Contents

1 Mathematical Preliminaries

$e^{\alpha+j\theta} = e^{\alpha}\cos\theta + je^{\alpha}\sin(\theta) \quad e^{\alpha-j\theta} = e^{\alpha}\cos\theta - je^{\alpha}\sin(\theta)$

$\cos(\theta) = \frac{1}{2}e^{j\theta} + \frac{1}{2}e^{-j\theta} \quad \sin(\theta) = \frac{1}{2j}e^{j\theta} - \frac{1}{2j}e^{-j\theta} = -\frac{j}{2}e^{j\theta} + \frac{j}{2}e^{-j\theta}$

$\delta(at+b) = \frac{1}{|a|}\delta\left(t + \frac{b}{a}\right)$

$\int t\cos(t)dt = \cos(t) + t\sin(t)$

$\int t^2\cos(t)dt = 2t\cos(t) + (t^2-2)\sin(t)$

$\int t\sin(t)dt = \sin(t) - t\cos(t)$

$\int t^2\sin(t)dt = 2t\sin(t) - (t^2-2)\cos(t)$

$\int te^{at}dt = \left(\frac{t}{a} - \frac{1}{a^2}\right)e^{at}$

$\int t^2e^{at}dt = \left(\frac{t^2}{a} - \frac{2t}{a^2} - \frac{2}{a^3}\right)e^{at}$

2 Signal Basics

2.1 Even and odd signal

$x(t) = x_{\text{even}} + x_{\text{odd}} \qquad x_{\text{even}} = \frac{1}{2}(x(t) + x(-t)) \quad x_{\text{odd}} = \frac{1}{2}(x(t) - x(-t))$

2.2 Average value, enery and power

2.2.1 definition

Average value: $A = \lim_{L\to\infty}\left[\frac{1}{2L}\int_{-L}^{L} x(t)dt\right] = c_0$

Energy: $E = \int_{-\infty}^{\infty} |x(t)|^2dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2d\omega$

Power: $P = A = \lim_{L\to\infty}\left[\frac{1}{2L}\int_{-L}^{L} |x(t)|^2dt\right] = \sum_{k=-\infty}^{\infty} |c_k|^2$

2.2.2 signal classification

Energy signal: $E < \infty \quad P = 0$

Power signal: $E = \infty \quad P > 0$

2.2.3 common signals

$x(t) = a\cos(\omega t + \theta) \quad x(t) = \ a\sin(\omega t + \theta) + b \longrightarrow A = 0 \quad E = \infty \quad P = \dfrac{a^2}{2}$

$x(t) = e^{-at}u(t) \longrightarrow A = 0 \quad E = \dfrac{1}{2a} \quad P = 0$

$x(t) = ae^{j(\omega t + \theta)} \longrightarrow A = 0 \quad E = \infty \quad P = a^2$

$x(t) = \displaystyle\sum_{k=-\infty}^{\infty} a_k e^{jk\omega t} \longrightarrow A = 0 \quad E = \infty \quad P = \sum_{k=-\infty}^{\infty} a_k^2$

periodic signal $x(t) \longrightarrow A = \dfrac{1}{T}\displaystyle\int_0^T x(t)dt \quad E = \infty \quad P = \dfrac{1}{T}\int_0^T |x(t)|^2 dt$

2.3 Rectangular Function

$x(t) = \mathrm{rect}(at + b) = u(x_1) - u(x_2) \longrightarrow x_1 = -\dfrac{2b+1}{2a} \quad x_2 = -\dfrac{2b-1}{2a}$

$x_1 = r_1 \quad x_2 = r_2 \longrightarrow x(t) = \mathrm{rect}\left(\dfrac{1}{r_2 - r_1}t - \dfrac{r_2 + r_1}{2(r_2 - r_1)}\right)$

periodic function from r_1 to r_2 with original function $x(t)$:

$$\begin{aligned} y(t) &= \sum_{k=-\infty}^{\infty} \left[x(t - kT) \cdot \mathrm{rect}(a(t - kT) + b)\right] \\ &= \sum_{k=-\infty}^{\infty} \left[x\left(t - k(r_2 - r_1)\right) \cdot \mathrm{rect}\left(\frac{1}{r_2 - r_1}t - \frac{r_2 + r_1}{2(r_2 - r_1)} - k\right)\right] \end{aligned}$$

