Chapter 12 – Rigid Body Mechanics II: Dynamics

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Physics I (Summer 2019)
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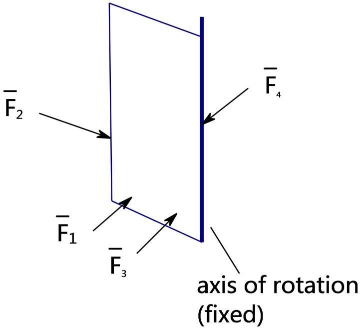
Agenda

1. Introduction
   - Torque
2. Dynamics of rotational motion of a rigid body about a fixed axis
   - Torque and angular acceleration for a rigid body
   - Second law of dynamics
   - Examples
3. Rigid body rotation about a moving axis
   - Kinetic energy in the combined motion
   - Rolling with/without slipping. Examples
   - Energy in combined motion. Examples
   - Dynamics of combined motion. Examples

Introduction

Torque. Motivation

Example: revolving door



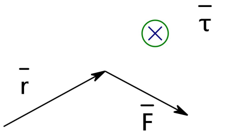
Even though all forces are of the same magnitude, their effect is different.

- $\bar{F}_1$ vs $\bar{F}_3$: both cause the door to revolve, but $\bar{F}_1$ "more effective" than $\bar{F}_3$
- $\bar{F}_2$ and $\bar{F}_4$: neither can make the door revolve

Torque. Definition

$$\boxed{\bar{\tau} = \bar{r} \times \bar{F}}$$

Note. The torque $\bar{\tau}$ is always defined with respect to a point/axis of rotation.



right-hand rule

units: N·m (=joule)

Decomposing

$$\bar{r} = \underbrace{\bar{r}_{||} + \bar{r}_{\perp}}_{\text{parallel/perpendicular to } \bar{F}}$$

we have

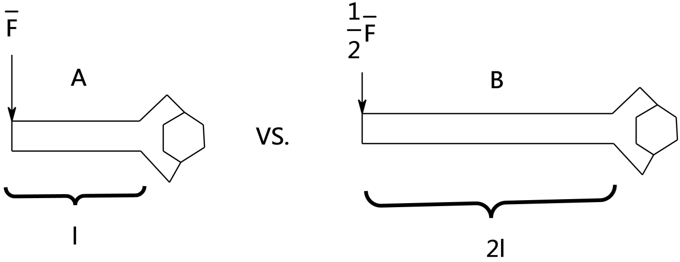
$$\bar{\tau} = (\bar{r}_{||} + \bar{r}_{\perp}) \times \bar{F} = \bar{r}_{\perp} \times \bar{F}, \qquad \text{and} \qquad |\bar{\tau}| = \underbrace{|\bar{r}_{\perp}|}_{\text{lever arm}} \; |\bar{F}|$$

Torque. Applications

Magnitude of the torque (alternative ways to calculate)

$$|\bar{\tau}| = |\bar{r}||\bar{F}|sin\angle(\bar{r},\bar{F}) = |\bar{r}_{\perp}||\bar{F}| = |\bar{r}||\bar{F}_{\perp}|$$

Example. Wrench

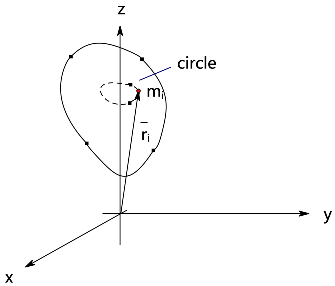


Smaller force needed to rotate the screw

Dynamics of rotational motion of a rigid body about a fixed axis

Torque on a rigid body

(rotation about a fixed axis; choose the z-axis to be the axis of rotation)

