
Question1 (4 points)

Find an antiderivative for each of the following functions

(a) (1 point) $y = \pi + \dfrac{x^2 + x^3}{\sqrt{x}}$

(b) (1 point) $y = 3\sin x - 2\sec^2 x$

(c) (1 point) $y = \dfrac{1}{2\sqrt{1-x^2}} - \dfrac{3}{1+x^2}$

(d) (1 point) $y = x^x(\ln x + 1)$

Solution:

(a) Rearranging, we have

$$y = \pi + x^{3/2} + x^{5/2}$$

using the linear property and $(x^n)' = nx^{n-1}$, we have

$$\left(\pi x + \frac{2}{5}x^{5/2} + \frac{2}{7}x^{7/2}\right)' = \pi + x^{3/2} + x^{5/2}$$

thus any function of the following form is an antiderivative of y for $x > 0$

$$Y = \pi x + \frac{2}{5}x^{5/2} + \frac{2}{7}x^{7/2} + C, \qquad \text{where } C \text{ is a constant.}$$

(b) Using the linear property, and $(\cos x)' = -\sin x$ and $(\tan x)' = \sec^2 x$, we have

$$(-3\cos x - 2\tan x)' = 3\sin x - 2\sec^2 x$$

thus any function of the following form is an antiderivative of y

$$Y = -3\cos x - 2\tan x + C, \qquad \text{where } C \text{ is a constant.}$$

(c) This again is relying on our knowledge regarding derivatives of various functions.

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}} \qquad (\arctan x)' = \frac{1}{x^2+1}$$

thus any function of the following form is an antiderivative of y for $-1 < x < 1$

$$Y = \frac{1}{2}\arcsin x - 3\arctan x + C, \qquad \text{where } C \text{ is a constant.}$$

(d) Recall this is the derivative function that you should have found in assignment 3 for

$$(x^x)' = x^x(\ln x + 1)$$

thus any function of the following form is an antiderivative of y for $x > 0$

$$Y = x^x + C, \qquad \text{where } C \text{ is a constant.}$$

Question2 (1 points)

Find the indefinite integral of

$$f(x) = x^x (\ln x + 1)$$

Solution:

1M This question is my way of getting you to think about the possible difference between antiderivative and indefinite integral. In terms of what we are actually writing down, they are very similar. However, it is useful to draw a distinction between a particular function, antiderivative, such that

$$F' = f$$

and a set of functions, indefinite integral, such that the derivative of each member is

$$f$$

An indefinite integral is a like general expression, while an antiderivative is one of many particular possibilities that the general expression can give. However, there isn't a consensus in the mathematical community about the usage of antidervative and indefinite integral. For example, I am aware of the fact that your textbook actually does not distinguish the two terms.

Question3 (3 points)

(a) (1 point) For the function below,

$$f(x) = x^2 - x^3$$

find a formula for the Riemann sum obtained by dividing the interval $[-1, 0]$ into n equal subintervals and using the right-hand endpoint for each x_k^*. Then take a limit of the sum as $n \to +\infty$ to calculate the area under the curve $y = f(x)$ over $[-1, 0]$.

Solution:

1M The norm here is $\Delta x_k = \Delta x = \frac{0-(-1)}{n} = \frac{1}{n}$ and the sample points are given by

$$x_k^* = a + k\Delta x = -1 + \frac{k}{n}$$

The corresponding Riemann sum is given by

$$\begin{aligned}
\sum_{k=1}^{n} f(x_k^*)\Delta x_k &= \sum_{k=1}^{n} \Big((-1+\frac{k}{n})^2 - (-1+\frac{k}{n})^3\Big)\frac{1}{n} \\
&= \frac{1}{n^4}\sum_{k=1}^{n} (k-n)^2 (2n-k) \\
&= -\frac{1}{n^4}\sum_{k=1}^{n} k^3 + \frac{4}{n^3}\sum_{k=1}^{n} k^2 - \frac{5}{n^2}\sum_{k=1}^{n} k + \frac{2}{n}\sum_{k=1}^{n} 1 \\
&= -\frac{n^2(n+1)^2}{4n^4} + \frac{2}{3}\frac{(2n+1)(n+1)}{n^2} - \frac{5}{2}\frac{n+1}{n} + 2
\end{aligned}$$

Taking the limit $n \to \infty$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = -\frac{1}{4} + \frac{4}{3} - \frac{5}{2} + 2 = \frac{7}{12}$$

(b) (1 point) Find two different tagged partitions

$$P$$

such that the following limit has two different values, one for each tagged partition.

$$\lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

where

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1] \cap \mathbb{Q}, \\ 0 & \text{if } x \in [0,1] \cap \mathbb{Q}^{\complement}. \end{cases}$$

Solution:

1M Since every interval of non-zero length contains both rational and irrational numbers. Thus it is sufficient to consider two equally spaced partitions, where the first one uses any rational number in each subinterval to be the sample points, while the second uses any irrational number in each subinterval to be the sample points. The first will result the limit of the corresponding Riemann sum to be 1, while the second will result the limit to be 0.

