta w = 0.$$

$$\boxed{E_{1t} = E_{2t} \qquad (\text{V/m}),}$$

The tangential component of an **E** field is continuous across an interface.

# B.C.: Tangential Component

- A conductor/free space interface:

$$E_{2t,\text{conductor}}=0 \Rightarrow E_{1t,\text{free space}}=0$$

- Two dielectrics:

$$\boxed{E_{1t} = E_{2t} \qquad (\text{V/m}),} \Rightarrow \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}.$$

# B.C.: Normal Component

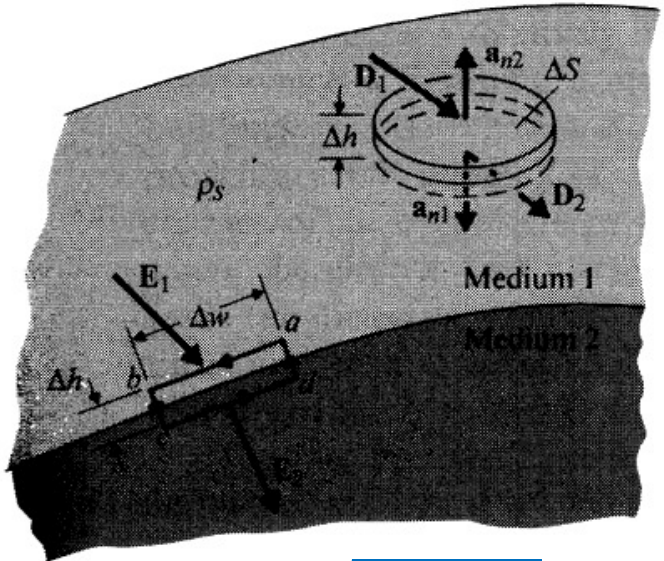



**FIGURE 3–23**
An interface between two media.

$\Delta h \rightarrow 0$

Gauss's law:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = (\mathbf{D}_1 \cdot \mathbf{a}_{n2} + \mathbf{D}_2 \cdot \mathbf{a}_{n1})\Delta S$$
$$= \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2)\Delta S$$
$$= \rho_s \Delta S,$$

$$\boxed{\mathbf{a}_{n2} = -\mathbf{a}_{n1}}$$

$$\boxed{\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s} \qquad \boxed{D_{1n} - D_{2n} = \rho_s \qquad (\mathrm{C/m^2}),}$$

Ref.: $\mathbf{a}_{n2}$

The normal of **D** field is **discontinuous** across an interface where **a surface charge** exists—the amount of discontinuity being equal to the surface charge density.

# B.C.: Normal Component

- For a dielectric (Medium 1)/conductor (Medium 2) interface:

$$\mathbf{D}_2 = 0 \quad \Rightarrow \quad D_{1n} = \epsilon_1 E_{1n} = \rho_s,$$

- For no charge existing at the interface

$$\rho_s = 0, \quad \Rightarrow \quad \begin{aligned} D_{1n} &= D_{2n} \\ \epsilon_1 E_{1n} &= \epsilon_2 E_{2n}. \end{aligned}$$

# B.C.

The tangential component of an **E** field is continuous across an interface.

| | |
|---|---|
| Tangential components, | $E_{1t} = E_{2t}$; |
| Normal components, | $\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$. |

The normal of **D** field is discontinuous across an interface where a surface charge exists—the amount of discontinuity being equal to the surface charge density.

# Continuity of $D_n$ and $E_n$

$$(\boldsymbol{D_1} - \boldsymbol{D_2}) \cdot \hat{\boldsymbol{n}} = (\boldsymbol{P_1} + \varepsilon_0 \boldsymbol{E_1} - \boldsymbol{P_2} - \varepsilon_0 \boldsymbol{E_2}) \cdot \hat{\boldsymbol{n}} = \rho_s$$

$$(\varepsilon_0 \boldsymbol{E_1} - \varepsilon_0 \boldsymbol{E_2}) \cdot \hat{\boldsymbol{n}} = \rho_s - \underline{(\boldsymbol{P_1} - \boldsymbol{P_2}) \cdot \hat{\boldsymbol{n}}} = \rho_s - \underline{\rho_{ps}}$$

$$(\boldsymbol{E_1} - \boldsymbol{E_2}) \cdot \hat{\boldsymbol{n}} = \frac{\rho_s - \rho_{ps}}{\varepsilon_0}$$

- The normal component of the D field is discontinuous by the amount of TRUE charge on the surface.
- The normal component of the E field is discontinuous by the amount of TOTAL charge (true plus polarization charge).
- Therefore, **the $D_n$ field may be continuous (when $\rho_s$=0)**.

