Vv156 Lecture 14

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Definition

A function F is called an antiderivative of f on an interval $\mathcal{I}$ if

$$F'(x) = f(x) \qquad \text{for all} \qquad x \in \mathcal{I}$$

Q: Find an antiderivative for each of the following functions

1. $f(x) = 2x$
2. $g(x) = \cos x$
3. $h(x) = \frac{1}{x} + 2e^{2x}$

- We need to think backward here:

  What function do we know has a derivative equal to the given function?

1. $F(x) = x^2$
2. $G(x) = \sin x$
3. $H(x) = \ln|x| + e^{2x}$

Defintion

The process, by which we determine functions from their derivatives, is called antidifferentiation or integration, it reverses the operation of differentiation.

- Every differentiation formula is also an integration formula, however,

$$\frac{d}{dx} c = 0 \qquad \boxed{\frac{d}{dx} x^n = n x^{n-1}} \qquad F(x) = x^2 \Longleftarrow f(x) = 2x$$

$$\frac{d}{dx} e^x = e^x \qquad \frac{d}{dx} a^x = a^x \ln a \qquad \frac{d}{dx}\left(x^2 + 1\right) = 2x$$

$$\frac{d}{dx} \ln|x| = \frac{1}{x} \qquad \frac{d}{dx} \log_a x = \frac{1}{x \ln a} \qquad \boxed{\frac{d}{dx}\left(x^n + \boxed{c}\right) = n x^{n-1}}$$

$$\frac{d}{dx} \sin x = \cos x \qquad \frac{d}{dx} \cos x = -\sin x \qquad \frac{d}{dx} \tan x = \sec^2 x$$

$$\frac{d}{dx} \csc x = -\csc x \cot x \qquad \frac{d}{dx} \sec x = \sec x \tan x \qquad \frac{d}{dx} \cot x = -\csc^2 x$$

$$\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}} \qquad \frac{d}{dx} \tan^{-1} x = \frac{1}{1 + x^2}$$

$$\frac{d}{dx} \csc^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}} \qquad \frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}} \qquad \frac{d}{dx} \cot^{-1} x = -\frac{1}{1 + x^2}$$

- For a given function f, the antiderivative of f is not unique, there is actually a set of them for a given f.

Definition

The set of all antiderivatives of f is known as the indefinite integral of f with respect to x, denoted by

$$\int f(x)\,dx$$

The function f is known as the integrand of the integral and the variable x is known as the variable of integration.

- It is clear that functions, which differ only by an additive constant, belong to the same set of antiderivatives.

Q: Is it true that the set of all antiderivatives of a function only contains functions differed by an additive constant?

Theorem

If functions f and g are differentiable on an open interval $\mathcal{I}$, and if $f'(x) = g'(x)$ for all x in $\mathcal{I}$, then $f - g$ is constant; that is, there is a constant k such that

$$f(x) - g(x) = k \qquad \text{for all } x \in \mathcal{I}.$$

Proof

- Suppose x_1 and x_2 are any two points in the interval such that

$$x_1 < x_2$$

- We know f and g are differentiable in $\mathcal{I}$, since $[x_1, x_2]$ is a subinterval

$$f \text{ and } g \text{ are continuous on } [x_1, x_2] \text{ and differentiable on } (x_1, x_2).$$

- Moreover, the following function is also continuous and differentiable

$$\begin{aligned} h(x) &= f(x) - g(x) \\ \implies h'(x) &= f'(x) - g'(x) = 0 \quad \text{for all } x \in (x_1, x_2). \end{aligned}$$

- This implies has a stationary point at every $x \in (x_1, x_2)$,

$$h(x) = f(x) - g(x)$$

Since x_1 and x_2 are arbitrary, f and g must differ only by a single constant.

- According to the last theorem, once we have found one antiderivative F of f, other antiderivatives of f differ from F only by a constant.
- We indicate this in integral notation in the following way:

$$\int f(x)\,dx = F(x) + c$$

where c is called the constant of integration, aka an arbitrary constant.