(c) (1 point) The following function is continuous on $[0, 4]$, thus integrable on this interval.

$$g(x) = \sqrt{x}$$

Using the definition of definite integral to evaluate

$$\int_0^4 g(x)\,dx$$

Solution:

1M Consider the following partition with an unequal length

$$0 < 4(1)^2/n^2 < 4(2)^2/n^2 < \cdots < 4(n-1)^2/n^2 < 4$$

if we use the right endpoints to be sample points, the Riemann sum is

$$\begin{aligned}\sum_{k=1}^{n} f(x_k^*)\Delta x_k &= \sum_{k=1}^{n} \left(4\left(\frac{k}{n}\right)^2\right)^{1/2} \left(4\left(\frac{k}{n}\right)^2 - 4\left(\frac{k-1}{n}\right)^2\right) \\ &= \frac{8}{n^3}\sum_{k=1}^{n} 2k^2 - k = \frac{8}{n^3}\left(\frac{2}{3}n^3 + \frac{n^2}{2} - \frac{n}{6}\right) = \frac{16}{3} + \frac{4}{n} - \frac{4}{3n^2}\end{aligned}$$

upon taking the limit as $n \to \infty$, we have

$$\int_0^4 g(x)\,dx = \lim_{n\to\infty}\left(\frac{16}{3} + \frac{4}{n} - \frac{4}{3n^2}\right) = \frac{16}{3}$$

Question4 (3 points)
Use the fundamental theorem of calculus to evaluate the following definition integrals.

(a) (1 point) $\displaystyle\int_0^{2\pi} \sin x\,dx$

(b) (1 point) $\displaystyle\int_0^1 \sqrt{x}\,dx$

(c) (1 point) $\displaystyle\int_{-1}^4 f(x)\,dx$, where $f(x) = \begin{cases} x & x < 2, \\ x^2 & x \geq 2. \end{cases}$

Solution:

(a) Sine is everywhere continuous, hence we can apply FTC directly,

$$\int_0^{2\pi} \sin x\,dx = [-\cos x]_0^{2\pi} = -\cos 2\pi + \cos 0 = -1 + 1 = 0$$

which is not a surprise since half of the area is above x-axis, the other below x-axis.

(b) The square root function is continuous on the interval of integration $[0, 1]$, so

$$\int_0^1 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}$$

so it is clear that we don't need the function to be everywhere continuous for FTC to work. Having continuity on the interval of integration, here $[0, 1]$, is sufficient. The evaluation part of FTC is actually true without strict continuity.

(c) If you have a closer look at the proof for FTC, you will find we actually only need F to be differentiable, not continuously differentiable. So $f(x)$ need not be continuous. The reason for assuming f to be continuous at this stage is to avoid unnecessary complications which are known as improper integrals. We will talk about improper integrals next week. For this example, we can apply the property of definite integrals splitting the interval of integration at $x = 2$, which can be proven by splitting the Riemann sum

$$\lim_{\|P\|\to 0}\sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{\|P_1\|\to 0}\sum_{k=1}^{(m-1)} f(x_k^*)\Delta x_k + \lim_{\|P_2\|\to 0}\sum_{k=m}^{n} f(x_k^*)\Delta x_k$$

Then consider the functions

$$g(x) = x \qquad \text{and} \qquad h(x) = x^2$$

Both are continuous on $[-1, 2]$ and $[2, 4]$, respectively. So, we can apply FTC to each

$$\begin{aligned}\int_{-1}^4 f(x)\,dx &= \int_{-1}^2 f(x)\,dx + \int_2^4 f(x)\,dx \\ &= \int_{-1}^2 x\,dx + \int_2^4 x^2\,dx = \frac{3}{2} + \frac{56}{3} = \frac{121}{6}\end{aligned}$$

Question5 (24 points)

Find the followings. Show all your workings.

