Name and ID: ______________________________

1. Planes and Lines

Question1 (1 point)

Basic Parametric: From Known to Unknown

There is a line $\begin{cases} x = x_0 + t\cos\alpha \\ y = y_0 + t\sin\alpha \end{cases}$ (t as parameter). The parameters corresponding to point A, B are t_1, t_2. If point P divide the segment $\overline{AB}$ that $\overline{AP} = \lambda\overline{PB}$, find the parameter of point P.

Solution:

1M we have

$$x_A = x_0 + t_1\cos\alpha$$
$$x_B = X_0 + t_2\cos\alpha$$

since $\overline{AP} = \lambda\overline{PB}$

then $x_P - x_A = \lambda(X_B - X_P)$ that

$$x_A - (x_P) = \lambda(x_B - (x_P))$$

so

$$x_P = \frac{(x_0 + t_1\cos\alpha) + \lambda(x_0 + t_2\cos\alpha)}{1+\lambda}$$
$$= x_0 + \frac{t_1 + \lambda t_2}{1+\lambda}\cos\alpha$$

so the parameter for point P is $\frac{t_1+\lambda t_2}{1+\lambda}$

Question2 **(1 point)**

Basic Parametric: Substitution

Find the maximum and minimum distance between the ellipse $\dfrac{x^2}{25} + \dfrac{y^2}{81} = 1$ and the line $3x + 4y - 64 = 0$

Solution:

1M We transfer the ellipse to parameter function

$\begin{cases} x = 5\cos\theta \\ y = 9\sin\theta \end{cases}$ $(0 \le \theta \le 2\pi)$

then every point on the ellipse can be presented as $P(5\cos\theta, 9\sin\theta)$

so the distance between point P and the line $3x + 4y - 64 = 0$ is

$$d = \frac{|3\times 5\cos\theta + 4\times 9\sin\theta - 64|}{5}$$
$$= \frac{|39\sin(\theta+\phi) - 64|}{5}$$

so when $\sin(\theta+\phi) = -1$, $d_{max} = \frac{103}{5}$; when $\sin(\theta+\phi) = 1$, $d_{min} = 5$

Question3 (1 point)

Basic Parametric: From Known to Unknown

If line $\begin{cases} x = t\cos\theta \\ y = t\sin\theta \end{cases}$ is tangential to a circle $\begin{cases} x = 4 + 2\cos\phi \\ y = 2\sin\phi \end{cases}$, find the slope of the line

Solution:

1M If $\theta = \frac{\pi}{2}$, then the line is $x = 0$, which is clearly not tangential to the circle

Else if $\theta \neq \frac{\pi}{2}$, line: $y = \tan\theta x$

Because circle is $(x-4)^2 + y^2 = 4$, when they are tangential, the distance to the origin d=r(radius).

so

$$\begin{aligned} d &= \frac{|4\tan\theta - 0|}{\sqrt{1+\tan^2\theta}} \\ &= 2 \end{aligned}$$

we get $\tan\theta = \pm\frac{\sqrt{3}}{3}$

so the slope is $\pm\frac{\sqrt{3}}{3}$

2. Exercise 2 & Exercise 3

Question1 (1 point)

If $z = f(t)$, where $t = x - y$, show that $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$

Solution:

1M Take the left-hand side of equation and apply Chain Rule

$$\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = \frac{dz}{dt}\frac{\partial t}{\partial x} + \frac{dz}{dt}\frac{\partial t}{\partial y} = \frac{dz}{dt}\cdot 1 + \frac{dz}{dt}\cdot(-1) = 0$$

which is the right-hand side of the desired equation.

Question2 (1 point)

Errors

Suppose that

$$z = x(e^x + e^{-y})$$

where x and y are found to be 2 and ln2 with maximum possible errors of

$$|\Delta x| = |dx| = 0.1, \textit{and } |\Delta y| = |dy| = 0.02$$

Solution:

1M Using the linear approximation, we have

$$\begin{aligned} dz &= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy \\ &= \left(e^{x} + e^{-y} + xe^{x}\right)dx + \left(-xe^{-y}\right)dy \end{aligned}$$

Thus the maximum possible error is

$$\left|e^{x} + e^{-y} + xe^{x}\right||dx| + \left|-xe^{-y}\right||dy| = \left|e^{2} + \frac{1}{2} + 2e^{2}\right||0.1| + \left|2\frac{1}{2}\right||0.02| = 2.287$$

