vv255:Functions of several variables.

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Today 05-27-2019

1. Functions of several variables.
2. Contour maps.
3. Limits and continuity.

Functions of several variables

Definition

Let $n > 1$ be a natural number. A real-valued function of n independent variables or just a function of n variables is a function $f : D \longrightarrow \mathbb{R}$ such that $D \subseteq \mathbb{R}^n$. We will systematically abuse notation and write $f(x_1, \ldots, x_n)$ for the value that f takes on $(x_1, \ldots, x_n) \in D$.

So, a real-valued function with $n > 1$ independent variables is a function that maps points in n-dimensional space to real numbers. In particular, a function of two variables is a function that maps points 2D space to real numbers. This means that a function of two variables, $f : D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^2$, can be visualised is 3D space by:

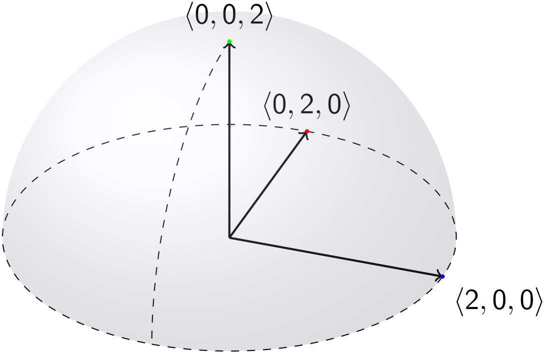
$$z = f(x, y)$$

This means that functions of two variables often describe surfaces in $\mathbb{R}^3$.

Functions of several variables

Example

The function $f(x,y) = \sqrt{4 - x^2 - y^2}$ *with domain* $D = \{(x,y) \mid x^2 + y^2 \leq 4\}$ *describes a hemisphere centred at* $(0,0,0)$ *of radius* 2*:*



Functions of several variables

Definition

Let $f : D \longrightarrow \mathbb{R}$ be a function of n variables where $n \geq 1$. The graph of f is collection of points in $\mathbb{R}^{n+1}$ defined by

$$\{(x_1, \ldots, x_n, y) \mid y = f(x_1, \ldots, x_n)\}$$

Definition

Let $f : D \longrightarrow \mathbb{R}$ be a function of n variables where $n \geq 1$ with independent variables $x_1, \ldots, x_n$. The function f is linear if there exists $a_0, \ldots a_n \in \mathbb{R}$ such that

$$f(x_1, \ldots, x_n) = a_0 + a_1 x_1 + \cdots + a_n x_n$$

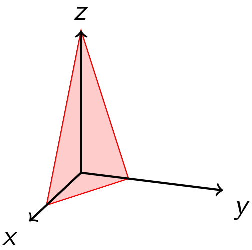
Linear functions of two variables specify planes in 3D space.

Functions of several variables

Example

Consider $f(x,y) = \frac{-3x-6y}{2} + 1$. *The graph of this function is the plane*

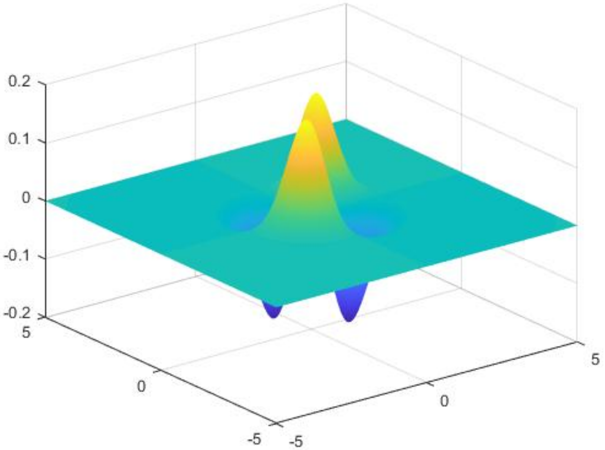
$$2z + 3x + 6y = 2$$



Functions of several variables

Example

Consider $f(x,y) = -xye^{-x^2-y^2}$. The graph of this function can be plotted using MatLab:



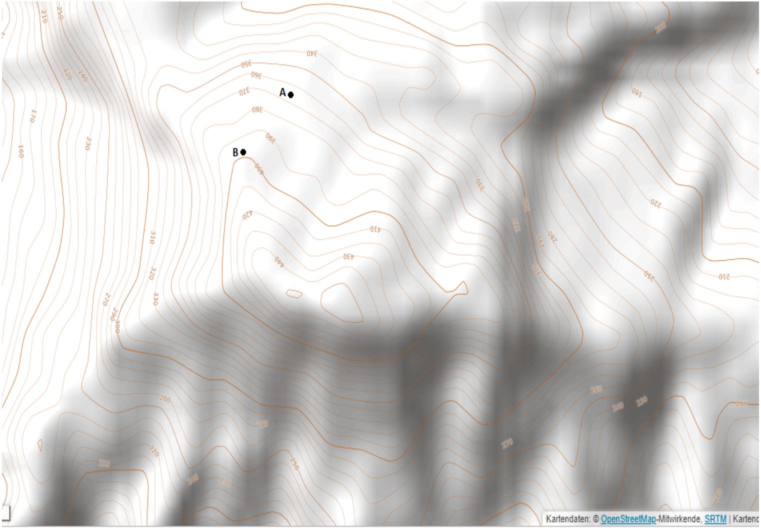
Functions of several variables

The following code was used to generate the plot above:

```
>> x=-5:0.01:5;
>> y=-5:0.01:5;
>> [X, Y]= meshgrid(x, y);
>> Z=X.*Y.*exp(-(X.^2+Y.^2));
>> surf(X,Y,Z,'EdgeColor','none')
```

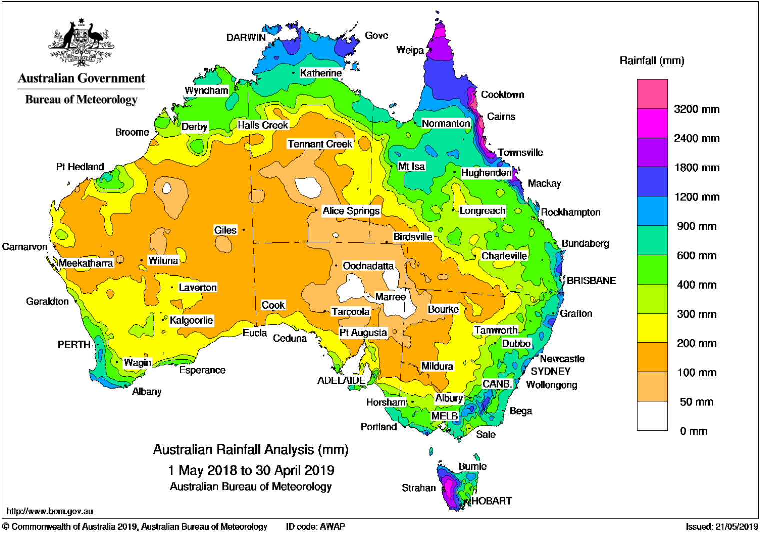
Another way of visualising functions of two variables is using a contour plot on the xy-plane (or on another plane if this is helpful).
A contour plot on the xy-plane is a plot of the relationship $f(x, y) = k$ for different fixed values of k. This yields the shape of the cross-sections of the graph of f in the plane $z = k$. This is the same method that is used to represtent height on a topographical map.