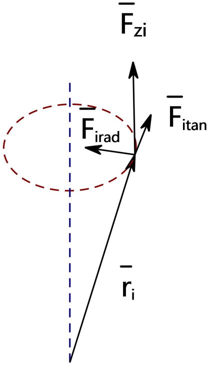


2nd law of dynamics for the element of mass m_i

$$\underbrace{\bar{F}_i}_{\text{net force}} = m_i \bar{a}_i, \qquad \Longrightarrow \qquad \underbrace{\bar{r}_i \times \bar{F}_i}_{\text{torque of net force w.r.t the origin}} = m_i \bar{r}_i \times \bar{a}_i$$

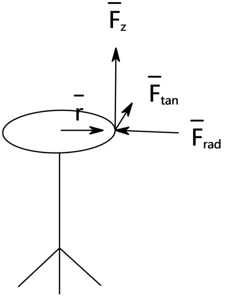
The net force can be decomposed into three components: radial, tangential and along the z-axis $\bar{F}_i = \bar{F}_{i,rad} + \bar{F}_{i,tan} + \bar{F}_{i,z}$

$$\bar{F}_i = \bar{F}_i = \underbrace{\bar{F}_{i,rad}}_{\text{radial}} + \underbrace{\bar{F}_{i,tan}}_{\text{tangential}} + \underbrace{\bar{F}_{i,z}}_{\text{along the z-axis}}$$



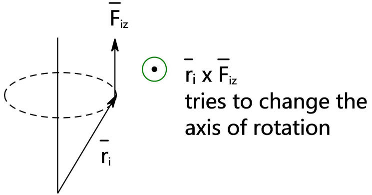
What is the role of these components?

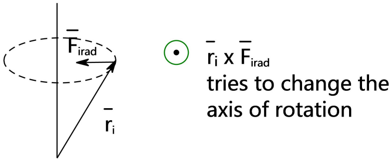
Experiment. Rotating chair (fixed axis)



Observations

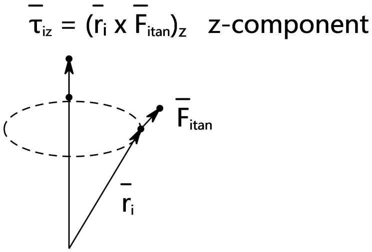
- → Only $\bar{F}_{tan}$ makes the chair rotate. The torque corresponding to $\bar{F}_{tan}$ is in the direction along the rotation axis.
- → $\bar{F}_z$ and $\bar{F}_{rad}$ try to tilt the axis of rotation, but it is fixed!





Conclusion. The only component of the net force that can effect rotational motion about the (fixed) z-axis is due to the tangential component of the force $\bar{F}_{i,tan}$.

Consequently,



$$[\bar{r}_i \times \bar{a}_i]_z = \underbrace{[\bar{r}_i \times \bar{a}_{i,rad}]_z}_{=0} + \underbrace{[\bar{r}_i \times \bar{a}_{i,z}]_z}_{=0} + [\bar{r}_i \times \bar{a}_{i,tan}]_z$$

$$= [\bar{r}_i \times \bar{a}_{i,tan}]_z$$

Hence, the equation $\bar{r}_i \times \bar{F}_i = m_i \bar{r}_i \times \bar{a}_i$ for rotation about a fixed axis (z-axis) simplifies to the form

$$\tau_{i,z} = (\bar{r}_i \times \bar{F}_{i,tan})_z = m_i(\bar{r}_i \times \bar{a}_{i,tan})_z$$

Recalling that $\bar{a}_{i,tan} = \bar{\varepsilon} \times \bar{r}_{i\perp}$ (see the previous lecture), we can rewrite the rhs as

$$\tau_{i,z} = [\bar{r}_i \times (\bar{\varepsilon} \times \bar{r}_{i\perp})]_z = [(\bar{r}_{i\parallel} + \bar{r}_{i\perp}) \times (\bar{\varepsilon} \times \bar{r}_{i\perp})]_z$$

$$\tau_{i,z} = \underbrace{[\bar{r}_{i\|} \times (\bar{\varepsilon} \times \bar{r}_{i\perp})]_z}_{\text{zero, because has no z-component}} + [\bar{r}_{i\perp} \times (\bar{\varepsilon} \times \bar{r}_{i\perp})]_z$$

