vv255: Surface Integrals.

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Surfaces Described by Vector Functions

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- If $\bar{r}$ is a paramaterisation of the surface $\mathcal{S}$ with

$$\bar{r}(u, v) = \alpha(u, v)\bar{i} + \beta(u, v)\bar{j} + \gamma(u, v)\bar{k},$$

  then we call the equations

$$x = \alpha(u, v) \qquad y = \beta(u, v) \qquad z = \gamma(u, v)$$

  the parametric equations of $\mathcal{S}$.

Surfaces Described by Vector Functions

Example

Consider the surface described by the vector function:

$$\bar{r}(u, v) = u \cos v \bar{i} + u \sin v \bar{j} + v \bar{k}, \quad 0 \le u \le 2,\ 0 \le v \le 2\pi$$

- *When* $u_0 \in [0, 2]$ *is held constant and* v *is allowed to vary,* $\bar{r}(u_0, v)$ *describes a "helix" with radius* u_0*.*

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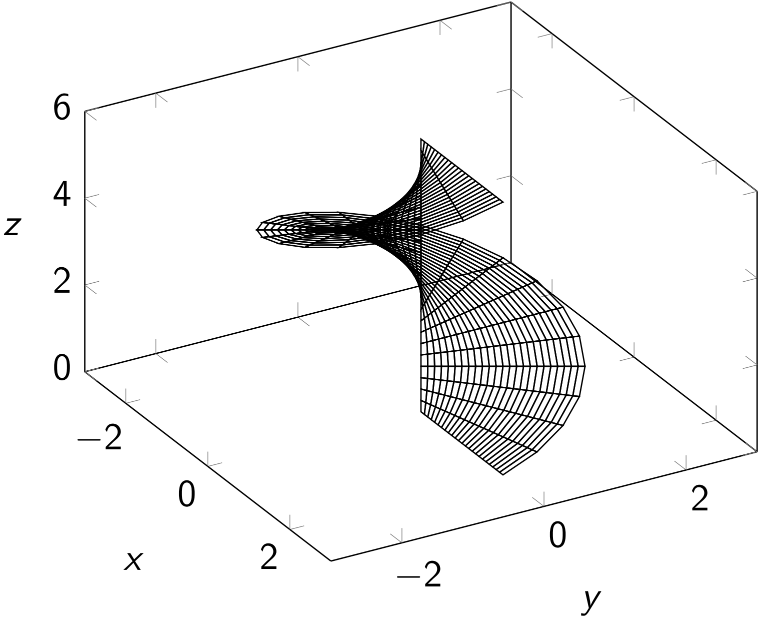
- *Putting these mesh lines together we see that the surface described by* $\bar{r}$ *looks like:*

Surfaces Described by Vector Functions



Surfaces Described by Vector Functions

Let $f : D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^2$. The surface described by the graph $z = f(x, y)$ is described by the vector function $\bar{r} : D \longrightarrow \mathbb{R}^3$ defined by: for all $(u, v) \in D$,

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Example

Let $\mathcal{S}$ be the surface described by $z = 2\sqrt{x^2 + y^2}$. A parameterisation of $\mathcal{S}$ is

$$\bar{r}(u, v) = u\bar{i} + v\bar{j} + 2\sqrt{u^2 + v^2}\bar{k}$$

Surfaces Described by Vector Functions

Example

(Continued.) Again, we can visualise this surface by observing that:

- *If v_0 is held constant, then the vector function*

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  describes a curve in the plane $y = v_0$. In this case, this curve is a conic section in the shape of a hyperbola.

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These curves can then be seen as forming a mesh, or analogues of longditudinal and latitudinal lines, that describe the parameterised surface.

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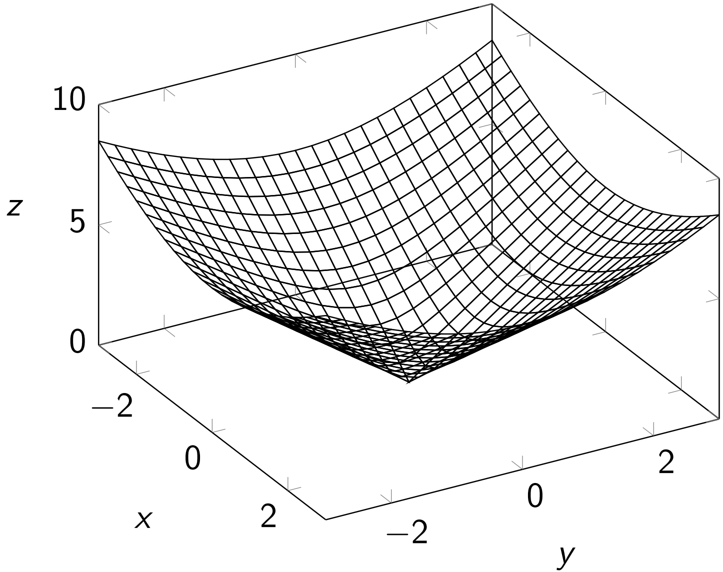

Surfaces Described by Vector Functions

Example

(Continued.) As with parameterised curves, parameterisations of surfaces are not unique.
In this example, we can also use polar coordinates to parameterise the surface using the vector function

$$\bar{s}(r,\theta) = r\cos\theta\,\bar{i} + r\sin\theta\,\bar{j} + 2r\,\bar{k}, \quad 0 \le r < \infty,\ 0 \le \theta \le 2\pi$$