Topographical Maps: Elevation above the sea level



Xuedou Mountains, Zhejiang Province

Isothermals: locations with the same temperature



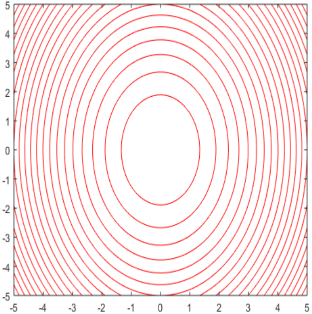
Functions of several variables

Example

Consider $f(x,y) = 2x^2 + y^2 + 3$. *If* $f(x,y) = k$, *then*

$$\frac{2x^2}{k-3} + \frac{y^2}{k-3} = 1$$

which for $k > 3$ *describes a family of ellipses. These ellipses are the cross-sections of the graph of* f *in the plane* $z = k$.



Functions of several variables

Example

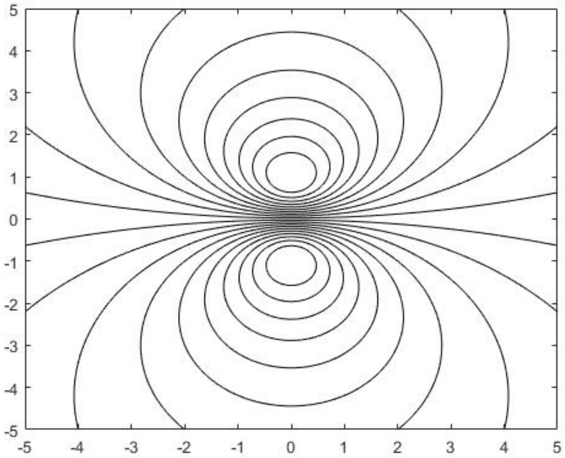
Consider

$$f(x, y) = \frac{-3y}{x^2 + y^2 + 1}.$$

The contours of this function can be plotted using MatLab:

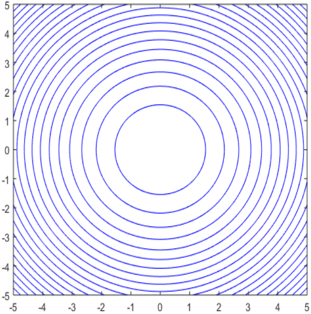
```
>> x=-5:0.01:5;
>> y=-5:0.01:5;
>> [X, Y]= meshgrid(x, y);
>> Z= -3*Y./(X.^2+Y.^2+1);
>> contour(X,Y,Z,20,'k')
```

Functions of several variables



Examples

Two contour maps correspond to functions whose graphs are a cone and a paraboloid. Which is which, and why?



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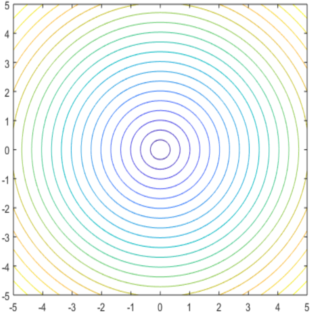


Euclidean Space

We now turn to doing calculus on functions with more than one independent variable. In order to do this we need to think about $\mathbb{R}^n$ as what is called a normed vector space. When thought of as a normed vector space $\mathbb{R}^n$ is called Euclidean Space. We have already seen that by thinking of each point in $\mathbb{R}^n$ as a vector we can coherantly define addition of two points in $\mathbb{R}^n$ (addition of vectors) and scalar multiplication (scalar multiplication of vectors). We also have a magnitude function $|\cdot|$. This function is called a norm and measures distance in $\mathbb{R}^n$ in the same way that $|\cdot|$ measures distance in $\mathbb{R}$. In order to make it clear when we are taking the magnitude of vectors rather than scalars (real numbers), we will start using $||\cdot||$ instead of $|\cdot|$ to denote vector magnitude (the Euclidean norm). The magnitude function can be represented using the dot product: if $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$, then

$$\bar{x} \cdot \bar{y} = \sum_{k=1}^{n} x_k y_k \text{ and } ||\bar{x}||^2 = \bar{x} \cdot \bar{x}$$

This is an example of what is called an inner product.

Euclidean Space

Theorem

Let $\bar{x}, \bar{y} \in \mathbb{R}^n$ and let $\alpha \in \mathbb{R}$. Then

1. $||\bar{x}|| \geq 0$, $||\bar{x}|| = 0$ *if and only if* $\bar{x} = \bar{0}$
2. $||\alpha\bar{x}|| = |\alpha|||\bar{x}||$
3. *(Cauchy-Schwarz Inequality)* $\bar{x} \cdot \bar{y} \leq ||\bar{x}||||\bar{y}||$
4. $||\bar{x} + \bar{y}|| \leq ||\bar{x}|| + ||\bar{y}||$

(5) is called the triangle inequality and corresponds to the triangle inequality in $\mathbb{R}$: for all $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$
There are subsets of $\mathbb{R}^n$ that are analogues of the open and closed intervals on $\mathbb{R}$.

Definition

Let $\bar{a} \in \mathbb{R}^n$ and let $r \geq 0$. The open ball centred at $\bar{a}$ with radius r is the set

$$B(\bar{a}, r) = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x} - \bar{a}\| < r\}$$

Euclidean Space

Definition

The closed ball centred at $\bar{a}$ with radius r is the set

$$C(\bar{a}, r) = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x} - \bar{a}\| \leq r\}$$

Definition

The punctured open ball centred at $\bar{a}$ with radius r is the set

$$P(\bar{a}, r) = \{\bar{x} \in \mathbb{R}^n \mid 0 < \|\bar{x} - \bar{a}\| < r\}$$

If A is one of these sets, then we say that A is a basic interval of $\mathbb{R}^n$.

The distance measure $\|\cdot\|$ in $\mathbb{R}^n$ allows us to define the notions of limit and continuity in the same way that we did in $\mathbb{R}$.

Limits

Definition

Let $f : D \longrightarrow \mathbb{R}$ be a fuction with $D \subseteq \mathbb{R}^n$. Let $\bar{a} \in \mathbb{R}^n$ and let $L \in \mathbb{R}$. We say that the limit of f at $\bar{x}$ approaches $\bar{a}$ is L and write

$$\lim_{\bar{x} \to \bar{a}} f(\bar{x}) = L$$

if for all $\epsilon > 0$, there exists a $\delta > 0$ such that for all $\mathbf{x} \in \mathbb{R}^n$, if $||\bar{x} - \bar{a}|| < \delta$, then $|f(\bar{x}) - L| < \epsilon$.

This says that we can ensure that $f(\bar{x})$ is arbitrarily close to L when $\bar{x}$ is arbitrarily close to $\bar{a}$. Instead of

$$\lim_{\bar{x} \to \bar{a}} f(\bar{x}) = L$$

we might also write $f(\bar{x}) \to L$ as $\bar{x} \to \bar{a}$.

Limits

Example

Consider

$$f(x,y) = \frac{5x^2y}{x^2+y^2}$$

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Recall that when we were doing calculus of functions of one real variable, we could show that the limit of $f(x)$ as $x \to a$ does not exist by showing that $f(x) \to L_1$ as $x \to a^-$ and $f(x) \to L_2$ as $x \to a^+$ with $L_1 \neq L_2$. Let $f : D \longrightarrow \mathbb{R}$ be a function with $D \subseteq \mathbb{R}^n$ and $\bar{a} \in D$. If $f(\bar{x}) \to L$ as $\bar{x} \to \bar{a}$, then $f(\bar{x})$ *must approach* L *on all paths through* $\mathbb{R}^n$ *that approach* $\bar{a}$. Therefore, we can show that the limit $f(\bar{x})$ does not exist by finding paths P_1 and P_2 approaching $\bar{a}$ such that $f(\bar{x}) \to L_1$ as $\bar{x} \to \bar{a}$ along P_1 and $f(\bar{x}) \to L_2$ as $\bar{x} \to \bar{a}$ along P_2, and $L_1 \neq L_2$.

