vv255: Lines and planes. Vector functions.

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Today 05-20-2019

1. Review: cross sections of surfaces, the cross product and the triple product.
2. Lines and planes in 3D. Normal vectors.
3. The distance between planes.
4. Vector functions: definition, limit, derivatives and integrals.

Cross product

Definition

Algebraic Definition: *Let* $\bar{a} = x_0\bar{i} + y_0\bar{j} + z_0\bar{k}$ *and* $\bar{b} = x_1\bar{i} + y_1\bar{j} + z_1\bar{k}$.

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \end{vmatrix}$$

Definition

Geometric Definition: $\bar{a} \times \bar{b} = S_{parall\ \bar{a},\bar{b}}\bar{n}$, *where* $\bar{n}$ *is a unit vector perpendicular to the parallelogram with direction given by the right hand rule.*

Definition

The cross product of vectors $\bar{a}$ *and* $\bar{b}$ *is a vector* $\bar{c} = \bar{a} \times \bar{b}$ *s.t.*

1. $\bar{c} \perp \bar{a}$, $\bar{c} \perp \bar{b}$
2. *The triple* $\bar{a}$, $\bar{b}$, $\bar{c}$ *is right hand oriented*

Exercises

1. Find $\bar{u} \cdot \bar{v}$, where $\bar{u} = 4\bar{i} - 6\bar{k}$ and $\bar{v} = -\bar{i} + \bar{j} + \bar{k}$.
2. Find $\bar{u} \cdot \bar{v}$ where $\bar{u} = 3\bar{i} + \bar{j} - \bar{k}$ is a vector of length 2 oriented at an angle of $\pi/4$ away from the direction of $\bar{u}$.
3. Using the geometric definition, what is $\bar{i} \times \bar{j}$ and $\bar{j} \times \bar{i}$?.
4. For $\bar{v} = 3\bar{i} - 2\bar{j} + 4\bar{k}$, $\bar{w} = \bar{i} + 2\bar{j} - \bar{k}$, find $\bar{v} \times \bar{w}$ using the algebraic and geometric definitions.

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Check your results in Matlab:

Command Window

```
>> vecu = [4,0,-6]; vecv = [-1,1,1];
dot(vecu,vecv) %exercise 1
vecu = [3,1,-1];
norm(vecu)*2*cos(pi/4) %exercise 2
cross([1,0,0],[0,1,0]) %exercise 3
cross([0,1,0],[1,0,0])
vecv = [3,-2,4]; vecw = [1,2,-1];
cross(vecv,vecw) %exercise 4
```

Properties of the cross product

Theorem

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be 3D vectors, and let d be a scalar. Then

1. $\bar{a} \times \bar{b} = -\bar{b} \times \bar{a}$
2. $(d\bar{a}) \times \bar{b} = d(\bar{a} \times \bar{b}) = \bar{a} \times (d\bar{b})$
3. $\bar{a} \times (\bar{b} + \bar{c}) = \bar{a} \times \bar{b} + \bar{a} \times \bar{c}$
4. $(\bar{b} + \bar{c}) \times \bar{a} = \bar{b} \times \bar{a} + \bar{c} \times \bar{a}$
5. $\bar{a} \cdot (\bar{b} \times \bar{c}) = (\bar{a} \times \bar{b}) \cdot \bar{c}$
6. $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \cdot \bar{c})\bar{b} - (\bar{a} \cdot \bar{b})\bar{c}$

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Note that the cross product is NOT associative. I.e. There exists 3D vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ such that

$$\bar{a} \times (\bar{b} \times \bar{c}) \neq (\bar{a} \times \bar{b}) \times \bar{c}$$

Applications of the cross product

Example

Consider the points $P(1,3,2)$*,* $Q(3,-1,6)$ *and* $R(5,2,0)$*. The cross product*

$$\overrightarrow{PQ} \times \overrightarrow{PR}$$

is perpendicular to the plane that passes through P*,* Q *and* R*. The value*

$$|\overrightarrow{PQ} \times \overrightarrow{PR}|$$

is the area of the parallelogram with adjacent sides $\overline{PQ}$ *and* $\overline{PR}$*. Therefore the area of the triangle* $\triangle PQR$ *is*

$$\frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}|(2,-4,4) \times (4,-1,-2)| = \frac{\sqrt{12^2 + 20^2 + 18^2}}{2}$$

Vector triple product

Definition

Let $\bar{a}$, $\bar{b}$ and $\bar{c}$ be 3D vectors. The scalar triple product of $\bar{a}$, $\bar{b}$ and $\bar{c}$ is the value

$$\bar{a} \cdot (\bar{b} \times \bar{c})$$

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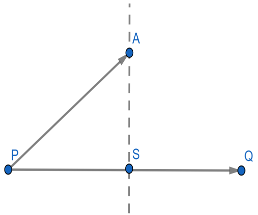
The value $|\bar{a} \cdot (\bar{b} \times \bar{c})|$ is the volume of the parallelepiped determined by the vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$.