3 LTI System

3.1 Convolution

3.1.1 general property

$x(t - t_1) * h(t - t_2) = y(t - t_1 - t_2)$
$x(at) * h(at) = \dfrac{1}{a}y(at)$

3.1.2 common convolution

$x(t) * u(t - t_0) = \displaystyle\int_{-\infty}^{t - t_0} x(\tau)d\tau$

$x(t) * \mathrm{rect}(at + b) = \displaystyle\int_{t - x_2}^{t - x_1} x(\tau)d\tau = \int_{t + \frac{2b-1}{2a}}^{t + \frac{2b+1}{2a}} x(\tau)d\tau$

$u(t) * u(t) = tu(t)$

$\mathrm{rect}(t) * \mathrm{rect}(t) = \mathrm{tri}(t)$

$$e^{-\alpha t} * e^{-\beta t} = \begin{cases} te^{-\alpha t}u(t) & \alpha = \beta \\ \dfrac{1}{\beta - \alpha}\left(e^{-\beta t} - e^{-\alpha t}\right) & \alpha \neq \beta \end{cases}$$

3.2 impulse response

$$y(t) = \int_{-\infty}^{t-t_0} w(t-\tau)x(\tau)d\tau \longrightarrow h(t) = w(t)u(t-t_0)$$

$$y(t) = \int_{t-x_2}^{t-x_1} w(t-\tau)x(\tau)d\tau = \int_{t+\frac{2b-1}{2a}}^{t+\frac{2b+1}{2a}} w(t-\tau)x(\tau)d\tau \longrightarrow h(t) = w(t)\mathrm{rect}(at+b)$$

4 Fourier Series

4.1 Definition

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega t} \quad c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega t}dt$$

4.2 Real form

$$x(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega t) + \sum_{k=1}^{\infty} b_k \sin(k\omega t) \quad \begin{cases} a_k = \dfrac{2}{T}\displaystyle\int_0^T x(t)\cos(k\omega t)dt = c_k + c_{-k} \\ \\ b_k = \dfrac{2}{T}\displaystyle\int_0^T x(t)\sin(k\omega t)dt = j(c_k - c_{-k}) \end{cases}$$

5 Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt \qquad f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

6 Filtering

RMS bandwidth: $\omega_{\mathrm{rms}} = \sqrt{\dfrac{\int_{-\infty}^{\infty} \omega^2 |X(\omega)|^2 d\omega}{\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}}$

RMS time duration: $\tau_{\mathrm{rms}} = \sqrt{\dfrac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{\int_{-\infty}^{\infty} |x(t)|^2 dt}}$

Time-bandwidth product: $\omega_{\mathrm{rms}} \cdot \tau_{\mathrm{rms}}$

7 Sampling

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \longleftrightarrow P(\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$x_s(t) = x(t)p(t) \longleftrightarrow X_s(\omega) = \frac{1}{T_s} \textstyle\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$$

i) sinc interpolation

$$h(t) = \frac{\omega_s T_s}{\pi} \mathrm{sinc}\left(\frac{\omega_c}{\pi} t\right) \longleftrightarrow H(\omega) = T_s \mathrm{rect}\left(\frac{\omega}{2\omega_s}\right) \quad x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \mathrm{sinc}\left(\frac{t - nT_s}{T_s}\right)$$

ii) linear interpolation

$$h(t) = \mathrm{tri}\left(\frac{t}{T_s}\right) \longleftrightarrow H(\omega) = T_s \mathrm{sinc}^2\left(\frac{\omega}{\omega_s}\right) \quad x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \mathrm{tri}\left(\frac{t - nT_s}{T_s}\right)$$

iii) zero-order interpolation

$$h(t) = \mathrm{rect}\left(\frac{t}{T_s} - \frac{1}{2}\right) \longleftrightarrow H(\omega) = T_s \mathrm{sinc}\left(\frac{\omega}{\omega_s}\right) e^{-j\omega\frac{T_s}{2}} \quad x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \mathrm{rect}\left(\frac{t - nT_s}{T_s} - \frac{1}{2}\right)$$

