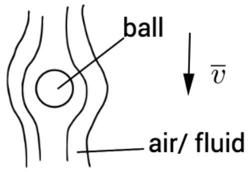
1 Motion with air/ fluid resistance

$*$ Stokes (linear) drag

$$\boxed{\overline{F}_{\text{drag}} = -k\overline{v} = -kv\left(\frac{\overline{v}}{v}\right)} \quad \rightarrow \text{small objects/ low speeds} \tag{1}$$



where $k = \sigma\pi\eta r_s = \text{const}$
η is viscosity (property of the fluid)
r_s is Stoke's radius (property of the object)
$*$ Quadratic drag

$$\boxed{\overline{F}_{\text{drag}} = -bv^2\left(\frac{\overline{v}}{v}\right)} \quad \rightarrow \text{large objects/ high speeds} \tag{2}$$

where $b = \frac{1}{2}\varrho C_d A = \text{const}$
ϱ is fluid density, C_d is drag coefficient (e.g. cars 0.25-0.5)
A is cross-sectional area perpendicular to the direction of motion

$\sim \circ \sim$

General problem and solving strategy (1D)

$$a_y = \frac{F(v_y)}{m} \qquad \rightarrow \text{Newton's } 2^{\text{nd}} \text{ law (equation of motion)}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = \frac{F(\frac{\mathrm{d}y}{\mathrm{d}t})}{m} \qquad \rightarrow \begin{array}{l} 2^{\text{nd}} \text{ order ODE} \\ \text{can be reduced to } 1^{\text{st}} \text{ order ODE} \\ \text{with seperable variables by } v_y = \frac{\mathrm{d}y}{\mathrm{d}t} \end{array}$$

$$\frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{F(v_y)}{m}$$

$$m\frac{\mathrm{d}v_y}{F(v_y)} = \mathrm{d}t$$

$$\int\limits_{v_{y0}}^{v_y(t)} \frac{m}{F(v_y)}\mathrm{d}v_y = \int\limits_0^t \mathrm{d}t$$

$$\boxed{\int\limits_{v_{y0}}^{v_y(t)} \frac{m}{F(v_y)}\mathrm{d}v_y = t} \quad \overset{\text{yields}}{\Longrightarrow} \quad v_y(t) = ... \quad \Rightarrow$$

acceleration $\quad a = \frac{\mathrm{d}v}{\mathrm{d}t}$

position $\quad \frac{\mathrm{d}y}{\mathrm{d}t} = v_y(t) \Rightarrow y(t) = \int\limits_0^t v_y(t)\mathrm{d}t + y_0$

Qualitative analysis (fall with air drag, no initial velocity, linear drag assumed)

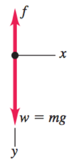
initial phase

$$
\begin{array}{ccc}
t \approx 0 & \Rightarrow & v \approx 0 \\
F_{\text{drag}} & \propto & v = 0 \\
 & \Downarrow & \\
 & a \approx g &
\end{array}
$$

In the initial phase
the particle moves as it was free-falling

final phase

speed increases $\Rightarrow$ air drag increases



net force: $\quad mg - kv_\infty = 0$

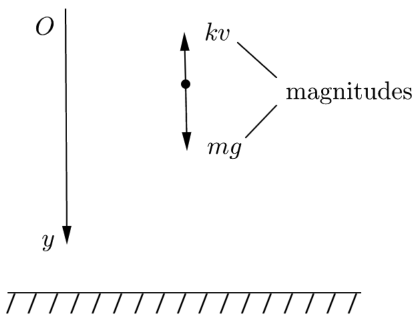
terminal speed: $v_\infty = \dfrac{mg}{k}$

$\rightarrow$ conclusion: heavier objects tend to fall fast

$\rightarrow$ in the final phase, the drag balances weight

$\Rightarrow$ particle moves with constant speed

Fall with Linear Drag



Initial conditions:
$y(0) = 0$
$v_y(0) = 0$

Newton's second law (equation of motion)

$$
\begin{aligned}
ma_y &= mg - kv_y \quad \rightarrow \text{ net force} \\
a_y &= g - \frac{k}{m} v_y
\end{aligned}
$$

But $a_y = \dfrac{\mathrm{d}v_y}{\mathrm{d}t}$

$$
\begin{aligned}
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= g - \frac{k}{m} v_y \\
\frac{\mathrm{d}v_y}{\mathrm{d}t} &= -\frac{k}{m}\left(v_y - \frac{mg}{k}\right) \\
\frac{\mathrm{d}v_y}{v_y - \frac{mg}{k}} &= -\frac{k}{m}\mathrm{d}t \\
\int_0^{v_y(t)} \frac{\mathrm{d}v_y}{v_y - \frac{mg}{k}} &= -\frac{k}{m}\int_0^t \mathrm{d}t \\
\ln\left|\frac{v_y(t) - \frac{mg}{k}}{-\frac{mg}{k}}\right| &= -\frac{k}{m} t
\end{aligned}
$$

