Comparison of Two Variances

Comparing Two Variances

Two Normally-Distributed Populations:

- $X^{(1)} \sim N(\mu_1, \sigma_1^2)$,
- $X^{(2)} \sim N(\mu_2, \sigma_2^2)$.

Goal: Develop a test to compare σ_1^2 and σ_2^2.

Taking samples of sizes n_1 and n_2 from the populations, we know that

$$\frac{(n_1-1)S_1^2}{\sigma_1^2} \sim \chi^2_{n_1-1}, \qquad \frac{(n_2-1)S_2^2}{\sigma_2^2} \sim \chi^2_{n_2-1}.$$

The difference of two chi-squared distributions is hard to analyze, so we may be better off looking at the quotient:

$$\sigma_1^2 = \sigma_2^2 \qquad \text{if and only if} \qquad \frac{\sigma_1^2}{\sigma_2^2} = 1$$

The F-Distribution

20.1. Definition. Let $X^2_{\gamma_1}$ and $X^2_{\gamma_2}$ be independent chi-squared random variables with γ_1 and γ_2 degrees of freedom, respectively.

The random variable

$$F_{\gamma_1,\gamma_2} = \frac{X^2_{\gamma_1}/\gamma_1}{X^2_{\gamma_2}/\gamma_2}$$

is said to follow an ***F-distribution with γ_1 and γ_2 degrees of freedom***.

20.2. Remark. From the definition, it is clear that

$$P[F_{\gamma_1,\gamma_2} < x] = P\left[\frac{1}{F_{\gamma_1,\gamma_2}} > \frac{1}{x}\right] = 1 - P\left[F_{\gamma_2,\gamma_1} < \frac{1}{x}\right],$$

so the density functions of the F_{γ_1,γ_2} and F_{γ_2,γ_1}-distributions are related.

Density of the F-Distribution

Using Theorem 10.2 for the density of the quotient of two independent variables we can calculate the density function explicitly:

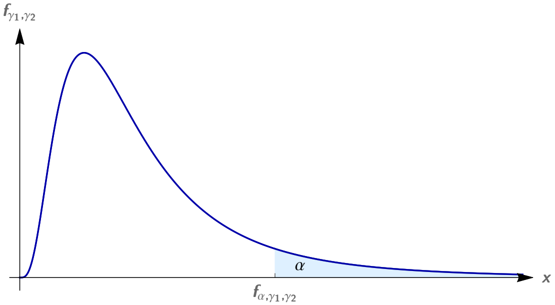
20.3. Lemma. The density of a random variable following an F-distribution with γ_1 and γ_2 degrees of freedom is given by

$$f_{\gamma_1,\gamma_2}(x) = \gamma_1^{\gamma_1/2}\gamma_2^{\gamma_2/2}\frac{\Gamma(\frac{\gamma_1+\gamma_2}{2})}{\Gamma(\frac{\gamma_1}{2})\Gamma(\frac{\gamma_2}{2})}\frac{x^{\gamma_1/2-1}}{(\gamma_1 x+\gamma_2)^{(\gamma_1+\gamma_2)/2}}$$

for $x \geq 0$ and $f_{\gamma_1,\gamma_2}(x) = 0$ for $x < 0$.

Critical Points of the F-Distribution

For $0 < \alpha < 1$, we define the point $f_{\alpha,\gamma_1,\gamma_2}$ by $P[F_{\gamma_1,\gamma_2} > f_{\alpha,\gamma_1,\gamma_2}] = \alpha$.



Selected critical values $f_{\alpha,\gamma_1,\gamma_2}$ are tabulated. Since the F distribution has two parameters γ_1 and γ_2, often only the values $f_{0.1,\gamma_1,\gamma_2}$ and $f_{0.05,\gamma_1,\gamma_2}$ are listed for various values of γ_1 and γ_2.

Critical Points of the F-Distribution

From Remark 20.2 we see that

$$\begin{aligned}1-\alpha &= P[F_{\gamma_1,\gamma_2} \geq f_{1-\alpha,\gamma_1,\gamma_2}] \\ &= 1 - P[F_{\gamma_1,\gamma_2} < f_{1-\alpha,\gamma_1,\gamma_2}] \\ &= P[F_{\gamma_2,\gamma_1} < 1/f_{1-\alpha,\gamma_1,\gamma_2}] \\ &= 1 - P[F_{\gamma_2,\gamma_1} \geq 1/f_{1-\alpha,\gamma_1,\gamma_2}] \\ &\overset{!}{=} 1 - P[F_{\gamma_2,\gamma_1} \geq f_{\alpha,\gamma_2,\gamma_1}]\end{aligned}$$

so

$$f_{1-\alpha,\gamma_1,\gamma_2} = \frac{1}{f_{\alpha,\gamma_2,\gamma_1}}. \tag{20.1}$$

It follows that the values of the "right-tail" critical point $f_{\alpha,\gamma_1,\gamma_2}$ are sufficient to find corresponding "left-tail" points.

The F-Distribution

20.4. Theorem. Let S_1^2 and S_2^2 be sample variances based on independent random samples of sizes n_1 and n_2 drawn from normal populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively.

If $\sigma_1^2 = \sigma_2^2$, then the statistic

$$S_1^2/S_2^2$$

follows an F-distribution with $n_1 - 1$ and $n_2 - 1$ degrees of freedom.