Question3 (1 point)
Tangent Planes

Find an equation of the tangent plane to the following ellipsoid at $(2, 1, -3)$.

$$\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$$

Solution:

1M Recall your total differential

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

gives the change in z on the tangent plane, so an equation of the tangent plane with the point of tangency being $P(x_0, y_0, z_0)$ is

$$z - z_0 = \left.\frac{\partial z}{\partial x}\right|_P (x - x_0) + \left.\frac{\partial z}{\partial y}\right|_P (y - y_0)$$

It is clear that the ellipsoid is smooth, we expect the equation implicitly defines z as a differentiable function of x and y, and we can differentiate implicitly when $z \neq 0$,

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x^2}{4} + y^2 + \frac{z^2}{9}\right) = \frac{\partial}{\partial x}(3) &\implies \frac{x}{2} + \frac{2z}{9}\frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{9x}{4z} \\ \frac{\partial}{\partial y}\left(\frac{x^2}{4} + y^2 + \frac{z^2}{9}\right) = \frac{\partial}{\partial y}(3) &\implies 2y + \frac{2z}{9}\frac{\partial z}{\partial y} = 0 \implies \frac{\partial z}{\partial y} = -\frac{9y}{z} \end{aligned}$$

Hence an equation for the tangent plane is

$$z + 3 = -\frac{9 \cdot 2}{4 \cdot (-3)}(x - 2) - \frac{9 \cdot 1}{-3}(y - 1) \implies -\frac{3}{2}(x - 2) - 3(y - 1) + (z + 3) = 0$$

Question4 (1 point)
Horribleeee Chain

Solution:

Let $f(x, y)$ be differentiable everywhere. Suppose $f(a, a) = a$, $f_x(a, a) = b$, $f_y(a, a) = c$, where a, b and c are constants. Find $g(a)$ and $h'(a)$ in terms of a, b and c, where

$$g(x) = f\Big(x, f\big(x, f(x, x)\big)\Big) \qquad \text{and} \qquad h(x) = [g(x)]^2$$

Solution:

1M It is clear that

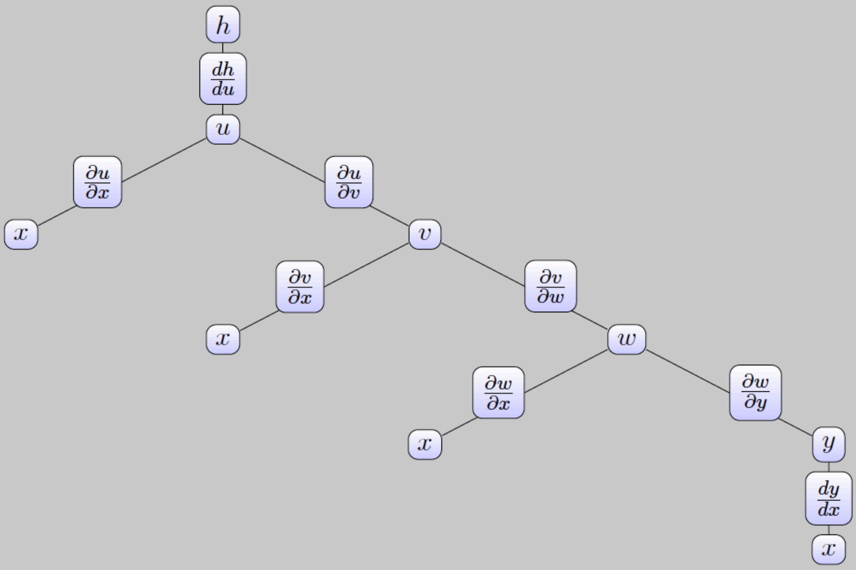
$$g(a) = a$$

In fact, any such composition is always a. We will use this fact in the next part.