Exercise: Find the volume of the parallelepiped with sides parallel to $\bar{u} = (3, 4, 5)$, $\bar{v} = (5, 4, 3)$, $\bar{w} = (1, 1, 0)$

Examples from Physics

- The work done by the force that moves the object from P to Q pointing in the direction of the vector $\overline{PA}$ is the product of the component of the force along the displacement vector $\overline{PQ}$ and the distance moved:

$$W = \left(|\overline{PA}|\cos\left(\overline{PQ},\ \overline{PA}\right)\right)|\overline{PQ}| = \overline{PA}\cdot\overline{PQ}$$



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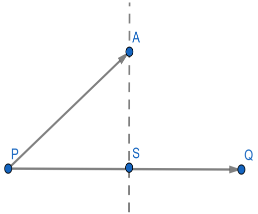


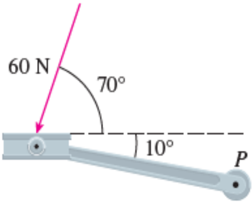
- Exercise: Let $\bar{v} = 3\bar{i} + 4\bar{j}$ and $\bar{F} = 4\bar{i} + \bar{j}$. Find the component of the force vector $\bar{F}$ parallel to $\bar{v}$:
  a. Find the unit vector $\hat{v}$.
  b. Find $\bar{F} \cdot \hat{v}$ the length of the component of $\bar{F}$ parallel to $\bar{v}$.
  c. Construct the vector $\bar{F}_{parallel}$.

Examples from Physics

- Consider a force F acting on a rigid body at a point given by a position vector r. The torque $\bar{\tau}$ measures the tendency of the body to rotate about the origin. It is defined as the cross product of the position and force vectors

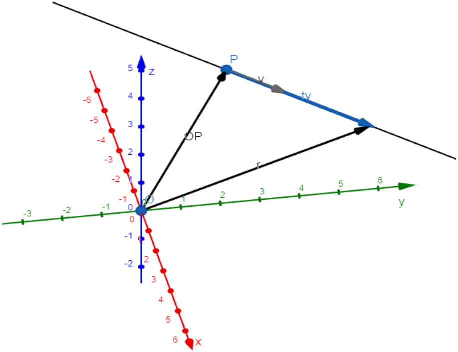
$$\bar{\tau} = \bar{r} \times \bar{F}$$

  The direction of the torque vector indicates the axis of rotation.
- **Example (Stewart):** A bicycle pedal is pushed by a foot with a 60-N force as shown. The shaft of the pedal is 18 cm long. Find the magnitude of the torque about P.



Lines

Let L be a line in 3D space. Let P be a point on L and let $\bar{v}$ be a vector that is parallel to L.



For all $t \in \mathbb{R}$,

$$\bar{r}(t) = \overrightarrow{OP} + t\bar{v} \tag{1}$$

is a vector that points from the origin (O) to a point on L. Equation (1) is called the vector equation of L.

Lines

Therefore if $P(x_0, y_0, z_0)$ and $\bar{v} = a\bar{i} + b\bar{j} + c\bar{k}$, then for all $t \in \mathbb{R}$, the point $Q(x, y, z)$ where

$$x = x_0 + ta \qquad y = y_0 + tb \qquad z = z_0 + tc \tag{2}$$

lies on L. (2) are called the parametric equations of L.

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$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c} \tag{3}$$

These are called the **symmetric equations** of L.

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Definition

We say two lines L_1 and L_2 is 3D space are skew if L_1 and L_2 are not parallel and don't intersect.

Planes

We want to find the equation of a plane perpendicular to the vector $\bar{n} = \bar{i} + \bar{j} - \bar{k}$ and passing through the point $(0, 0, -1)$.

- We are looking for points (x, y, z) that sit in the plane. Create a displacement vector, $\bar{v}$ between a point (x, y, z) and the point $(0, 0, -1)$.

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- We are looking for points (x, y, z) that sit in the plane. Create a displacement vector, $\bar{v}$ between a point (x, y, z) and the point $(0, 0, -1)$.
- We want this displacement vector to be perpendicular to $\bar{n}$, so we want $\bar{v} \cdot \bar{n} = 0$: Plug your displacement vector and the information for $\bar{n}$ into this dot product. Expand and simplify.

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- You have found an equation for a plane. Show that it passes through $(0, 0, -1)$.

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- We want this displacement vector to be perpendicular to $\bar{n}$, so we want $\bar{v} \cdot \bar{n} = 0$: Plug your displacement vector and the information for $\bar{n}$ into this dot product. Expand and simplify. You should get $z = x + y - 1$ for the displacement vector to be perpendicular to $\bar{n}$.
- You have found an equation for a plane. Show that it passes through $(0, 0, -1)$.
- Is any vector parallel to this plane perpendicular to $\bar{n}$? Choose two points on the plane and convince yourself that the vector between those points is perpendicular to $\bar{n}$. This can be shown to hold in general, but just choose enough pairs of points to convince yourself.

Planes: now we are to generalize the previous example

A plane $\mathcal{P}$ in $\mathbb{R}^3$ is completely determined by a point P that lies on the plane and a vector $\bar{n}$, called a/the normal vector, that points in a direction which is perpendicular to $\mathcal{P}$.

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$$\bar{n} \cdot (\bar{r} - \overrightarrow{OP}) = 0 \tag{4}$$

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(4) is called the vector equation of $\mathcal{P}$. If $P(x_0, y_0, z_0)$ and $\bar{n} = a\bar{i} + b\bar{j} + c\bar{k}$, then this yields

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \tag{5}$$

which is called the scalar equation of $\mathcal{P}$.

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which is called the scalar equation of $\mathcal{P}$. Therefore a plane $\mathcal{P}$ with normal vector $\bar{n} = a\bar{i} + b\bar{j} + c\bar{k}$ is described by the equation

$$ax + by + cz = d$$

Today 05-22-2019

1. Review: lines and planes in 3D, distance between planes.
2. Vector functions: definition, limit, derivatives and integrals.
3. Arc length and curvature.

