Momentum and Collisions

Newton $\rightarrow$ "quantity/ amount of motion" - his idea

1 Momentum

2^{nd} law of dynamics

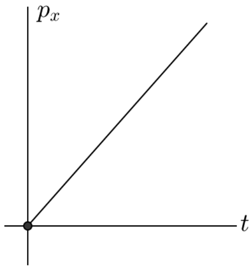
$$\overline{F} = m\overline{a} = m\frac{\mathrm{d}\overline{v}}{\mathrm{d}t} \overset{(\text{true if } v \ll c = 3\times 10^8 m/s)}{\underset{m=\text{const}}{=\!=\!=\!=\!=\!=\!=\!=\!=\!=}} \frac{\mathrm{d}}{\mathrm{d}t}(m\overline{v}) = \frac{\mathrm{d}\overline{p}}{\mathrm{d}t}$$

$$\overline{p} = m\overline{v} \quad \text{- momentum(linear momentum) units } [kg \cdot m/s^2]$$

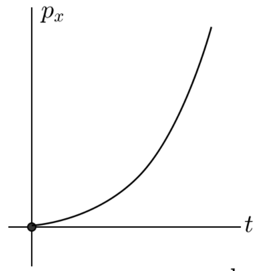
Newton's 2^{nd} law in terms of linear momentum $\overline{p} = m\overline{v}$

$$\overline{F} = \frac{\mathrm{d}\overline{p}}{\mathrm{d}t} \quad \rightarrow \quad \text{the net force acting on a particle is equal to the time rate of change of particle's momentum} \tag{1}$$

Example



constant force $F_x = \text{const} > 0$



increasing force $F_x = \frac{\mathrm{d}p_x}{\mathrm{d}t} > 0$

$$F_x = \frac{\mathrm{d}p_x}{\mathrm{d}t}$$

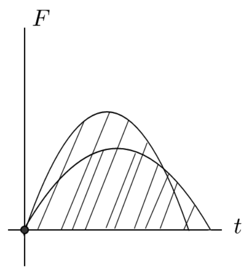
$$F_x \mathrm{d}t = \mathrm{d}p_x \quad \Rightarrow \quad \int_{t_1}^{t_2} F_x \mathrm{d}t = \int_{p_1}^{p_2} \mathrm{d}p_x$$

momentum-impulse theorem (1D)

$$p_2 - p_1 = \underbrace{\int_{t_1}^{t_2} \overset{\text{net force}}{F_x} \mathrm{d}t}_{\text{impulse (1D)}} \tag{2}$$

Observation:

the same net change in momentum can be achieved in various ways (different forces acting over different intervals of time), if the areas under both curves are the same, the resulting net changes of the momentum are equal



momentum-impulse theorem (3D)

$$\overline{p}_2 - \overline{p}_1 = \int_{t_1}^{t_2} \overline{F} \mathrm{d}t \quad \rightarrow \text{impulse } \overline{\tau} \tag{3}$$

1.1 Kinetic Energy vs. Momentum

(both are meassures of the "amount of motion")

Kinetic Energy

* scalar
* changes defined by work-kinetic energy theorem

$$\mathrm{d}K = \overline{F} \cdot \mathrm{d}\overline{r} \quad \rightarrow \text{force over distance}$$

Momentum

* vector
* changes defined by momentum - impulse theorem

$$\mathrm{d}\overline{p} = \overline{F} \cdot \mathrm{d}t \quad \rightarrow \text{force over period of time}$$

Example:

(a) Bring two objects to a stop

$$\begin{aligned} v_A &= 10m/s & v_B &= 5m/s \\ m_A &= 400kg & m_B &= 800kg \\ p_A &= 4000kgm/s \quad = & p_B &= 4000kgm/s \\ K_A &= 20kJ & K_B &= 10kJ \end{aligned}$$

What is the average stopping force if both A and B have stopped

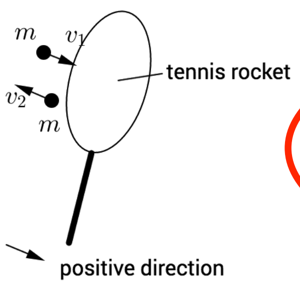
* after 100m? $\triangle K = -F \cdot s$

