Vv156 Lecture 12

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Definition

A limit of the form

$$\lim_{x\to a}\frac{f(x)}{g(x)}, \qquad \text{in which} \quad f(x)\to 0 \quad \text{and} \quad g(x)\to 0, \qquad \text{as} \quad x\to a,$$

is called an indeterminate form of type $\dfrac{0}{0}$.

- Some examples encountered earlier are

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2, \qquad \lim_{x\to 0}\frac{\sin x}{x}=1$$

- The first limit was obtained algebraically by considering $x+1$, and the 2nd limits were obtained using a geometric argument and the squeeze theorem.
- However, there are many limits that have indeterminate forms for which neither algebraic nor geometric methods will simplify the limit, so we need to develop a more general method of evaluating such limits.

- Suppose that we have an indeterminate form $\lim_{x \to a} \frac{f(x)}{g(x)}$ of type $\frac{0}{0}$, in which
    1. $f'(x)$ and $g'(x)$ are continuous at $x = a$
    2. $g'(a) \neq 0$.
- Then the following limits are equivalent,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \lim_{x \to a} \frac{\frac{f(x)-f(a)}{x-a}}{\frac{g(x)-g(a)}{x-a}}$$

- Let $h = x - a$, so $x = h + a$ and clearly $h \to 0$ as $x \to a$,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{h \to 0} \frac{\frac{f(h+a)-f(a)}{h}}{\frac{g(a+h)-g(a)}{h}} = \frac{\lim\limits_{h \to 0} \dfrac{f(h+a) - f(a)}{h}}{\lim\limits_{h \to 0} \dfrac{g(a+h) - g(a)}{h}} = \frac{f'(a)}{g'(a)} = \frac{\lim\limits_{x \to a} f'(x)}{\lim\limits_{x \to a} g'(x)}$$

$$= \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

- Although this assumes f and g have continuous derivatives at $x = a$ and that $g'(a) \neq 0$, the result is true under less stringent conditions.

L'Hospital's rule for the form $\frac{0}{0}$

Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$,

and that

$$\lim_{x \to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to a} g(x) = 0$$

If $\displaystyle\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, or if this limit is $+\infty$ or $-\infty$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as

$$x \to a^- \qquad x \to a^+ \qquad x \to -\infty \qquad x \to +\infty$$

Exercise

Find

$$\lim_{x\to 0} \frac{e^x - 1}{x^3}$$

- Applying L'Hospital's rule to limits that are not in indeterminate forms can lead to incorrect results. For example, the computation

$$\lim_{x\to 0} \frac{x+6}{x+2} = \frac{6}{2} \neq \lim_{x\to 0} \frac{(x+6)'}{(x+2)'} = \lim_{x\to 0} \frac{1}{1} = 1$$

Q: Can we apply L'Hospital's rule to

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}},$$

where clearly

$$\lim_{x\to\infty} \ln x = \infty \quad \text{and} \quad \lim_{x\to\infty} \sqrt[3]{x} = \infty$$

Definition

The limit of a ratio, $\lim_{x \to a} \frac{f(x)}{g(x)}$, is called an indeterminate form of type $\frac{\infty}{\infty}$ if

$$f(x) \to \infty \quad \text{and} \quad g(x) \to \infty \qquad \text{as} \quad x \to a$$

L'Hospital's rule for the form $\frac{\infty}{\infty}$

Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$,

and that

$$\lim_{x \to a} f(x) = \infty \qquad \text{and} \qquad \lim_{x \to a} g(x) = \infty$$

If $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, or if this limit is $+\infty$ or $-\infty$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$

Moreover, this statement is also true in the case of a limit as

$$x \to a^- \qquad x \to a^+ \qquad x \to -\infty \qquad x \to +\infty$$

Exercise

Find

$$\lim_{x\to\infty} \frac{\ln x}{\sqrt[3]{x}}$$

- If n is any positive integer, then

$$x^n \to \infty \qquad \text{as} \qquad x \to \infty$$

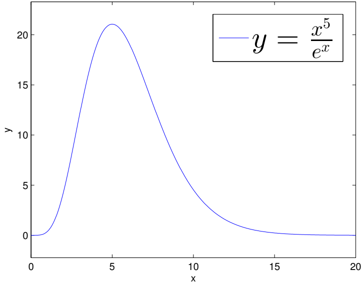
- Such integer powers of x are sometimes used as "measuring sticks" to describe how rapidly other functions grow. For example,

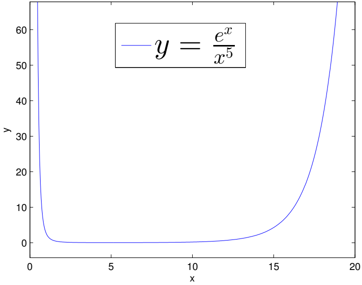
$$e^x \to \infty \qquad \text{and} \qquad x \to \infty$$

- It is reasonable to ask whether x^n grow faster or e^x grow faster.
- One way to investigate this is to examine the behaviour of the ratio

$$\frac{x^n}{e^x} \text{ or } \frac{e^x}{x^n}.$$

- For a given n, graphically it is clear that e^x grows faster for large values of x,





