vv214: Introduction: systems of linear equations, graphs, even/oddtown, Markov chains.

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UM-SJTU Joint Institute

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This week

Today

1. Even/oddtown problems.
2. Card shuffling/Markov processes.
3. Graphs.
4. Systems of linear equations: motivation with two historical problems, solutions of a SLE, geometrical interpretations of linear systems.
5. Gauss-Jordan elimination.
6. Simplest input-output Leontief models.

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Next class

1. Matrices: coefficient, augmented, square, upper and lower triangular, identity.
2. Elementary transformations of a matrix.
3. Reduced row-echelon form of a matrix.
4. Rank of a matrix.

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  $\Rightarrow$ infinitely many clubs
- **Club rule 1:** No two clubs have exactly the same membership

$$\forall i \neq j \quad C_i \neq C_j \quad (\Rightarrow m \leq 2^n)$$

- **Club rule 2:** Every club must have an even number of people

$$\forall i \quad |C_i| = \text{even}$$

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- **Club rule 3:** Every two clubs must have an even number of common members

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Example: Let $n = 16$.Then 16 residents can form $255 = 2^8 - 1$ clubs in an eventown.

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Q: What happens if we shuffle the deck multiple times?

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Definition: The adjacency matrix of the graph G with vertices $v_1, \ldots, v_n$ is an $n \times n$ matrix $A = A_G = (a_{ij})_{n \times n}$ defined by

$$a_{ij} = \begin{cases} 1 & v_i \sim v_j \\ 0 & v_i \nsim v_j \end{cases}$$

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Theorem(Handshake theorem): The number of people who have made an odd number of hand shakes must be even.

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Theorem (Graph Coloring): If $deg(G) \leq d$, then G can be colored with $d + 1$ colors.

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It is assumed that each field initially provides the same amount x of grass that the daily growth y of the fields remains constant, and that all the cows eat the same amount z each day. Quantities x, y and z are measured by weight.
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$$N_{fields}x + N_{fields}N_{days}y = N_{cows}N_{days}z$$

Gauss-Jordan Elimination

1. Proceed from equation to equation, from top to bottom.
2. For the ith equation: let x_j be the leading (pivot) variable. If x_j does not appear in the ith equation, swap the ith equation with the first equation below that does contain x_j.
3. Multiply the ith equation by the appropriate scalar so that the coefficient of the leading variable becomes 1.
4. Eliminate x_j from all the other equations, above and below the ith by subtracting suitable multiples of the ith equation from the others.
5. Now proceed to the next equation.
6. If an equation zero $=$ nonzero emerges in this process, then the system fails to have solutions; the system is inconsistent.

Elementary Row Operations

Recall, that the following algebraic operations do not affect the solution(s) of systems of linear equations:

1. Swap equations.
2. Divide/multiply an equation by a nonzero scalar.
3. Add/substract a multiple of an equation to/from another one.

$\Rightarrow$ we can define the following **elementary row operations**:

1. Interchange two rows.
2. Multiply a row by a nonzero constant,
3. Add a multiple of one row to another one.

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Metrodorus, Greek Anthology, 6th century A.D.
"Make me a crown weighing 60 minae, mixing gold, bronze, tin, and wrought iron. Let the gold and bronze together form two-thirds, the gold and tin together three-fourths, and the gold and iron three-fifths. Tell me how much gold, tin, bronze, and iron you must put in."

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Hint:

$$\begin{cases} x+ & y & & & = \frac{2}{3}60 \\ x+ & & z & & = \frac{3}{4}60 \\ x+ & & & t & = \frac{3}{5}60 \\ x+ & y & +z & +t & = 60 \end{cases}$$

Simplest Leontief input-output model

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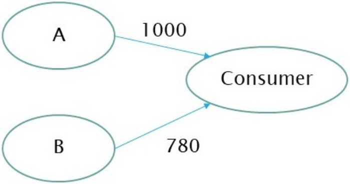

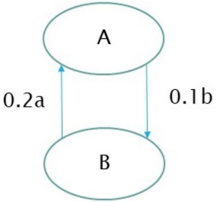

Assume that the consumer demand for their products is, respectively, 1,000 and 780, in millions of dollars per year. What outputs a and b (in millions of dollars per year) should the two industries generate to satisfy the demand?

Reduced Row-Echelon Form. Rank of a Matrix

Definition: A matrix is in reduced row-echelon form *rref* if it satisfies all of the following conditions:

1. If a row has nonzero entries, then the first nonzero entry is a 1, called the leading 1 in this row.
2. If a column contains a leading 1, then all the other entries in that column are 0.
3. If a row contains a leading 1, then each row above it contains a leading 1 further to the left

$\Rightarrow$ rows of 0's, if any, appear at the bottom of the matrix.

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Definition: The number of leading 1's in the *rref* A is called the rank of the matrix A.

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Questions

1. How many solutions may a system of linear equations have? What is a consistent system?
2. What is a matrix? What is a vector?
3. What are the elementary transformations of matrices?
4. When do we say that a matrix is in reduced row-echelon form?
5. What is the Gauss-Jordan elimination?
6. What is a rank of a matrix?
7. How is the rank of a coefficient matrix related to the number of solutions of a linear system?

The number of solutions and the rank of the coefficient matrix

Consider a linear system of equations n with m variables $\Rightarrow$ the coefficient matrix A of the system is $A_{n\times m}$.

1. $rank\ A \leq n$, $rank\ A \leq m$
2. If $rank\ A = n$ then the system is consistent.
3. If $rank\ A = m$ then the system has at most one solution.
4. If $rank\ A < m$ then the system either has infinitely many solutions OR inconsistent.

Remarks:

1. If $n < m$ then $rank\ A \leq n < m \Rightarrow$ infinitely many OR no solutions
2. If $n = m$ and

a. $rank\ A = n \Rightarrow$ there exists a unique solution

b. $rank\ A < n \Rightarrow$ infinitely many OR no solutions

Matrix Algebra

1. The sum of two matrices $A_{n \times m}$ and $B_{n \times m}$ is the matrix $C_{n \times m}$ s.t.
$$c_{ij} = a_{ij} + b_{ij},\ i = \overline{1,\ n}, j = \overline{1,\ m}.$$
2. The scalar product $\alpha A_{n \times m} = (\alpha a_{ij}), i = \overline{1,\ n}, j = \overline{1,\ m}.$
3. The product of a row-matrix $\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$ and a column-matrix $\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ is
$$\begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Matrix Algebra

4. The product of a matrix $A_{n\times m}$ and a vector $\bar{x} \in \mathbb{R}^m$ is

$$A_{n\times m}\bar{x} = \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \\ \vdots \\ \bar{w}_n \end{pmatrix} \bar{x} = (def) \begin{pmatrix} (\bar{w}_1, \bar{x}) \\ (\bar{w}_2, \bar{x}) \\ \vdots \\ (\bar{w}_n, \bar{x}) \end{pmatrix}$$

$$= (prop)(\bar{a}_1 \quad \bar{a}_2 \ldots \bar{a}_m)\bar{x} = x_1\bar{a}_1 + x_2\bar{a}_2 + \ldots + x_m\bar{a}_m$$

Next Week

- More practice with linear systems/rref/rank.
- Matrix Algebra.
- Number fields and linear spaces.
- Abelian groups and linear spaces. Cyclic groups.
- Linear combinations and linear dependence/independence.
- Structure of a linear space: basis, dimension.
- Lots of examples!