Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 1

Date Due: 12:55 PM, Monday, the 3rd of March 2021

Discussion Class Preparation

Please (re-)view Video files 1-12 and/or finish reading the section "Distributions" in the lecture slides. You should be able to answer the following questions:

i) Explain what a test function in $\mathbb{R}^n$ is.

ii) Explain what a null sequence of test functions is and give an example and a counter-example.

iii) Explain what a distribution is.

iv) What is a locally integrable function? Give examples and counter-examples.

v) Define what a regular and a singular distribution is. Give examples.

vi) Explain how operations for functions are extended to distributions "by duality".

vii) What are Green's first and second identities?

viii) Why is the function $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = 1/x$ not a distribution? Explain what the principal value of g is.

Exercises (16 Marks)

Exercise 1.1
Consider the test function on $\mathbb{R}^n$,

$$\phi(x) = \begin{cases} c_n e^{1/(|x|^2-1)} & |x| < 1, \\ 0 & |x| \geq 1. \end{cases}$$

where c_n has been chosen so that the integral of ϕ over the unit ball is 1. Thus c_n is unambiguously determined by the condition $\int_{\mathbb{R}^n} \phi = 1$. For $\varepsilon > 0$ and $\xi \in \mathbb{R}^n$ the function $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^n)$,

$$\varphi_\varepsilon(x-\xi) = \frac{1}{\varepsilon^n}\phi\left(\frac{x-\xi}{\varepsilon}\right),$$

vanishes for $|x-\xi| \geq \varepsilon$ and has the property $\int_{\mathbb{R}^n} \phi_\varepsilon(x-\xi)\,dx = 1$.

Now let $u \in C(\mathbb{R}^n)$ be a function of compact support (note that u need only be continuous).

i) Show that the convolution

$$(u * \phi_\varepsilon)(x) := \int_{\mathbb{R}^n} \phi_\varepsilon(x-\xi)u(\xi)\,d\xi$$

is a test function.
(2 Marks)

ii) Show that $u * \phi_\varepsilon \to u$ uniformly as $\varepsilon \to 0$, i.e.,

$$\sup_{x\in\mathbb{R}} |u * \phi_\varepsilon(x) - u(x)| \xrightarrow{\varepsilon\to 0} 0.$$

(2 Marks)

iii) What changes if the condition that u have compact support is replaced by requiring that $\lim\limits_{|x|\to\infty} u(x) = 0$?
(2 Marks)

Exercise 1.2

Identify each of the following maps $T: \mathcal{D}(\mathbb{R}) \to \mathbb{C}$ as a regular distribution, a singular distribution, or not a distribution at all. Give a short reason for your answer.

i) $T: \varphi \mapsto \int_{-1}^{1} \varphi(x)\,dx$

ii) $T: \varphi \mapsto \int_{-\infty}^{\infty} x\varphi(x)\,dx$

iii) $T: \varphi \mapsto 0$

iv) $T: \varphi \mapsto \varphi'(0)$

v) $T: \varphi \mapsto \varphi(0) \cdot \varphi(1)$

(10 Marks)