Chapter 7 – Dynamics of Circular Motion
Dynamics in Non-Inertial Frames of Reference

UM-SJTU Joint Institute
Physics I (Summer 2019)
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Agenda

1. Dynamics of Circular Motion
   - Uniform Motion
   - Non-Uniform Motion
2. Dynamics in Non-Inertial Frames of Reference. Forces of Inertia
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   - Equation of Motion in a Non-Inertial FoR
   - Pseudoforces or Forces of Inertia
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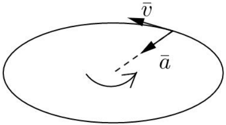
Dynamics of Circular Motion

Dynamics of Uniform Circular Motion



$|\bar{v}| = \text{const, but}$
$\bar{v} \neq \text{const}$

centripetal acceleration $\overset{2^{\text{nd}}\ \text{law}}{\Longleftrightarrow}$ **force**

(towards the center, always)

(called **centripetal force**)

here radial $\parallel$ normal

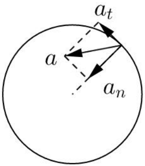
$$\bar{a}_r = -\frac{v^2}{R}\hat{n}_r = -\omega^2 R\hat{n}_r$$

$$\bar{F}_r = m\bar{a}_r = -m\frac{v^2}{R}\hat{u}_r = -m\omega^2 R\hat{u}_r$$

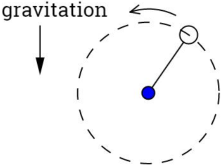
Various forces may play the role of the centripetal force: tension in a cord (contact force); gravitational force; electromagnetic force (field forces),...

Dynamics of Non-Uniform Circular Motion

$|\bar{v}| \neq \text{const} \quad \Rightarrow \quad a_t \neq 0 \quad \Rightarrow \quad$ force in the tangential direction



E.g. "vertical" circular motion

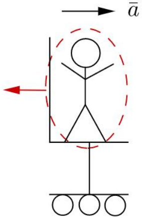


Dynamics in Non-Inertial Frames of Reference. Forces of Inertia

Motivation

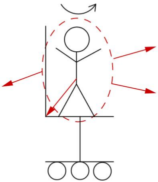
Examples/ demonstrations

(1) accelerating chair: moving along a straight line



feels a "force" pushing him in the direction opposite to the direction of acceleration

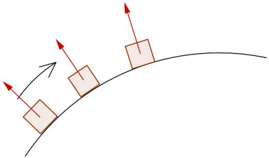
(2) rotating chair



feels a centrifugal "force"

Motivation

(3) chair moving along a curved path



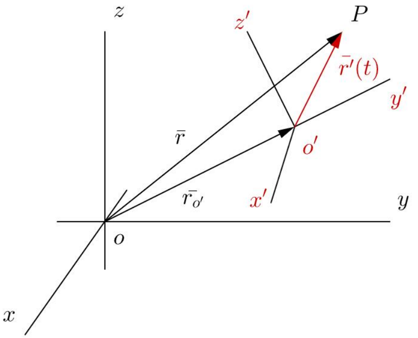
(view from the top)

feels a centrifugal "force"
(directed outwards, along the instantaneous radius of curvature)

These "forces" cannot be regular forces: real forces are always of a material origin and always appear in pairs (Newton's third law).

What is the nature and the origin of these "forces"?

Equation of Motion in a Non-Inertial FoR

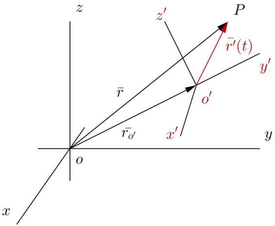


xyz — inertial FoR

$x'y'z'$ — another FoR that moves arbitrarily w.r.t. *xyz* (may be accelerating, rotating with varying angular velocity,. . .)

Previously, we have considered a situation when $x'y'z'$ was moving with a constant velocity w.r.t *xyz* (then $x'y'z'$ is inertial as well). Now, it moves arbitrarily, hence is not inertial.

Goal: Derive a (kinematic) relationship between accelerations of a particle *P* in both frames of reference.



Relationship between position vectors in both FoR

$$\bar{r} = \bar{r}_{O'} + \bar{r'}$$

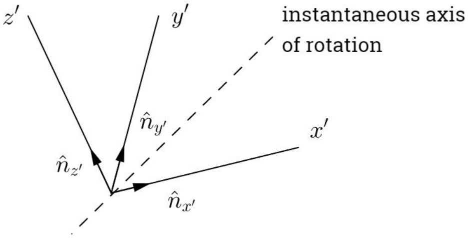
where $\bar{r}$ – position of the particle in xyz (we use fixed unit vectors $\hat{n}_x, \hat{n}_y, \hat{n}_z$ here); $\bar{r}_{O'}$ — position of the origin of $x'y'z'$ as seen from xyz; $\bar{r}'$ — position of the particle in $x'y'z'$ (here, we have to use $\hat{n}_{x'}, \hat{n}_{y'}, \hat{n}_{z'}$, that are not fixed; in particular may rotate)

Complication: Motion of $x'y'z'$ is arbitrary, will need to take into account that $\hat{n}_{x'}, \hat{n}_{y'}, \hat{n}_{z'}$ are not fixed (i.e. will need to know how to calculate their derivatives w.r.t. time).

Relation between velocities (mathematical details skipped here)

$$\boxed{\bar{v} = \bar{v}_{O'} + \bar{v}' + (\bar{\omega} \times \bar{r}')}$$

Comment: The arbitrary motion of $x'y'z'$ can be decomposed into a translational motion and a rotational motion about an instantaneous axis of rotation; the last term is due to the latter.



