vv255:Functions of several variables.

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UM-SJTU Joint Institute

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Today

1. Extreme values of functions of several variables: critical points, second derivative test.
2. Extreme values theorem.
3. Extreme values subject to constrains: Lagrange multipliers.

Extreme Values

Q: What are the maximum and minimum values of a function $f(x, y)$ of two independent real variables?

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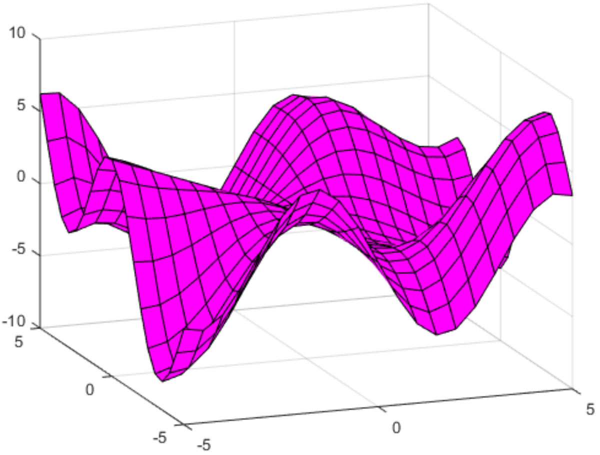
Maximum and minimum values are peaks and troughs in the surface defined by the graph $z = f(x, y)$.



Extreme Values: Definitions

Let $f : D \longrightarrow \mathbb{R}$ be a function with $D \subseteq \mathbb{R}^2$. We say that

- $\bar{a} \in D$ is a local maximum of f

$$\exists r > 0 \quad \forall \bar{x} \in B(\bar{a}, r) \quad f(\bar{x}) \leq f(\bar{a})$$

  and we call $f(\bar{a})$ a local maximum value.

- $\bar{a} \in D$ is a global maximum or absolute maximum of f if

$$\forall \bar{x} \in D \quad f(\bar{x}) \leq f(\bar{a})$$

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- $\bar{a} \in D$ is a local minimum of f if

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Extreme Values: Example

The function

$$f(x,y) = \sqrt{9 - x^2 - y^2}$$

with domain $D = \{(x,y) :\in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$ describes a hemisphere of radius 3.

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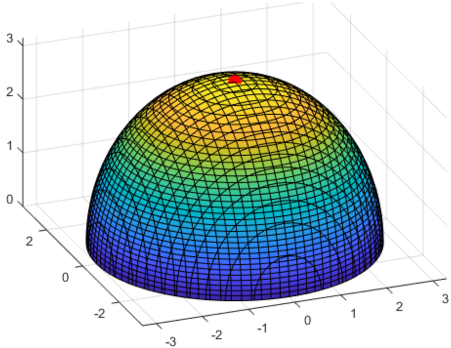

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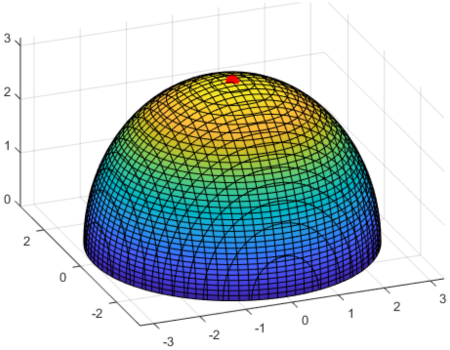

$\Rightarrow$ a local maximum is at the point $(0, 0, 3)$, and every point $(x, y, 0)$ with $x^2 + y^2 = 9$ is a local minimum. The point $(0, 0, 3)$ is also a global maximum, and all of the points $(x, y, 0)$ with $x^2 + y^2 = 9$ are global minima.

Extreme values

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Definition

Let $f : D \longrightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}^2$ and let $\bar{a} \in D$ be such that f is differentiable at $\bar{a}$. We say that $\bar{a}$ is a critical point of f if $f_x(\bar{a}) = f_y(\bar{a}) = 0$.