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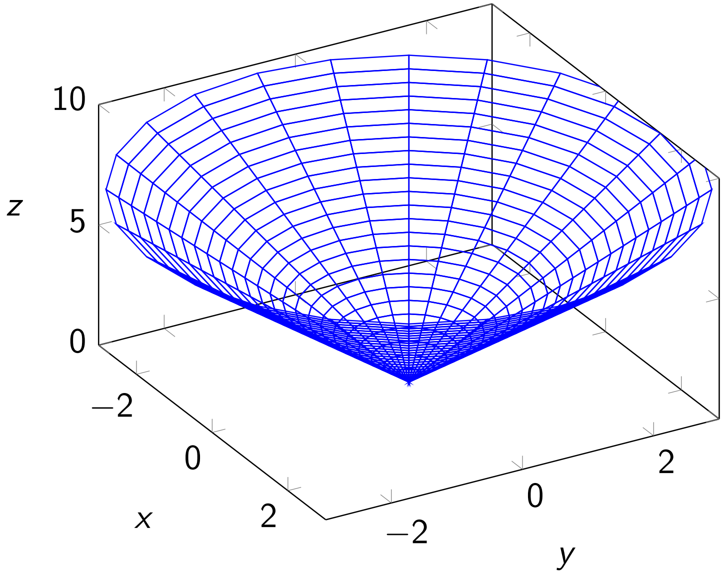
Plotting the surface parameterised by $\bar{s}(r, \theta)$*, we can see that we get the same surface that is parameterised by* $\bar{r}(u, v)$*, but the mesh lines that make up this surface are different.*
In particuluar, the curves produced by $\bar{s}(r, \theta)$ *when* r *is held contstant are circles, and*
the curves $\bar{s}(r, \theta)$ *when* θ *is held contstant are straight lines.*

Surfaces Described by Vector Functions



Parameterisation of a Sphere

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Let $\mathcal{S} : x^2 + y^2 + z^2 = a^2$ *(a sphere centred at the origin with radius a).*

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$\Rightarrow$ *the sphere can be parameterised by the vector function*

$$\bar{r}(\theta, \phi) = a \sin\phi \cos\theta \bar{i} + a \sin\phi \sin\theta \bar{j} + a \cos\phi \bar{k}$$

$$0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi$$

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$$\bar{r}(x,\theta) = x\bar{i} + a\cos\theta\bar{j} + a\sin\theta\bar{k}$$

$$0 \leq x \leq b, \quad 0 \leq \theta \leq 2\pi$$

Parameterisation of Surfaces: More Examples

Example

Let $\mathcal{S}$ be the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

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$$x = 2\sin\phi\cos\theta \quad y = 2\sin\phi\sin\theta \quad z = 2\cos\phi \quad 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi$$

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$$x = x,\ y = y,\ z = \sqrt{4 - x^2 - y^2},\ x^2 + y^2 \le 2$$

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$$\bar{r}(x, \theta) = x\bar{i} + f(x)\cos\theta\bar{j} + f(x)\sin\theta\bar{k}$$

where $x \in D$ and $\phi_1 \leq \theta \leq \phi_2$.

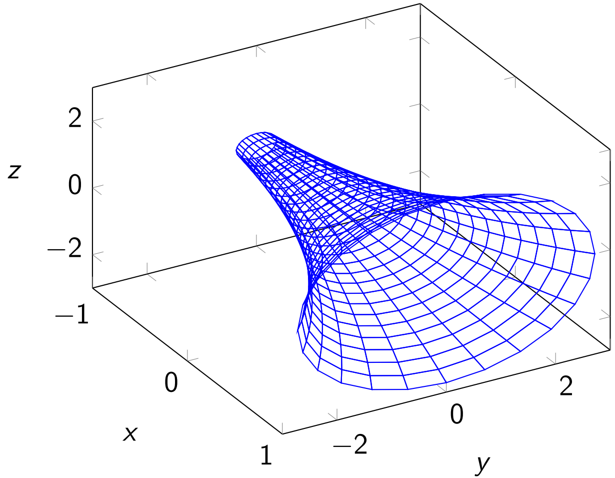
Example

Consider $f : [-1, 1] \longrightarrow \mathbb{R}$ defined by $f(x) = e^x$ and let $\mathcal{S}$ be the surface that is obtained by rotating the curve $y = f(x)$ through 2π radians about the x-axis. Then $\mathcal{S}$ is parameterised by

$$\bar{r}(x, \theta) = x\bar{i} + e^x\cos\theta\bar{j} + e^x\sin\theta\bar{k}$$

where $-1 \leq x \leq 1$ and $0 \leq \theta \leq 2\pi$.

Rotation Surfaces: Example



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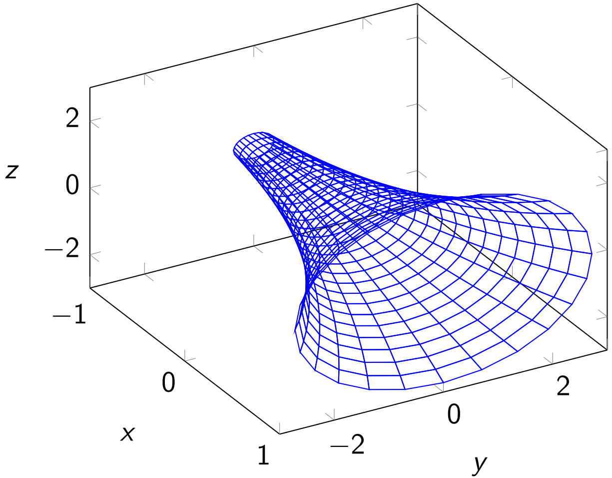

The curves described by $\bar{r}(x, \theta)$ when x is held constant are circles. And the curves described by $\bar{r}(x, \theta)$ when θ is held constant are rotations of the graph $y = e^x$.