Limits

Example

Consider

$$f(x,y)=\frac{xy}{x^2+y^2}$$

The limit of $f(x,y)$ *as* $(x,y)\to(0,0)$ *does not exist.*

Example

Consider

$$f(x,y)=\frac{xy^2}{x^2+y^4}$$

The limit of $f(x,y)$ *as* $(x,y)\to(0,0)$ *does not exist.*

One can easily see that if $f(x_1,\ldots,x_n)=x_i$ for $1\le i\le n$ and $\bar{a}\in(a_1,\ldots,a_n)\in\mathbb{R}^n$, then

$$\lim_{\bar{x}\to\bar{a}} f(\bar{x})=a_i$$

Similarly, if $f(x_1,\ldots,x_n)=c$ where $c\in\mathbb{R}$ is constant, then $\forall\bar{a}\in\mathbb{R}^n$,

$$\lim_{\bar{x}\to\bar{a}} f(\bar{x})=c$$

Continuity

All of the basic properties of limits of functions of one variable such as respecting sums, products and quotients can be generalised to limits of functions of more than one variable. These generalisations follow from the fact that $\mathbb{R}^n$ is equipped with a norm $\|\cdot\|$ that satisfies the triangle inequality.

Definition

($\epsilon\delta$ definition of continuity) Let $f : D \longrightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ and let $\bar{a} \in D$. We say that f is continuous at $\bar{a}$ if for all $\epsilon > 0$, there $\exists \delta > 0$ such that for all $\bar{x} \in D$, if $\|\bar{x} - \bar{a}\| < \delta$, then $|f(\bar{x}) - f(\bar{a})| < \epsilon$.

Definition

(Limit definition of continuity) Let $f : D \longrightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ and let $\bar{a} \in D$. We say that f is continuous at $\bar{a}$ if

$$\lim_{\bar{x} \to \bar{a}} f(\bar{x}) = f(\bar{a})$$

Continuity

Again, all of the basic properties of continuous functions of a single real variable generalise to functions of more than one real variable. In particular:

Theorem

Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^n$ be functions that are continuous at $\bar{a} \in D$. Let $\alpha \in \mathbb{R}$. Then

1. *$f + g$ is continuous at $\bar{a}$*
2. *αf is continuous at $\bar{a}$*
3. *fg is continuous at $\bar{a}$*
4. *if $g(\bar{a}) \neq 0$ then $\frac{f}{g}$ is continuous at $\bar{a}$*

Definition

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ that is the sum of terms in the form $\alpha \prod_{1 \leq i \leq n} x_i^{k_i}$ where the x_is are the independent variables, $k_i \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, is called a polynomial function of n variables. A function that is the quotient of polynomial functions is called a rational function.

Continuity

Theorem

A polynomial function of n is continuous at every point in $\mathbb{R}^n$. A rational function is continuous at every point in its domain.

Theorem

Let $f : D \longrightarrow \mathbb{R}$ and $g : E \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^n$ and $E \subseteq \mathbb{R}$, be functions. Let $\bar{a} \in D$ be such that $f(\bar{a}) \in E$. If f is continuous at $\bar{a}$ and g is continuous at $f(\bar{a})$, then $g \circ f : D \longrightarrow \mathbb{R}$ is continuous at $\bar{a}$.

Example

Consider

$$f(x, y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2} & \textit{if } \langle x, y\rangle \neq (0,0) \\ 0 & \textit{if } (x, y) = (0,0) \end{cases}$$

Since f is defined and a rational function at all points except $(x, y) = (0, 0)$, f is continuous everywhere except possibly $(x, y) = (0, 0)$.

Continuity

Example

Since

$$\lim_{(x,y)\to\bar{0}} f(x,y) \neq 0,$$

f is not continuous at $(x,y) = (0,0)$.

Example

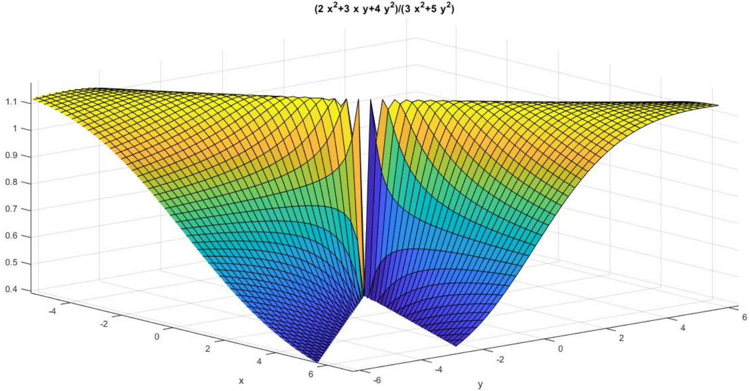
Consider $f(x,y) = \arctan\left(\frac{x}{y}\right)$. The rational function $\frac{x}{y}$ is continuous everywhere except for the line $y = 0$. The function $\arctan(x)$ is continuous everywhere. Therefore f is continous at all points that do not lie on the line $y = 0$. Note that f is not defined on the line $y = 0$!

Continuity

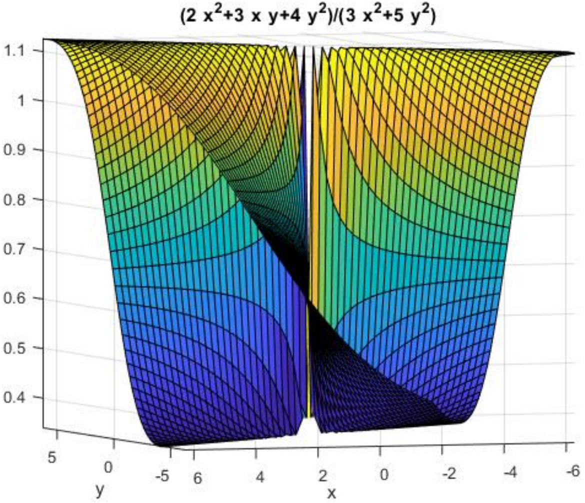
Example

Consider

$$f(x, y) = \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$$



Continuity




```
>> fh= @(x, y) (2*x.^2+3*x.*y+4*y.^2)./(3*x.^2+5*y.^2);
>> ezsurf(fh)
```

Continuity

There is no way of defining $f(x, y)$ at $\bar{0}$ to obtain a continuous function because

$$\lim_{(x,y)\to\bar{0}} f(x, y)$$

does not exist!

Next Class

1. Partial derivatives.
2. Tangent plane.
3. Gradient.

Today 2019-29-5

1. Review: limits and continuity.
2. Partial derivatives.
3. Tangent plane.
4. The Chain rule.

Continuity

All of the basic properties of limits of functions of one variable such as respecting sums, products and quotients can be generalised to limits of functions of more than one variable. These generalisations follow from the fact that $\mathbb{R}^n$ is equipped with a norm $\|\cdot\|$ that satisfies the triangle inequality.

Definition

($\epsilon\delta$ definition of continuity) Let $f : D \longrightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ and let $\bar{a} \in D$. We say that f is continuous at $\bar{a}$ if for all $\epsilon > 0$, there $\exists \delta > 0$ such that for all $\bar{x} \in D$, if $\|\bar{x} - \bar{a}\| < \delta$, then $|f(\bar{x}) - f(\bar{a})| < \epsilon$.