(a) (1 point) $\displaystyle\int_4^9 2x\sqrt{x}\,dx$

(b) (1 point) $\displaystyle\int_{\sqrt{2}}^2 \frac{dx}{x\sqrt{x^2-1}}$

(c) (1 point) $\displaystyle\int \frac{x^2}{x+1}\,dx$

(d) (1 point) $\displaystyle\int_0^1 xe^{-x^2/2}\,dx$

(e) (1 point) $\displaystyle\int x\ln x\,dx$

(f) (1 point) $\displaystyle\int \sin(\ln x)\,dx$

(g) (1 point) $\displaystyle\int_1^3 \sqrt{x}\arctan\sqrt{x}\,dx$

(h) (1 point) $\displaystyle\int \frac{x^2}{\sqrt{16-x^2}}\,dx$

(i) (1 point) $\displaystyle\int \frac{dx}{(4x^2-9)^{3/2}}$

(j) (1 point) $\displaystyle\int_1^3 \frac{dx}{x^4\sqrt{x^2+3}}$

(k) (1 point) $\displaystyle\int \frac{dx}{x^2-3x-4}$

(l) (1 point) $\displaystyle\frac{d}{dx}\int_1^x \sin\left(t^2\right)\,dt$

(m) (1 point) $\displaystyle\frac{d}{dx}\int_x^0 t\sec t\,dt$

(n) (1 point) $\displaystyle\int \sin x\cos(x/2)\,dx$

(o) (1 point) $\displaystyle\int_0^{\pi/3} \sin^4 3x\cos^3 3x\,dx$

(p) (1 point) $\displaystyle\int \tan^4 x\sec x\,dx$

(q) (1 point) $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,dx$

(r) (1 point) $\displaystyle\int \frac{x^3-2x^2+2x-2}{x^2+1}\,dx$

(s) (1 point) $\displaystyle\int \frac{xe^{2x}}{(1+2x)^2}\,dx$

(t) (1 point) $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x-\cos x}$

(u) (2 points) $\displaystyle\lim_{x\to\infty}\frac{1}{\sqrt{x}}\int_1^x \frac{dt}{\sqrt{t}}$

(v) (2 points) $\displaystyle\frac{d^2}{dx^2}\int_0^x\left(\int_1^{\sin t}\sqrt{1+u^4}\,du\right)dt$

Solution:

(a) Simplifying and applying FTC, we have

$$\int_4^9 2x\sqrt{x}\,dx=\int_4^9 2x^{3/2}\,dx=2\left[\frac{2}{5}x^{5/2}\right]_4^9=\frac{4}{5}\left(3^5-2^5\right)=\frac{844}{5}$$

(b) Note the integrand has

$$\sqrt{x^2-1}$$

in the denominator, so the natural domain of the integrand is

$$x<-1 \qquad \text{and} \qquad x>1$$

The interval of integration is $\sqrt{2}\le x\le 2$, thus we need to use the trig substitution

$$x=g(\theta)=\sec\theta \qquad \text{for} \qquad 0<\theta<\frac{\pi}{2}$$

The domain of g will ensure the interval $x>1$, thus

$$\sqrt{2}\le x\le 2$$

is in the range of g. The derivative of g is given by

$$g' = \sec\theta\tan\theta \qquad \text{for} \qquad 0 < \theta < \frac{\pi}{2}$$

Note both $f(x)$ and $g'(\theta)$ are continuous on their domains, so we expect to be able to apply the substitution without any trouble

$$\theta = g^{-1}(x) = \sec^{-1} x \implies \begin{array}{l} \theta_1 = \sec^{-1}\sqrt{2} = \dfrac{\pi}{4} \\ \theta_2 = \sec^{-1} 2 = \dfrac{\pi}{3} \end{array}$$

Applying the u-substitution formula in reverse

$$\int_{x_1}^{x_2} f(g(x))g'(x)\,dx = \int_{u_1}^{u_2} f(u)\,du \quad ; \quad \int_{\theta_1}^{\theta_2} f(g(\theta))g'(\theta)\,d\theta = \int_{x_1}^{x_2} f(x)\,dx$$

