Vv156 Lecture 26

Jing Liu

UM-SJTU Joint Institute

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- It is clear that within its interval of convergence a power series is a continuous function with derivatives of all orders.

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \implies f'(x) = \sum_{n=1}^{\infty} n c_n (x-a)^{n-1} \implies \cdots$$

- What about the other way around?

If a function $f(x)$ has derivatives of all orders on an interval $\mathcal{I}$, can it be expressed as a power series on $\mathcal{I}$? If it can, what will its coefficients be?

- The last question can be readily answered if we assume that $f(x)$ has a power series representation with a positive radius of convergence, that is,

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \cdots + c_n(x-a)^n + \cdots = f(x)$$

- Repeatedly differentiate term-by-term within the interval of convergence I,

we will find the coefficient c_n.

- For example, if there exists a series such that $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots ,$$
$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + \cdots ,$$
$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \cdots ,$$

- The nth derivative has the following form,

$$f^{(n)}(x) = n!c_n + \text{a sum of terms with } (x-a) \text{ as one of the factors.}$$

- These equations all hold at $x = a$, so evaluating the derivatives at $x = a$,

$$f'(a) = c_1, \qquad f''(a) = 2c_2, \qquad f'''(a) = 2\cdot 3c_3$$

- In general,

$$f^{(n)}(a) = n!c_n \implies c_n = \frac{f^{(n)}(a)}{n!}$$

- This formula gives a unique set of coefficients if there is such a series for f.

Existence $\implies$ Uniqueness

- If f has has a power series representation at $x = a$, then the series must be

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

Q: If we start with an arbitrary function

$$f(x)$$

that is infinitely differentiable on an interval $\mathcal{I}$ centered at $x = a$, will the series on the right always converge to $f(x)$ at each x in the interior of $\mathcal{I}$?

- If a power series representation exists for a function, then this representation must be unique and is given by the above the formula.
- However, there is no guarantee that we have a power series representation for a given function in the first place.
- Since the series on the right might diverge or not converge to the function.

- Nevertheless the last power series is a very important power series for $f(x)$.

Taylor series

Suppose that $f(x)$ is a function with derivatives of all orders throughout some interval I containing a as an interior point, then

- The Taylor series generated by f at $x = a$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots .$$

- The Maclaurin series generated by f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots .$$

which is the Taylor series generated by f at $x = 0$.

Exercise

(a) Find the Taylor series generated by $f(x) = \frac{1}{x}$ centered at $x = 2$.

(b) Where, if anywhere, does the series converge to $f(x)$?

Solution

- The coefficients $c_n = \frac{f^{(n)}(a)}{n!}$ partly define the series, in this case, they are

$$c_0 = \frac{f(2)}{0!} = \frac{1}{2},\ c_1 = \frac{f'(2)}{1!} = \frac{-2^{-2}}{1},\ c_2 = \frac{f''(2)}{2!} = \frac{2!2^{-3}}{2!}, \cdots c_n = \frac{(-1)^n}{2^{n+1}}$$

- It is centered at $x = 2$, so

$$\frac{1}{2} - \frac{(x-2)}{4} + \frac{(x-2)^2}{8} - \cdots + (-1)^n \frac{(x-2)^n}{2^{n+1}} + \cdots$$

- It is a geometric series with $a = \frac{1}{2}$ and $r = \frac{-x}{2} + 1$, which converges to $\frac{1}{x}$

$$|x-2| < 2 \implies \text{for} \quad 0 < x < 4$$

- Recall the linearization of a differentiable function f at a point a is

$$L(x) = f(a) + f'(a)(x-a) = P_1(x)$$

which is a polynomial of degree one.

- We used this linear approximation for a differentiable $f(x)$ at x near a.
- If higher-order derivatives of f exit at a, then it has higher-order polynomial approximations as well, one for each available derivative.