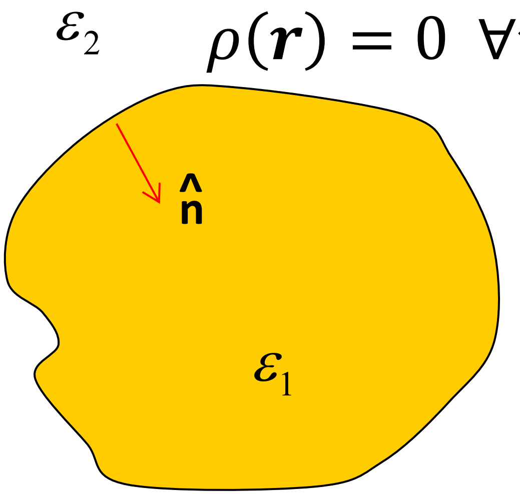
# Continuity of $D_t$ and $E_t$

$$(\boldsymbol{E_1}-\boldsymbol{E_2})\times\hat{\boldsymbol{n}}=\left(\frac{\boldsymbol{D_1}-\boldsymbol{P_1}}{\varepsilon_0}-\frac{\boldsymbol{D_2}-\boldsymbol{P_2}}{\varepsilon_0}\right)\times\hat{\boldsymbol{n}}=\boldsymbol{0}$$

$$(\boldsymbol{D_1}-\boldsymbol{D_2})\times\hat{\boldsymbol{n}}=(\boldsymbol{P_1}-\boldsymbol{P_2})\times\hat{\boldsymbol{n}}=\varepsilon_0\underline{(\Delta\chi_e)}\boldsymbol{E}\times\hat{\boldsymbol{n}}$$

- The tangential component of the E field is continuous across EVERY interface.
- The tangential component of the D field is discontinuous by the amount of **susceptibility difference** of the interface.

# An extreme case: $\varepsilon_1 >> \varepsilon_2$




$$\rho(\boldsymbol{r}) = 0 \;\; \forall \boldsymbol{r} \in S \longrightarrow (\boldsymbol{D_1} - \boldsymbol{D_2}) \cdot \widehat{\boldsymbol{n}} \; = 0 \;\; \forall \boldsymbol{r} \in S$$

$$\boldsymbol{D_1} = \varepsilon_1 \boldsymbol{E_1}(\boldsymbol{r})$$

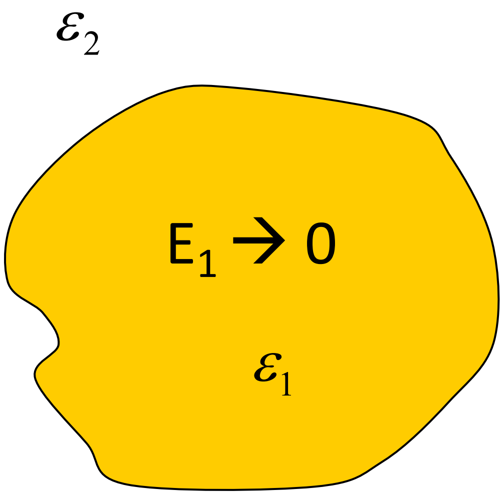
$$\boldsymbol{D_2} = \varepsilon_2 \boldsymbol{E_2}(\boldsymbol{r})$$

$$(\varepsilon_1 \boldsymbol{E_1} - \varepsilon_2 \boldsymbol{E_2}) \cdot \widehat{\boldsymbol{n}} \; = 0 \;\; \forall \boldsymbol{r} \in S$$

if $|\varepsilon_1| \gg |\varepsilon_2| \longrightarrow |\boldsymbol{E_2} \cdot \widehat{\boldsymbol{n}}| \gg |\boldsymbol{E_1} \cdot \widehat{\boldsymbol{n}}| \;\; \forall \boldsymbol{r} \in S$

$$\lim_{\varepsilon_1 \to \infty} \boldsymbol{E_1}(\boldsymbol{r}) = 0$$

# An extreme case: $\varepsilon_1 >> \varepsilon_2$

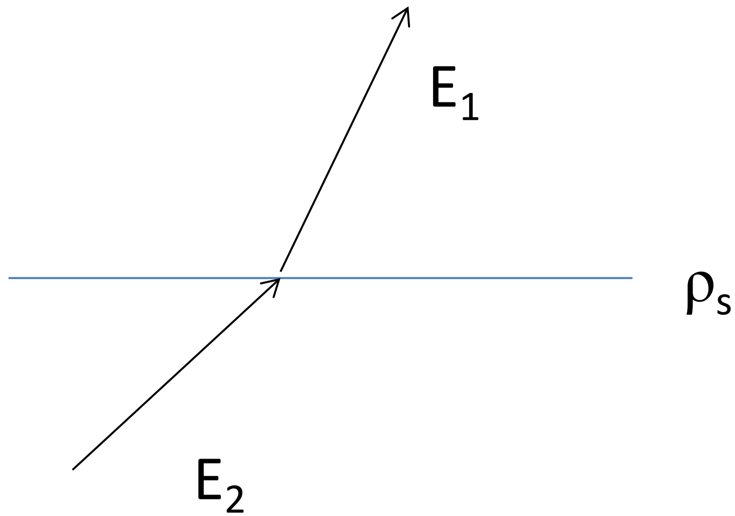



- Thus, if the source of the electric field is outside region 1, a **strong dielectric will expel the electric field from within itself**.

