Vv156 Lecture 8

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- In terms of computing derivatives, the definition is not very friendly

$$\frac{d}{dx}\sqrt{2x^2+1}$$

- Based on the formal definition of derivative, we want to derive a set of laws.

Theorem

The derivative of a constant function is 0; that is, if c is any real number, then

$$\frac{d}{dx}\left(c\right)=0$$

Proof

- By definition, $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}0=0$ $\square$

- Geometrically, the graph of $f(x)=c$ is a horizontal line, which has a slope of 0 everywhere, so the tangent line must have a slope of 0 everywhere as well.

The Power Rule

If n is a positive integer, then

$$\frac{d}{dx}\left(x^n\right) = nx^{n-1}$$

Proof

$$\frac{d}{dx}\left(x^n\right) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h\to 0}\frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h\to 0}\left[nx^{n-1} + \boxed{\frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}}\right] = nx^{n-1}$$

$\square$

- Although our proof of the power rule applies only to positive integer powers of x, it is not difficult to show that the same formula holds for all real number r.

The General Power Rule

If r is any real number, then

$$\frac{d}{dx}\left(x^r\right) = rx^{r-1}$$

- In words, to differentiate a power function,

$$y = x^r$$

1. We take the exponent, and multiply it into the coefficient,

2. then reduce the exponent by 1.

Theorem

If f is differentiable at x and c is any real number, then cf is also differentiable at x and

$$\frac{d}{dx}\Big(cf\Big) = c\frac{d}{dx}\Big(f\Big)$$

Proof

- By definition,

$$\frac{d}{dx}\Big(cf\Big) = \lim_{h\to 0}\frac{cf(x+h)-cf(x)}{h} = c\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = c\frac{df}{dx}$$

$\square$

- In words, a constant factor can be moved through a differential operator.

Theorem

If f and g are differentiable at x, then so are $f+g$ and $f-g$, and

$$\frac{d}{dx}\Big(f \pm g\Big) = \frac{d}{dx}\Big(f\Big) \pm \frac{d}{dx}\Big(g\Big)$$

Proof

- $$\begin{aligned}\frac{d}{dx}\Big(f \pm g\Big) &= \lim_{h\to 0} \frac{\Big[f(x+h) \pm g(x+h)\Big] - \Big[f(x) \pm g(x)\Big]}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \frac{d}{dx}\Big(f\Big) \pm \frac{d}{dx}\Big(g\Big)\end{aligned}$$

- In words, the derivative of a sum equals the sum of the derivatives, and the derivative of a difference equals the difference of the derivatives.
- Although this theorem is stated for for sums and differences of two functions, they can be easily extended to any finite number of functions.

Exercise

(a) Find $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ and $\left.\frac{d^3y}{dx^3}\right|_{x=1}$ where

$$y = 3x^8 - 2x^5 + 6x + 1$$

(b) At what points, if any, does the graph of

$$y = x^3 - 3x + 4$$

have a horizontal tangent line?

(c) Evaluate the limit by first converting it to a derivative at a particular x-value.

$$\lim_{x \to 1} \frac{x^{50} - 1}{x - 1}$$

- You might be tempted to conjecture that the derivative of a product of two functions is the product of their derivatives. However, consider the following

$$
\begin{aligned}
f(x) &= x & \qquad g(x) &= x^2 \\
\implies f'(x) &= 1 & \qquad \implies g'(x) &= 2x
\end{aligned}
$$

$$f'(x)\cdot g'(x) = 1\cdot 2x = 2x \qquad \Big(f(x)\cdot g(x)\Big)' = (x^3)' = 3x^2$$

$$f'(x)\cdot g'(x) \neq \Big(f(x)\cdot g(x)\Big)'$$

The product rule

If f and g are differentiable at x, then so is the product $f\cdot g$, and

$$\frac{d}{dx}\Big(f\cdot g\Big) = \frac{df}{dx}\cdot g + f\cdot\frac{dg}{dx}$$

Proof

By definition,

$$\begin{aligned}
\frac{d}{dx}\Big[f \cdot g\Big] &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} f(x+h) \boxed{\frac{g(x+h)-g(x)}{h}} + \lim_{h\to 0} g(x) \boxed{\frac{f(x+h)-f(x)}{h}} \\
&= \boxed{f'} \cdot g + f \cdot \boxed{g'}
\end{aligned}$$

Exercise

Find $f'(x)$ for

$$f(x) = 3\sqrt[3]{x} \cdot (\frac{1}{2}x^2 + x)$$

- Just as the derivative of a product is not the product of the derivatives, so the derivative of a quotient is not generally the quotient of the derivatives.