Tangent Planes of Parameterised Surfaces

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Let (u_0, v_0) be in the domain of $x(u,v)$, $y(u,v)$ and $z(u,v)$. Consider

$$\bar{r}_u(u,v) = \frac{\partial x}{\partial u}\bar{i} + \frac{\partial y}{\partial u}\bar{j} + \frac{\partial z}{\partial u}\bar{k} \text{ and } \bar{r}_v(u,v) = \frac{\partial x}{\partial v}\bar{i} + \frac{\partial y}{\partial v}\bar{j} + \frac{\partial z}{\partial v}\bar{k}$$

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Therefore $\bar{r}_u(u_0, v_0)$ and $\bar{r}_v(u_0, v_0)$ form adjacent edges of a parallelogram intersecting at $\bar{r}(u_0, v_0)$ that describes the tangent plane to the surface parameterised by $\bar{r}(u, v)$ at the point $\bar{r}(u_0, v_0)$.

Tangent Planes of Parameterised Surfaces

Therefore, if $\mathcal{S}$ is a smooth surface that is parameterised by the vector function

$$\bar{r}(u,v) = x(u,v)\bar{i} + y(u,v)\bar{j} + z(u,v)\bar{k}$$

and (u_0, v_0) is in the domain of $x(u,v)$, $y(u,v)$ and $z(u,v)$, then the tangent plane of $\mathcal{S}$ at the point $\bar{r}(u_0,v_0)$ is the plane with normal vector

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where

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Example

Let $\mathcal{S}$ be the surface parameterised by

$$\bar{r}(u,v) = (u+v)\bar{i} + 3u^2\bar{j} + (u-v)\bar{k}$$

Find the tangent plane to the surface described by $\bar{r}(u,v)$ at the point $(2,3,0)$.

Surface Area of Parameterised Surfaces

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Recall that when we were computing the surface area of a smooth surface $\mathcal{S}$ defined by the graph of a function of two variable $z = f(x, y)$ we showed that this surface area could be obtained by inegrating over the description of the tangent plane $|\bar{r}_x(x, y) \times \bar{r}_y(x, y)|$,
where

$$\bar{r}_x(x, y) = \bar{i} + \left(\frac{\partial f}{\partial x}\right)\bar{k}, \quad \bar{r}_y(x, y) = \bar{j} + \left(\frac{\partial f}{\partial y}\right)\bar{k}$$

Surface Area of Parameterised Surfaces

Recall that when we were computing the surface area of a smooth surface $\mathcal{S}$ defined by the graph of a function of two variable $z = f(x, y)$ we showed that this surface area could be obtained by inegrating over the description of the tangent plane $|\bar{r}_x(x, y) \times \bar{r}_y(x, y)|$,
where

$$\bar{r}_x(x, y) = \bar{i} + \left(\frac{\partial f}{\partial x}\right)\bar{k}, \quad \bar{r}_y(x, y) = \bar{j} + \left(\frac{\partial f}{\partial y}\right)\bar{k}$$

We can now obtain a formula for the surface area of a smooth surface $\mathcal{S}$ parameterised by

$$\bar{r}(u, v) = x(u, v)\bar{i} + y(u, v)\bar{j} + z(u, v)\bar{k}, \quad (u, v) \in D \subseteq \mathbb{R}^2$$

by replacing the description of the tangent plane above with the description of the tangent plane that we obtain from the parameterisation $\bar{r}(u, v)$.

Surface Area of Parameterised Surfaces

Definition

Let $\mathcal{S}$ be a smooth surface that is parameterised by

$$\bar{r}(u,v) = x(u,v)\bar{i} + y(u,v)\bar{j} + z(u,v)\bar{k}, \quad (u,v) \in D \subseteq \mathbb{R}^2$$

and $\bar{r}(u,v)$ points to each point on $\mathcal{S}$ exactly once (the parameterisation is injective). Then the surface area of $\mathcal{S}$ is given by

$$A(\mathcal{S}) = \iint_D |\bar{r}_u \times \bar{r}_v| \; dA$$

where

$$\bar{r}_u(u,v) = \frac{\partial x}{\partial u}\bar{i} + \frac{\partial y}{\partial u}\bar{j} + \frac{\partial z}{\partial u}\bar{k} \text{ and } \bar{r}_v(u,v) = \frac{\partial x}{\partial v}\bar{i} + \frac{\partial y}{\partial v}\bar{j} + \frac{\partial z}{\partial v}\bar{k}$$

Surface Area of Parameterised Surfaces

Example

Find the surface area of the sphere of radius a centered at the origin $x^2 + y^2 + z^2 = a^2$.

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Surface Area of Parameterised Surfaces

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Find the surface area of the sphere of radius a centered at the origin $x^2 + y^2 + z^2 = a^2$.

Example

Find the surface area of the part of the surface $z = xy$ *that lies within the cylinder* $x^2 + y^2 = 1$.

Surface Integrals

Surface Integrals

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Surface Integrals

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- We will now generalise this idea by defining a surface integral that simulates the double integral of a function of two variables for a function of three variables over a surface in 3D space.
- Recall that the line integral of a function $\gamma(x, y)$ over a curve $\mathcal{C}$ with parametric equations
$$x = f(t) \quad y = g(t)$$
with domain $[a, b]$ is given by
$$\int_{\mathcal{C}} \gamma \; ds = \int_a^b \gamma(f(t), g(t)) \sqrt{(f'(t))^2 + (g'(t))^2} \; dt$$

Surface Integrals

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with domain $[a, b]$ is given by
$$\int_{\mathcal{C}} \gamma \; ds = \int_a^b \gamma(f(t), g(t)) \sqrt{(f'(t))^2 + (g'(t))^2} \; dt$$
- One way of thinking about this is that the infinitesimal volume component "dx" is replaced by "$\sqrt{(f'(t))^2 + (g'(t))^2} \; dt$" that represents a scaling of this volume component by the length of a straight line that approximates $\mathcal{C}$.

