Dynamics of circular motion

(1) uniform motion

$|\bar{v}| = \text{const}$, but
$\bar{v} \neq \text{const}$

centripetal acceleration
(toward the center, always)
here radial ∥ normal

$$\bar{a}_r = -\frac{v^2}{R}\hat{u}_r = -\omega^2 R\hat{u}_r$$

Newton's 2nd law

$\Longleftrightarrow$

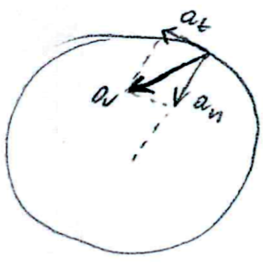
force
(called centripetal force)

$$\bar{F}_r = m\bar{a}_r = -m\frac{v^2}{R}\hat{u}_r = -m\omega^2 R\hat{u}_r$$

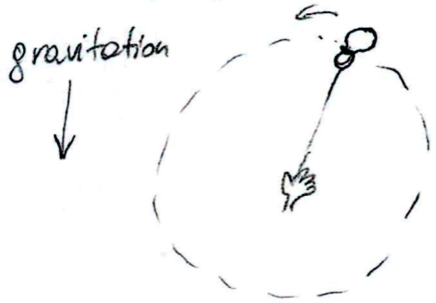
Various forces can play the role of a centripetal force
(e.g. tension in a cord - contact force,
gravitational force - field force)

(2) non-uniform motion

$|\bar{v}| \neq \text{const} \Rightarrow a_t \neq 0 \Rightarrow$ force in the tangential direction



Eg. "vertical" circular motion



DYNAMICS IN NON-INERTIAL FRAMES OF REFERENCE

Examples / demonstrations:

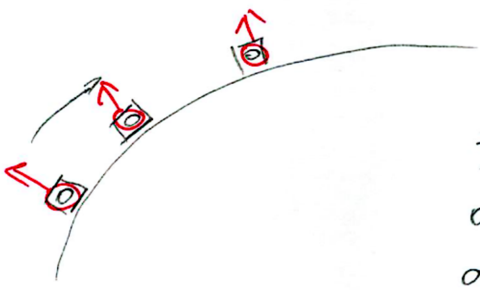
(1) accelerating chair moving along a straight line

feels a "force" pushing him in the direction opposite to the direction of acceleration

(2) rotating chair

feels a "centrifugal force"

(3) chair moving along a curved path



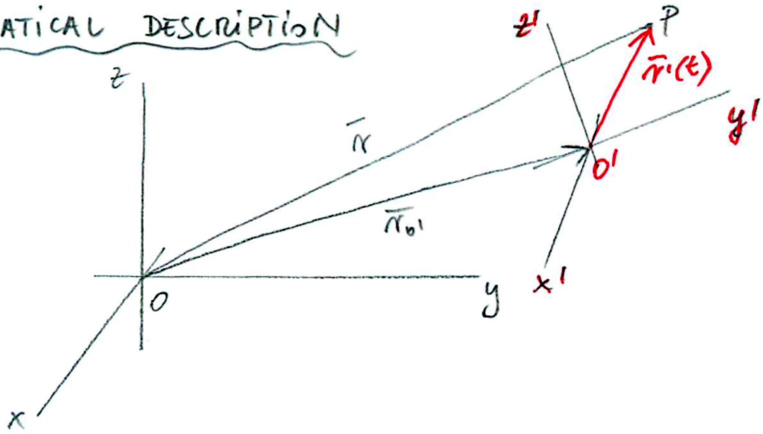
(view from the top)

feels a "centrifugal force" (directed outward, along the instantaneous radius of curvature)

These "forces" cannot be regular forces: real forces are <u>always</u> of material origin and always appear in pairs (Newton's third law)

What is the nature of these "forces"?

MATHEMATICAL DESCRIPTION



xyz – inertial FoR

$x'y'z'$ – moves arbitrarily (can accelerate, rotate with variable angular velocity, ...)

