Periodic Motion and The Harmonic Oscillator

So far we have discussed

* constant forces (e.g. free fall)
* velocity dependent forces (e.g. air/ fluid resistance; fall with air drag)
* time dependent forces (cf. Assignment 3)
* $\boxed{\text{position dependent forces}}$ ← now

In general

$$\overline{F} = \overline{F}(\overline{v}, \overline{\gamma}, t) \qquad \text{3D}$$
$$F_x = F(v_x, x, t) \qquad \text{1D}$$

↪velocity along the x-direction (one-component vector)

Newton's 2^{nd} law $\quad m\overline{a} = \overline{F} \quad$ or $\quad \overline{a} = \dfrac{\overline{F}}{m}$

In the 1-D case ($a_x = \dfrac{\mathrm{d}^2x}{\mathrm{d}t^2}$, $v_x = \dfrac{\mathrm{d}x}{\mathrm{d}t}$)

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = \frac{F_x(\frac{\mathrm{d}x}{\mathrm{d}t}, x, t)}{m} \qquad 2^{\text{nd}} \text{ order ordinary differential equation (ODE)}$$

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First, we will study a simple case $F_x = -kx$ Then

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -\frac{k}{m}x \qquad \text{an example of a } 2^{\text{nd}} \text{ order ODE with constant coefficients}$$

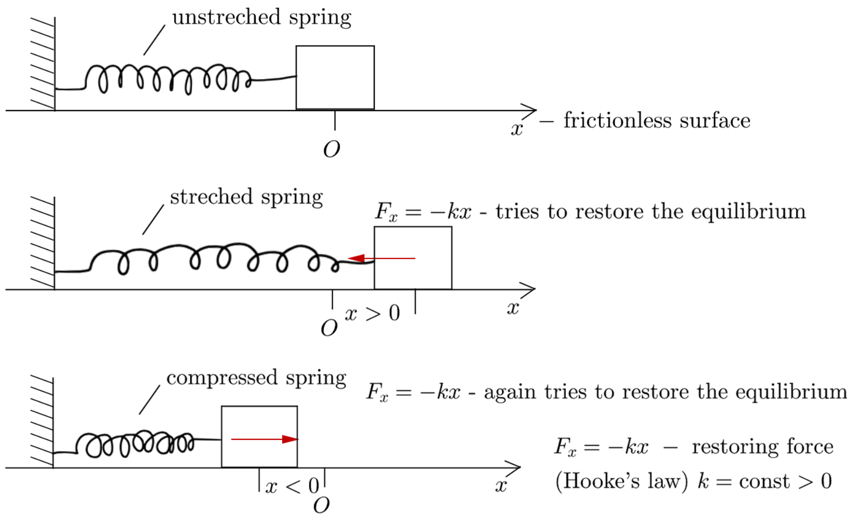
Note:

system for which the equation of motion is of the force $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} + \dfrac{k}{m}x = 0$ is called the harmonic oscillator

Where can we come across this kind of position-dependent force?

Examples

(a)



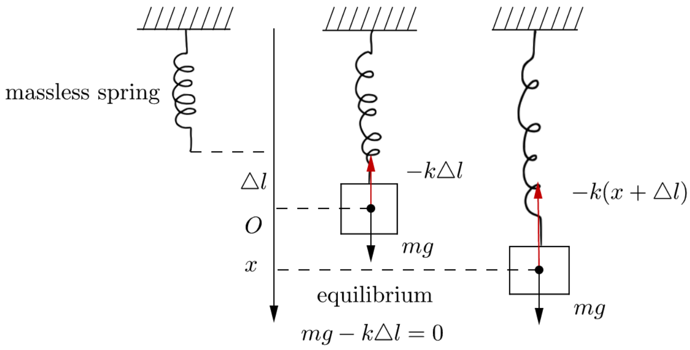
$F_x = -kx$ – restoring force

(Hooke's law) $k = \text{const} > 0$

Equation of motion

$$ma = -kx$$
$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{k}{m}x = 0 - \text{harmonic oscillator}$$

