Mid 2 Review Part II

Wu Zheng

wuzheng0214@sjtu.edu.cn

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Overview

Reference: VV255 Lecture Slides by Professor Jing, Professor Olga, VV255 TA Group18SU, Demidovich, and Stewart's Textbook

1. Tips for Mid2
2. Change of Variables
3. Integrable
4. Double Integral

Typical Questions

1. Calculate an integral.
2. Change from double integral to iterated representation to calculate the integral.
3. Change the order of the integrals.
4. Tell the region bounded by the provided curves. Integrate over the region to find region area/surface area/volume.
5. Physical Applications: Calculate total mass, centroid, moment of inertia.
6. ★★★★
   Most important tool: calculate a volume or area bounded by given functions using change of coordinates. Introduce intermediate variables to calculate an integral (with Jacobian).

Jacobian: Double case

Definition

The Jacobian of the coordinate transformation $x = x(u, v)$, $y = x(u, v)$ is

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

It gives how much the transformation is expanding or contracting an infinitesimal area at a point in uv-plane as the point is transformed into xy-plane.

Theorem

If $f(x, y)$, and $x(u, v)$ and $y(u, v)$ have continuous partial derivatives and $J(u, v)$ is zero only at isolated points, if at all, then

$$\iint_R f(x,y)\,dA = \iint_S f(g(u,v), h(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

- The ABSOLUTE VALUE of the Jacobian severs to correct the distortion.

- Similar procedures can be applied to substitutions in triple integrals.

Definition

For an one-to-one transformation that maps a region in $\mathbb{R}^3$ onto a region in $\mathbb{R}^3$,

$$x = g(u, v, w) \quad y = h(u, v, w) \quad z = k(u, v, w)$$

the Jacobian is

$$J(u, v, w) = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix}$$

This determinant measures how much the volume near a point is being expanded or contracted by the transformation from (u, y, w) to (x, y, z) coordinates.

Change of Variables: Useful Lemma

- For cylindrical coordinates r, θ, and z,

$$\iiint\limits_{E} F(x,y,z)\,dV = \iiint\limits_{G} H(r,\theta,z)\,|r|\,dr\,d\theta\,dz$$

- For spherical coordinates, ρ, θ, and ϕ,

$$\iiint\limits_{E} F(x,y,z)\,dV = \iiint\limits_{G} H(\rho,\theta,\phi)\,|\rho^2\sin\phi|\,d\rho\,d\theta\,d\phi$$

Integrable

Notations. Regions

Definition. Closed Rectangle

$$[a_1, b_1] \times \cdots \times [a_n, b_n] = \{(x_1, \ldots, x_n) \mid a_1 \leq x_1 \leq b_1, \ldots, a_n \leq x_n \leq b_n\}$$

is called a closed rectangle in $\mathbb{R}^n$. We also write: $R \subseteq \mathbb{R}^n$ is a closed rectangle.

Definition. n-dimensional volume of R

Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a closed rectangle in $\mathbb{R}^n$. The n-dimensional volume of R, $V(R)$, is defined by

$$V(R) = \prod_{k=1}^{n} (b_k - a_k)$$

Integrable

Notations. Darboux Integrals

Definition. upper and lower Darboux integral

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle.
$f\colon R \longrightarrow \mathbb{R}$ be a bounded function.
The upper Darboux integral of f over R

$$\overline{\int_R} f = \inf\{U(f, P) \mid P \text{ is a partition of } R\}$$

The lower Darboux integral of f over R

$$\underline{\int_R} f = \sup\{L(f, P) \mid P \text{ is a partition of } R\}$$

Integrable

Sufficient and Necessary Condition, Darboux's Definition

Definition. Darboux

We say that f is Darboux integrable or just integrable over R if $\overline{\int_R} f = \underline{\int_R} f$, and if this is the case then we use

$$\int_R f \ or \ \int_R f \ dV$$

- Having continuity is sufficient for a functions f to be integrable, but it is not a necessary condition.

Theorem. Integrable for Double

If $z = f(x, y)$ is continuous in $\mathcal{D}$, except on a finite number of smooth curves on which $f(x, y)$ is bounded, then f is integrable over $\mathcal{D}$, where $\mathcal{D}$ is some union of type I-II regions.

Darboux Integrable: Simple Example

Example

Show a function f is Darboux-integrable over a region D.

Sol

1. Choose the proper partitions to divide the region into several parts.
2. Calculate the upper and lower Darboux sum with respect to the partition.
3. Show that

$$U_{f,P_n} - L_{f,P_n} < \epsilon$$

Double Integrals

Definition. Net signed volume

If region $D \subseteq \mathbb{R}^2$ and $f(x, y)$ is an integrable function of two variables, then

$$\iint_{\mathcal{D}} f(x, y)dA$$

gives the difference between the volume above the xy-plane and the volume below.