Q: How can we show for any positive integer n

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0$$

- We have discussed indeterminate forms of type

$$\frac{0}{0} \qquad \text{and} \qquad \frac{\infty}{\infty}$$

- However, these two forms are not the only possibilities; in general, the limit of an expression that has one of the forms

$$\frac{f(x)}{g(x)}, \qquad f(x)g(x), \qquad f(x)^{g(x)}, \qquad f(x) \pm g(x)$$

is called an indeterminate form if the limits of $f(x)$ and $g(x)$ individually exert conflicting influences on the limit of the entire expression. For example,

$$\lim_{x \to 0^+} x \ln x$$

is an indeterminate form of type $0 \cdot \infty$ because the limit of the first factor is 0, the limit of the second factor is $-\infty$, so they have conflicting influences on the value of the limit, therefore the limit in this form is indeterminate

Exercise

Evaluate $(a)\quad \lim_{x\to 0^+} x\ln x$

- A limit problem that leads to one of the expressions

$$\infty-\infty, \qquad -\infty-(-\infty), \qquad \infty+(-\infty), \qquad -\infty+\infty$$

  is an indeterminate form of type $\infty-\infty$.

- However, limit problems that lead to one of the expressions

$$\infty+\infty, \qquad \infty-(-\infty), \qquad -\infty+(-\infty), \qquad -\infty-\infty$$

  is not an indeterminate form.

- Indeterminate forms of $\infty-\infty$ can sometimes be evaluated by combining the terms and manipulating the result to be an indeterminate form of type

$$\frac{0}{0} \quad \text{or} \quad \frac{\infty}{\infty}$$

Exercise

Evaluate $(b)\quad \lim_{x\to 1}\left(\frac{x}{x-1}-\frac{1}{\ln x}\right)$

- Limits of the form

$$\lim_{x\to a}\Big(f(x)\Big)^{g(x)}$$

can give rise to indeterminate forms of the types ∞^0, 0^0, 1^∞. For example,

$$\lim_{x\to 0}(1+\sin x)^{\frac{1}{x}}$$

- Such indeterminate forms can be evaluated by first introducing a variable y

$$y = f(x)^{g(x)}$$

$$\implies \ln y = g(x)\ln f(x)$$

and then compute the limit of $\ln y$.

Exercise

Find

$$\lim_{x\to 0}(1+\sin x)^{1/x}$$

No.	Form	Indeterminate	Technique
1	$(0/0)$	Yes	Direct
2	(∞/∞)		
3	$0 \cdot \infty$		$(0/(1/\infty))$ or $(\infty/(1/0))$
4	$\infty - \infty$		factorize or exponential transformation
5	0^0		logarithmic transformation
6	∞^0		
7	1^∞		
8	0^∞	No	
9	$(0/\infty)$		$0 \cdot (1/\infty)$
10	$(\infty/0)$		$\infty \cdot (1/0)$
11	$\infty \cdot \infty$		∞
12	$+\infty + (+\infty)$		∞
13	$+\infty - (-\infty)$		∞
14	$-\infty + (-\infty)$		$-\infty$
15	$-\infty - (+\infty)$		$-\infty$

Table: Table of Indeterminate forms

When NOT to use L'Hospital's rule

1. When it is making the problem worse. For example,

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{x}{(\ln x)^{-1}} = \lim_{x\to 0^+} \frac{1}{-\frac{1/x}{(\ln x)^2}} = \lim_{x\to 0^+} \left[-x(\ln x)^2\right]$$

which is more complicated than the original problem

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/x} = \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} (-x) = 0$$

2. When there is a better way to get the answer.

$$\lim_{x\to\infty} \frac{x^2-1}{2x^2+1} = \lim_{x\to\infty} \frac{\frac{x^2-1}{x^2}}{\frac{2x^2+1}{x^2}} = \lim_{x\to\infty} \frac{1-\frac{1}{x^2}}{2+\frac{1}{x^2}} = \frac{1}{2}$$

3. It isn't an indeterminate form!

$$\lim_{x\to 0^+} \frac{\cos x}{x} = +\infty \neq \lim_{x\to 0^+} \frac{(\cos x)'}{(x)'} = \lim_{x\to 0^+} \frac{-\sin x}{1} = 0$$

- In many problems, the properties of interest in the graph of a function are:

Properties of curves

- symmetries
- periodicity
- x-intercepts
- y-intercept
- relative extrema
- concavity
- intervals of increase and decrease
- inflection points
- asymptotes
- behavior as $x \to \infty$ or as $x \to -\infty$

- Some of these properties may not be relevant in certain cases. For example,

asymptotes are characteristic of rational functions

$$\lim_{x\to a} f(x) = \infty \qquad \text{or} \qquad \lim_{x\to a} f(x) = -\infty$$

$$\lim_{x\to\infty} f(x) = b \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = b$$

- Here we discuss how to find the features of polynomial and rational functions

$$y = P_n(x) \qquad \text{and} \qquad y = \frac{P(x)}{Q(x)}$$

however, similar procedures can be used for other functions.