Definition

(Limit definition of continuity) Let $f : D \longrightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}^n$ and let $\bar{a} \in D$. We say that f is continuous at $\bar{a}$ if

$$\lim_{\bar{x} \to \bar{a}} f(\bar{x}) = f(\bar{a})$$

Continuity

Theorem

Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^n$ be functions that are continuous at $\bar{a} \in D$. Let $\alpha \in \mathbb{R}$. Then

1. $f + g$ *is continuous at* $\bar{a}$
2. αf *is continuous at* $\bar{a}$
3. fg *is continuous at* $\bar{a}$
4. *if* $g(\bar{a}) \neq 0$ *then* $\frac{f}{g}$ *is continuous at* $\bar{a}$

Definition

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ that is the sum of terms in the form $\alpha \prod_{1 \leq i \leq n} x_i^{k_i}$ where the x_is are the independent variables, $k_i \in \mathbb{N}$ and $\alpha \in \mathbb{R}$, is called a polynomial function of n variables. A function that is the quotient of polynomial functions is called a rational function.

Continuity

Theorem

A polynomial function of n is continuous at every point in $\mathbb{R}^n$. A rational function is continuous at every point in its domain.

Theorem

Let $f : D \longrightarrow \mathbb{R}$ and $g : E \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^n$ and $E \subseteq \mathbb{R}$, be functions. Let $\bar{a} \in D$ be such that $f(\bar{a}) \in E$. If f is continuous at $\bar{a}$ and g is continuous at $f(\bar{a})$, then $g \circ f : D \longrightarrow \mathbb{R}$ is continuous at $\bar{a}$.

Example

Consider

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & \textit{if } \langle x, y \rangle \neq (0, 0) \\ 0 & \textit{if } (x, y) = (0, 0) \end{cases}$$

Since f is defined and a rational function at all points except $(x, y) = (0, 0)$, f is continuous everywhere except possibly $(x, y) = (0, 0)$.

Continuity

Example

Since

$$\lim_{(x,y)\to\bar{0}} f(x,y) \neq 0,$$

f *is not continuous at* $(x,y) = (0,0)$.

Example

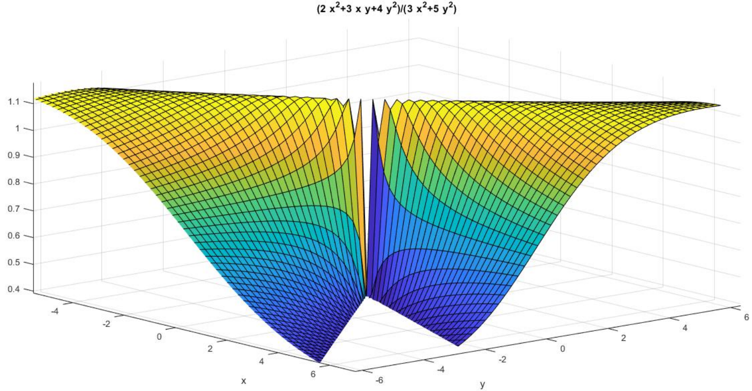
Consider $f(x,y) = \arctan\left(\frac{x}{y}\right)$. *The rational function* $\frac{x}{y}$ *is continuous everywhere except for the line* $y = 0$. *The function* $\arctan(x)$ *is continuous everywhere. Therefore* f *is continous at all points that do not lie on the line* $y = 0$. *Note that* f *is not defined on the line* $y = 0$*!*

Continuity

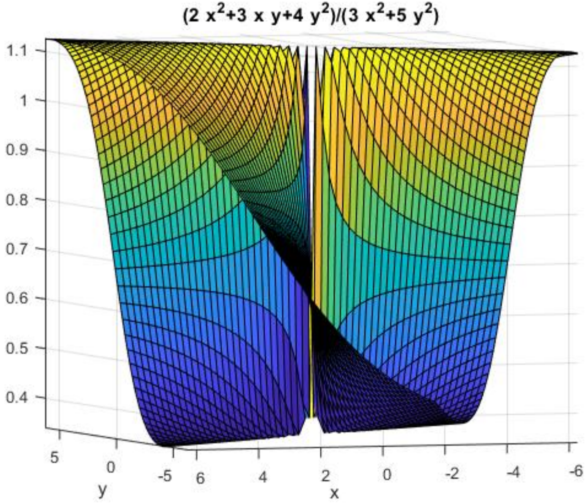
Example

Consider

$$f(x,y) = \frac{2x^2 + 3xy + 4y^2}{3x^2 + 5y^2}$$



Continuity




```
>> fh= @(x, y) (2*x.^2+3*x.*y+4*y.^2)./(3*x.^2+5*y.^2);
>> ezsurf(fh)
```

Continuity

There is no way of defining $f(x, y)$ at $\bar{0}$ to obtain a continuous function because

$$\lim_{(x,y)\to\bar{0}} f(x, y)$$

does not exist!

Differentiation

Definition

Let $f : D \longrightarrow \mathbb{R}$ be a function where D is an open ball of $\mathbb{R}^n$. Let $\bar{a} \in D$. The function f is differentiable at $\bar{a}$ with derivative $Df(\bar{a}) \in \mathbb{R}^n$ if

$$\frac{\left\| f(\bar{a}+\bar{h}) - f(\bar{a}) - Df(\bar{a}) \cdot \bar{h} \right\|}{\left\| \bar{h} \right\|} \to 0 \ as \ \bar{h} \to \bar{0}$$

The intuition is that the derivative of a function f gives the best linear approximation of f at the point $\bar{a}$.

Note that the derivative of a function, f, of n variables is an n-dimensional vector (or point in $\mathbb{R}^n$). This makes the "derivative function", if it exists, look like a function $Df : \mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Differentiation

Theorem

Let $f : D \longrightarrow \mathbb{R}$ and $g : D \longrightarrow \mathbb{R}$, where D is an open ball of $\mathbb{R}^n$, be functions that are differentiable at $\bar{a} \in D$. Let $\alpha \in \mathbb{R}$. Then

1. *$f + g$ is differentiable at $\bar{a}$ with $D(f + g)(\bar{a}) = Df(\bar{a}) + Dg(\bar{a})$*
2. *αf is differentiable at $\bar{a}$ with $D(\alpha f)(\bar{a}) = \alpha Df(\bar{a})$*

Let $f : D \longrightarrow \mathbb{R}$, where D is an open ball of $\mathbb{R}^n$, be a function with independent variables $x_1, \ldots, x_n$ that is differentiable at $\bar{a} = (a_1, \ldots, a_n) \in D$. Then

$$Df(\bar{a}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

Differentiation

Since

$$\frac{\left\|f(\bar{a}+\bar{h})-f(\bar{a})-Df(\bar{a})\cdot\bar{h}\right\|}{\left\|\bar{h}\right\|}\to 0 \text{ as } \bar{h}\to\bar{0},$$

we can consider $\bar{h}=(0,\ldots,0,h,0,\ldots,0)$, where h appears in i^{th} place of $\bar{h}$, and we get

$$\frac{|f(a_1,\ldots,a_i+h,\ldots,a_n)-f(a_1,\ldots,a_i,\ldots,a_n)-\alpha_i h|}{|h|}$$

$$=\left|\frac{f(a_1,\ldots,a_i+h,\ldots,a_n)-f(a_1,\ldots,a_i,\ldots,a_n)}{h}-\alpha_i\right|\to 0 \text{ as } h\to 0$$

Therefore α_i is the derivative of the function of one variable $f(a_1,\ldots,x_i,\ldots,a_n)$ at the point a_i.

