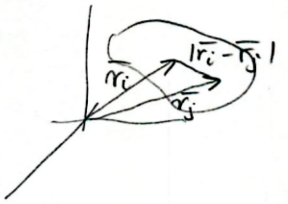
Rigid body mechanics

Rigid body – an object for which the distances between its parts do not change.



$|\bar{r}_i - \bar{r}_j| = \text{const}$ for any $\bar{r}_i, \bar{r}_j$ pointing to points of the object

Examples

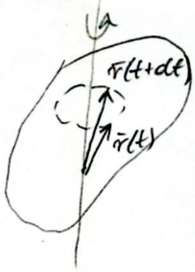
cannot bend, change length

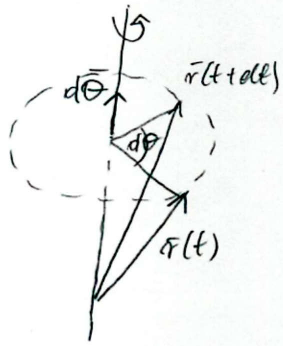
non-deformable material

Angular velocity and angular acceleration (rotation about a fixed axis)

Angular displacement and angular velocity

exaggerated ($d\theta$ - small)



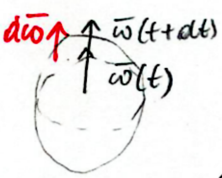


$d\bar{\theta}$ – infinitesimal angular displacement
direction determined by the right hand rule; magnitude: the angle swept by the radius

$\bar{\omega} \overset{def}{=} \frac{d\bar{\theta}}{dt}$ → rate of change of the angular displacement
$[rad/s]$

Consequently $\bar{\varepsilon} = \frac{d\bar{\omega}}{dt}$ $[rad/s^2]$ rate of change of the angular velocity

Direction of $\bar{\varepsilon}$ (fixed axis of rotation, i.e. only the magnitude of $\bar{\omega}$ can change)



$d\bar{\omega} = \bar{\omega}(t+dt) - \bar{\omega}(t)$
points upward
↳ $\bar{\varepsilon} = \frac{d\bar{\omega}}{dt} \parallel \bar{\omega}$
(rotates faster and faster)

$d\bar{\omega}$
$\bar{\varepsilon} = \frac{d\bar{\omega}}{dt}$ } point downward
(rotates slower and slower)

What if axis of rotation is not fixed, i.e. changes its direction?

$d\bar{\omega} \parallel \bar{\varepsilon} \nparallel \bar{\omega}(t)$

$\bar{\varepsilon}$ is not parallel to the axis of rotation

Example: Rotation with constant angular acceleration $\varepsilon = const$
(e.g. about z-axis - fixed) ↳ z - component

Initial conditions: $\omega(0) = \omega_0$
$\theta(0) = \theta_0$

$$\varepsilon = \frac{d\omega}{dt} \Rightarrow d\omega = \varepsilon\, dt \Rightarrow \int_{\omega_0}^{\omega(t)} d\omega = \int_0^t \varepsilon\, dt$$

$$\boxed{\omega(t) = \omega_0 + \varepsilon t}$$

$$\omega = \frac{d\theta}{dt} \Rightarrow d\theta = \omega\, dt \Rightarrow \int_{\theta_0}^{\theta(t)} d\theta = \int_0^t (\omega_0 + \varepsilon t)\, dt$$

$$\boxed{\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\varepsilon t^2}$$

Compare with $a = const$
$$v(t) = v_0 + at$$
$$x(t) = x_0 + v_0 t + \frac{1}{2} a t^2$$

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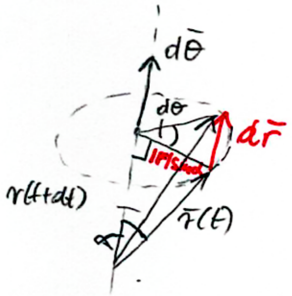
Linear and angular quantities in rotational motion

(fixed axis of rotation)