Surface Integrals

More formally,

- a partition of the domain of the parmeterisation $[a, b]$ (as in the definition of the Darboux integral) induces a partition of the curve $\mathcal{C}$,
- the "volume" (width) of each piece of this partition of $\mathcal{C}$ is approximated by a straight line joining the two end-points.
- The length of the straight line approximation of the piece of $\mathcal{C}$ corresponding to the piece $[t_i, t_{i+1}]$ of $[a, b]$ gets arbitrarily close to

$$\sqrt{(f(t_{i+1}) - f(t_i))^2 + (g(t_{i+1}) - g(t_i))^2}$$

$$=$$

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$$\sqrt{(f(t_{i+1}) - f(t_i))^2 + (g(t_{i+1}) - g(t_i))^2}$$

$$= (t_{i+1} - t_i) \sqrt{\left(\frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}\right)^2 + \left(\frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i}\right)^2}$$

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$$= (t_{i+1} - t_i)\sqrt{\left(\frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}\right)^2 + \left(\frac{g(t_{i+1}) - g(t_i)}{t_{i+1} - t_i}\right)^2}$$

$$= \sqrt{(f'(t_i))^2 + (g'(t_i))^2}(t_{i+1} - t_i) \text{ as } t_{i+1} - t_i \to 0$$

and these straight lines become arbitrarily good approximations of $\mathcal{C}$.

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  and these straight lines become arbitrarily good approximations of $\mathcal{C}$.

We can now use exactly the same idea to simulate the double integral over a surface in 3D space.

Surface Integrals

Let $\mathcal{S}$ be a smooth surface that is parameterised by

$$\bar{r}(u, v) = x(u, v)\bar{i} + y(u, v)\bar{j} + z(u, v)\bar{k} \text{ for } (u, v) \in R$$

where R is a closed rectangle in $\mathbb{R}^2$.

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$$\bar{r}_u(u,v) = \frac{\partial x}{\partial u}\bar{i} + \frac{\partial y}{\partial u}\bar{j} + \frac{\partial z}{\partial u}\bar{k} \text{ and } \bar{r}_v(u,v) = \frac{\partial x}{\partial v}\bar{i} + \frac{\partial y}{\partial v}\bar{j} + \frac{\partial z}{\partial v}\bar{k}$$

- The vectors $\bar{r}_u(u,v)$ and $\bar{r}_v(u,v)$ describe the tangent plane at the point $\bar{r}(u,v)$.

Surface Integrals

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- The vectors $\bar{r}_u(u,v)$ and $\bar{r}_v(u,v)$ describe the tangent plane at the point $\bar{r}(u,v)$.
- So, to get a simulation of the double integral over $\mathcal{S}$, we replace the infinitesimal volume (area) component "dA" of the double integral with the scaled area component "$|\bar{r}_u \times \bar{r}_v|\ du\ dv$".
- Note that $|\bar{r}_u \times \bar{r}_v|$ corresponds to the area of the parallelogram formed by the vectors $\bar{r}_u$ and $\bar{r}_v$.

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Surface Integrals

More formally,

- any partition of R (as in the definition of the Darboux integral) induces a partition of $\mathcal{S}$.
- If $[u_i, u_{i+1}] \times [v_i, v_{i+1}]$ is a piece of the partition of R, then the area of the the piece of $\mathcal{S}$ corresponding to this piece of R can be approximated by the parallelogram formed by its vertices.
- The area of this parallelogram gets arbitrarily close to

  $$|\bar{r}_u(u_i, v_i) \times \bar{r}_v(u_i, v_i)|(u_{i+1} - u_i)(v_{i+1} - v_i) \text{ as } u_{i+1} - u_i, v_{i+1} - v_i \to 0$$

  and, moreover, the parallelogram becomes an arbitrarily good approximation of $\mathcal{S}$.

Surface Integrals

Definition

Let $\mathcal{S}$ be a smooth surface that is parameterised by

$$\bar{r}(u,v) = x(u,v)\bar{i} + y(u,v)\bar{j} + z(u,v)\bar{k} \ \textit{for} \ (u,v) \in \mathcal{R} \subseteq \mathbb{R}^2$$

Let $f : D \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^3$, be continuous such that $\mathcal{S}$ is contained in D. The ***surface integral*** *of f over $\mathcal{S}$ is defined by*

$$\iint_{\mathcal{S}} f(x,y,z)\ dS = \iint_{\mathcal{R}} f(\bar{r}(u,v))|\bar{r}_u \times \bar{r}_v|\ dA$$

where

$$\bar{r}_u(u,v) = \frac{\partial x}{\partial u}\bar{i} + \frac{\partial y}{\partial u}\bar{j} + \frac{\partial z}{\partial u}\bar{k} \ \textit{and} \ \bar{r}_v(u,v) = \frac{\partial x}{\partial v}\bar{i} + \frac{\partial y}{\partial v}\bar{j} + \frac{\partial z}{\partial v}\bar{k}$$

Surface Integrals

Note that, as is the case with the double integral, computing the integral of the function that is constantly 1 yields the area of the region being integrated over. Let $\mathcal{S}$ be a smooth surface that is parameterised by

$$\bar{r}(u, v) = x(u, v)\bar{i} + y(u, v)\bar{j} + z(u, v)\bar{k} \text{ for } (u, v) \in \mathcal{R} \subseteq \mathbb{R}^2$$

$$\boxed{A(\mathcal{S}) = \iint_{\mathcal{R}} |\bar{r}_u \times \bar{r}_v| \, dA = \iint_{\mathcal{S}} \, dS}$$