Earlier, we considered a situation when $x'y'z'$ was moving with a constant velocity wrt xyz (we have shown that $x'y'z'$ was inertial then). Now, $x'y'z'$ moves arbitrarily (so that $\bar{v}_{O'} \neq const$) hence is not inertial.

$$\bar{r} = \bar{r}_{O'} + \bar{r}'$$

$\bar{r}$ → position in xyz (we use $\hat{n}_x, \hat{n}_y, \hat{n}_z$ here) → fixed!

$\bar{r}_{O'}$ → position of the origin of $x'y'z'$ as seen from the system xyz

$\bar{r}'$ → position in $x'y'z'$ (here, we have to use $\hat{n}_{x'}, \hat{n}_{y'}, \hat{n}_{z'}$) → not fixed: travel (in particular, rotate) with $x'y'z'$

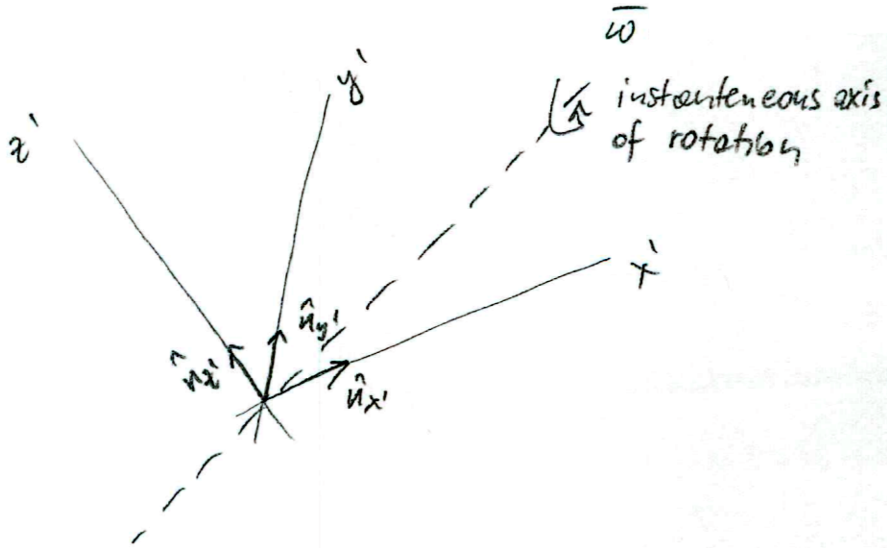
Goal: derive a (kinematic) relation between accelerations of P in both frames of reference

$\boxed{\text{Problem}}$ motion of $x'y'z'$ is arbitrary, will need to take into account that $\hat{n}_{x'}, \hat{n}_{y'}, \hat{n}_{z'}$ are not fixed (i.e. will need to know how to calculate their derivatives w.r.t. time)

Relation between velocities (detailed calculations skipped here)

$$\boxed{\bar{v} = \bar{v}_{0'} + \bar{v}' + (\bar{\omega} \times \bar{r}')}$$

Comment: The arbitrary motion of $x'y'z'$ can be decomposed into translational motion and rotational motion, the last term is due to the latter



Eventually, the relation for accelerations

$$\boxed{\bar{a} = \bar{a}_{0'} + \bar{a}' + 2\bar{\omega} \times \bar{v}' + \frac{d\bar{\omega}}{dt} \times \bar{r}' + \bar{\omega} \times (\bar{\omega} \times \bar{r}')}$$

Multiply by m and rearrange (leave $m\bar{a}'$ on one side, all other terms move to the other side)

$$m\bar{a}' = m\bar{a} - m\bar{a}_{0'} - m\frac{d\bar{\omega}}{dt} \times \bar{r}' - 2m(\bar{\omega} \times \bar{v}') - m\bar{\omega} \times (\bar{\omega} \times \bar{r}')$$

But $m\bar{a} = \underbrace{\bar{F}}_{}$ because xyz is an inertial FoR (can use Newton's laws here)

↳ real force (i.e. of material origin)

$$\boxed{m\bar{a}' = \bar{F} \underbrace{- m\bar{a}_{01} - m\frac{d\bar{\omega}}{dt}\times\bar{r}' - 2m(\bar{\omega}\times\bar{v}') - m\bar{\omega}\times(\bar{\omega}\times\bar{r}')}}$$

eqn. of motion in a non-inertial frame of reference

pseudo forces (also called fictitious forces or forces of inertia) - kinematic corrections (have units of [N]) that are due to the fact that we describe dynamics in a non-inertial FoR.

