

## Chapter 5a – Motion with Air Drag

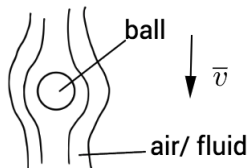
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Physics I (Summer 2019)  
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# Introduction. Models for Air Resistance

**Stokes (or linear) drag** [small objects/low speeds]

$$\bar{F}_{\text{drag}} = -k\bar{v} = -k\nu\left(\frac{\bar{v}}{\nu}\right)$$

where  $k = 6\pi\eta R_s = \text{const}$ ,  $\eta$  is the fluid viscosity,  $R_s$  is the Stokes' radius (property of the object)



**Quadratic drag** [large objects/high speeds]

$$\bar{F}_{\text{drag}} = -b\nu^2\left(\frac{\bar{v}}{\nu}\right)$$

where  $b = \frac{1}{2}\varrho C_d A = \text{const}$ ,  $\varrho$  is the fluid density,  $C_d$  is a drag coefficient (e.g. for cars 0.25-0.5),  $A$  is the cross-sectional area of the object perpendicular to the direction of motion

# Qualitative Analysis

[fall with linear air drag, no initial velocity]

initial phase

$$t \approx 0 \Rightarrow v \approx 0$$

$$F_{\text{drag}} \propto v = 0$$

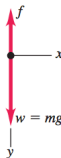


$$a \approx g$$

In the initial phase  
the particle moves as it was  
free-falling

final phase

speed increases  $\Rightarrow$  air drag  
increases



$$\text{net force: } mg - kv_{\infty} = 0$$

$\rightarrow$  in the final phase, the drag  
balances the weight

$\Rightarrow$  particle moves with constant  
speed

$$\text{terminal speed: } v_{\infty} = \frac{mg}{k}$$

**Observation:** Heavier objects  
reach greater terminal speeds  
(that is, fall down faster).

# General Problem: Strategy

## General problem

$$a_y = \frac{F(v_y)}{m} \quad \rightarrow \text{Newton's 2}^{\text{nd}} \text{ law (equation of motion)}$$

$$\frac{d^2 y}{dt^2} = \frac{F\left(\frac{dy}{dt}\right)}{m} \quad \rightarrow \text{Problem: the second derivative...}$$

$$\text{Idea: Try the substitution } v_y = \frac{dy}{dt}$$

Separate the variables...

$$\frac{dv_y}{dt} = \frac{F(v_y)}{m} \quad \Rightarrow \quad m \frac{dv_y}{F(v_y)} = dt$$

...and integrate

$$\int_{v_{y0}}^{v_y(t)} \frac{m}{F(v_y)} dv_y = \int_0^t dt$$

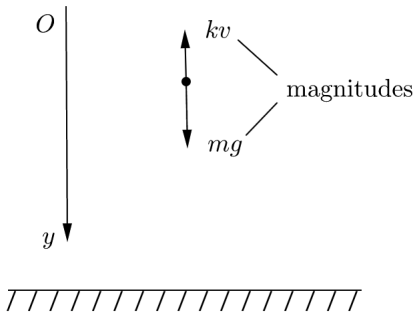
$$\boxed{\int_{v_{y0}}^{v_y(t)} \frac{m}{F(v_y)} dv_y = t} \quad \xRightarrow{\text{yields}} \quad v_y(t) = \dots$$

Once the velocity as a function of time is known, both the acceleration and the position as functions of time can be found as well

$$\text{acceleration} \quad a_y = \frac{dv_y}{dt}$$

$$\text{position} \quad \frac{dy}{dt} = v_y(t) \quad \Rightarrow \quad y(t) = \int_0^t v_y(t) dt + y_0$$

## Example: Fall with Linear Drag



Initial conditions

$$\begin{cases} y(0) = 0 \\ v_y(0) = 0 \end{cases}$$

Newton's second law (equation of motion)

$$ma_y = mg - kv_y \quad (\text{net force}) \quad \implies \quad a_y = g - \frac{k}{m}v_y$$