Partial Derivatives

Let $f : D \longrightarrow \mathbb{R}$, where D is an open ball of $\mathbb{R}^n$, be a function $f(x_1, \ldots, x_n)$ that is differentiable at $\bar{a} = (a_1, \ldots, a_n) \in D$. If

$$Df(\bar{a}) = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$

then we call α_i the partial derivative of f with respect to x_i at $\bar{a}$ and we write

$$\left.\frac{\partial f}{\partial x_i}\right|_{\bar{a}} \text{ or } f_{x_i}(\bar{a})$$

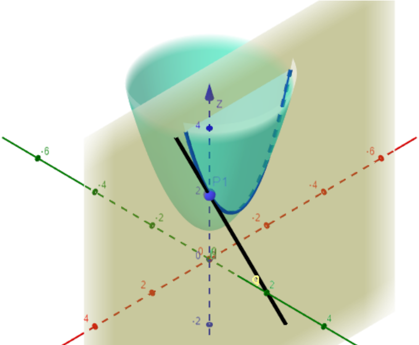
We write $\frac{\partial f}{\partial x_i}$ or $f_{x_i}(x_1, \ldots, x_n)$ for the function of n variables, if it exists, $\bar{a} \mapsto f_{x_i}(\bar{a})$.

Partial Derivatives

For functions of two variables $f(x, y)$ we can interpret the partial derivatives geometrically. Let (a, b, c) a point such that $c = f(a, b)$. Let C_1 be the curve that is obtained by intersecting the graph $z = f(x, y)$ with the plane $y = b$.
The partial derivative

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}$$

is the slope of the tangent line of C_1 in the plane $y = b$.

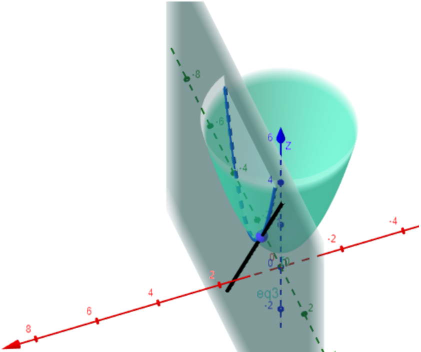


Geometrical Interpretation

Let C_2 be the curve that is obtained by intersecting the graph $z = f(x, y)$ with the plane $x = a$.

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}$$

is the slope of the tangent line of C_2 in the plane $x = a$. I.e. the partial derivatives represent the slopes of the tangent lines of the curves obtained by intersecting the graph $z = f(x, y)$ by the planes that run parallel to the coordinate axes.



Partial Derivatives

Example

Consider $f(x, y) = (2x + 3y)^{10}$*. We have*

$$f_x(x, y) = 20(2x + 3y)^9 \; and \; f_y(x, y) = 30(2x + 3y)^9$$

Example

Consider

$$f(x, y) = \frac{ax + by}{cx + dy}$$

$$f_x(x, y) = \frac{(ad - cb)y}{(cx + dy)^2}, \quad f_y(x, y) = \frac{(bc - ad)x}{(cx + dy)^2}$$

Higher Partial Derivatives

Let $f : D \longrightarrow \mathbb{R}$, where D is an open ball of $\mathbb{R}^n$, be a function $f(x_1, \ldots, x_n)$ that is differentiable. If the partial derviative $f_{x_i}(x_1, \ldots, x_n)$ is differentiable, then we can find the partial $f_{x_i}(x_1, \ldots, x_n)$ with respect to one of the independent varuables x_j. A partitial derivative of a partial derivative is called a second partial derivative. We write

$$\frac{\partial^2 f}{\partial x_i \partial x_j} \text{ or } f_{x_i x_j}(x_1, \ldots, x_n)$$

for the partial derivative of $f_{x_i}(x_1, \ldots, x_n)$ with respect to the variable x_j (this becomes $\frac{\partial^2 f}{\partial x_i^2}$ when we are taking the partial derivative twice with respect to the same variable). The process and notation generalises to third partial derivatives, fourth partial derivatives, ...

Partial Derivatives

Example

Consider $f(x, y, z) = \sin(3x + yz)$*. Then*

$$\frac{\partial^4 f}{\partial x^2 \partial y \partial z} = f_{xxyz}(x, y, z) = -9\cos(3x + yz) + 9yz\sin(3x + yz)$$

Partial Derivatives

Theorem

(Clairaut's Theorem) Let $f : D \longrightarrow \mathbb{R}$*, where* D *is an open ball of* $\mathbb{R}^2$*, be a function* $f(x_1, x_2)$ *and let* $(a, b) \in D$*. If* $f_{x_1 x_2}$ *and* $f_{x_2 x_1}$ *are both continuous on* D*, then*

$$f_{x_1 x_2}(a, b) = f_{x_2 x_1}(a, b)$$

Partial Derivatives

Theorem

Let $f : D \longrightarrow \mathbb{R}$, where D is an open ball of $\mathbb{R}^n$, be a function $f(x_1, \ldots, x_n)$. If for all $1 \leq i \leq n$, $\dfrac{\partial f}{\partial x_i}$ exists and is continuous on D, then

$$\Delta f = f_{x_1}(\bar{x})\Delta x_1 + \ldots + f_{x_n}(\bar{x})\Delta x_n + \varepsilon_1 \Delta x_1 + \ldots + \varepsilon_n \Delta x_n,$$

where $\varepsilon_i \to 0$, $i = 1, \ldots, n$ as $(\Delta x_1, \ldots, \Delta x_n) \to (0, \ldots, 0)$.

Proof.

Let $f = f(x_1, x_2)$.

$$\Delta f = f(x_1 + \Delta x_1, x_2 + \Delta x_2) - f(x_1, x_2)$$

$\square$

Proof.
Represent

$$\Delta f = [f(x_1 + \Delta x_1, x_2 + \Delta x_2) - f(x_1, x_2 + \Delta x_2)]$$

$$+ [f(x_1, x_2 + \Delta x_2) - f(x_1, x_2)] \, .$$

Let $g(t) = f(t, x_2 + \Delta x_2),\ t \in [x_1,\ x_1 + \Delta x_1] \Rightarrow g'(t) = f_t(t, x_2 + \Delta x_2)$.
Mean Value Theorem:

$$\exists c \in [x_1,\ x_1 + \Delta x_1] \quad g(x_1 + \Delta x_1) - g(x_1) = g'(c)\Delta x_1$$

$$\Rightarrow f(x_1 + \Delta x_1, x_2 + \Delta x_2) - f(x_1, x_2 + \Delta x_2) = f_{x_1}(c, x_2 + \Delta x_2)\Delta x_1$$

Similarly, $\exists b \in [x_2,\ x_2 + \Delta x_2]$:

$$f(x_1, x_2 + \Delta x_2) - f(x_1, x_2) = f_{x_2}(x_1, b)\Delta x_2$$

□

Proof.