- Indefinite integral of a function is not a function in the normal sense, it is a family of functions or a set of closely related functions.

$$\int f(x)\,dx$$

- So the indefinite integrals of

$1.\, f(x) = 2x$ $\quad$ $2.\, g(x) = \cos x$ $\quad$ $3.\, h(x) = \frac{1}{x} + 2e^{2x}$

$$\int f(x)\,dx = x^2 + c \qquad \int g(x)\,dx = \sin x + c \qquad \int h(x)\,dx = \ln|x| + e^{2x} + c$$

Properties of indefinite integrals

Suppose that $F(x)$ and $G(x)$ are antiderivatives of $f(x)$ and $g(x)$, respectively.

1. A constant factor can be moved through an integral sign; that is,

$$\int kf(x)\,dx = kF(x) + C, \qquad \text{where } k \text{ is a constant.}$$

2. An antiderivative of a sum is the sum of the antiderivatives; that is,

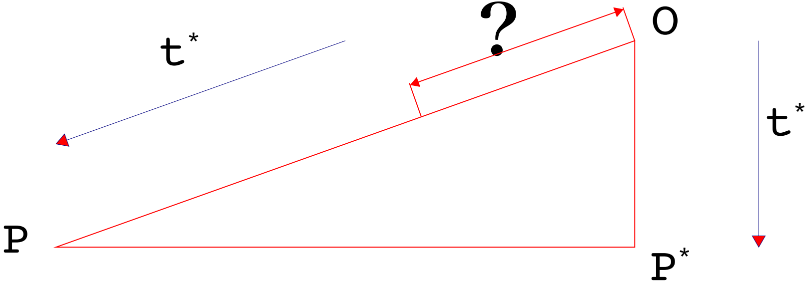
$$\int \Big[f(x) + g(x)\Big]\,dx = F(x) + G(x) + C$$

3. Combining the two properties and extended to more than two functions

$$\int \Big[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)\Big]\,dx$$
$$= c_1 \int f_1(x)\,dx + c_2 \int f_2(x)\,dx + \cdots + c_n \int f_n(x)\,dx$$

Exercise

Suppose t^* is the time required for a stationary object to free fall from a point O in the air to a point P^* on the ground. Now suppose the object, instead of free falling from O, slides down an inclined plane OP starting from rest at O. Find the position of the object at time t^*. Assume there is no friction and air resistance.



Definition

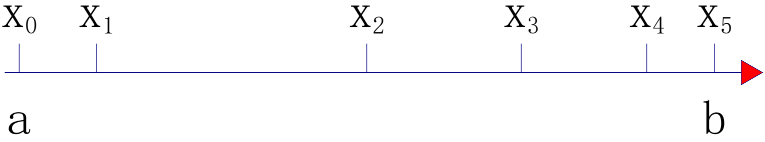
A partition P of $[a, b]$ is a collect of distinct points such that

$$P = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$$

which divides the interval into subintervals

$$[x_{i-1}, x_i]$$

- For example,

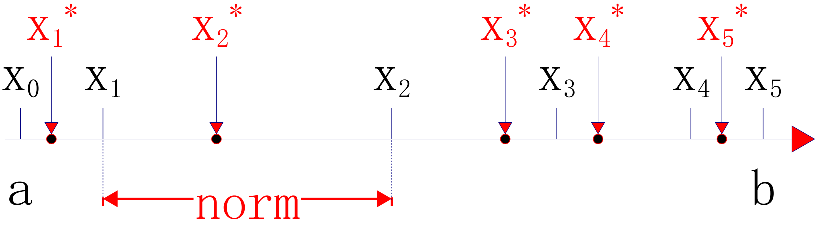


Definition

A tagged partition $P(x, x^*)$ of an interval $[a, b]$ is a partition together with a collect of numbers x^*, which is known as sample points

$$\{[x_0 = a, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n = b]\}$$

$$x_1^*, x_2^*, x_3^*, \cdots, x_n^* \qquad x_i^* \in [x_{i-1}, x_i]$$



Definition

The norm of a partition is defined to be the length of the longest subinterval,

$$\|P\| = \max \Delta x_i = \max (x_i - x_{i-1}), \quad i = 1, \ldots, n.$$

Definition

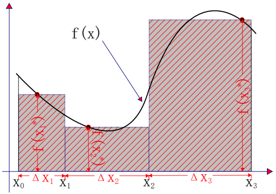
The Riemann sum of f over $[a,b]$ with $P(x,x^*)$ is defined to be

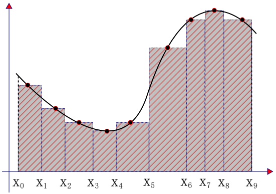
$$S = \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

