Vv156 Lecture 9

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- So far we have been concerned with differentiating functions given by

$$y = f(x)$$

Definition

A function in which the dependent variable is written explicitly in terms of the independent variable is called an explicit function. We say

$$y \text{ is explicitly defined by } \quad y = f(x)$$

- Functions can be defined by equations in which y is not alone on one side,

$$\text{e.g.} \qquad xy + y + 1 = x \tag{1}$$

is not of the form $y = f(x)$, but equation (1) defines y as a function of x,

$$xy + y + 1 = x \implies y(x+1) = x - 1 \implies y = \frac{x-1}{x+1}$$

- Here we say y is implicitly defined as a function of x by equation (1).

Definition

An implicit equation is a relation between variables, which cannot, in general, be isolated on their own, or solved in terms of other variables. An implicit function is a function that is defined implicitly by an implicit equation.

- An implicit equation can implicitly define more than one function of x, e.g.

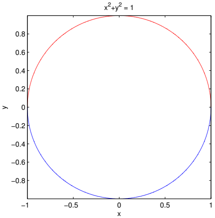
$$x^2 + y^2 = 1$$

Matlab

```
>> syms x y
>> obj = ezplot('x^2+y^2=1', [-1,1,0,1]);
>> set(obj, 'color','red'); clear obj
>> hold on
>> obj = ezplot('x^2+y^2=1', [-1,1,-1,0]);
>> set(obj, 'color','blue'); clear obj
>> hold off
>> axis([-1,1,-1,1])
>> axis equal tight
```

$$y = \sqrt{1-x^2}$$

$$y = -\sqrt{1-x^2}$$



- So here we have two functions implicitly defined by the equation

- In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly. To illustrate this, consider

$$xy = 1$$

- One way to find $\frac{dy}{dx}$ is to rewrite this equation as

$$xy = 1 \implies y = \frac{1}{x} \implies \frac{dy}{dx} = \frac{-1}{x^2}$$

- Another way is to differentiate both sides of the original equation

$$xy = 1 \implies \frac{d}{dx}(xy) = \frac{d}{dx}(1) \implies x\frac{dy}{dx} + y\frac{d}{dx}(x) = 0$$

$$\implies x\frac{dy}{dx} + y \cdot (1) = 0 \implies \frac{dy}{dx} = -\frac{y}{x}$$

- Then solve for y in terms of x, and make a substitution

$$\frac{dy}{dx} = -\frac{1/x}{x} = \frac{-1}{x^2}$$

- This is known as the implicit differentiation.

Exercise

(a) Use implicit differentiation to find y' for y defined by $5y^2 + \sin y = x^2$.

(b) Find an equation of the tangent line to the circle at the point $(3, 4)$.

$$x^2 + y^2 = 25$$

(c) Show that if a normal line to each point on an ellipse passes through the centre of an ellipse, then the ellipse is circle.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

- When differentiating implicitly, we assume that y represents a differentiable function of x. If it is not so, then the resulting calculations may be nonsense.

(d) Use implicit differentiation to find y' if

$$x^2 + y^2 + 1 = 0$$

- In order to discuss the derivative of logarithmic, exponential, and inverse of a differentiable function in general, we need the next two results.

Theorem

The following limit exists

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

and can be used to define Euler's number

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = 2.718281828459045235360287 4\ldots$$

and the following two limits are equal

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$$

Proof

- We have shown the following limit exists as an exercise in L2P18

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

  and it is actually one of several definitions of e.

- For any $x \in \mathbb{R}_+$, we can find $n \in \mathbb{N}$ such that $n \le x < n+1$, and

$$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{n}\right)^{n+1}$$

- Since $\displaystyle\lim_{n\to\infty}\left(1+\frac{1}{1+n}\right)^n = \lim_{n\to\infty}\frac{(1+\frac{1}{n+1})^{n+1}}{1+\frac{1}{n+1}} = e,$ there must exist an $N_l \in \mathbb{N}$ such that $e-\epsilon < \left(1+\frac{1}{1+n}\right)^n$ for $n > N_l$ for a given ϵ.