Exercise

Sketch a graph for each of the following functions and specify the locations of the intercepts, relative extrema, inflection points and asymptotes.

(a) Polynomial function

$$y = x^3 - 3x + 2$$

(b) Rational function

$$y = \frac{x^2 - 1}{x^3}$$

- Rational functions of which the degree of P did not exceed the degree of Q,

$$f(x) = \frac{P(x)}{Q(x)}$$

have either vertical asymptotes or horizontal asymptotes.

- Suppose P of a rational function has greater degree than Q, then other "asymptotes" are possible. For example, consider the rational function

$$f(x) = \frac{x^2 + 1}{x}$$

- By division we can rewrite it as

$$f(x) = x + \frac{1}{x}$$

- The second terms approach 0 as $x \to \infty$ or as $x \to -\infty$, then

$$\Big(f(x) - x\Big) \to 0$$

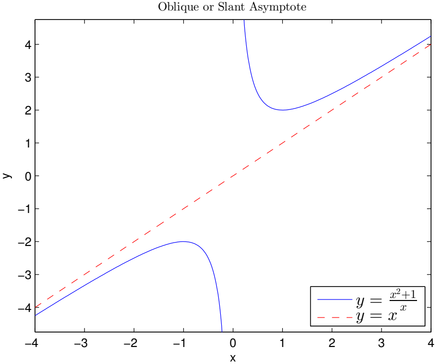
- Geometrically, this means that the graph of

  $y = f(x)$ and the line $y = x$ eventually gets closer and closer

  as $x \to \infty$ or as $x \to -\infty$.

Definition

The line $y = x$ is called an oblique or slant asymptote of f.



- Similarly, consider the rational function

$$g(x) = \frac{x^3 - x^2 - 8}{x - 1}$$

we can rewrite it as

$$g(x) = x^2 - \frac{8}{x - 1}$$

The second terms approach 0 as $x \to \infty$ or as $x \to -\infty$, then

$$\left(g(x) - x^2\right) \to 0$$

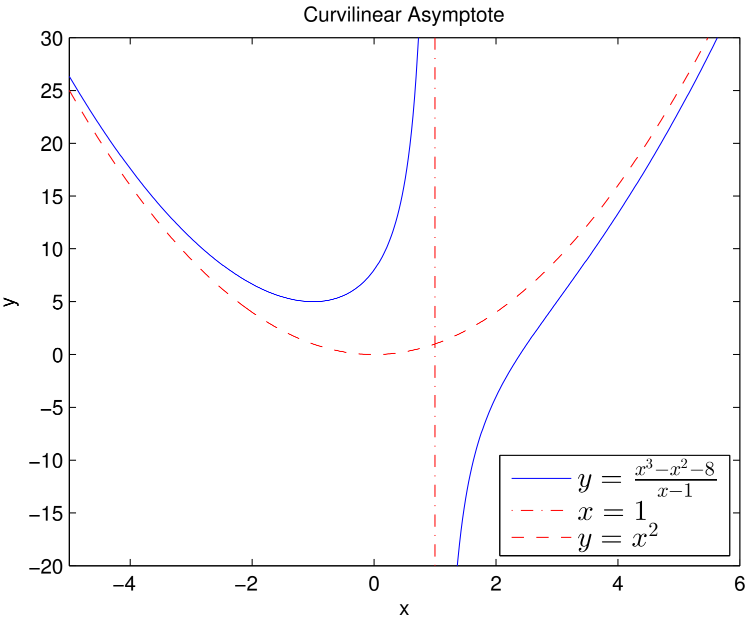
So the graph of

$y = g(x)$ eventually gets closer and closer to the parabola $y = x^2$

as $x \to \infty$ or as $x \to -\infty$.

Definition

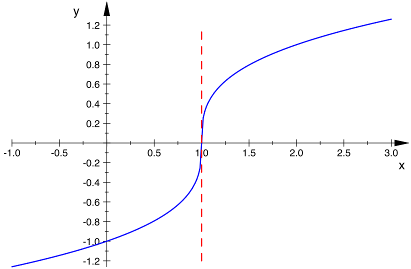
The parabola is called a curvilinear asymptote of g.


Curvilinear Asymptote
30
25
20
15
10
5
0
−5
−10
−15
−20
y
−4
−2
0
2
4
6
x
$y = \frac{x^3-x^2-8}{x-1}$
$x = 1$
$y = x^2$

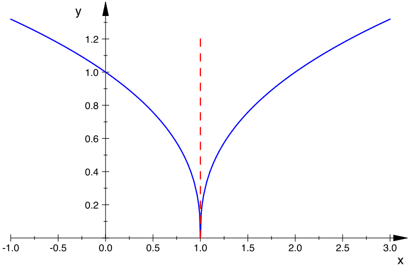
- Vertical tangents are commonly found in graphs of functions

$$f(x) = (x - a)^{p/q}$$

that involve radicals or fractional exponents.



$y = (x - 1)^{1/3}$



$(x - 1)^{2/5}$