Name and ID: ______________________________

1. Supplement Jing's course

Question1 (1 point)
How to understand the case in class (L2, P7) involving Harry and Hermione by applying the method of Mathematical Induction?

Solution:

Overall path: First notice that only one of A and B's number is 1, at the beginning. It is somewhat the concept of <reference point>. Following the inductive path, the relative point 1 can be then changed to any natural number.

Beginning Principle of induction
Middle
(a) The Base Case
They know all numbers are larger than 0, Then when A ask B whether B knows the number of B, If B doesn't know, it means the number of A is larger than 1 (A's memory refreshing$\cdots$),
Then when B ask A whether A knows the number of A, If A doesn't know, it means the number of B is larger than 2 (B's memory refreshing$\cdots$).
(b) The Inductive Case
If they know the numbers of A is larger than 2n−1, that of B is larger than 2n,
Then when A ask B whether B knows the number of B, If B doesn't know, it means the number of A is larger than 2n+1 (A's memory refreshing$\cdots$),
Then when B ask A whether A knows the number of A, If A doesn't know, it means the number of B is larger than 2n+2 (B's memory refreshing$\cdots$).
End
by the principle of induction, we can conclude that the game does end.

Question2 (1 point)
How do we derive the precise definition of the limit of a convergent sequence? (Show the general strategy (Thinking pattern) to instantiate an intuitive idea.)

Solution:

Intuitive/Vague Understanding ⇒ Strict/Precise Definition

Question3 (1 point)
How to avoid confusion when we are discussing the items about the limits?

Solution:

Try your best to summarize and memorize every theorem and its applicable **CONDITION**.

2. Useful Identities

Question1 (1 point)
Show the basic conclusions involving the limit of a sequence.

Solution:

(a) <Sufficient> The sequence $\{a_n\}$ being convergent implies it being bounded.
Yet the converse statement is **NOT** true.
Hint for proof: Similar strategy as the proof for product law as we discussed in class.

(b) <Theorem on uniqueness of limits> A sequence can have at most one limit.
Hint for proof: By applying the method of proof by contradiction (Reduction to Absurdity).

(c) <Sign-Preserving Theorem of Limit> If $\lim_{n\to\infty} x_n = a, a > 0$, then there must exists N $\in \mathbb{N}$, s.t.$x_n > 0$ when n>N.
<Lemma> If for a sequence $\{x_n\}$, there exists $N_1 \in \mathbb{N}$, when $n > N_1$, $x_n \geqslant 0$ (or $x_n \leqslant 0$), and $\lim_{n\to\infty} x_n = a$, then a $\geqslant 0$ (or a $\leqslant 0$)
Hint: Notice carefully the inequality sign.

(d) When applying the limit law, NO need to elaborate your steps too much. Just show you **KET STEPS**.

(e) Important inequality: $n! \geqslant 2^{n-1}$

Question2 (1 point)
Show the definition of subsequence and the related theorems on it.

Solution:

(a) <Definition of Subsequence> a subsequence is a sequence that can be derived from another sequence by deleting some or no elements without changing the order of the remaining elements.

(b) <Basic Theorem> If a sequence converges at a, then its arbitrary subsequence also converges at a.

(c) <Bolzano–Weierstrass Theorem/Sequential Compactness Theorem>
Each bounded sequence has a convergent subsequence.
<Equivalent formulation of M.S.T>
Hint: Well apply the method by using the method of **NESTED INTERVALS**.

(d) <Stolz–Cesàro theorem/O'stolz theorem>
Let $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ be two sequences of real numbers. Assume that $(b_n)_{n\geq 1}$ is strictly monotone and divergent sequence (i.e. strictly increasing and approaches $+\infty$ or strictly decreasing and approaches -∞ and the following limit exists:

$$\lim_{n\to\infty} \frac{a_{n+1} - an}{b_{n+1} - b_n} = l$$

Then, the limit

$$\lim_{n\to\infty} \frac{a_n}{b_n} = l$$

Tips: O'Stolz theorem is also called L'Hôpital's rule for sequences. (Consider the relationship between continuous function and discrete sequence)

Question3 (1 point)
State the Nested-Interval Theorem.

Solution:
Consider a nested family of intervals $I_n := [a_n, b_n]$, n $\in \mathbb{N}$, where for each n we have $a_n \leqslant a_{n+1} \leqslant b_{n+1} \leqslant b_n$, then a= sup_n a_n and $b = inf_n$ b_n both exist and,

$$\bigcap_{n=1}^{\infty} I_n = [a, b] \neq 0$$

Moreover, if

$$\lim_{n\to\infty} (b_n - a_n) = 0$$

then a = b and the intersection will consist of only the point {a}
To be noticed, the most popular formulation we applied in the Calculus course is that

$$|I_{n+1}| = \frac{1}{2}|I_n|$$

, where the notation $|I_n|$ stands for the length of that interval.
For the application, see the following proof question.

