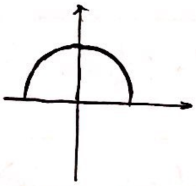
Parametric

	Cartesian formula	Parametric formula.
① Area	$\int_{x_1}^{x_2} y\,dx$	$\int_{t_1}^{t_2} y(t)\,x'(t)\,dt.$
② Arclength.	$\int_{x_1}^{x_2}\sqrt{1+(\frac{dy}{dx})^2}\,dx = \int_{x_1}^{x_2}\sqrt{dx^2+dy^2}$	$\int_{t_1}^{t_2}\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt$
③ Volume. Revolving about x-axis	$\int_{x_1}^{x_2} \pi y^2\,dx$	$\int_{t_1}^{t_2} \pi y^2(t)\,x'(t)\,dt.$
④. Surface Area. Revolving about x-axis	$\int_{x_1}^{x_2} 2\pi y\sqrt{1+(\frac{dy}{dx})^2}\,dx.$	$\int_{t_1}^{t_2} 2\pi y(t)\sqrt{(\frac{dx}{dt})^2+(\frac{dy}{dt})^2}\,dt.$

Ex: ① find the volume of a ball with radius R. also surface area



$x = R\cos t$

$y = R\sin t$

$$V = \int_{\pi}^{0} \pi R^2\sin^2 t \cdot R(-\sin t)\,dt$$
$$= \pi R^3\int_0^{\pi}\sin^3 t\,dt$$
$$= \pi R^3 \times \frac{2}{3}\times 2$$
$$= \frac{4}{3}\pi R^3$$

$$A = \int_{\pi}^{0} 2\pi R\sin t\sqrt{R^2}\,dt$$
$$= 2\pi R^2\left[-\cos t\right]_{\pi}^{0}$$
$$= 4\pi R^2.$$

$$*:\quad \int_0^{\frac{\pi}{2}}\sin^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2} & n\text{: even number} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3} & n\text{: odd number.} \end{cases}$$

also. with symmetry

$\int_0^{\frac{\pi}{2}}\cos^n x\,dx$ is the same. $\quad \int_0^{\pi}\sin^4 x\,dx = 2\cdot\int_0^{\frac{\pi}{2}}\sin^4 x\,dx$ while $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3 x\,dx = 0$ ——

If you have interest, plz use reduction of order to prove it. :)

② How to remember and derive the formula.

Chain rule. & Substitution.

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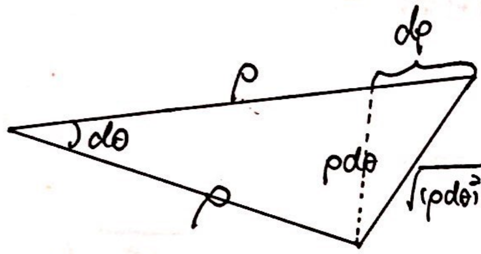
Special Case. Polar coordinate.

Arc length $\displaystyle\int_{\theta_1}^{\theta_2} \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta.$

Area. $\displaystyle\int_{\theta_1}^{\theta_2} \frac{1}{2}\rho^2\, d\theta$

How to derive or understand?

Intuitively.



$\sqrt{(\rho d\theta)^2 + (d\rho)^2} = \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta.$

Remember.

Be more careful?

$$dl = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

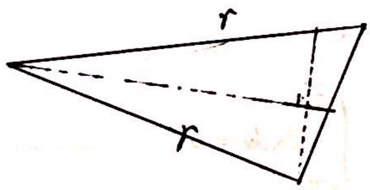
$$\begin{cases} x = r(\theta)\cos\theta \\ y = r(\theta)\sin\theta. \end{cases}$$

$$dl = \sqrt{\left(\frac{dr}{d\theta}\cos\theta - r\sin\theta\right)^2 + \left(\frac{dr}{d\theta}\sin\theta + r\cos\theta\right)^2}$$

$$= \sqrt{\left(\left(\frac{dr}{d\theta}\right)^2 + r^2\right)\cancel{\left(\sin^2\theta + \cos^2\theta\right)}}.$$

Believe.

Intuitively



$$dA = \frac{1}{2} \cdot r d\theta \cdot r$$
$$= \frac{1}{2} r^2 d\theta.$$

More Precisely.