Use the vector identity $\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \circ \bar{C}) - \bar{C}(\bar{A} \circ \bar{B})$ to rewrite the latter term

$$\tau_{i,z} = (\bar{\varepsilon}\, \bar{r}_{i\perp}^2)_z - \bar{r}_{i\perp}(\bar{r}_{i\perp} \circ \bar{\varepsilon}) = r_{i\perp}^2 \varepsilon_z.$$

Where we have used the fact that $\bar{r}_{i\perp} \circ \bar{\varepsilon} = 0$, because $\bar{r}_{i\perp} \perp \bar{\varepsilon}$. Hence, $\tau_{iz} = m_i r_{i\perp}^2 \varepsilon_z$. Adding all contributions

$$\tau_z = \sum_{i=1}^{N} \tau_{i,z} = \underbrace{\left(\sum_{i=1}^{N} m_i r_{i\perp}^2 \right)}_{I_z} \varepsilon_z = I_z \varepsilon_z$$

Second law of dynamics

Eventually, the **second law of dynamics for a rigid body rotating about a fixed axis**

$$\boxed{\tau_z = I_z \varepsilon_z}$$

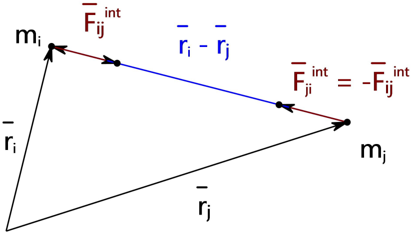
where τ_z — torque, I_z — moment of inertia and ε_z — angular acceleration; all with respect to the axis of rotation (here the z axis).

Compare with the 2nd law of dynamics for (1D) motion of a particle

- particle: *net force* $\Rightarrow$ *acceleration* which is $\propto$ *force*, and $\propto \frac{1}{mass}$
- rigid body: net torque along axis of rotation $\Rrightarrow$ angular acceleration which is $\propto$ torque, and $\propto \frac{1}{moment\ \ of\ \ inertia}$

Comments

- The above discussion is valid for a rigid body; we assumed the same $\bar{\varepsilon}$ for all elements of mass.
- Internal forces between the elements of mass do not contribute to the torque.



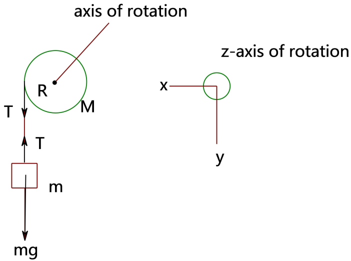
Contribution due to this pair is zero, because

$$\bar{r}_i \times \bar{F}_{ij}^{int} + \bar{r}_j \times \bar{F}_{ji}^{int} = \bar{r}_i \times \bar{F}_{ij}^{int} - \bar{r}_j \times \bar{F}_{ij}^{int} = (\bar{r}_i - \bar{r}_j) \times \bar{F}_{ij}^{int} = \mathbf{0}$$

since $(\bar{r}_i - \bar{r}_j) \parallel \bar{F}_{ij}^{int}$.

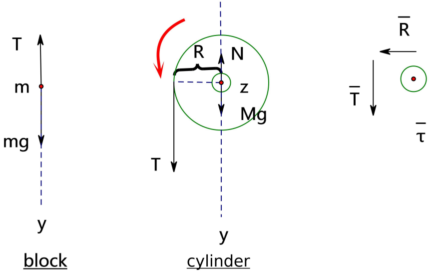
Example (a): unwinding string

The cylinder has mass M, radius R, , can rotate about its axis of symmetry ($I_z = \frac{1}{2}MR^2$. Find the acceleration of the block a_y and the tension T in the string as it unwinds from the cylinder.



Free-body diagrams

For rotational motion FBDs do not treat a rigid body as a point particle. Remember to draw the force vectors at the points where they act onto!