The variation Δf becomes

$$\begin{aligned}\Delta f &= f_{x_1}(c, x_2 + \Delta x_2)\Delta x_1 + f_{x_2}(x_1, b)\Delta x_2 \\ &= f_{x_1}(x_1, x_2)\Delta x_1 + \left[f_{x_1}(c, x_2 + \Delta x_2) - f_{x_1}(x_1, x_2)\right]\Delta x_1 \\ &\quad + f_{x_2}(x_1, x_2)\Delta x_2 + \left[f_{x_2}(x_1, b) - f_{x_2}(x_1, x_2)\right]\Delta x_2 \\ &= f_{x_1}(x_1, x_2)\Delta x_1 + f_{x_2}(x_1, x_2)\Delta x_2 + \varepsilon_1 \Delta x_1 + \varepsilon_2 \Delta x_2,\end{aligned}$$

with $\varepsilon_1 = f_{x_1}(c, x_2 + \Delta x_2) - f_{x_1}(x_1, x_2)$ and $\varepsilon_2 = f_{x_2}(x_1, b) - f_{x_2}(x_1, x_2)$.

$$c \in [x_1,\ x_1 + \Delta x_1],\ b \in [x_2,\ x_2 + \Delta x_2] \Rightarrow \varepsilon_1,\ \varepsilon_2 \to 0 \quad \text{as} \quad \Delta x_1,\ \Delta x_2 \to 0$$

$\square$

Remark

We can define the concept of a differentiable function using this theorem and say that a function $f \to D \subset \mathbb{R}^n \to \mathbb{R}$ is differentiable at the point $\bar{a}$ if the variation Δf is represented as

$$\Delta f = f_{x_1}(\bar{a})\Delta x_1 + \ldots + f_{x_n}(\bar{a})\Delta x_n + \varepsilon_1 \Delta x_1 + \ldots + \varepsilon_n \Delta x_n,$$

where $\varepsilon_i \to 0,\ i = 1, \ldots, n$ as $(\Delta x_1, \ldots, \Delta x_n) \to (0, \ldots, 0)$.

Definition

The total differential of function $f \to D \subset \mathbb{R}^n \to \mathbb{R}$ is

$$df = f_{x_1}(\bar{x})dx_1 + \ldots + f_{x_n}(\bar{x})dx_n,$$

where $dx_1 = \Delta x_1, \ldots, dx_n = \Delta x_n$.

Remark

Let

$$z = f(x, y) = x^2 - 3xy - y^2$$

The total differential

$$dz = f_x dx + f_y dy = (2x + 3y)dx + (3x - 2y)dy$$

If x changes from 2 to 2.05 and y changes from 3 to 2.96, then$dx = \Delta x = 0.05$, $dy = \Delta y = -0.04$

$$dz|_{(2,3)} = (2 \cdot 2 - 3 \cdot 3)0.05 + (3 \cdot 2 - 2 \cdot 3)(-0.04) = 0.65$$

While

$$\Delta z = f(2.05, 2.96) - f(2, 3) = 0.6449 \Rightarrow \Delta z \approx dz$$

Tangent Planes

Recall, that a function $f(x)$ of a single variable can be approximated at a point a by a tangent line whose slope is given by $f'(a)$. The derivatives of functions of two variables (and functions of more than two variables) also give rise to a linear approximation of the function. For a function $f(x, y)$, the linear approximation of $f(x, y)$ around a point (a, b) will be a plane instead of a line.
Let $f(x, y)$ be a function that is differentiable at a point (a, b). The tangent plane of f at (a, b) is the plane that passes through (a, b) and is parallel to both:

1. the tangent line of the curve obtained by intersecting the graph $z = f(x, y)$ with the plane $y = b$
2. the tangent line of the curve obtained by intersecting the graph $z = f(x, y)$ with the plane $x = a$

We saw that the slopes of the tangent lines described by 1. and 2. are given by the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ respectively.

Tangent Planes

It follows that the equation of the tangent plane $f(x,y)$ at (a,b) is:

$$z - f(a,b) = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

Example

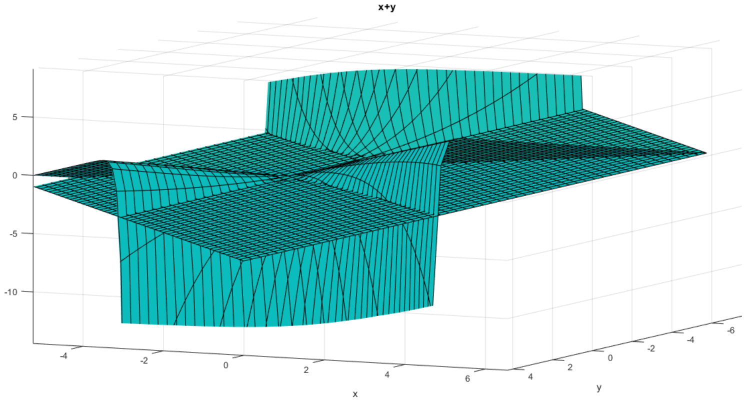
Consider $f(x,y) = xe^{xy}$ *at the point* $(1,0)$. $(a,b,f(a,b)) = (1,0,1)$

$$\Rightarrow z - 1 = (e^{xy} + xye^{xy})|_{(1,0)}(x-1) + (x^2 e^{xy})_{(1,0)}(y-0)$$

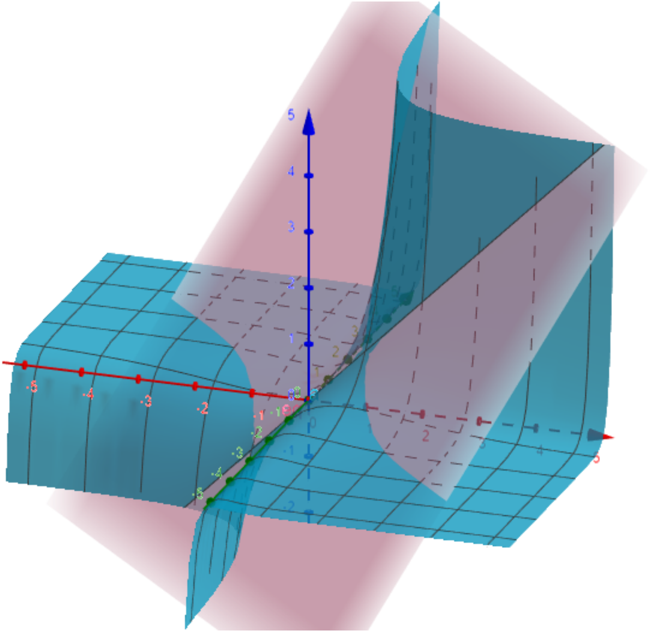
$$z = x + y$$

```
>> fh= @(x, y) x.*exp(x.*y);
>> fp= @(x, y) x+y;
>> ezsurf(fh)
>> hold on
>> ezsurf(fp)
```

Tangent Planes



Tangent Planes



Tangent Planes

Example

Consider $f(x,y) = \sqrt{xy}$ *at the point* $\langle 1,1,1\rangle$. *We have*

$$f_x(x,y) = \frac{y}{2\sqrt{xy}} \text{ and } f_y(x,y) = \frac{x}{2\sqrt{xy}}$$

So, $f_x(1,1) = f_y(1,1) = \frac{1}{2}$. *Therefore the tangent plane is described by the equation*