Extreme Values

Theorem

Let $f : D \longrightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}^2$ and let $\bar{a} \in D$ such that f is differentiable in an open ball around $\bar{a}$. If $\bar{a}$ is local maximum or minimum of f, then $\bar{a}$ is a critical point of f.

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Note that saying that f is differentiable in an open ball around $\bar{a}$ precludes $\bar{a}$ frome being on the boundary of D.

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As is the case with functions of a single variable, critical points are not necessarily local minima or maxima.

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Consider the hyperbolic paraboloid $f(x, y) = y^2 - x^2$.

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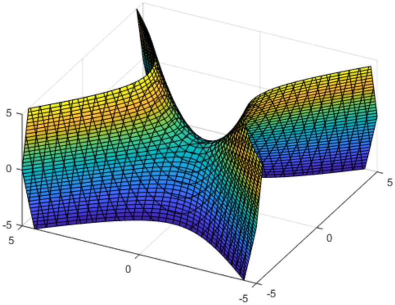

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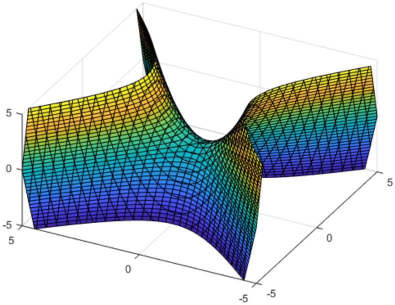

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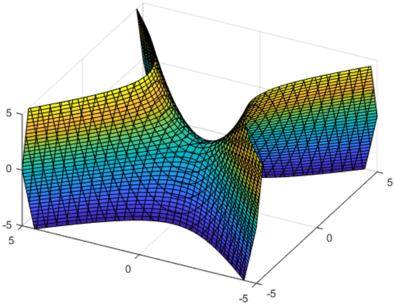

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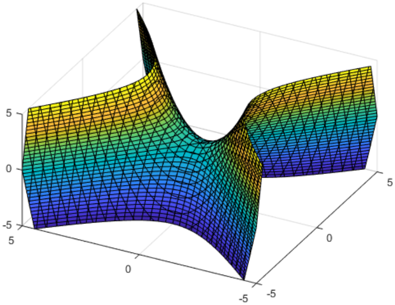

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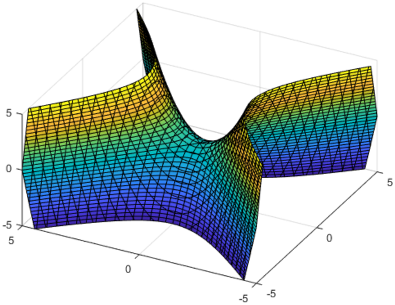

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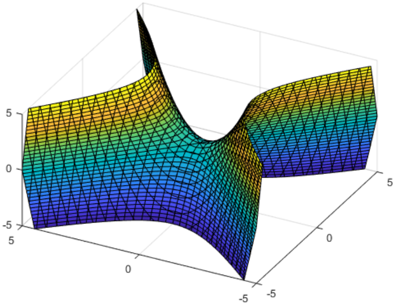

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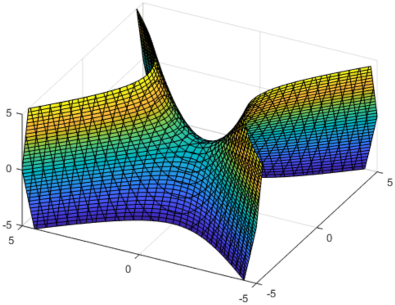

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$\Rightarrow (0, 0)$ is not a local minimum or local maximum of f.

