

Chapter 5: Steady Electric Currents

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JOINT INSTITUTE
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**Everything you need to know
about steady electric currents
in this course...**



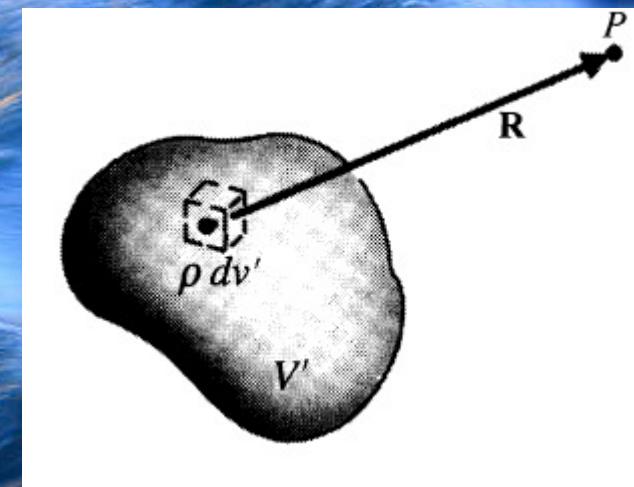
Static electric charge density
Electric field



Static electric charge density Electric field



$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{a}_R \frac{\rho}{R^2} dv' \quad (\text{V/m}),$$



Static electric charge density

Electric field



Inside a Conductor
(Under Static Conditions)

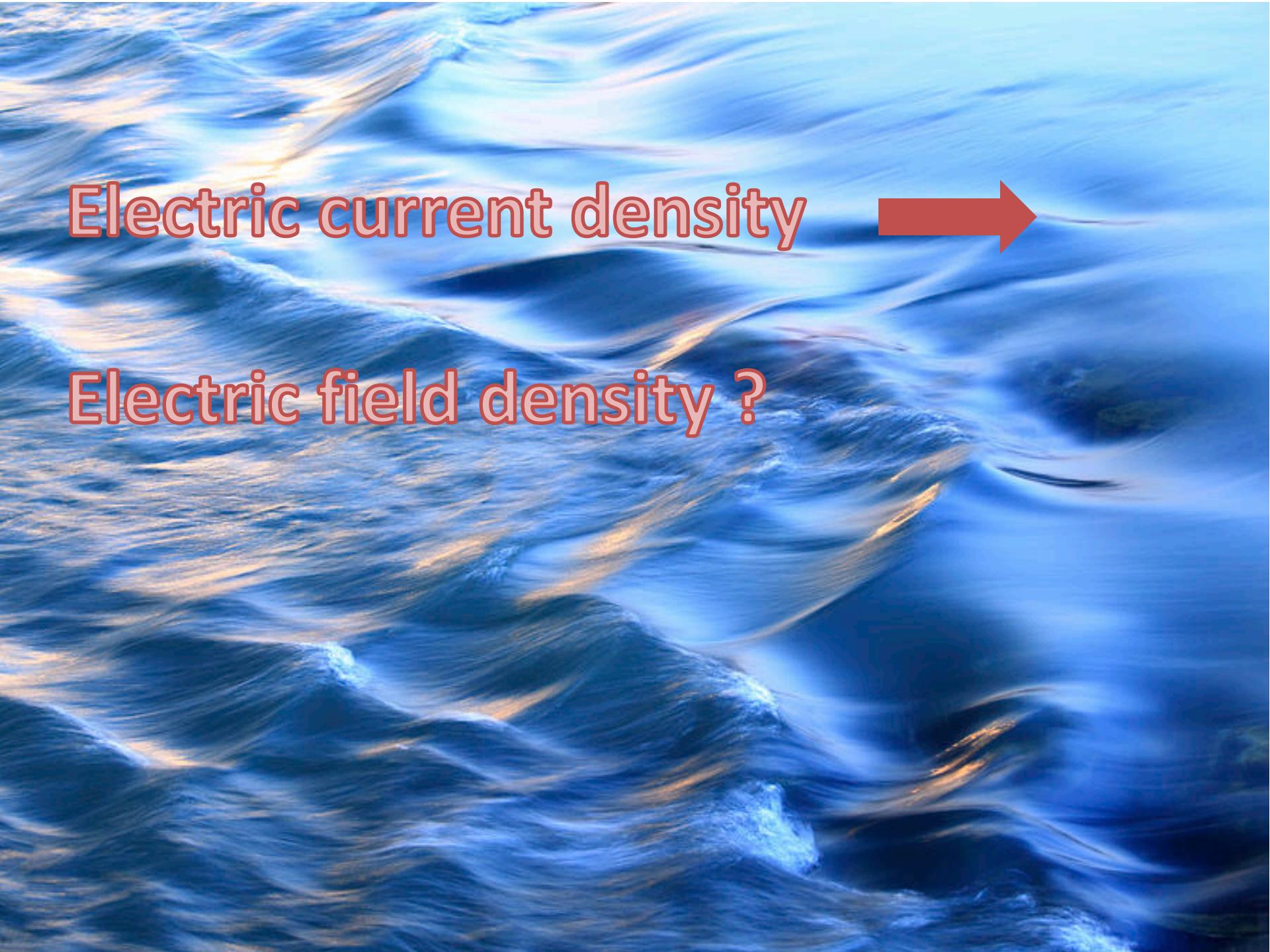
$$\rho = 0$$

$$\mathbf{E} = 0$$

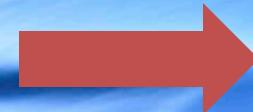
Boundary Conditions
at a Conductor/Free Space Interface

$$E_t = 0$$

$$E_n = \frac{\rho_s}{\epsilon_0}$$

The background of the image is a photograph of ocean waves at sunset. The water is a deep blue, with white foam and sunlight reflecting off the crests of the waves, creating a golden glow.

Electric current density



Electric field density ?

The background of the slide features a photograph of ocean waves at sunset. The water is a deep blue, with white foam and sunlight reflecting off the crests of the waves, creating a warm, golden glow.

Electric current density



Electric field density ?

Deriving Ohm's Law:

Relating speed, current,
resistivity/conductivity

Electric current density



Electric field density

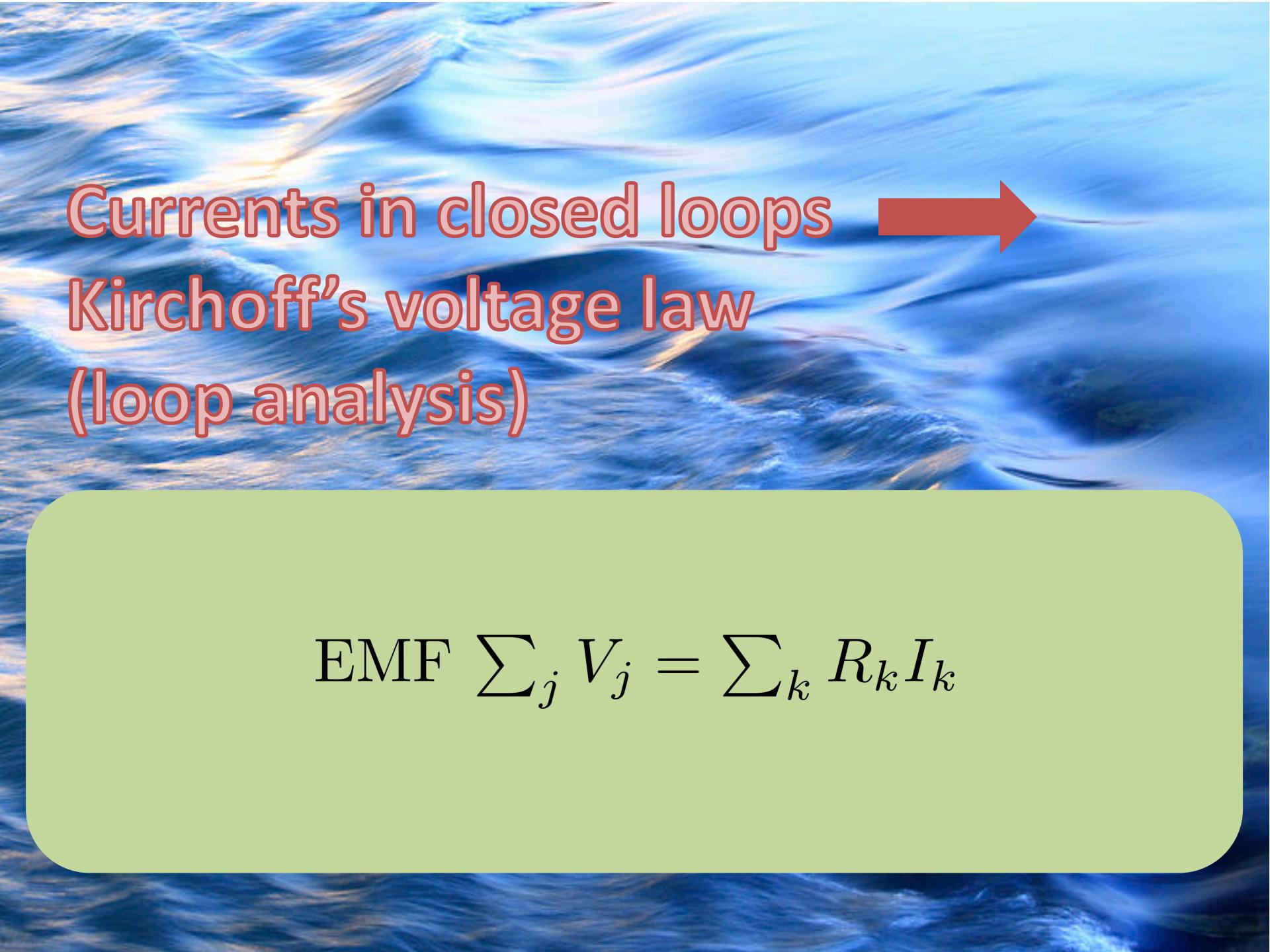
Deriving Ohm's Law:

$$V = IR$$

$$J = \sigma E$$

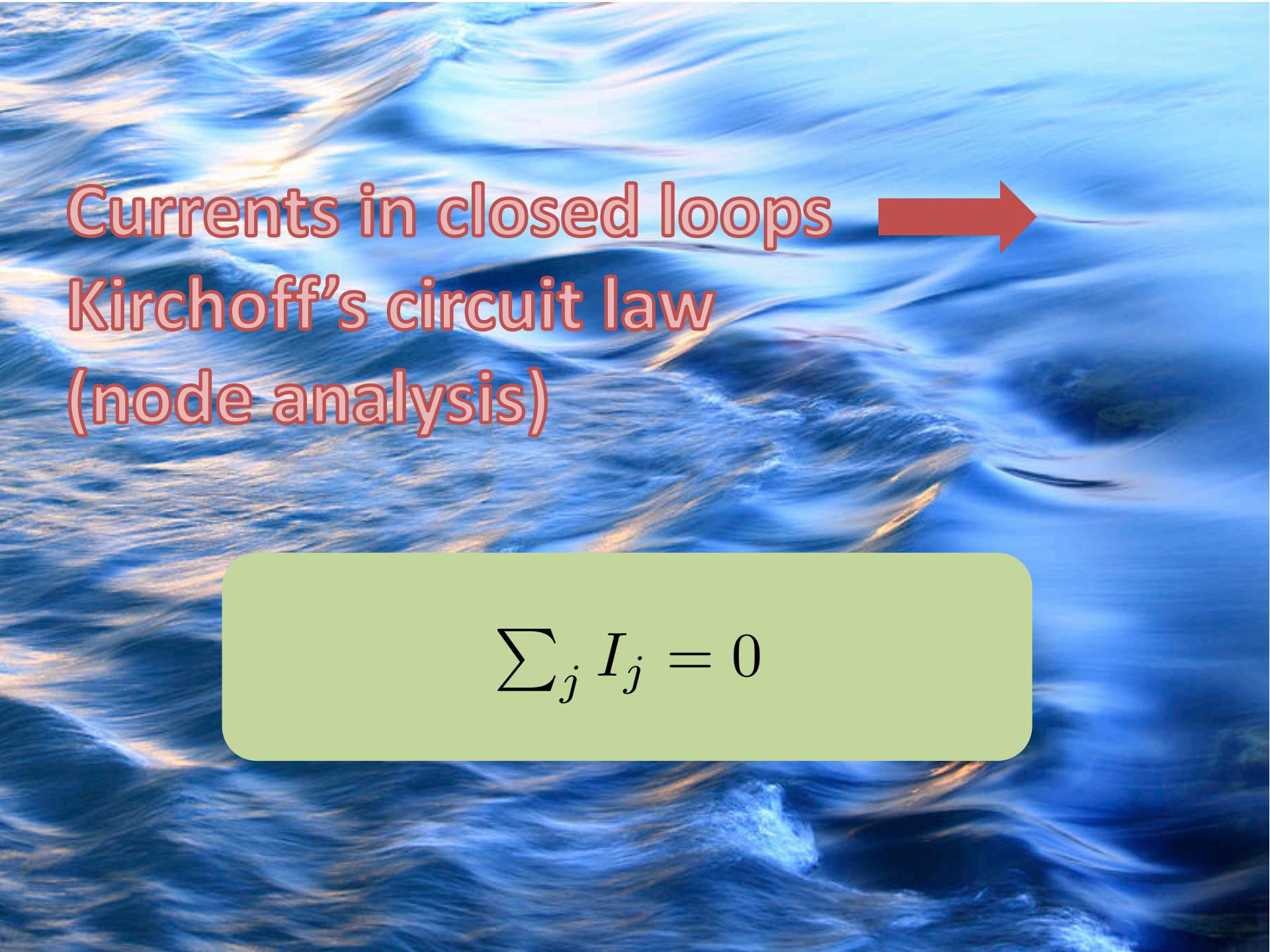


Currents in closed loops →
Conservation laws



Currents in closed loops →
Kirchoff's voltage law
(loop analysis)

$$\text{EMF } \sum_j V_j = \sum_k R_k I_k$$

A photograph of ocean waves with sunlight reflecting off the water, creating bright highlights on the crests.

Currents in closed loops →
Kirchoff's circuit law
(node analysis)

$$\sum_j I_j = 0$$

Ohm's law versus Kirchoff's circuit law

Ohm's Law describes the relationship between voltage and current across a resistive element.

Kirchhoff's Law describes the behaviour of current and voltage respectively in a circuit branch.

Ohm's Law states that voltage across a conductor is proportional to the current flows through it.

KCL states that the sum of current flows to a node is equal to zero while KVL states that the sum of voltages in a closed loop is zero.

Ohm's Law is applicable to a single resistive element or set of resistive circuits as a whole.

KCL and KVL are applicable to a series of resistive elements in a circuit.

Steady currents governing equations

Governing Equations for Steady Current Density

Differential Form	Integral Form
$\nabla \cdot \mathbf{J} = 0$	$\oint_s \mathbf{J} \cdot d\mathbf{s} = 0$
$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0$	$\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\ell = 0$

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$$\oint_s \mathbf{J} \cdot d\mathbf{s} = 0$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0$$

$$\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\ell = 0$$

Steady currents governing equations

Normal
component

$$\nabla \cdot \mathbf{J} = 0$$

Tangential
component

$$J_{1n} = J_{2n} \quad (\text{A/m}^2).$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0$$

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.$$

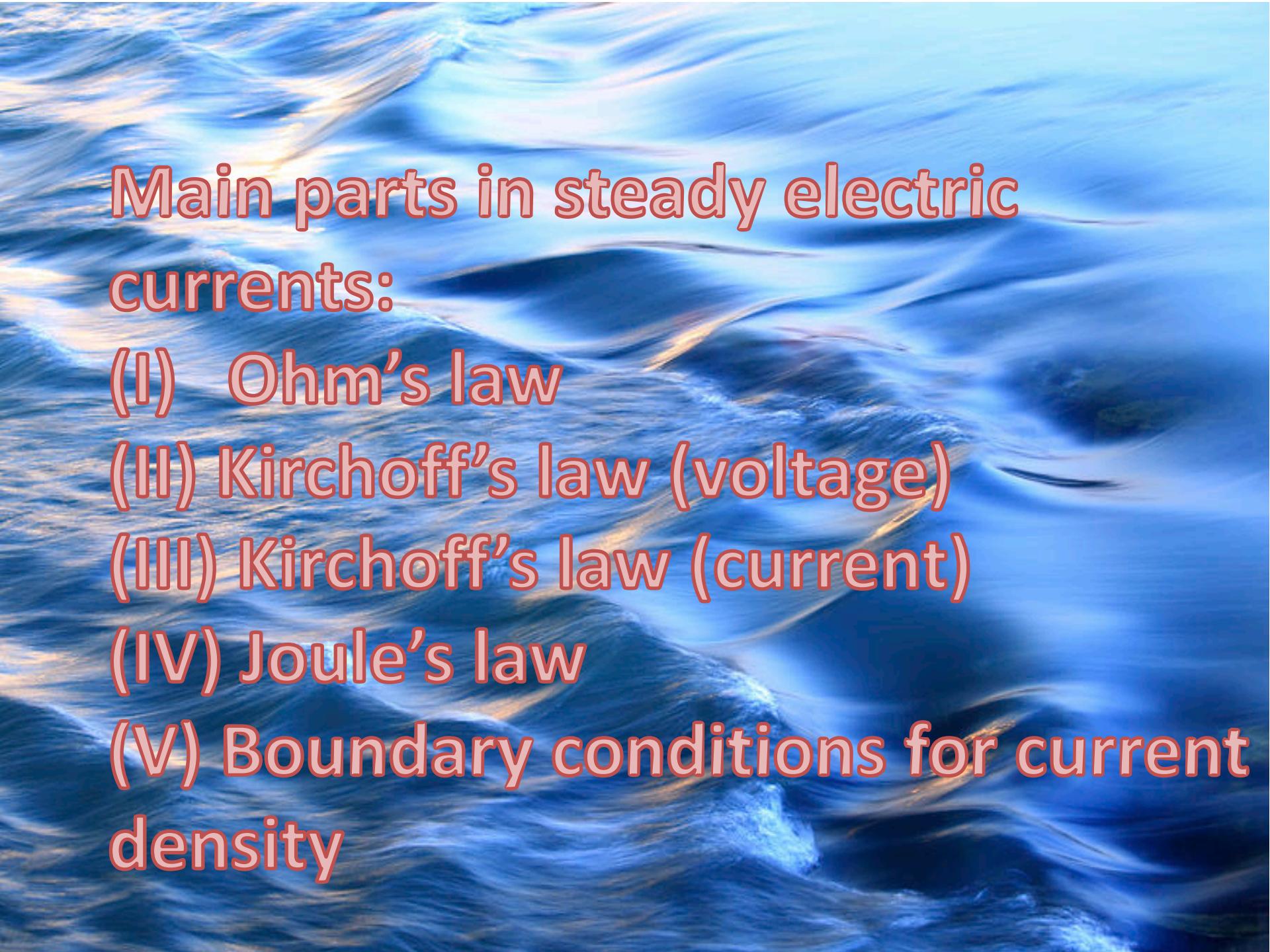
Analogy of physics at the boundary

Dielectric

$$D \quad \epsilon$$

Conductor

$$J \quad \sigma$$



Main parts in steady electric currents:

- (I) Ohm's law
- (II) Kirchoff's law (voltage)
- (III) Kirchoff's law (current)
- (IV) Joule's law
- (V) Boundary conditions for current density

5-1 Introduction

- Charges at rest (Ch3 and Ch4); Charges in motion (Ch5)
- Different types of currents
 - **Conduction currents:**
 - in conductors and semiconductors
 - electrons and/or holes
 - **Electrolytic currents:**
 - essentially in a liquid medium
 - Ions (e.g., Li-ion batteries)
 - **Convection currents:**
 - in vacuum or rarefied gas
 - electrons and/or ions

Average drift velocity of electrons for good conductors

Option	Speed in one direction	Number of votes
A	$10^3 - 10^4$ m/s	
B	$10^{-4} - 10^{-3}$ m/s	
C	$10^{-14} - 10^{-13}$ m/s	

Average drift velocity of electrons for good conductors

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A	$10^3 - 10^4$ m/s	
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Topics for Conduction Currents

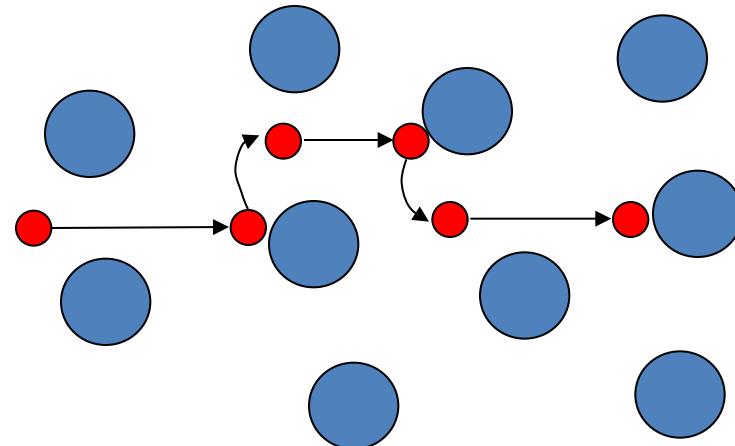
- Point form of Ohm's law
- Kirchhoff's voltage law
- Kirchhoff's current law
 - Conservation of charge
 - Equation of continuity
- Boundary conditions for current density

Conduction Currents

- For a conductor, atoms consist of positively charged nuclei surrounded by electrons
 - Inner shell: tightly bound charges
 - Outermost shell: loosely bound charges (valance or conduction electrons)
- Without external E: conduction electrons wander randomly → **no net drift motion** of conduction electrons

Conduction Currents

- With external E : **organized motion** of conduction electrons
 - Very low drift velocity due to collision with atoms
 - Conductor remains electrically neutral (electric forces prevent excess electrons from accumulating at any point of a conductor)



5-2 Current Density and Ohm's Law

Charge q across surface Δs with a velocity \mathbf{u}
N: #/volume

The amount of charge passing Δs $\Delta Q = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s \Delta t$ (C).

differential volume with Δs along \mathbf{a}_n



$$\Delta I = \frac{\Delta Q}{\Delta t} = Nq\mathbf{u} \cdot \mathbf{a}_n \Delta s = Nq\mathbf{u} \cdot \Delta s \quad (\text{A}).$$

$$\mathbf{J} = Nq\mathbf{u} \quad (\text{A/m}^2),$$

$$\mathbf{J} = \rho\mathbf{u} \quad (\text{A/m}^2),$$

Defined as **volume** current density

$$\Delta I = \mathbf{J} \cdot \Delta s.$$

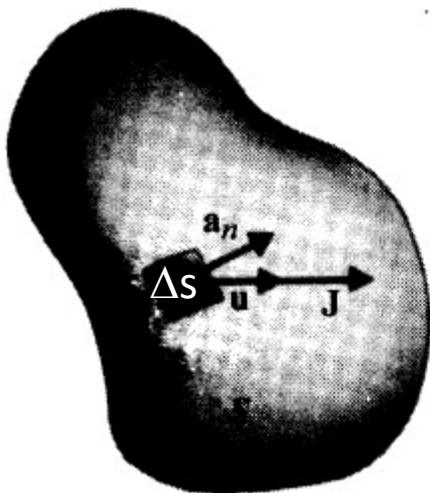


FIGURE 5-1
Conduction current due to drift motion of charge carriers across a surface.