Laws of dynamics

1. $0 = Mg + T - N$ [the axis of the cylinder is fixed — statics]
2. $ma_y = mg - T$ [block — translational motion]
3. $I_z \varepsilon_z = \tau_z$ [cylinder — rotational motion]
4. $a_y = R\,\varepsilon_z$ [linear acceleration of the string = tangential acceleration of the point on the rim of the cylinder; string does not slip and is inextensible]

But $\tau_z = +RT$ – use in (3). And substitute (4) into (2)

$$\begin{cases} mR\varepsilon_z = mg - T & (2') \\ RT = I_z\varepsilon_z & (3') \end{cases}$$

Solving the system of equations we obtain

$$\boxed{\begin{aligned} \varepsilon_z &= \frac{mg}{mR + \frac{I_z}{R}} = \frac{mg}{(m + \frac{1}{2}M)R} \\ a_y &= \frac{m}{m + \frac{1}{2}M} g \\ T &= \frac{Mmg}{2m + M} \end{aligned}}$$

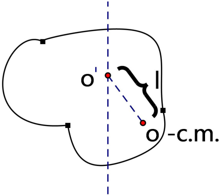
Discussion

$$\boxed{\begin{aligned} \varepsilon_z &= \frac{mg}{(m + \frac{1}{2}M)R} \\ a_y &= \frac{m}{m + \frac{1}{2}M} g \\ T &= \frac{Mmg}{2m + M} \end{aligned}}$$

- If $M = 0$ (massless pulley) $\Rightarrow T = 0 \quad a_y = g \quad \varepsilon_z = g/R$
- If $M \gg m$ ($\frac{m}{M} \to 0$; i.e. very heavy pulley) $\Rightarrow T \to mg$, a_y, $\varepsilon_z \to 0$
- If $M = m \Rightarrow T = \frac{1}{3}mg$, $a_y = \frac{2}{3}g$, $\varepsilon_z = \frac{2}{3}\frac{g}{R}$

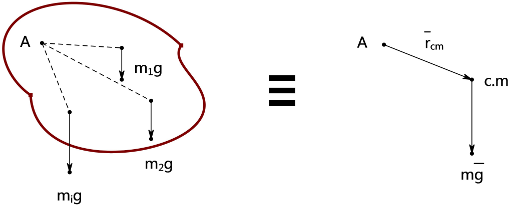
Example (b): physical pendulum

Analyze motion of a rigid body with mass m that can rotate about an axis O' parallel to axis O through the center of mass (moment of inertia I_O. The distance between the axes O and O' is l.

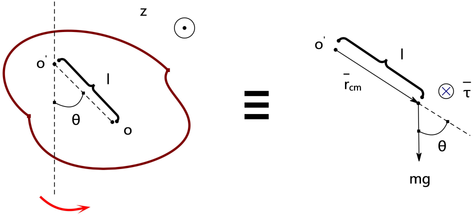


Observation: If the external force acting on a rigid body is the gravitational force, then its torque about an axis of rotation A is

$$\boxed{\bar{\tau}} = \sum_{i=1}^{N} \bar{r}_i \times m_i \bar{g} = \underbrace{\sum_{i=1}^{N} (m_i \bar{r}_i)}_{m\,\bar{r}_{cm}} \times \bar{g} = \boxed{\bar{r}_{cm} \times m\bar{g}}$$



Use this observation



Torque of the gravity $\bar{\tau} = \bar{r}_{cm} \times m\bar{g}$, and $\tau_z = -mgl\sin\theta$.
Equation of motion

$$I_{O'}\varepsilon_z = \tau_z \qquad \Longrightarrow \qquad I_{O'}\varepsilon_z = -mgl\sin\theta$$

But $\varepsilon_z = \ddot{\theta}$, hence

$$I_{O'}\ddot{\theta} = -mgl\sin\theta \qquad \Longrightarrow \qquad \ddot{\theta} + \frac{mgl}{I_{O'}}\sin\theta = 0$$

If the amplitude of oscillations is small $\sin\theta \approx \theta$ and

$$\boxed{\ddot{\theta} + \frac{mgl}{I_{O'}}\theta = 0.}$$

This is equation of motion of a simple harmonic oscillator with $\omega_0 = \sqrt{\frac{mgl}{I_{O'}}}$ and $T = 2\pi\sqrt{\frac{I_{O'}}{mgl}}$, where $I_{O'} = I_O + ml^2$ (parallel axis theorem).