Proof.
We know that $(n_1 - 1)S_1^2/\sigma_1^2$ and $(n_2 - 1)S_2^2/\sigma_2^2$ follow chi-squared distributions with $n_1 - 1$ and $n_2 - 1$ degrees of freedom, respectively. Then

$$F_{n_1-1,n_2-1} = \frac{[(n_1-1)S_1^2/\sigma_1^2]/(n_1-1)}{[(n_2-1)S_2^2/\sigma_2^2]/(n_2-1)} = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2}.$$

If $\sigma_1^2 = \sigma_2^2$, this reduces to S_1^2/S_2^2. $\square$

The F-Test

20.5. F-Test. Let S_1^2 and S_2^2 be sample variances based on independent random samples of sizes n_1 and n_2 drawn from normal populations with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Then a test based on the statistic

$$F_{n_1-1,n_2-1} = \frac{S_1^2}{S_2^2}$$

is called an ***F-test***.

We reject at significance level α

- $H_0\colon \sigma_1 \le \sigma_2$ if $\dfrac{S_1^2}{S_2^2} > f_{\alpha,n_1-1,n_2-1}$,
- $H_0\colon \sigma_1 \ge \sigma_2$ if $\dfrac{S_2^2}{S_1^2} > f_{\alpha,n_2-1,n_1-1}$,
- $H_0\colon \sigma_1 = \sigma_2$ if $\dfrac{S_1^2}{S_2^2} > f_{\alpha/2,n_1-1,n_2-1}$ or $\dfrac{S_2^2}{S_1^2} > f_{\alpha/2,n_2-1,n_1-1}$

Remarks on the F-Test

20.6. Remarks.

- We have used (20.1) and written the critical regions in terms of right-tailed points; note the subscripts of the critical points carefully!
- For the F-test to be applicable, it is essential that the populations are normally distributed.
- If possible, the sample sizes n_1 and n_2 should be equal.
- It turns out that the F-test is not very powerful; β can be quite large. In order to keep β small, one often tests at $\alpha = 0.1$ or $\alpha = 0.2$ level of significance.
- When testing to see whether two population variances are equal for the purposes of later applying other tests, such as a comparison of their means, one ***hopes to not reject H_0***! In that case, the probability of committing a (Type II) error is given by β and a small β is more important than a small α.

Comparing Two Variances - The F-Test

20.7. Example. Chemical etching is used to remove copper from printed circuit boards. X_1 and X_2 represent process yields in % when two different concentrations are used. Suppose that we wish to test

$$H_0 \colon \sigma_1^2 = \sigma_2^2.$$

Two samples of sizes $n_1 = n_2 = 8$ yield $s_1^2 = 4.02$ and $s_2^2 = 3.89$, and

$$\frac{s_1^2}{s_2^2} = \frac{4.02}{3.89} = 1.03.$$

From Table IX we see that $f_{0.1;7,7} = 2.785$. Since our test statistic is much smaller than this value, the P-value of the (two-tailed) test is significantly greater than $2 \cdot 0.1 = 20\%$. There is not enough evidence to reject H_0.

OC Curves for the F-Test

For the case $n_1 = n_2 = n$ there are OC curves plotting β against the parameter

$$\lambda = \frac{\sigma_1}{\sigma_2}.$$

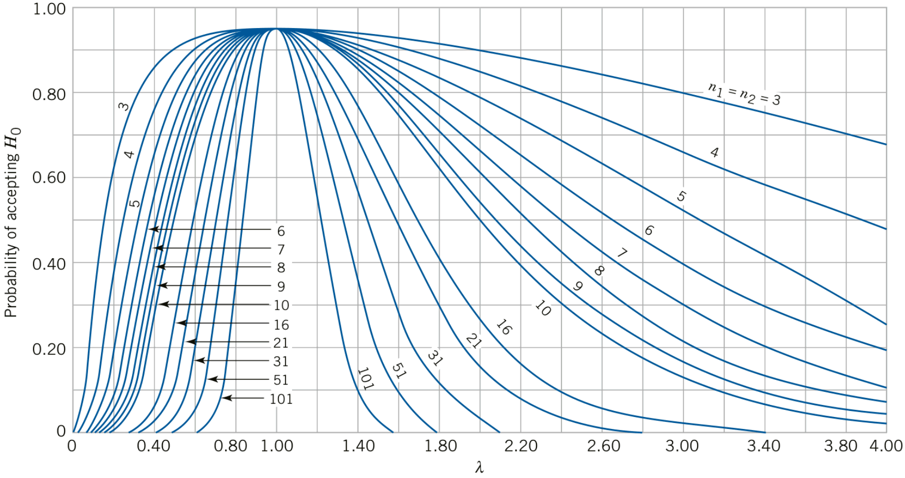
20.8. Example. Continuing from Example 20.7, suppose that one of the concentrations affected the variance of the yield so that one of the variances was four times the other and we wished to detect this with probability at least 0.80. What sample size should be used?

For this situation, a Neyman-Pearson test should be used:

$$H_0\colon \sigma_1^2 = \sigma_2^2, \qquad\qquad H_1\colon \max\left(\frac{\sigma_1^2}{\sigma_2^2}, \frac{\sigma_1^2}{\sigma_2^2}\right) \geq 4$$

If one variance is four times the other, then $\lambda = \sigma_1/\sigma_2 = 2$.

OC Curves for the F-Test



From the OC chart, we see that a sample size of about 20 will be sufficient.