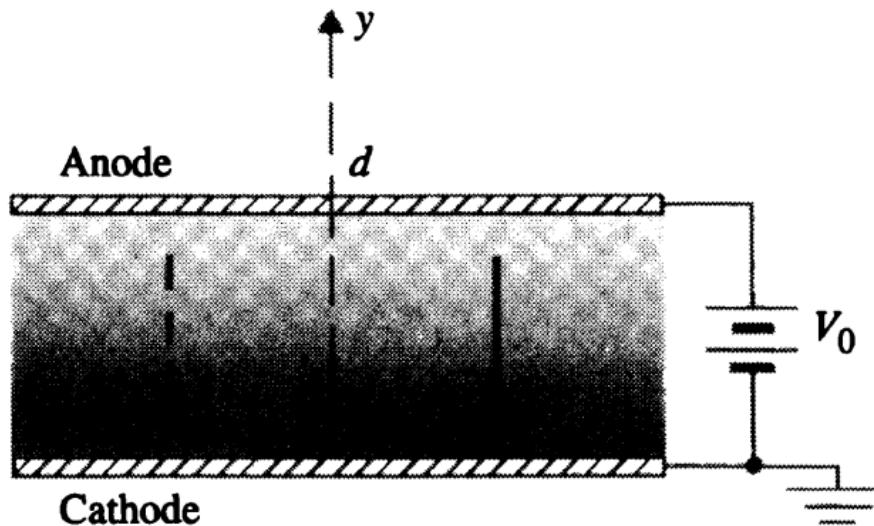
$$\Delta I = \mathbf{J} \cdot \Delta \mathbf{s}.$$



Total current I flowing
through a surface S

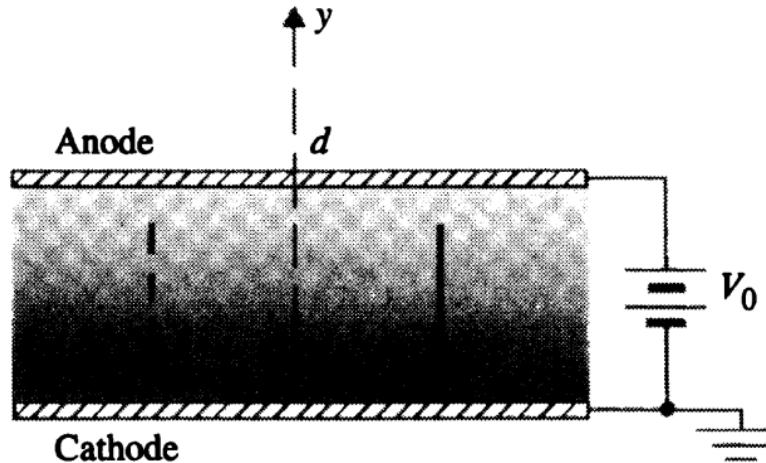
$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \quad (\text{A}).$$

Example



In vacuum-tube diodes, electrons are emitted from a hot cathode at zero potential and collected by an anode maintained at a potential V_0 , resulting in a convection current flow. Assuming that the cathode and the anode are parallel conducting plates and that the electrons leave the cathode with a zero initial velocity (space-charge limited condition), find the relation between the current density J and V_0 .

Example



In other words, the net electric field at the cathode is zero. Neglecting fringing effects, we have

$$\mathbf{E}(0) = \mathbf{a}_y E_y(0) = -\mathbf{a}_y \left. \frac{dV(y)}{dy} \right|_{y=0} = 0.$$

In the steady state the current density is constant, independent of y :

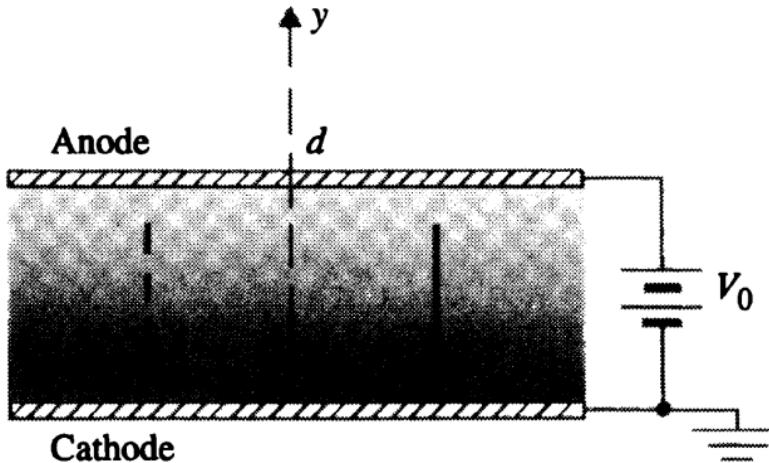
$$\mathbf{J} = -\mathbf{a}_y J = \mathbf{a}_y \rho(y) \mathbf{u}(y),$$

where the charge density $\rho(y)$ is a negative quantity. The velocity $\mathbf{u} = \mathbf{a}_y u(y)$ is related to the electric field intensity $\mathbf{E}(y) = \mathbf{a}_y E(y)$ by Newton's law of motion:

$$m \frac{du(y)}{dt} = -eE(y) = e \frac{dV(y)}{dy},$$

where $m = 9.11 \times 10^{-31}$ (kg) and $-e = -1.60 \times 10^{-19}$ (C)

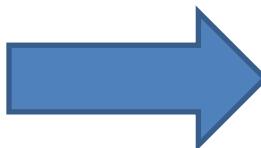
Example



$$m \frac{du}{dt} = m \frac{du}{dy} \frac{dy}{dt} = mu \frac{du}{dy}$$
$$= \frac{d}{dy} \left(\frac{1}{2} mu^2 \right),$$

$$\frac{1}{2} mu^2 = eV,$$

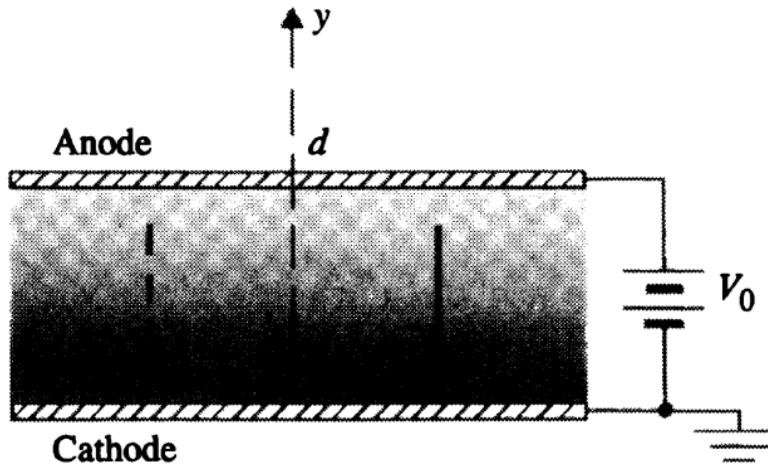
$$\frac{d}{dy} \left(\frac{1}{2} mu^2 \right) = e \frac{dV}{dy}.$$



$$u = \left(\frac{2e}{m} V \right)^{1/2}$$

where the constant of integration has been set to zero because at $y = 0, u(0) = V(0) = 0$.

Example



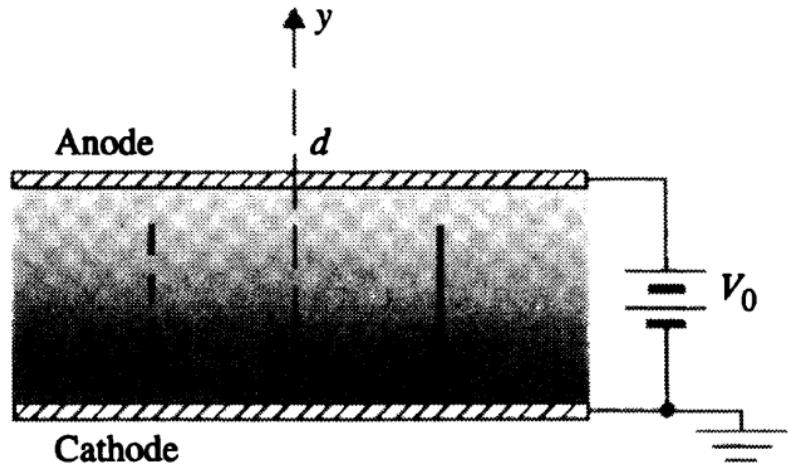
In order to find $V(y)$ in the interelectrode region we must solve Poisson's equation with ρ expressed in terms of $V(y)$

$$\rho = -\frac{J}{u} = -J \sqrt{\frac{m}{2e}} V^{-1/2}.$$

Poisson's equation

$$\frac{d^2V}{dy^2} = -\frac{\rho}{\epsilon_0} = \frac{J}{\epsilon_0} \sqrt{\frac{m}{2e}} V^{-1/2}$$

Example



Poisson's equation

$$\frac{d^2V}{dy^2} = -\frac{\rho}{\epsilon_0} = \frac{J}{\epsilon_0} \sqrt{\frac{m}{2e}} V^{-1/2}$$

$$\left(\frac{dV}{dy}\right)^2 = \frac{4J}{\epsilon_0} \sqrt{\frac{m}{2e}} V^{1/2} + c$$

At $y = 0$, $V = 0$, and $dV/dy = 0$

$$E(0) = \mathbf{a}_y E_y(0) = -\mathbf{a}_y \frac{dV(y)}{dy} \Big|_{y=0} = 0$$



$$c = 0$$

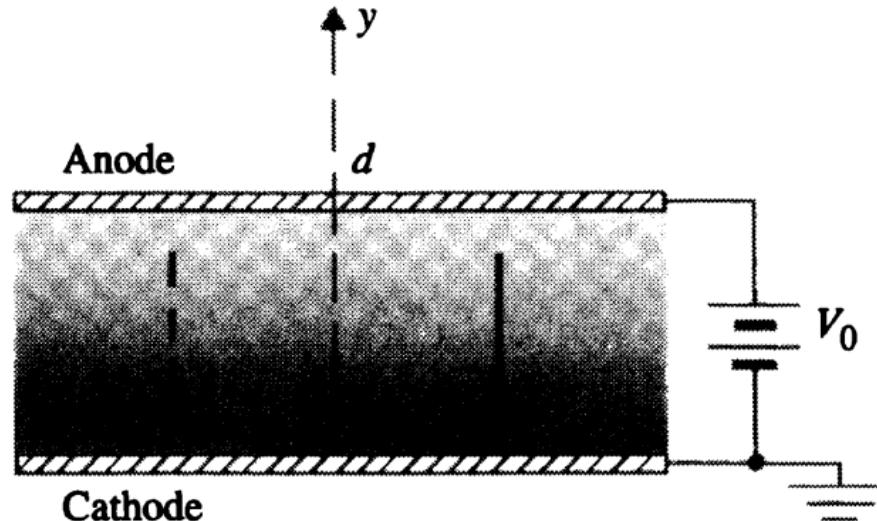
$$V^{-1/4} dV = 2 \sqrt{\frac{J}{\epsilon_0}} \left(\frac{m}{2e}\right)^{1/4} dy$$

Integrate V from $V = 0$ to $V = V_0$

Integrate y from $y = 0$ to $y = d$

$$\frac{4}{3} V_0^{3/4} = 2 \sqrt{\frac{J}{\epsilon_0}} \left(\frac{m}{2e}\right)^{1/4} d$$

Example



Space-charge-limited vacuum diode

$$J = \frac{4\epsilon_0}{9d^2} \sqrt{\frac{2e}{m}} V_0^{3/2} \quad (\text{A/m}^2)$$

Child-Langmuir Law

Conduction Currents

- For more than one kind of charge carriers (electrons, holes, and ions) drifting, current density:

$$\boxed{\mathbf{J} = Nq\mathbf{u} \quad (\text{A/m}^2)}, \quad \rightarrow \quad \mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \quad (\text{A/m}^2).$$

- For most conducting materials,

$$\mathbf{u} = -\mu_e \mathbf{E} \quad (\text{m/s}),$$

\mathbf{u} : averaged drift velocity

$-\mu_e$: electron mobility ($\text{m}^2/\text{V}\cdot\text{s}$)

Conduction Currents

- For most conducting materials,

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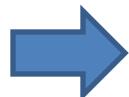
$-\mu_e$: electron mobility ($\text{m}^2/\text{V}\cdot\text{s}$)

copper $3.2 \times 10^{-3} (\text{m}^2/\text{V}\cdot\text{s})$

aluminum $1.4 \times 10^{-4} (\text{m}^2/\text{V}\cdot\text{s})$

silver $5.2 \times 10^{-3} (\text{m}^2/\text{V}\cdot\text{s})$

$$\left. \begin{array}{l} \mathbf{J} = Nq\mathbf{u} \quad (\text{A/m}^2), \\ \mathbf{u} = -\mu_e \mathbf{E} \quad (\text{m/s}), \end{array} \right\}$$



$$\mathbf{J} = -\rho_e \mu_e \mathbf{E},$$

where $\rho_e = -Ne$



$$\boxed{\mathbf{J} = \sigma \mathbf{E} \quad (\text{A/m}^2)},$$

Such materials are called ohmic media

where $\underline{\sigma} = -\rho_e \mu_e$ $\rho_e < 0$

σ : conductivity (A/V·m or S/m)

For conductors, $\sigma = -\rho_e \mu_e$

For semiconductors, $\sigma = -\rho_e \mu_e + \rho_h \mu_h$,

In general $\mu_e \neq \mu_h$

Conductivities

unit for σ is ampere per volt-meter ($A/V \cdot m$) or siemens per meter (S/m).