We have

$$\begin{aligned} \int_{\sqrt{2}}^{2} \frac{1}{x\sqrt{x^2-1}}\,dx &= \int_{\pi/4}^{\pi/3} \frac{1}{\sec\theta\sqrt{\sec^2\theta - 1}} \sec\theta\tan\theta\,d\theta \\ &= \int_{\pi/4}^{\pi/3} \frac{\tan\theta}{\tan\theta}\,d\theta = \Big[\theta\Big]_{\pi/4}^{\pi/3} = \frac{\pi}{12} \end{aligned}$$

Note there is no need to take the absolute value of

$$\tan\theta = \sqrt{\sec^2\theta - 1} = \sqrt{\tan^2\theta}$$

since it is always positive for $0 < \theta < \dfrac{\pi}{2}$

(c) The integrand is an improper rational function, so applying polynomial division,

$$\begin{array}{r|l} & \quad\; x - 1 \\ \hline x+1 & x^2 \\ & -x^2 - x \\ \hline & \quad -x \\ & \quad\; x + 1 \\ \hline & \qquad 1 \end{array}$$

Therefore, we can rewrite the integral as the following

$$\int \frac{x^2}{x+1}\,dx = \int \left(x - 1 + \frac{1}{x+1}\right) dx = \frac{1}{2}x^2 - x + \ln|x+1| + C$$

where C is an arbitrary constant.

(d) Applying u-substitution with $u = g(x) = \frac{1}{2}x^2$, which has $g'(x) = x$, and

$$u_1 = \frac{1}{2} \cdot 0^2 = 0 \qquad \text{and} \qquad u_2 = \frac{1}{2}1^2 = \frac{1}{2}$$

thus we have

$$\int_0^1 xe^{-x^2/2}\,dx = \int_0^{1/2} e^{-u}\,du = \Big[-e^{-u}\Big]_0^{1/2} = -e^{-1/2} + 1$$

(e) Applying integration by parts and following the LIATE rule, we let

$$f(x) = \ln x \implies f'(x) = \frac{1}{x} \qquad \text{and} \qquad g(x) = x \implies G(x) = \frac{1}{2}x^2$$

then

$$\begin{aligned}\int x \ln x \, dx = f(x)G(x) - \int f'(x)G(x)\, dx &= \frac{1}{2}x^2 \ln x - \int \frac{1}{2}x\, dx \\ &= \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + C\end{aligned}$$

where C is an arbitrary constant.

(f) A composition made of sine and logarithmic function is very difficult to deal. So let

$$u = g(x) = \ln x \implies g'(x)\frac{1}{x} \qquad \text{and} \qquad x = e^u$$

Applying u-substitution, we have

$$\int \sin(\ln x)\, dx = \int x \sin(\ln x) \frac{1}{x}\, dx = \int e^u \sin u\, du$$

Applying integration by parts and following the LIATE rule, we let

$$f(u) = \sin u \implies f'(u) = \cos u \qquad \text{and} \qquad g(u) = e^u \implies G(x) = e^u$$

then

$$\begin{aligned}\int \sin(\ln x)\, dx = \int e^u \sin u\, du &= f(u)G(u) - \int f'(u)G(u)\, du \\ &= \sin(u)e^u - \int \cos(u)e^u\, du\end{aligned}$$

Applying integration by parts once more, we have

$$\begin{aligned}\int \sin(\ln x)\, dx = \int e^u \sin u\, du &= \sin(u)e^u - \int \cos(u)e^u\, du \\ &= \sin(u)e^u - \cos(u)e^u - \int \sin(u)e^u\, du\end{aligned}$$

Solving for $\int \sin(u)e^u\, du$, we have

$$\int \sin(u)e^u\, du = \frac{\sin u - \cos u}{2} \cdot e^u + C$$

Therefore,

$$\int \sin(\ln x)\, dx = \int \sin(u)e^u\, du = \frac{\sin(\ln x) - \cos(\ln x)}{2} \cdot x + C$$

(g) The integrand is a composition function of inverse tangent and square root function, so let us consider $u = g(x) = \sqrt{x}$ to simplify it, thus we have

$$\implies \begin{aligned} u_1 &= \sqrt{1} = 1 \\ u_2 &= \sqrt{3} \end{aligned} \qquad \text{and} \qquad g'(x) = \frac{1}{2}x^{-1/2}$$

Applying u-substitution, we have

$$\int_1^3 \sqrt{x}\arctan\sqrt{x}\,dx = 2\int_1^3 x\arctan\sqrt{x}\frac{1}{2}x^{-1/2}\,dx = 2\int_1^{\sqrt{3}} u^2\arctan u\,du$$

