1. Angular Momentum

Definition

The angular momentum with respect to the origin

$$\overline{L} = \overline{r} \times \overline{p}$$

Unit: $[kg \cdot \frac{m^2}{s}]$

1.1 Angular Momentum of single particle

$\overline{r} \times \overline{F} = \frac{d\overline{L}}{dt}$ or $\overline{\tau} = \frac{d\overline{L}}{dt}$

1.2 Angular Momentum of Rigid Body

Rotation about a symmetry axis:

$$\overline{L} = I \times \overline{\omega}$$

where I is the moment of inertia around the symmetry axis.
In general,valid for any axis of rotation:

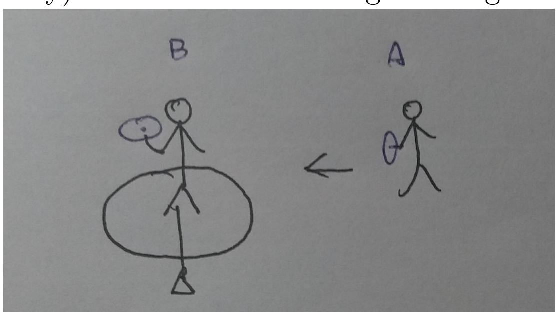
$$\frac{d\overline{L}}{dt} = \overline{\tau}^{ext}$$

1.3 Conservation of Angular Momentum

When the net external torque on a system is zero, then the total angular momentum of the system is conserved. For rotation about an axis of symmetry (about z-axis)

$$\frac{d\overline{L_z}}{dt} = \overline{\tau}^{ext} = \overline{0} \;\Rightarrow\; L_z = I_z\omega_z = const$$

e.g. A person is holding a rotating disk at its horizonal rotational axis at state A. When he jumps onto a big disk which can rotate around its vertial axis(but not rotates initially) and turns the holding rotating disk's axis vertial.



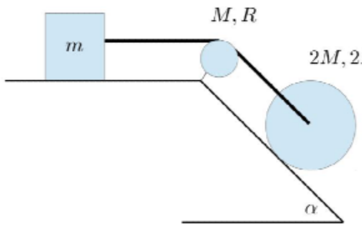
if the hoding disk is rotating clockwise, the person will rotate couterclockwise with the bottom big disk;if the hoding disk is rotating counterclockwise, the person will rotateclockwise with the bottom big disk.

2. Additional Examples of Rigid Body Dynamics

2.1 Inclined Plane

A light inextensible string is attached to an axle going through the center of mass of a ball with radius $2R$ and mass $2M$, placed on a plane inclined at an angle α. The string is parallel to the incline and goes over a cylindrical pulley with radius R and mass M, that can rotate freely about the axis of symmetry. The other end of the string is attached to a block with mass m placed on a smooth horizontal surface. There is no slipping anywhere in the system.

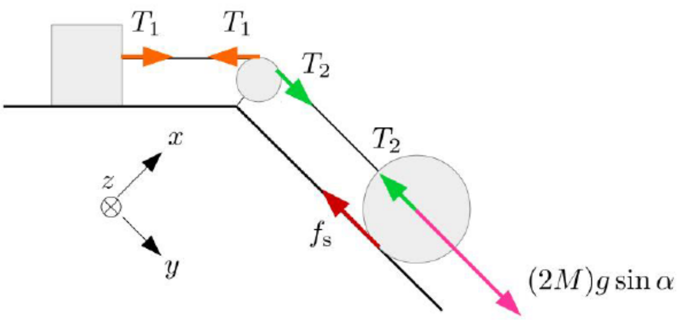
Find the linear acceleration of the center of mass of the ball.



Solution

Step 1: Set up an appropriate coordinate system

Step 2: Draw free body diagram for each object



Step 3: List equations based on

1. translational motion of the c.m.

$$(2M)a_y = (2M)gsin\alpha - f_s - T_2$$

2. rotation about the axis through the c.m.