8 Communication System

8.1 DSB/SC-AM

$$c(t) = \cos(\omega_c t + \theta_c) \quad y(t) = x(t)c(t) \quad w(t) = y(t)c(t)$$

$$Y(\omega) = \frac{1}{2} e^{j\theta_c} X(\omega - \omega_c) + \frac{1}{2} e^{-j\theta_c} X(\omega + \omega_c) \quad W(\omega) = \frac{1}{4} e^{j2\theta_c} X(\omega - 2\omega_c) + \frac{1}{2} X(\omega) + \frac{1}{4} e^{-j2\theta_c} X(\omega + 2\omega_c)$$

8.2 DSB/WC-AM

$$c(t) = \cos(\omega_c t) \quad y(t) = (A + x(t))c(t)$$

$$Y(\omega) = A\pi[\delta(\omega + \omega_c) + \delta(\omega - \omega_c)] + \frac{1}{2}[X(\omega + \omega_c) + X(\omega - \omega_c)]$$

9 Laplace Transform

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-st} dt$$

$$H(s) = G \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)}$$

$$\Rightarrow |H(\omega)| = |G| \frac{|j\omega - z_1| \cdots |j\omega - z_m|}{|j\omega - p_1| \cdots |j\omega - p_n|}$$

$$\Rightarrow \angle H(\omega) = \angle G + \angle(j\omega - z_1) + \cdots \angle(j\omega - z_m) - \angle(j\omega - p_1) - \cdots - \angle(j\omega - p_n)$$

10 Appendix

10.1 Collections of common Fourier Series

10.1.1 Common Fourier Series Representation and Fourier Transform

Sawtooth

$$f(t) = \frac{1}{T} \quad t \in [0, T]$$

$$c_k = \begin{cases} \frac{1}{2} & k = 0 \\ j\frac{1}{2\pi k} & k \neq 0 \end{cases}$$

$$F(\omega) = \pi\delta(\omega) + j\sum_{k\neq 0}\left[\frac{1}{k}\delta\left(\omega - \frac{2\pi}{T}k\right)\right]$$

Impulse Train

$$f(t) = \delta(t + nT + b) \quad t \in [0, T]$$

$$c_k = \frac{1}{T}e^{\frac{2\pi b}{T}k}$$

$$F(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} e^{j\frac{2\pi b}{T}k}\delta\left(\omega - \frac{2\pi}{T}k\right) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} e^{-j\frac{2\pi(T-b)}{T}k}\delta\left(\omega - \frac{2\pi}{T}k\right)$$

Rectangular Wave

$$f(t) = \mathrm{rect}\left(\frac{1}{T_0}t\right) \quad t \in \left[-\frac{T}{2}, \frac{T}{2}\right]$$

$$c_k = \begin{cases} \frac{T_0}{T} & k = 0 \\ \frac{T_0}{T}\mathrm{sinc}\left(\frac{T_0}{T}k\right) & k \neq 0 \end{cases}$$

$$F(\omega) = 2\pi\frac{T_0}{T}\delta(\omega) + \frac{T_0}{T}\sum_{k\neq 0}\left[\mathrm{sinc}\left(\frac{T_0}{T}k\right)\delta\left(\omega - \frac{2\pi}{T}k\right)\right]$$

Square Wave

$$f(t) = \begin{cases} 1 & t \in \left[0, \frac{T}{2}\right] \\ -1 & t \in \left[-\frac{T}{2}, T\right] \end{cases}$$

$$c_k = \begin{cases} 0 & k = 2m \\ -j\frac{2}{\pi k} & k = 2m+1 \end{cases}$$

$$F(\omega) = -j\frac{2}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{2k+1}\delta\left(\omega - \frac{2\pi}{T} - \frac{4\pi}{T}k\right)$$

Triangular Wave

$$f(t) = \begin{cases} \frac{2}{T}t & t \in \left[0, \frac{T}{2}\right] \\ 2 - \frac{2}{T}t & t \in \left[-\frac{T}{2}, T\right] \end{cases}$$

$$c_k = \begin{cases} \frac{1}{2} & k = 0 \\ 0 & k = 2m, m \neq 0 \\ -\frac{2}{\pi^2 k^2} & k = 2m+1 \end{cases}$$

$$F(\omega) = \pi\delta(\omega) - \frac{2}{\pi^2}\sum_{k\neq 0}\frac{1}{k^2}\delta\left(\omega - \frac{2\pi}{T} - \frac{4\pi}{T}k\right)$$