Applying integration by parts and following the LIATE rule, we let

$$f(u) = \arctan(u) \implies f'(u) = \frac{1}{1+u^2} \qquad \text{and} \qquad g(u) = u^2 \implies G(u) = \frac{1}{3}u^3$$

Thus we have the following by integration by parts

$$\begin{aligned} \int_1^3 \sqrt{x}\arctan\sqrt{x}\,dx &= 2\int_1^{\sqrt{3}} u^2\arctan u\,du \\ &= \frac{2}{3}\left[u^3\arctan u\right]_1^{\sqrt{3}} - \frac{2}{3}\int_1^{\sqrt{3}}\frac{u^3}{1+u^2}\,du \end{aligned}$$

Applying polynomial division, the remaining integral becomes

$$\begin{aligned} \int_1^{\sqrt{3}}\frac{u^3}{1+u^2}\,du &= \int_1^{\sqrt{3}} u\,du - \int_1^{\sqrt{3}}\frac{u}{1+u^2}\,du \\ &= \frac{1}{2}\left[u^2\right]_1^{\sqrt{3}} - \int_1^{\sqrt{3}}\frac{u}{1+u^2}\,du \end{aligned}$$

Applying another substitution $v = 1+u^2$, the remaining integral becomes

$$\int_1^{\sqrt{3}}\frac{u}{1+u^2}\,du = \frac{1}{2}\int_2^4\frac{1}{v}\,dv = \frac{1}{2}\left[\ln v\right]_2^4$$

Putting everything together, we have

$$\begin{aligned} \int_1^3 \sqrt{x}\arctan\sqrt{x}\,dx &= \frac{2}{3}\left[u^3\arctan u\right]_1^{\sqrt{3}} - \frac{1}{3}\left[u^2\right]_1^{\sqrt{3}} + \frac{1}{3}\left[\ln v\right]_2^4 \\ &= \frac{2\sqrt{3}\pi}{3} - \frac{\pi}{6} - \frac{2}{3} + \frac{1}{3}\ln 2 \end{aligned}$$

(h) Let us consider the trigonometric substitution

$$x = g(\theta) = a\sin\theta$$

Note if we choose the domain to be $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $x = g(\theta)$ is between

$$-a \le g(\theta) \le a$$

which covers all possible values of x in the natural domain of

$$\sqrt{a^2 - x^2}$$

In the interval of $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, the derivative function

$$g'(\theta) = a\cos\theta$$

is continuous and always positive, so there is no need to take the absolute value of it

$$a\cos\theta = \sqrt{a^2 - a\sin^2\theta} = a\sqrt{\cos^2\theta}$$

So if the integrand is continuous, then we can apply the trigonometric substitution as it is without any additional adjustment. If the integrand is not continuous at certain values of x inside the natural domain of

$$\sqrt{a^2 - x^2}$$

then technically we have to further modify the domain of

$$g(\theta) = a\sin\theta$$

For example, in this case, $x \neq \pm 4$ since

$$\sqrt{4^2 - x^2}$$

is in the denominator of the integrand. In terms of θ, we need to remove

$$\theta = -\frac{\pi}{2} \qquad \text{and} \qquad \theta = \frac{\pi}{2}$$

from the domain of $g(\theta)$. Thus technically we consider the trigonometric substitution

$$x = g(\theta) = 4\sin\theta \qquad \text{for} \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

which leads us to

$$x^2 = 16\sin^2\theta, \qquad g'(\theta) = 4\cos\theta \qquad \text{and} \qquad \theta = \arcsin\frac{x}{4}$$

Applying the following formula $\int f(x)\,dx = \int f(g(\theta))g'(\theta)\,d\theta$, we have

$$\begin{aligned}
\int \frac{x^2}{\sqrt{16-x^2}}\,dx &= \int \frac{16\sin^2\theta}{\sqrt{16-16\sin^2\theta}} 4\cos\theta\,d\theta \\
&= \int \frac{16\sin^2\theta 4\cos\theta}{4\cos\theta}\,d\theta \\
&= \int 16\sin^2\theta\,d\theta \\
&= 16\left(-\frac{1}{2}\sin\theta\cos\theta + \frac{1}{2}\theta\right) + C \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= -8\frac{x}{4}\cos\left(\arcsin\frac{x}{4}\right) + 8\arcsin\frac{x}{4} + C \qquad \text{for} \quad -4 < x < 4 \\
&= -\frac{x\sqrt{16-x^2}}{2} + 8\arcsin\frac{x}{4} + C \qquad \text{for} \quad -4 < x < 4
\end{aligned}$$

where C is an arbitrary constant.