$\cancel{dA = ydx.}$

$\cancel{= r\sin\theta}$

Just use Riemum Sum.

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Improper Integral.

Type 1. Infinite Intervals

$$\int_a^{\infty} f(x)dx = \lim_{t\to+\infty}\int_a^t f(x)dx$$

Type 2 Discontinuous integrands

1) If f is discontinuous at b

$$\int_a^b f(x)dx = \lim_{t\to b^-}\int_a^t f(x)dx$$

2) If f is discontinuous at a.

$$\int_a^b f(x)dx = \lim_{t\to a^+}\int_t^b f(x)dx$$

3) If $f(x)$ is discontinuous at c $(a<c<b)$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx.$$

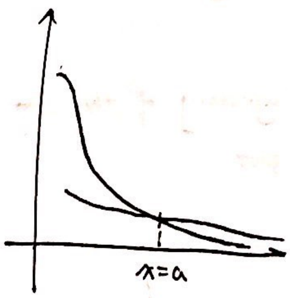
Comparison test.

If f & g are continuous

$f(x) \geq g(x) \geq 0$ for $x \geq a$.

$\int_a^{\infty} f(x)dx$ converges $\Rightarrow$ $\int_a^{\infty} g(x)dx$ converges

$\int_a^{\infty} g(x)dx$ diverges $\Rightarrow$ $\int_a^{\infty} f(x)$ diverges.



* Convergent: finite area.

Divergent: Infinite area.

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Ex. Find the arclength of $r = a\theta \quad 0 \le \theta \le 6\pi$

$$\int_0^{6\pi} \sqrt{r^2+a^2}\, d\theta = \int_0^{6\pi} a\sqrt{1+\theta^2}\, d\theta$$

$\underline{\underline{\theta = \tan u}}$

$$\int_0^{\arctan 6\pi} a\sec^3 u\, du = \frac{a}{2}\left[6\pi\sqrt{1+36\pi^2} + \ln\left| 6\pi + \sqrt{1+36\pi^2} \right| \right]$$

$$* \int \sec^3 x\, dx = \tan x \sec x - \int \tan^2 x \sec x\, dx$$

$$= \tan x \sec x - \int (\sec^2 x - 1)\sec x\, dx$$

$$\therefore 2\int \sec^3 x\, dx = \tan x \sec x + \int \sec x\, dx$$

$$\therefore \int \sec^3 x\, dx = \frac{1}{2}\tan x \sec x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

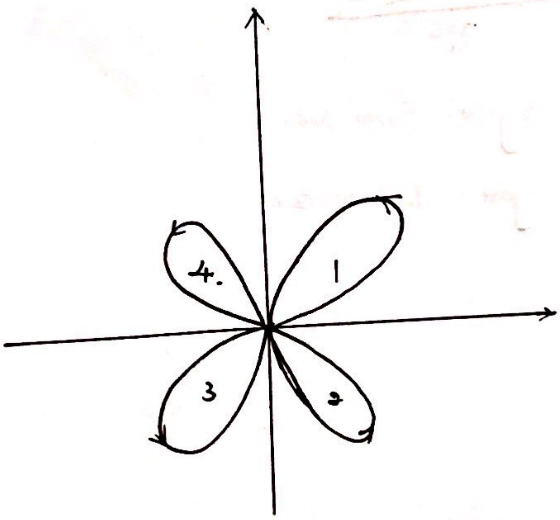
Ex. Find the area of $r = \sin 2\theta$. $\theta \in [0, 2\pi]$.

$$\int_0^{2\pi} \frac{1}{2}\sin^2 2\theta\, d\theta = \int_0^{2\pi} \frac{1}{2}\cdot\frac{1-\cos 4\theta}{2}\, d\theta = \frac{1}{4}\times 2\pi = \frac{\pi}{2}$$

Suppose an ant is moving on this curve. and the position of it is given by:

$\theta = t$

determine the trajectory of the ant (including the direction).
on the following plot.