Exercises

1. Let points $(0, 1, 2), (2, -1, 3)$ and $(0, 0, 1)$ form a triangle that lies in a plane.

a. Find a normal vector to the plane and construct an equation for the plane.

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$$\bar{n} = (3/2, -1, 1), \quad -3x - 2y + 2z - 2 = 0$$

b. Find the area of the triangle.

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$$E(0, -1, 2)$$

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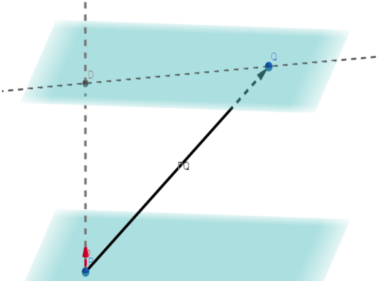
If $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel planes, then the normal vector, $\bar{n}$, of either of these planes describes the direction of the shortest path between $\mathcal{P}_1$ and $\mathcal{P}_2$. Therefore, if P lies on $\mathcal{P}_1$ and Q lies on $\mathcal{P}_2$, then the shortest distance between $\mathcal{P}_1$ and $\mathcal{P}_2$ is given by

$$D = |\mathrm{comp}_{\bar{n}}(\overrightarrow{PQ})| = \frac{|\overrightarrow{PQ} \cdot \bar{n}|}{|\bar{n}|}$$



Planes

Similarly, if $\mathcal{P}$ is a plane with normal vector $\bar{n}$, P is a point on $\mathcal{P}$ and Q is a point that does not lie on $\mathcal{P}$, then the shortest distance between $\mathcal{P}$ and Q is given by

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Example

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Example

Let $\mathcal{P}: 5x + y - z + 10 = 0$ *and* $Q(1, 2, 3)$.

$$D(Q, \mathcal{P}) = \frac{5 + 2 - 3 + 10}{\sqrt{25 + 1 + 1}} = 2.6943$$

Vector functions

Definition

A vector-valued function or vector function is a function whose domain is a subset of the reals and range is a set of vectors, i.e we say that $\bar{r}$ is a vector function if $\bar{r} : A \longrightarrow \mathbb{R}^3$ where $A \subseteq \mathbb{R}$.

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By interpreting vectors as arrows that point from the origin to a point in $\mathbb{R}^3$, we can interpret vector functions as describing a curve in $\mathbb{R}^3$. That is, if $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$, then $\bar{r}(t)$ describes the curve in $\mathbb{R}^3$ with parametric equations

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

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Definition

A vector-valued function or vector function is a function whose domain is a subset of the reals and range is a set of vectors, i.e we say that $\bar{r}$ is a vector function if $\bar{r} : A \longrightarrow \mathbb{R}^3$ where $A \subseteq \mathbb{R}$.

By interpreting vectors as arrows that point from the origin to a point in $\mathbb{R}^3$, we can interpret vector functions as describing a curve in $\mathbb{R}^3$. That is, if $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$, then $\bar{r}(t)$ describes the curve in $\mathbb{R}^3$ with parametric equations

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

Example

We have already seen how to compute vector-valued functions that describe lines in $\mathbb{R}^3$.

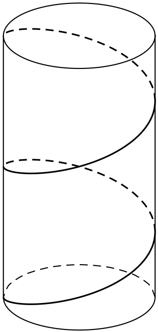
Vector functions

Example

The vector function

$$\bar{r}(t) = \cos(t)\bar{i} + \sin(t)\bar{j} + t\bar{k}$$

describes a spiral around the surface of an infinitely long cylinder of radius 1 *centred around the z-axis. This curve is called a helix.*



Vector functions

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Let $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ *and let* $a \in \mathbb{R}$*. If the limits* $\lim_{t \to a} f(t)$, $\lim_{t \to a} g(t)$ *and* $\lim_{t \to a} h(t)$ *exist, then* $\lim_{t \to a} \bar{r}(t)$ *exists and*

$$\lim_{t \to a} \bar{r}(t) = \left(\lim_{t \to a} f(t)\right)\bar{i} + \left(\lim_{t \to a} g(t)\right)\bar{j} + \left(\lim_{t \to a} h(t)\right)\bar{k}$$

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Definition

Let $A \subseteq \mathbb{R}$. *A vector function* $\bar{r} : A \longrightarrow \mathbb{R}^3$ *is continuous at a point* $a \in \mathbb{R}$ *if* $a \in A$ *and*

$$\lim_{t\to a} \bar{r}(t) = \bar{r}(a)$$

We say that $\mathbf{r} : A \longrightarrow \mathbb{R}^3$ *is continuous on an interval* I *if* $\bar{r}$ *is continuous at all points* $a \in I$.

Vector functions

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$$\lim_{t\to a} \bar{r}(t) = \left(\lim_{t\to a} f(t)\right)\bar{i} + \left(\lim_{t\to a} g(t)\right)\bar{j} + \left(\lim_{t\to a} h(t)\right)\bar{k}$$

Definition

Let $A \subseteq \mathbb{R}$. A vector function $\bar{r} : A \longrightarrow \mathbb{R}^3$ is continuous at a point $a \in \mathbb{R}$ if $a \in A$ and

$$\lim_{t\to a} \bar{r}(t) = \bar{r}(a)$$

We say that $\mathbf{r} : A \longrightarrow \mathbb{R}^3$ is continuous on an interval I if $\bar{r}$ is continuous at all points $a \in I$.