$|d\bar{r}| = |\bar{r}| \sin\alpha\, d\theta$

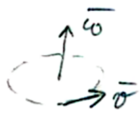
(α - angle between $\bar{r}(t)$ and axis of rotation i.e. $d\bar{\theta}$)

introducing vector $d\bar{\theta}$ as before (right hand rule)
we have

$$d\bar{r} = d\bar{\theta} \times \bar{r}$$

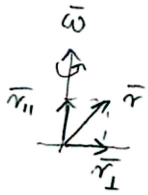
$$\underbrace{\frac{d\bar{r}}{dt}}_{\bar{v}} = \underbrace{\frac{d\bar{\theta}}{dt}}_{\bar{\omega}} \times \bar{r}$$

$$\boxed{\bar{v} = \bar{\omega} \times \bar{r}}$$



Note: decompose $\bar{r}$ as

$$\bar{r} = \bar{r}_{\parallel} + \bar{r}_{\perp}$$

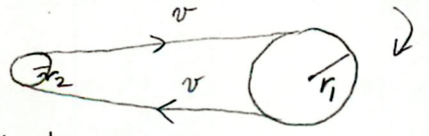


so

$$\bar{v} = \bar{\omega} \times (\bar{r}_{\parallel} + \bar{r}_{\perp}) = \underbrace{\bar{\omega} \times \bar{r}_{\parallel}}_{=0} + \bar{\omega} \times \bar{r}_{\perp} = \bar{\omega} \times \bar{r}_{\perp}$$

Magnitude: $|\bar{v}| = |\bar{\omega} \times \bar{r}_{\perp}| = |\bar{\omega}|\, |\bar{r}_{\perp}|$
↳ distance from the axis of rotation

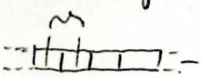
Example: Bicycle gears



$$v = r_1\omega_1 = r_2\omega_2$$

$$\frac{\omega_2}{\omega_1} = \frac{r_1}{r_2}$$ larger sprocket rotates slower

teeth spacing (has to be equal for both sprockets)



\- chain

$$\frac{2\pi r_1}{N_1} = \frac{2\pi r_2}{N_2} \Rightarrow \frac{r_1}{r_2} = \frac{N_1}{N_2}$$

Hence $\frac{\omega_2}{\omega_1} = \frac{N_1}{N_2}$

The rear wheel rotates with the largest angular velocity (given a pedaling rate ω_1) when N_1 - max (largest front sprocket)
N_2 - min (smallest rear sprocket)

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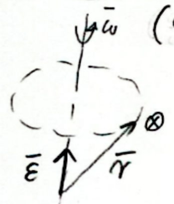
Acceleration (fixed axis of rotation)

$$\bar{v} = \bar{\omega} \times \bar{r}$$

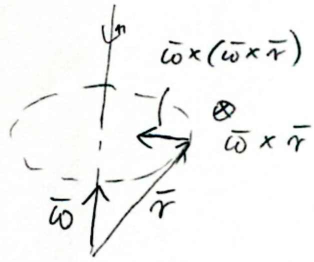
differentiate to get $\bar{a} = \frac{d\bar{v}}{dt} = \frac{d\bar{\omega}}{dt} \times \bar{r} + \bar{\omega} \times \underbrace{\frac{d\bar{r}}{dt}}_{\bar{v}\text{ and use previous result}} = \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})$

$$\boxed{\bar{a} = \bar{\varepsilon} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})}$$

Direction of these terms



$\bar{\varepsilon} \times \bar{r}$ points in the tangential direction (tangential acceleration)



$\bar{\omega} \times (\bar{\omega} \times \bar{r})$ points in the radial direction (centripetal acc.)

$$\bar{a} = \bar{a}_{tan} + \bar{a}_{centripetal}$$

Note: again $\bar{r} = \bar{r}_{\parallel} + \bar{r}_{\perp}$

$$\bar{a} = \bar{\varepsilon} \times (\bar{r}_{\parallel} + \bar{r}_{\perp}) + \bar{\omega} \times (\bar{\omega} \times (\bar{r}_{\parallel} + \bar{r}_{\perp})) = \bar{\varepsilon} \times \bar{r}_{\perp} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{\perp})$$

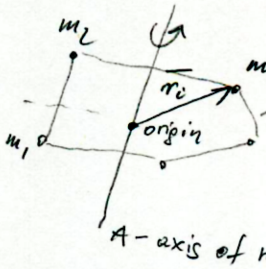
Magnitudes

$$|\bar{a}_{tan}| = |\bar{\varepsilon} \times \bar{r}_{\perp}| = |\bar{\varepsilon}|\,|\bar{r}_{\perp}|$$

$$|\bar{a}_{centripetal}| = |\bar{\omega} \times (\bar{\omega} \times \bar{r}_{\perp})| = |\bar{\omega}| \cdot |\bar{\omega} \times \bar{r}_{\perp}| = \omega^2 |\bar{r}_{\perp}|$$