These "forces" never appear in inertial FoRs

$-m\bar{a}_{01}$ d'Alembert "force"

$-m\frac{d\bar{\omega}}{dt}\times\bar{r}'$ Euler "force"

$-2m\,\bar{\omega}\times\bar{v}'$ Coriolis "force"

$-m\,\bar{\omega}\times(\bar{\omega}\times\bar{r}')$ centrifugal "force"

[always proportional to the mass "m" of a particle]

Note. If you discuss dynamics in a non-inertial frame of reference and include these "forces" in your free body diagrams, please make sure you remember that these are pseudo-forces (and distinguish them from any real forces)

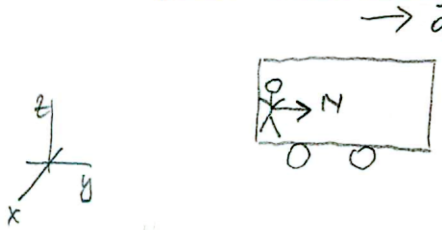
Note. You cannot have the centripetal and the centrifugal force on the same free-body diagram!

Examples

$$m\bar{a}' = \bar{F} - m\bar{a}_{o'} - m\,\bar{\omega}\times(\bar{\omega}\times\bar{r}') - 2m(\bar{\omega}\times\bar{v}') - m\frac{d\bar{\omega}}{dt}\times\bar{r}'$$

(1) $\bar{a}_{o'} \neq 0$, $\bar{\omega} = 0$, $\bar{v}' = 0$; e.g. accelerating car moving along a straight line

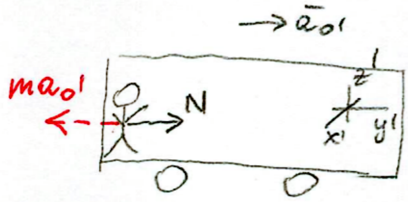
INERTIAL OBSERVER



$$m a_{o'} = N$$

"He moves with acceleration $\bar{a}_{o'}$ because there is a force (normal force $\bar{N}$ due to the wall) acting upon him"

NON-INERTIAL OBSERVER



$$0 = N - m a_{o'}$$

"He is at rest in my FoR because the normal force is balanced by the d'Alembert «force» "

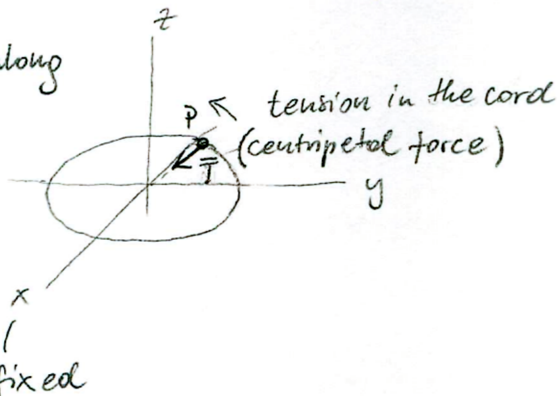
mathematically we have <u>identical</u> equs.
but their physical interpretetation is <u>different</u>

(2) $\bar{a}_{o'} = 0$; $\bar{\omega} = $ const ; $\bar{v}' = 0$; e.g. uniform circular motion