$F_A = 200N \qquad F_B = 100N$

* after 10s? $\triangle p = -F \cdot t$

$F_A = 400N \qquad F_B = 400N$

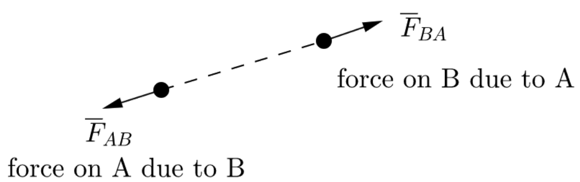
(b)



$$\frac{\triangle p}{\triangle t} = \frac{m(-v_2 - v_1)}{\triangle t} = F_{av}$$

where $\triangle t$ is period of time when ball was in contact with rocket, F_{av} is average force exerted on ball.

1.2 Conservation of momentum



isolated system (only interactions between particles present)

Newton's 3rd law

$$\overline{F}_{AB} = -\overline{F}_{BA}$$

Newton's 2nd law

$$\overline{F}_{AB} = \frac{\mathrm{d}\overline{p}_A}{\mathrm{d}t}; \qquad \overline{F}_{BA} = \frac{\mathrm{d}\overline{p}_B}{\mathrm{d}t}$$

$$\frac{\mathrm{d}\overline{p}_A}{\mathrm{d}t} = -\frac{\mathrm{d}\overline{p}_B}{\mathrm{d}t} \quad \Rightarrow \quad \frac{\mathrm{d}\overline{p}_A}{\mathrm{d}t} + \frac{\mathrm{d}\overline{p}_B}{\mathrm{d}t} = 0$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}(\overline{p}_A + \overline{p}_B) = 0$$

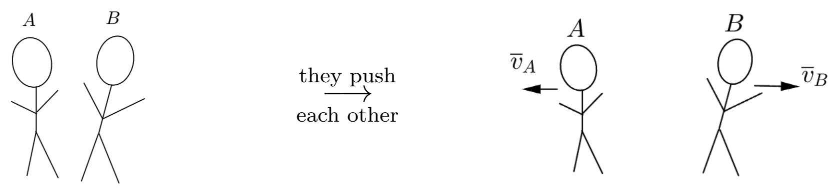
$$\Rightarrow \boxed{\overline{p}_A + \overline{p}_B = \mathrm{const}} \qquad \text{only interaction between particles present (net \underline{external} force is zero)} \qquad (4)$$

Law of conservation of momentum:

⌈ If the sum of all external forces on the system is equal to zero, then the total momentum of the system is constant ⌋

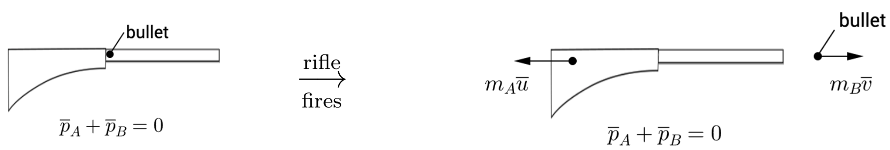
Examples:

(a) two astronauts in space (far away from any planets)



initial $\overline{p}_A + \overline{p}_B = 0$ still $\overline{p}_A + \overline{p}_B = 0$, i.e., $m_A\overline{v}_A = -m_B\overline{v}_B$

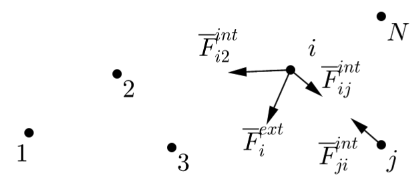
(b) recoil of a rifle



$m_A\overline{u} = -m_B\overline{v}$

1.3 Conservation of momentum (general case of N particles)

system of N particles interacting with each other



$$\overline{F}_i = \overline{F}_i^{ext} + \underbrace{\overline{F}_i^{int}}_{= \sum\limits_{j=1, j\neq i}^{N} \overline{F}_{ij}^{int}}$$

