
Question1 (1 points)

Find the derivative function of the following function using the definition of derivative.

$$g(t) = \frac{1}{\sqrt{t}}$$

State the natural domain of $g(t)$ and the natural domain of its derivative function.

Question2 (1 points)

Show that

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is differentiable but that f' is not continuous at $x = 0$.

Question3 (1 points)

Let f be defined on $(-\infty, \infty)$, and c denote a constant. Consider the following

$$f(c+h) = f(c) + Ah + \varepsilon(h)$$

where A is not a function of h. Show that $f(x)$ is continuous at c if and only if

$$\lim_{h \to 0} \varepsilon(h) = 0$$

Question4 (4 points)

Find the derivative y', show all your workings.

(a) (1 point) $y = x^8 - 3\sqrt{x} + 5x^{-3}$

(b) (1 point) $y = \sin x + 2\cos^3 x$

(c) (1 point) $y = \dfrac{\sin x \cos x}{\sqrt{x}}$

(d) (1 point) $y = x^x$

Question5 (1 points)

Consider the function

$$f(x) = (x^{156} - 1)g(x)$$

where $g(x)$ is continuous at $x = 1$, and $g(1) = 1$. Find the derivative of $f(x)$ at $x = 1$.

Question6 (1 points)

For a positive integer n, consider

$$f(x) = a_1 \sin(x) + a_2 \sin(2x) + \cdots + a_n \sin(nx)$$

where $a_1, a_2, ..., a_n$ are real numbers such that $|f(x)| \leq |\sin(x)|$ for all $x \in \mathbb{R}$. Show that

$$|a_1 + 2a_2 + 3a_3 + ... + na_n| \leq 1$$

Question7 (1 points)

Suppose there is a bowl in the shape of a hemisphere of radius a meters. Water is pouring into the bowl with a constant rate of $5\pi a^3$ cubic meters per second. Find the rate at which the water level is rising in the bowl.

Question8 (0 points)

(a) (1 point (bonus)) Suppose

$$x = a(\theta - \sin\theta) \qquad \text{and} \qquad y = a(1 - \cos\theta),$$

where a is a constant. Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) (1 point (bonus)) In a series LC circuit the charge

$$q(t)$$

on the capacitor at time $t > 0$ satisfies the equation

$$L\ddot{q} + \frac{1}{C}q = 0$$

where L and C are constants, known as the inductance and capacitance, respectively.

$$q(t) = A\cos\left(\frac{t}{\sqrt{LC}}\right) + B\sin\left(\frac{t}{\sqrt{LC}}\right)$$

Show the above function satisfies the equation for any constants A and B.

$$q(0) = 0 \qquad \text{and} \qquad \dot{q}(0) = 0$$

Find A and B that satisfy the above conditions.

(c) (1 point (bonus)) Let $f, g \in \mathcal{C}^\infty$. Show the nth-order derivative of their product is

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} f^{(n-k)} g^{(k)}$$

(d) (1 point (bonus)) Suppose f is m times continuously differentiable on (a, b), that is,

$$f \in \mathcal{C}^m(a, b)$$

The function $f(x)$ is said to have a root of order m at $x^* \in (a, b)$ if and only if

$$f(x^*) = 0, \quad f'(x^*) = 0, \quad f''(x^*) = 0, \quad \cdots \quad f^{(m-1)}(x^*) = 0, \quad \text{and} \quad f^{(m)}(x^*) \neq 0$$

The positive integer m is known as the multiplicity of the root. A root of order

$$m = 1$$

is often called a simple root, and when

$$m > 1$$

the root is called a multiple root. Show there exists a continuous function

$$h(x)$$

so that $f(x)$ can be expressed as the product

$$f(x) = (x - x^*)^m g(x), \qquad \text{where} \qquad g(x^*) \neq 0$$

if the function $f(x)$ has a root of order m at $x = x^*$.