UM-SJTU JOINT INSTITUTE

VV156 RC

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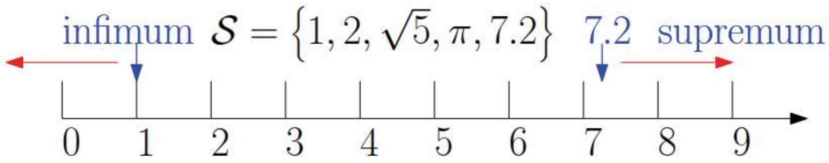
Lecture 1

Concepts

1. upper bound$\Leftrightarrow$ bounded from above
2. lower bound$\Leftrightarrow$ bounded from below
bounded from above and below$\longrightarrow$ bounded; else unbounded
3. supremum/least upper bound
4. infimum/greatest lower bound
 * The supremum and infimum do not necessarily belong to the set.



5. δ - neighbourhood of x $(x-\delta, x+\delta)$. 6. neighbourhood: a set that contains a δ - neighbourhood of x. * A neighbourhood can be a closed interval while a $delta$ - neighbourhood cannot.
7. open set: a set in which every point has a δ - neighbourhood in S. In other words, every point in S is an interior point.
* The union of open sets is open. The intersection of finite open sets is open. The intersection of infinite open sets may be closed.
8. closed sets: complements of open sets
* The intersection of closed sets is closed. The union of finite closed sets is closed. The union of infinite closed sets may be open.
* $\varnothing$ and $\mathbb{R}$ are both open and closed sets.
* Some sets are neither open or closed sets, like [2,3). Therefore, for a set S where there exists a point $x \in S$ that is not an interior point, S may be a closed set or neither.

Definition of various points

1. interior point: a point $x \in S$ s.t. $(x-\delta$,$x+\delta) \subset S$
2. boundary point: a point $x \in \mathbb{R}$ s.t. $(x-\delta$,$x+\delta)$ contains at least one point in S and at least one point out of S.
3. limit point: a point $x \in \mathbb{R}$ s.t. every neighbourhood $(x-\delta$,$x+\delta)$ contains a point in S other than x itself. That is, a point in S arbitrarily close to x.
* A limit point is either an interior point or boundary point.
4. isolated point: a point $x \in S$ is an isolated point if there exists δ s.t. x is the only point belonging to S in the neighbourhood $(x-\delta,\ x+\delta)$. That is, there isn't any point in S arbitrarily close to x, which is contrary to the limit point.
* Interior points and isolated points should belong to S, while boundary points and limit points are only required to belong to $\mathbb{R}$.
5. compact set: closed and bounded.

Lecture 2

Definition of the limit of a sequence

If for every $\epsilon > 0$, there is a corresponding integer N s.t.
if $n > N$, then $|a_n - L| < \epsilon$
then $\lim_{n\to\infty} a_n = L$.
* The definition is used to prove that the limit of the sequence equals a given number, not to evaluate the limit.
* Triangle inequality may be helpful in the proof.

$$||a| - |b|| \leq |a \pm b| \leq |a| + |b|$$

* Use sufficient conditions to find N.
* converge: L exists.
* diverge: L does not exist.

Property of limits

Limit laws

provided $\lim_{n\to\infty} a_n = L_a$ and $\lim_{n\to\infty} b_n = L_b$

1. $\lim\limits_{n\to\infty} a = a$
2. $\lim\limits_{n\to\infty} (a_n \pm b_n) = \lim\limits_{n\to\infty} a_n \pm \lim\limits_{n\to\infty} b_n = L_a + L_b$
3. $\lim\limits_{n\to\infty} (a_n b_n) = (\lim\limits_{n\to\infty} a_n)(\lim\limits_{n\to\infty} b_n) = L_a L_b$
4. $\lim\limits_{n\to\infty} (\frac{a_n}{b_n}) = \frac{\lim_{n\to\infty} a_n}{\lim n \to \infty b_n} = \frac{L_a}{L_b}, when b_n \neq 0, L_b \neq 0$

Monotonicity

1. increasing: $a_{n+1} \geq a_n$ for all n
2. decreasing: $a_{n+1} \leq a_n$ for all n

Monotonic Sequence Theorem
A monotonic sequence converges if and only if it is bounded.
* A sequence that converges must be bounded
* An unbounded sequence must be divergent.
* A sequence that is bounded may be divergent.