(b) vertical oscillator

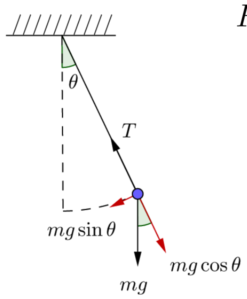


Equation of motion

$$ma = mg - k(x + \triangle l)$$
$$ma = \underbrace{mg - k\triangle l}_{=0} - kx$$
$$ma = -kx \quad \Rightarrow \quad \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \frac{k}{m}x = 0$$

-harmonic oscillator

(c) simple pendulum

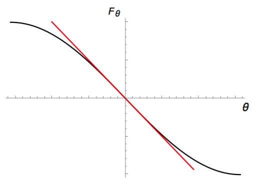


Tangential component

$$F_\theta = -mg\sin\theta \overset{\text{small angles only}}{\approx} -mg\theta$$

(small angles:
$\sin\theta = \underbrace{\theta}_{\text{keep the 1st term}} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!}$)

$\theta = 0.1 \quad (6^\circ), \quad \sin\theta = 0.0998$



Motion along the arc (tangential components)

$$ma_\theta = F_\theta$$
$$s = l\theta$$
$$\Rightarrow m\frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \approx -mg\frac{s}{l}$$
$$\Rightarrow \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} = -\frac{g}{l}s \quad - \text{harmonic oscillator}$$

first case

$\boxed{\text{A}}$ only the restoring force acts $\quad m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -kx$

Mathematical problem to solve

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = 0 \tag{1}$$

How to solve?

* use methods of the theory of differential equations

* look for solutions $\propto e^{i\tilde{\omega}t}$, $\tilde{\omega}$ -complex, then take the real part of this complex solution

$\boxed{*}$ guess $\boxed{x(t) = \cos\omega_0 t}$ and check

Check:

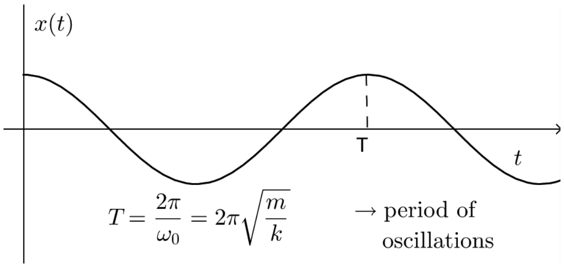
$$
\begin{aligned}
\dot{x}(t) &= -\omega_0 \sin \omega_0 t \\
\ddot{x}(t) &= -\omega_0^2 \cos \omega_0 t = -\omega_0^2 x(t) \\
&\Rightarrow \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + \omega_0^2 x = 0
\end{aligned}
$$

↱natural angular frequency

Conclusion: $x(t) = \cos \omega_0 t$ solves (1) if $\omega_0 = \sqrt{k/m}$

↪angular frequency ($vs.$frequency $f_0 = \omega_0 / 2\pi$)

Periodic behavior (oscillations) →



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Have we found the most general solution?
Observations:

(1)

$$
\begin{aligned}
x(t) &= A \cos \omega_0 t \\
\dot{x}(t) &= -\omega_0 A \sin \omega_0 t \\
\ddot{x}(t) &= -\cos^2 A \cos \omega_0 t = -\omega_0^2 x(t) \qquad \rightarrow \text{also solves(1)}
\end{aligned}
$$

(2)

$$
\begin{aligned}
x(t) &= A \cos(\omega_0 t + \varphi) \\
\dot{x}(t) &= -\omega_0 A \sin(\omega_0 t + \varphi) \\
\ddot{x}(t) &= -\cos^2 A \cos(\omega_0 t + \varphi) = -\omega_0^2 x(t) \qquad \rightarrow \text{solves(1), too}
\end{aligned}
$$

The most general solution

$$
\boxed{x(t) = \underset{\text{amplitude}}{A} \cos(\omega_0 t + \underset{\text{phase shift}}{\varphi})} \tag{2}
$$