Therefore a vector function $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ is continuous at a if and only if $a \in \mathrm{dom}(\bar{r})$, and f, g, h are each continuous at a.

Vector functions

Definition

Let $A \subseteq \mathbb{R}$ and $\bar{r} : A \longrightarrow \mathbb{R}^3$. Let $t \in A$. If the limit

$$\bar{r}'(t) = \lim_{h \to 0} \frac{\bar{r}(t+h) - \bar{r}(t)}{h}$$

exists, then we say that $\bar{r}$ is differentiable at t and call $\bar{r}'(t)$ the derivative of $\bar{r}$ at t. Using Leibniz's notation the function corresponding to the derivative of $\bar{r}$ will also sometimes be denoted $\frac{d\bar{r}}{dt}$.

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Theorem

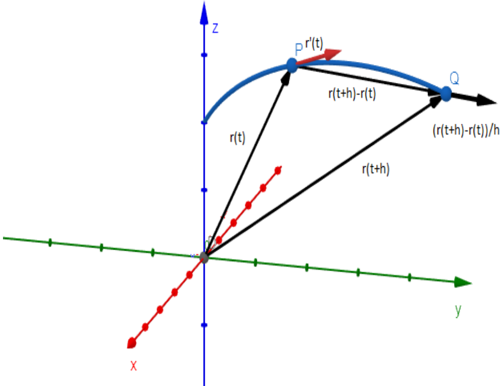
If $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ where f, g and h are functions that are differentiable on an interval I, then $\bar{r}$ is differentiable at every point in I and

$$\bar{r}'(t) = f'(t)\bar{i} + g'(t)\bar{j} + h'(t)\bar{k}$$

Vector functions

Let $\bar{r}$ be a vector function and let $t \in \mathrm{dom}(\bar{r})$. Let P be the point described by the vector $\bar{r}(t)$.

If $\bar{r}'(t)$ exists and $\bar{r}'(t) \neq 0$, then $\bar{r}'(t)$ is called the tangent vector to the curve defined by $\bar{r}$ at the point P. The tangent line to the curve described by $\bar{r}$ at the point P is the line that is parallel to the vector $\bar{r}'(t)$.



Vector functions

The unit tangent vector, sometimes denoted $\bar{T}(t)$, is the unit vector of $\bar{r}'(t)$.

$$T(t) = \frac{\bar{r}'(t)}{|\bar{r}'(t)|}$$

Example

The derivative of the vector function $\bar{r} = (e^{-t}\cos t,\ e^{-t}\sin t, e^{-t})$ *is*

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The tangent line through the point $(1, 0, 1)$ *parallel to the vector* $(-1, 1, -1)$ *is* $x = 1 - t, \quad y = t, \quad , z = 1 - t$

Vector functions

Theorem

Let $\bar{u}$ and $\bar{v}$ be differentiable vector functions, let $c \in \mathbb{R}$ and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Then

1. $\frac{d}{dt}\left[\bar{u}(t) + \bar{v}(t)\right] = \bar{u}'(t) + \bar{v}'(t)$
2. $\frac{d}{dt}\left[c\bar{u}(t)\right] = c\bar{u}'(t)$
3. $\frac{d}{dt}\left[f(t)\bar{u}(t)\right] = f'(t)\bar{u}(t) + f(t)\bar{u}'(t)$
4. $\frac{d}{dt}\left[\bar{u}(t) \cdot \bar{v}(t)\right] = \bar{u}'(t) \cdot \bar{v}(t) + \bar{u}(t) \cdot \bar{v}'(t)$
5. $\frac{d}{dt}\left[\bar{u}(t) \times \bar{v}(t)\right] = \bar{u}'(t) \times \bar{v}(t) + \bar{u}(t) \times \bar{v}'(t)$
6. *(Chain rule)* $\frac{d}{dt}\left[\bar{u}(f(t))\right] = f'(t)\bar{u}'(f(t))$

Vector functions

Theorem

Let $\overline{r}(t)$ be a vector function that is differentiable on an interval I. If for all $t \in I$, $|\overline{r}(t)|$ is constant, then for all $t \in I$, $\overline{r}(t)$ and $\overline{r}'(t)$ are perpendicular.

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Proof.

Suppose that for all $t \in I$, $|\overline{r}(t)| = c$.

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Therefore $\bar{r}'(t) \cdot \bar{r}(t) = 0$ and

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Geometrically this says that if a curve lies on the surface of a sphere, then the position vector of the curve is perpendicular to the tangent vector.

Vector functions

We have just seen that if a vector function $\bar{r}$ is defined by differentiable functions f, g and h in each of its coordinates, then the derivative of $\bar{r}$ is vector function defined by the coordinate functions f', g' and h'. This leads us to define the integral of a vector $\bar{r}$ as the vector function that is obtained by integrating the functions that define $\bar{r}$ in each of its coordinates.