Total momentum of the system

$$\overline{P} = \sum_{i=1}^{N} \overline{p}_i$$

change of the total momentum (time-rate of change)

$$\begin{aligned}
\frac{\mathrm{d}\overline{P}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\sum_{i=1}^{N}\overline{p}_i\Big) = \sum_{i=1}^{N}\frac{\mathrm{d}\overline{p}_i}{\mathrm{d}t} \overset{2^{\text{nd}}\text{ law}}{=\!=\!=} \sum_{i=1}^{N}\overline{F}_i = \sum_{i=1}^{N}(\overline{F}_i^{ext} + \overline{F}_i^{int}) \\
&= \sum_{i=1}^{N}\overline{F}_i^{ext} + \sum_{i=1}^{N}\sum_{j=1, j\neq i}^{N} \underset{\hookrightarrow \text{force on 'i' due to 'j'}}{\overline{F}_{ij}^{int}} \\
&= \sum_{i=1}^{N}\overline{F}_i^{ext} + \sum_{i=1}^{N}\sum_{j=1, j<1}^{N} \underbrace{(\overline{F}_{ij}^{int} + \overline{F}_{ji}^{int})}_{=0} \underset{\overline{F}_{ij}^{int} = -\overline{F}_{ji}^{int}}{\overset{3^{\text{rd}}\text{law}}{=\!=\!=\!=}} \sum_{i=1}^{N}\overline{F}_i^{ext}
\end{aligned}$$

Hence

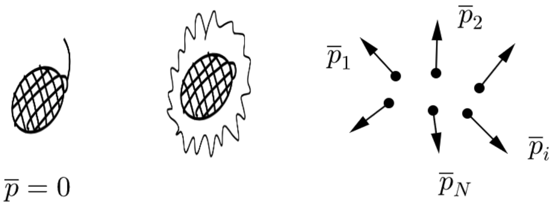
$$\boxed{\frac{\mathrm{d}\overline{p}}{\mathrm{d}t} = \sum_{i=1}^{N}\overline{F}_i^{ext} = \overline{F}^{ext}} \tag{5}$$

Conclusion: the total momentum of a system can be only changed by external forces or:

> If the sum of external forces on a system is equal to zero, then the total momentum of the system is constant (i.e. is conserved)

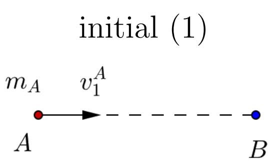
Example

(a) grenade

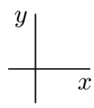


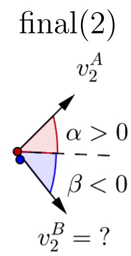
$\overline{p} = \sum_{i=1}^{N}\overline{p}_i = 0$

(b) collision



$\overline{p}_1 = (m_A v_1^A, 0)$





$$\begin{aligned}
\overline{p}_2 =& \overline{p}_2^A + \overline{p}_2^B \\
=&(m_A v_2^A \cos\alpha + m_B v_2^B \cos\beta, \\
& m_A v_2^A \sin\alpha + m_B v_2^B \sin\beta)
\end{aligned}$$

No external forces $\quad \Rightarrow \quad \overline{p}_1 = \overline{p}_2$

$$\begin{cases} m_A v_1^A = m_A v_2^A \cos\alpha + m_B v_2^B \cos\beta \\ \qquad 0 = m_A v_2^A \sin\alpha + m_B v_2^B \sin\beta \end{cases}$$

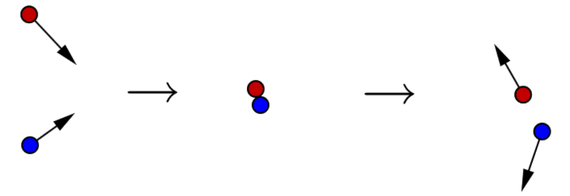
$$\begin{cases} m_A(v_1^A - v_2^A \cos\alpha) = m_B v_2^B \cos\beta \\ -m_A v_2^A \sin\alpha = m_B v_2^B \sin\beta \end{cases}$$

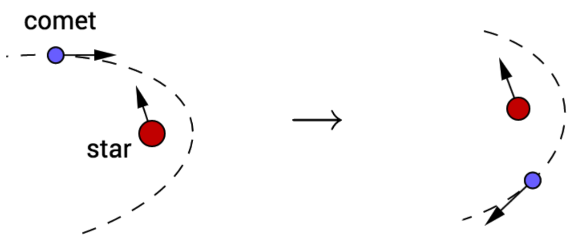
$$\Rightarrow \boxed{\begin{aligned} \tan\beta &= -\frac{v_2^A \sin\alpha}{v_1^A - v_2^A \cos\alpha} \\ v_B^2 &= -\frac{m_A v_2^A \sin\alpha}{m_B \sin\beta} \end{aligned}} \tag{6}$$

If we know: v_2^A, α and m_A, m_B, we can find β and v_2^B.