$$\begin{cases} x = \sin 2t \cos t \\ y = \sin 2t \sin t \end{cases} \Rightarrow \begin{cases} \dot{x} = 2\cos 2t \cos t - \sin 2t \sin t \\ \dot{y} = 2\cos 2t \sin t + \sin 2t \cos t \end{cases}$$

when $t = \frac{\pi}{4}$.	$\frac{3}{4}\pi$.	$\frac{5}{4}\pi$	$\frac{7}{4}\pi$.
$x > 0$	$x > 0$	$x < 0$	$x < 0$
$y > 0$	$y < 0$	$y < 0$	$y > 0$
$\dot{x} < 0$	$\dot{x} > 0$	$\dot{x} > 0$	$\dot{x} < 0$
$\dot{y} > 0$	$\dot{y} > 0$	$\dot{y} < 0$	$\dot{y} < 0$

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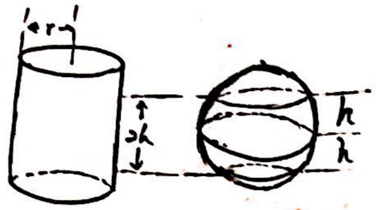
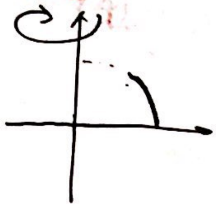
Exercise:

1).





Show that the area is the same.

$A_1 = 2\pi R \cdot 2h.$

$$A_2 = 2 \cdot \int_{x_1}^{x_2} 2\pi x \sqrt{dx^2 + dy^2}$$

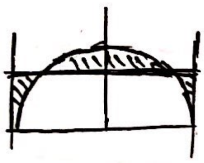
$$= 2 \cdot \int_{0}^{\arcsin\frac{h}{R}} 2\pi \cdot R\cos\theta \cdot R \, d\theta.$$

$$= 4\pi R^2 \left[\sin\theta\right]_0^{\arcsin\frac{h}{R}}$$

$$= 4\pi R^2 \frac{h}{R}$$

$$= 4\pi h R$$

2) determine h to minimize the area.



Idea I. it must happen when $\frac{dA}{dh} = 0$ or at boundary.

$$\Rightarrow \quad x = \frac{R}{2} \quad \Rightarrow \quad h = \sqrt{R^2 - \frac{R^2}{4}} = \frac{\sqrt{3}}{2} R.$$

Idea 2.

$$A(h) = \int_{-R}^{R} |h - y| \, dx \qquad \begin{cases} x = R\cos\theta \\ y = R\sin\theta. \end{cases}$$

$$= 2 \cdot \int_0^R |h - y| \, dx$$

$$= 2\left\{ \int_{\frac{\pi}{2}}^{\arcsin\frac{h}{R}} (-h + R\sin\theta)(-R\sin\theta)\, d\theta + \int_{\arcsin\frac{h}{R}}^{0} (-R\sin\theta + h)(-R\sin\theta)\, d\theta. \right\}$$

$$= 2\left\{ \left[hR\cos\theta + R^2\frac{\theta}{2} - R^2\frac{1}{4}\sin 2\theta \right]_{\arcsin\frac{h}{R}}^{0} + \left[-hR\cos\theta - R^2\frac{\theta}{2} + R^2\frac{1}{4}\sin 2\theta \right]_{\frac{\pi}{2}}^{\arcsin\frac{h}{R}} \right\}$$

$$= 2\left\{ hR - hR\sqrt{1 - \left(\frac{h}{R}\right)^2} - R^2\frac{1}{2}\arcsin\frac{h}{R} + \cancel{\frac{R^2}{4} 2 \frac{h}{R}\sqrt{1 - \frac{h^2}{R^2}}} - hR\sqrt{1 - \frac{h^2}{R^2}} - R^2\frac{1}{2}\sin\frac{h}{R} + \cancel{R^2\frac{1}{4} 2\frac{h}{R}\sqrt{1 - \frac{h^2}{R^2}}} + R^2\frac{1}{2}\left(\frac{\pi}{2}\right) \right\}$$

$$= 2\left\{ hR - R^2\arcsin\frac{h}{R} - hR\sqrt{1 - \frac{h^2}{R^2}} + \frac{\pi}{4}R^2 \right\}$$