INERTIAL OBSERVER

P- moves along a circle in xyz

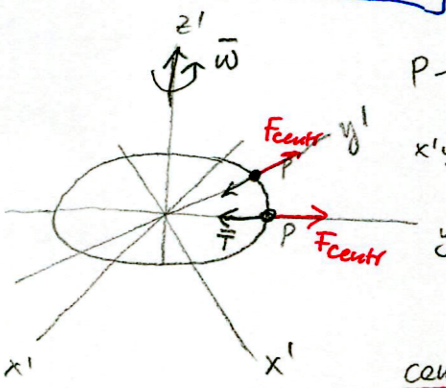
xyz - at rest



$$m\bar{a} = \bar{T}$$

$$-m\omega^2 R\,\hat{n}_r = \bar{T}$$

NON-INERTIAL OBSERVER

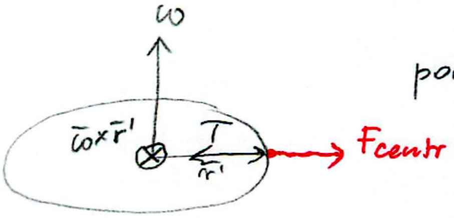


P - at rest in x'y'z'

x'y'z' - rotates

$$0 = \bar{F}_{centr} + \bar{T}$$

<u>cannot</u> call it centripetal



points outward (double cross-product)

since $\bar{r}' \perp \bar{\omega}$ and $\bar{\omega} \perp (\bar{\omega}\times\bar{r}')$ then

$$|\bar{F}_{centr}| = |-m\,\bar{\omega}\times(\bar{\omega}\times\bar{r}')| =$$

$$= m\omega^2 r' = m\omega^2 R$$

~ o ~

<u>or</u> use the formula for the double cross product

$$\bar{\omega}\times(\bar{\omega}\times\bar{r}') = \bar{\omega}\underbrace{(\bar{\omega}\circ\bar{r}')}_{=0} - \bar{r}'\underbrace{(\bar{\omega}\circ\bar{\omega})}_{=\omega^2}$$

$$\bar{F}_{centr} = m\omega^2\bar{r}' = m\omega^2 R\,\hat{n}_r$$

~ o ~

So eventually

$$-m\omega^2 R\,\hat{n}_r = \bar{T}$$

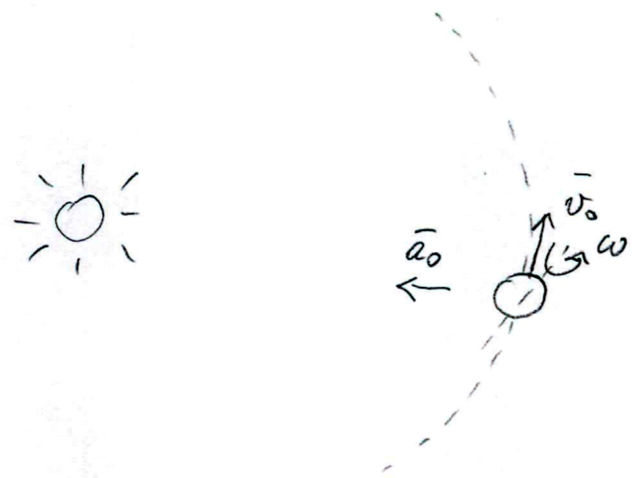
$$0 = m\omega^2 R\,\hat{n}_r + \bar{T}$$

"He moves along a circle with centripetal acceleration $-\omega^2 R\,\hat{n}_r$, because there is tension in the cord that plays a role of a centripetal force"

"He rests in my frame of reference, because the tension in the cord is balanced by the «centrifugal force»"

again both eqns. are algebraically identical but they are formulated by both observers using different language

The Earth as a frame of reference



orbital motion + rotational motion

$v_0 \sim 30$ km/s

$\omega = \frac{2\pi}{T} = \frac{2\pi}{24h} \sim 7 \cdot 10^{-5} \frac{1}{s}$

} ⇒ non-inertial FoR

For an (non-inertial) observer on the Earth

$$m\bar{a}' = \bar{F} - m\bar{a}_0 - m\bar{\omega} \times (\bar{\omega} \times \bar{r}') - 2m(\bar{\omega} \times \bar{v}')$$