Kinetic energy of a rotating rigid body (fixed axis)

Discrete distribution of mass



– massless rods

$i = 1, 2, \dots, N$

A – axis of rotation

Total kinetic energy

$$K = \sum_{i=1}^{N} K_i = \sum_{i=1}^{N} \frac{1}{2} m_i v_i^2$$

But $\bar{v}_i = \bar{\omega} \times \bar{r}_i$ and $|\bar{v}_i| = |\bar{\omega}| \, |\bar{r}_{i\perp}|$ – distance from the axis of rotation

↳ same for all m_i's – rigid body

Hence

$$K = \sum_{i=1}^{N} \frac{1}{2} m_i \omega^2 r_{i\perp}^2 = \frac{1}{2} \underbrace{\left(\sum_{i=1}^{N} m_i r_{i\perp}^2 \right)}_{\text{moment of inertia for this rigid body and this particular axis of rotation}} \omega^2$$

$$\boxed{K = \frac{1}{2} I \omega^2} \quad \text{(fixed axis)}$$

↳ moment of inertia about axis of rotation

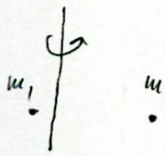
Examples: $I = \sum_{i=1}^{N} m_i r_{i\perp}^2$ – depends on the distribution (arrangement) of mass

(a) ①

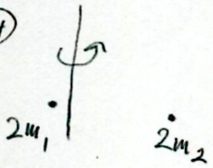
$I_1 > I_2$

②

(b) ③

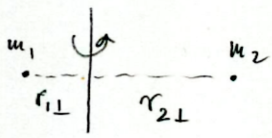


④



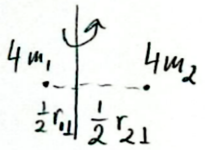
$I_3 < I_4 = 2 I_3$

(c) ⑤



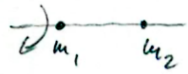
$$I_5 = m_1 r_{1\perp}^2 + m_2 r_{2\perp}^2$$

⑥



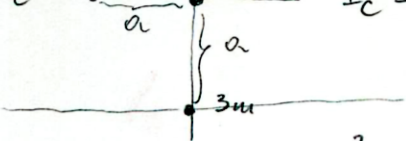
$$I_6 = 4m_1 \left(\tfrac{1}{2} r_{1\perp}\right)^2 + 4m_2 \left(\tfrac{1}{2} r_{2\perp}\right)^2 = I_5$$

(d) ⑦



$I_7 = 0$

(e)



$I_C = 3ma^2$

$I_B = 2ma^2 + ma^2$

$I_A = ma^2$

Moment of inertia about an axis for a continuous distribution of mass

Contribution to the kinetic energy due to the element of mass dm

$$dK = \frac{1}{2}(dm)\,v^2 = \frac{1}{2}\,\omega^2 r_\perp^2\, dm$$

Total kinetic energy (add all contributions)

$$K = \int_{object} dK = \int_{object} \frac{1}{2}\,\omega^2 r_\perp^2\, dm =$$

$$= \frac{1}{2}\underbrace{\left(\int_{object} r_\perp^2\, dm\right)}\omega^2$$

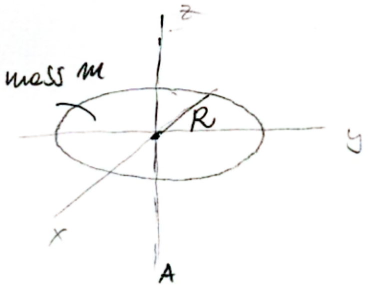
I_A - moment of inertia of the object about the axis A

$$\boxed{I_A = \int_{object} r_\perp^2\, dm}$$

distance from axis A

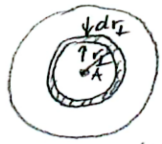
Calculations of moments of inertia about an axis