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Definition

Let $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ where f, g and h are functions that are integrable on $[a, b]$.

$$\int_a^b \bar{r}(t)\ dt = \left(\int_a^b f(t)\ dt\right)\bar{i} + \left(\int_a^b g(t)\ dt\right)\bar{j} + \left(\int_a^b h(t)\ dt\right)\bar{k}$$

$$\int \bar{r}(t)\ dt = \left(\int f(t)\ dt\right)\bar{i} + \left(\int g(t)\ dt\right)\bar{j} + \left(\int h(t)\ dt\right)\bar{k}$$

Vector functions

Example

$$\int_0^1 \left(\frac{4}{1+t^2}\bar{j} + \frac{2t}{1+t^2}\bar{k} \right) dt = [4\tan^{-1} t\bar{j} + \ln(1+t^2)\bar{k}]_0^1$$

$$= 4\tan^{-1} 1\bar{j} + \ln 2\bar{k} - [4\tan^{-1} 0\bar{j} + \ln 1\bar{k}] = \pi\bar{j} + \ln 2\bar{k}$$

Arc length

In Calculus II you discussed the arc length of curves in 2D space. Now, consider a curve $\mathcal{C}$ defined parametrically on the interval $[a, b]$ by

$$x = f(t) \qquad y = g(t) \qquad z = h(t)$$

The same reasoning that you discussed in Calculus II can be used to show that the arc length of $\mathcal{C}$, L, is given by

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Noting that the curve $\mathcal{C}$ is described by the vector function $\overline{r}(t) = f(t)\overline{i} + g(t)\overline{j} + h(t)\overline{k}$ yields

$$L = \int_a^b |\overline{r}'(t)|\; dt$$

Arc length

Definition

Let $\mathcal{C}$ be the curve described by the vector function $\bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k}$ on $[a, b]$. The arc length function of $\mathcal{C}$, denoted $s(t)$, is defined on $[a, b]$ by

$$s(t) = \int_a^t |\bar{r}'(u)| \, du$$

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- The function $s(t)$ describes the distance along $\mathcal{C}$ that the point described by $\bar{r}(t)$ is away from the point described by $\bar{r}(a)$
- The Second Fundamental Theorem of Calculus tell us that

$$\frac{ds}{dt} = |\bar{r}'(t)|$$

Arc length

- If $s(t)$ is invertible, then then the vector function

$$\bar{r}(t) = f(s^{-1}(t))\bar{i} + g(s^{-1}(t))\bar{j} + h(s^{-1}(t))\bar{k}$$

describes the curve $\mathcal{C}$ on $[0, L]$ where L is the arc length of $\mathcal{C}$.

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Example

Reparametrize the helix $\bar{r}(t) = \cos t\bar{i} + \sin t\bar{j} + t\bar{k}$ *with respect to arc length measured from* $(1, 0, 0)$ *in the direction of increasing* t*.*

$(1, 0, 0) \Rightarrow\ t = 0$

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$$\Rightarrow t = \frac{s}{\sqrt{2}} \Rightarrow$$

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Today 5/24/2019

1. Review: vector functions, arc length, curvature.
2. Motion in space.
3. Functions of several variables.
4. The Euclidean space.

Vector Functions

Match the equations of the curves defined by the vector functions with the graphs.

$$A.\, \bar{r} = (\sin t,\, \cos t,\, \sin 2t),\; B.\, \bar{r} = (1/(t^2+2),\, \cos t,\, \sin t),$$

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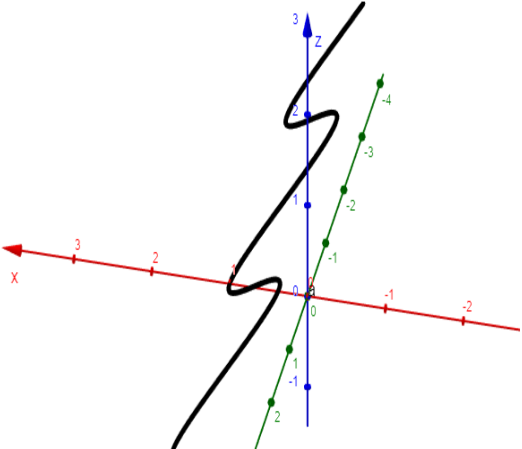

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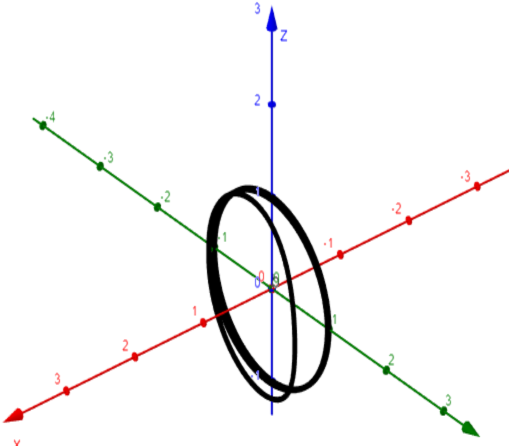

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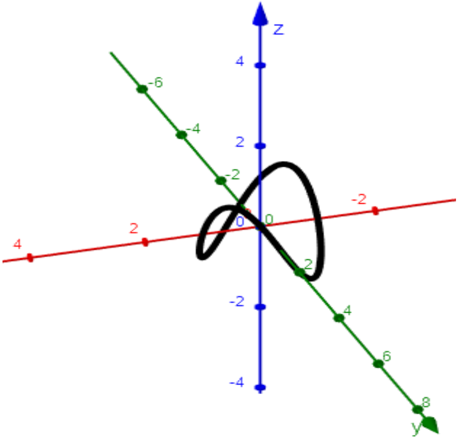