$\bar{r}'$, $\bar{v}'$ } in the Earth's FoR

$\bar{F}$: material force (gravitation) $\bar{F}_{Earth} + \bar{F}_{Sun}$

$-\bar{a}_0$ "orbital" acceleration

In the non-inertial FoR associated with the Earth $\bar{F}_{Sun} - m\bar{a}_0 \cong 0$

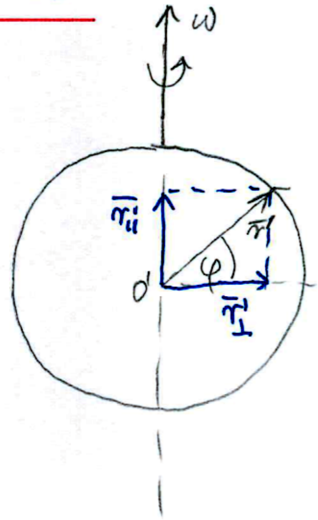
$-\bar{a}_0$ acceleration due to orbital motion

$$\boxed{m\bar{a}' = \underbrace{\bar{F}_{Earth}}_{\text{weight}} - m\bar{\omega} \times (\bar{\omega} \times \bar{r}') - 2m(\bar{\omega} \times \bar{v}')}$$

2nd law of dynamics on the Earth (including kinematic corrections due to fact that the Earth is a non-inertial FoR)

Estimation of "fictitious" forces

(*) centrifugal "force"



φ - latitude

($\varphi > 0$ for the N hemisphere
$\varphi < 0$ for the S hemisphere)

$$\bar{r}' = \bar{r}'_{\perp} + \bar{r}'_{\parallel}$$

$\bar{r}'_{\perp}$: perpendicular to the axis of rotation

$\bar{r}'_{\parallel}$: parallel to the axis of rotation

$$\bar{F}_{centr} = -m\,\bar{\omega} \times (\bar{\omega} \times (\bar{r}'_{\perp} + \bar{r}'_{\parallel})) = -m\,\bar{\omega} \times (\bar{\omega} \times \bar{r}'_{\perp} + \underbrace{\bar{\omega} \times \bar{r}'_{\parallel}}_{=0})$$

$$= -m\,\bar{\omega} \times (\bar{\omega} \times \bar{r}'_{\perp})$$

The magnitude

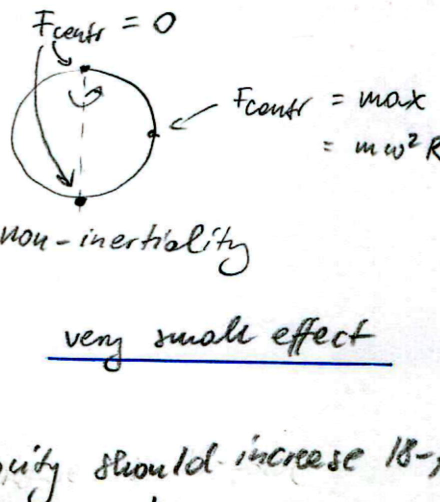
$$F_{centr} = m\,\omega^2 r'_{\perp} \qquad \text{because } \bar{\omega} \perp \bar{r}'_{\perp} \text{ and } \bar{\omega} \perp (\bar{\omega} \times \bar{r}'_{\perp})$$

But $|\bar{r}'_{\perp}| = |\bar{r}| \cos\varphi = R\cos\varphi$

R → Earth's radius $\sim$ 6357 km (pole) ÷ 6378 km (the Equator)

so

$$\boxed{F_{centr} = m\,\omega^2 R \cos\varphi}$$

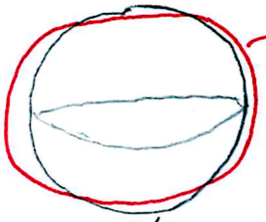


Estimation of the contribution of this "force" to non-inertiality

$$\frac{(F_{centr})_{max}}{mg} = \frac{\omega^2 R}{g} \approx \underline{0{,}003} \qquad \text{very small effect}$$