Silicon



Copper



Germanium

$5.80 \times 10^7 \text{ (S/m)}$

2.2 (S/m)

$1.6 \times 10^{-3} \text{ (S/m)}$

Votes	Right answer
Copper	
Germanium	
Silicon	

Conductivities



Silicon



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Germanium

5.80×10^7 (S/m)

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Votes	Right answer
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- Ohm's law $V_{12} = RI.$
- Point form of Ohm's law
 - Holds at all points
 - σ can be a function of space

$$\boxed{\mathbf{J} = \sigma \mathbf{E} \quad (\text{A/m}^2),}$$

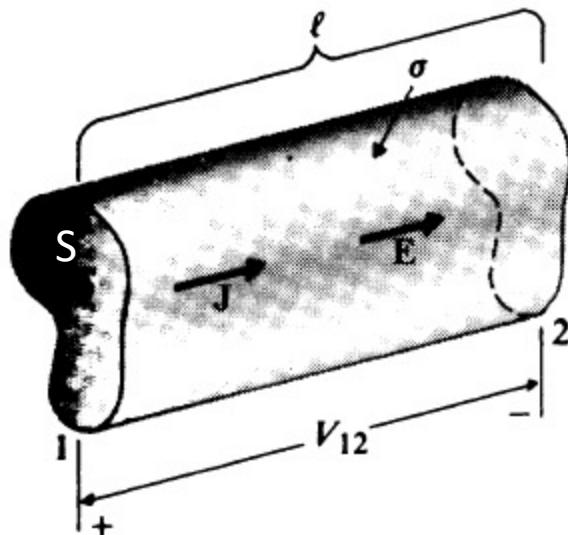
$$V_{12} = RI.$$



Integration?

$$\mathbf{E} = (1/\sigma)\mathbf{J}$$

Ohm's Law: Point Form to Circuit Form



$$V_{12} = E\ell \quad \text{or} \quad E = \frac{V_{12}}{\ell}.$$

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} = JS \quad \text{or} \quad J = \frac{I}{S}.$$

FIGURE 5–3
Homogeneous conductor with a constant cross section.

$$\mathbf{J} = \sigma \mathbf{E},$$



$$J = \frac{I}{S}. \quad E = \frac{V_{12}}{\ell}.$$

$$\frac{I}{S} = \sigma \frac{V_{12}}{\ell}$$



$$V_{12} = \left(\frac{\ell}{\sigma S} \right) I = RI,$$

The resistance

$$R = \frac{\ell}{\sigma S} \quad (\Omega).$$

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

Determine the d-c resistance of 1-(km) of wire having a 1-(mm) radius

(a) if the wire is made of copper, and (b) if the wire is made of aluminum

Quick classroom exercise

Break up into groups of 5 and spend 5-10 minutes working out:

a) For copper wire, $\sigma_{cu} = 5.80 \times 10^7$ (S/m):

$$\ell = 10^3 \text{ (m)}, \quad S = \pi(10^{-3})^2 = 10^{-6}\pi \text{ (m}^2\text{)}.$$

We have

$$R_{cu} = \frac{\ell}{\sigma_{cu}S} = \frac{10^3}{5.80 \times 10^7 \times 10^{-6}\pi} = 5.49 \text{ (\Omega)}.$$

b) For aluminum wire, $\sigma_{al} = 3.54 \times 10^7$ (S/m):

$$R_{al} = \frac{\ell}{\sigma_{al}S} = \frac{\sigma_{cu}}{\sigma_{al}} R_{cu} = \frac{5.80}{3.54} \times 5.49 = 8.99 \text{ (\Omega)}.$$

Conductance

The **conductance**, G , or the reciprocal of resistance, is useful in combining resistances in parallel. The unit for conductance is (Ω^{-1}), or siemens (S).

$$G = \frac{1}{R} = \sigma \frac{S}{\ell} \quad (\text{S}).$$

From circuit theory we know the following:

- a) When resistances R_1 and R_2 are connected in series (same current), the total resistance R is

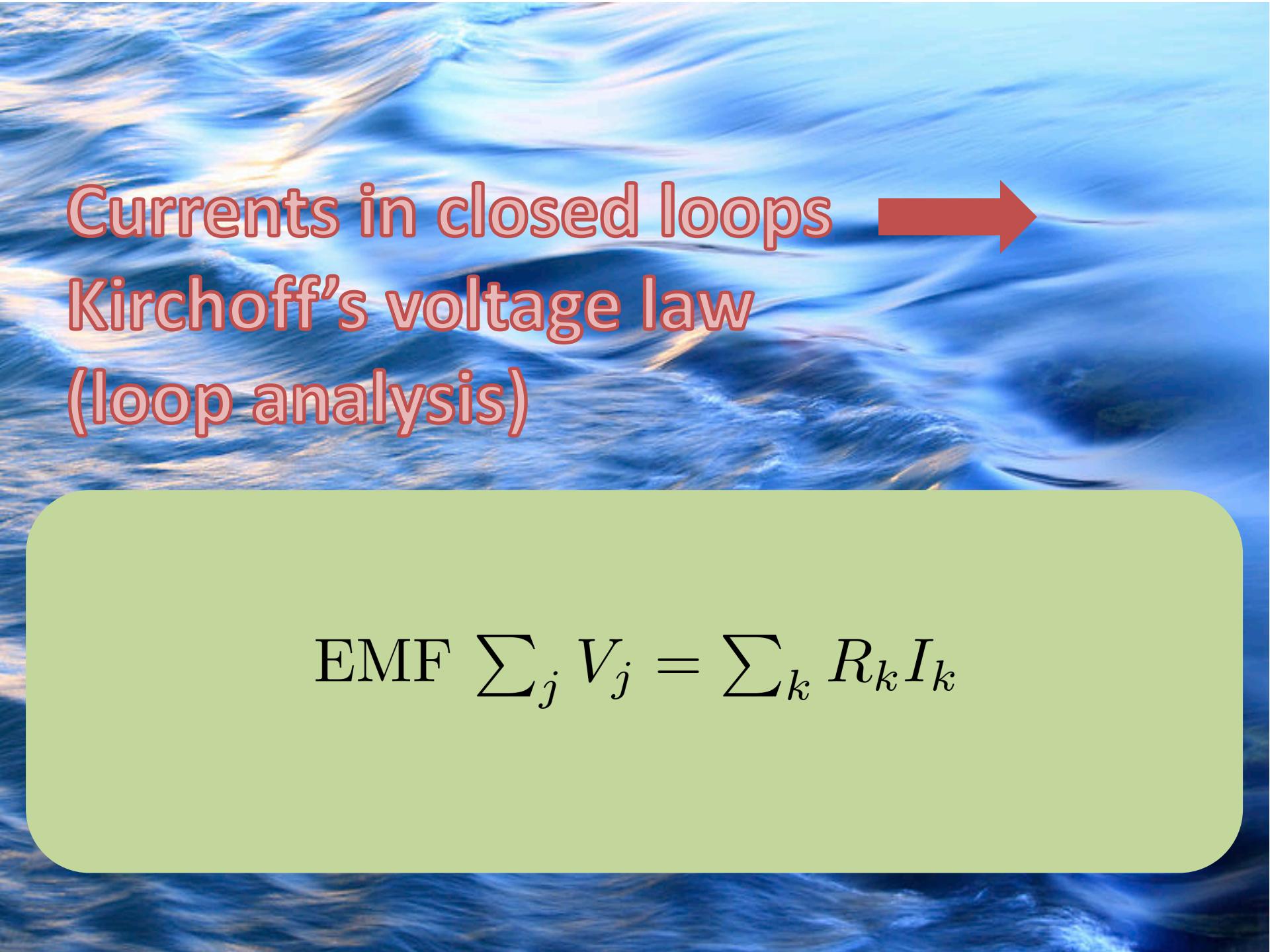
$$R_{sr} = R_1 + R_2.$$

- b) When resistances R_1 and R_2 are connected in parallel (same voltage), we have

$$\frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2}$$

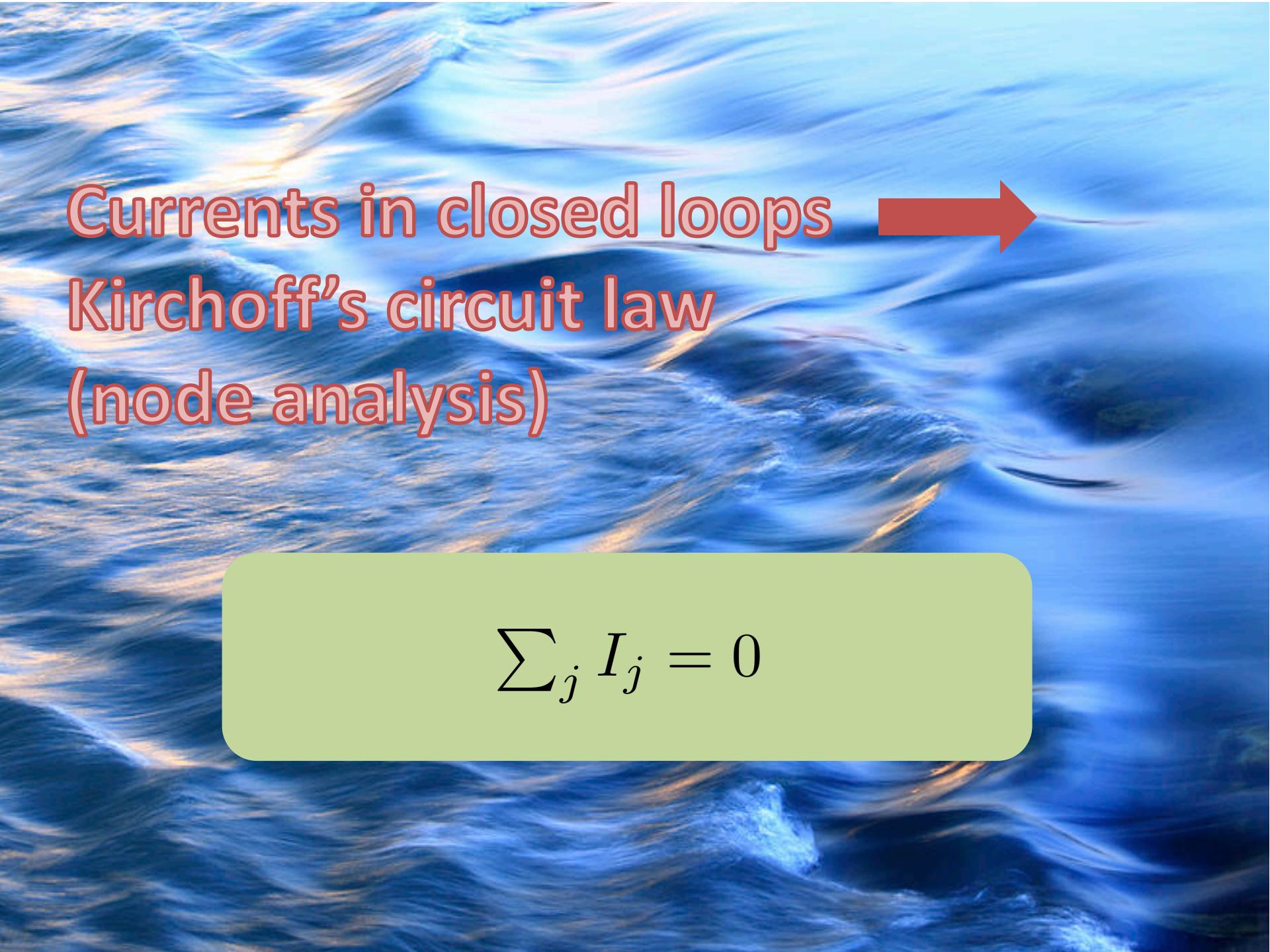
or

$$G_{||} = G_1 + G_2.$$



Currents in closed loops →
Kirchoff's voltage law
(loop analysis)

$$\text{EMF } \sum_j V_j = \sum_k R_k I_k$$

A photograph of ocean waves with sunlight reflecting off the water, creating bright highlights on the crests.