Equivalently (see Problem Set 4) the most general solution can be written as

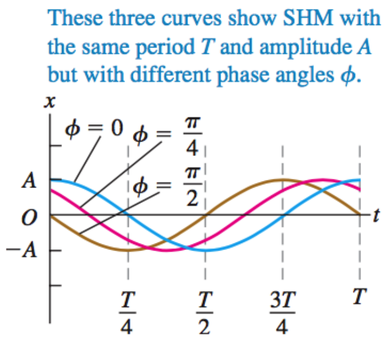
$$
x(t) = B \cos \omega_0 t + C \sin \omega_0 t \tag{3}
$$

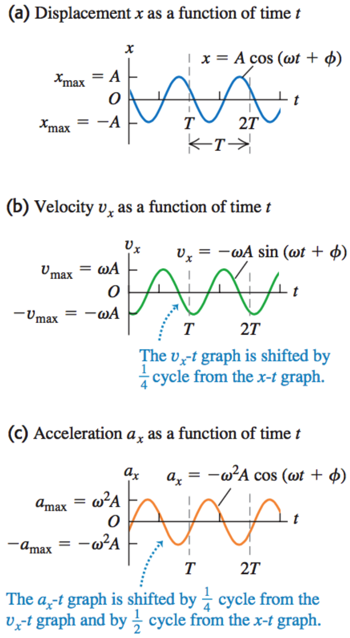
Note: B and C again are two constants; 2^{nd} order ODEs have general solutions depending on two parameters

The constants A and φ (or B and C) are found by applying the initial conditions:

$$
\begin{cases}
x(0) = x_0 \\
v_x(0) = v_{0x}
\end{cases}
$$

then the problem has a unique solution.





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Position, velocity, and acceleration in the simple harmonic motion (only the restoring force acts)

Comment:

* velocity shifted by 1/4 of the cycle ($\pi/2$) with respect to position

* acceleration shifted by 1/2 of the cycle (π) with respect to position

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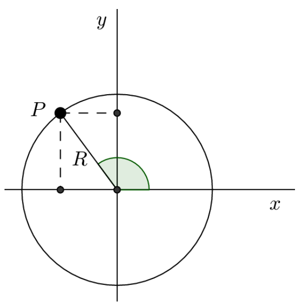
Comment: Harmonic motion and uniform circular motion



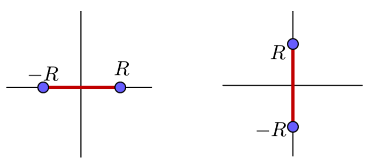
$$\frac{\mathrm{d}\varphi}{\mathrm{d}t} = \omega_0 = \frac{v}{R} = \text{const} \quad \Rightarrow \quad \varphi = \omega_0 t \quad [\text{assume } \varphi(0) = 0]$$

$$\begin{cases} x = R\cos \overbrace{\omega_0 t}^{\varphi} \\ y = R\sin \underbrace{\omega_0 t}_{\varphi} \end{cases}$$

Differentiate twice w.r.t. time

$$\begin{cases} a_x = -R\omega^2 \cos\omega_0 t = -\omega_0^2 x \\ a_y = -\omega_0^2 y \end{cases}$$

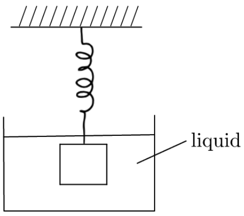
Conclusion the projection of P onto the x axis (or the y axis) moves as if it was in a harmonic motion.



