Name and ID: ______________________________

1. Psychological construction.

Question1 (0 points)
How could I start the project?

Solution:

This term we designed to use Piazza for the project related issue discussion and the class discussion.
Find our class page at https://piazza.com/sjtu.org/summer2019/umjivv255/home
Rather than emailing questions to the teaching staff, we encourage and require you to post your questions on Piazza. Questions posted on Piazza benefit everyone. Students like knowing that others have the same question, and when it's answered on Piazza, it's answered for everybody. If you feel shy to ask questions in public, post them anonymously.

Question2 (0 points)
Have you mastered all the conceptions related to the integrals?

Solution:

Before get started for the technique part of the into integration, always ask yourself that have you understand all the conceptions related to double integration, especially the definition introduce by Darbox and the Riemann, the properties and the similarity compared to the integration of 1D.
Only then you will be well equipped to become a master of double integration and don't get distracted.
Never start applying and investigating the specific technique before you get understood the theoretical part!

Question3 (0 points)
How shall I deal with the tedious proof provided in the slides?

Solution:

Whenever you meet the similar problem, always bear it in mind that the proof provided serves always as the <reference> for your improvement of understanding the specific technique.
So don' panic even if you are confused about some specific points of the proof.

Question4 (0 points)
What could I do if I am struggling to finish the recent assignment?

Solution:

Actually, some of the problems in the recent assignment is far beyond the requirement of the course. And it is the exact reason why we have prepared the expected result for you to check the validity of your algorithm. Besides, we encourage you to work out the elegant solution to deal with those tedious problems!
The potential **BONUS POINTS** will be given to those who figure out the impressive approach to the tedious problems.

2. Integration by I don't know why but anyway useful.

Question1 (1 point)
Prove: If

$$(a-c)^2+b^2\neq 0$$

then

$$\int \frac{a_1\sin x+b_1\cos x}{a\sin^2 x+2b\sin x\cos x+c\cos^2 x}dx = A\int\frac{du_1}{k_1u_1^2+\lambda_1}+B\int\frac{du_2}{k_2u_2^2+\lambda_2}$$

where λ_1, λ_2 is the root of the equation

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0, (\lambda_1\neq\lambda_2)$$

and

$$u_i=(a-\lambda_i)\sin x+b\cos x,\;\; k_i=\frac{1}{a-\lambda_i}\;\;(i=1,2)$$

A and B are constants to be determined.
Hint: Try first to reform the denominator according to the given determinate and then the numerator.

Solution:

Notice the target of our expression goes that

$$A\int\frac{du_1}{k_1u_1^2+\lambda_1}+B\int\frac{du_2}{k_2u_2^2+\lambda_2}$$

If we can express the numerator and the denominator, respectively as

$$(a_1\sin x+b_1\cos x)dx=Adu_1+Bdu_2$$

$$a^2\sin^2 x+2b\sin x\cos x+c\cos^2 x=k_iu_i^2+\lambda_i$$

For the fact that u_i is the linear combination of $\sin x$ & $\cos x$, the differential du_i is also the linear combination of them.
First we can investigate the denominator:

$$a^2\sin^2 x+2b\sin x\cos x+c\cos^2 x$$

which could be simplified as

$$\begin{aligned} LHS &= (a-\lambda_i)\sin^2 x+2b\sin x\cos x+(c-\lambda_i)\cos^2 x+\lambda_i \\ &= \frac{1}{a-\lambda_i}((a-\lambda_i)^2\sin^2 x+2b\cdot(a-\lambda_i)\sin x\cos x \\ &+ (c-\lambda_i)\cdot(a-\lambda_i)\cos^2 x)+\lambda_i \\ &= k_iu_i^2+\lambda_i \end{aligned}$$

We want to obtain the perfect square here, so we need the constraint

$$(c-\lambda_i)\cdot(a-\lambda_i)=b^2$$

i.e.