Critical Points

Definition

Let $f : D \longrightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}^2$ and let $\bar{a} \in D$ such that f is differentiable in an open ball around $\bar{a}$. We say that $\bar{a}$ is saddle point *of f if $\bar{a}$ is a critical point and $\bar{a}$ is not a local minimum or local maximum.*

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We get a version of the second derivative test for functions of two variables:

The second derivative test

Theorem

(Second Derivative Test) Let $f : D \longrightarrow \mathbb{R}$ with $D \subseteq \mathbb{R}^2$ and let $\bar{a} \in D$ such that the second derivatives of f are continuous on an open ball around $\bar{a}$. Suppose that $\bar{a}$ is a critical point of f and

$$Q(\bar{a}) = \begin{vmatrix} f_{xx}(\bar{a}) & f_{xy}(\bar{a}) \\ f_{yx}(\bar{a}) & f_{yy}(\bar{a}) \end{vmatrix}$$

Then

(I) *If $Q(\bar{a}) > 0$ and $f_{xx}(\bar{a}) > 0$, then $\bar{a}$ is a local minimum*

(II) *If $Q(\bar{a}) > 0$ and $f_{xx}(\bar{a}) < 0$, then $\bar{a}$ is a local maximum of f*

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Note that if $Q(\bar{a}) = 0$, then this result gives us no information about whether $\bar{a}$ is a local minimum or maximum, or saddle point of f.

The second derivative test

Proof.

(sketch of part (I) only) Suppose that $\bar{a}$ is a critical point of f, $Q(\bar{a}) > 0$ and $f_{xx}(\bar{a}) > 0$.

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$$D_{\bar{u}}f = \nabla f \cdot \bar{u} = f_x h + f_y k$$

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$$= f_{xx}h^2 + 2f_{xy}hk + f_{yy}k^2$$

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Proof.

Now, completing the square yields:

$$D_{\bar{u}}^2 f = f_{xx}\left(h + \frac{f_{xy}}{f_{xx}}\right)^2 + \frac{k^2}{f_{xx}}\left(f_{xx}f_{yy} - (f_{xy})^2\right)$$

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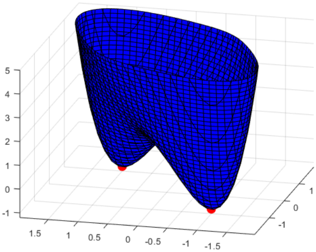
The second derivative test

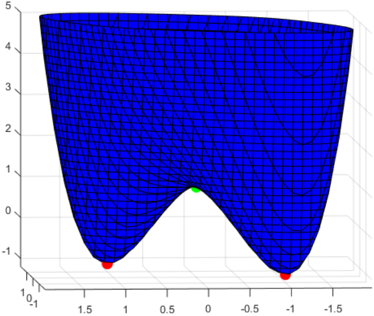
Example

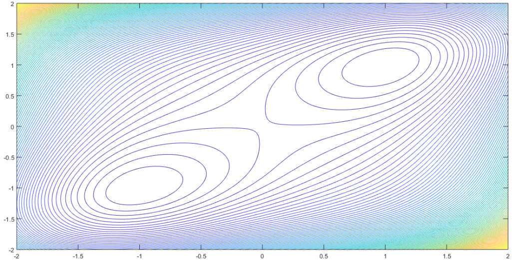
Find and classify the critical points of

$$f(x, y) = x^4 + y^4 - 4xy + 1$$

The second derivative test







Extreme Value Theorem

The second derivative test allows us to find and classify points within the interior of the domain of a sufficiently differentiable function of more than one variable that correspond to local minima or local maxima. If the global maximum or global minimum of a function of more than one variable occurs within the interior of domain, then this global minimum or global maximum must coincide with a critical point. However, as was the case with functions of a single real variable, it may also be the case that the global maximum or minimum of a function does not exist, or occurs at a boundary point of the domain of the function.

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For functions of a single variable, every continuous function whose domain is a closed and bounded set achieves its maximum and minimum value. This is the **Extreme Value Theorem** for functions of a single variable.

Extreme Value Theorem

The obvious generalization of this is also true for functions of more than one variable.