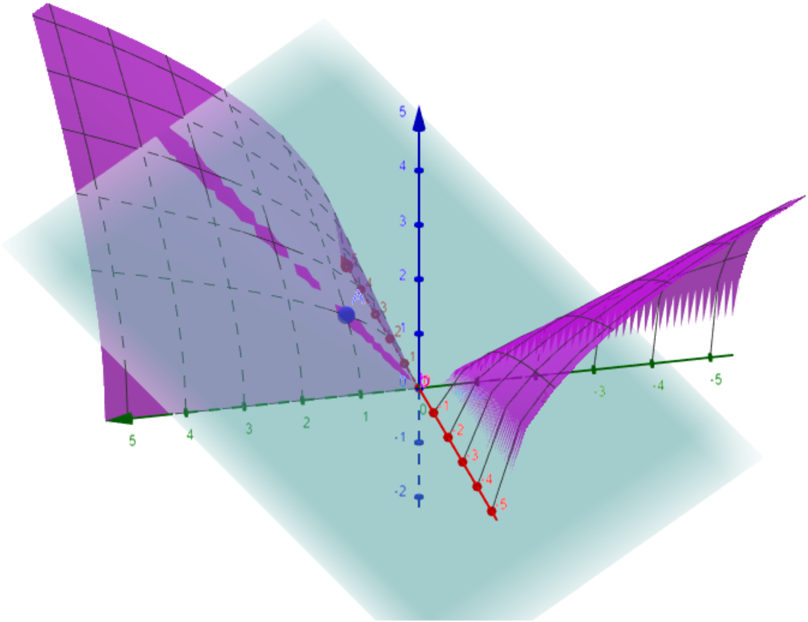
$$z - 1 = \frac{1}{2}(x-1) + \frac{1}{2}(y-1)$$

Or

$$z = \frac{1}{2}x + \frac{1}{2}y$$

$$\Rightarrow f(1.1, 0.95) \approx \frac{1}{2}(1.1) + \frac{1}{2}(0.8) \approx 1.025$$

Tangent Planes



The chain rule

The chain rule for functions of a single real variable tells us that if $y = f(x)$ and $x = g(t)$ where f and g are differentiable functions, then

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

This tool for differentiating composite functions generalises to functions of more than one variable.

Theorem

(Chain Rule I) Let $x = g(t)$ and $y = h(t)$ be functions that are differentiable on an interval I. Let $z = f(x, y)$ be a function that is differentiable at the points with x-coordinate in the range of g restricted to I, and y-coordinates in the range of h restricted to I. Then $f(x, y)$ is differentiable with respect to t on the interval I and

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

The chain rule

Proof.

$$\text{The increment } t \to \Delta t \Rightarrow \begin{array}{c} x = g(t) \to \Delta x = g(t+\Delta t) - g(t) \\ y = h(t) \to \Delta y = h(t+\Delta t) - h(t) \end{array}$$

$$\Delta z = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

with $\varepsilon_1,\, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$.

$$\frac{\Delta z}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

$$\Delta t \to 0 \Rightarrow \Delta x = \underbrace{g(t+\Delta t) - g(t) \to 0,}_{\text{continuity}} \Delta y = \underbrace{h(t+\Delta t) - h(t) \to 0}_{\text{continuity}}$$

$$\Rightarrow \varepsilon_1,\, \varepsilon_2 \to 0$$

□

The chain rule

Proof.

(Cont.)

$$\frac{dz}{dt} = \lim_{\Delta t \to 0} \frac{\Delta z}{\Delta t} = f_x \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + f_y \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

$$+ \underbrace{\left(\lim_{\Delta t \to 0} \varepsilon_1 \right)}_{0} \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} + \underbrace{\left(\lim_{\Delta t \to 0} \varepsilon_2 \right)}_{0} \lim_{\Delta t \to 0} \frac{\Delta y}{\Delta t}$$

$$\frac{dz}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt}$$

□

The chain rule

Example

Consider $f(x, y) = \sqrt{1 + x^2 + y^2}$ *where* $x = \ln(t)$ *and* $y = \cos(t)$.

Theorem

(Chain Rule II) Suppose that $z = f(x, y)$ *is a differentiable function of* x *and* y, *where* $x = g(s, t)$ *and* $y = h(s, t)$ *are differentiable functions. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example

Consider $z = \sin(\theta)\cos(\phi)$ *where* $\theta = st^2$ *and* $\phi = s^2 t$.

The chain rule

Theorem

(General chain rule) Let u be a differentiable function of n variables $x_1, \ldots, x_n$ such that for all $1 \leq i \leq n$, x_i is a differentiable function of m variables $t_1, \ldots, t_m$. Then u is a function of $t_1, \ldots, t_m$ and for all $1 \leq j \leq m$,

$$\frac{\partial u}{\partial t_j} = \sum_{1 \leq i \leq n} \frac{\partial u}{\partial x_i} \frac{\partial x_i}{\partial t_j}$$

Example

Consider $u = x^4 y + y^2 z^3$ where $x = rse^t$, $y = rs^2 e^{-t}$ and $z = r^2 s \sin(t)$. Find

$$\frac{\partial u}{\partial s}$$

The chain rule

Example

If $z = f(x, y)$ has continuous second partial derivatives, and $x = r^2 + s^2$ and $y = 2rs$, then find

$$\frac{\partial z}{\partial r} \ and \ \frac{\partial^2 z}{\partial r^2}$$

Implicit differentiation

Assume that an equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x. The exact conditions on F that ensures that this occurs is given by a result known as the **Implicit Function Theorem** that, unfortunately, is outside the scope of this course. If $F(x, y)$ is differentiable, then the chain rule tells us that

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Therefore, if $\frac{\partial F}{\partial y} \neq 0$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

The **Implicit Function Theorem** says that one can find an open ball around a point (a, b) where this derivation is valid if there is an open ball $B(\langle a, b\rangle, r)$ on which $F(x, y)$ is defined, where $F(x, y) = 0$, $F_y(x, y) \neq 0$, and F_x and F_y are continuous on $B(\langle a, b\rangle, r)$.

Implicit differentiation

Similiarily, if $F(x, y, z) = 0$ defines z implicitly as a differentiable function of x and y, then the chain rule tells us that

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, we get

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

So, if $\frac{\partial F}{\partial z} \neq 0$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Implicit differentiation

Again, the Implicit Function Theorem is gives conditions under which this derivation is valid. If $F(x, y, z)$ is defined on $B(\langle a, b, c\rangle, r)$, where $F(x, y, z) = 0$, $F_z(x, y, z) \neq 0$, and F_x, F_y and F_z are continuous on $B(\langle a, b, c\rangle, r)$, then there is an open ball around $\langle a, b, c\rangle$ in which $F(x, y, z) = 0$ implicitly defines z as a differentiable function of x and y and thus the above derivations are valid.

Example

Find $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ *if* $e^z = xyz$.

The gradient and directional derivatives

If $f : D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^n$ and $\bar{a} \in D$ is a function that is differentiable at $\bar{a}$, then we call the derivative of f at $\bar{a}$— $Df(\bar{a})$— the gradient of f at $\bar{a}$, and write $\nabla f(\bar{a})$, when we are thinking about this entity as an n-dimensional vector. I.e. if $f(x, y, z)$ is differentiable on an open ball $D \subseteq \mathbb{R}^3$, then the gradient of f is given by

$$\nabla f(x,y,z) = f_x(x,y,z)\bar{i} + f_y(x,y,z)\bar{j} + f_z(x,y,z)\bar{k} = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j} + \frac{\partial f}{\partial z}\bar{k}$$

And, if $f(x, y)$ is differentiable on an open ball $D \subseteq \mathbb{R}^2$, then the gradient of f is given by

$$\nabla f(x,y) = f_x(x,y)\bar{i} + f_y(x,y)\bar{j} = \frac{\partial f}{\partial x}\bar{i} + \frac{\partial f}{\partial y}\bar{j}$$

Directional derivative

We have seen that the partial derivatives of a function f with more than one independent variables are the components of the the derivative of f in the direction of each of the coordinate axes. Given any unit vector $\bar{u}$, we can also find the component of the derivative in the direction of $\bar{u}$.

