Name and ID: ____________________

1. Beta and Gamma function.

Question1 (0 points)
List the definition form for Beta and Gamma function.

Solution:

Beta Function:

$$B\left(a,b\right)=\int_0^1 x^{a-1}\cdot(1-x)^{b-1}\,dx$$

Gamma Function:

$$\Gamma(\alpha)=\int_0^{\infty} x^{\alpha-1}\cdot e^{-x}\,dx,\quad \alpha>0$$

Question2 (0 points)
Specify the transformation between Beta and Gamma function.

Solution:

$$B(a,b)=\frac{\Gamma(a)\cdot\Gamma(b)}{\Gamma(a+b)},\quad \Rightarrow B(a,b)=B(b,a)$$

$$\bigstar\bigstar\bigstar\ B(a,1-a)=\Gamma(a)\cdot\Gamma(1-a)=\frac{\pi}{\sin a\pi}$$

Question3 (0 points)
Specify the other useful formula for applying beta and gamma funcrion

Solution:

The recursion expression: (a,b $\in\mathbb{R}$)

$$\Gamma(a+1)=a\cdot\Gamma(a)$$

$$B(a,b)=\frac{b-1}{a+b-1}\cdot B(a,b-1)$$

$$B(a,b)=\frac{a-1}{a+b-1}\cdot B(a-1,b)$$

Question4 (0 points)
Use Beta and Gamma function to represent

$$\int_0^{\pi/2}\sin^{a-1}\varphi\cdot\cos^{b-1}\varphi\,d\varphi,\quad (a,b>0)$$

Comment: It is a conclusion of extreme use! If you couldn't imagine how it is derived, at least memorize it!

Solution:

Consider $x=\sin\varphi$, then

$$\frac{1}{2}\cdot B(\frac{a}{2},\frac{b}{2})=\frac{1}{2}\cdot\frac{\Gamma(\frac{a}{2})\cdot\Gamma(\frac{b}{2})}{\Gamma(\frac{a+b}{2})}$$

Comment: Don't forget the $\frac{1}{2}$ here. Also, when apply the transformation, be aware of the upper and lower bound for integration should match exactly as is illustrated above (from 0 to $\frac{\pi}{2}$)!

Question5 (1 point)
Apply the conclusion in Question 5 and 3, calculate

$$\int_0^{\pi/2} (\sin\theta + \cos\theta)\cdot(\sin^{3/2}\theta\cdot\cos^{3/2}\theta)d\theta$$

Solution:

$$\begin{aligned} LHS &= \int_0^{\pi/2} (\sin^{5/2}\theta\cdot\cos^{3/2}\theta + \sin^{3/2}\theta\cdot\cos^{5/2}\theta)d\theta \\ &= 2\times\frac{1}{2}\cdot B(\frac{1}{2}\times(\frac{5}{2}+1), \frac{1}{2}\times(\frac{3}{2}+1)) \\ &= \frac{\Gamma(\frac{7}{4})\times\Gamma(\frac{5}{4})}{\Gamma(3)} = \frac{3}{16}\times\frac{\pi}{\sin\frac{1}{4}\pi}\cdot\frac{1}{2!} = \frac{3\sqrt{2}}{32}\pi \end{aligned}$$

2. Integration by I don't know why but anyway useful.

Question1 (1 point)
Prove: If

$$(a-c)^2 + b^2 \neq 0$$

then

$$\int \frac{a_1\sin x + b_1\cos x}{a\sin^2 x + 2b\sin x\cos x + c\cos^2 x}dx = A\int\frac{du_1}{k_1u_1^2+\lambda_1} + B\int\frac{du_2}{k_2u_2^2+\lambda_2}$$

where λ_1, λ_2 is the root of the equation

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0, (\lambda_1 \neq \lambda_2)$$

and

$$u_i = (a-\lambda_i)\sin x + b\cos x,\ \ k_i = \frac{1}{a-\lambda_i}\ \ (i = 1,2)$$

A and B are constants to be determined.
Hint: Try first to reform the denominator according to the given determinate and then the numerator.

Solution:

Notice the target of our expression goes that

$$A\int\frac{du_1}{k_1u_1^2+\lambda_1} + B\int\frac{du_2}{k_2u_2^2+\lambda_2}$$

If we can express the numerator and the denominator, respectively as

$$(a_1\sin x + b_1\cos x)dx = Adu_1 + Bdu_2$$

$$a\sin^2 x + 2b\sin x\cos x + c\cos^2 x = k_iu_i^2 + \lambda_i$$

For the fact that u_i is the linear combination of $\sin x$ & $\cos x$, the differential du_i is also the linear combination of them.