$\boxed{\text{B}}$ add a linear drag to the model

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -b\frac{\mathrm{d}x}{\mathrm{d}t} - kx$$

where $b > 0$ is constant



or

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + \frac{b}{m}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{k}{m}x = 0$$

How to solve? Look for solutions of the form (complex!)

$$\boxed{\tilde{x}(t) = Ae^{i(\tilde{\omega}t+\varphi)}}$$

where $\tilde{\omega} = \alpha + i\beta$, α, β -real

Then the solution with a physical meaning

$$x(t) = Re\tilde{x}(t)$$

Why can we do so?
Equation is linear, so the real & imaginary parts do not mix with each other.
Derivatives:

$$\dot{\tilde{x}}(t) = i\tilde{\omega}Ae^{i(\tilde{\omega}t+\varphi)} = i\tilde{\omega}\tilde{x}(t) = (i\alpha - \beta)\tilde{x}(t)$$
$$\ddot{\tilde{x}}(t) = (i(\tilde{\omega})^2\tilde{x}(t) = -(\alpha^2 + 2i\beta\alpha - \beta^2)\tilde{x}(t)$$

Now, the equation $\dfrac{\mathrm{d}^2\tilde{x}}{\mathrm{d}t^2} + \dfrac{b}{m}\dfrac{\mathrm{d}\tilde{x}}{\mathrm{d}t} + \dfrac{k}{m}\tilde{x} = 0$

$$\left[(\beta^2 - 2i\alpha\beta - \alpha^2) + \frac{b}{m}(i\alpha - \beta) + \frac{k}{m}\right]\tilde{x}(t) = 0$$
$$\Updownarrow$$
$$(\beta^2 - \alpha^2 - \frac{b}{m}\beta + \frac{k}{m}) - i(2\alpha\beta - \frac{b}{m}\alpha) = 0$$

Hence

$$\boxed{\beta^2 - \alpha^2 - \frac{b}{m}\beta + \frac{k}{m} = 0} \quad \text{and} \quad \boxed{2\alpha\beta = \frac{b}{m}\alpha}$$

Note: A differential equation turned into two algebraic equations

Possible cases:

(a) $\alpha \neq 0$, then

$$\beta = \frac{b}{2m} \quad \text{and} \quad \alpha = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

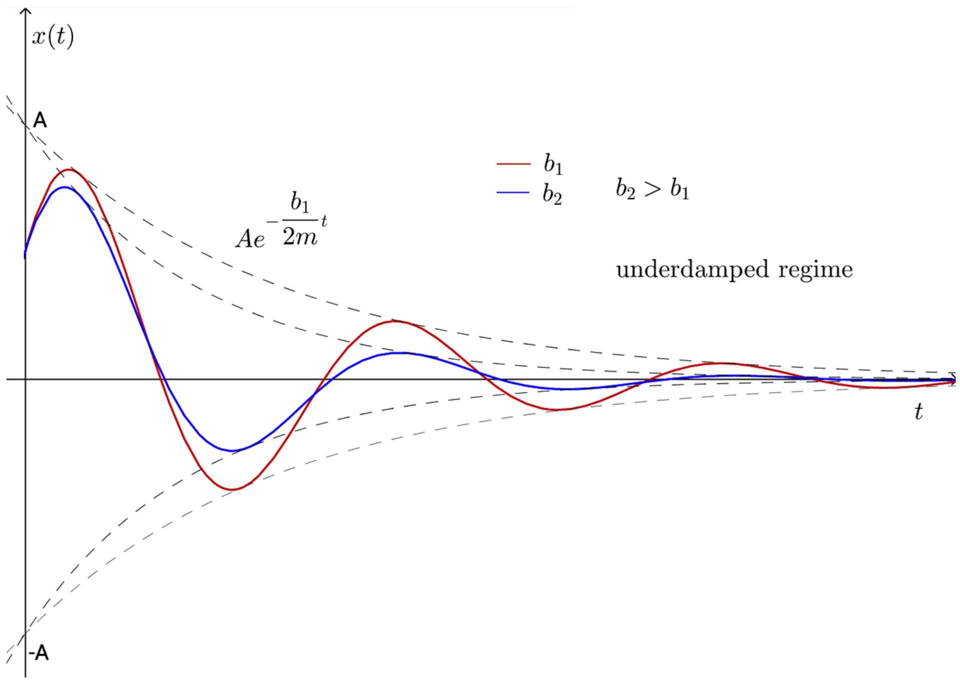
where $\dfrac{k}{m} > \dfrac{b^2}{4m^2}$ (α is real)

and

$$\tilde{x}(t) = Ae^{i[(\alpha+i\beta)t+\varphi]} = Ae^{i(\alpha t+\varphi)}e^{-\beta t}$$
$$\tilde{x}(t) = Ae^{i\left(\sqrt{\frac{k}{m}-\frac{b^2}{4m^2}}t+\varphi\right)}e^{-\frac{b}{2m}t}$$