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Therefore

$$\bar{r}_x(x, y) = \bar{i} + \frac{\partial g}{\partial x}\bar{k} \text{ and } \bar{r}_y(x, y) = \bar{j} + \frac{\partial g}{\partial y}\bar{k}$$

We have

$$\bar{r}_x \times \bar{r}_y = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 1 & 0 & \frac{\partial g}{\partial x} \\ 0 & 1 & \frac{\partial g}{\partial y} \end{vmatrix} = -\frac{\partial g}{\partial x}\bar{i} - \frac{\partial g}{\partial y}\bar{j} + \bar{k}$$

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$$|\bar{r}_x \times \bar{r}_y| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

Surface Integrals

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$$|\bar{r}_x \times \bar{r}_y| = \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}$$

Therefore, if $f(x, y, z)$ is a continuous function that is defined on $\mathcal{S}$, then

$$\iint_{\mathcal{S}} f(x, y, z)\ dS = \iint_D f(x, y, g(x, y)) \sqrt{\left(\frac{\partial g}{\partial x}\right)^2 + \left(\frac{\partial g}{\partial y}\right)^2 + 1}\ dA$$

Surface Integrals

Example

Compute

$$\iint_{\mathcal{S}} x^2yz \, dS$$

where $\mathcal{S}$ *is the part of the plane* $z = 1 + 2x + 3y$ *that lies above the rectangle* $[0,3] \times [0,2]$.

Surface Integrals

Example

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Example

Compute

$$\iint_{\mathcal{S}} z \; dS$$

where $\mathcal{S}$ *is the part of the cylinder* $x^2 + y^2 = 1$ *that lies above the plane* $z = 0$ *and below the plane* $z = 1 + x$.

Surface Integrals

Example

Compute

$$\iint_{\mathcal{S}} y \; dS$$

where $\mathcal{S}$ *is the surface parameterised by*

$$\bar{r}(u, v) = u \cos v \bar{i} + u \sin v \bar{j} + v \bar{k}$$

for $0 \leq u \leq 1$ *and* $0 \leq v \leq \pi$*.*

Oriented and Orientable Surfaces

Oriented and Orientable surfaces

Consider the unit sphere

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

Oriented and Orientable surfaces

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Using spherical coordinates, this surface can be described by

$$\mathcal{S} = \{(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \in \mathbb{R}^3 \mid 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

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$$\mathcal{S} = \{(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) \in \mathbb{R}^3 \mid 0 \leq \phi \leq \pi, 0 \leq \theta \leq 2\pi\}$$

Let (x_0, y_0, z_0) be any point on $\mathcal{S}$. Then the vector $x_0\bar{i} + y_0\bar{j} + z_0\bar{k}$ is a normal vector to the tangent plane of $\mathcal{S}$ at the point (x_0, y_0, z_0) (verify this if it is not obvious).

Oriented and Orientable surfaces

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$$\hat{\bar{n}}(x_0, y_0, z_0) = \frac{x_0\bar{i} + y_0\bar{j} + z_0\bar{k}}{\sqrt{x_0^2 + y_0^2 + z_0^2}} = x_0\bar{i} + y_0\bar{j} + z_0\bar{k}$$

is a continuous vector function that describes unit normal vectors to the surface $\mathcal{S}$.

Oriented and Orientable surfaces

Consider the unit sphere

$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

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is a continuous vector function that describes unit normal vectors to the surface $\mathcal{S}$. Note that

$$-\hat{\bar{n}}(x_0, y_0, z_0) = -x_0\bar{i} - y_0\bar{j} - z_0\bar{k}$$

is also a continuous vector function that describes unit normal vectors to the surface $\mathcal{S}$.

Oriented and orientable surfaces

The unit normal vectors $-\hat{\bar{n}}(x_0, y_0, z_0)$ point towards the centre of the sphere, while the unit normal vectors $\hat{\bar{n}}(x_0, y_0, z_0)$ emanate out from the sphere.

Oriented and orientable surfaces

The unit normal vectors $-\hat{\bar{n}}(x_0, y_0, z_0)$ point towards the centre of the sphere, while the unit normal vectors $\hat{\bar{n}}(x_0, y_0, z_0)$ emanate out from the sphere.

In contrast, consider the surface described by the parametric equations:

$$x(r, \theta) = 2\cos\theta + r\cos\frac{\theta}{2}$$

$$y(r, \theta) = 2\sin\theta + r\cos\frac{\theta}{2}$$

$$z(r, \theta) = r\sin\frac{\theta}{2}$$

for $-\frac{1}{2} \leq r \leq \frac{1}{2}$ and $0 \leq \theta \leq 2\pi$.

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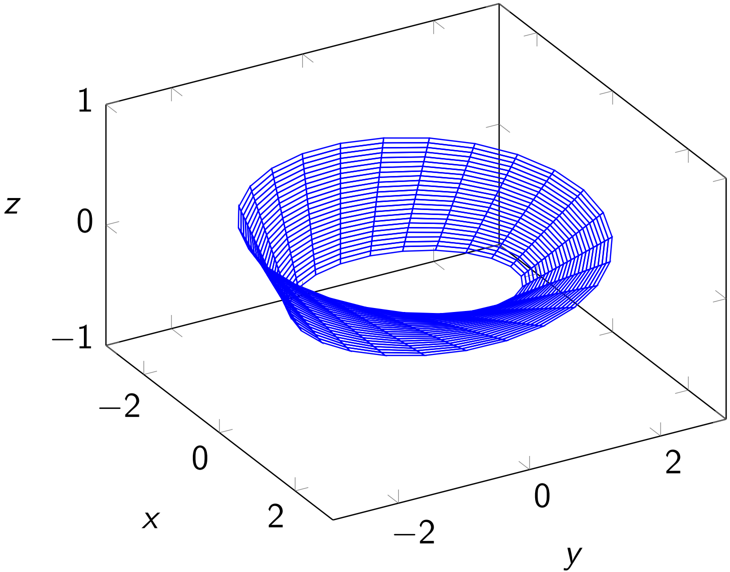
for $-\frac{1}{2} \leq r \leq \frac{1}{2}$ and $0 \leq \theta \leq 2\pi$. This surface is called the Möbius Strip.