The detail construction could refer to the integration worksheet. Here I just list the conclusion. The differential could be derived as

$$du_1 = ((a-\lambda_1)\cos x - b\sin x)dx$$
$$du_2 = ((a-\lambda_2)\cos x - b\sin x)dx$$

And the coefficient

$$A = -\frac{a_1(a-\lambda_2)+bb_1}{b(\lambda_1-\lambda_2)}$$
$$B = \frac{a_1(a-\lambda_1)+bb_1}{b(\lambda_1-\lambda_2)}$$

Question2 (1 point)
Change variables and find the area of regions bounded by the following curves:

$$\left(x^3+y^3\right)^2 = x^2+y^2, \quad x \geq 0, \quad y \geq 0$$

(Hint: Apply polar transformation and utilize the conclusion in question 1)

Solution:

3. Double Integral

Question1 (1 point)
Non-Trivial $f(x,y)$
Compute

$$\int_0^1 \int_0^1 x \max(x,y) dydx$$

Hint: Well tell the meaning of the double integral and interpret it as the net signed volume! You could possibly meet the form of $f(x,y)$ as a more abstract form than $max(x,y)$ in exam!

Solution:

$$\int_{x=0}^1 \int_{y=0}^x x^2 dydx + \int_{x=0}^1 \int_{y=x}^1 xydydx = \int_0^1 \left(x^3 + x\frac{1-x^2}{2}\right)dx$$
$$= \frac{1}{4} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}$$

Question2 (1 point)
Double and iterated integral
Express each double integral over the given region R as an iterated integral, using the given order of integration.
a) R is the triangle with vertices at the origin, (0, 2), and (-2, 2).
b) R lies in the first quadrant $(x > 0, y > 0)$, which is the sector of the circle with center at the origin and radius 2 lying between the x-axis and the line y = x.
The given order:

$$\text{i) } \iint_R dydx \qquad \text{ii) } \iint_R dxdy$$

Hint: For some of them, it may be necessary to break the integral up into two parts. In each case, begin by sketching the region.

Solution:

$$\text{a)} \quad \text{(i)} \iint_R dydx = \int_{-2}^{0}\int_{-x}^{2} dydx \quad \text{(ii)} \iint_R dxdy = \int_0^2\int_{-y}^{0} dxdy$$

$$\text{b)} \quad \text{(i)} \iint_R dydx = \int_0^{\sqrt{2}}\int_0^{x} dydx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} dydx$$

$$\text{(ii)} \iint_R dxdy = \int_0^{\sqrt{2}}\int_y^{\sqrt{4-y^2}} dxdy$$

Question3 (1 point)

Change the order

Compute:

$$\int_0^4\int_{\sqrt{x}}^{2} 1/\left(1+y^3\right) dydx$$

Hint: Change the order to make life easier.

Solution:

If follows the original order to calculate the double we will struggle to doubt the life!

Instead, swapping integration to integrate x first, then we have

$$\int_0^2\int_0^{y^2} \frac{1}{1+y^3} dxdy = \int_0^2 \frac{y^2}{1+y^3} dy$$

which can be easily dealt with.

Question4 (1 point)

Change the order

Evaluate each of the following iterated integrals,by changing the order of integration (begin by figuring out what the region R is, and sketching it).

$$\text{a)} \int_0^2\int_x^2 e^{-y^2} dy\, dx \quad \text{b)} \int_0^{1/4}\int_{\sqrt{t}}^{1/2} \frac{e^u}{u} du\, dt \quad \text{c)} \int_0^1\int_{x^{1/3}}^{1} \frac{1}{1+u^4} du\, dx$$

Solution:

$$\int_0^2\int_x^2 e^{-y^2} dydx = \int_0^2\int_0^y e^{-y^2} dxdy = \int_0^2 e^{-y^2} ydy = -\frac{1}{2}e^{-y^2}]_0^2 = \frac{1}{2}\left(1-e^{-4}\right)$$

$$\int_0^{\frac{1}{4}}\int_{\sqrt{t}}^{\frac{1}{2}}\frac{e^u}{u}dudt=\int_0^{\frac{1}{2}}\int_0^{u^2}\frac{e^u}{u}dtdu=\int_0^{\frac{1}{2}}ue^u du=(u-1)e^u]_0^{\frac{1}{2}}=1-\frac{1}{2}\sqrt{e}$$

$$\int_0^1\int_{x^{1/3}}^1\frac{1}{1+u^4}dudx=\int_0^1\int_0^{u^3}\frac{1}{1+u^4}dxdu=\int_0^1\frac{u^3}{1+u^4}du=\frac{1}{4}\ln\left(1+u^4\right)]_0^1$$
$$=\frac{\ln 2}{4}$$