(a) uniform disk (2D) about the axis $\perp$ to the disk through its center



x, y, z - principal axes

top view



build the disk from concentric rings

$$I_A = \int_{\text{object}} r_\perp^2 \, dm$$

here a double integral

$$dm = 2\pi r_\perp \sigma \, dr_\perp$$

surface density of mass $\sigma = \frac{m}{\pi R^2}$

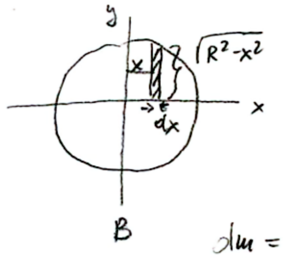
$$dI_A = r_\perp^2 \, dm \quad \Rightarrow \quad I_A = \int_{\text{object}} r_\perp^2 \, dm$$

$$I_A = \int_0^R 2\pi\sigma r_\perp^3 \, dr_\perp = \sigma 2\pi \frac{R^4}{4} = \frac{1}{2}\sigma\pi R^4$$

$$= \frac{1}{2} m R^2$$

$$\boxed{I_A = \frac{1}{2} m R^2}$$

(b) the same disk about the axis contained in the plane of the disk



$$dm = \sigma\sqrt{R^2-x^2}\, dx$$

(I) physicist's method

$$I_B = 4 \cdot \int_{\substack{\text{quarter}\\\text{of disk}}} x^2 \, dm = 4\sigma \int_0^R x^2\sqrt{R^2-x^2}\, dx =$$

$$= \left\{ \begin{aligned} x &= R\sin u \\ dx &= R\cos u \, du \end{aligned} \right\} 4\sigma R^4 \int_0^{\frac{\pi}{2}} \sin^2 u \cos^2 u \, du =$$

$$= 4\sigma R^4 \int_0^{\frac{\pi}{2}} \left(\tfrac{1}{2}\sin 2u\right)^2 du = \sigma R^4 \int_0^{\frac{\pi}{2}} \tfrac{1}{2}[1-\cos 4u]\, du$$

$$= \sigma R^4 \frac{\pi}{4} = \frac{1}{4} m R^2$$

$$\boxed{I_B = \frac{1}{4} m R^2}$$

(II) mathematician's method - iterated integral

$$I_B = 4 \iint_{D_q} x^2 \sigma \, dA = 4\sigma \int_0^R dx \int_0^{\sqrt{R^2-x^2}} x^2 \, dy = 4\sigma \int_0^R x^2\sqrt{R^2-x^2}\, dx$$

D_q: quarter of the disk

dA: element of surface

$\hookrightarrow$ iterated integral

Note: Due to symmetry, moments of inertia about axes y and x are both equal to $\frac{1}{4} mR^2$

Note: $I_z = I_x + I_y$? a general rule for planar objects? (see problem set)

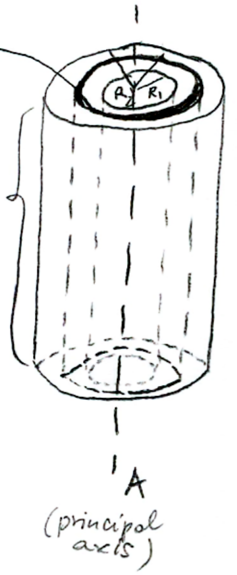
(c) hollow cylinder (uniform, mass $-m$, inner radius $-R_1$, outer radius $-R_2$)

about vertical symmetry axis

bulk density of mass $\rho = \frac{m}{V}$

thin walled cylinder

$dm = 2\pi r_\perp H\, dr_\perp$



$$I_A = \int_{cylinder} r_\perp^2\, dm = \int_{R_1}^{R_2} r_\perp^2 \rho H 2\pi r_\perp\, dr_\perp =$$

$$= \frac{2\pi\rho H}{4}\left(R_2^4 - R_1^4\right)$$

Express in terms of mass and radii

$$m = \rho \cdot V = \rho\pi(R_2^2 - R_1^2)H$$

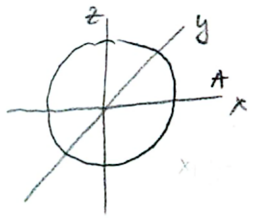
Hence $I_A = \frac{\pi\rho H}{2}\underbrace{(R_2^2 - R_1^2)}_{m}(R_2^2 + R_1^2) = \frac{1}{2}m(R_2^2 + R_1^2)$

$$\boxed{I_A = \frac{1}{2}m\left(R_1^2 + R_2^2\right)}$$

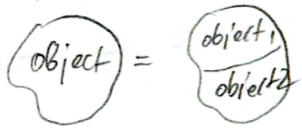
In particular for a full cylinder ($R_1 = 0$) $I_A = \frac{1}{2}mR_2^2$

with the same mass

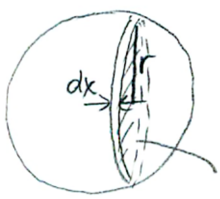
(d) ball with mass $-m$, radius $-R$, about any axis of symmetry