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Theorem

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In fact, the Extreme Value Theorem completely generalizes to functions of more than one variable. That is, if $f : D \longrightarrow \mathbb{R}^n$, where $D \subseteq \mathbb{R}^n$ is a closed and bounded set, is continuous, then f achieves its minimum and maximum values on D.
The "geometric version" of the Extreme Value Theorem for functions of two variables:

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The "geometric version" of the Extreme Value Theorem for functions of two variables:

- A region $\mathcal{R}$ in $\mathbb{R}^2$ is bounded if there exists an open ball B that contains every point in $\mathcal{R}$. Intuitively, a region is bounded if it has finite area.

Extreme Value Theorem

- Let $\mathcal{R}$ be a bounded region in $\mathbb{R}^2$. We say that $(x, y) \in \mathbb{R}^2$ is a boundary point of $\mathcal{R}$ if for all open balls B with $(x, y) \in B$, B contains both points from $\mathcal{R}$ and points from $\mathbb{R}^2 \backslash \mathcal{R}$. Intuitively, a boundary point of a region $\mathcal{R}$ is a point that is touching points that are in $\mathcal{R}$ and points that are outside of $\mathcal{R}$.

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Theorem

Let $f : D \longrightarrow \mathbb{R}$ be a function where $D \subseteq \mathbb{R}^2$ is a bounded region that contains all of its boundary points. If f is continuous, then f achieves its minimum and maximum values on D. I.e. there exists $\bar{a} \in D$ such that $\bar{a}$ is a global minimum for f and there exists $\bar{b} \in D$ such that $\bar{b}$ is a global maximum for f.

Extreme Value Theorem

Example

- *Consider* $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 4\}$. *This is a filled circle of radius* 2. D *is completely contained in the open ball* $B(\bar{0}, 2.1)$ *and so is a bounded region. The boundary points of* D *are the points* $P = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 4\}$. *Every point in* P *is in* D*, so* D *contains all of its boundary points.*

Extreme Value Theorem

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- *Consider* $D = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$*. This is the infinitely long horizontal strip with width* 2 *centred about the* x*-axis. There is now open ball* B *such that* D *is contained in* B *so* D *is not a bounded region.*

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- *Consider* $D = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$. *This is the infinitely long horizontal strip with width* 2 *centred about the* x*-axis. There is now open ball* B *such that* D *is contained in* B *so* D *is not a bounded region.*
- *Consider* $D = \{(x, y) \in \mathbb{R}^2 \mid 0 < x^2 + y^2 \leq 9\}$. *This is a punctured filled circle of radius* 3. D *is completely contained in the open ball* $B(\bar{0}, 3.1)$ *and so is a bounded region. The boundary points of* D *are the points*

$$P = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 9\} \cup \{0\}$$

  Since 0 *is not in* D, D *does not contain all of its boundary points.*

Optimization problems

Example

Find the global minimum and maximum values of $f : D \longrightarrow \mathbb{R}$ *defined by*

$$f(x, y) = x^2 + y^2 + x^2 y + 4$$

where $D = \{(x, y) \in \mathbb{R}^2 \mid -1 \leq x, y \leq 1\}$.

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Example

Find the maximum volume of a rectangular box without a lid made from 12 *square meters of cardboard.*

Optimization problems

Example

Find the global minimum and maximum values of $f : D \longrightarrow \mathbb{R}$ *defined by*

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Example

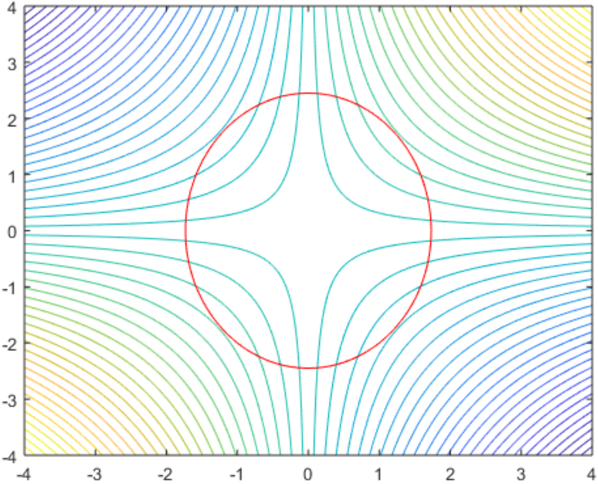
Find the maximum volume of a rectangular box without a lid made from 12 *square meters of cardboard.*