Such an oscillating rigid body is called the *physical pendulum*.

Rigid body rotation about a moving axis

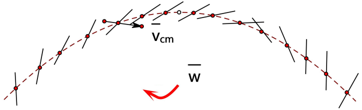
Translational and rotational motion of a rigid body

motion of a rigid-body

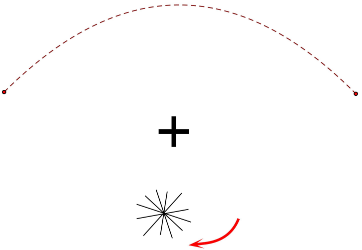
translational motion of the center of mass

$+$

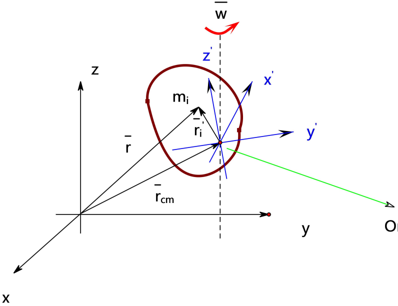
rotational motion about an instantaneous axis of rotation
(through the center of mass)



axis does not change direction



Kinetic energy in the combined motion



Origin placed at the center of mass

Velocity of element of mass m_i

$$\bar{v}_i = \bar{v}_{cm} + \bar{\omega} \times \bar{r}_{i'}$$
$$= \bar{v}_{cm} + \bar{\omega} \times \bar{r}_{i'_{\perp}}$$

$$\begin{aligned}
K &= \sum_i K_i = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i \bar{v}_i \circ \bar{v}_i \\
&= \frac{1}{2}\sum_i m_i \cdot (v_{cm}^2 + 2\bar{v}_{cm} \circ (\bar{\omega} \times \bar{r}_{i\perp}') + (\bar{\omega} \times \bar{v}_{i\perp}')^2) \\
&= \frac{1}{2}\sum_i m_i v_{cm}^2 + \bar{v}_{cm} \circ (\bar{\omega} \times \underbrace{\sum_i m_i \bar{r}_{i\perp}'}_{=0}) + \frac{1}{2}\sum_i m_i(\bar{\omega} \times \bar{r}_{i\perp}')^2
\end{aligned}$$

But

1. $\sum_i m_i \bar{r}'_{i\perp} = M\,\bar{r}'_{cm\perp}$ and the origin of $x'y'z'$ is placed at the center of mass if the body, so $\bar{r}'_{cm} = 0$ and $\bar{r}'_{cm\perp} = 0$
2. $\bar{\omega} \perp \bar{r}'_{i\perp}$, hence $(\bar{\omega} \times \bar{r}'_{i\perp})^2 = \omega^2 \bar{r}'^2_{i\perp}$

Eventually,

$$K = \sum_i K_i = \frac{1}{2}\sum_i m_i v_{cm}^2 + \frac{1}{2}\sum_i m_i \bar{r}^2_{i\perp}\omega^2.$$

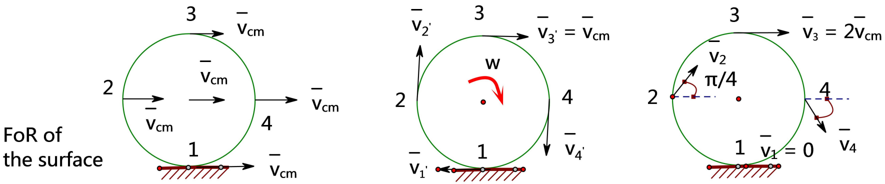
Note that $\sum_i m_i = M$ — the mass of the rigid body, and $\sum_i m_i r^2_{i\perp}$ is the moment of inertia I_{cm} of the body about the axis of rotation through the center of mass.

$$\boxed{K = \frac{1}{2}Mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2}$$

Conclusion

The rigid body's kinetic energy is the sum of $\frac{1}{2}Mv_{cm}^2$, associated with the translational motion of the center of mass, and $\frac{1}{2}I_{cm}\omega^2$ associated with rotation about an axis through the center of mass.