Currents in closed loops →
Kirchoff's circuit law
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$$\sum_j I_j = 0$$

Ohm's law versus Kirchoff's circuit law

Ohm's Law describes the relationship between voltage and current across a resistive element.

Kirchhoff's Law describes the behaviour of current and voltage respectively in a circuit branch.

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5-3 Electromotive Force and Kirchhoff's Voltage Law

$$\oint_c \mathbf{E} \cdot d\ell = 0.$$



For an ohmic material
 $\mathbf{J} = \sigma \mathbf{E}$,

$$\oint_c \frac{1}{\sigma} \mathbf{J} \cdot d\ell = 0.$$

A steady current cannot be maintained in the same direction in a closed circuit by **an electrostatic field (conservative field)**

That is, to maintain a steady current in a closed circuit, there must be **non-conservative field** (e.g., electric batteries, etc.), which termed as **impressed electric field intensity E_i**

Electromotive Force

- Chemical action (E_i) → cumulation of + and – charges on electrodes due to E_i →
 - Inside: E and E_i
 - $E = -E_i$ due to $I=0$ for open circuit
 - Outside: E only

E : electrostatic field
 E_i : nonconservative field

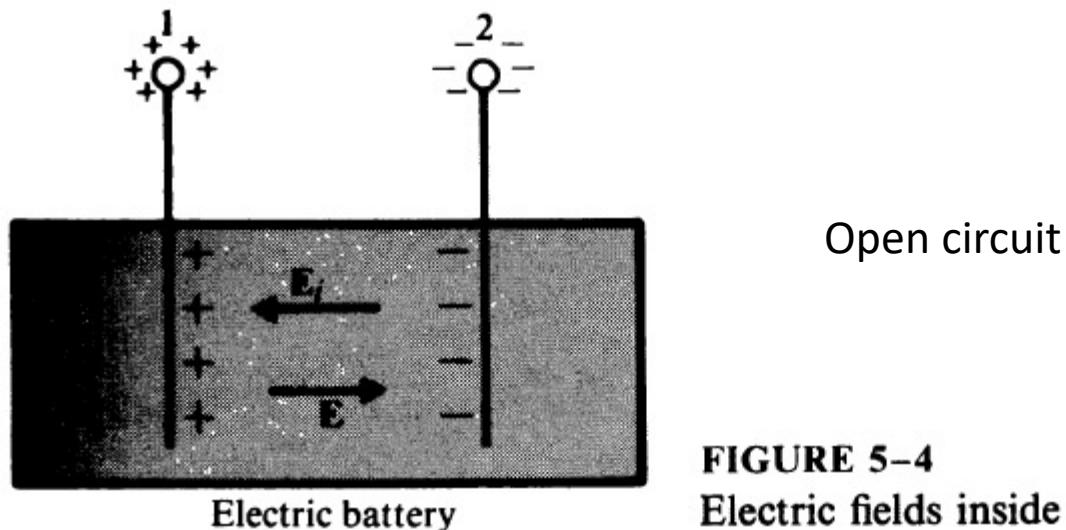


FIGURE 5-4
Electric fields inside an electric battery.

Electromotive Force

Against E_i from 1 to 2

• E_i

$$\gamma = \int_2^1 E_i \cdot d\ell = - \int_2^1 E \cdot d\ell.$$

Inside
the source

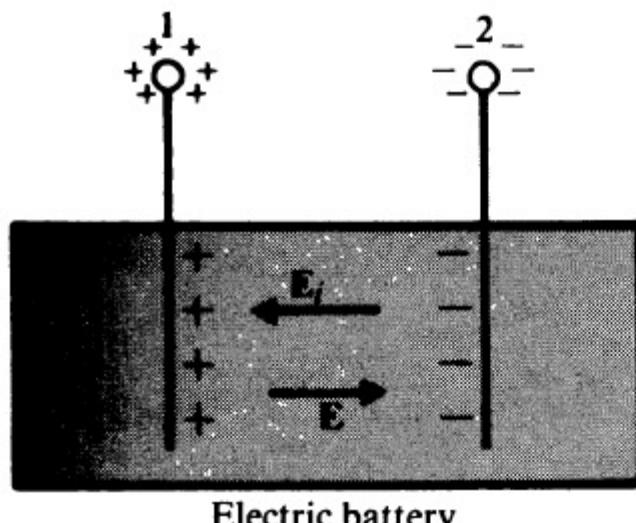
Open circuit: $E_i = -E$

The electromotive force is a measure of the strength of the nonconservative source, **not a force**

• E

$$\oint_C E \cdot d\ell = \int_{\text{Outside}}^2 E \cdot d\ell + \int_{\text{Inside}}^1 E \cdot d\ell = 0.$$

Because of conservative field



Open circuit

FIGURE 5–4

Electric fields inside an electric battery.

$$\mathcal{V} = \int_2^1 \mathbf{E}_i \cdot d\ell = - \int_2^1 \mathbf{E} \cdot d\ell.$$

Inside
the source

$$\oint_C \mathbf{E} \cdot d\ell = \int_1^2 \mathbf{E} \cdot d\ell + \int_2^1 \mathbf{E} \cdot d\ell = 0.$$

Outside
the source Inside
the source

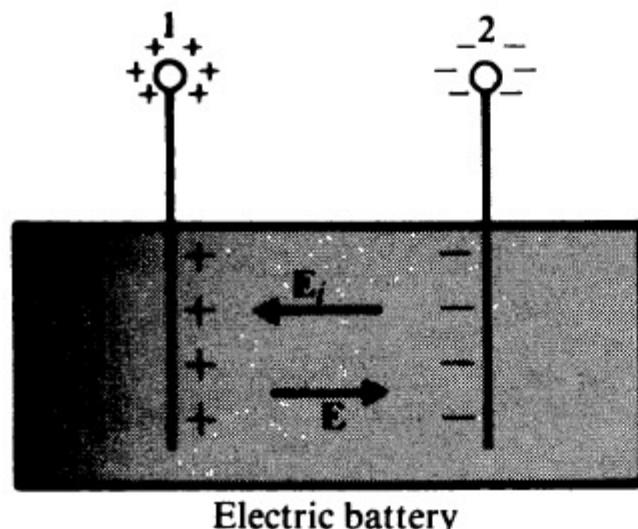
$$\mathcal{V} = \int_1^2 \mathbf{E} \cdot d\ell$$

Outside
the source

or

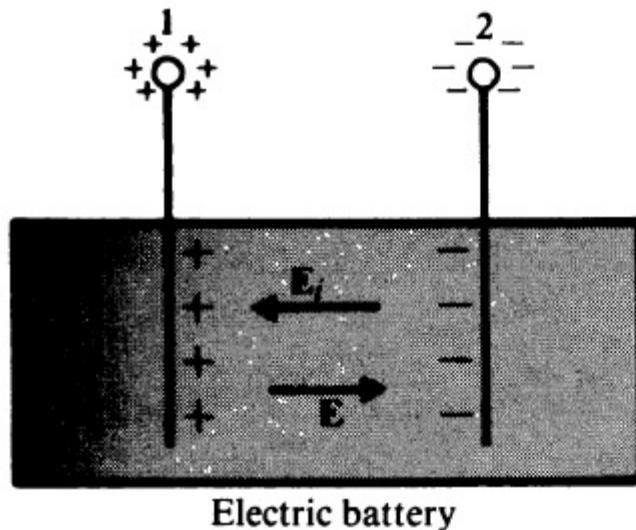
$$\mathcal{V} = V_{12} = V_1 - V_2.$$

emf = voltage rise between + and – terminals



Open circuit

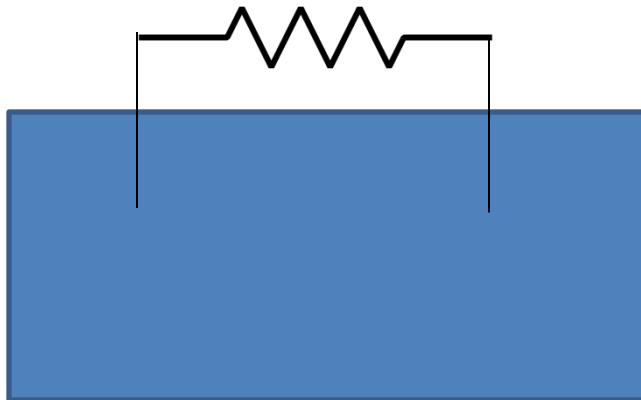
FIGURE 5–4
Electric fields inside an electric battery.



Open circuit → No currents

FIGURE 5–4
Electric fields inside an electric battery.