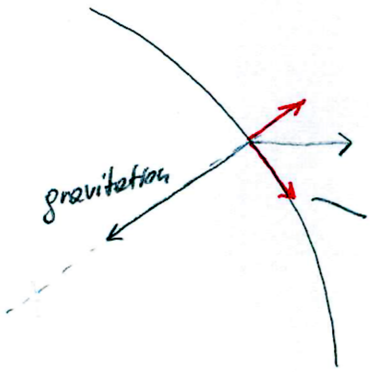
Note. To make $(F_{centr})_{max} \sim mg$, the angular velocity should increase 18-fold. Then the day would last only 1^h 19 mins!

Digression: the shape of the Earth and the centrifugal "force"



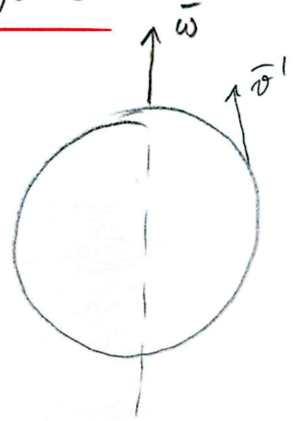
$R_{pole} \approx 6357$ km

$R_{equator} \approx 6378$ km



the Earth is not rigid, the centrifugal "force" pushes the rocks toward the Equator; result - the equilibrium shape is the shape of the GEOID (approximately ellipsoid)

(*) the Coriolis' force

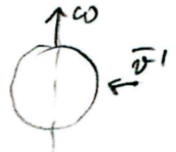


$$\bar{F}_c = -2m\left(\bar{\omega} \times \bar{v}'\right)$$

Its magnitude

$$F_c = |\bar{F}_c| = 2m|\bar{\omega}|\,|\bar{v}|\,\sin\measuredangle(\bar{\omega}, \bar{v}')$$

↳ maximum if $\bar{\omega} \perp \bar{v}'$
(e.g. free fall at the Equator)



Estimation of its contribution to the corrections to Newton's 2^{nd} law

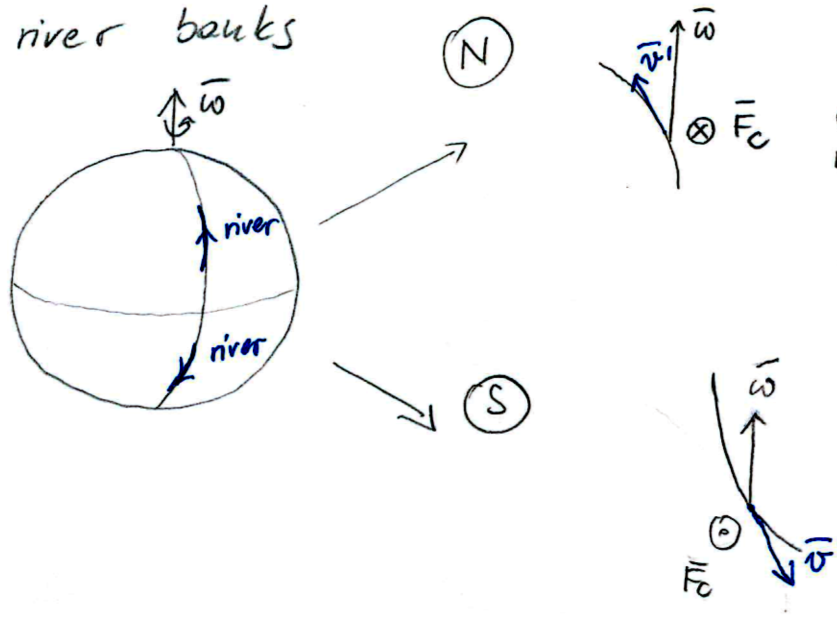
$$\frac{(F_c)_{max}}{mg} = \frac{2\omega}{g} v' \approx 1.50 \times 10^{-5} \left[\frac{s}{m}\right] v'$$

To make this ratio ~ 1 we should have our object moving @ speed 66 km/s, i.e. 2,2 times faster than the Earth moves in its orbital motion

Conclusion: The Earth can be treated, with a good accuracy, as an inertial frame of reference.

