
Question1 (1 points)
Find the area of the region enclosed by the curves

$$x = 1 - y^2 \qquad \text{and} \qquad x = y^2 - 1$$

Question2 (1 points)
Find the volume of the solid that is generated by revolving the region bounded by

$$y = 1 - \frac{1}{4}x^2$$

and the x-axis about the y-axis.

Question3 (1 points)
For what values of m do the line $y = mx$ and the curve $y = \dfrac{x}{x^2+1}$ enclose a region?

Question4 (1 points)
Consider the triangle with vertices $(1,1)$, $(1,2)$ and $(2,2)$. Find the volume of the region generated by rotating this triangle about the line $x = \dfrac{10}{3}$ and the line $y = 1$.

Question5 (1 points)
Let $\mathcal{A}$ be the region that is enclosed by the x-axis and the curve

$$y = 1 - x^2$$

Find the volume of the solid generated when the region $\mathcal{A}$ is rotated around the line $x = 1$.

Question6 (1 points)
Find the length of the curve $y = \ln(\sec x)$ for $0 \le x \le \pi/4$.

Question7 (1 points)
Find the area of the surface generated by revolving the following curve about x-axis.

$$y = \cos x, \qquad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Question8 (1 points)
Find the volume common to two circular cylinders, each with radius r, if the axes of the cylinders intersect at right angles.

Question9 (1 points)
Find the volume common to two spheres, each with radius r, if the centre of each sphere lies on the surface of the other sphere.

Question10 (3 points)

(a) (1 point) A 20 meter long steel cable has density 2 kilograms per meter, and is hanging straight down. How much work is required to lift the entire cable to the height of its top end?

(b) (1 point) Now suppose the cable in the previous question has a 100kg bucket of water attached to its lower end. How much work is required to lift the entire cable and bucket to the height of its top end?

(c) (1 point) Consider again the cable and bucket of the previous question. How much work is required to lift the bucket 10 meters by raising the cable 10 meters? i.e. The top half of the cable ends up at the height of the top end of the cable, while the bottom half of the cable is lifted 10 meters.

Question11 (1 points)
A lamina occupies the region between the circle $x^2 + y^2 = 4$ and the circle $x^2 + y^2 = 1$ in the first quadrant. Find the centroid.

Question12 (1 points)
Find the centroid of a thin wire shaped like an open semicircle of radius r.

Question13 (1 points)
Consider a pyramid with a square base of side a and a sphere with its centre on the base of the pyramid. Let the sphere be tangent to all eight sides of the pyramid. Find the height of the pyramid and then find the volume of the intersection of the pyramid and the sphere.

Question14 (0 points)

(a) (1 point (bonus)) Consider three curves in the first quadrant

$$C_1\colon y = x^2, \quad C_2\colon y = 2x^2 \qquad \text{and} \quad C_3\colon x = f(y)$$

Suppose for every point (m, n) on the curve C_2, the area of the region bounded by $y = 0$, $y = n$, C_2 and C_3 is equal to to the area of the region bounded by $x = 0$, $x = m$, C_1 and C_2. Find the curve C_3.

(b) (1 point (bonus)) Consider a bullet that is travelling in a straight line with an initial velocity of v_0. Suppose the drag force, which acts in the opposite direction to the motion, is proportional to the cube of the velocity of the bullet. Find the velocity function in terms of time t.

(c) (1 point (bonus)) Consider a plastic cylinder that is glued to the origin of some xyz-coordinate system so that the x-axis is the axis of the cylinder. Suppose a massless string is completed wrapped around the cylinder and there is a tiny metal ball tied to the end of the string and resting upon the cylinder. Imagine the ball receives a sudden force in the direction the y-axis so that the ball starts moving with a velocity of v in the direction of the y-axis. Since the string is attached to the ball, it starts to be unwrapped from the cylinder. Suppose the string was resting in a plane that is parallel to the yz-plane. Find the acceleration and velocity of the ball, and the trajectory equation of the ball.

(d) (1 point (bonus)) Suppose you are a human resource officer having n candidates applying for a single opening, where n is very very large. You are instructed by some eccentric CEO to make the decision about whether to recruit the candidate or not right after each interview, that is, you are not allowed to interview the next candidate unless reject the current one. Assuming all applicants are lined up in a queue, and the quality of applicants are distributed randomly. Assuming also it is your first day as a human resource officer, so you have know no idea on the distribution of the quality of the people in the market. How many applicants should you reject no matter what to form a reference sample before you start to consider giving out the position, that is, how big shall the learning batch be?

[Hint: You might find the note on the next page useful for some of the bonus questions.]

Definition 1. *A differential equation is an equation of*

an unknown function $y(t)$ and its derivatives.

Definition 2. *A first-order differential equation is called separable if it can be written in the form*

$$\dot{y} = GF$$

where G is a known function of y and F is a known function of t.

Theorem 1. *If $G(y)$ and $F(x)$ are continuous, then a separable equation has the solution,*

$$\int \frac{1}{G}\, dy = \int F\, dt$$

Proof. The proof is based on the chain rule and the fundamental theorem of calculus (FTC).

- Given a separable equation
$$\dot{y} = GF$$
- we rearrange to obtain
$$\frac{1}{G(y)} \frac{dy}{dt} = F(t)$$
- Write $\dfrac{1}{G(y)}$ and $F(t)$ as the derivative of their antiderivative using FTC,
$$\frac{d}{dy}\left(\int_{y_0}^{y} \frac{1}{G(\eta)}\, d\eta\right) \frac{dy}{dt} = \frac{d}{dt}\left(\int_{t_0}^{t} F(\tau)\, d\tau\right)$$
- Use the chain rule in reverse for the left-hand side
$$\frac{d}{dt}\left(\int_{y_0}^{y} \frac{1}{G(\eta)}\, d\eta\right) = \frac{d}{dt}\left(\int_{t_0}^{t} F(\tau)\, d\tau\right)$$
- Two functions have the same derivative must only differ by an additive constant
$$\int_{y_0}^{y} \frac{1}{G(\eta)}\, d\eta = \int_{t_0}^{t} F(\tau)\, d\tau + C \iff \int \frac{1}{G}\, dy = \int F\, dt$$

$\square$