But  $a_y = \frac{dv_y}{dt}$ , so that

$$\frac{dv_y}{dt} = g - \frac{k}{m}v_y \quad \implies \quad \frac{dv_y}{dt} = -\frac{k}{m}\left(v_y - \frac{mg}{k}\right).$$

Separating the variables  $v_y$  and  $t$ ,

$$\frac{dv_y}{v_y - \frac{mg}{k}} = -\frac{k}{m} dt$$

and integrating with the given initial conditions

$$\int_0^{v_y(t)} \frac{dv_y}{v_y - \frac{mg}{k}} = -\frac{k}{m} \int_0^t dt \quad \Rightarrow \quad \ln \left| \frac{v_y(t) - \frac{mg}{k}}{-\frac{mg}{k}} \right| = -\frac{k}{m} t.$$

But  $v_y(t) < \frac{mg}{k}$  (terminal speed), so that  $\ln \frac{\frac{mg}{k} - v_y(t)}{\frac{mg}{k}} = -\frac{k}{m} t$

or, equivalently,

$$\frac{mg}{k} - v_y(t) = \frac{mg}{k} e^{-\frac{k}{m} t}.$$

Solving for  $v_y(t)$

$$v_y(t) = \frac{mg}{k} \left( 1 - e^{-\frac{k}{m}t} \right).$$

## Discussion

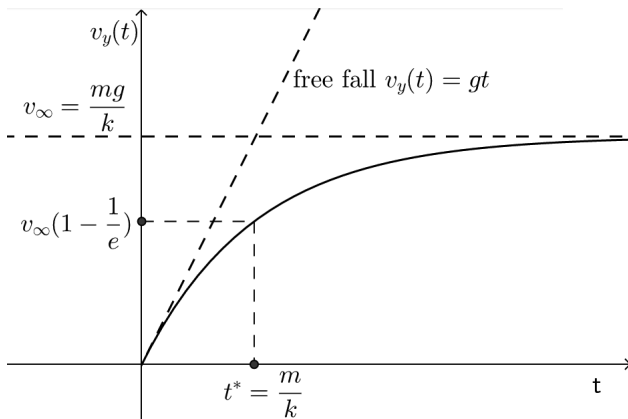
- 1° For short times, i.e. ,  $t \ll \frac{m}{k}$ , then  $\frac{k}{m}t \ll 1$ , we can expand the exponential function in the Taylor (Maclaurin) series  $e^u = 1 + \frac{u}{1!} + \frac{u^2}{2!} + \dots$  and approximate by keeping the first two terms in the expansion

$$v_y(t) = \frac{mg}{k} (1 - e^{-\frac{k}{m}t}) \approx \frac{mg}{k} (1 - 1 + \frac{k}{m}t) = gt$$
$$v_y(t) \approx gt \quad \text{constant acceleration}$$

- 2° Long times,  $t \gg \frac{m}{k}$ , i.e.  $t \rightarrow \infty$

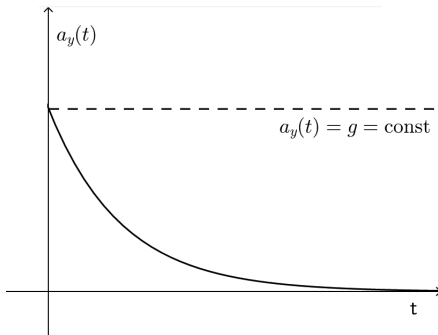
$$v_\infty = \lim_{t \rightarrow \infty} \frac{mg}{k} (1 - e^{-\frac{k}{m}t}) = \frac{mg}{k} = \text{const} \quad (\text{constant velocity})$$