Digression: role of the Coriolis' force in the nature

(*) erosion of river banks



eastern bank more eroded

eastern bank more eroded

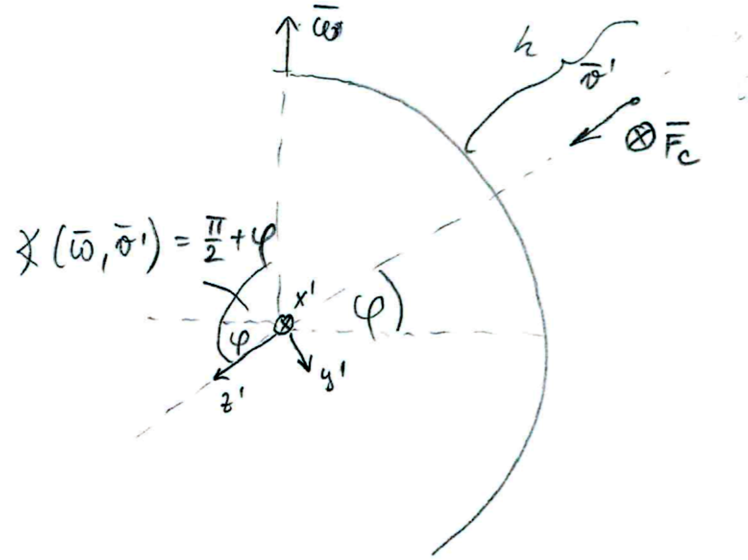
General observation: (*) on the N hemisphere objects are deflected to the right (with respect to the direction of motion)

(*) on the S hemisphere objects are deflected to the left

(*) winds: the prevaling westerlies, cyclones and anticyclons $\rightarrow$ RECITATION CLASS

Example: Free fall with the Coriolis force (simplified/approximate discussion)

Observation: Free-falling objects are deflected toward east



!!! sketch (exaggerated): we still assume constant gravitational force) close to Earth's surface

φ - latitude

The Coriolis force:

$$\bar{F}_c = -2m(\bar{\omega} \times \bar{v}')$$

$$F_c = |\bar{F}_c| = 2m\,\omega v' \sin \sphericalangle(\bar{\omega}, \bar{v}')$$

But $\sphericalangle(\bar{\omega}, \bar{v}') = \frac{\pi}{2} + \varphi$

$$F_c = 2m\omega v' \cos\varphi \qquad (!)$$

In free fall $v' = gt$, so

$$F_c = 2mgt\omega \cos\varphi$$

The acceleration due to this force (directed to the east)

$$a_{x'} = \frac{F}{m} = 2g\omega t \cos\varphi$$

So the velocity toward the east:

$$v_{x'} = \int_0^t a_c \, dt = g\omega t^2 \cos\varphi$$

and deflection

$$x' = \int_0^t v_{x'} \, dt = \frac{1}{3} g\omega t^3 \cos\varphi$$

But $h = \frac{g t_f^2}{2} \Rightarrow t_f = \sqrt{\frac{2h}{g}} \Rightarrow x_{defl} = x'(t_f) = \boxed{\frac{1}{3} g\omega \left(\frac{2h}{g}\right)^{3/2} \cos\varphi}$

E.g. $h = 642\,m$

$x_{defl} = 22\,cm$

Discussion:

→ a pole (N or S), then $\varphi = \frac{\pi}{2}$ – no deflection

→ the equator, then $\varphi = 0$ – maximum deflection

$$(x_{defl})_{max} = \frac{1}{3} g \omega \left(\frac{2h}{g}\right)^{3/2}$$

Note!!! This is an approximate analysis. In fact $\bar{v} = \bar{v}_{z'} + \bar{v}_{x'}$ and this total velocity should be used in the formula for the Coriolis force.

We have only included the z'-component (see step (!)).

The full analysis of the problem is complicated.