Oriented and orientable surfaces



Oriented and orientable surfaces

If you were to slide the letter "E" along this surface, then you would eventually be able to slide the "E" back to the opposite side of the strip at the same point that it started from, and your “E” would look something like "$\exists$".

Oriented and orientable surfaces

If you were to slide the letter "E" along this surface, then you would eventually be able to slide the "E" back to the opposite side of the strip at the same point that it started from, and your "E" would look something like "$\exists$". Similarily, if you started with an arrow pointing in the direction of the normal to this surface at a point on the strip, then you could slide this arrow along the surface of the strip until you got back to the same point (but on the opposite side) and your arrow would be pointing in the opposite direction.

Oriented and orientable surfaces

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Definition

Let $\mathcal{S}$ be a surface in $\mathbb{R}^3$. We say that $\mathcal{S}$ is orientable if there exists a continuous vector function $\bar{n}$ that is defined on $\mathcal{S}$, and on input (x, y, z) on $\mathcal{S}$, outputs a unit normal vector to $\mathcal{S}$ at (x, y, z). An oriented surface is a surface $\mathcal{S}$ and a vector function $\bar{n}$ that witnesses the fact that $\mathcal{S}$ is orientable. If $\bar{n}$ witnesses the fact that $\mathcal{S}$ is orientable, then we say that $\bar{n}$ is an orientation of $\mathcal{S}$.

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Most of the surfaces we discuss will be orientable!

Surface Integrals of Vector Fields

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- To give you some physical context, imagine a vector field F that measures the direction and speed of the flow of a fluid through 3D space and imagine that $\mathcal{S}$ describes a perfectly porous membrane, like and idealised grill or net. Then the surface integral of F over $\mathcal{S}$ will measure the rate at which the fluid flows through $\mathcal{S}$.

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It should now be clear how we are going to define the surface integral of a vector field. Let F be a vector field in $\mathbb{R}^3$. Let $\mathcal{S}$ be an oriented surface in $\mathbb{R}^3$ with normal vector field $\bar{n}$, then, at each point on $\mathcal{S}$ the magnitude F flowing through $\mathcal{S}$ at this point is given by $F \cdot \bar{n}$.

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The surface integral of the vector field F is also called the flux of F across the surface.

Surface Integrals of Vector Fields

Example

Evaluate

$$\iint_{\mathcal{S}} F \cdot d\bar{S},$$

where $F(x, y, z) = x\bar{i} - z\bar{j} + y\bar{k}$ *and* $\mathcal{S}\colon x^2 + y^2 + z^2 = 4$, $x, y, z \geq 0$ *oriented towards the origin.*

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In spherical coordinates, we can parameterise $\mathcal{S}$ *by*

$$\bar{r}(\phi, \theta) = 2\sin\phi\cos\theta\bar{i} + 2\sin\phi\sin\theta\bar{j} + 2\cos\phi\bar{k}$$

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Surface Integrals of Vector Fields

Example

(Continued.)

$$\bar{r}_\phi \times \bar{r}_\theta = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2\cos\phi\cos\theta & 2\cos\phi\sin\theta & -2\sin\phi \\ -2\sin\phi\sin\theta & 2\sin\phi\cos\theta & 0 \end{vmatrix}$$

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At this point we observe that for $0 \leq \phi \leq \frac{\pi}{2}$ *and* $0 \leq \theta \leq \frac{\pi}{2}$, $\bar{r}_\phi \times \bar{r}_\theta$ *points away from the origin. Therefore, in order to ensure that* $\mathcal{S}$ *is oriented correctly, we need to compute the surface integral with* $-\bar{r}_\phi \times \bar{r}_\theta$.

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$$F(\bar{r}(\phi,\theta)) \cdot (-\bar{r}_\phi \times \bar{r}_\theta) = -8\sin^3\phi\cos^2\theta + 8\cos\phi\sin^2\phi\sin\theta$$

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$$= -8\sin^3\phi\cos^2\theta$$

Example

(Continued.)

Surface Integrals of Vector Fields

Example

(Continued.) Now, observing that

$$\int_0^{\frac{\pi}{2}} \sin^3 \phi \ d\phi = \frac{2}{3} \ \textit{and} \ \int_0^{\frac{\pi}{2}} \cos^2(\theta) \ d\theta = \frac{\pi}{4}$$

We see that

$$\iint_{\mathcal{S}} F \cdot d\bar{S} = -8 \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \sin^3(\phi) \cos^2(\theta) \ d\phi d\theta = -\frac{4\pi}{3}$$

Surface integrals of vector fields

Let $g : D \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^2$, be a smooth function. Let $\mathcal{S}$ be an oriented surface described by the graph $z = g(x, y)$ and the normal vector field $\bar{n}$.

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with

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$$\bar{r}_x \times \bar{r}_y = -\frac{\partial g}{\partial x}\bar{i} - \frac{\partial g}{\partial y}\bar{j} + \bar{k}$$

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where the sign depends on the orientation of $\mathcal{S}$.