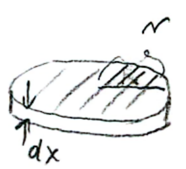


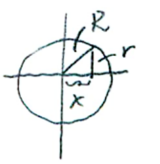
Note that $\int_{object} r_\perp^2\, dm = \int_{object\,1} r_\perp^2\, dm + \int_{object\,2} r_\perp^2\, dm$



Idea: cut the ball into slices (cylinders of infinitesimal height), add the contributions.







$r = \sqrt{R^2 - x^2}$

$dm = \rho\pi r^2\, dx =$

$= \rho\pi(R^2 - x^2)\, dx$

$$dI_A = \frac{1}{2}r^2\, dm$$ contribution of one cylindrical slice

contribution of one half of the ball

$$I_A = \underbrace{2}_{\text{two halves}} \cdot \int_0^R \frac{1}{2}\underbrace{(R^2 - x^2)}_{r^2}\, \underbrace{\rho\pi(R^2 - x^2)\, dx}_{dm} =$$

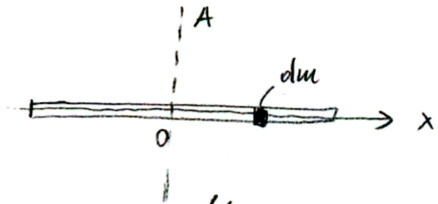
$$= \pi\rho\int_0^R \left(R^4 - 2R^2x^2 + x^4\right) dx =$$

$$= \pi\rho\left(R^5 - \frac{2}{3}R^2R^3 + \frac{1}{5}R^5\right) = \pi\rho\,\frac{15 - 10 + 3}{15}R^5 =$$

$$= \frac{8}{15}R^5\pi\rho \quad \Rightarrow \quad \boxed{I_A = \frac{2}{5}mR^2}$$

(e) uniform slender rod with mass m and length ℓ about perpendicular axis of symmetry

$\lambda = \frac{m}{\ell}$ - linear density of mass

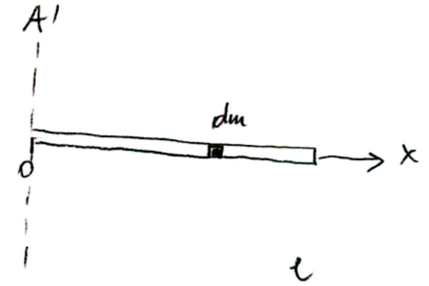


$dm = \lambda\, dx$

$$I_A = \int_{object} r_\perp^2\, dm = \int_{-\ell/2}^{\ell/2} x^2 \lambda\, dx = \frac{1}{3}\lambda \ell^3\left(\frac{1}{8}+\frac{1}{8}\right) = \frac{1}{12}\lambda\ell^3 = \boxed{\frac{1}{12} m\ell^2}$$

sometimes we may need to know the moment of inertia about an axis A' that is not a principal axis, e.g. throug one of ends of the rod

Note. axis A' ∥ axis A



$$I_{A'} = \int_{object} r_\perp^2\, dm = \int_0^\ell x^2 \lambda\, dx = \frac{1}{3}\lambda\ell^3 = \boxed{\frac{1}{3} m \ell^2}$$

Compare: $I_{A'} - I_A = \left(\frac{1}{3} - \frac{1}{12}\right) m\ell^2 = \frac{1}{4} m\ell^2 = m\left(\frac{\ell}{2}\right)^2$

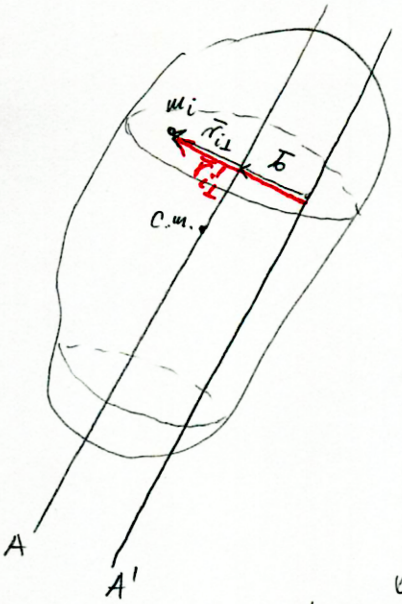
$$I_{A'} = I_A + m\left(\frac{\ell}{2}\right)^2$$

↳ distance between parallel axes A and A'

Is there a universal relation between I_A and $I_{A'}$?

