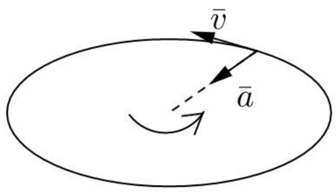
1 Dynamics of circular motion

1.1 uniform motion



$|\bar{v}| = \text{const}$, but
$\bar{v} \neq \text{const}$

centripetal acceleration $\overset{\text{Newton's 2}^{\text{nd}}\text{ law}}{\Longleftrightarrow}$ force

(toward the center, always) (called centripetal force)

here radial $\parallel$ normal

$$\bar{a}_r = -\frac{v^2}{R}\hat{u}_r = -\omega^2 R\hat{u}_r$$

$$\bar{F}_r = m\bar{a}_r = -m\frac{v^2}{R}\hat{u}_r = -m\omega^2 R\hat{u}_r$$

various forces can play the role

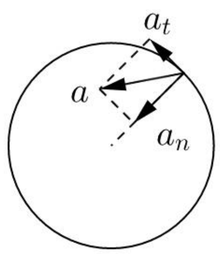
of a centripetal force

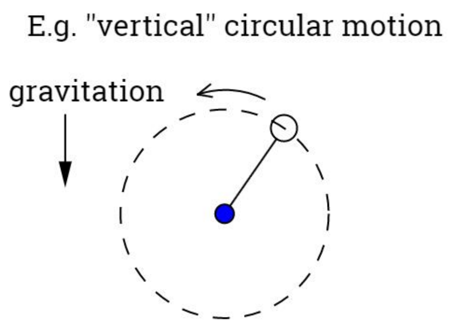
(e.g. tension of a cord $-$ contact force,

gravitational force $-$ field force)

1.2 non-uniform motion

$$|\bar{v}| \neq \text{const} \quad \Rightarrow \quad a_t \neq 0 \quad \Rightarrow \quad \text{force in the tangential direction}$$

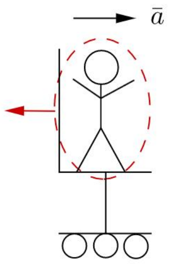




2 Dynamics in non-inertial frames of reference

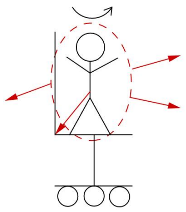
Examples/ demonstrations:

(1) accelerating chair: moving along a straight line



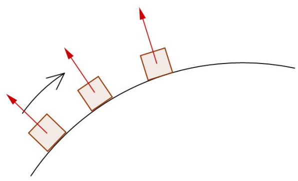
feels a "force" pushing him in the direction opposite to the direction of acceleration

(2) rotating chair



feels a "centrifugal force"

(3) chair moving along a curved path



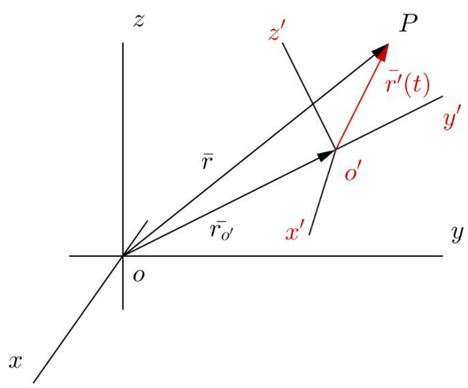
(view from the top)

feels a "centrifugal force" (directed outward, along the instantaneous radius of curvature)

These "forces" cannot be regular forces: real forces are always of material origin and always appear in pairs (Newton's third law)

What is the nature of these "forces"?

3 Mathematical description



xyz - inertial FoR
$x'y'z'$ - moves arbitrarily (can accelerate, rotate with variable angular velocity,...)

Earlier, we considered a situation when $x'y'z'$ was moving with a constant velocity w.r.t xyz (we have shown that $x'y'z'$ was inertial then). Now, $x'y'z'$ moves arbitrarily (so that $\bar{v}_{o'} \neq$ const) hence is not inertial.

$$\bar{r} = \bar{r}_{o'} + \bar{r'}$$

$\swarrow$ $\downarrow$ $\searrow$

position in xyz (we use $\hat{n}_x, \hat{n}_y, \hat{n}_z$ here) $\downarrow$ fixed !

position of the origin of $x'y'z'$ as seen from the system xyz

position in $x'y'z'$ (here, we have to use $\hat{n}_{x'}, \hat{n}_{y'}, \hat{n}_{z'}$) $\downarrow$ not fixed: travel with $x'y'z'$ (in particular rotate)