But $v_y(t) < \dfrac{mg}{k}$ (terminal speed)

$$\ln \frac{\frac{mg}{k} - v_y(t)}{\frac{mg}{k}} = -\frac{k}{m}t$$

$$\frac{mg}{k} - v_y(t) = \frac{mg}{k} e^{-\frac{k}{m}t}$$

$$\boxed{v_y(t) = \frac{mg}{k}(1 - e^{-\frac{k}{m}t})} \tag{3}$$

Analysis

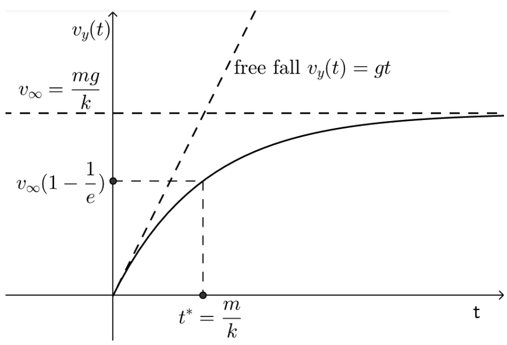
1° Short times, i.e. , $t \ll \dfrac{m}{k}$, then $\dfrac{k}{m}t \ll 1$, can approximate $exp(...)$ with Taylor (Maclaurin) polynomial $e^n = 1 + \dfrac{n}{1!}$

$$v_y(t) = \frac{mg}{k}(1 - e^{-\frac{k}{m}t}) \approx \frac{mg}{k}(1 - 1 + \frac{k}{m}t) = gt$$

$$v_y(t) \approx gt \qquad \text{constant acceleration}$$

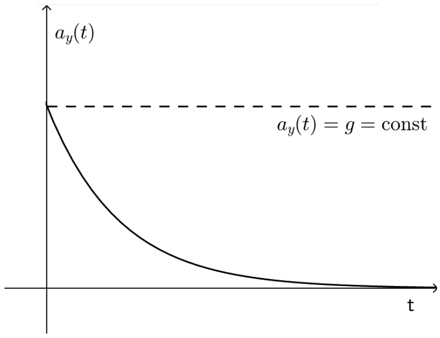
2° Long times, i.e. $t \to \infty$

$$v_\infty = \lim_{t\to\infty} \frac{mg}{k}(1 - e^{-\frac{k}{m}t}) = \frac{mg}{k} = \text{ const} \quad \text{(constant velocity)}$$



Acceleration

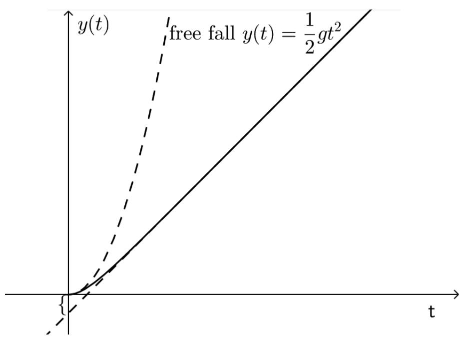
$$a_y(t) = \frac{\mathrm{d}v_y(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{mg}{k}\Big(1 - e^{-\frac{k}{m}t}\Big)\Big] \tag{4}$$



Position

$$
\begin{aligned}
v_y(t) &= \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \\
\int_0^{y(t)} \mathrm{d}y &= \frac{mg}{k}\int_0^t \left(1 - e^{-\frac{k}{m}t}\right)\mathrm{d}t \\
y(t) &= \frac{mg}{k}\left(t - \left(-\frac{m}{k}\right)e^{-\frac{k}{m}t}\right)\Big|_0^t = \frac{mg}{k}\left(t + \frac{m}{k}e^{-\frac{k}{m}t} - \frac{m}{k}\right)
\end{aligned}
$$

$$
\Rightarrow \quad \boxed{y(t) = \frac{mg}{k}\left(t + \frac{m}{k}\left(e^{-\frac{k}{m}t} - 1\right)\right)} \tag{5}
$$



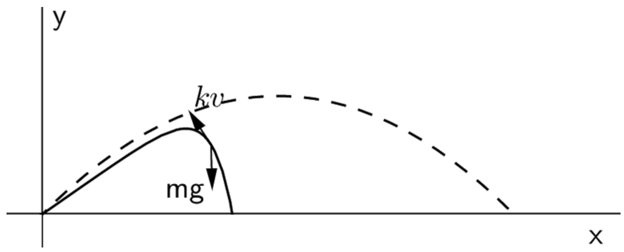
1° Short times (now add one more term in Taylor polynomial)

$$
\begin{aligned}
y(t) &\approx \frac{mg}{k}\left[t + \frac{m}{k}\left(\cancel{1} - \frac{k}{m}t + \frac{1}{2}\left(\frac{k}{m}t\right)^2 - \cancel{1}\right)\right] \\
&= \frac{mg}{k}\left[t - t + \frac{m}{k}\frac{1}{2}\left(\frac{k}{m}t\right)^2\right] = \frac{1}{2}gt^2 \\
y(t) &\approx \frac{1}{2}gt^2
\end{aligned}
$$

2° Long times ($t \gg \frac{m}{k}$)

$$
e^{-\frac{k}{m}t} \approx 0 \qquad \text{and} \qquad y(t) \approx \frac{mg}{k}\left(t - \frac{m}{k}\right)
$$

Example Projectile motion with linear air drag $\overline{F}_{\text{drag}} = -k\overline{v} = -kv_x\hat{n}_x - kv_y\hat{n}_y$



Equations of motion $\begin{cases} ma_x = -kv_x \\ ma_y = -mg - kv_y \end{cases}$ $+$ initial conditions $\begin{cases} \overline{\gamma}(0) = 0 \\ \overline{v}(0) = \overline{v}_0 \end{cases}$

solution strategy as before.

Effects of the air drag :

* reduces the maximum height
* shortens the range