2 Collisions

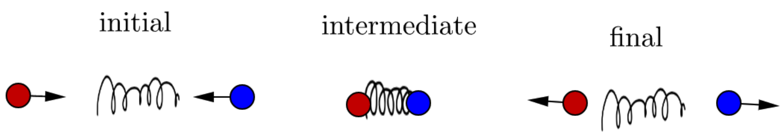
Two objects interact (directly or non-directly) over a finite time-interval.





Examples

(a) elastic collision



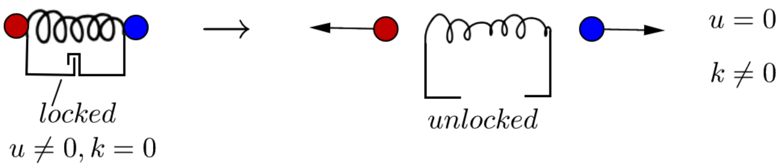
$u = \text{const}(=0)$ $\quad$ $k \neq 0$

$u \neq 0$ $\quad$ $k = 0$ $\quad$ $(E = k + u = \text{const})$

$u = 0$ $\quad$ $k \neq 0$

$$K_{\text{initial}} = K_{\text{final}}$$

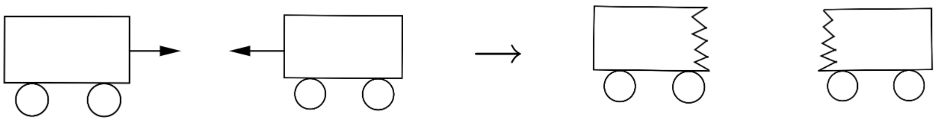
(b) superelastic collision



$$K_{\text{final}} > K_{\text{initial}}$$

(c) inelastic collision

irereversible deformation + heat
(non-conservative forces present)



$$W_{non-cons} = \triangle E = -\triangle u_{int}$$

$$K_{\text{final}} < K_{\text{initial}}$$

In general:

$$K_{\text{final}} = K_{\text{initial}} + Q \quad \begin{cases} = 0 & \text{(elastic)} \\ > 0 & \text{(superelastic)} \\ < 0 & \text{(inelastic)} \end{cases}$$

elastic

internal forces involved are potential (conservative) hence mechanical energy is conserved $\triangle E = 0$ (usually in our examples $u = \text{const}$, hence $K = \text{const}$)

inelastic

internal forces - non-conservative, mechanical energy not conserved (total energy is conserved, but a part of mechanical energy is transformed irreversibly into internal energy)

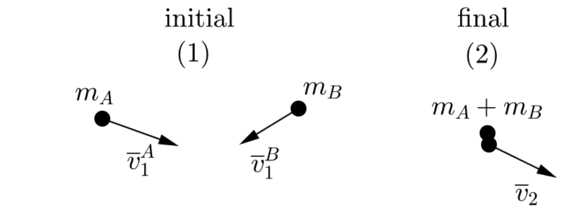
* completely inelastic collisions (colliding particles move as one object after the collision, stick to each other)
* partially inelastic

In both cases, the total momentum of the system is conserved

2.1 Inelastic collisions

Example

(a) completely inelastic collision ($Q < 0$)

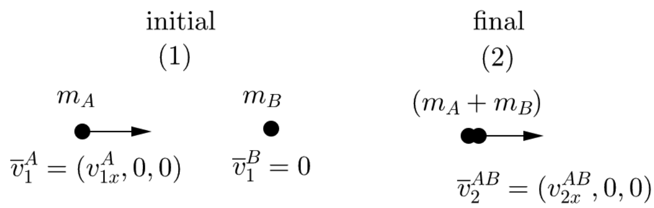


$$m_A\overline{v}_1^A + m_B\overline{v}_1^B \quad = \quad (m_A + m_B)\overline{v}_2$$