Proof

- Since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^{n}\cdot\left(1+\frac{1}{n}\right)\right] = e,$ there must exist an $N_u \in \mathbb{N}$ such that $\left(1+\frac{1}{n}\right)^{n+1} < e+\epsilon$ for $n > N_u$ for a given ϵ.
- Thus if choose $x > \max(N_l, N_u)$, then

$$e-\epsilon < \left(1+\frac{1}{n+1}\right)^{n} < \left(1+\frac{1}{x}\right)^{x} < \left(1+\frac{1}{n}\right)^{n+1} < e+\epsilon$$

$$\Longrightarrow \left|\left(1+\frac{1}{x}\right)^{x} - e\right| < \epsilon \Longrightarrow \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = e$$

- Similarly, $\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\left(\frac{n}{n-1}\right)^{n}$

$$\Longrightarrow \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^{x} = \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)\left(1+\frac{1}{n-1}\right)^{n-1} = e$$

Theorem

If $f(x)$ is continuous at $x = b$ and $\lim_{x \to a} g(x) = b$, then

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(b)$$

Proof

- For $\epsilon > 0$, we need to show that there is a $\delta > 0$ such that

$$|f(g(x)) - f(b)| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

- Since $f(x)$ is continuous at $x = b$, there must be a $\delta_f > 0$ so that

$$|f(x) - f(b)| < \epsilon \qquad \text{if} \qquad 0 < |x - b| < \delta_f$$

  and since $g(x) \to b$ as $x \to a$, there must be a $\delta > 0$ so that

$$|g(x) - b| < \delta_f \qquad \text{if} \qquad 0 < |x - a| < \delta$$

- So using this δ ensures $|f(g(x)) - f(b)| < \epsilon$ through the existence of δ_f. $\square$

Theorem

The natural logarithmic function $f(x) = \ln x$ is differentiable, and moreover

$$f'(x) = \frac{1}{x}, \qquad \text{for} \qquad x > 0.$$

Proof

- By definition, we have

$$\frac{d}{dx}(\ln x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln(x)}{h} = \lim_{h\to 0} \frac{1}{h} \ln\left(1 + \frac{h}{x}\right)$$

- Manipulating further, we have

$$\frac{d}{dx}(\ln x) = \frac{1}{x} \lim_{h\to 0} \frac{x}{h} \ln\left(1 + \frac{1}{x/h}\right) = \frac{1}{x} \lim_{h\to 0} \ln\left(1 + \frac{1}{x/h}\right)^{x/h}$$

- Notice $\dfrac{x}{h} \to \infty$ as $h \to 0^+$ and $\dfrac{x}{h} \to -\infty$ as $h \to 0^-$ for $x > 0$.

Proof

- Let $u = \dfrac{x}{h}$, then

$$\lim_{h\to 0} \ln\left(1+\frac{1}{x/h}\right)^{x/h} = \lim_{u\to\pm\infty} \ln\left(1+\frac{1}{u}\right)^{u}$$

- Since $\ln x$ is continuous, we have

$$\lim_{h\to 0} \ln\left(1+\frac{1}{x/h}\right)^{x/h} = \ln\left(\lim_{u\to\pm\infty} \left(1+\frac{1}{u}\right)^{u}\right) = 1$$

- Therefore

$$\begin{aligned}\frac{d}{dx}(\ln x) &= \frac{1}{x}\lim_{h\to 0}\frac{x}{h}\ln\left(1+\frac{1}{x/h}\right) = \frac{1}{x}\lim_{h\to 0}\ln\left(1+\frac{1}{x/h}\right)^{x/h} \\ &= \frac{1}{x} \quad \square\end{aligned}$$