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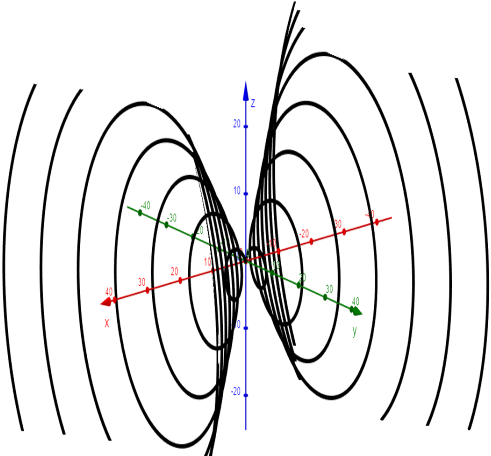

Vector Functions: Exercise

Let a vector function

$$\bar{r}(t) = \Big(3t - \cos t - 6,\, \sin^2 t - 3t,\, (\cos^2 t + \cos t + 1)/2\Big)$$

describes a curve that lies in a plane. Find the equation of the plane.

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We need to know either

- a point on the plane and a normal vector OR
- three points on the plane.

Arc Length

- Let $\mathcal{C}: \bar{r}(t) = f(t)\bar{i} + g(t)\bar{j} + h(t)\bar{k},\ t \in [a, b]$. The arc length function of $\mathcal{C}$ is
$$s(t) = \int_a^t |\bar{r}'(u)|\ du$$
- The function $s(t)$ describes the distance along $\mathcal{C}$ that the point described by $\bar{r}(t)$ is away from the point described by $\bar{r}(a)$

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- The Second Fundamental Theorem of Calculus tell us that

$$\frac{ds}{dt} = |\bar{r}'(t)|$$

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The curve approaches but does not include the point $(-1,0)$, *since* $\cos s = -1$ *for* $s = \pi + 2\pi k$ *but then* $t = \frac{s}{\sqrt{2}}$ *is undefined.*

Curvature

Definition

Let $A \subseteq \mathbb{R}$. We say that a vector function $\bar{r} : A \longrightarrow \mathbb{R}^3$ is ***smooth*** *on an interval $I \subseteq A$ if $\bar{r}'$ is continuous on I and for all $t \in I$, $\bar{r}'(t) \neq 0$.*
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Let $\bar{r} : A \longrightarrow \mathbb{R}^3$ be a vector function that is smooth on the interval I. The curvature of the curve $\mathcal{C}$ described by $\bar{r}$ is the function defined by

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The function κ measures the rate at which the direction of the vector function $\bar{r}$ is changing

Curvature

The chain rule tells us that

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Theorem

Let $\bar{r} : A \longrightarrow \mathbb{R}^3$ be a vector function that is smooth on the interval I and such that $\bar{r}'$ is differentiable on I. Then for all $t \in I$,

$$\kappa(t) = \frac{|\bar{r}'(t) \times \bar{r}''(t)|}{|\bar{r}'(t)|^3}$$

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Example

Find the curvature of the curve $\overline{r}(t) = t\overline{i} + t^2\overline{j} + e^t\overline{k}$.

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$$\bar{r}'(1) \times \bar{r}''(1) = (2, 0, -4) \Rightarrow \quad |\bar{r}'(1) \times \bar{r}''(1)| = 2\sqrt{5}$$

Curvature

Example

Find the curvature of the curve $\bar{r}(t) = (t^2, \ln t, t \ln t)$ *at the point* $(1, 0, 0)$.

$$\bar{r}'(t) = (2t, 1/t, 1+\ln t), \quad \bar{r}''(t) = (2, -1/t^2, 1/t) \Rightarrow at\,(1,0,0)\ t = 1$$

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$$\kappa(1) = \frac{|\bar{r}'(1) \times \bar{r}''(1)|}{|\bar{r}'(1)|^3} = \frac{2\sqrt{5}}{6\sqrt{6}}$$

The TBN Frame

Definition

The plane orthogonal to the (unit) tangent vector $\bar{T}$ of the curve at the point P is called the normal plane of the curve at the point P.

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Q: How can one define the normal plane?
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A: The plane must contain the tangent vector $\bar{T}$ at the point P and the unit vector $\bar{N}(t) = \frac{\bar{T}'(t)}{|\bar{T}'(t)|}$ which indicates the direction in which the curve is turning at the point P.

The TBN Frame

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$\bar{N}(t) = \frac{\bar{T}'(t)}{|\bar{T}'(t)|}$ *which indicates the direction in which the curve is turning at the point* P.

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The set of vectors $\bar{T}$, $\bar{N}$, $\bar{B}$ *which start at various points of the curve is called the* $\bar{T}\bar{N}\bar{B}$ *frame.*

The TBN Frame

Example

The principal vector and the binormal vector of the helix $\bar{r}(t) = \cos t\bar{i} + \sin t\bar{j} + t\bar{k}$ *are*

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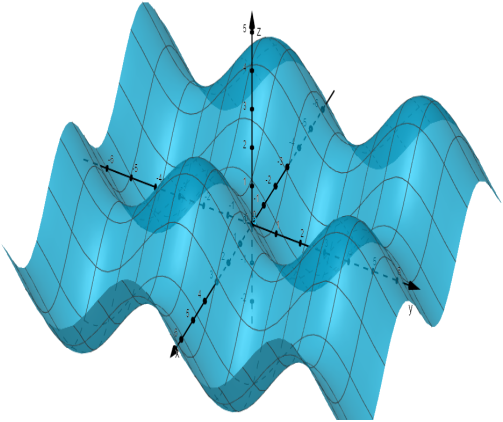
2. Exercise: Consider the same problem for a particle moving alone the curve $\bar{r}(t) = t\bar{i} + 2\cos t\bar{j} + \sin t\bar{k},\ t = 0.$

Next

Functions of several variables



Functions of several variables

Definition

Let $n > 1$ be a natural number. A real-valued function of n independent variables or just a function of n variables is a function $f : D \longrightarrow \mathbb{R}$ such that $D \subseteq \mathbb{R}^n$. We will systematically abuse notation and write $f(x_1, \ldots, x_n)$ for the value that f takes on $(x_1, \ldots, x_n) \in D$.