If we know: m_A, m_B and $\overline{v}_1^A$, $\overline{v}_2^B$, we can final

$$\overline{v}_2 = \frac{m_A\overline{v}_1^A + m_B\overline{v}_1^B}{m_A + m_B}$$

(b) more specific example - central, completely inelastic collision ($Q < 0$)



$$\overline{p}_1 = \overline{p}_2$$
$$\Downarrow$$
$$v_{2x}^{AB} = \frac{m_A v_{1x}^A}{m_A + m_B}$$

Compare kinetic energy before and after collision (we assume u =const)

$$K_1 = \frac{1}{2}m_A(v_{1x}^A)^2$$
$$K_2 = \frac{1}{2}(m_A + m_B)(v_{2x}^{AB})^2 = \frac{1}{2}(m_A + m_B)\frac{m_A^2(v_{1x}^A)^2}{(m_A + m_B)^2}$$
$$= \frac{1}{2}\frac{m_A^2}{m_A + m_B}(v_{1x}^A)^2 = \frac{m_A}{m_A + m_B}K_1$$
$$\triangle K = K_2 - K_1 = K_1(\frac{m_A}{m_A + m_B} - 1) < 0$$

(*)limit case $\quad m_A \gg m_B$

$\overline{v}_2^{AB} \approx (v_{1x}^A, 0, 0)$

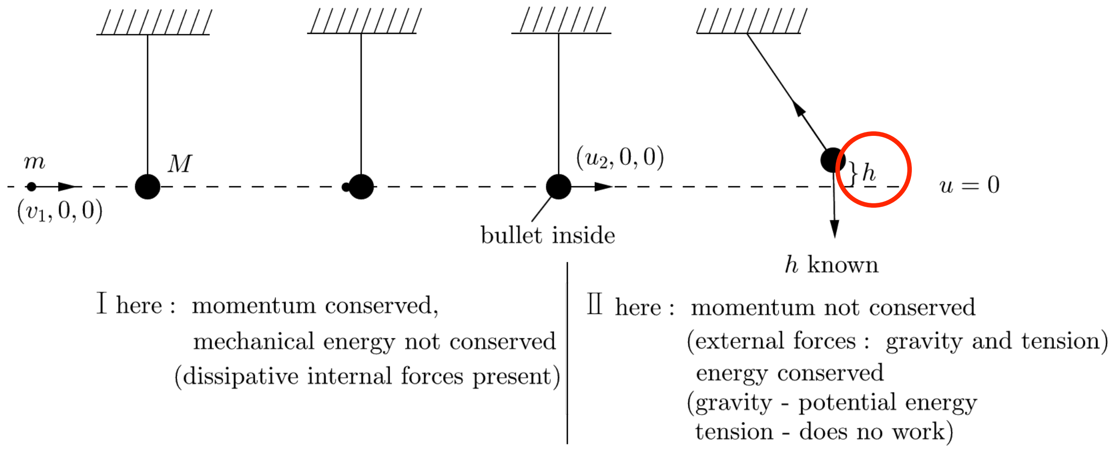
$\triangle K$ – negligible

(*)limit case $\quad m_A \ll m_B$

$\overline{v}_2^{AB}$ – negligible

$\triangle K \approx -K_1$ almost all initial kinetic energy dissipated

Example: Ballistic pendulum - used to estimate speed of a bullet



I use conservation of momentum to find u_2

$$
\begin{aligned}
\overline{p}_1 &= \overline{p}_2 \\
m v_1 &= (M+m) u_2 \\
\Rightarrow u_2 &= \frac{m}{M+m} v_1
\end{aligned}
$$

II use conservation of energy to relate h with u_2 (and hence with v_1)