Eventually, the **relation between accelerations**

$$\boxed{\bar{a} = \bar{a}_{O'} + \bar{a}' + 2\bar{\omega} \times \bar{v}' + \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' + \bar{\omega} \times (\bar{\omega} \times \bar{r}')}$$

Multiply by m and rearrange (leave $m\bar{a}'$ on one side; move all other terms move to the other side)

$$m\bar{a}' = m\bar{a} - m\bar{a}_{O'} - m\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' - 2m(\bar{\omega} \times \bar{v}') - m\bar{\omega} \times (\bar{\omega} \times \bar{r}')$$

But $m\bar{a} = \bar{F}$ because xyz is an inertial FoR (can use 2^{nd} law)

$\downarrow$

real force (i.e. of material origin)

Equation of Motion in a Non-Inertial FoR

$$m\bar{a}' = \bar{F} \underbrace{- m\bar{a}_{o'} - m\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' - 2m(\bar{\omega} \times \bar{v}') - m\bar{\omega} \times (\bar{\omega} \times \bar{r}')}$$

Pseudo forces (also called fictitious forces or forces of inertia) — kinematic corrections (have units of $[N]$) that are due to the fact that we describe dynamics in a non-inertial FoR.

These "forces" must **never** appear in inertial FoRs!

$$\begin{cases} -m\bar{a}_{o'} & \text{d}'\text{Alembert "force"} \\ -m\dfrac{\mathrm{d}\omega}{\mathrm{d}t} \times \bar{r}' & \text{Euler "force"} \\ -2m\bar{\omega} \times \bar{v}' & \text{Coriolis "force"} \\ -m\bar{\omega} \times (\bar{\omega} \times \bar{r}') & \text{centrifugal "force"} \end{cases}$$

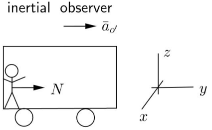
Always proportional to the particle's mass!

Note. If you discuss dynamics in a non-inertial frame of reference and include these "forces" in your free body diagrams, please make sure you remember that these are pseudo forces (and distinguish them from any real forces)

Example I. Accelerating Car Moving along a Straight Line

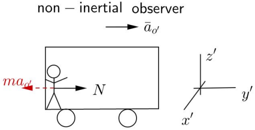
$$\boxed{m\bar{a}' = \bar{F} - m\bar{a}_{o'} - m\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' - 2m(\bar{\omega} \times \bar{v}') - m\bar{\omega} \times (\bar{\omega} \times \bar{r}')}$$

Assume $\bar{a}_{o'} \neq 0$, $\bar{\omega} = 0$, $\bar{v}' = 0$; e.g. acclerating car moving along a straight line.



$$ma_{o'} = N$$

"He moves with acceleration $\bar{a}_{o'}$, because there is a force (normal force $\bar{N}$ due to the wall) acting upon him"



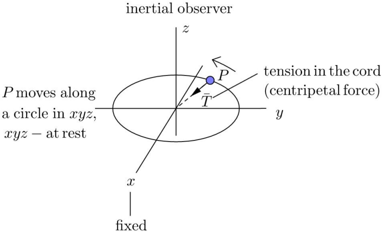
$$0 = N - ma_{o'}$$

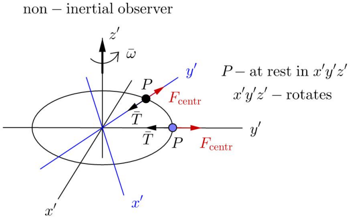
"He is at rest in my FoR because the normal force is balanced by the d'Alembert 'force' "

Mathematically, both equations are **identical**,
but their physical interpretation is **different**!

Example II. Uniform Circular Motion

(2) $\bar{a}_{o'} = 0$; $\bar{\omega} = \mathrm{const}$; $\bar{v'} = 0$; e.g. uniform circular motion





$$m\bar{a} = \bar{T}$$

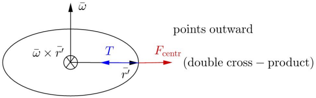
$$0 = \bar{F}_{\mathrm{centr}} + \boxed{\bar{T}}$$

$\downarrow$

cannot call it centripetal

$$-m\omega^2 R\hat{n}_r = \bar{T}$$

"He moves in a circle with acceleration $-\omega^2 R\hat{n}_r$, because there is tension in the cord that plays the



Since $\bar{r'} \perp \bar{\omega}$ and $\bar{\omega} \perp (\bar{\omega} \times \bar{r'})$ then

$$
\begin{aligned}
|\bar{F}_{\mathrm{centr}}| &= |-m\bar{\omega} \times (\bar{\omega} \times \bar{r'})| \\
&= m\omega^2 r' = m\omega^2 R
\end{aligned}
$$

Or use the formula for the double cross product

$$\bar{\omega} \times (\bar{\omega} \times \bar{r'}) = \bar{\omega}(\underbrace{\bar{\omega} \circ \bar{r'}}_{=0}) - \bar{r'}(\underbrace{\omega \circ \bar{\omega}}_{=\omega^2})$$

$$\bar{F}_{\mathrm{centr}} = m\omega^2 \bar{r'} = m\omega^2 R \hat{n}_r$$

So eventually

$$0 = m\omega^2 R\hat{n}_r + \bar{T}$$

"He rests in my frame of reference, because the tension in the cord is balanced by the 'centrifugal force' "

Again, both equations are algebraically identical but they are formulated by the both observers using different language

Earth as a Frame of Reference

[see the blackboard/lecture notes]