$$I_A\epsilon_{A,z} = 2Rf_s(ball), I_B\epsilon_{B,z} = (T_2 - T_1)R(pulley)$$

3. 2nd law of dynamics-relate force with acceleration

$$ma_y = T_1$$

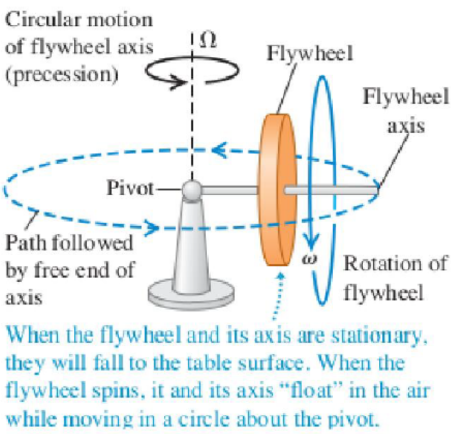
4. ball does not slip while rolling

$$a_y = 2R\epsilon_{A,z}$$

5. string inextensible and does not slip on the pulley

$$a_y = R\epsilon_{B,z}$$

2.2 Gyroscopic effect and Precession



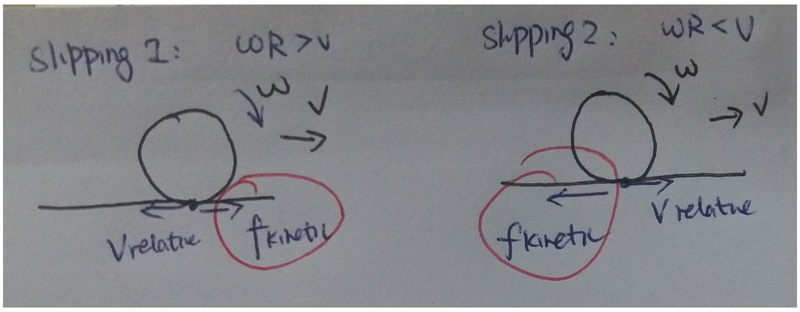
$$\Omega = \frac{d\phi}{dt} = \frac{\frac{|d\mathbf{L}|}{|\mathbf{L}|}}{dt} = \frac{\tau^{\text{ext}}}{L} = \frac{mgr}{I\omega}$$

3. Direction and Type of friction force

3.1 Slipping

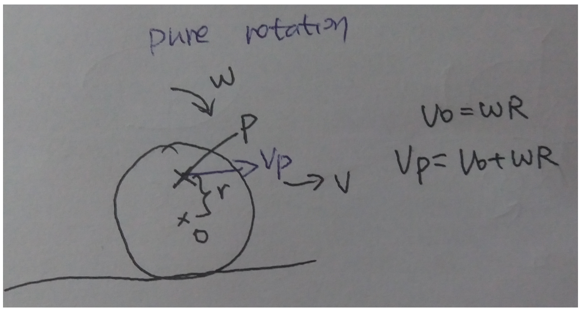
if an object is purely slipping or is slipping combined with rotating, the friction force is kinetic friction force, decided by the relative motion of the point connected with the ground.

1. Slipping 1: rotation is much faster than translational velocity ($\omega R > v$ for wheel)
2. Slipping 2: rotation is much slower than translational velocity ($\omega R < v$ for wheel)



3.2 Pure Rotating

1. if a wheel is pure rotating, the friction force is static friction force, decided by the relative motion trend of the point connected with the ground.
2. for pure rotating, the static friction force doesn't do any work.
3. for pure rotating, it is always true that $\omega R = v$.
4. for the point that is not at the mass center of the wheel P, $v_p = v_0 + \omega R$.

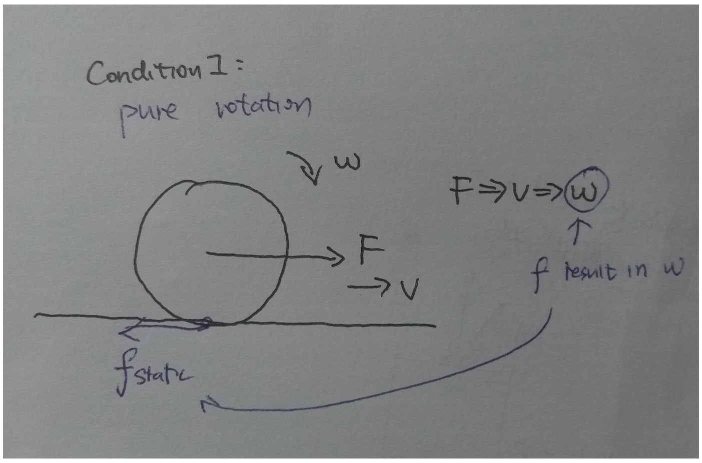


The direction of static friction force

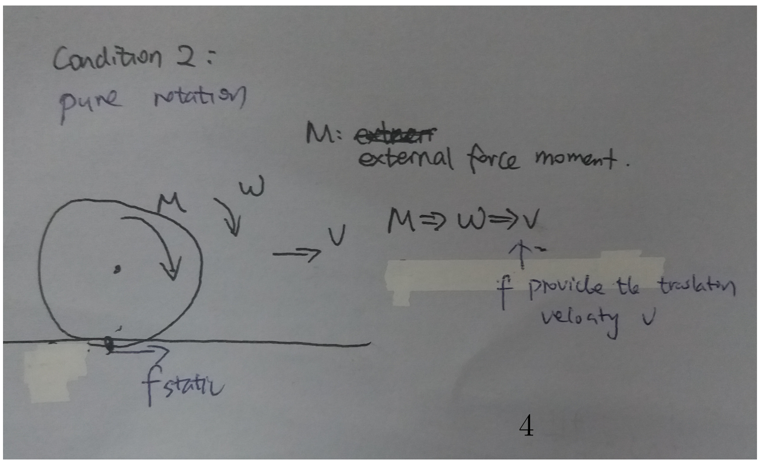
method:

decide the direction of relative motion trend.