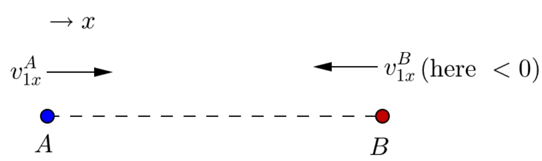
$$
\begin{aligned}
\frac{1}{2}(M+m)u_2^2 &= (M+m)gh \\
\frac{1}{2}(M+m)\frac{m^2}{(M+m)^2}v_1^2 &= (M+m)gh
\end{aligned}
$$

$$\boxed{v_1 = \frac{M+m}{m}\sqrt{2gh}} \tag{7}$$

2.2 Elastic collisions (Q=0)

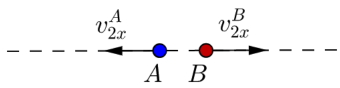
General 1D case (central collision) → before/ after collision velocities of particles are colinear

(1) momentum conserved



$$m_A v_{1x}^A + m_B v_{1x}^B = m_A v_{2x}^A + m_B v_{2x}^B$$

(2) energy conserved



$$\frac{1}{2}m_A(v^A_{1x})^2 + \frac{1}{2}m_B(v^B_{1x})^2 = \frac{1}{2}m_A(v^A_{2x})^2 + \frac{1}{2}m_B(v^B_{2x})^2$$

More specific example: B initially at rest ($v^B_{1x} = 0$). Then

$$\begin{cases} m_A v^A_{1x} = m_A v^A_{2x} + m_B v^B_{2x} \\ \frac{1}{2}m_A(v^A_{1x})^2 = \frac{1}{2}m_A(v^A_{2x})^2 + \frac{1}{2}m_B(v^B_{2x})^2 \end{cases}$$

$$\begin{cases} m_A(v^A_{1x} - v^A_{2x}) = m_B v^B_{2x} \\ m_A \underbrace{(v^A_{1x} - v^A_{2x})(v^A_{1x} + v^A_{2x})}_{((v^A_{1x})^2 - (v^A_{2x})^2} = m_B (v^B_{2x})^2 \end{cases} \tag{8}$$

Divide to get

$$v^A_{1x} + v^A_{2x} = v^B_{2x} \tag{9}$$

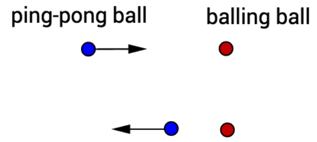
substitute v^B_{2x} in (8) with (9) and solve for v^A_{2x}, substitute back in (9) to get v^B_{2x}

$$\Rightarrow \boxed{\begin{aligned} v^A_{2x} &= \frac{m_A - m_B}{m_A + m_B} v^A_{1x} \\ v^B_{2x} &= \frac{2m_A}{m_A + m_B} v^A_{1x} \end{aligned}} \tag{10}$$

Discussion of this result

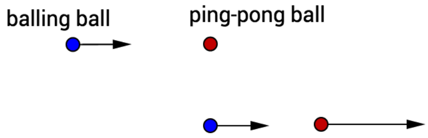
(a) $m_A \ll m_B$

$$\begin{aligned} v_{2x}^A &\approx -v_{1x}^A \\ v_{2x}^B &\ll v_{1x}^A \end{aligned}$$



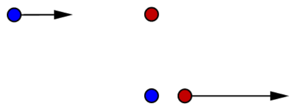
(b) $m_A \gg m_B$

$$\begin{aligned} v_{2x}^A &\approx v_{1x}^A \\ v_{2x}^B &\approx 2v_{1x}^A \end{aligned}$$



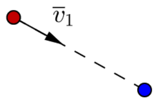
(c) $m_A = m_B$

$$\begin{aligned} v_{2x}^A &= 0 \\ v_{2x}^B &= v_{1x}^A \end{aligned}$$



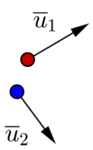
Example (for pool players) Show that in a non-central, (perfectly) elastic collision of two identical balls, one of which is initially at rest, the balls after the collision move in mutually perpendicular directions

(1)



$$\begin{aligned} \text{momentum: } & \left\{ \begin{array}{l} m\overline{v}_1 = m\overline{u}_1 + m\overline{u}_2 \\ \frac{1}{2}mv_1^2 = \frac{1}{2}m_A u_1^2 + \frac{1}{2}m_A u_2^2 \end{array} \right. \\ \text{mechanical energy } (u = \text{const}) : & \end{aligned}$$