Definition

Let $f : D \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, be a function and let $\bar{a} \in D$ be such that f is differentiable at $\bar{a}$. Let $\bar{u}$ be an n-dimensional unit vector ($\|u\| = 1$). The directional derivative of f in the direction of $\bar{u}$ at $\bar{a}$, written $D_{\bar{u}}f(\bar{a})$, is defined by

$$D_{\bar{u}}f(\bar{a}) = \lim_{h \to 0} \frac{f(\bar{a} + h\bar{u}) - f(\bar{a})}{h}$$

Directional derivative

- If $f : D \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, is differentiable at a point $\bar{a} \in D$, then for all unit vectors $\bar{u}$, $D_{\bar{u}} f(\bar{a})$ exists. The limit that defines $D_{\bar{u}} f(\bar{a})$ is just a special case of the limit that defines the derivative!
- Note that if $f : D \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, is differentiable on an open ball $B \subseteq D$ and $\bar{u}$ is a unit vector, then $D_{\bar{u}} f$ looks like a function with n independent variables defined on B.

Directional derivatives

If $f(x, y, z)$ is differentiable on $D \subseteq \mathbb{R}^3$, then

$$D_{\bar{i}} f(x, y, z) = \frac{\partial f}{\partial x} \qquad D_{\bar{j}} f(x, y, z) = \frac{\partial f}{\partial y} \qquad D_{\bar{k}} f(x, y, z) = \frac{\partial f}{\partial z}$$

If $f(x, y)$ is differentiable on $D \subseteq \mathbb{R}^2$, (a, b) and $\bar{u}$ is a 2D unit vector, then the directional derivative of f in the direction of $\bar{u}$ at the point (a, b) ($D_{\bar{u}} f(a, b)$) can be interpreted geometrically.

- Let $\mathcal{S}$ be the surface defined by the graph $z = f(x, y)$ and let $P = (a, b, f(a, b))$.
- Let $\mathcal{P}$ be the plane that passes through P and is parallel to both $\bar{u}$ and $\bar{k}$.
- Let $\mathcal{C}$ be the curve defined by the intersection of $\mathcal{P}$ and $\mathcal{S}$.
- Then $D_{\bar{u}} f(a, b)$ is the slope of the tangent line of $\mathcal{C}$ at P that runs parallel to $\mathcal{P}$.

Note that this is consistent with our geometric interpretation of the partial derivatives of $f(x, y)$.

Directional derivative

The following result shows that the directional derivative can be easily computed from the derivative (gradient):

Theorem

Let $f : D \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^n$, be such that f is differentiable on D. Let $\bar{u}$ be an n dimensional unit vector. Then for all $\bar{x} \in D$,

$$D_{\bar{u}} f(\bar{x}) = \nabla f(\bar{x}) \cdot \bar{u}$$

Proof.

Let $\bar{a} = (a_1, \ldots, a_n) \in D$. Define a function $g(h)$ such that

$$g(h) = f(\underbrace{a_1 + hu_1}_{x_1}, \underbrace{a_2 + hu_2}_{x_2}, \ldots, \underbrace{a_n + hu_n}_{x_n})$$

$$g'(h) = \frac{\partial f}{\partial x_1} \frac{dx_1}{dh} + \ldots + \frac{\partial f}{\partial y} \frac{dx_n}{dh} = f_{x_1}(\bar{x}) u_1 + \ldots + f_{x_n}(\bar{x}) u_n$$

$\square$

Directional derivative

Proof.

$$\Rightarrow g'(0) = f_{x_1}(\bar{a})u_1 + \ldots + f_{x_n}(\bar{a})u_n = \nabla f(\bar{a}) \cdot \bar{u}$$

On another hand,

$$g'(0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(a_1 + hu_1, a_2 + hu_2, \ldots, a_n + hu_n) - f(a_1, a_2, \ldots, a_n)}{h}$$

$$= D_{\bar{u}} f(a_1, \ldots, a_n) = D_{\bar{u}} f(\bar{a})$$

$$\Rightarrow \forall \bar{a} \in D \quad D_{\bar{u}} f(\bar{a}) = \nabla f(\bar{a}) \cdot \bar{u}$$

□

Directional derivative

Example

Find the directional derivative of $f(x, y) = x^3 - 3xy + 4y^2$ *in the direction of the unit vector described by the angle* $\theta = \frac{\pi}{6}$*.*

Example

Find the directional derivative of $f(x, y, z) = xe^{2yz}$ *in the direction of the vector* $\bar{v} = (2, -2, 1)$ *at the point* $P = (3, 0, 2)$*.*

$$\nabla f = (e^{2yz}, 2xze^{2yz}, 2xye^{2yz}) \Rightarrow \nabla f(P) = (1, 12, 0)$$

The unit vector $\bar{u}$ *in the direction of the vector* $\bar{v}$ *is*

$$\bar{u} = \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right)$$

$$D_{\bar{v}} f(P) = \nabla f(P) \cdot \bar{u} = (1, 12, 0) \cdot \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right) = -\frac{22}{3}$$

Maximizing the directional derivative

The directional derivative of a function f in the direction $\bar{u}$ is maximised when $\bar{u}$ points in the same direction as ∇f:

Theorem

Let $f(\bar{x})$ be a differentiable function. The maximum value of $D_{\bar{u}} f(\bar{x})$ is $\|\nabla f(\bar{x})\|$ and this value is achieved when $\bar{u}$ points in the same direction as ∇f (i.e. there exists nonnegative $\lambda \in \mathbb{R}$ such that $\nabla f = \lambda \bar{u}$).

Maximizing the directional derivative

Proof.

Using the Cauchy-Schwarz Inequality

$$D_{\bar{u}} f(\bar{x}) = \nabla f(\bar{x}) \cdot \bar{u} \leq \|\nabla f(\bar{x})\| \, \|\bar{u}\| = \|\nabla f(\bar{x})\|$$

If $\nabla f(\bar{x})$ points in the same direction as $\bar{u}$, then $\nabla f(\bar{x}) = \|\nabla f \bar{x})\| \, \bar{u}$ and

$$D_{\bar{u}} f(\bar{x}) = \nabla f(\bar{x}) \cdot \bar{u} = \|\nabla f(\bar{x})\| \, \bar{u} \cdot \bar{u} = \|\nabla f(\bar{x})\| \, \|\bar{u}\|^2 = \|\nabla f(\bar{x})\|$$

Observe that

$$D_{\bar{u}} f(\bar{x}) = \nabla f(\bar{x}) \cdot \bar{u} = \|\nabla(\bar{x})\| \, \|\bar{u}\| \cos(\theta) = \|\nabla f\| \cos(\theta)$$

where θ is the angle between $\nabla f(\bar{x})$ and $\bar{u}$. $\square$

Maximizing the directional derivative

Example

Suppose that the temperature at a point (x, y, z) in space is described by the function

$$T(x, y, z) = \frac{80}{1 + x^2 + 2y^2 + 3z^2}$$

Then at the point $(1, 1, -2)$ the temperature is increasing most rapidly in the direction

$$\nabla T(1, 1, -2) = \frac{160}{256}(-\bar{i} - 2\bar{j} + 6\bar{k})$$

and the rate of increase in this direction is

$$\|\nabla T(1, 1, -2)\| = \frac{160}{256}\sqrt{41}$$