This is math, everything is made up. But question is whether or not it turns out to be a useful construct for modelling the world.

Corollary 6. *If (x_n) is a Cauchy sequence of real numbers, then (x_n) is convergent.*

Proof. Recall the following facts.
(i) Any Cauchy sequence is bounded.
(ii) If (x_n) is a Cauchy sequence and (x_{n_k}) is a subsequence convergent to a real number α, then (x_n) converges to α.

The proof is an easy consequence of these two facts along with the last theorem. The reader should complete the proof on his own.

Let (x_n) be a Cauchy sequence of real numbers. By the first fact, (x_n) is bounded. By Bolzano-Weierstrass theorem, there exists a convergent subsequence (x_{n_k}). It follows from the second fact that the given Cauchy sequence is convergent. □

Theorem 5 (Bolzano-Weierstrass). *Let (x_n) be a bounded sequence of real numbers. Then there exists a convergent subsequence (x_{n_k}) of (x_n).*

Proof. Recall that a real sequence is a function $x\colon \mathbb{N} \to \mathbb{R}$ and that we let $x_n := x(n)$. Thus the image of the sequence is the subset $x(\mathbb{N}) \equiv \{x_n : n \in \mathbb{N}\} \subset \mathbb{R}$.

If the image $x(\mathbb{N})$ is finite, then there exists a real number α such that $x_n = \alpha$ for infinitely many $n \in \mathbb{N}$, say, for all $k \in S \subset \mathbb{N}$, an infinite subset. Using the well-ordering principle, we can exhibit the elements of S as an increasing sequence of natural numbers:

$$n_1 < n_2 < \cdots < n_k < \cdots$$

Then the subsequence (x_{n_k}) is the constant sequence α and hence is convergent.

So, we now assume that $x(\mathbb{N})$ is infinite. Since (x_n) is bounded, there exists a positive real number M such that

$$-M \le x_n \le M, \qquad \text{for all } n \in \mathbb{N}.$$

We bisect the interval $J_0 := [-M, M]$ into two subintervals of equal length, $[-M, 0]$ and $[0, M]$. Since $x(\mathbb{N}) \subset [-M, M] = [-M, 0] \cup [0, M]$, at least one of the subintervals will contain infinitely many elements of $x(\mathbb{N})$, that is, infinitely many terms of the given sequences. Call one such subinterval as J_1. Now we again bisect J_1 into two subintervals of equal length. Since $x(\mathbb{N}) \cap J_1$ is infinite by our choice of J_1, one of the subintervals of J_1 must have infinite number of elements from $x(\mathbb{N}) \cap J_1$. Call it J_2. Thus, $J_2 \subset J_1$, $\ell(J_2) = 2^{-1}\ell(J_1) = 2^{-2}\ell(J_0) = 2^{-2}2M$. Also, J_2 has infinitely many terms from (x_n).

Proceeding inductively, we construct a nested sequence of intervals J_n with the following properties:
(i) $J_{n+1} \subset J_n$ for all $n \in \mathbb{N}$.
(ii) $\ell(J_n) = 2^{-n+1}M$ for all $n \in \mathbb{N}$.
(iii) For each $n \in \mathbb{N}$, the interval J_n contains infinitely many terms of the sequence (x_n).

Using the nested interval theorem, we get a real number α such that $\alpha \in \cap_n J_n$. Now we inductively define a subsequence which will converge to α. Since J_1 contains infinitely many terms of the sequence (x_n), there exists $n_1 \in \mathbb{N}$ such that $x_1 \in J_1$. Assume that for all $1 \leq i \leq k$, we have found x_{n_i} such that $x_{n_i} \in J_i$. Now, J_{k+1} contains infinitely many terms of the given sequence and hence there exists n_k such that $n_{k+1} > n_i$ for $1 \leq i \leq k$ with the property that $x_{n_{k+1}} \in J_{k+1}$. Thus we get a subsequence (x_{n_k}).

We claim that $x_{n_k} \to \alpha$ as $k \to \infty$. For since $\alpha, x_{n_k} \in J_k$, we have

$$|x_{n_k} - \alpha| \leq \ell(J_k) = 2^{-k+1}M \to 0, \text{ as } k \to \infty.$$

This completes the proof of the theorem. $\square$