(2)



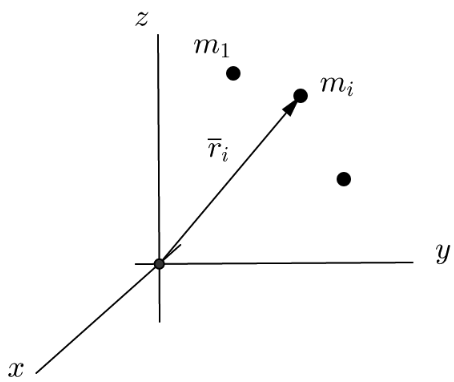
$$\left\{ \begin{array}{l} \overline{v}_1 \cdot \overline{v}_1 = \overline{v}_1^2 = (\overline{u}_1 + \overline{u}_2) \cdot (\overline{u}_1 + \overline{u}_2) \\ \qquad v_1^2 = u_1^2 + u_2^2 \end{array} \right.$$

$$\left\{ \begin{array}{l} v_1^2 = u_1^2 + u_2^2 + \underbrace{\overline{u}_1 \cdot \overline{u}_2 + \overline{u}_2 \cdot \overline{u}_1}_{2\overline{u}_1 \cdot \overline{u}_2} \\ \\ v_1^2 = u_1^2 + u_2^2 \end{array} \right.$$

$$\Downarrow \text{subtract}$$

$$2\overline{u}_1 \cdot \overline{u}_2 = 0 \Rightarrow \boxed{\overline{u}_1 \perp \overline{u}_2}$$

3 Center of Mass



position vector of the center of mass $\quad \overline{r}_{cm} = \dfrac{\sum_{i=1}^{N} m_i \overline{r}_i}{\sum_{i=1}^{N} m_i} \rightarrow$ discrete distribution of mass (11)

Note: the sum is replaced by a triple integral for a continuous distribution of mass in 3D

Properties:

$$\boxed{(\underbrace{\sum_{i=1}^{N} m_i}_{\text{total mass of the system}})\overline{r}_{cm} = \sum_{i=1}^{N} m_i \overline{r}_i} \qquad \Big/ \frac{\mathrm{d}}{\mathrm{d}t}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\sum_{i=1}^{N} m_i)\overline{r}_{cm} = \frac{\mathrm{d}}{\mathrm{d}t}\sum_{i=1}^{N} m_i \overline{r}_i$$

$$(\sum_{i=1}^{N} m_i)\frac{\mathrm{d}\overline{r}_{cm}}{\mathrm{d}t} = \sum_{i=1}^{N} \underbrace{m_i \underbrace{\frac{\mathrm{d}\overline{r}_i}{\mathrm{d}t}}_{\overline{v}_i}}_{\overline{p}_i}$$

Hence

$$\boxed{M\overline{v}_{cm} = \sum_{i=1}^{N} \overline{p}_i = \overline{p}} \tag{12}$$

$M = \sum_{i=1}^{N} m_i$

Conclusion:

The total momentum of the system is equal to the momentum of a hypothetical particle of mass M moving with velocity $\overline{v}_{cm}$

Differentiate again

$$\frac{\mathrm{d}\overline{p}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(M\overline{v}_{cm}) = M\frac{\mathrm{d}\overline{v}_{cm}}{\mathrm{d}t}$$

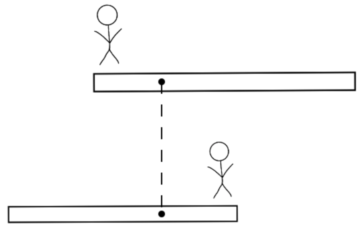
But $\dfrac{\mathrm{d}\overline{p}}{\mathrm{d}t} = \sum_{i=1}^{N} \overline{F}_i^{ext} = \overline{F}^{ext}$ $\boxed{\text{If } \overline{F}^{ext} = 0 \Rightarrow \overline{v}_{cm} = \text{const}}$

Conclusion:

If the sum of all external forces acting on the system is equal to zero, the center of mass moves with a constant velocity.