Theorem

For the general logarithmic function

$$\frac{d}{dx}\Big(\log_b x\Big) = \frac{1}{x \ln b}, \qquad \text{for } x > 0, \text{ and } b > 0.$$

Proof

- Starting from the left-hand side,

$$\begin{aligned} \frac{d}{dx}\Big(\log_b x\Big) &= \frac{d}{dx}\left(\frac{\ln x}{\ln b}\right) && \text{Property of logarithmic function} \\ &= \frac{1}{\ln b}\frac{d}{dx}(\ln x) \\ &= \frac{1}{x \ln b} \quad \square \end{aligned}$$

Exercise

(a) Find the derivative function for

$$y = \frac{x^2 \sqrt[3]{7x - 14}}{(1 + x^2)^4}$$

(b) Find the values of h, k, and a that make the circle

$$(x - h)^2 + (y - k)^2 = a^2$$

tangent to the parabola $y = x^2 + 1$ at the point $(1, 2)$, that is, they share the same tangent line, and that also make the second derivatives y'' have the same value on both curves there. Such circles are called osculating circles (from the Latin *osculari*, meaning "to kiss").

(c) Suppose that f is an one-to-one differentiable function such that

$$f(2) = 1 \qquad \text{and} \qquad f'(2) = 3/4$$

Evaluate $\left(f^{-1}\right)'(1)$.

Theorem

Let f be a continuous one-to-one function defined on an interval, and suppose f is differentiable at $f^{-1}(b)$, then f^{-1} is differentiable at b, and

$$\left(f^{-1}\right)'(b) = \frac{1}{f'\left(f^{-1}(b)\right)} \qquad \text{provided} \quad f'\left(f^{-1}(b)\right) \neq 0$$

Proof

- Let $b = f(a)$, then

$$\left(f^{-1}\right)'(b) = \lim_{h \to 0} \frac{f^{-1}(b+h) - f^{-1}(b)}{h} \lim_{h \to 0} \frac{f^{-1}(b+h) - a}{h}$$

- Now every number $b + h$ in the domain of f^{-1} can be written in the form

$$b + h = f(a + \Delta a)$$

$$\Longrightarrow \left(f^{-1}\right)'(b) = \lim_{h \to 0} \frac{f^{-1}\left(f(a+\Delta a)\right) - a}{f(a+\Delta a) - b} = \lim_{h \to 0} \frac{\Delta a}{f(a+\Delta a) - f(a)}$$

Proof

- Recall

$$\begin{aligned} b &= f(a) \\ b+h &= f(a+\Delta a) \end{aligned} \implies \Delta a = f^{-1}(b+h) - f^{-1}(b)$$

- Since f is continuous, f^{-1} is continuous,

$$\Delta a = \left(f^{-1}(b+h) - f^{-1}(b)\right) \to 0 \quad \text{as } h \to 0.$$

- Therefore

$$\left(f^{-1}\right)'(b) = \lim_{h\to 0} \frac{\Delta a}{f(a+\Delta a) - f(a)} = \lim_{\Delta a \to 0} \frac{1}{\frac{f(a+\Delta a) - f(a)}{\Delta a}} = \frac{1}{f'\left(f^{-1}(b)\right)}$$

- For an invertible differentiable function with nonvanishing derivative, it says

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

- Our next objective is to show the following theorem

Theorem

The general exponential function

$$f(x) = b^x, \qquad \text{where } b > 0,$$

is differentiable everywhere, and moreover

$$\frac{d}{dx}\left(b^x\right) = b^x \ln b$$

Proof

- The function $f(x) = b^x$ is differentiable since it is the inverse of

$$y = \log_b x$$

which is differentiable, and satisfies other requirements of theorem P14.