Take Re and note that $\dfrac{k}{m} = \omega_0^2$

$$\boxed{x(t) = Re\tilde{x}(t) = Ae^{-\frac{b}{2m}t}\cos\left(\sqrt{\omega_0^2 - \frac{b^2}{4m^2}}t + \varphi\right)} \tag{4}$$



Consequences of damping:

* motion still periodic (if $\frac{b^2}{4m^2} < \omega_0^2$, i.e. damping not too strong)
* the amplitude of oscillations decreases with time
* the angular frequency $\omega^2 = \omega_0^2 - \frac{b^2}{4m^2} < \omega_0^2$, so it is smaller than in the undamped case (consequently the period increases, $T = \frac{2\pi}{\omega}$)

Note We could have chosen $\alpha < 0$, it wouldn't change the result, since φ has to be choosen accordingly

(b) $\alpha = 0$, then

$$\beta = \frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \omega_0^2} \qquad \text{or} \qquad \beta = \frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \omega_0^2}$$

where $\frac{b^2}{4m^2} > \omega_0^2$

$$\tilde{x}(t) = A_1 e^{i\varphi} e^{-\left(\frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t} + A_2 e^{i\varphi} e^{-\left(\frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t}$$

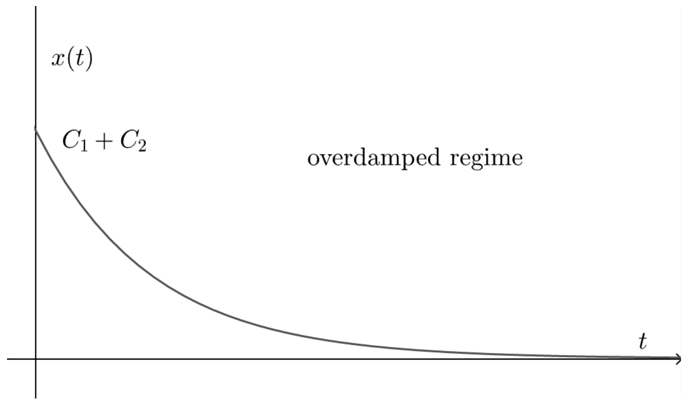
Two linearly independent solution, have to take their linear combination

$$\boxed{X(t) = Re\tilde{x}(t) = C_1 e^{-\left(\frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t} + C_2 e^{-\left(\frac{b}{2m} - \sqrt{\frac{b^2}{4m^2} - \omega_0^2}\right)t}} \tag{5}$$

$C_k = A_k \cos\varphi, k = 1, 2$

Consequence:

* strong damping ($\frac{b^2}{4m^2} > \omega_0^2$) results in aperiodic motion: the particle returns aperiodically to the equilibrium



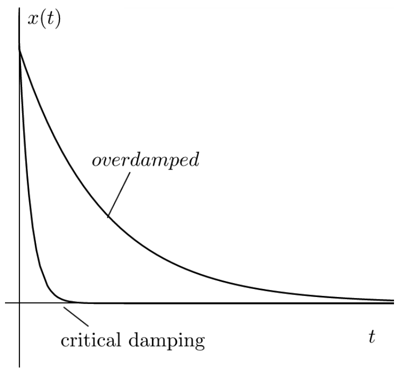
(c) if $\frac{b^2}{4m^2} = \frac{k}{m} (= \omega_0^2)$, then we have $\alpha = 0 \quad \beta = \frac{b}{2m}$

$$\boxed{x(t) = D_1 e^{-\frac{b}{2m}t} + D_2 t e^{-\frac{b}{2m}t}} \tag{6}$$

t added in the second term to generate second solution linearly independent from $e^{-\frac{b}{2m}t}$