Rolling without slipping



translation of c.m. & rotation about axis through c.m (no slipping) = combined motion

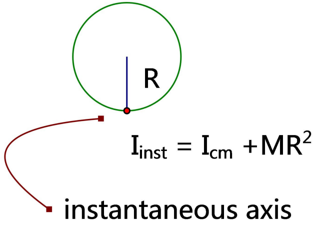
The rolling object, e.g. a cylinder, does not slip (is *rolling without slipping*) if the point of the object which is in contact with the surface is *instantaneously at rest with respect to that surface*.

Hence, the condition for rolling without slipping is $v_1' = v_{cm}$. But, on the other hand $v_1' = \omega R$. Therefore, the condition for rolling without slipping is

$$\boxed{v_{cm} = \omega R}$$

Comment

We may also treat this motion as rotational motion about an *instantaneous axis of rotation* through the point of contact with the surface. This rotation is with the same angular speed ω as the rotation about the axis through the center of mass. Then

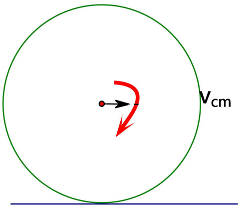


$$\begin{aligned} K &= \frac{1}{2} I_{inst}\omega^2 = \frac{1}{2}(I_{cm} + MR^2)\omega^2 = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M(\omega R)^2 \\ &= \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}^2 \end{aligned}$$

Example (a): Rolling with slipping

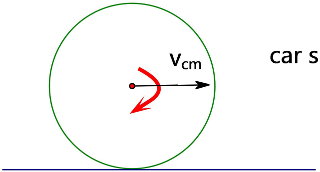
What happens if the equality $v_{cm} = \omega R$ does not hold?

"fast start"



$\omega R > v_{cm}$

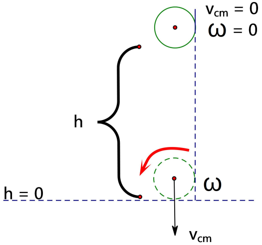
braking without ABS



car slides

$\omega R < v_{cm}$

Example (b): Thread unwinding from cylinder (no slipping)



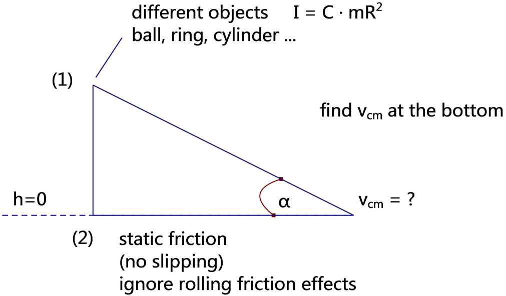
Use conservation of energy (*why can we use it?*)

$$K_1 + U_1 = K_2 + U_2$$

$$\begin{aligned} mgh &= \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{cm}\omega^2 \\ &= \frac{1}{2}mv_{cm}^2 + \frac{1}{4}mv_{cm}^2 = \frac{4}{3}mv_{cm}^2 \end{aligned}$$

$$\boxed{v_{cm} = \sqrt{\frac{3}{4}gh}}$$

Example (b): Rigid body race



Use conservation of energy (*why can we use it?*)