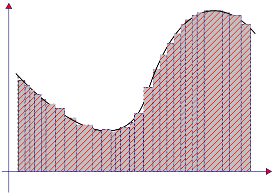
Definition

If the function f is continuous on $[a,b]$ and if $f(x) \geq 0$ for all x in $[a,b]$, then the area A under the curve $y = f(x)$ over the interval $[a,b]$ is defined by

$$A = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$







Definition

A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

exists and does not depend on the choice of partitions or on the choice of the sample points x_i^*. When this is the case we denote the limit by

$$\int_a^b f(x)\, dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*) \Delta x_i$$

which is called the definite integral, or the Riemann integral, of f from a to b.

- The numbers a and b are called the lower limit of integration and the upper limit of integration, respectively, and $f(x)$ is called the integrand.

- The following states that the common functions are in fact integrable.

Theorem

If f is continuous on $[a, b]$, or if f has finitely many discontinuities but is bounded on $[a, b]$, then f is integrable on $[a, b]$; that is

$$\int_a^b f(x)\,dx \text{ exists.}$$

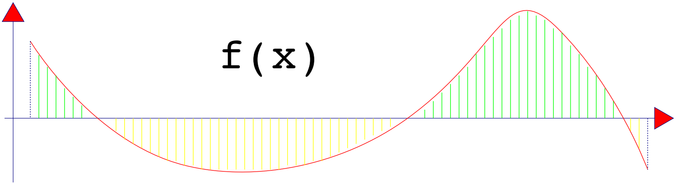
Q: Which the following functions are integrable on $[-3, 3]$?

A. $f(x) = x^2,$

B. $g(x) = \begin{cases} x & -3 \le x \le 2 \\ x^2 & 2 < x \le 3 \end{cases},$

C. $h(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$

- If f takes on both positive and negative values, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis



- A definite integral thus can be interpreted as a net area, a difference of areas

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, while A_2 is the area of the region below it.

Properties of definite integrals

If $f(x)$ and $g(x)$ are integrable on $[a, b]$.

1. A constant factor can be moved through an integral sign; that is,

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx, \qquad \text{where } k \text{ is a constant.}$$

2. An integral of a sum is the sum of the integrals; that is,

$$\int_a^b \Big[f(x) + g(x)\Big]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

3. Combining the two properties and extended to more than two functions

$$\int_a^b \Big[c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x)\Big]\,dx$$
$$= c_1\int_a^b f_1(x)\,dx + c_2\int_a^b f_2(x)\,dx + \cdots + c_n\int_a^b f_n(x)\,dx$$

- Some properties can be understood by interpreting the integral as an area.

Properties of definite integrals

4. If f is integrable on a closed interval containing the three points a, b, and c, then

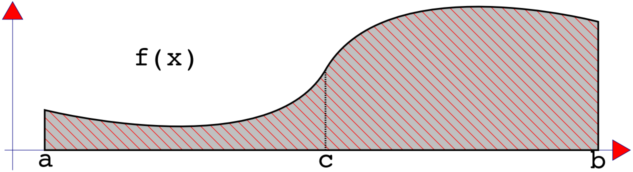
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

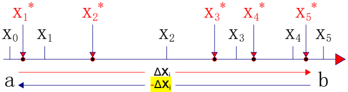
5. If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x)\,dx \geq 0$$

6. If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$,

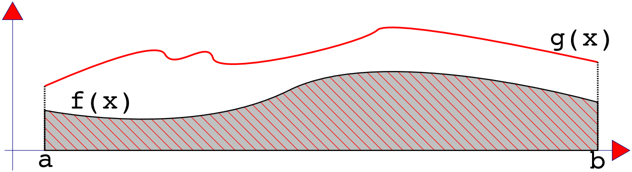
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$





$$\int_{b}^{a} f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(x_i^*)(-\Delta x_i)$$

$$= -\int_a^b f(x)\,dx$$



Q: Is it true that, if $m \le f(x) \le M$ for $a \le x \le b$, then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

Exercise

(a) Use the definition of area

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Find the area between $f(x) = x^2$ and x-axis over the interval $[0,1]$.

(b) Use the definition of area

$$A = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

Find the area between $f(x) = \sqrt{x}$ and x-axis for $0 \le x \le 1$.