Example

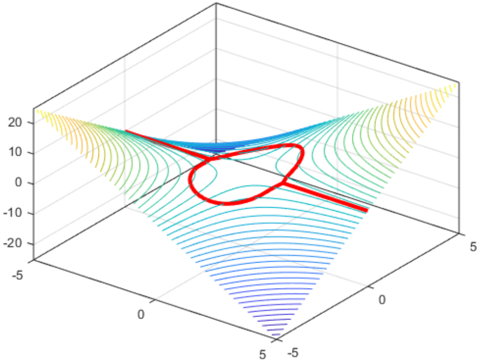
Find the points on the cone $z^2 = x^2 + y^2$ *that are closest to the points* $(4, 2, 0)$.

Lagrange Multipliers

Consider a function of two variables $f(x, y) = xy$. Suppose that we want to maximize or minimize this function given the constraint $3x^2 + y^2 = 6$. We can visualize both of these functions on plot on the Cartesian plane:



Lagrange Multipliers




```
>> x=-5:.1:5; y=-5:.1:5;
[X,Y] = meshgrid(x,y);
f = X.*Y;
contour3( X, Y, f, 50);
hold on;
y1 = sqrt(6 -3*x.^2);
y2 = -sqrt(6 -3*x.^2);
plot3( x, y1, x.*y1, '-r', x, y2, x.*y2, '-r', 'LineWidth', 3);
```

Lagrange Multipliers

The plot on the previous slide shows the constraint equation $3x^2 + y^2 = 6$ (red) and contours of $f(x, y)$: $f(x, y) = c$. Points where contour line intersect the red constraint line represent points on that contour that satisfy the constraint. It should be clear from the plot that minimum or maximum values of $f(x, y)$ must occur on contour line that exactly touch the red constraint line, because if this were not the case then one could move to a higher or lower adjacent contour line and find the points of intersection between that contour line and the constraint line.

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$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \text{ and } g(x_0, y_0) = 6$$

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Note that

$$\nabla f(x, y) = f_x(x, y)\bar{i} + f_y(x, y)\bar{j} \text{ and } \nabla g(x, y) = g_x(x, y)\bar{i} + g_y(x, y)\bar{j}$$

Lagrange Multipliers

Where $f_x(x, y) = y$, $f_y(x, y) = x$, $g_x(x, y) = 6x$ and $g_y(x, y) = 2y$.

Lagrange Multipliers

Where $f_x(x, y) = y$, $f_y(x, y) = x$, $g_x(x, y) = 6x$ and $g_y(x, y) = 2y$. Therefore, by equating the component, we would obtain a solution to the equations

$$\nabla f(x, y) = \lambda \nabla g(x, y) \text{ and } g(x, y) = 6$$

if we had a solution to the system of equations

$$y = \lambda \cdot 6x \qquad x = \lambda \cdot 2y \qquad 3x^2 + y^2 = 6$$

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Solving these equations we get that $\lambda = \pm \frac{1}{\sqrt{12}}$ and the possible locations of minimums and maximums of f satisfying the constraint are

$$(1, \sqrt{3}) \qquad (-1, \sqrt{3}) \qquad (1, -\sqrt{3}) \qquad (-1, -\sqrt{3})$$

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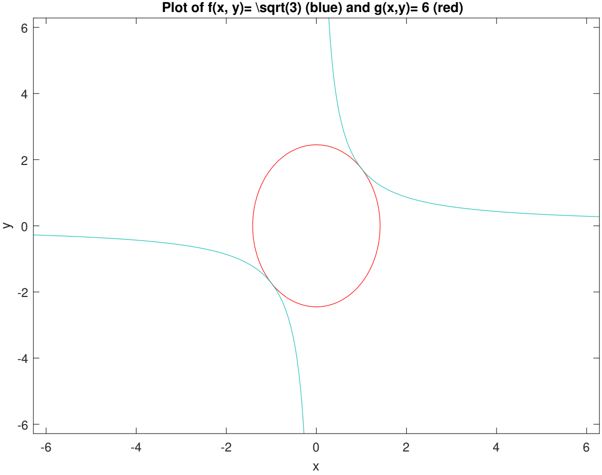
The maximum value of f satisfying the constraint occur at the points $(\pm 1, \pm\sqrt{3})$, and the minimum values of f satisfying the constraint occur at $(\pm 1, \mp\sqrt{3})$.