1M We rely on the chain rule to find $h'(a)$, let

$$u = g(x) = f(x, v) \quad \text{where} \quad v = f\big(x, f(x, x)\big) = f(x, w) \quad \text{and} \quad w = f(x, x)$$

The tree diagram for the dependency



this suggests the following chain rule

$$\begin{aligned}
h'(x) &= \frac{dh}{dx} \\
&= \frac{dh}{du}\frac{du}{dx} \\
&= \frac{dh}{du}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial v}\frac{dv}{dx}\right) \\
&= \frac{dh}{du}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial w}\frac{dw}{dx}\right)\right) \\
&= \frac{dh}{du}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial w}\left(\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y}\frac{dy}{dx}\right)\right)\right) \\
&= 2g(a)\left(f_x(a, v) + f_y(a, v)\left(f_x(a, w) + f_y(a, w)\left(f_x(a, a) + f_y(a, a)\cdot 1\right)\right)\right) \\
&= 2g(a)\left(f_x(a, a) + f_y(a, a)\left(f_x(a, a) + f_y(a, a)\left(f_x(a, a) + f_y(a, a)\cdot 1\right)\right)\right) \\
&= 2a\left(b + c\left(b + c\left(b + c\right)\right)\right)
\end{aligned}$$

Question5 (1 point)
Basic Evaluation of directional derivatives

Solution:

1M If $\cos\theta \neq 0$

$$\begin{aligned}\frac{df}{de}|_{(0,0)} &= \lim_{t\to 0}\frac{f(\rho\cos\theta,\rho\sin\theta)-f(0,0)}{\rho}\\ &= \lim_{\rho\to 0}\frac{\cos\theta\cdot\sin^2\theta}{\cos^2\theta+\rho^2\sin^4\theta}\\ &= \frac{\sin^2\theta}{\cos\theta}\end{aligned}$$

If $\cos\theta = 0$

$$\frac{df}{de}|_{(0,0)} = 0$$

Question6 (1 point)
The Gradient of the function and the geometric meaning

Solution:
At $(0,0,1)$, we have

$$t = 1$$

and

$$\dot{\mathbf{r}}(1) = \frac{1}{1}\mathbf{e}_x + (1+0)\mathbf{e}_y + \mathbf{e}_z = \mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z$$

Let $F(x,y,z) = xz^2 - yz + \cos xy$, then the surface is a level surface of F,

$$\nabla F(0,0,1) = \Big(z^2 - y\sin(xy)\Big)\mathbf{e}_x + \Big(-z - x\sin(xy)\Big)\mathbf{e}_y + (2xz - y)\mathbf{e}_z = \mathbf{e}_x - \mathbf{e}_y$$

which is a vector normal to the surface at $(0,0,1)$. Since

$$\dot{\mathbf{r}}(1)\cdot\nabla F(0,0,1) = 0$$

thus the curve is tangent to the surface at $(0,0,1)$.

Question7 (1 point)
Directional Derivatives and the gradient

Solution:

Suppose that the directional derivatives of $f(x, y)$ are known at a given point in two non-parallel directions given by unit vectors $\mathbf{u}$ and $\mathbf{v}$. Is it possible to find ∇f at this point? If so, how would you do it?

Solution:

Suppose $\mathbf{u} =< ab >$ and $\mathbf{v} =< cd >$.
Then $\frac{\partial f}{\partial x}a + \frac{\partial f}{\partial y}b$ and $\frac{\partial f}{\partial x}c + \frac{\partial f}{\partial y}d$ are known.
Since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, there is a unique solution for $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$.
Therefore, it is possible to find ∇f at this point.

Question8 (1 point)
The implication of the gradient

(1 point) Is there a direction $\mathbf{v}$ in which the rate of change of

$$F(x, y) = x^2 - 3xy + 4y^2$$

at $(1, 2)$ equals 14? Justify your answer.

Solution:

1M The gradient is given by

$$\nabla F = (2x - 3y)\mathbf{e}_x + (8y - 3x)\mathbf{e}_y$$

At $(1, 2)$, we have

$$\nabla F = -4\mathbf{e}_x + 13\mathbf{e}_y$$

which means the maximum rate of increase can only be

$$|\nabla F| = \sqrt{16 + 169} = \sqrt{185} < 14$$

So no direction at $(1, 2)$ gives the rate of change of F equal to 14.

Question9 (1 point)
Total derivatives and the gradient

(1 point) Express $\dfrac{dw}{dt}$ as a dot product, where

$$w = Q(x, y, z),\ x = f(t),\ y = g(t), \text{ and } z = h(t)$$

are differentiable functions.

Solution:

1M If all the functions are differentiable, then the chain rule gives

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = \nabla f \cdot \dot{\mathbf{r}}$$

where $\mathbf{r}(t) = f(t)\mathbf{e}_x + g(t)\mathbf{e}_y + h(t)\mathbf{e}_z$. So total derivatives as well as directional derivatives are dot products between the gradient vector and the tangent vector.