Vv156 Lecture 21

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Curves

- So far we have studied curves as the graphs of
  - Functions involving the two variables x and y

$$y = f(x) \qquad \text{or} \qquad x = g(y)$$

  - Equations involving the two variables x and y

$$F(x, y) = 0$$

- Curves described by functions can be described by equations as well

$$y = f(x) \implies y - f(x) = 0 \qquad \text{and} \qquad x = g(y) \implies x - g(y) = 0$$

- However, curves described by equations are not necessarily functions. e.g.

$$x^2 + y^2 = 1$$

- So an equation $F(x, y) = 0$ describes a broader class of curves.

Ant walking on a large table

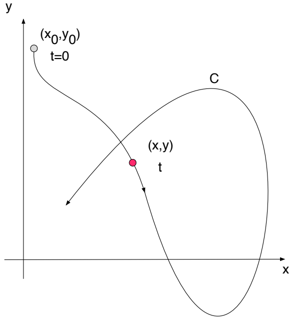
- Suppose we have the following curve, and imagine it is the trail of an energetic ant on a large table.

1. It fails the vertical line test, so

   y is not a function of x.

2. However, the position (x, y) of the ant changes according to time, thus both x and y-coordinates are functions of t.

$$x = f(t); \qquad y = g(t)$$
$$x = \boxed{x}(t); \qquad y = \boxed{y}(t);$$



- Such a pair of equations is often a convenient way of describing more general curves than those defined by

$$F(x, y) = 0$$

Definition

Suppose that x and y are both given as functions of a third variable t

$$x = f(t), \qquad y = g(t)$$

over an interval I of t-values, then the set of points $(f(t), g(t))$ defined by these equations is known as a parametric curve.

- The equations are parametric equations for the curve.
- The variable t is known as a parameter for the curve.
- Its domain I is the parameter interval. If I is a closed interval, $a \leq t \leq b$,
  the point $(f(a), g(a))$ is the initial point of the curve, and
  the point $(f(b), g(b))$ is the terminal point of the curve.
- When we give parametric equations and a parameter interval for a curve, we say that we have parametrized the curve.
- The parametric equations and the parameter interval together constitute

  a parametrization of the curve.

Q: What curve is represented by the following parametric equations?

$$x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi$$
$$x^2 = \cos^2 t, \qquad y^2 = \sin^2 t, \qquad x^2 + y^2 = \cos^2 t + \sin^2 t = 1.$$

- Thus the set of equations represents the unit circle centered at origin.

Q: What curve is represented by the following parametric equations?

$$x = \cos 2t, \qquad y = \sin 2t, \qquad 0 \le t \le 2\pi$$
$$x^2 = \cos^2 2t, \qquad y^2 = \sin^2 2t, \qquad x^2 + y^2 = \cos^2 2t + \sin^2 2t = 1.$$

- For a given curve, parametric equations are not unique.

Q: What is the usual parametrization for the circle of radius r with center (h, k)

A: One of many possible parametrization is

$$x = h + r\cos t, \qquad y = k + r\sin t, \qquad \text{where} \quad 0 \le t \le 2\pi$$

Exercise

A curve C is defined by the parametric equations

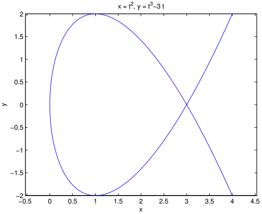
$$x = t^2, \qquad y = t^3 - 3t$$

(a) Find the equation of the tangent at the point $(3, 0)$.

(b) Find the points on C where the tangent is horizontal or vertical.

(c) Determine where the curve is concave upward or downward.

Q: Why there are two equations for the tangent line?

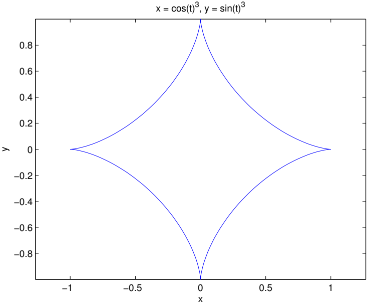


Exercise

(a) Find the area enclosed by the astroid.

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

(b) Find the length of the astroid.



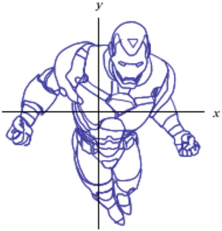
(c) Find the surface area generated by revolving the astroid in the first quadrant about the x-axis.