(i) In general, having

$$\sqrt{x^2-a^2}$$

involves trigonometric substitution

$$x = g(\theta) = a\sec\theta$$

Note if we choose the domain to be

$$\begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if} \quad x \ge a, \\ \pi \le \theta < \dfrac{3\pi}{2} & \text{if} \quad x \le -a. \end{cases}$$

the substitution $x = g(\theta)$ will cover all possible values of x in the natural domain of

$$\sqrt{x^2-a^2}$$

In this domain, the derivative function of $g(\theta)$

$$g'(\theta) = a\sec\theta\tan\theta$$

is always continuous and positive, so there is no need to take the absolute value of it

$$a\tan\theta = \sqrt{\sec^2\theta - a^2} = a\sqrt{\tan^2\theta}$$

So if the integrand is continuous, then we apply the trigonometric substitution as it is without any additional modification. If the integrand is not continuous at some values of x inside the natural domain of

$$\sqrt{x^2-a^2}$$

then we might have to modify the domain of

$$g(\theta) = a\sec\theta$$

For example, in this case, $x = \pm\frac{3}{2}$ since

$$\sqrt{4x^2 - 9}$$

is in the denominator of the integrand. In terms of θ, we need to remove

$$\theta = 0 \qquad \text{and} \qquad \theta = \pi$$

from the domain of $g(\theta)$. So in this case we consider the trigonometric substitution

$$x = g(\theta) = \frac{3}{2}\sec\theta \qquad \text{for} \qquad \begin{cases} 0 < \theta < \dfrac{\pi}{2} & \text{if} \quad x > \frac{3}{2}, \\ \pi < \theta < \dfrac{3\pi}{2} & \text{if} \quad x < -\frac{3}{2}. \end{cases}$$

Applying the following formula and the identity $\sec^2\theta - 1 = \tan^2\theta$, we have

$$\int f(x)\,dx = \int f(g(\theta))g'(\theta)\,d\theta$$

$$\int \frac{1}{(4x^2-9)^{3/2}}\,dx = \int \left(4\frac{9}{4}\sec^2\theta - 9\right)^{-3/2} \frac{3}{2}\sec\theta\tan\theta\,d\theta = \frac{1}{18}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta$$

Applying the u-substitution $u = g(\theta) = \sin\theta \implies g'(\theta) = \cos\theta$, we have

$$\int \frac{1}{(4x^2-9)^{3/2}}\,dx = \frac{1}{18}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{18}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{18}\int \frac{1}{u^2}\,du = \frac{-1}{18u} + C$$

Back substituting, we have

$$\begin{aligned} \int \frac{1}{(4x^2-9)^{3/2}}\,dx &= \frac{-1}{18u} + C \\ &= \frac{-1}{18\sin\theta} + C \\ &= \frac{-1}{18}\left(\sqrt{1 - \frac{1}{\sec^2\theta}}\right)^{-1} + C \\ &= \frac{-1}{18}\left(\sqrt{1 - \frac{9}{4x^2}}\right)^{-1} + C \end{aligned}$$

where C is an arbitrary constant. Both of the following intervals

$$x < -\frac{3}{2} \qquad \text{and} \qquad x > \frac{3}{2}$$

are valid intervals for the antiderivatives, so technically it is better to write

$$\int \frac{1}{(4x^2-9)^{3/2}}\,dx = \begin{cases} \frac{-1}{18}\left(\sqrt{1-\frac{9}{4x^2}}\right)^{-1} + C_1 & \text{for} \quad x < -\frac{3}{2} \\ \frac{-1}{18}\left(\sqrt{1-\frac{9}{4x^2}}\right)^{-1} + C_2 & \text{for} \quad x > \frac{3}{2} \end{cases}$$

However, often the domain of an indefinite integral is not emphasised in an applied calculus course, specially when there are multiple intervals over which it is defined. So I will not be too picky about it.