Lagrange Multipliers

The following plot shows the contours of $f(x, y)$ corresponding to $f(x, y) = \sqrt{3}$ (blue), and the constraint curve $3x^2 + y^2 = 6$ (red).



Lagrange Multipliers

In Matlab, one may use

```
fsolve(f,x)
```

to solve a nonlinear system $\bar{f}(\bar{x}) = \bar{0}$ for $\bar{x}$ numerically. x in the command gives the initial approximation to $\bar{x}$

Represent the system $y = \lambda 6x \quad x = \lambda 2y \quad 3x^2 + y^2 = 6$ in the vector form

$$\bar{h}(\bar{x}) = \begin{pmatrix} x_2 - 6x_3x_1 \\ x_1 - 2x_3x_2 \\ 3x_1^2 + x_2^2 - 6 \end{pmatrix} = 0, \quad x = x_1,\, y = x_2,\, \lambda = x_3$$

```
>> h=@(x) [x(2)-6*x(3)*x(1); x(1)-2*x(3)*x(2); 3*x(1)^2+x(2)^2-6];
fsolve(h,[1 1 1])

Equation solved.

fsolve completed because the vector of function values is near zero
as measured by the default value of the function tolerance, and
the problem appears regular as measured by the gradient.

<stopping criteria details>


ans =

    1.0000    1.7321    0.2887

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Lagrange Multipliers

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Method of Lagrange Multipliers

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- Suppose that $f(x, y, z)$ has an extreme value on $\mathcal{S}$ at the point (x_0, y_0, z_0) (a maximum or minimum value of f on $\mathcal{S}$).
- Let $\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$ be a smooth curve that lies on $\mathcal{S}$ and such that $\bar{r}(t_0) = x_0\bar{i} + y_0\bar{j} + z_0\bar{k}$.

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- Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(t) = f(x(t), y(t), z(t))$. Since f has an extreme value at (x_0, y_0, z_0), h will have an extreme value at t_0 and $h'(t_0) = 0$. So, using the chain rule:

$$
\begin{aligned}
0 &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\
&= \nabla f(x_0, y_0, z_0) \cdot \bar{r}'(t_0)
\end{aligned}
$$

Method of Lagrange Multipliers

- At an extreme value of f on the surface $\mathcal{S}$, the vector ∇f must be perpendciular to the tangent plane of $\mathcal{S}$ at this point.

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- If $\langle x_0, y_0, z_0 \rangle$ is the point of an extreme value of f, then $\nabla f(x_0, y_0, z_0)$ must be parallel to the normal vector $\nabla g(x_0, y_0, z_0)$ of the tangent plane of $\mathcal{S}$ at $(x_0, y_0, z_0) \Rightarrow \exists \lambda \in \mathbb{R}$ such that

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- **Method of Lagrange Multipliers:** To maximize or minimize the value of a differentiable function $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$, $k \in \mathbb{R}$, assume that extreme values of f exist on the surface defined by $g(x, y, z) = k$, and $\nabla g \neq \bar{0}$ on this surface, then the extreme values of f can be identified by finding x, y, z such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \text{ and } g(x, y, z) = k$$

  Evaluating f at these solutions then reveals if the solution corresponds to a minimum or maximum value.

Lagrange Multipliers

Example

Find the largest volume of a rectangular box with sides parallel to the coordinate planes that is contained in the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, \quad a, b, c > 0$$

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$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, \quad a, b, c > 0$$

Let one of the corners be at the point (x, y, z)

Lagrange Multipliers

Example

Find the largest volume of a rectangular box with sides parallel to the coordinate planes that is contained in the solid ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, \quad a, b, c > 0$$