If connected with a resistor
→ Currents



Point form of Ohm's law

$$\mathbf{J} = \sigma(\mathbf{E} + \mathbf{E}_i),$$



$$\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}.$$

$$\mathcal{V} = \oint_C (\mathbf{E} + \mathbf{E}_i) \cdot d\ell = \oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\ell.$$

Integration due to 1st integrand = 0

Integration due to \mathbf{E}_i outside the source = 0

$$\oint_C \mathbf{E} \cdot d\ell = \int_1^2 \mathbf{E} \cdot d\ell + \int_2^1 \mathbf{E} \cdot d\ell = 0.$$

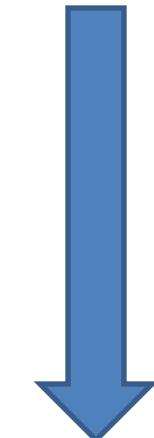
Outside the source Inside the source

$$\mathcal{V} = \int_2^1 \mathbf{E}_i \cdot d\ell = - \int_2^1 \mathbf{E} \cdot d\ell.$$

Inside the source

$$\int_1^2 \mathbf{E}_i \cdot dl = 0$$

$$\int_1^2 \mathbf{E} \cdot dl + \int_2^1 \mathbf{E} \cdot dl = 0$$



$$\mathbf{E} + \mathbf{E}_i = \frac{\mathbf{J}}{\sigma}.$$

For a resistor with uniform cross section:

$$J = I/S$$

$$R = \frac{\ell}{\sigma S}$$

$$\mathcal{V} = RI.$$

$$\int_2^1 \mathbf{E}_i \cdot dl + 0 + 0 = \oint_c (\mathbf{E} + \mathbf{E}_i) \cdot dl$$

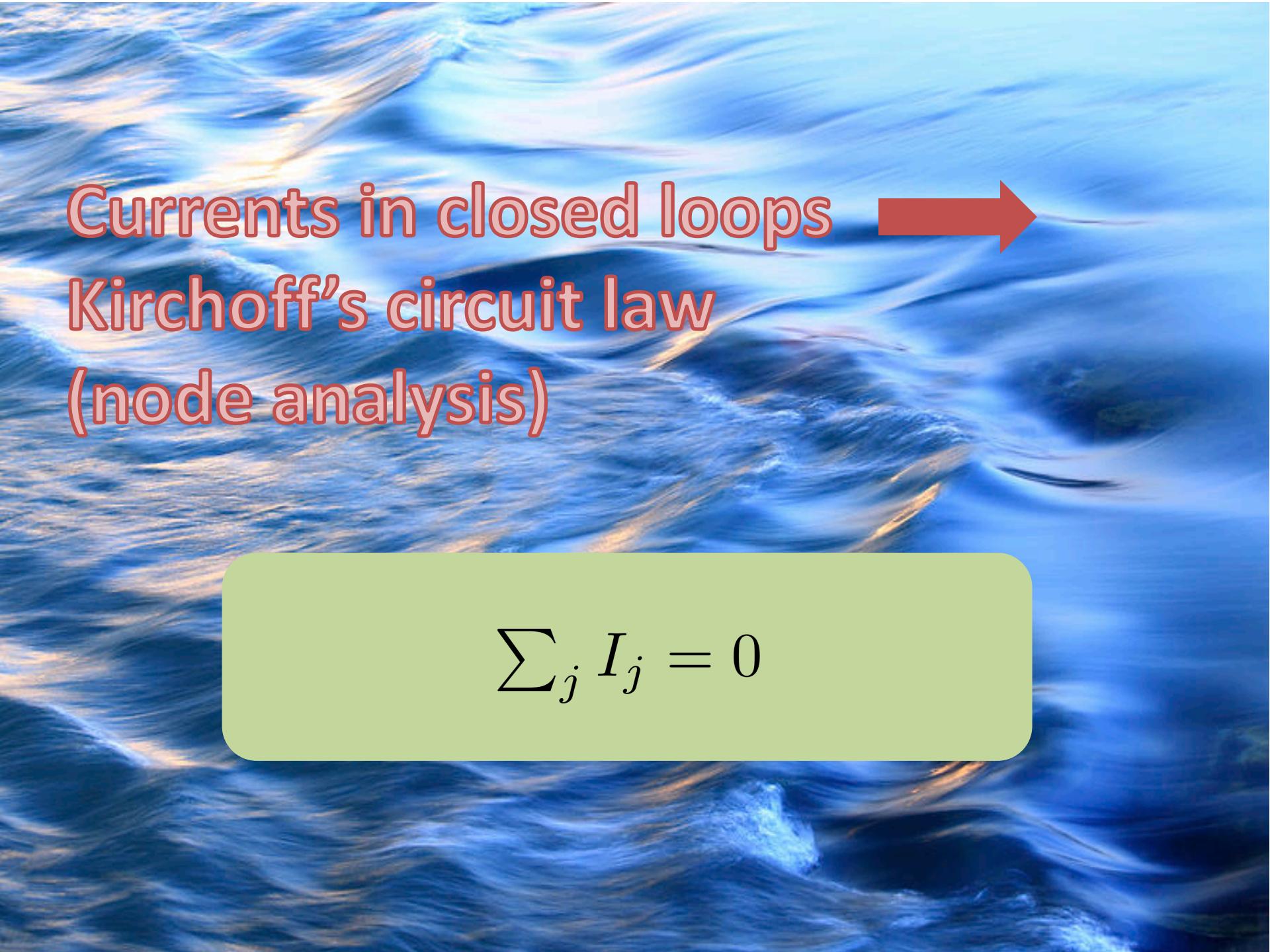
$$\mathcal{V} = RI.$$

For more-than-one emf and more-than-one resistor connected in series



$$\sum_j \mathcal{V}_j = \sum_k R_k I_k \quad (\text{V}).$$

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistances

A photograph of ocean waves with sunlight reflecting off the water, creating bright highlights on the crests.

Currents in closed loops →
Kirchoff's circuit law
(node analysis)

$$\sum_j I_j = 0$$

5-4 Equation of Continuity and Kirchhoff's Current Law

- Principle of conservation of charge: electric charges may not be created or destroyed



$$\underline{I = \oint_s \mathbf{J} \cdot d\mathbf{s}} = \underline{-\frac{dQ}{dt}} = -\frac{d}{dt} \int_V \rho dv.$$

Current leaving a volume

Rate of charge decrease in the volume



$$\int_V \nabla \cdot \mathbf{J} dv = - \int_V \frac{\partial \rho}{\partial t} dv.$$

$$\int_V \nabla \cdot \mathbf{J} dv = - \int_V \frac{\partial \rho}{\partial t} dv.$$

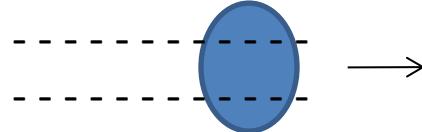


Holds for arbitrary volume V

Equation of continuity

$$\nabla \cdot \mathbf{J} = - \frac{\partial \rho}{\partial t} \quad (\text{A/m}^3).$$

- For **steady current ($I=\text{constant}$)**, charge density does not vary with time (or charge in a volume is a constant over time although charge is moving): $\partial\rho/\partial t = 0$.



$\nabla \cdot \mathbf{J} = 0$. (divergenceless: streamlines of steady currents close upon themselves)

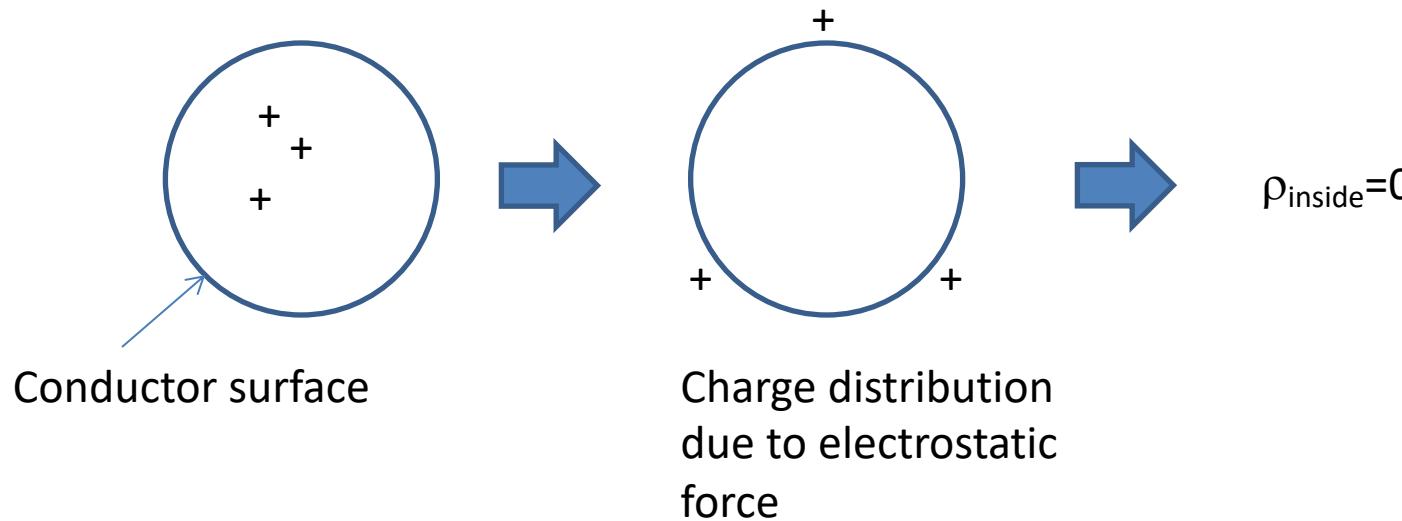
$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0, \quad (\text{integral form})$$

↓ closed $S \rightarrow 0$;
That is, volume (enclosed by S) $\rightarrow 0$

$$\sum_j I_j = 0 \quad (\text{A}).$$

Kirchhoff's current law: the algebraic sum of all the currents flowing **out of a junction** (a small volume) in an electric circuit is zero.

\mathbf{E} and ρ inside a Conductor: time to equilibrium



Inside a Conductor (Under Static Conditions)
$\rho = 0$
$\mathbf{E} = 0$

By Gauss's law

Time to Reach Equilibrium in a Conductor

- Inside a conductor, $\rho=0$, $\mathbf{E}=0$ under equilibrium conditions
- Time to reach equilibrium?

Equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \quad (\text{A/m}^3).$$



Assume a constant σ

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}.$$

$$\sigma \nabla \cdot \mathbf{E} = -\frac{\partial \rho}{\partial t}.$$



In a simple medium
 $\nabla \cdot \mathbf{E} = \rho/\epsilon$

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\epsilon} \rho = 0.$$

Solution:

$$\rho = \rho_0 e^{-(\sigma/\epsilon)t} \quad (\text{C/m}^3),$$

Charge density inside a conductor will decrease with time exponentially.

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \quad (\text{s}).$$

Relaxation time for copper

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \quad (\text{s}).$$



Option	Speed in one direction	Number of votes
A	$1.5 \times 10^{-3} \text{ s}$	
B	$1.5 \times 10^{-8} \text{ s}$	
C	$1.5 \times 10^{-19} \text{ s}$	

Relaxation time for copper

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \quad (\text{s}).$$



Option	Speed in one direction	Number of votes
A	$1.5 \times 10^{-3} \text{ s}$	
	1/2 time for a housefly's wing flap	
	Sound to travel 30 cm	
 A close-up photograph of a housefly resting on a bright green leaf. The fly has a striped body and transparent wings with prominent veins.	Typical response time in high-end LCD computer monitors	

Relaxation time for copper



Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \quad (\text{s}).$$

Option	Speed in one direction	Number of votes
	Half-life of Lithium-12	A photograph of a nuclear explosion, showing a bright fireball at the center with a distinct mushroom cloud rising into the sky.
B	$1.5 \times 10^{-8} \text{ s}$	Half the time in fusion reaction in hydrogen bomb Light to travel 15 m in vacuum

Relaxation time for copper



Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \quad (\text{s}).$$

Option	Speed in one direction	Number of votes
	Time it takes light to travel diameter of hydrogen atom	
	More than 200 times shorter than the shortest pulse of laser light ever created	
C	10 attoseconds to 1 second is 1 second to 32 billion years	

