Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 9

Date Due: 12:55 PM, Wednesday, the 28th of April 2021

Exercises (9 Marks)

Exercise 9.1
Consider the half-disk in $\mathbb{R}^2$,

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon |x|^2 < 1,\ x_2 > 0\}.$$

i) Use the method of images to find the Green's function for the Dirichlet problem on the half-disk.

ii) Use this expression for Green's function and the solution formula obtained in the previous assignment to solve the Dirichlet problem

$$-\Delta u = 0 \quad \text{on } \Omega, \qquad u(x_1, 0) = 0 \quad \text{for } x_1 \in [-1, 1], \qquad u(x_1, \sqrt{1 - x_1^2}) = 1 \quad \text{for } x_1 \in (-1, 1).$$

iii) Plot the solution, e.g., using Mathematica.

iv) Compare the solution qualitatively with the series expansion obtained in the previous assignment.

(6 Marks)

Exercise 9.2
Consider the boundary value problem for the heat equation on a finite interval $(0, L) \subset \mathbb{R}$:

$$\begin{aligned} u_t - c^2 u_{xx} &= F(x,t), && 0 < x < L, \\ u(0,t) &= \gamma_1, && 0 < t < T, \\ u(L,t) &= \gamma_2, && 0 < t < T, \\ u(x,0) &= f(x), && 0 < x < L. \end{aligned} \tag{$*$}$$

where $T > 0$ is some fixed time, $\gamma_1, \gamma_2 \in \mathbb{R}$, and $f \colon [0, L] \to \mathbb{R}$, $F \colon [0, L] \times \mathbb{R} \to \mathbb{R}$ suitably smooth functions. A causal fundamental solution for the heat equation on $\mathbb{R}$ is given by

$$E(x, t; \xi, \tau) = \frac{H(t - \tau)}{\sqrt{4\pi c^2 (t - \tau)}} e^{-\frac{(x-\xi)^2}{4c^2(t-\tau)}}.$$

Use the method of images to find an infinite series representation of $g(x, t; \xi, \tau)$ using suitable image charges. Draw a sketch!
(3 Marks)