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = 0$$

Notice the Δ for the equation of λ is

$$(a-c)^2+4b^2\geq(a-c)^2+b^2\neq 0\Rightarrow\lambda_1\neq\lambda_2$$

Also, notice that to ensure the existence of k_i, we here need to assume

$$b\neq 0$$

To be noticed, in the case I will leave the case when b=0 for you to fill in the blank. Then we can derive

$$u_1 = (a-\lambda_1)\sin x + b\cos x,\ \ k_1 = \frac{1}{a-\lambda_1}$$
$$u_2 = (a-\lambda_2)\sin x + b\cos x,\ \ k_2 = \frac{1}{a-\lambda_2}$$

Also the differential could be derived as

$$du_1 = ((a-\lambda_1)\cos x - b\sin x)dx$$
$$du_2 = ((a-\lambda_2)\cos x - b\sin x)dx$$

And we need to equalize the coefficient for $\sin x$ & $\cos x$ as:

$$-b(A+B) = a_1$$
$$A(a-\lambda_1) + B(a-\lambda_2) = b_1$$

Applying Cramer's rule we have

$$A = -\frac{a_1(a-\lambda_2)+bb_1}{b(\lambda_1-\lambda_2)}$$
$$B = \frac{a_1(a-\lambda_1)+bb_1}{b(\lambda_1-\lambda_2)}$$

Question2 (1 point)
Change variables and find the area of regions bounded by the following curves:

$$\left(x^3+y^3\right)^2 = x^2+y^2, \quad x \geq 0, \quad y \geq 0$$

(Hint: Apply polar transformation and utilize the conclusion in question 1)

Solution:

3. Double Integral

Question1 (1 point)
Let $R: 0 \leq x \leq t, 0 \leq y \leq t$ and

$$F(t) = \iint_R e^{-\frac{tx}{y^2}}\,dxdy$$

Compute $F'(t)$
Hint: Try to write down $F'(t)$ in terms of $F(t)$ And the transformation T:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} tu \\ tv \end{pmatrix}$$

Solution:

Question2 (1 point)

Use Generalized Polarize, compute

$$\left(\frac{x}{a}+\frac{y}{b}\right)^4=\frac{x^2}{h^2}+\frac{y^2}{k^2}\ (x>0,y>0)$$

Solution:

Question3 (1 point)

Change variables to find the area of regions bounded by the following curves:

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=1,\quad \sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=2,\quad \frac{x}{a}=\frac{y}{b},\quad 4\frac{x}{a}=\frac{y}{b}\quad (a>0,b>0)$$

Hint: consider

$$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}=u,\quad \frac{x}{y}=v$$

Solution:

Apply the transformation, and then we have

$$x=\frac{u^2v}{\left(\sqrt{\frac{v}{a}}+\frac{1}{\sqrt{b}}\right)^2},\quad y=\frac{u^3}{\left(\sqrt{\frac{v}{a}}+\frac{1}{\sqrt{b}}\right)^2}$$

$$1\leqslant u\leqslant 2,\ \frac{a}{4b}\leqslant v\leqslant\frac{a}{b}$$

and

$$|J|=\frac{2u^3}{\left(\sqrt{\frac{v}{a}}+\frac{1}{\sqrt{b}}\right)^4}$$

$$S=\int_1^2 2u^3du\int_{\frac{a}{4b}}^{\frac{a}{b}}\frac{dv}{\left(\sqrt{\frac{v}{a}}+\frac{1}{\sqrt{b}}\right)^4}$$

$$=\frac{15}{2}\cdot\int_{\frac{1}{2\sqrt{b}}}^{\frac{1}{\sqrt{b}}}\frac{2atdt}{\left(t+\frac{1}{\sqrt{b}}\right)^4}\ ,v=at^2$$

$$=15a\cdot\left(\frac{7b}{72}-\frac{37b}{648}\right)=\frac{65ab}{108}$$