Suppose $\{a_n\}$ and $\{b_n\}$ are convergent, and

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

If for some $N \in \mathbb{N}$,

$$a_n \le c_n \le b_n \qquad \text{for all} \quad n > N$$

then the sequence $\{c_n\}$ is convergent. Moreover,

$$\lim_{n\to\infty} c_n = L$$

Squeeze Theorem

Lecture 3

Definition of limit

Let f be a function defined on some open interval that contains the number a, except possibly at a itself. The value of L is the limit of $f(x)$ as x approaches a,

$$\lim_{x\to a} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{if} \qquad 0 < |x - a| < \delta$$

* Note that $|x - x_0|$ >0. The limit of $f(x)$ at $x = a$ has nothing to do with $f(a)$.
* If L fails to exist, the limit of $f(x)$ when $x \to x_0$ does not exist.
* The way to find δ corresponding to ϵ is similar to the method of finding N for the limit of sequence.
* Sometimes, we first set δ=1 to simplify calculations take the smaller value (1 and another calculated one) of δ. Likewise, in limit of sequence, we take the bigger one as N (often comparison involved).
* We can assign special values to ϵ when we are asked to prove a statement.

Limit Laws

* Actually, law 5 can be extended. As long as $f(x)$ is basic elementary function, the limit at every point x_0 in its domain always exists and is equal to $f(x_0)$. *

$$lim[f(x)]^{g(x)} = K^L$$

Assume that $\lim_{x \to a} f(x) = K$ and $\lim_{x \to a} g(x) = L$, and that c is constant,

1 The limit of a constant is the constant itself.

$$\lim_{x \to a} c = c$$

2 The limit of a sum/difference is the sum/difference of the limits.

$$\lim_{x \to a} \left[f(x) \pm g(x) \right] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x) = K \pm L$$

3 The limit of a product is the product of the limits.

$$\lim_{x \to a} \left[f(x)g(x) \right] = \left[\lim_{x \to a} f(x) \right] \left[\lim_{x \to a} g(x) \right] = KL$$

4 The limit of a quotient is the quotient of the limits.

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{K}{L}, \qquad \text{provided } \lim_{x \to a} g(x) \neq 0$$

5 If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$

6 If $f(x) = g(x)$ for all x near a, possibly except at $x = a$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x), \qquad \text{provided the limits exist}$$

If $g(x) \leq f(x) \leq h(x)$ when x is near a, except possibly at a, and if

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L, \quad \text{then } \lim_{x \to a} f(x) = L$$

The Squeeze Theorem

One-sided limit

1. right-hand limit

$$\lim_{x \to a^+} f(x)$$

The range of $f(x)$ is $a < x < \delta + a$.
2. left-hand limit

$$\lim_{x \to a^-} f(x)$$

The range of $f(x)$ is $a - \delta < x < a$.
* The limit of $f(x)$ when $x \to a$ exists only when the right-hand limit and the left-hand limit both exist and are equal.

Lecture 4

Definition of limit at infinity and infinite limit

1. The limit of $f(x)$ approaches infinity

$$\lim_{x \to a} f(x) = \infty$$

if for every number $M > 0$ there exists a number $\delta > 0$ such that $f(x) > M$ if $0 < |x - a| < \delta$.
* If the limit exists, then $f(x)$ has a vertical asymptote $x = a$.
2.

$$\lim_{x \to \infty} f(x) = L$$

if for every number $\epsilon > 0$ there exists a number $Min(a, \infty)$ such that $|f(x) - L| < \epsilon$ if $x > M$. * Here M is similar to N in the limit of sequence.
* Difference: M can be real numbers, while N should be a positive integer.
* If the limit exists, then $f(x)$ has a horizontal asymptote $y = L$.

Special cases in limit laws

Suppose f and g are functions such that $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = L$.

1. The limit of the sum/difference is infinity

$$\lim_{x \to a} [f(x) \pm g(x)] = \infty$$

2. The limit of the product is infinity if $L > 0$ and negative infinity if $L < 0$

$$\lim_{x \to a} [f(x)g(x)] = \pm\infty$$

3. The limit of the quotient is infinity if $L > 0$ and negative infinity if $L < 0$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \pm\infty \qquad \text{and} \qquad \lim_{x \to a} \frac{g(x)}{f(x)} = 0$$

* Still hold when a is replaced by ∞.

Three Theorems

1. If r is a positive rational number, then

$$\lim_{x \to \infty} \frac{1}{x^r} = 0$$

2. If $p(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_0$ is a polynomial of degree n, then

$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} a_n x^n$$

3. The limit of a rational function as $x \to \infty$ is the limit of the quotient of the terms of highest degree in the numerator and the denominator as $x \to \infty$.
* This is the extension of the theorem we learnt in high school to functions (where x are real numbers instead of only positive integers).
* Only when $x \to \infty$ the theorem holds true.

Some techniques

* Convert $tanx$ to the quotient of $sinx$ and $cosx$.
* Substitute $sinx$ with 1.
* Double angle formula
* Product to sum formula, sum to product formula
* u-substitution

Two important limits

$$\lim_{n \to \infty} n^{\frac{1}{n}} = 1$$

n is an integer.

$$\lim_{x \to \infty} (1 + \frac{1}{x})^x = e$$

Both can be proved using squeeze theorem.

Continuity

Definition of continuity

$$\lim_{x \to c} f(x) = f(c)$$

* The judgement of continuity is actually identifying the value of the limit.