- A positive value for the double integral of f over $\mathcal{D}$ means that there is more volume above $\mathcal{D}$ than below, and vice versa.

Iterated Integral: Fubini's Theorem

Fubini's Theorem

Let $R = [a, b] \times [c, d]$

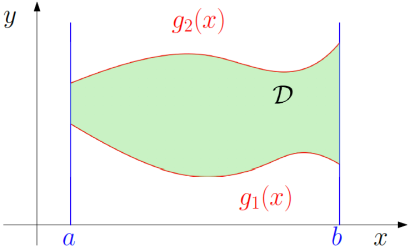
If $f(x, y)$ is continuous on this rectangle, then

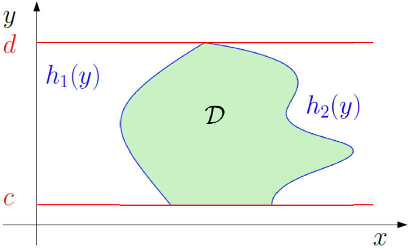
$$\iint_{\mathcal{R}} f(x,y)\, dA = \int_c^d \int_a^b f(x,y)\, dx\, dy = \int_a^b \int_c^d f(x,y)\, dy\, dx$$

These integrals are called Iterated Integrals.

Two types of regions for Double

A type I region is bounded on the left and right by vertical lines $x = a$ and $x = b$ and is bounded below and above by continuous curves $y = g_1(x)$ and $y = g_2(x)$, where $g_1(x) \leq g_2(x)$ for $a \leq x \leq b$.





A type II region is bounded below and above by horizontal lines $y = c$ and $y = d$ and is bounded on the left and right by continuous curves $x = h_1(y)$, $x = h_2(y)$ satisfying $h_1(y) \leq h_2(y)$ for $c \leq y \leq d$.

Theorem

Assume f integrable on region R

$$\textit{Type}-\textit{I}\ \textit{Region}\quad \iint_{\mathcal{R}} f dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$

$$\textit{Type}-\textit{II}\ \textit{Region}\quad \iint_{\mathcal{R}} f dA = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$$

Double Integral: Properties

Assume that all of the following integrals exist.

- Let c be a constant, then

$$\iint\limits_{\mathcal{D}} cf(x,y)\,dA = c\iint\limits_{\mathcal{D}} f(x,y)\,dA$$

$$\iint\limits_{\mathcal{D}} [f(x,y)+g(x,y)]\,dA = \iint\limits_{\mathcal{D}} f(x,y)\,dA + \iint\limits_{\mathcal{D}} g(x,y)\,dA$$

- If $f(x,y) \geq g(x,y)$ for all (x,y) in $\mathcal{D}$, then

$$\iint\limits_{\mathcal{D}} f(x,y)\,dA \geq \iint\limits_{\mathcal{D}} g(x,y)\,dA$$

- If $\mathcal{D}$ is partitioned into $\mathcal{D}_1$ and $\mathcal{D}_2$, then

$$\iint_{\mathcal{D}} f(x,y)\, dA = \iint_{\mathcal{D}_1} f(x,y)\, dA + \iint_{\mathcal{D}_2} f(x,y)\, dA$$

- If we integrate the constant function $f(x,y) = h$ over a region $\mathcal{D}$, we have

$$\iint_{\mathcal{D}} h\, dA = h \iint_{\mathcal{D}} 1\, dA = V = h \cdot A$$

where A is the area of the region $\mathcal{D}$.

- If f is bounded, that is, $m \le f(x,y) \le M$ for all (x,y) in $\mathcal{D}$, then

$$mA \le \iint_{\mathcal{D}} f(x,y)\, dA \le MA$$

Surface Area

Definition

Area of a smooth surface with equation $z = f(x, y)$ over a region $\mathcal{D}$,

$$S = \iint\limits_{D} \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Exercises

Iterated

(1 point) Let $f(x,y)$ be continuous. Find a single iterated integral that is equal to

$$\int_0^1 \int_0^{x^2} f(x,y)\,dy\,dx + \int_1^3 \int_0^{\frac{3-x}{2}} f(x,y)\,dy\,dx$$

by changing the order of integration.

Surface Area

(1 point) Find the area of the part of the sphere

$$x^2 + y^2 + z^2 = 4z$$

that lies inside the paraboloid

$$z = x^2 + y^2$$

Surface Area: Alternative form

(1 point) Find the area of the portion of the paraboloid

$$\mathbf{r}(u,v) = \begin{bmatrix} u\cos v \\ u\sin v \\ u^2 \end{bmatrix},$$

for which $1 \le u \le 2$ and $0 \le v \le 2\pi$.

Volume

(1 point) Find the volume of the solid region lying below the surface

$$f(x,y) = \frac{xy}{1 + x^2y^2}$$

and above the plane region bounded by $xy = 1$, $xy = 4$, $x = 1$, and $x = 4$.