Yes! Steiner's theorem (parallel axis thm.)

Steiner's theorem (parallel axis theorem)

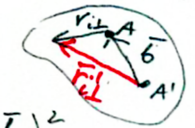


A - axis through the center of mass

A' - any axis parallel to A

For an element of mass m_i

view from the top



$$\bar{r}'_{i\perp} = \bar{r}_{i\perp} + \bar{b}$$

$$\underset{\substack{\downarrow \\ \text{moment of inertia about axis } A'}}{I_{A'}} = \sum_{i=1}^{N} m_i \, r'^{\,2}_{i\perp} = \sum_{i=1}^{N} m_i \, (\bar{r}_{i\perp} + \bar{b})^2 =$$

$$= \sum_{i=1}^{N} m_i \left(r_{i\perp}^2 + 2\,\bar{r}_{i\perp}\cdot\bar{b} + b^2 \right) =$$

$$= \underbrace{\sum_{i=1}^{N} m_i r_{i\perp}^2}_{I_A} + 2\bar{b}\underbrace{\sum_{i=1}^{N} m_i \bar{r}_{i\perp}}_{m\,\bar{r}_{cm\perp}} + b^2 \underbrace{\sum_{i=1}^{N} m_i}_{m}$$

But, since A passes through the c.m., then $|\bar{r}_{cm\perp}| = 0$ (c.m. lies on this axis)

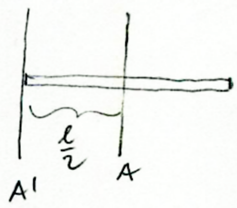
Eventually

$$\boxed{I_{A'} = I_A + m\,b^2}$$

I_A → A - axis through the center of mass

b → distance between axes A, A'

Example (a) slender uniform rod (mass - m, length - ℓ)



$$I_{A'} = I_A + m\left(\frac{\ell}{2}\right)^2 = I_A + \frac{m\ell^2}{4}$$

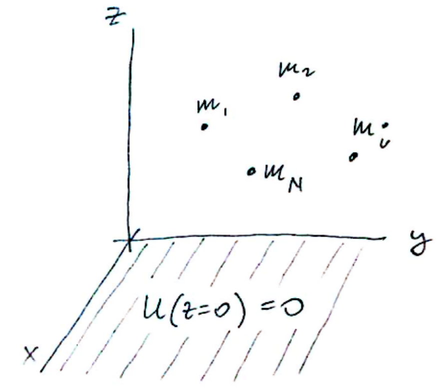
$$I_{A'} = \frac{1}{3} m\ell^2$$

$$I_A = I_{A'} - \frac{m\ell^2}{4} = \frac{1}{3} m\ell^2 - \frac{1}{4} m\ell^2 = \underline{\frac{1}{12} m\ell^2}$$

(b) ball (mass - m, radius - R)

$$I_{A'} = I_A + mR^2 = \frac{2}{5} mR^2 + mR^2 = \frac{7}{5} mR^2$$

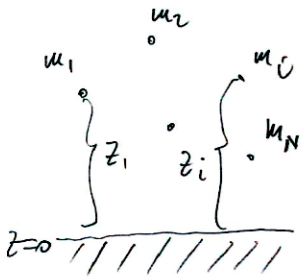
Gravitational potential energy of a rigid body



$$U = \sum_{i=1}^{N} m_i g z_i = g \underbrace{\sum_{i=1}^{N} m_i z_i}_{= z_{cm} \cdot \underbrace{\sum_{i=1}^{N} m_i}_{=M}}$$

Recall: $$\bar{r}_{cm} = \frac{\sum_{i=1}^{N} m_i \bar{r}_i}{\sum_{i=1}^{N} m_i}$$

Hence $$U = M g z_{cm}$$



has the same potential energy as

$$M = \sum_{i=1}^{N} m_i$$

Note. The same of course holds for a continuous distribution of mass.