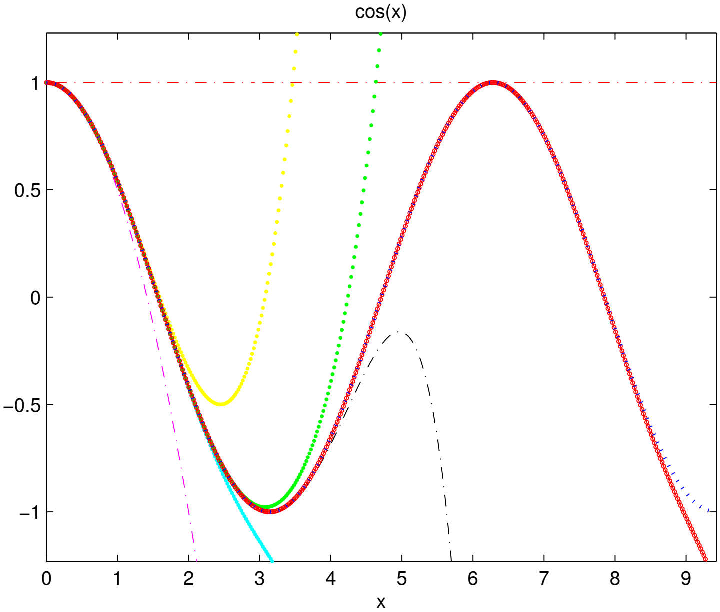
Definition

Suppose $f(x)$ is a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval $\mathcal{I}$ containing a as an interior point, then for any integer n from 0 to N, the Taylor polynomial of degree n generated by $f(x)$ at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Exercise

Find the first two Taylor polynomial of $f(x) = \cos x$ at $x = 0$.


cos(x)
1
0.5
0
−0.5
−1
0
1
2
3
4
5
6
7
8
9
x

Q: Is there always a power series representation of

$$y = f(x)$$

with positive radius of convergence? How about $f \in \mathcal{C}^\infty(a, b)$?

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n$$

Q: Given f has a power series representation, is this representation unique?

Q: Given f has a power series representation, is it unique?

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n \qquad \text{where} \quad x \in (a, b)$$

Q: Given $f \in \mathcal{C}^\infty$, do all Taylor polynomials of f exist?

Q: Given f has a Taylor series expansion at $x = x_0$, is the Taylor series always a power series presentation of f with positive radius of convergence?

Exercise

Find the Maclaurin series generated by

$$f(x) = \begin{cases} 0, & x = 0 \\ e^{-1/x^2}, & x \neq 0 \end{cases}$$

and determine the convergence of it.

Solution

- It can be shown easily that derivatives of all order at $x = 0$ is zero

$$f^{(n)}(0) = 0, \qquad \text{for all } n.$$

- So the Maclaurin series converges for every x but converges to $f(x)$ only for

$$x = 0$$

Q: What do we know and can use when f is many times differentiable?

Taylor's theorem

If f and its first n derivatives, f', f'', $\ldots$, $f^{(n)}$, are continuous on $[a,b]$, and $f^{(n)}$ is differentiable on (a,b), then there exists a number c in (a,b) such that,

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

The Mean-Value theorem

Let f be a function that satisfies the following conditions

1. f is continuous on the closed interval $[a,b]$.
2. f is differentiable on the open interval (a,b).

Then there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

- If we hold a fixed and treat b as an independent variable, then Taylor's formula is easier to use in cases like these and we simply replace b by x.

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \\ &= P_n(x) + R_n(x). \end{aligned}$$

where $R_n(x) = \dfrac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x.

Definition

The function $R_n(x)$ is called the remainder of order n or the error term for the approximation of $f(x)$ by $P_n(x)$ over I.

Definition

$$\text{If} \qquad R_n(x) \to 0 \qquad \text{as} \qquad n \to \infty \qquad \text{for all } x \in I,$$

we say the Taylor series generated by f at $x = a$ converges to f on I, and write,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Exercise

Show that the Taylor series generated by $f(x) = e^x$ at $x = 0$ converges to $f(x)$ for every real value of x.