## Acceleration

$$\boxed{a_y(t)} = \frac{dv_y(t)}{dt} = \frac{d}{dt} \left[ \frac{mg}{k} \left( 1 - e^{-\frac{k}{m}t} \right) \right] = \boxed{ge^{-\frac{k}{m}t}}$$



## Position

$$v_y(t) = \frac{dy}{dt} = \frac{mg}{k} \left(1 - e^{-\frac{k}{m}t}\right)$$

$$\int_0^{y(t)} dy = \frac{mg}{k} \int_0^t \left(1 - e^{-\frac{k}{m}t}\right) dt$$

$$y(t) = \frac{mg}{k} \left( t - \left( -\frac{m}{k} \right) e^{-\frac{k}{m}t} \right) \Big|_0^t = \frac{mg}{k} \left( t + \frac{m}{k} e^{-\frac{k}{m}t} - \frac{m}{k} \right)$$

$$\Rightarrow \boxed{y(t) = \frac{mg}{k} \left[ t + \frac{m}{k} \left( e^{-\frac{k}{m}t} - 1 \right) \right]}$$

## Discussion

- 1° Short times (now, add one more term in the Taylor polynomial)

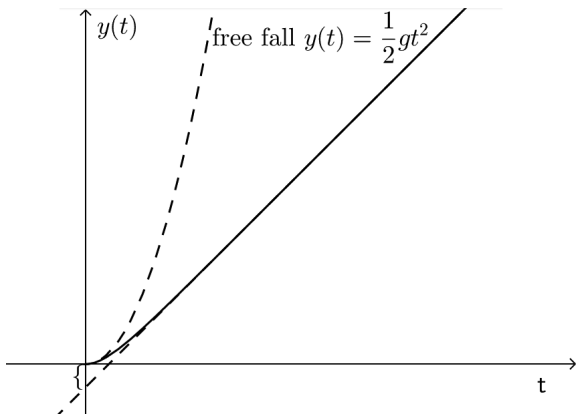
$$\begin{aligned}y(t) &\approx \frac{mg}{k} \left[ t + \frac{m}{k} \left( 1 - \frac{k}{m}t + \frac{1}{2} \left( \frac{k}{m}t \right)^2 - 1 \right) \right] \\&= \frac{mg}{k} \left[ t - t + \frac{m}{k} \frac{1}{2} \left( \frac{k}{m}t \right)^2 \right] = \frac{1}{2}gt^2\end{aligned}$$

$$y(t) \approx \frac{1}{2}gt^2$$

- 2° Long times ( $t \gg \frac{m}{k}$ )

$$e^{-\frac{k}{m}t} \approx 0 \quad \text{and}$$

$$y(t) \approx \frac{mg}{k} \left( t - \frac{m}{k} \right)$$



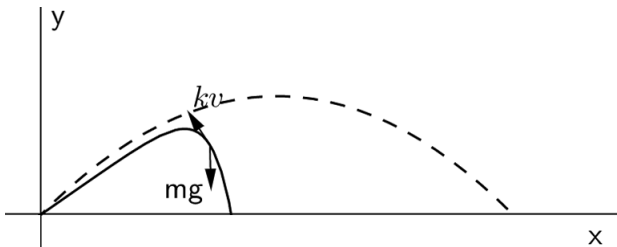
## Example: Projectile motion with linear air drag

Drag force  $\vec{F}_{\text{drag}} = -k\vec{v} = -kv_x\hat{n}_x - kv_y\hat{n}_y$

Equation of motion  $m\vec{a} = \vec{F}_{\text{grav}} + \vec{F}_{\text{drag}}$  and initial conditions

$$\begin{cases} ma_x = -kv_x \\ ma_y = -mg - kv_y \end{cases} \quad \begin{cases} \vec{r}(0) = 0 \\ \vec{v}(0) = \vec{v}_0 \end{cases}$$

**Solution strategy:** same as before (equations for both components can be solved independently)



### Effects of air drag

- \* reduces the maximum height
- \* shortens the range