Goal: derive a (kinematic) relation between accelerations of P in both frames of reference

$\boxed{\text{Problem}}$ motion of $x'y'z'$ is arbitrary, will need to take into account that $\hat{n}_{x'}, \hat{n}_{y'}, \hat{n}_{z'}$ are not fixed (i.e. will need to know how to calculate their derivatives w.r.t. time)

4 Relation between velocities

Started with

$$\bar{r}(t) = \bar{r}_{o'}(t) + \bar{r'}(t) \quad (*)$$

We will use Einstein's notation in the following calculations

$\bar{r}(t) = r_\alpha \hat{n}_\alpha \quad \rightarrow \quad$ unit vectors in xyz

$\bar{r'}(t) = r_{\alpha'} \hat{n}_{\alpha'} \quad \rightarrow \quad$ unit vectors in $x'y'z'$

$r_\alpha \hat{n}_\alpha = \sum_{\alpha=x,y,z} r_\alpha \hat{n}_\alpha$
$r_x = x$
$r_y = y$
$r_z = z$
(skip the "$\sum$", summation runs over any repeated Greek indices)
Einstein notation

Differentiate (*) w.r.t. time

$$\frac{\mathrm{d}\bar{r}}{\mathrm{d}t} = \bar{v} = \frac{\mathrm{d}\bar{r}_{o'}(t)}{\mathrm{d}t} + \frac{\mathrm{d}\bar{r'}(t)}{\mathrm{d}t} = \bar{v}_{o'} + \frac{\mathrm{d}\bar{r'}(t)}{\mathrm{d}t} \quad (**)$$

Now

$$\frac{\mathrm{d}\bar{r'}(t)}{dt} = \frac{\mathrm{d}}{\mathrm{d}t}(r_{\alpha'}, \hat{n}_{\alpha'}) = \boxed{\dot{r}_{\alpha'}\hat{n}_{\alpha'}} + r_{\alpha'}\dot{\hat{n}}_{\alpha'}$$

$\downarrow$

velocity of P observed from the non-inertial frame of reference

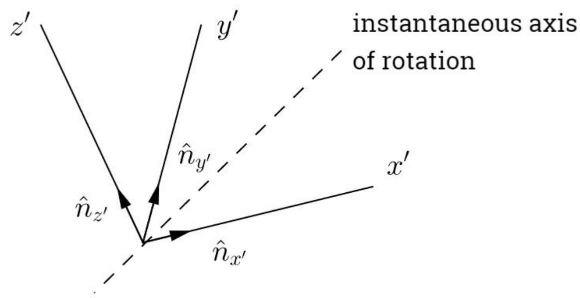
$$\dot{r}_{\alpha'}\hat{n}_{\alpha'} = v_{\alpha'}\hat{n}_{\alpha'} = \bar{v}'$$

So

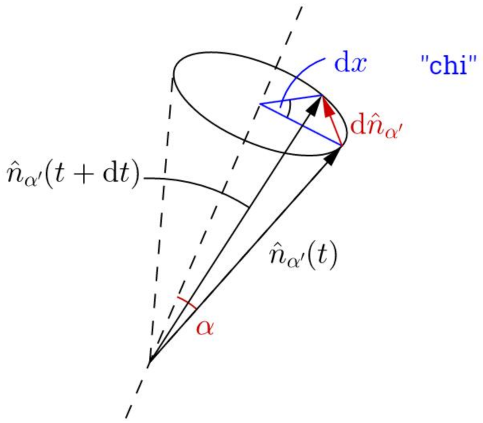
$$\frac{\mathrm{d}\bar{r'}(t)}{\mathrm{d}t} = \bar{v'} + r_{\alpha'}\boxed{\dot{\hat{n}}_{\alpha'}}$$

$\downarrow$

need to learn how to differentiate unit vectors ($\dot{\hat{n}}_{\alpha'}$ is related to rotational motion of $x'y'z'$ about an instantaneous axis of rotation)