The Quotient Rule

If f and g are differentiable at x and if $g(x) \neq 0$, then $\dfrac{f}{g}$ is differentiable at x

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{\dfrac{df}{dx} \cdot g - f \cdot \dfrac{dg}{dx}}{(g)^2}$$

- You can prove this rule by adding and subtracting $f \cdot g$ in the right place.
- Sometimes it is better to simplify a function first than to apply the quotient rule immediately. For example, it is easier to differentiate

$$f(x) = \frac{x^{3/2} + x}{\sqrt{x}} = x + x^{1/2}$$

- If we assume that the variable x in the trigonometric functions $\sin x$, $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\csc x$ is measured in radians, we derive formulae of trigonometric function using the definition of derivative. For example,

$$\begin{aligned}
\frac{d}{dx}(\sin x) &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\sin h\cos x-\sin x}{h} \\
&= \lim_{h\to 0}\left[\sin x\left(\frac{\cos h-1}{h}\right)+\cos x\frac{\sin h}{h}\right] \\
&= \underbrace{\left[\lim_{h\to 0}\sin x\left(\frac{\cos h-1}{h}\right)\right]}_{[\lim_{h\to 0}\sin x]\left[\lim_{h\to 0}\frac{\cos h-1}{h}\right]}+\left[\lim_{h\to 0}\cos x\right]\left[\lim_{h\to 0}\frac{\sin h}{h}\right] \\
&= \cos x.
\end{aligned}$$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1; \qquad \lim_{\theta\to 0}\frac{1-\cos\theta}{\theta}=0$$

Exercise

Find the derivatives: $(a)\ \dfrac{d}{dx}\left(\cos\left(x+\frac{\pi}{2}\right)\right)$ $(b)\ \dfrac{d}{dx}(\sin 2x)$

- Other derivatives of trigonometric function can be found in a similar fashion, or using known derivatives or rules of differentiation, especially the next rule.

$$\frac{d}{dx}c = 0 \qquad \frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x.

Moreover, if

$$y = f(g(x)) \qquad \text{and} \qquad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Exercise

Find the derivatives using the chain rule:

$$(a)\, \frac{d}{dx}\left(\cos\left(x + \frac{\pi}{2} \right) \right) \qquad\qquad (b)\, \frac{d}{dx}\left(\sin 2x\right)$$

Proof

- We want to compute

$$\lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \tag{1}$$

- Let $v = \dfrac{g(x+h)-g(x)}{h} - g'(x)$, notice

$$v \to 0 \qquad \text{as} \qquad h \to 0$$

make $g(x+h)$ the subject

$$g(x+h) = \underbrace{g(x)}_{u} + \underbrace{\Big[g'(x)+v\Big]h}_{k}$$

- Notice

$$k \to 0 \qquad \text{as} \qquad h \to 0$$

Proof

- Let $w = \dfrac{f(u+k) - f(u)}{k} - f'(u)$, then $w \to 0$ as $k \to 0$.
- Make $f(u+k)$ the subject, we have

$$f(u+k) = f(u) + \Big[f'(u) + w\Big]k$$

$$f(g(x+h)) = f(g(x)) + \Big[f'(g(x)) + w\Big]\Big[g'(x) + v\Big]h$$

Collect terms in equation (1),

$$\Longrightarrow \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0} \Big[f'(g(x)) + w\Big]\Big[g'(x) + v\Big]$$

$$= f'(g(x)) \cdot g'(x)$$

$$= \frac{dy}{du} \cdot \frac{du}{dx} \quad \square$$

A commom flawed proof for the chain rule

- Use the alternative definition of derivative

$$
\begin{aligned}
\frac{dy}{dx} &= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a} \\
&= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{x-a} \frac{g(x)-g(a)}{g(x)-g(a)} \\
&= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \cdot \frac{g(x)-g(a)}{x-a} \\
&= \lim_{x\to a} \frac{f(g(x)) - f(g(a))}{g(x)-g(a)} \cdot \lim_{x\to a} \frac{g(x)-g(a)}{x-a} \\
&= \frac{dy}{du} \cdot \frac{du}{dx}
\end{aligned}
$$

- The major problem with this proof is that we cannot be sure that we didn't multiply and divide by zero.
- The definition of limit guarantees that (as $x \to a$) x will not equal a; but the same cannot be be said about $g(x)$ and $g(a)$.

Exercise

Find the derivative $\dfrac{d}{dx}\left(\sin^2\left(\sqrt{2x^2+1}\right)\right)$.

Q: Is the derivative of

$$\sin(u), \qquad \text{where } u \text{ is measured in degrees,}$$

equal to the derivative of $\sin(x)$ where x is the same angle but in radians?

- In radians,

$$\frac{d}{dx}(\sin x) = \cos x$$

- In degrees,

$$\begin{aligned}\frac{d}{du}(\sin u) = \frac{d}{du}\left(\sin\frac{180^\circ}{\pi}x\right) &= \frac{d}{dx}\left(\sin\frac{180^\circ}{\pi}x\right)\frac{dx}{du}\\ &= \frac{180^\circ}{\pi}\left(\cos\frac{180^\circ}{\pi}x\right)\frac{\pi}{180^\circ} = \cos u\end{aligned}$$

- However, $\dfrac{d}{dx}\sin u = \dfrac{d}{du}(\sin u)\dfrac{du}{dx} = \dfrac{180^\circ}{\pi}\cos u = \dfrac{180^\circ}{\pi}\cos\left(\dfrac{180^\circ}{\pi}x\right)$