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$$R - \frac{R}{\sqrt{1-(\frac{h}{R})^2}} + \frac{\frac{h^2}{R}}{\sqrt{1-(\frac{h}{R})^2}} = 0$$

$$R^2\left(1-\frac{h^2}{R^2}\right) = \left(R-\frac{h^2}{R}\right)^2$$

$$(R^2-h^2) = R^2 - 2h^2 + \frac{h^4}{R^2}$$

$$\frac{h^4}{R^2} - h^2 = 0$$

$$A = 2\left\{hR - R^2\arcsin\frac{h}{R} - \sqrt{h^2R^2-h^4} + \frac{\pi}{2}R^2\right\}$$

$$\frac{dA}{dh} = 2\left\{R - R^2\cdot\frac{\frac{1}{R}}{\sqrt{1-(\frac{h}{R})^2}} - \frac{1}{2}\frac{2R^2h-4h^3}{\sqrt{h^2R^2-h^4}}\right\}$$

$$= 2\left\{R - \frac{R}{\sqrt{1-(\frac{h}{R})^2}} + \frac{2h^3-R^2h}{\sqrt{h^2R^2-h^4}}\right\}$$

$$R = \frac{R}{\sqrt{1-(\frac{h}{R})^2}} - \frac{2\frac{h^2}{R}-R}{\sqrt{1-(\frac{h}{R})^2}}$$

$$\therefore R^2\left(1-\frac{h^2}{R^2}\right) = 4\left(R-\frac{h^2}{R}\right)^2$$

$$R^2-h^2 = 4\left(R^2-2h^2+\frac{h^4}{R^2}\right)$$

$$(R^2-h^2) = 4(R^2-h^2)\frac{\left(R-\frac{h^2}{R}\right)}{R}.$$

$$\therefore 1 = 4\left(1-\frac{h^2}{R^2}\right)$$

$$\therefore \frac{h^2}{R^2} = \frac{\sqrt{3}}{2}$$

$$\therefore h = \frac{\sqrt{3}}{2}R$$

same answer.

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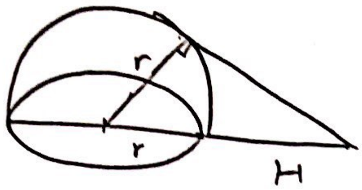
Show that an observer at a Height H above the north pole of a sphere of radius r can see a part of the sphere that has area.

$$\frac{2\pi r^2 H}{r+H}.$$

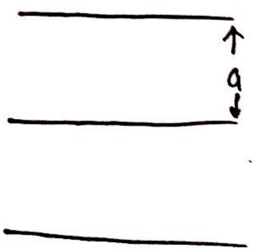


$$A = \int 2\pi y \sqrt{dx^2+dy^2} = \int_0^{\arccos\frac{r}{H+r}} 2\pi R\sin\theta \cdot R\, d\theta$$

$$= 2\pi R^2 \left[-\cos\theta\right]_0^{\arccos\frac{r}{H+r}}$$

$$= 2\pi R^2 \frac{H+r-r}{H+r}$$

$$= 2\pi r^2 \frac{H}{H+r}.$$



throw a needl, with length ℓ.

find the probability that the needles fall on the line.

① 走向

$$P = \frac{\int_0^{\ell} \frac{(\frac{\pi}{2}-\arcsin\frac{y}{\ell})\,dy}{\frac{\pi}{2}} + \int_{\ell}^{a} 0\, dy}{a.} \overset{*}{=} \frac{\ell}{\frac{\pi}{2}a} = \frac{2\ell}{\pi a}.$$

$$* \int (\frac{\pi}{2} - \arcsin\frac{y}{\ell})\, dy = \frac{\pi}{2}y - y\arcsin\frac{y}{\ell} + \int y \cdot \frac{\frac{1}{\ell}}{\sqrt{1-(\frac{y}{\ell})^2}}\, dy.$$

$$-\frac{2y}{\ell^2}$$

$$= \frac{\pi}{2}y - y\arcsin\frac{y}{\ell} + \int \frac{d(1-\frac{y^2}{\ell^2})}{\sqrt{1-(\frac{y}{\ell})^2}} (-\frac{\ell}{2})$$

$$= \left[\frac{\pi}{2}y - y\arcsin\frac{y}{\ell} - \frac{\ell}{2}\cdot 2\sqrt{1-(\frac{y}{\ell})^2}\right]_0^{\ell}$$

$$= \frac{\pi}{2}\ell - \frac{\pi}{2}\ell + \ell$$

$$= \ell$$

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