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So, a real-valued function with $n > 1$ independent variables is a function that maps points in n-dimensional space to real numbers. In particular, a function of two variables is a function that maps points 2D space to real numbers. This means that a function of two variables, $f : D \longrightarrow \mathbb{R}$ where $D \subseteq \mathbb{R}^2$, can be visualised is 3D space by:

$$z = f(x, y)$$

This means that functions of two variables often describe surfaces in $\mathbb{R}^3$.

Functions of several variables

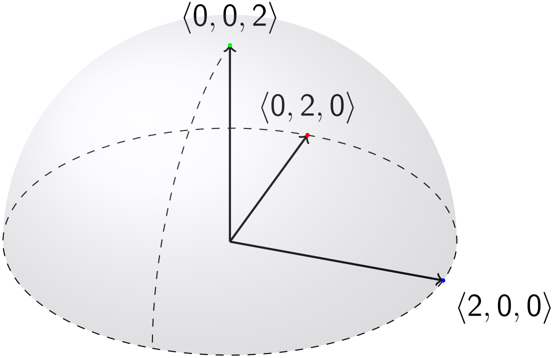
Example

The function $f(x, y) = \sqrt{4 - x^2 - y^2}$ *with domain*

Functions of several variables

Example

The function $f(x,y) = \sqrt{4 - x^2 - y^2}$ *with domain* $D = \{(x,y) \mid x^2 + y^2 \leq 4\}$ *describes a hemisphere centred at* $(0,0,0)$ *of radius* 2*:*



Functions of several variables

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Let $f : D \longrightarrow \mathbb{R}$ be a function of n variables where $n \geq 1$. The graph of f is collection of points in $\mathbb{R}^{n+1}$ defined by

$$\{(x_1, \ldots, x_n, y) \mid y = f(x_1, \ldots, x_n)\}$$

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Let $f : D \longrightarrow \mathbb{R}$ be a function of n variables where $n \geq 1$ with independent variables $x_1, \ldots, x_n$. The function f is linear if there exists $a_0, \ldots a_n \in \mathbb{R}$ such that

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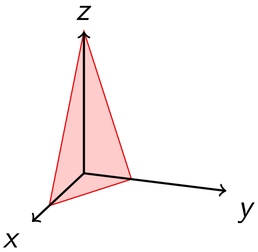
Linear functions of two variables specify planes in 3D space.

Functions of several variables

Example

Consider $f(x, y) = \frac{-3x-6y}{2} + 1$*. The graph of this function is the plane*

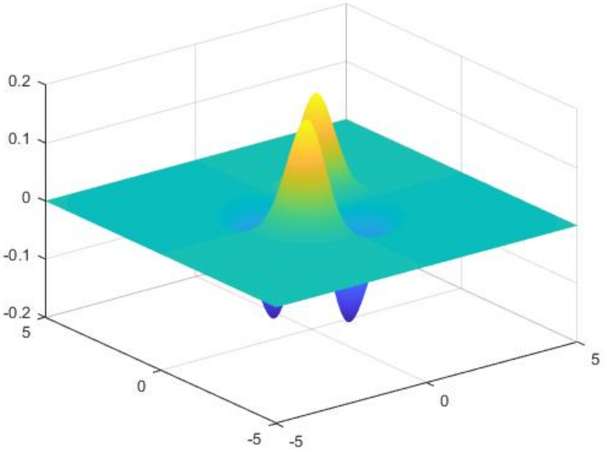
$$2z + 3x + 6y = 2$$



Functions of several variables

Example

Consider $f(x,y) = -xye^{-x^2-y^2}$. *The graph of this function can be plotted using MatLab:*



Functions of several variables

The following code was used to generate the plot above:

```
>> x=-5:0.01:5;
>> y=-5:0.01:5;
>> [X, Y]= meshgrid(x, y);
>> Z=X.*Y.*exp(-(X.^2+Y.^2));
>> surf(X,Y,Z,'EdgeColor','none')
```

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```

Another way of visualising functions of two variables is using a contour plot on the xy-plane (or on another plane if this is helpful). A contour plot on the xy-plane is a plot of the relationship $f(x, y) = k$ for different fixed values of k. This yields the shape of the cross-sections of the graph of f in the plane $z = k$. This is the same method that is used to represtent height on a topographical map.

Functions of several variables

Example

Consider $f(x,y) = 2x^2 + y^2 + 3$. *If* $f(x,y) = k$, *then*

$$\frac{2x^2}{k-3} + \frac{y^2}{k-3} = 1$$

which for $k > 3$ *describes a family of ellipses. These ellipses are the cross-sections of the graph of* f *in the plane* $z = k$.