10.1.2 Fourier Series Properties

$y(t)$	ω_d	d_k
$ax(t) + b$	ω_c	$d_0 = b + ac_0, d_k = ac_k$
$x(at+b), a > 0$	$a\omega_c$	$d_k = c_k e^{jk\omega_c b}$
$x(-t)$	ω_c	$d_k = c_{-k}$
$x^*(t)$	ω_c	$d_k = c^*_{-k}$
$x(t)e^{jn\omega_c}$	ω_c	$d_k = c_{k-n}$
$\frac{d}{dt}x(t)$	ω_c	$d_k = jk\omega_c c_k$

$z(t)$	ω_z	z_k
$ax(t) + by(t), \omega_c = \omega_d$	ω_c	$d_0 = b + ac_0, d_k = ac_k$
$x(t)y(t)$	ω_c	$z_k = \sum_{n=-\infty}^{\infty} a_n b_{k-n}$

10.2 Collections of common Fourier Transform

10.2.1 Fourier Transform Pairs

$f(t)$	$F(\omega)$
$\delta(at+b)$	$\frac{1}{\lvert a\rvert}e^{j\frac{b}{a}\omega}$
$u(at+b)$	$e^{j\frac{b}{a}\omega}\left(\pi\delta(\omega)+\frac{\mathrm{sgn}(a)}{j\omega}\right)$
$\cos(\omega_0 t+\phi)$	$\pi e^{j\phi}\delta(\omega-\omega_0)+\pi e^{-j\phi}\delta(\omega+\omega_0)$
$\sin(\omega_0 t+\phi)$	$-j\pi e^{j\phi}\delta(\omega-\omega_0)+j\pi e^{-j\phi}\delta(\omega+\omega_0)$
$\mathrm{rect}(at+b)$	$\frac{1}{a}e^{j\frac{b}{a}\omega}\mathrm{sinc}(\frac{\omega}{2\pi a})$
$\mathrm{tri}(at+b)$	$\frac{1}{\lvert a\rvert}e^{j\frac{b}{a}\omega}\mathrm{sinc}^2(\frac{\omega}{2\pi a})$
$\mathrm{sinc}(at+b)$	$\frac{1}{a}e^{j\frac{b}{a}\omega}\mathrm{rect}(\frac{\omega}{2\pi a})$
$\mathrm{sinc}^2(at+b)$	$\frac{1}{\lvert a\rvert}e^{j\frac{b}{a}\omega}\mathrm{tri}(\frac{\omega}{2\pi a})$
$\mathrm{sgn}(at+b)$	$\mathrm{sgn}(a)e^{j\frac{b}{a}\omega}\frac{2}{j\omega}$
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\frac{j}{\pi(at+b)}$	$\frac{1}{\lvert a\rvert}e^{j\frac{b}{a}\omega}\mathrm{sgn}(\omega)$
$t^{n-1}e^{-at}u(t)$	$\frac{(n-1)!}{(j\omega+a)^n}$
e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\frac{1}{4a}\omega^2}$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$
$\frac{1}{a^2+t^2}$	$\frac{\pi}{a}e^{-a\lvert t\rvert}$

$F(\omega)$	$f(t)$
$e^{ja\omega}$	$\delta(t+a)$
$\delta(\omega+\omega_0)$	$\frac{1}{2\pi}e^{-j\omega_0 t}$
$\pi\delta(\omega)+\frac{1}{j\omega}$	$u(t)$
$\mathrm{rect}(a\omega)$	$\frac{1}{2\pi a}\mathrm{sinc}(\frac{1}{2\pi a}t)$
$\mathrm{tri}(a\omega)$	$\frac{1}{2\pi\lvert a\rvert}\mathrm{sinc}^2(\frac{1}{2\pi a}t)$
$\mathrm{sinc}(a\omega)$	$\frac{1}{2\pi a}\mathrm{rect}(\frac{1}{2\pi a}t)$
$\mathrm{sinc}^2(a\omega)$	$\frac{1}{2\pi\lvert a\rvert}\mathrm{tri}(\frac{1}{2\pi a}t)$
$\frac{1}{j\omega}$	$\frac{1}{2}\mathrm{sgn}(t)$
$\mathrm{sgn}(\omega)$	$\frac{j}{\pi t}$
$\frac{1}{(j\omega+a)^n}$	$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$
$\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$	$\frac{1}{2\sqrt{a\pi}}e^{-\frac{1}{4a}t^2}$
$\frac{1}{a^2+\omega^2}$	$\frac{1}{2a}e^{-a\lvert t\rvert}$
$e^{-a\lvert t\rvert}$	$\frac{a}{\pi(a^2+t^2)}$