Relaxation time for copper



Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \quad (\text{s}).$$

Option	Speed in one direction	Number of votes
A	$1.5 \times 10^{-3} \text{ s}$	
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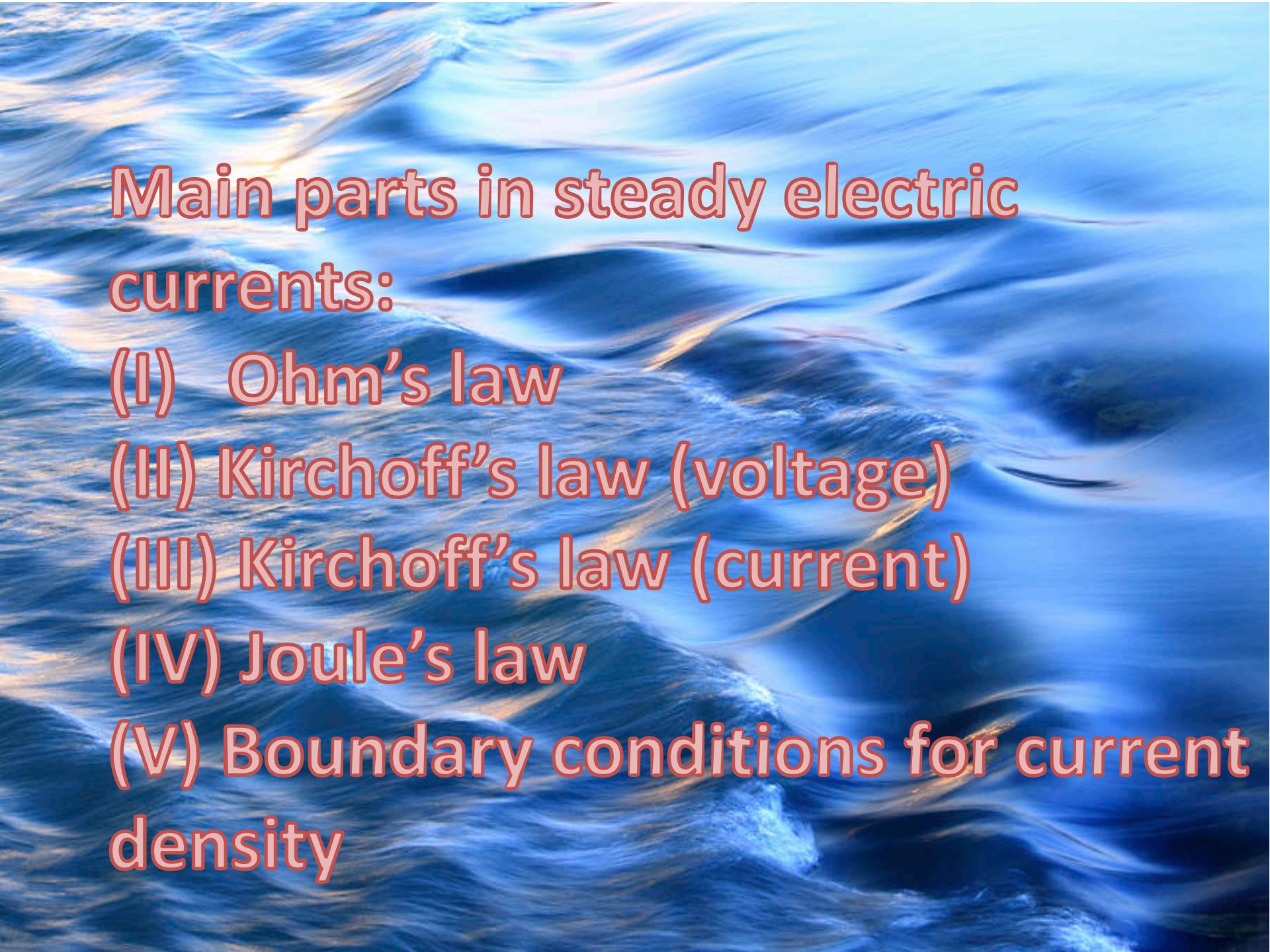
Relaxation times

Relaxation time: time for ρ_0 to decay to $1/e$

$$\tau = \frac{\epsilon}{\sigma} \quad (\text{s}).$$

CHARGE RELAXATION TIMES OF TYPICAL MATERIALS

Conductivity	ϵ/ϵ_0	Relaxation time s
Water, distilled	2×10^{-4}	3.6×10^{-6}
Corn oil	5×10^{-11}	0.55
Mica	$10^{-11} - 10^{-15}$	$5.1 - 5.1 \times 10^4$



Main parts in steady electric currents:

- (I) Ohm's law
- (II) Kirchoff's law (voltage)
- (III) Kirchoff's law (current)
- (IV) Joule's law
- (V) Boundary conditions for current density

5-5 Power Dissipation and Joule's Law

- Power dissipation:

External $\mathbf{E} \rightarrow$ drift motion of electrons, which collide with atoms on lattice sites \rightarrow thermal energy

- Power by \mathbf{E} to move a charge q

$$p = \lim_{\Delta t \rightarrow 0} \frac{\Delta w}{\Delta t} = \underline{q \mathbf{E} \cdot \mathbf{u}}, \quad \mathbf{u}: \text{drift velocity}$$

Total power in a volume dv $dP = \sum_i p_i = \mathbf{E} \cdot \left(\sum_i \underline{N_i q_i \mathbf{u}_i} \right) dv,$

Total Q in a volume dv

$$dP = \sum_i p_i = \mathbf{E} \cdot \left(\sum_i N_i q_i \mathbf{u}_i \right) dv,$$



$$\mathbf{J} = \sum_i N_i q_i \mathbf{u}_i \quad (\text{A/m}^2).$$

$$dP = \mathbf{E} \cdot \mathbf{J} dv$$



$$\frac{dP}{dv} = \mathbf{E} \cdot \mathbf{J} \quad (\text{W/m}^3).$$

Power density



Joule's law

$$P = \int_V \mathbf{E} \cdot \mathbf{J} dv \quad (\text{W}).$$

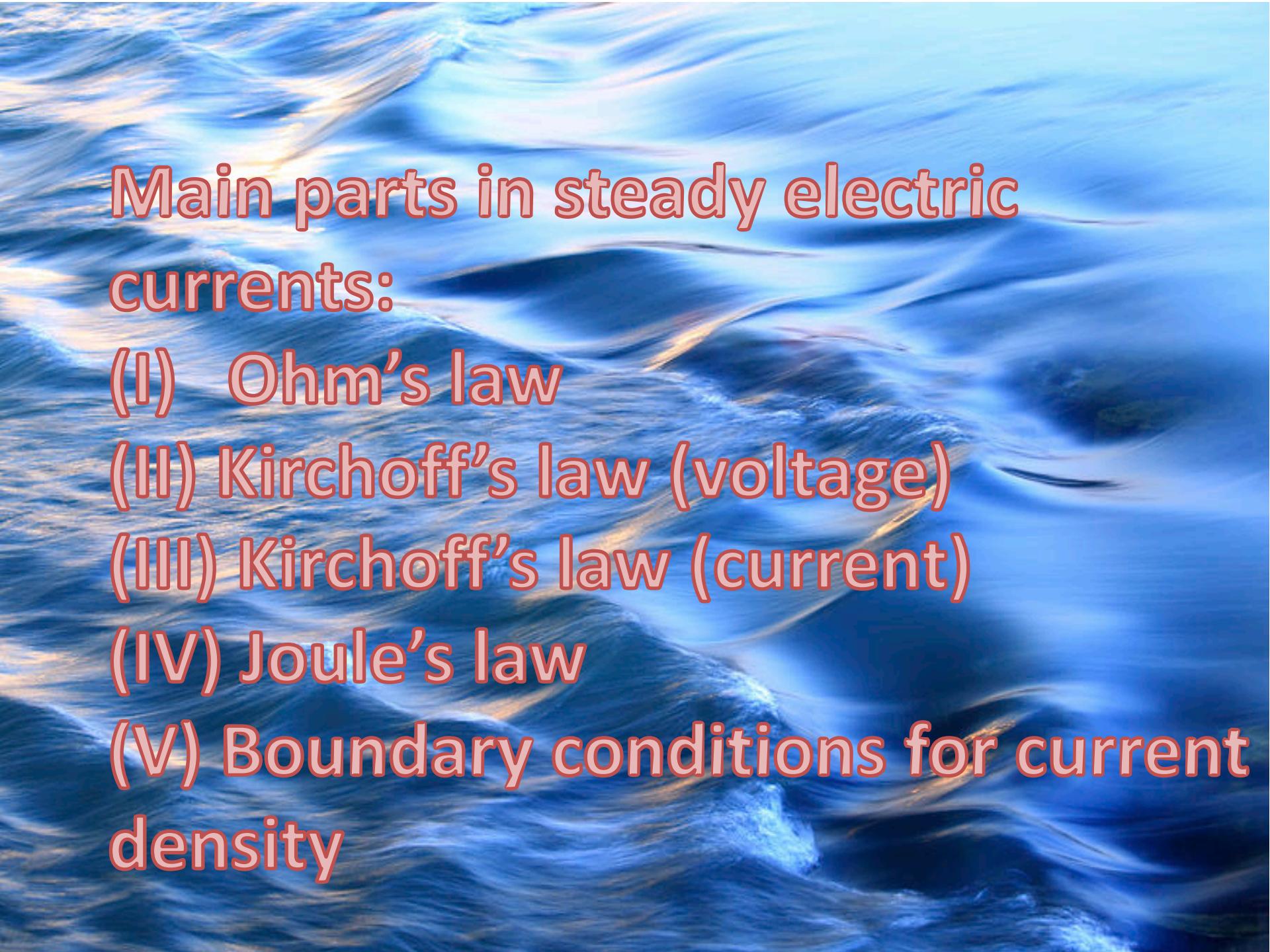
$$P = \int_V \mathbf{E} \cdot \mathbf{J} dv \quad (\text{W}).$$



In a conductor with constant cross section
 $dv = ds d\ell$

$$P = \int_L E d\ell \int_S J ds = VI,$$

We get the familiar expression $P = I^2 R$



Main parts in steady electric currents:

- (I) Ohm's law
- (II) Kirchoff's law (voltage)
- (III) Kirchoff's law (current)
- (IV) Joule's law
- (V) Boundary conditions for current density

5-6 Boundary Conditions for Current Density

- Steady current density \mathbf{J} on boundaries without nonconservative energy source

Governing Equations for Steady Current Density	
Differential Form	Integral Form
$\nabla \cdot \mathbf{J} = 0$	$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$
$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0$	$\oint_C \frac{1}{\sigma} \mathbf{J} \cdot d\ell = 0$

The **normal** component of a **divergenceless** vector field is continuous

$$\nabla \cdot \mathbf{D} = \rho$$

$$D_{1n} - D_{2n} = \rho_s \quad (\text{C/m}^2),$$

$$\nabla \cdot \mathbf{J} = 0$$

$$J_{1n} = J_{2n} \quad (\text{A/m}^2).$$

$$\nabla \times \mathbf{E} = 0$$

$$E_{1t} = E_{2t} \quad (\text{V/m}),$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0$$

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.$$

The ratio of J_t at two sides of an interface is equal to **the ratio of the conductivities**

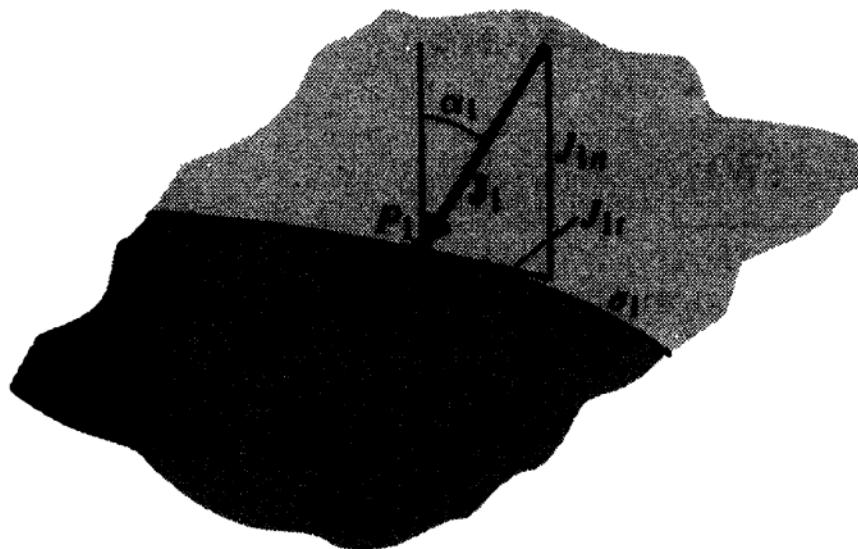
Example

Two conducting media with conductivities σ_1 and σ_2 are separated by an interface.

The steady current density in medium 1 at point

P_1 has a magnitude J_1 and makes an angle α_1 with the normal.

Determine the magnitude and direction of the current density at point P_2 in medium 2.

