Lecture 6

Definition of derivative

The derivative of $f(x)$ at a point c inside the interval $[a, b]$ is

$$\lim_{h\to 0}[\frac{f(c+h)-f(c)}{h}] = f'(c)$$

* $f(x)$ is differentiable at c if $f'(c)$ exists.
* Geometric meaning: the slope of $f(x)$ at $x = c$.
* The geometric meaning of the definition of derivative.
* Indicate the rate of change of $f(x)$ with respect to x.
* The difference between $f'(x_0)$ and $f'(x)$:
$f'(x_0)$ is a value, which equals the derivative of $f(x)$ at $x = x_0$. For example, $f'(x_0)$ is the value of $f(x)$ at $x = x_0$.
$f'(x)$ is a function, which can assign a corresponding value $f'(c)$ to different c (where c can vary in the domain).
* When we use the definition to calculate derivative, note that h is a variable, and x is considered as a constant.
* Limits can be interpreted by the derivative of a function. Therefore, evaluating the limit can be transformed into evaluating the derivative.

Differentiability and Continuity

1. differentiable$\Longrightarrow$continuous
2. if $f(x)$ is continuous, it may not be differentiable (like corner or cusp). A function can be everywhere continuous but nowhere differentiable.
3. not continuous$\Longrightarrow$not differentiable

One-sided derivative

1. right-hand derivative

$$f'(c^+) = \lim_{h\to 0^+}\frac{f(c+h)-f(c)}{h}$$

2. left-hand derivative

$$f'(c^-) = \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h}$$

* The derivative of $f(x)$ at $x = c$ exists only when the right-hand derivative and the left-hand derivative both exist and are equal. Similar to the right-hand limit and left-hand limit.
* If the derivative of continuous function $f(x)$ at $x = c$ fails to exist, then there are following possibilities.
1. one-sided derivatives exist but are not equal. (corner)
2. Both of the one-sided derivatives do not exist, and the limits approach infinity when $x \to c$. (cusp or functions with a vertical tangent line)
* $f(x)$ is differentiable on $[a, b]$ if
1. in (a, b), all the derivatives exist.
2. The right-hand derivative of $f'(a)$ exists, and the left-hand derivative of $f'(b)$ exists.
* Note that on the boundary points, only one-sided derivative exists.

Vertical Tangent

$f(x)$ has a vertical tangent at c if

$$\lim_{x \to c} |f'(x)| = \infty$$

* A vertical cusp
* If $f(x)$ is continuous at $x = x_0$ but not differentiable, and $f'(x_0) \neq \infty$, then it does not have tangent lines at $x = x_0$. That is to say, $f(x)$ either has no tangent line or has vertical tangent.
* f has a horizontal tangent line at $x = x_0$ if $f'(x_0) = 0$.

Lecture 7

* differentiable on an open interval I
* differentiable
* differentiable on a closed interval $[a, b]$
* common notations: $f'(x), y', \frac{dy}{dx}, \frac{df}{dx}$

Second derivative

Second derivative is the derivative of $f'(x)$.
* Notation: $f''(x), y'', \frac{d^2y}{dx^2}, \frac{d^2f}{dx^2}$
* n th derivative: $f^{(n)}$

Definition of continuously differentiable

f is continuously differentiable on (a, b): f' is continuous on (a, b).
* Notation: $f \in C^1(a, b)$
* $f \in C^k$ means $f', f'', ..., f^{(k)}$ are continuous.
* smooth: $f \in C^m$, m ranges from 2 to infinity.

Linear approximation & linearization

The linear approximation of f at a:

$$f(x) \approx f(a) + f'(a)(x - a)$$

Linearization of f at a

$$L(x) = f(a) + f'(a)(x - a)$$

Differentiability

Suppose $f(x)$ is defined for $a \leq x \leq b$, then f is differentiable at $c \in (a, b)$ if and only if there exists a constant A and a function $\varepsilon(h)$ such that

$$f(c+h) = f(c) + Ah + \epsilon(h), \qquad \text{where} \quad \lim_{h\to 0} \frac{\varepsilon(h)}{h} = 0$$

* The rate of change of $\epsilon(h)$ must be faster than the rate of change of h.
* A is equal to $f'(c)$.
* One interpretation of the definition is: first, find $f'(c)$. If we can prove that

$$\lim_{h\to 0} \frac{\epsilon(h)}{h} = 0$$

where $\epsilon(h) = f(c+h) - f(c) - hf'(c)$, then $f(x)$ is differentiable. Otherwise, $f(x)$ is not differentiable.
* Another interpretation is : for every $A \in \mathbb{R}$, none can satisfy $lim = 0$.

Laws of derivative computation

Suppose $f(x)$ and $g(x)$ are differentiable.
1. $\frac{d}{dx}(c) = 0$ where c is a constant
2. $\frac{d}{dx}(x^r) = rx^{r-1}$ where r is any real number
3. $\frac{d}{dx}(cf) = c\frac{d}{dx}f$ where c is a constant

4. $\frac{d}{dx}(f \pm g) = \frac{d}{dx}(f) \pm \frac{d}{dx}(g)$
* applicable to the sum or the difference of finite number of functions
5. $\frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx}$
6. $\frac{d}{dx}\frac{f}{g} = \frac{\frac{df}{dx} \cdot g - f \cdot \frac{dg}{dx}}{g^2}$
* If $f(x) = 1$, then $[\frac{1}{g(x)}]' = -\frac{g'}{g^2}$

Lecture 8

Derivative of common functions

$\frac{d}{dx}c = 0$	$\frac{d}{dx}x^n = nx^{n-1}$	
$\frac{d}{dx}e^x = e^x$	$\frac{d}{dx}a^x = a^x \ln a$	
$\frac{d}{dx}\ln\lvert x\rvert = \frac{1}{x}$	$\frac{d}{dx}\log_a x = \frac{1}{x \ln a}$	
$\frac{d}{dx}\sin x = \cos x$	$\frac{d}{dx}\cos x = -\sin x$	$\frac{d}{dx}\tan x = \sec^2 x$
$\frac{d}{dx}\csc x = -\csc x \cot x$	$\frac{d}{dx}\sec x = \sec x \tan x$	$\frac{d}{dx}\cot x = -\csc^2 x$
$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}$
$\frac{d}{dx}\csc^{-1} x = -\frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$	$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}$

The Chain Rule

$$y = f(g(x))$$
$$u = g(x)$$
$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

* When we are familiar with the chain rule, we need not explicitly write down u.
* Note that the formulas of common derivatives only work for simple variable x, for expressions like $2x, 3x+1$,we need to apply the chain rule to derive its derivative rather than simply substitute the x in the formula with $2x, 3x+1$. The factor multiplied x matters.
eg. $\frac{d}{dx}(sin^2(\sqrt{2x^2+1}))$ (L8 p8)

Explicit and implicit function

explicit function: $y = f(x)$
implicit function: $F(x, y) = 0$