10.2.2 Fourier Transform Property

$f(t)$	$F(\omega)$
$f(at+b)$	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
$f_1(t) * f_2(t)$	$F_1(\omega)F_2(\omega)$
$f_1(t)f_2(t)$	$\frac{1}{2\pi}F_1(\omega) * F_2(\omega)$
$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
$f(t)\cos(\omega_0 t)$	$\frac{1}{2}(F(\omega+\omega_0)+F(\omega-\omega_0))$
$f(t)\sin(\omega_0 t)$	$\frac{j}{2}(F(\omega+\omega_0)-F(\omega-\omega_0))$
$\frac{d^n}{dt^n}f(t)$	$(j\omega)^n F(\omega)$
$(-jt)^n f(t)$	$\frac{d^n}{d\omega^n}F(\omega)$
$t^n f(t)$	$j^n \frac{d^n}{d\omega^n}F(\omega)$
$\int_{-\infty}^{t} f(t)dt$	$\pi F(0)\delta(\omega)+\frac{1}{j\omega}F(\omega)$
$f^*(t)$	$F^*(-\omega)$
$F(t)$	$2\pi f(-\omega)$

10.2.3 Period Signal Fourier Transform

complex form FS: $f(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t} \longleftrightarrow F(\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k \delta(\omega - k\omega_0)$

real form FS: $f(t) = c_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + \sum_{k=1}^{\infty} b_k \sin(k\omega_0 t) \longleftrightarrow$

$F(\omega) = 2\pi c_0 \delta(\omega) + \pi \sum_{k=1}^{\infty}(a_k - jb_k)\delta(\omega - k\omega_0) + \pi \sum_{k=1}^{\infty}(a_k + jb_k)\delta(\omega + k\omega_0)$

10.3 Collections of common Laplace Transform

10.3.1 Laplace Transform Pairs

$f(t)$	$F(s)$	ROC	$f(t)$	$F(s)$	ROC
$\delta(t)$	1	$\mathrm{Re}(s) \in \mathbb{R}$	$\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$u(t)$	$\frac{1}{s}$	$\mathrm{Re}(s) > 0$	$\cos(\omega_0 t)u(t)$	$\frac{s}{s^2+\omega_0^2}$	$\mathrm{Re}(s) > 0$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > 0$	$e^{-at}\sin(\omega_0 t)u(t)$	$\frac{\omega_0}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > \mathrm{Re}(-a)$
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) > \mathrm{Re}(-a)$	$e^{-at}\cos(\omega_0 t)u(t)$	$\frac{s+a}{(s+a)^2+\omega_0^2}$	$\mathrm{Re}(s) > \mathrm{Re}(-a)$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\mathrm{Re}(s) < \mathrm{Re}(-a)$	$\frac{d^n}{dt^n}\delta(t)$	s^n	$\mathrm{Re}(s) \in \mathbb{R}$
$t^n e^{-at}u(t)$	$\frac{n!}{s^{n+1}}$	$\mathrm{Re}(s) > \mathrm{Re}(-a)$	$u(t) * \cdots * u(t)$	$\frac{1}{s^n}$	$\mathrm{Re}(s) > 0$

10.3.2 Laplace Transform Property

$f(t)$	$F(\omega)$	ROC
$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$	$\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
$f(at+b)$	$\frac{1}{\lvert a\rvert} e^{bs} F\left(\frac{s}{a}\right)$	$a \cdot \mathrm{ROC}$
$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$	$\mathrm{ROC}_1 \cap \mathrm{ROC}_2$
$f(t)e^{s_0 + j\omega_0 t}$	$F(s - s_0 - j\omega_0)$	$\mathrm{ROC} + \mathrm{Re}(s_0)$
$\frac{d^n}{dt^n} f(t)$	$s^n F(s)$	ROC
$(-t)^n f(t)$	$\frac{d^n}{d\omega^n} F(\omega)$	ROC
$\int_{-\infty}^{t} f(t)dt$	$\frac{1}{s} F(s)$	$\mathrm{ROC} \cap \mathrm{Re}(s) > 0$