Solution

- The Taylor series generated by by e^x at $x = 0$ is

$$1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + R_n(x), \qquad \text{where } R_n(x) = \frac{e^c}{(n+1)!}x^{n+1}$$

for some $c \in (0, x)$. We need to show $R_n(x) \to 0$ as $n \to \infty$ for all x

Solution

- Although $R_n(x)$ depends on c, we don't usually need to know the value of c.
- For example, in this case, since e^x is an increasing function of x, so

$$c \text{ is between } 0 \text{ and } x \text{ implies } \quad e^c \text{ lies between } 1 \text{ and } e^x.$$

- When x is zero,

$$R_n(x) = \frac{e^c}{(n+1)!}x^{n+1} = 0$$

- When x is negative, so is c, and $e^c < 1$.

$$|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}, \qquad \text{when} \quad x \leq 0$$

- When x is positive, so is c,

$$|R_n(x)| < e^x \frac{x^{n+1}}{(n+1)!}, \qquad \text{when} \quad x > 0$$

Solution

- If we can show that $\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} = 0$, then $R_n(x) \to 0$ as $n \to \infty$.
- Effectively, we need to show that $\lim_{n\to\infty} \frac{x^n}{n!} = 0$.
- Since $-\frac{|x|^n}{n!} \le \frac{x^n}{n!} \le \frac{|x|^n}{n!}$, all we need to show is $\frac{|x|^n}{n!} \to 0$ as $n \to \infty$.

$$
\begin{aligned}
0 \le \frac{|x|^n}{n!} &= \frac{|x|^n}{1 \cdot 2 \cdot 3 \cdots M \cdot (M+1) \cdots n} \\
&\le \frac{|x|^n}{M! M^{(n-M)}} \\
&= \frac{|x|^n}{M!} \frac{M^M}{M^n} = \frac{M^M}{M!} \frac{|x|^n}{M^n} = \frac{M^M}{M!} \left(\frac{|x|}{M} \right)^n
\end{aligned}
$$

- As $n \to \infty$, M can always be chosen such that $M > |x|$, so $R_n(x) \to 0$. □

Taylor's Inequality

If there is a positive constant M such that

$$|f^{(n+1)}(x)| \leq M \qquad \text{for} \qquad |x-a| \leq d,$$

then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \qquad \text{for} \qquad |x-a| \leq d$$

If this inequality holds for every n and the other conditions of Taylor's theorem are satisfied by $f(x)$, then the series converges to $f(x)$.

Exercise

Show that the Taylor series generated by

$$f(x) = e^x$$

at $x = 0$ converges to $f(x)$ for every real value of x using Taylor's inequality.

Solution

- Since

$$f^{(n)}(x) = e^x \qquad \text{for all } n.$$

- For any positive number d such that $|x| \leq d$, we have

$$\begin{aligned} \left|f^{n+1}(x)\right| &= e^x \leq e^d = M \\ \Longrightarrow |R_n(x)| &\leq \frac{e^d}{(n+1)!} x^{n+1} \quad \text{for } |x| \leq d \text{ and all } n. \end{aligned}$$

Exercise

Find the Taylor series generated by the exponential function at $x = 2$

$$f(x) = e^x$$

Solution

- The Taylor series generated by e^x at $x = 2$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = \sum_{k=0}^{\infty} \frac{e^2}{k!}(x-2)^k$$

- It can be shown that the radius of convergence is ∞, moreover, we can verify

$$\lim_{n\to\infty} R_n(x) = 0$$

- Thus the Taylor series converges to e^x for all x

$$e^x = \sum_{k=0}^{\infty} \frac{e^2}{k!}(x-2)^k$$

- Notice now we have two power series representations converge to e^x, the first is better for x near 0, and the second is better for x near 2.

Exercise

Show that Euler's identity is consistent with Taylor series.

Solution

- We consider $x = \mathrm{i}\theta$ in the Taylor series for e^x.

$$
\begin{aligned}
e^{\mathrm{i}\theta} = e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} &= 1 + \frac{\mathrm{i}\theta}{1!} + \frac{\mathrm{i}^2\theta^2}{2!} + \frac{\mathrm{i}^3\theta^3}{3!} + \frac{\mathrm{i}^4\theta^4}{4!} + \cdots + \frac{\mathrm{i}^k\theta^k}{k!} + \cdots \\
&= 1 + \frac{\mathrm{i}\theta}{1!} + \frac{-\theta^2}{2!} + \frac{-\mathrm{i}\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + \mathrm{i}\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) \\
&= \cos\theta + \mathrm{i}\sin\theta
\end{aligned}
$$