1. Condition1: exerted force causes the wheel to pure rotate.



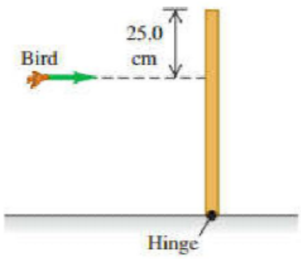
2. Condition2: exerted moment causes the wheel to pure rotate



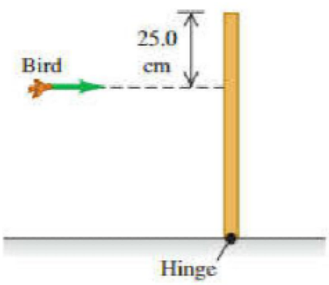
4. Exercise

Ex1.

A 500.0-g bird is flying horizontally at 2.25 m/s, not paying much attention, when it suddenly flies into a stationary vertical bar, hitting it 25.0 cm below the top. The bar is uniform, 0.750 m long, has a mass of 1.50 kg, and is hinged at its base. The collision stuns the bird so that it just drops to the ground afterward (but soon recovers to fly happily away). What is the angular velocity of the bar (a) just after it is hit by the bird and (b) just as it reaches the ground?



Solution



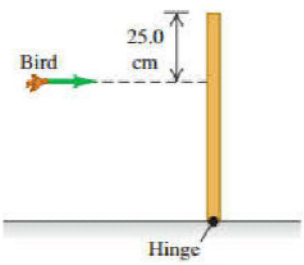
a)

$$I\omega = mvd$$

$$\frac{1}{3}m_{\text{bar}}\omega l^2 = m_{\text{bird}}v \cdot 0.5m$$

$$\Rightarrow \ \omega = 2\text{rad/s}$$

Direction: clockwise



b)

$$\frac{1}{2}I\omega^2 + mgh = \frac{1}{2}I\omega'^2$$

$$\frac{1}{2}\cdot\frac{1}{3}ml^2\omega^2 + mg\cdot\frac{1}{2}l = \frac{1}{2}\cdot\frac{1}{3}ml^2\omega'^2$$

$$\Rightarrow \ \omega' = 6.57\text{rad/s}$$

Direction: clockwise

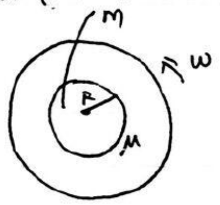
Ex2.

ex1. Put a disk A with radium=R rest onto a constant-speed-rotating disk B,

① Calculate the torque caused by friction force $\vec{\tau}_f$ when the disk A start to rotate.

② Calculate the time needed for the disk A to have the same ω with disk B.

alway know: disk B's angular velocity $=\omega$, friction coefficient μ. A's radium: R
disk A's mass: m

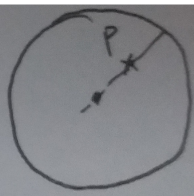


Solution

$$P:\ d\tau_f = r\cdot df = r\cdot\mu g\cdot\rho s\cdot r\cdot d\theta\cdot dr$$

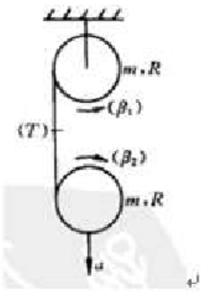
$$\tau = \int_0^{2\pi}\int_0^R d\tau = \frac{2}{3}\mu m g R$$

$$\varepsilon = \frac{\tau}{I} = \frac{4\mu g}{3R} \Rightarrow t = \frac{\omega}{\varepsilon} = \frac{3R\omega}{4\mu g}$$

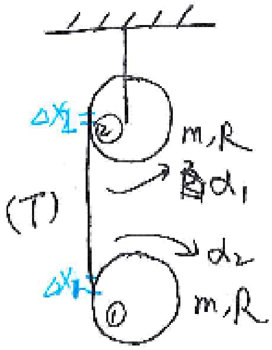


Ex3.

Two frictionless pulleys are connected by a very light, flexible cord. Find the acceleration of the center of the mass (a_{cm}) of the lower pulley when we release the lower pulley



Solution



Recall:

$$\begin{cases} \sum \bar{F} = m \sum \bar{a}_c \\ \bar{\tau} = \bar{r} \times \bar{F} \\ \text{Link:} \end{cases}$$

$$mg - T = m a_c \quad ①$$

$$\begin{cases} T R = \frac{1}{2} m R^2 \cdot \alpha_1 \quad ② \\ T \cdot R = \frac{1}{2} m R^2 \cdot \alpha_2 \quad ③ \end{cases}$$

consider a small time interval Δt after release

$$\begin{cases} \Delta x_1 = \frac{1}{2} \alpha_2 R \Delta t^2 \\ \Delta x_2 = \frac{1}{2} \alpha_1 R \Delta t^2 \\ \frac{1}{2} a_c \Delta t^2 = \Delta x = \Delta x_1 + \Delta x_2 \end{cases}$$

$$\Rightarrow \quad a_c = (\alpha_1 + \alpha_2) R \quad ④$$

$$\Rightarrow a_{cm} = \frac{4}{5} g.$$