Question5 (1 point)

Properties of Double

Show the following estimates are valid.

a)

$$\iint_R\frac{dA}{1+x^4+y^4}\leq \quad \text{area of } R$$

b)

$$\iint_R\frac{xdA}{1+x^2+y^2}<.35$$

R is the square $0\leq x,y\leq 1$

Hint: use the inequality $f\leq g$ on $R\Rightarrow \iint_R fdA\leq\iint_R gdA$

Solution:

$$a)\ x^4+y^4\geq 0\Rightarrow\frac{1}{1+x^4+y^4}\leq 1$$

$$b)\iint_R\frac{xdA}{1+x^2+y^2}\leq\int_0^1\int_0^1\frac{x}{1+x^2}dxdy=\frac{1}{2}\ln\left(1+x^2\right)]_0^1=\frac{\ln 2}{2}<\frac{.7}{2}$$

Question6 (1 point)

Change the variables.

Consider the region R bounded by the circles:

$$\begin{aligned}x^2+y^2&=Ax\\x^2+y^2&=Bx\\x^2+y^2&=Cy\\x^2+y^2&=Dy\end{aligned}$$

evaluate the integral

$$\iint_R\frac{dxdy}{(x^2+y^2)^3}$$

Hint: Properly change the variables to make life easier.

Solution:

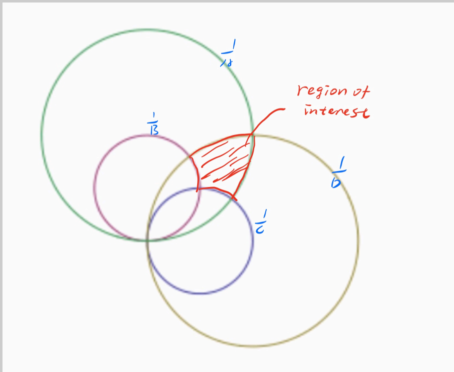
Consider the substitution as

$$\begin{cases}u=\dfrac{x}{x^2+y^2}\\v=\dfrac{y}{x^2+y^2}\end{cases}$$

Then the region R can be expressed as

$$R(u,v) = \left\{ (u,v) \mid \frac{1}{B} \le u \le \frac{1}{A}, \frac{1}{D} \le u \le \frac{1}{C} \right\}$$

And the figure below shows the case.



$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{(u^2+v^2)^2}$$

Then the integral equals

$$\int_{1/B}^{1/A} \int_{1/D}^{1/C} (u^2+v^2)^3 \frac{dvdu}{(u^2+v^2)^2} = \int_{1/B}^{1/A} \int_{1/D}^{1/C} (u^2+v^2) dvdu$$

This is very easy to integrate.

Question7 (1 point)

Use Generalized Polarize, compute

$$\left(\frac{x}{a} + \frac{y}{b}\right)^4 = \frac{x^2}{h^2} + \frac{y^2}{k^2} \ (x > 0, y > 0)$$

Solution:

$$S = \frac{ab}{6}\left(\frac{a^2}{h^2} + \frac{b^2}{k^2}\right)$$

Question8 (1 point)

Change variables to find the area of regions bounded by the following curves:

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 1, \quad \sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = 2, \quad \frac{x}{a} = \frac{y}{b}, \quad 4\frac{x}{a} = \frac{y}{b} \quad (a > 0, b > 0)$$

Hint: consider

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = u, \quad \frac{x}{y} = v$$

Solution:

Apply the transformation, and then we have

$$x = \frac{u^2 v}{\left(\sqrt{\frac{v}{a}} + \frac{1}{\sqrt{b}}\right)^2}, \quad y = \frac{u^3}{\left(\sqrt{\frac{v}{a}} + \frac{1}{\sqrt{b}}\right)^2}$$

$$1 \leqslant u \leqslant 2, \ \frac{a}{4b} \leqslant v \leqslant \frac{a}{b}$$

and

$$|J| = \frac{2u^3}{\left(\sqrt{\frac{v}{a}} + \frac{1}{\sqrt{b}}\right)^4}$$

$$S = \int_1^2 2u^3 du \int_{\frac{a}{4b}}^{\frac{a}{b}} \frac{dv}{\left(\sqrt{\frac{v}{a}} + \frac{1}{\sqrt{b}}\right)^4}$$

$$= \frac{15}{2} \cdot \int_{\frac{1}{2\sqrt{b}}}^{\frac{1}{\sqrt{b}}} \frac{2atdt}{\left(t + \frac{1}{\sqrt{b}}\right)^4} \ , v = at^2$$

$$= 15a \cdot \left(\frac{7b}{72} - \frac{37b}{648}\right) = \frac{65ab}{108}$$

Besides, you can also try the transformation

$$\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} = u, \sqrt{\frac{ay}{bx}} = v$$

the calculation of which may be of more elegance.

4. Application of double integral

Question1 (1 point)

Surface Area

Find the area of the part of the sphere

$$x^2 + y^2 + z^2 = 4z$$

that lies inside the paraboloid

$$z = x^2 + y^2$$

Solution:

The intersection is given by

$$z(z-3) = 0 \Longrightarrow z_1 = 0 \qquad \text{and} \qquad z_2 = 3$$

thus the region over which we shall integrate is

$$\mathcal{D} = \left\{(x, y) | x^2 + y^2 \leq 3\right\}$$

The partial derivatives are given by

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = \frac{x}{2-z}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = \frac{y}{2-z}$$

The surface area is given by

$$\begin{aligned} S &= \iint_{\mathcal{D}} \sqrt{1 + \frac{x^2}{(2-z)^2} + \frac{y^2}{(2-z)^2}} dA \\ &= \iint_{\mathcal{D}} \sqrt{1 + \frac{x^2+y^2}{4-(x^2+y^2)}} dA \\ &= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \frac{r^2}{4-r^2}} r dr d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{\frac{4}{4-r^2}} r dr d\theta = 4\pi \end{aligned}$$

Question2 (1 point)
Surface Area. Alternative form
Find the area of the portion of the paraboloid

$$\mathbf{r}(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ u^2 \end{bmatrix}$$

for which

$$1 \le u \le 2 \text{ and } 0 \le v \le 2\pi$$

Solution:

When the surface is defined parametrically

$$\mathbf{r}(u,v)$$

there is an easy way to find the surface area. Since the partial derivatives

$$\frac{\partial \mathbf{r}}{\partial u} = \cos(v)\mathbf{e}_x + \sin(v)\mathbf{e}_y + 2u\mathbf{e}_z$$

$$\frac{\partial \mathbf{r}}{\partial v} = -u\sin(v)\mathbf{e}_x + u\cos(v)\mathbf{e}_y$$

gives the tangent vectors in direction of increasing u and v,

$$\begin{aligned} S &= \int_1^2 \int_0^{2\pi} \left| \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v} \right| dv du \\ &= \int_1^2 \int_0^{2\pi} u\sqrt{1+4u^2} dv du = \frac{(17\sqrt{17} - 5\sqrt{5})\pi}{6} \end{aligned}$$

5. Physical Application of double integral

Question1 (1 point)
Find the mass M of the upper half of the annulus

$$1 < x^2 + y^2 < 9 (y \ge 0)$$

with density

$$\sigma(x,y) = \frac{y}{x^2+y^2}$$

Solution:

$$dm = \sigma dA = \frac{r\sin\theta}{r^2} r dr d\theta = \sin\theta dr d\theta$$

$$M = \iint_R dm = \int_0^\pi \int_1^3 \sin\theta dr d\theta = \int_0^\pi 2\sin\theta d\theta = [-2\cos\theta]_0^\pi = 4$$

Question2 (1 point)
Express the x-coordinate of the center of mass, $\bar{x}$, as an iterated integral in the former question. (Write explicitly the integrand and limits of integration.)
Without evaluating the integral, explain why $\bar{x} = 0$

Solution:

$$\overline{x} = \frac{1}{M}\iint_R x\delta dA = \frac{1}{4}\int_0^\pi \int_1^3 r\cos\theta\sin\theta dr d\theta$$

And since the region and the density are symmetric with respect to the y-axis, i.e.

$$\sigma(x,y) = \sigma(-x,y)$$

The x coordinate of the center of mass is 0.