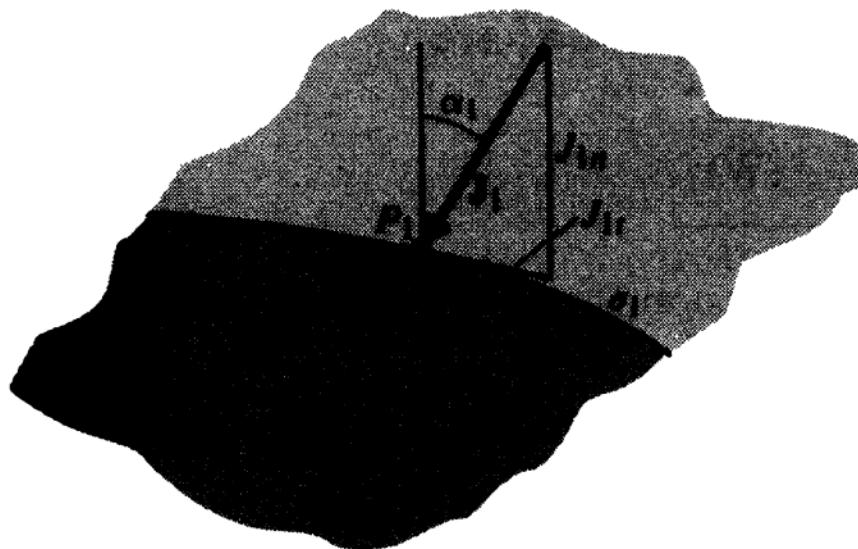


Example

Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. Determine the magnitude and direction of the current density at point P_2 in medium 2.

$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2$$

$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2.$$



Example

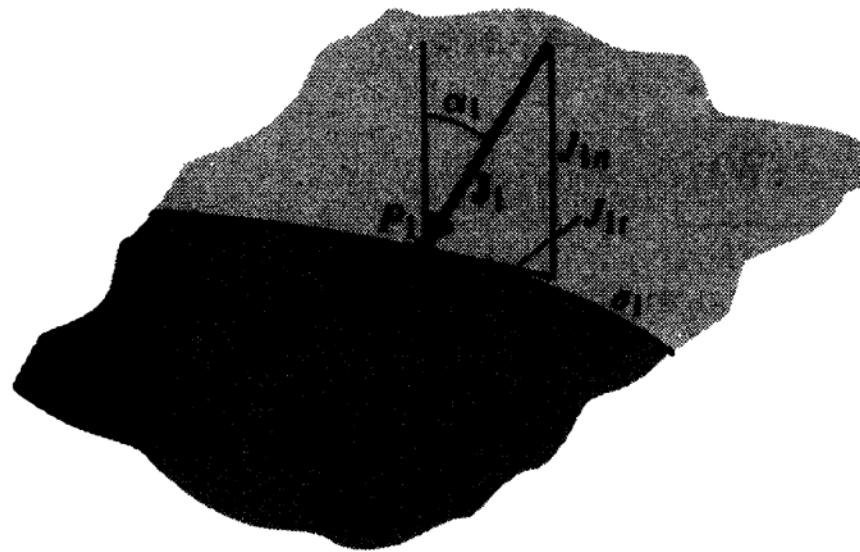
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$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2$$

$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2.$$



$$\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\sigma_2}{\sigma_1}.$$



Example

Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. Determine the magnitude and direction of the current density at point P_2 in medium 2.

$$J_1 \cos \alpha_1 = J_2 \cos \alpha_2$$



$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2.$$

$$\frac{\tan \alpha_2}{\tan \alpha_1} = \frac{\sigma_2}{\sigma_1}.$$

If medium 1 is a much better conductor than medium 2 ($\sigma_1 \gg \sigma_2$ or $\sigma_2/\sigma_1 \rightarrow 0$), α_2 approaches zero, and \mathbf{J}_2 emerges almost perpendicularly to the interface (normal to the surface of the good conductor). The magnitude of \mathbf{J}_2 is

$$\begin{aligned} J_2 &= \sqrt{J_{2t}^2 + J_{2n}^2} = \sqrt{(J_2 \sin \alpha_2)^2 + (J_2 \cos \alpha_2)^2} \\ &= \left[\left(\frac{\sigma_2}{\sigma_1} J_1 \sin \alpha_1 \right)^2 + (J_1 \cos \alpha_1)^2 \right]^{1/2} \end{aligned}$$

Example

Two conducting media with conductivities σ_1 and σ_2 are separated by an interface. Determine the magnitude and direction of the current density at point P_2 in medium 2.

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$$\sigma_2 J_1 \sin \alpha_1 = \sigma_1 J_2 \sin \alpha_2.$$

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$$J_2 = J_1 \left[\left(\frac{\sigma_2}{\sigma_1} \sin \alpha_1 \right)^2 + \cos^2 \alpha_1 \right]^{1/2}.$$

The **normal** component of a **divergenceless** vector field is continuous

$$\nabla \cdot \mathbf{D} = \rho$$

$$D_{1n} - D_{2n} = \rho_s \quad (\text{C/m}^2),$$

$$\nabla \cdot \mathbf{J} = 0$$

$$J_{1n} = J_{2n} \quad (\text{A/m}^2).$$

$$\nabla \times \mathbf{E} = 0$$

$$E_{1t} = E_{2t} \quad (\text{V/m}),$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0$$

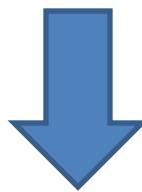
$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}.$$

The ratio of J_t at two sides of an interface is equal to **the ratio of the conductivities**

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0$$

$$\nabla \times (\mathbf{D}/\epsilon) = 0$$



Analogy

Interface of dielectric media

Interface of conducting media

\mathbf{D}

\mathbf{J}

ϵ

σ

A homogeneous conducting medium

Electrostatics analogy

$$\nabla \times \left(\frac{\mathbf{J}}{\sigma} \right) = 0$$



If σ is a constant
(homogeneous)

$$\nabla \times \mathbf{J} = 0.$$



By null identity

$$\mathbf{J} = -\nabla\psi.$$



Laplace's eq.:

$$\nabla^2\psi = 0.$$

$$\nabla \times \mathbf{E} = 0$$

$$\mathbf{E} = -\nabla V$$

$$\nabla^2 V = 0$$

$$\mathbf{J} = \sigma \mathbf{E}$$

$$\psi = \sigma V.$$

Boundary Condition between Two Lossy Dielectrics (for a Steady Current)

- Two **lossy** dielectrics: ϵ_1 and ϵ_2 σ_1 and σ_2

ϵ_1 and ϵ_2	σ_1 and σ_2
$E_{2t} = E_{1t}$	$J_{1t}/\sigma_1 = J_{2t}/\sigma_2$
$D_{1n} - D_{2n} = \rho_s \rightarrow \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s,$	$J_{1n} = J_{2n} \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n}$
	

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2 \right) E_{2n} = \left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2} \right) E_{1n}.$$

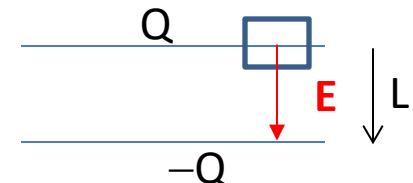
In most cases, a surface charge exists at the interface unless $\sigma_2/\sigma_1 = \epsilon_2/\epsilon_1$

What if medium 2 is a perfect conductor?

5-7 Resistance Calculations

- Capacitance between two conductors:

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\ell} = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\ell},$$

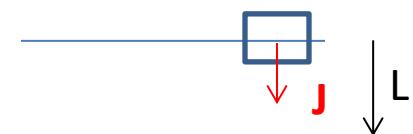


Numerator: surface integral over a surface enclosing the positive conductor

- Resistance between two conductors (medium between is lossy):

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\ell}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_L \mathbf{E} \cdot d\ell}{\oint_S \sigma \mathbf{E} \cdot d\mathbf{s}},$$

\uparrow
 $\mathbf{J} = \sigma \mathbf{E}$,



Denominator: the same surface as in the numerator of the above equation

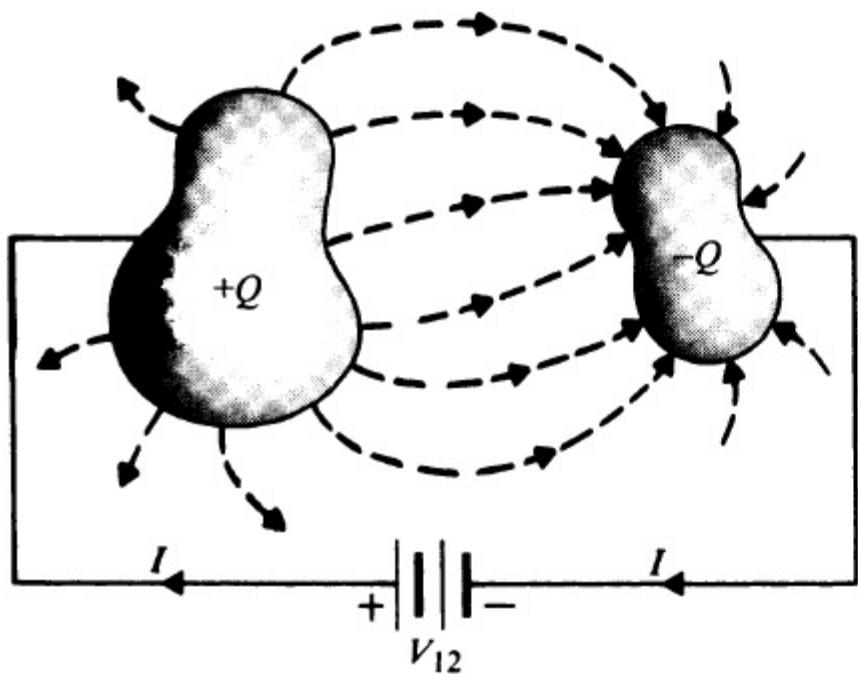


FIGURE 5-7
Two conductors in a lossy dielectric medium.

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\ell} = \frac{\oint_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_L \mathbf{E} \cdot d\ell},$$

\times

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E} \cdot d\ell}{\oint_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_L \mathbf{E} \cdot d\ell}{\oint_S \sigma \mathbf{E} \cdot d\mathbf{s}},$$



If ϵ and σ of the medium have the same space dependence or if the medium is homogeneous

$$RC = \frac{C}{G} = \frac{\epsilon}{\sigma}.$$

C_ℓ and R_ℓ

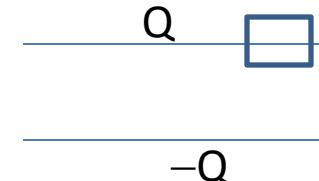
- C_ℓ : capacitance per unit length

$$C_\ell = C/\ell \text{ (F/m)}$$

(ℓ longer \rightarrow area S larger \rightarrow C larger)

$$C = \epsilon S/d$$

ℓ : in/out plane direction



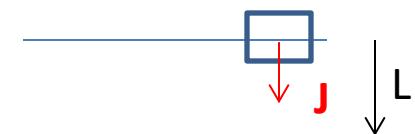
- R_ℓ : Resistance per unit length

$$R_\ell = R\ell \text{ (\Omega·m)}$$

(ℓ longer \rightarrow area S larger \rightarrow R smaller)

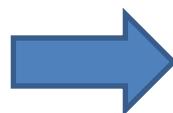
$$R = \rho L/S$$

ℓ : in/out plane direction



Note that ℓ and L are different!

$$RC = \frac{C}{G} = \frac{\epsilon}{\sigma}$$



$$R_\ell C_\ell = \epsilon / \sigma$$

Procedure to Compute R between Specified Equi-potential Surfaces

- Procedure 1: **V₀ to I**
 - 1. Choose a coordinate
 - 2. Assume potential difference V₀ between conductors
 - 3. Find E between conductors
 - $\nabla^2 V = 0 \rightarrow E = -\nabla V$
 - 4. Find current $I = \int_S \mathbf{J} \cdot d\mathbf{s} = \int_S \sigma E \cdot d\mathbf{s}$,
 - 5. Find R=V₀/I
- Procedure 2: **I to V₀**
 - Assume I → J → E → V₀ if J can be determined easily from I
 - R=V₀/I