- Another way to look at this is that the **strong dielectric is strongly polarized**. The discontinuity of polarization at the interface is equivalent to bound surface charge that acts to **reduce the electric field** within the dielectric to a small value. (strong $\mathbf{P}\cdot\mathbf{n}$ ➔ strong $\rho_{ps}$ ➔ strong $E_{induced}$)

$$\boxed{D_{1n} - D_{2n} = \rho_s \qquad (\mathrm{C/m^2}),}$$

Ref.: $\mathbf{a}_{n2}$



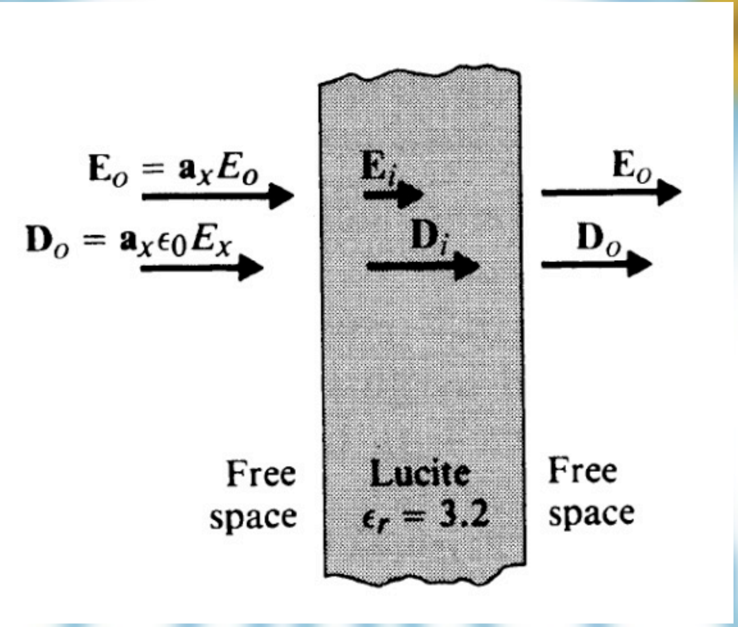

$D_{1n} - D_{2n} = \rho_s$

$D_{2n} - D_{1n} = \rho_s$

# Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

A lucite sheet ($\epsilon_r = 3.2$) is introduced perpendicularly in a uniform electric field $\mathbf{E}_o = \mathbf{a}_x E_o$ in free space. Determine $\mathbf{E}_i$, $\mathbf{D}_i$, and $\mathbf{P}_i$ inside the lucite.

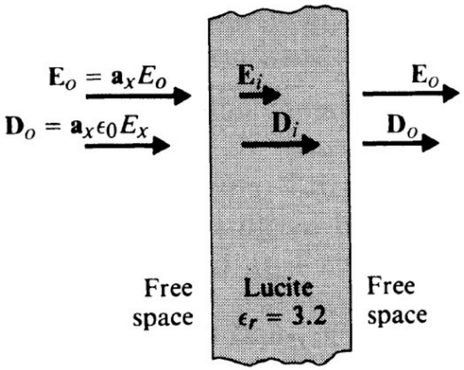


# Quick classroom exercise

When two dielectrics are in contact with *no free charges* at the interface we have

$$D_{1n} = D_{2n}$$

$$\epsilon_1 E_{1n} = \epsilon_2 E_{2n}.$$




$$\mathbf{D}_i = \mathbf{a}_x D_i = \mathbf{a}_x D_o$$

$$\mathbf{D}_i = \mathbf{a}_x \epsilon_0 E_o.$$

There is no change in electric flux density across the interface. The electric field intensity inside the lucite sheet is

$$\mathbf{E}_i = \frac{1}{\epsilon}\mathbf{D}_i = \frac{1}{\epsilon_o \epsilon_r}\mathbf{D}_i = \mathbf{a}_x \frac{E_o}{3.2}.$$

The polarization vector is zero outside the lucite sheet ($\mathbf{P}_o = 0$). Inside the sheet,

$$\mathbf{P}_i = \mathbf{D}_i - \epsilon_0 \mathbf{E}_i = \mathbf{a}_x\left(1 - \frac{1}{3.2}\right)\epsilon_0 E_o$$

$$= \mathbf{a}_x 0.6875 \epsilon_0 E_o \qquad (\mathrm{C/m^2}).$$

# A Lighting Arrester with a Sharp  Rod

- Example 3-13: The electric field intensities are inversely proportional to the radii. That is, E is higher at the surface with a larger curvature.

- E around sharp points >> E around a flat surface

# A Lighting Arrester with a Sharp Rod

- A cloud containing an abundance of electric charges
- ➔ charges of opposite sign are attracted from the ground to the tip
- ➔ E is very strong at the tip (sharp points)
- ➔ When E at tip > $E_{breakdown,wet\ air}$
- ➔ **Air** ionized, **becomes conducting**
- ➔ ⊖ in the cloud are discharged safely to the ground