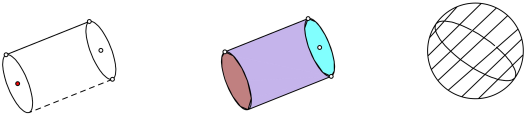
$$K_1 + U_1 = K_2 + U_2 \qquad \Longrightarrow \qquad mgh = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\left(\frac{v_{cm}}{R}\right)^2$$

$$gh = \frac{1}{2}(1+C)v_{cm}^2$$

Hence $\boxed{v_{cm} = \sqrt{\dfrac{2gh}{1+C}}}$

Conclusions from the result $\boxed{v_{cm} = \sqrt{2gh/(1+C)}}$

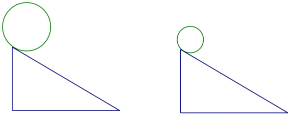
1. small C's give greater v_{cm}



m_1R	m_1R	m_1R
$I_1 = mR^2$	$I_2 = \frac{1}{2}mR^2$	$I_3 = \frac{2}{5}mR^2$

$v_{1cm} < v_{2cm} < v_{3cm}$ $\rightarrow$ less energy in rotational motion; more available for translational motion

2. v_{cm} does not depend on the size (R) or the mass (m); only the shape matters. E.g. all balls, irrespective of size and mass, have the same speed at the bottom.



Dynamics of combined motion

As we have discussed previously, for translational motion of the center of mass

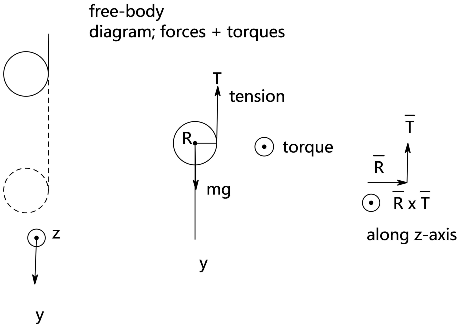
$$\sum_i \bar{F}_i^{ext} = \frac{d\bar{P}}{dt} = M\bar{a}_{cm}.$$

For rotational motion, the second law of dynamics,

$$\tau_z = I_z \varepsilon_z$$

was derived under the assumption that the axis of rotation is fixed.Detailed analysis shows that it is also true if the axis is moving if two conditions are satisfied: (1) the axis is a symmetry axis (through the center of mass); (2) the axis does not change its direction (orientation in space).

Example (a): thread unwinding from a cylinder



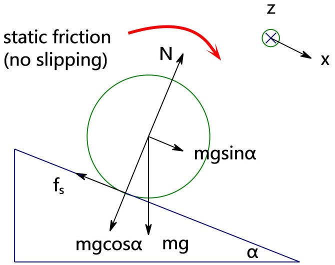
1. $mg - T = ma_{cm,y}$
2. $TR = I_{cm}\varepsilon_z = \frac{1}{2}mR^2\varepsilon_z$
3. $a_{cm,y} = R\varepsilon_z$ (no slipping; follows from $v_{cm,y} = \omega_z R$)

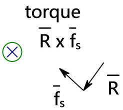
From (2) and (3): $T = \frac{1}{2}mR\varepsilon_z = \frac{1}{2}ma_{cm,y}$
From (1) and the above: $mg = \frac{3}{2}ma_{cm,y}$

These two equations yield

$$\boxed{a_{cm,y} = \frac{2}{3}g} \qquad \boxed{T = \frac{1}{3}mg}$$

Example (b): sphere rolling down an incline (no slipping)





$$\begin{cases} ma_{cm,x} = mg\sin\alpha - f_s \\ f_s R = I_{cm}\varepsilon_z = \frac{2}{5}mR^2\varepsilon_z \\ a_{cm,x} = \varepsilon_z R \end{cases}$$

Solving the system, one can find

$$\boxed{a_{cm,x} = \frac{5}{7}\,g\sin\alpha} \qquad \boxed{f_s = \frac{2}{7}\,mg\sin\alpha}$$

Comments

* Friction still in the same direction for a ball rolling uphill.
* In our examples we always neglect *rolling friction* (i.e. we assume perfect rigidity). Example: car tire.