(j) Consider the trigonometric substitution $x = \sqrt{3}\tan\theta$, we have

$$g'(\theta) = \sqrt{3}\sec^2\theta \qquad \text{and} \qquad \begin{aligned} \theta_1 &= \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6} \\ \theta_2 &= \arctan\frac{3}{\sqrt{3}} = \frac{\pi}{3} \end{aligned}$$

I will leave it to you to argue the correct interval to use is

$$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Applying the trigonometric substitution and the identity $\sec^2\theta = \tan^2\theta + 1$, we have

$$\begin{aligned} \int f(x)\,dx &= \int f(g(\theta))g'(\theta)\,d\theta \\ \int_1^3 \frac{1}{x^4\sqrt{x^2+3}}\,dx &= \int_{\pi/6}^{\pi/3} \frac{1}{9\tan^4\theta\sqrt{3\tan^2\theta+3}}\sqrt{3}\sec^2\theta\,d\theta \\ &= \frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{\sec\theta}{\tan^4\theta}\,d\theta \\ &= \frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{\cos^3\theta}{\sin^4\theta}\,d\theta \end{aligned}$$

Applying u-substitution $u = \sin\theta \implies u' = \cos\theta$, we have

$$\begin{aligned} \int_1^3 \frac{1}{x^4\sqrt{x^2+3}}\,dx = \frac{1}{9}\int_{\pi/6}^{\pi/3} \frac{1-\sin^2\theta}{\sin^4\theta}\cos\theta\,d\theta &= \frac{1}{9}\int_{1/2}^{\sqrt{3}/2} \frac{1-u^2}{u^4}\,du \\ &= \frac{1}{9}\int_{1/2}^{\sqrt{3}/2} \left(u^{-4} - u^{-2}\right)\,du \\ &= \frac{1}{9}\left[-\frac{1}{3}u^{-3} + u^{-1}\right]_{1/2}^{\sqrt{3}/2} \\ &= \frac{10\sqrt{3}}{243} + \frac{2}{27} \end{aligned}$$

(k) Factor the denominator, we see it has the following the partial fraction decomposition

$$\int \frac{1}{(x-4)(x+1)}\,dx = \frac{1}{5}\int \frac{1}{x-4} - \frac{1}{5}\int \frac{1}{x+1} = \frac{1}{5}\ln|x-4| - \frac{1}{5}\ln|x+1| + C$$

where C is an arbitrary constant.

(l) Applying FTC, we have

$$\frac{d}{dx}\int_1^x \sin(t^2)\,dt = \sin(x^2)$$

(m) Switching the upper limit with the lower limit, we have

$$\frac{d}{dx}\int_x^0 t\sec t\,dt = -\frac{d}{dx}\int_0^x t\sec t\,dt = -x\sec x$$

(n) Using the trigonometric identity $\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha-\beta)+\sin(\alpha+\beta)\right]$, we have

$$\int \sin x\cos\frac{x}{2}\,dx = \frac{1}{2}\int\left(\sin\frac{x}{2}+\sin\frac{3x}{2}\right)dx = -\cos\frac{x}{2}-\frac{1}{3}\cos\frac{3x}{2}+C$$

where C is an arbitrary constant.

(o) Applying u-substitution $u = g(x) = \sin 3x \implies g'(x) = 3\cos 3x$, we have

$$\int_0^{\pi/3}\sin^4 3x\cos^3 3x\,dx = \frac{1}{3}\int_0^0 u^4(1-u^2)\,du = 0$$

(p) Using the identity $\sec^2 x - 1 = \tan^2 x$ and applying reduction formula, we have

$$\begin{aligned}\int \tan^4 x\sec x\,dx &= \int(\sec^2 x-1)^2\sec x\,dx\\ &= \int\left(\sec^5 x - 2\sec^3 x+\sec x\right)dx\\ &= \frac{1}{4}\sec^3 x\tan x+\frac{3}{8}\ln|\sec x+\tan x| - \frac{5}{8}\sec x\tan x + C\end{aligned}$$

where C is an arbitrary constant.

(q) Considering the partial fraction decomposition of the integrand, we have

$$\begin{aligned}\int\frac{2x^2+3}{x(x-1)^2}\,dx &= \int\left(\frac{3}{x}-\frac{1}{x-1}+\frac{5}{(x-1)^2}\right)dx\\ &= 3\ln|x|-\ln|x-1|-\frac{5}{x-1}+C\end{aligned}$$

where C is an arbitrary constant.

(r) Computing the polynomial division, we have

$$\begin{aligned}\int\frac{x^3-2x^2+2x-2}{x^2+1}\,dx &= \int\left(x-2+\frac{x}{x^2+1}\right)dx\\ &= \frac{1}{2}x^2-2x+\frac{1}{2}\ln(x^2+1)+C\end{aligned}$$

where C is an arbitrary constant.