Surface integrals of vector fields

Example

Consider $F(x, y, z) = y\bar{i} + x\bar{j} + z\bar{k}$. *Compute*

$$\iint_{\mathcal{S}} F \cdot d\bar{S}$$

where $\mathcal{S}$ *is the part of the graph* $z = 1 - x^2 - y^2$ *that sits above the plane* $z = 0$ *and is oriented in the direction of the positive z-axis.*

Heat Flow

Let $u(x, y, z)$ be the temperature at a point $(x, y, z) \in E \subset \mathbb{R}^3$, where E is some region and u is a smooth enough function. Then

$$\nabla u = \frac{\partial u}{\partial x}\bar{i} + \frac{\partial u}{\partial y}\bar{j} + \frac{\partial u}{\partial z}\bar{k}$$

represents the temperature gradient, and heat "flows" with the vector field $-k\nabla u = F$ where K is a positive constant. Therefore, $\iint_{\mathcal{S}} F \cdot d\bar{S}$ is the total rate of heat flow or flux across the surface $\mathcal{S}$.

Example

Let $u(x, y, z) = x^2 + y^2 + z^2$ and $\mathcal{S}\colon x^2 + y^2 + z^2$ be outward oriented. Find the heat flux across the surface $\mathcal{S}$ if $k = 1$.

$F(x, y, z) = -\nabla u = -2x\bar{i} - 2y\bar{j} - 2z\bar{k}$

On $\mathcal{S}$, the unit outward normal vector to $\mathcal{S}$ at (x, y, z) is $\bar{n} = (x, y, z)$,

$$F \cdot \bar{n} = -2x^2 - 2y^2 - 2z^2 = -2(x^2 + y^2 + z^2) = -2$$

$$\iint_{\mathcal{S}} F \cdot d\bar{S} = \iint_{\mathcal{S}} F \cdot \bar{n} dS = -2 \iint_{\mathcal{S}} dS = -2A(S) = -8\pi$$

Stokes' Theorem

Stokes' Theorem

We now turn discussing a result, due to Sir. William Thomson (Lord Kelvin), that is known as Stokes' Theorem (it appears that this is because George Stokes asked students to prove this Theorem on a tripos examination [be grateful!]). Stokes' Theorem can be viewed a yet another generalisation of the First Fundamental Theorem of Calculus, and, less distantly, as a generalisation of Green's Theorem to oriented surfaces in $\mathbb{R}^3$, that is, given a sufficiently well-behaved bounded and oriented surface $\mathcal{S}$ in $\mathbb{R}^3$ and a smooth vector field F, Stokes' Theorem states that the surface integral of the microscopic circulation of F $(\mathrm{curl}(F))$ over $\mathcal{S}$ is equal to the circulation of F around the boundary of $\mathcal{S}$.

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Recall that we derived a vector version of Green's Theorem: Let $\mathcal{C}$ be a positively oriented, piecewise-smooth, simple closed curve in $\mathbb{R}^2$ and $\mathcal{R}$ the region enclosed by $\mathcal{C}$. Let F be a vector field in $\mathbb{R}^2$. Then

$$\oint_{\mathcal{C}} F \cdot d\bar{r} = \iint_{\mathcal{R}} (\mathrm{curl}(F)) \cdot \bar{k} \; dA$$

Stokes' Theorem

When F is is viewed as a vector field in $\mathbb{R}^3$, $\mathcal{C}$ a curve in $\mathbb{R}^3$ and $\mathcal{R}$ a surface in $\mathbb{R}^3$, we recognise that

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Theorem

(Stokes' Theorem) Let $\mathcal{C}$ be a positively oriented, piecewise-smooth, simple, closed curve in $\mathbb{R}^3$ and let $\mathcal{S}$ be a surface whose boundary is $\mathcal{C}$ oriented with respect to the orientation of $\mathcal{C}$ according the the right-hand rule. Let F be a vector field on $\mathbb{R}^3$ whose component have continuous partial derivatives on a domian that contains $\mathcal{S}$. Then

$$\int_{\mathcal{C}} F \cdot d\bar{r} = \iint_{\mathcal{S}} \mathrm{curl}(F) \cdot d\bar{S}$$

Stokes' Theorem

- Note that the curve $\mathcal{C}$ and the surface $\mathcal{S}$ in Stokes' Theorem must be oriented according to the right-hand rule: When the fingers of your right hand curl in the direction of $\mathcal{C}$ the thumb of your right hand is pointing in the direction of the normal vector to $\mathcal{S}$

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- The vector field $\mathrm{curl}(F)$ measures "rotation" at points on $\mathcal{S}$, that is, $\mathrm{curl}(F)$ measures how much a paddle wheel would spin if place into the fluid at that particular point. These vectors point in a direction that is normal to the plane of rotation.

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- Therefore

  $$\iint_{\mathcal{S}} \operatorname{curl}(F) \cdot d\bar{S}$$

  is the "continuous sum" of the microscopic rotation of F on the the surface $\mathcal{S}$.

Stokes' Theorem

- The integral
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Example

Let $\mathcal{C}$ be the closed curve formed by moving anticlockwise around the intersection of the plane $z = x + 4$ with the cylinder $x^2 + y^2 = 4$. Let

$$F = (x^3 + 2y)\bar{i} + (\sin y + z)\bar{j} + (x + \sin z^2)\bar{k}$$

and suppose that we want to compute

$$\oint_{\mathcal{C}} F \cdot d\bar{r}$$

Stokes' Theorem

Example

(Continued.) Let $\mathcal{S}$ be the part of the plane $z = x + 4$ that is contained in the cylinder $x^2 + y^2 = 4$.