More general cuves

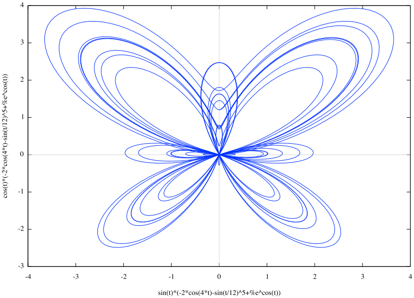
- Complicated relationship between variables can be model

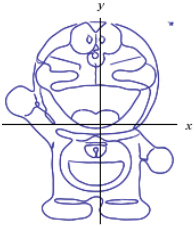
$$x = \sin(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$$

$$y = \cos(t)\left(e^{\cos(t)} - 2\cos(4t) - \sin^5\left(\frac{t}{12}\right)\right)$$



(plotted for t from 0 to 144π)





(plotted for t from 0 to 88π)

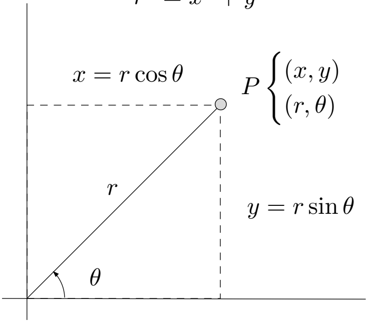
- Usually we use Cartesian coordinates system,

$$(x, y)$$

which are directed distances from two perpendicular axes.

- Polar coordinates system, (r, θ), is more convenient for many purposes.

$$r^2 = x^2 + y^2$$



where r and θ are called the radial and the angular coordinate, respectively.

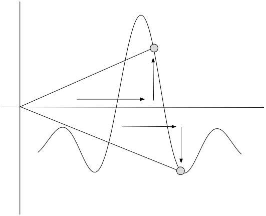
- Note the following represents the same point for $n \in \mathbb{Z}$

$$(r, \theta + 2n\pi)$$

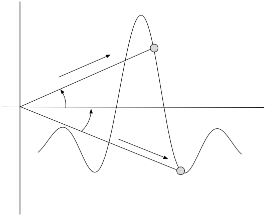
- Negative radial coordinates are defined as the reflection in the origin

$$(-r, \theta) \qquad \text{and} \qquad (r, \theta + \pi)$$

- Note how the same curve can be traced differently by the two systems



Cartesian



Polar

Q: What curve is represented by the polar equation $r = 2$?

circle of radius 2 centred at the origin.

Q: How about the polar equation $\theta = \frac{\pi}{3}$?

line that makes an angle of $\frac{\pi}{3}$ radians with the x-axis.

Q: Are they the same compared to $x = \frac{\pi}{3}$ or $y = 2$?

- Note that a Cartesian equation and a polar equation, e.g.

$$y = \cos 2x \qquad \text{and} \qquad r = \cos 2\theta$$

might share the same form, they represent completely different curves.

Four-leaved rose

```
>> syms x
>> ezplot('cos(2*x)',[0,2*pi])
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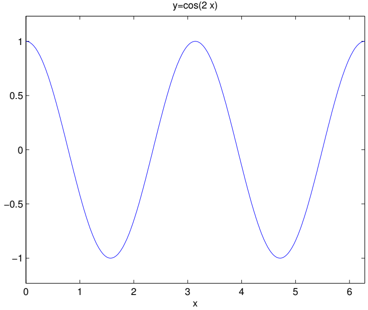

Cartesian

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>> syms t
>> ezpolar('cos(2*t)',[0,2*pi])
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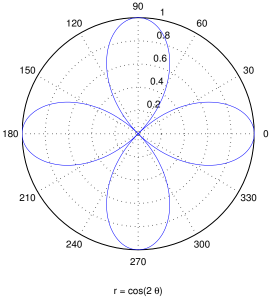


Polar

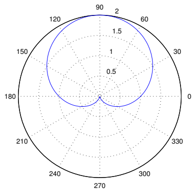
Exercise

(a) Find a Cartesian equation for the polar equation

$$r = 3\sin\theta$$

(b) Find the slope of the tangent line for the polar curve when $\theta = \pi/3$

$$r = 1 + \sin\theta$$



r = 1+sin(θ)

This curve is known as a cardioid.

(c) Find where the cardioid has horizontal or vertical tangent lines.

(d) Find the arc length of the cardioid.