Proof

- Once we know the function
$$f(x) = b^x$$
is differentiable, we can use implicit differentiation to obtain the formula
$$\begin{aligned} x &= \log_b y \\ \implies 1 &= \frac{1}{y \ln b} \frac{dy}{dx} \\ \implies \frac{dy}{dx} &= y \ln b = b^x \ln b \end{aligned}$$

- In the special case when $b = e$, we have
$$\ln e = 1$$
which leads to the special property
$$\frac{d}{dx} e^x = e^x \cdot 1 = e^x$$

- If u is differentiable function of x and $b > 0$, then

$$\frac{d}{dx}\left(b^u\right) = \frac{d}{du}\left(b^u\right) \cdot \frac{du}{dx} = b^u \ln b \frac{du}{dx}$$

- You might be tempted to use this result to find

$$\frac{d}{dx}\left[\left(x^2+1\right)^{\sin x}\right] = (x^2+1)^{\sin x} \ln(x^2+1) \frac{d}{dx} \sin x$$

- This is not correct! Because the base b is not a constant.
- The correct way, we let $y = (x^2+1)^{\sin x}$, then

$$\ln y = \sin x \ln(x^2+1)$$

- Differentiate implicitly with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln(x^2+1) + \frac{2x \sin x}{x^2+1}$$
$$\implies \frac{dy}{dx} = y\left[\cos x \ln(x^2+1) + \frac{2x \sin x}{x^2+1}\right]$$

- Theorem P14 is useful to find the derivative of inverse trig function, e.g.

$$\frac{d}{dx}\left(\sin^{-1} x\right)$$

- Since $\sin x$ is differentiable, the inverse is differentiable for $x \in [-1, 1]$ s.t.

$$\cos\left(\sin^{-1}(x)\right) \neq 0$$

that is

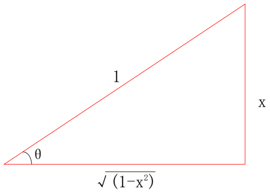
$$\sin^{-1}(x) \neq -\frac{\pi}{2} \qquad \text{and} \qquad \sin^{-1}(x) \neq \frac{\pi}{2}$$

so $\sin^{-1}(x)$ is differentiable on $(-1, 1)$

- By theorem P14

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\cos\left(\sin^{-1}(x)\right)} \qquad \text{for} \qquad -1 < x < 1$$

- Consider the triangle below,



- Notice

$$\sin\theta = \frac{x}{1} = x \implies \sin^{-1} x = \theta$$

- Also

$$\cos\theta = \sqrt{1-x^2} = \cos\left(\sin^{-1} x\right)$$

- Thus

$$\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}} \qquad \text{for} \qquad -1 < x < 1$$

Q: Can we prove the general power rule now?

The General Power Rule

If r is any real number, then

$$\frac{d}{dx}\left(x^r\right) = rx^{r-1}$$

Proof

- Let $y = x^r$, so $\ln y = r \ln x$, then we apply implicit differentiation,

$$\frac{1}{y}\frac{dy}{dx} = \frac{r}{x} \implies \frac{dy}{dx} = r\frac{y}{x} = rx^{r-1}$$

Q: Is there any hole in this proof?

$$\frac{d}{dx}\left(x^r\right) = \frac{d}{dx}e^{\ln x^r} = \frac{d}{dx}e^{r\ln x} = e^{r\ln x}\frac{d}{dx}\left(r\ln x\right) = e^{r\ln x}\frac{r}{x} = rx^{r-1}$$

Q: Are you satisfied by this? Is there any concern regarding this version?

Basic Differentiation Formulas

$$\frac{d}{dx}c = 0 \qquad \frac{d}{dx}x^n = nx^{n-1}$$

$$\frac{d}{dx}e^x = e^x \qquad \frac{d}{dx}a^x = a^x \ln a$$

$$\frac{d}{dx}\ln|x| = \frac{1}{x} \qquad \frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x \qquad \frac{d}{dx}\tan x = \sec^2 x$$

$$\frac{d}{dx}\csc x = -\csc x \cot x \qquad \frac{d}{dx}\sec x = \sec x \tan x \qquad \frac{d}{dx}\cot x = -\csc^2 x$$

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$$