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(Continued.) Let $\mathcal{S}$ be the part of the plane $z = x + 4$ that is contained in the cylinder $x^2 + y^2 = 4$. Let $\mathcal{R} = \{(x, y\rangle \in \mathbb{R}^2 \mid x^2 + y^2)\}$.
Using the fact that $\mathcal{S}$ is parameterised by

$$\bar{r}(x, y) = x\bar{i} + (x + 4)\bar{k}$$

we get

$$\bar{r}_x = \bar{i} + \bar{k} \textit{ and } \bar{r}_y = \bar{j}$$

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At this point, observe that the thumb of your right hand points in the direction of $\overline{r}_x \times \overline{r}_y$ when your fingers curl anticlockwise around $\mathcal{C}$.

Stokes' Theorem

Example

(Continued.) Therefore, Stokes' Theorem says that

$$\oint_{\mathcal{C}} F \cdot d\bar{r} = \iint_{\mathcal{S}} \mathrm{curl}(F) \cdot d\bar{S}$$

where $\mathcal{S}$ is oriented by the unit vector that points in the same direction as $\bar{r}_x \times \bar{r}_y$. Now,

$$\mathrm{curl}(F) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^3 + 2y) & (\sin y + z) & (x + \sin z^2) \end{vmatrix} = -\bar{i} - \bar{j} - 2\bar{k}$$

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Consider $F(x,y,z) = xz\bar{i} + yz\bar{j} + xy\bar{k}$. *Compute*

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If F is pointing in the opposite direction, then $\oint_{\mathcal{C}} F \cdot d\bar{r} < 0$ and particles tend to rotate clockwise.
If F is perpendicular to $\mathcal{C}$, then particles don't rotate on $\mathcal{C}$ at all and $\oint_{\mathcal{C}} F \cdot d\bar{r} = 0$.

Physical Intuition for curl

In general, $\oint_{\mathcal{C}} F \cdot d\bar{r}$ measures the amount of F that is moving counterclockwise around $\mathcal{C}$ (the amount that F rotates around $\mathcal{C}$).

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The curl of F in a given direction $\bar{n}$ at a point P is the "instantaneous" rotation of F in the plane perpendicular to $\bar{n}$ at the point P.

The Divergence Theorem

The Divergence Theorem

Recall that when we were discussing Green's Theorem we derived consequence of Green's Theorem that we called the "2D Divergence Theorem": Let $\mathcal{C}$ be a positively oriented, piecewise-smooth, simple closed curve and let $\mathcal{R}$ be the region that is enclosed by $\mathcal{C}$. Let $\bar{n}$ be the vector field such that for all points on $\mathcal{C}$, $\bar{n}$ is the unit normal vector that points away from the interior of the region enclosed by $\mathcal{C}$. If F is a vector field on $\mathbb{R}^2$ such that the partial derivatives of the components of F are continuous on an open region containing $\mathcal{R}$, then

$$\oint_{\mathcal{C}} F \cdot \bar{n}\, ds = \iint_{\mathcal{R}} \mathrm{div}(F)\, dA$$

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The Divergence Theorem Generalises this result to three dimensions: the surface integral of a vector field over the boundary surface of a 3D solid is equated with the triple integral of $\operatorname{div}(F)$ over that solid.

The Divergence Theorem

Theorem

Let $\mathcal{S}$ be a piecewise-smooth surface that encloses a solid $\mathcal{R}$ that is oriented so that the normal vectors point away from the interior of $\mathcal{S}$. Let F be a vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives on an open region that contains $\mathcal{R}$. Then

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Note that a version of the Divergence Theorem for solids that simultaneously type I, II and III can be proved using a similar argument to the one we used to prove a special case of Green's Theorem.

The Divergence Theorem

Example

Let $F(x, y, z) = z\bar{i} + y\bar{j} + z\bar{k}$. *Compute*

$$\iint_{\mathcal{S}} F \cdot d\bar{S}$$

where $\mathcal{S}$ *is the surface of the unit sphere* $x^2 + y^2 + z^2 = 1$ *oriented so that the normal vectors point away from the origin.*

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The Divergence Theorem can also be used to obtain physical intuition about the meaning of div in the same way that Stokes' Theorem allowed us to obtain physical intuition about the meaning of curl. Let F be a vector field that describes the rate of flow per unit area of a fluid through 3D space. Therefore, if $\mathcal{S}$ is a smooth surface, then

$$\iint_{\mathcal{S}} F \cdot d\bar{S}$$

measures the flow of the fluid through $\mathcal{S}$.

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measures the flow of the fluid through $\mathcal{S}$. Let $P = (x_0, y_0, z_0)$ be a point in space. Let S_a be the positively oriented sphere of radius a and let B_a be the ball enclosed by this sphere.

Physical intuition for div

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This indicates that $\mathrm{div}(F)$ is measuring the net "instantaneous" outward flow of the fluid at a given point.

The Divergence Theorem

Example

Let $F(x, y, z) = xye^z\bar{i} + xy^2z^3\bar{j} - ye^z\bar{k}$. *Compute*

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where $\mathcal{S}$ *is the surface of the box bounded by the coordinate planes and the planes* $x = 3$, $y = 2$ *and* $z = 1$ *that is oriented so that the normal vectors point away from the interior of the box.*

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Let $F(x,y,z) = x^2 \sin y\,\bar{i} + x \cos y\,\bar{j} - xz \sin y\,\bar{k}$. *Compute*

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where $\mathcal{S}$ *is the surface described by* $x^8 + y^8 + z^8 = 8$ *oriented so that the normal vectors point away from the origin.*