Consequence:

* aperiodic motion



$\boxed{\text{C}}$ forced oscillations and resonance

now: restoring force + linear drag + driving force F_{dr}

Simplest case to analyze: $F_{dr} = F_0 \cos \omega_{dr} t$ $\rightarrow$ sinusoidal time-dependence

Equation of motion

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -b\frac{\mathrm{d}x}{\mathrm{d}t} - kx + F_0 \cos \omega t$$

Observation:

After some time the oscillations stabilize and the particle oscillates with the angular frequency of the driving force (there may be a shift in phase between the drive and response though)

So steady-state solution should be of the form

$$\tilde{x}_s(t) = Ae^{i(\omega_{dr}t+\varphi)}$$

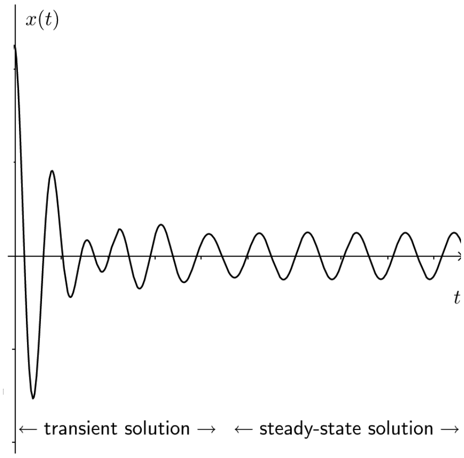
ω_{dr}-real, $\varphi < 0$ (phase lag)

Analysis similar to that in $\boxed{\text{B}}$ (details omitted) shows that

$$\boxed{\begin{aligned} A &= \frac{F_0}{m\sqrt{(\omega_0^2-\omega_{dr}^2)^2+\left(\frac{b\omega_{dr}}{m}\right)^2}} \\ \tan\varphi &= \frac{b\omega_{dr}}{m(\omega_{dr}^2-\omega_0^2)} \end{aligned}} \tag{7}$$

where $\omega_0 = \sqrt{\frac{k}{m}} \to natural frequency$

Note:



Discussion of the results

$$A = \frac{F_0}{m\sqrt{(\omega_0^2-\omega_{dr}^2)^2+\left(\frac{b\omega_{dr}}{m}\right)^2}}$$

$$\left[FIG.14.28/p.460\right]$$

Features:
For A:

* peak in the curve $A = A(\omega_{dr})$ at $\underbrace{\overset{=\omega_{dr}}{\omega_{res}}}_{\hookrightarrow\text{resonant frequency}} \approx \sqrt{\omega_0^2 - \frac{b^2}{2m^2}}$

sharp increase in the amplitude of oscillations when $\omega_{dr} \approx \omega_{res}$ is called the (mechanical) resonance

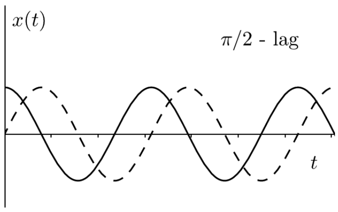
* if $\omega_{dr} \to 0$ (i.e., $T_{dr} \to \infty$, constant force), then $A \to \dfrac{F_0}{m\omega_0^2} = \dfrac{F_0}{k}$

For φ:

* if $\omega_{dr} \to \omega_0$ then

$$\tan\varphi = \frac{b\omega_{dr}}{m(\omega_{dr}^2 - \omega_0^2)} \qquad \Rightarrow \qquad \varphi \to -\pi/2$$

the response ($x(t)$) lags the drive ($F(t)$) by 1/4 of the cycle



* if $\omega_{dr} \to \infty$ (high frequencies)

$$\varphi \to \pi$$

the response lags the drive by 1/2 of the cycle (displacement and drive are in antiphase)