Examples:

(a) boat

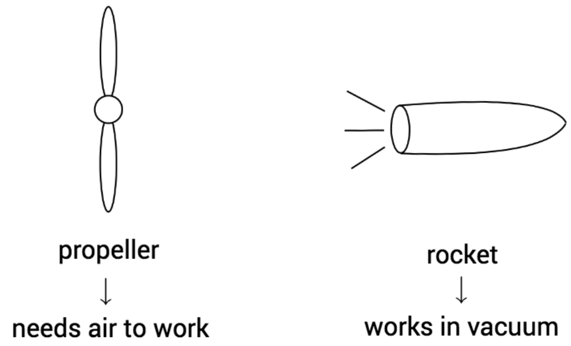


$$\overline{F}^{ext} = 0$$
$$\overline{v}_{cm} = 0$$

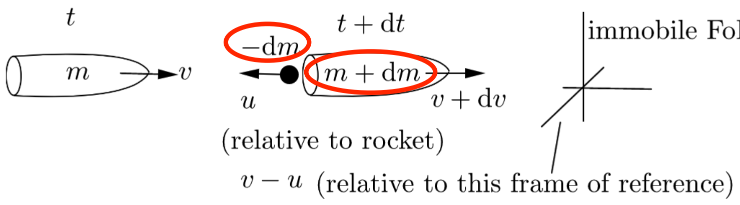
(b) grenade

$$\overline{F}^{ext} \neq 0$$
$$\frac{\mathrm{d}\overline{v}_{cm}}{\mathrm{d}t} = \frac{\overline{F}^{ext}}{m}$$
$$\Downarrow$$
$$\overline{r}_{cm} = \overline{r}_{cm}(t) \rightarrow \text{parabola}$$

Motion of objects with variable mass Rocket propulsion



Idea: burns fuel $\Rightarrow$ ejects burned fuel $\Rightarrow$ moves forward



Momentum conservation (all moments in the same (immobile) frame of reference!)

$$mv = \overbrace{(m+\mathrm{d}m)(v+\mathrm{d}v)}^{\text{rocket}}\overbrace{\underbrace{-dm}_{>0}\underbrace{(v-u)}_{<0}}^{\text{burned fuel}}$$

$$mv = mv + m\mathrm{d}v + v\mathrm{d}m + \underbrace{(\mathrm{d}m)(\mathrm{d}v)}_{\begin{array}{l}\text{much smaller than}\\ \text{the other terms}\\ \text{(product of 2 infinitesimally}\\ \text{small quantities)}\end{array}} \quad -v\mathrm{d}m + u\mathrm{d}m$$

Eventually

$$m\mathrm{d}v = -u\mathrm{d}m \tag{13}$$

$$\Downarrow$$

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -u\frac{\mathrm{d}m}{\mathrm{d}t}$$

$$\Downarrow$$

$$\boxed{a = \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{u}{m(t)}\frac{\mathrm{d}m}{\mathrm{d}t}} \tag{14}$$

Note: effective rocket $\begin{cases} \text{burns fuel at fast rate } \left(\left|\frac{\mathrm{d}m}{\mathrm{d}t}\right| \text{ large}\right) \\ \text{ejects burned fuel with large speed } (u \text{ - large}) \end{cases}$

Velocity
back to (13)

$$\boxed{\begin{aligned} \mathrm{d}v &= -u\frac{\mathrm{d}m}{m} \\ \int_{v_0}^{v(t)} \mathrm{d}v &= -u\int_{m_0}^{m(t)} \frac{\mathrm{d}m}{m} \end{aligned}}$$

$$\Rightarrow \boxed{v(t) = v_0 - u\ln\frac{m(t)}{m_0} = v_0 + u\ln\frac{m_0}{m(t)}} \tag{15}$$

Conclusions:

* The ratio of the initial mass of the rocket to the mass of the rocket after all fuel is burned should be as large as possible

* the final velocity greater in magnitude than u if $m_0/m(t_{final}) > e$ (for $v_0 = 0$)

Numerical example:

$$\begin{aligned} m(t_{final}) &= \frac{m_0}{4} \\ t_{final} &= 30s \\ u &= 2400m/s \\ v_0 &= 0 \end{aligned} \qquad \Rightarrow \qquad v_{final} = 3327m/s \tag{16}$$