Let one of the corners be at the point $(x, y, z) \Rightarrow V = (2x)(2y)(2z)$

Lagrange Multipliers

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$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1, \quad a,b,c>0$$

Let one of the corners be at the point $(x,y,z) \Rightarrow V=(2x)(2y)(2z)$

$$V=(2x)(2y)(2z)\to max, \quad g(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

$$8yz=\lambda\frac{2x}{a^2},\, 8xz=\lambda\frac{2y}{b^2},\, 8xy=\lambda\frac{2z}{c^2},\, \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

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$$8yz = \lambda\frac{2x}{a^2}, \, 8xz = \lambda\frac{2y}{b^2}, \, 8xy = \lambda\frac{2z}{c^2}, \, \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$x = \frac{a}{\sqrt{3}}, \, y = \frac{b}{\sqrt{3}}, \, z = \frac{c}{\sqrt{3}} \Rightarrow V_{max} = \frac{8abc}{3\sqrt{3}}$$

Lagrange Multipliers

Example

Consider the problem of finding the maximum volume of a rectangular box without a lid made from 12 *square meters of cardboard.*

Lagrange Multipliers

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Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and furthest from the point $(3, 1, -1)$.

Lagrange Multipliers

Example

Consider the problem of finding the maximum volume of a rectangular box without a lid made from 12 square meters of cardboard.

Example

Find the points on the sphere $x^2 + y^2 + z^2 = 4$ *that are closest to and furthest from the point* $(3, 1, -1)$. *The distance from a point* (x, y, z) *to the point* $(3, 1, -1)$ *is*

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$$

The problem is to maximize/minimize the function

$$f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$$

subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 4$.

$$\nabla f = \lambda \nabla g, g = 4 \Rightarrow 2(x-3) = 2\lambda x,\ 2(y-1) = 2\lambda y,\ 2(z+1) = 2\lambda z$$

Lagrange Multipliers

Example

$$x = \frac{3}{1-\lambda}, \; y = \frac{1}{1-\lambda} \; z = \frac{-1}{1-\lambda}$$

$$\frac{11}{(1-\lambda)^2} = 4 \Rightarrow \lambda = 1 \pm \frac{\sqrt{11}}{2}$$

$$\Rightarrow A\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right), \quad B\left(\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$$

Lagrange multipliers with two constraints

Let $f(x, y, z)$ be a differentiable function. Suppose that we wish to find the maximum or minimum value of $f(x, y, z)$ subject to two constraints $g(x, y, z) = k$ and $h(x, y, z) = c$ where $k, c \in \mathbb{R}$ are constants.

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$$\nabla f(x_0, y_0, z_0) \cdot \bar{r}'(t_0) = 0$$

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$$\nabla f(x_0, y_0, z_0) \cdot \bar{r}'(t_0) = 0$$

It follows that $\nabla f(x_0, y_0, z_0)$ lies in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$ (assuming that these vectors are not parallel and not zero). Therefore there exists $\lambda, \mu \in \mathbb{R}$ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$$

Lagrange multipliers with two constraints

Example

Consider $f(x, y, z) = x + 2y + 3z$*. Find the maximum value of* f *on the curve of intersection of the plane* $x - y + z = 1$ *and the cylinder* $x^2 + y^2 = 1$*.*

$$\nabla f = (1, 2, 3) = \lambda(1, -1, 1) + \mu(2x, 2y, 0)$$

$$\Rightarrow 2\mu x = 1 - \lambda,\ 2\mu y = 2 + \lambda,\ \lambda = 3$$

$$x = \frac{-1}{\mu},\ y = \frac{5}{2\mu} \Rightarrow \frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

$$\Rightarrow \mu = \pm\frac{\sqrt{29}}{2}$$

$$f(x_0, y_0, z_0) = 3 \pm \sqrt{29}$$