(s) The integrand is a product between a rational function and an exponential function,

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx = \int e^{2x}\frac{x}{(1+2x)^2}\,dx$$

Applying the partial fraction decomposition on the second factor, we have

$$\begin{aligned}\int \frac{xe^{2x}}{(1+2x)^2}\,dx &= \frac{1}{2}\int e^{2x}\left(\frac{1}{1+2x} - \frac{1}{(1+2x)^2}\right)dx \\ &= \frac{1}{2}\int e^{2x}\frac{1}{1+2x}\,dx - \frac{1}{2}\int e^{2x}\frac{1}{(1+2x)^2}\,dx\end{aligned}$$

Applying the u-substitution of $u = 2x \implies u' = 2$, we have

$$\int \frac{xe^{2x}}{(1+2x)^2}\,dx = \frac{1}{4}\int e^u\frac{1}{1+u}\,du - \frac{1}{4}\int e^u\frac{1}{(1+u)^2}\,du$$

Applying integration by parts and following the LIATE rule, we have

$$f(u) = \frac{1}{1+u} \implies f'(u) = \frac{-1}{(1+u)^2} \qquad \text{and} \qquad g(u) = e^u \implies G(u) = e^u$$

$$\implies \int e^u\frac{1}{1+u}\,du = \frac{e^u}{1+u} + \int \frac{e^u}{(1+u)^2}\,du$$

Therefore

$$\begin{aligned}\int \frac{xe^{2x}}{(1+2x)^2}\,dx &= \frac{1}{4}\int e^u\frac{1}{1+u}\,du - \frac{1}{4}\int e^u\frac{1}{(1+u)^2}\,du \\ &= \frac{1}{4}\frac{e^u}{1+u} + \frac{1}{4}\int\frac{e^u}{(1+u)^2}\,du - \frac{1}{4}\int e^u\frac{1}{(1+u)^2}\,du \\ &= \frac{1}{4}\frac{e^{2x}}{1+2x} + C\end{aligned}$$

where C is an arbitrary constant.

(t) Applying the Weierstrass substitution $u = g(x) = \tan\dfrac{x}{2}$, we have

$$\sin x = \frac{2u}{1+u^2}, \qquad \cos x = \frac{1-u^2}{1+u^2} \qquad \text{and} \qquad u = g'(x) = \frac{1+u^2}{2}$$

thus the original integral is equal to

$$\begin{aligned}\int_{\pi/3}^{\pi/2} \frac{1}{1+\sin x-\cos x}\,dx &= \int_{\sqrt{3}/3}^{1} \frac{2}{1+u^2}\frac{1}{1+\dfrac{2u}{1+u^2}-\dfrac{1-u^2}{1+u^2}}\,du \\ &= \int_{\sqrt{3}/3}^{1} \frac{1}{u(u+1)}\,du \\ &= \int_{\sqrt{3}/3}^{1} \left(\frac{1}{u}-\frac{1}{u+1}\right) du \\ &= \Big[\ln(u)-\ln(u+1)\Big]_{\sqrt{3}/3}^{1} = \ln\frac{\sqrt{3}+1}{2}\end{aligned}$$

(u) Notice there are two limit processes. Let us compute the inner limit first,

$$\begin{aligned}\lim_{x\to\infty}\frac{1}{\sqrt{x}}\int_1^x \frac{dt}{\sqrt{t}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}}\left[2t^{1/2}\right]_1^x &= \lim_{x\to\infty}\frac{1}{\sqrt{x}}\left(2\sqrt{x}-2\right) \\ &= \lim_{x\to\infty}\left(2-\frac{2}{\sqrt{x}}\right) = 2\end{aligned}$$

(v) Rewriting the expression and this time using FCT, we have

$$\begin{aligned}\frac{d^2}{dx^2}\int_0^x\left(\int_1^{\sin t}\sqrt{1+u^4}\,du\right)dt &= \frac{d}{dx}\left(\frac{d}{dx}\int_0^x\left(\int_1^{\sin t}\sqrt{1+u^4}\,du\right)dt\right) \\ &= \frac{d}{dx}\left(\int_1^{\sin x}\sqrt{1+u^4}\,du\right) \\ &= \frac{d}{dv}\left(\int_1^{v=\sin x}\sqrt{1+u^4}\,du\right)\frac{dv}{dx} \\ &= \cos x\sqrt{1+\sin^4 x}\end{aligned}$$