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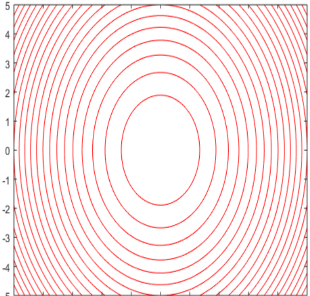


Functions of several variables

Example

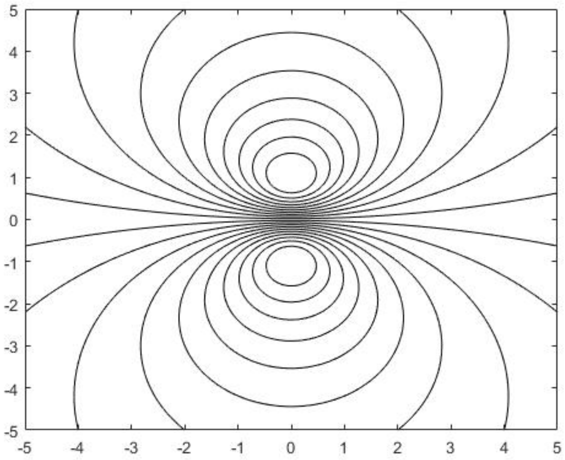
Consider

$$f(x, y) = \frac{-3y}{x^2 + y^2 + 1}.$$

The contours of this function can be plotted using MatLab:

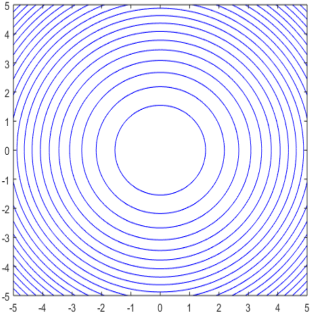
```
>> x=-5:0.01:5;
>> y=-5:0.01:5;
>> [X, Y]= meshgrid(x, y);
>> Z= -3*Y./(X.^2+Y.^2+1);
>> contour(X,Y,Z,20,'k')
```

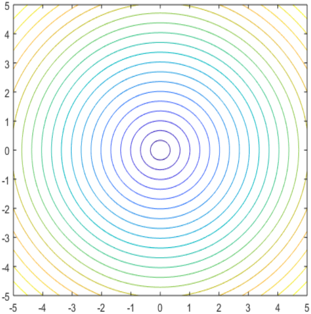
Functions of several variables



Exampes

Two contour maps correspond to functions whose graphs are a cone and a paraboloid. Which is which, and why?





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Euclidean Space

We now turn to doing calculus on functions with more than one independent variable. In order to do this we need to think about $\mathbb{R}^n$ as what is called a normed vector space. When thought of as a normed vector space $\mathbb{R}^n$ is called Euclidean Space. We have already seen that by thinking of each point in $\mathbb{R}^n$ as a vector we can coherantly define addition of two points in $\mathbb{R}^n$ (addition of vectors) and scalar multiplication (scalar multiplication of vectors). We also have a magnitude function $|\cdot|$. This function is called a norm and measures distance in $\mathbb{R}^n$ in the same way that $|\cdot|$ measures distance in $\mathbb{R}$. In order to make it clear when we are taking the magnitude of vectors rather than scalars (real numbers), we will start using $||\cdot||$ instead of $|\cdot|$ to denote vector magnitude (the Euclidean norm). The magnitude function can be represented using the dot product: if $\bar{x} = (x_1, \ldots, x_n)$ and $\bar{y} = (y_1, \ldots, y_n)$, then

$$\bar{x} \cdot \bar{y} = \sum_{k=1}^{n} x_k y_k \text{ and } ||\bar{x}||^2 = \bar{x} \cdot \bar{x}$$

Euclidean Space

Theorem

Let $\bar{x}, \bar{y} \in \mathbb{R}^n$ *and let* $\alpha \in \mathbb{R}$. *Then*

1. $||\bar{x}|| \geq 0$
2. $||\bar{x}|| = 0$ *if and only if* $\bar{x} = \bar{0}$
3. $||\alpha\bar{x}|| = |\alpha|||\bar{x}||$
4. *(Cauchy-Schwarz Inequality)* $\bar{x} \cdot \bar{y} \leq ||\bar{x}||||\bar{y}||$
5. $||\bar{x} + \bar{y}|| \leq ||\bar{x}|| + ||\bar{y}||$

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(5) is called the triangle inequality and corresponds to the triangle inequality in $\mathbb{R}$: for all $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$

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There are subsets of $\mathbb{R}^n$ that are analogues of the open and closed intervals on $\mathbb{R}$.

Definition

Let $\bar{a} \in \mathbb{R}^n$ and let $r \geq 0$. The open ball centred at $\bar{a}$ with radius r is the set

$$B(\bar{a}, r) = \{\bar{x} \in \mathbb{R}^n \mid ||\bar{x} - \bar{a}|| < r\}$$

Euclidean Space

Definition

The closed ball centred at $\bar{a}$ with radius r is the set

$$C(\bar{a}, r) = \{\bar{x} \in \mathbb{R}^n \mid \|\bar{x} - \bar{a}\| \leq r\}$$

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If A is one of these sets, then we say that A is a basic interval of $\mathbb{R}^n$.

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The distance measure $\|\cdot\|$ in $\mathbb{R}^n$ allows us to define the notions of limit and continuity in the same way that we did in $\mathbb{R}$.

Next Week

1. Limits and continuity of functions of several variables.
2. Derivatives and the chain rule.
3. Directional derivatives and gradient