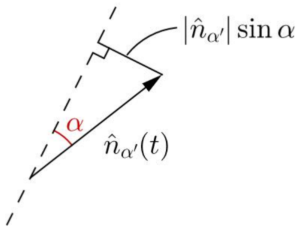


5 Derivative $\dot{\hat{n}}_{\alpha'}$

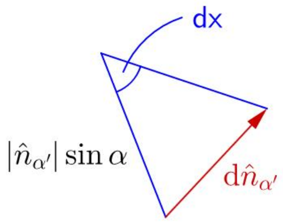


$\alpha-$ angle between the axis of rotation and $\hat{n}_{\alpha'}$

From the triangle



And from the triangle



Recall: for the cross product
$|\bar{u} \times \bar{l}| = |\bar{u}||\bar{l}| \sin \angle(\bar{u}, \bar{l})$

$$|\mathrm{d}\hat{n}_{\alpha'}| = \mathrm{d}x |\hat{n}_{\alpha'}| \sin \alpha$$

Define vector $\mathrm{d}\bar{x}$ as the vector along the instantaneous axis of rotation (use the right hand rule for direction), such that $\mathrm{d}x = |\mathrm{d}\bar{x}|$ is the angle that the tips of $\hat{n}_{\alpha'}(t)$, $\hat{n}_{\alpha'}(t+\mathrm{d}t)$ form over time $\mathrm{d}t$

Then

$$\mathrm{d}\hat{n}_{\alpha'} = \mathrm{d}\bar{x} \times \hat{n}_{\alpha'}$$

So

$$\frac{\mathrm{d}\hat{n}_{\alpha'}}{\mathrm{d}t} = \frac{\mathrm{d}\bar{x}}{\mathrm{d}t} \times \hat{n}_{\alpha'}$$

$\downarrow$

instantaneous angular velocity

$$\bar{\omega} = \frac{\mathrm{d}\bar{x}}{\mathrm{d}t}$$

Use this to complete the final step in calculating

$$\frac{\mathrm{d}\bar{r}'(t)}{\mathrm{d}t} = \bar{v}' + r_{\alpha'}\dot{\hat{n}}_{\alpha'} = \bar{v}' + r_{\alpha'}(\bar{\omega} \times \hat{n}_{\alpha'}) =$$
$$= \bar{v}' + \bar{\omega} \times \underbrace{r_{\alpha'}\hat{n}_{\alpha'}}_{\bar{r}' \text{ (Einstein's notation)}}$$

And eventually (substitute into (**))

$$\boxed{\bar{v} = \bar{v}_{o'} + \bar{v}' + (\bar{\omega} \times \bar{r}')}$$

Comment: The arbitrary motion of $x'y'z'$ can be decomposed into translational motion and rotational motion, the last term is due to the latter

Now, find the relation for accelerations

$$\bar{a} = \frac{\mathrm{d}\bar{v}}{\mathrm{d}t} = \frac{\mathrm{d}\bar{v}_{o'}}{\mathrm{d}t} + \frac{\mathrm{d}\bar{v}'}{\mathrm{d}t} + \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' + \bar{\omega} \times \frac{\mathrm{d}\bar{r}'}{\mathrm{d}t} \qquad (***)$$

$\downarrow$ $\qquad \searrow \qquad \swarrow$

$\bar{a}_{o'}$ $\quad$ use the derived rule here

First

$$\frac{\mathrm{d}\bar{v}'}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(v_{\alpha'}\hat{n}_{\alpha'}) = \dot{v}_{\alpha'}\hat{n}_{\alpha'} + v_{\alpha'}\dot{\hat{n}}_{\alpha'}$$
$$= \bar{a}' + v_{\alpha'}\bar{\omega} \times \hat{n}_{\alpha'}$$
$$= \bar{a}' + \bar{\omega} \times v_{\alpha'}\hat{n}_{\alpha'}$$
$$= \bar{a}' + \bar{\omega} \times \bar{v}'$$

Then

$$\bar{\omega} \times \frac{\mathrm{d}\bar{r}'}{\mathrm{d}t} = \bar{\omega} \times \frac{\mathrm{d}}{\mathrm{d}t}(r_{\alpha'}\hat{n}_{\alpha'})$$
$$= \omega \times (\dot{r}_{\alpha'}\hat{n}_{\alpha'} + r_{\alpha'}\dot{\hat{n}}_{\alpha'})$$
$$= \omega \times \bar{v}' + \bar{\omega} \times r_{\alpha'}(\bar{\omega} \times \hat{n}_{\alpha'})$$
$$= \omega \times \bar{v}' + \bar{\omega} \times (\bar{\omega} \times r_{\alpha'}\hat{n}_{\alpha'})$$
$$= \omega \times \bar{v}' + \bar{\omega} \times (\bar{\omega} \times \bar{r}')$$

Combine all terms into (***)

$$\bar{a} = \bar{a}_{o'} + \bar{a}' + \bar{\omega} \times \bar{v}' + \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r} + \bar{\omega} \times \bar{v}' + \bar{\omega} \times (\bar{\omega} \times \bar{r}')$$

Eventually

$$\boxed{\bar{a} = \bar{a}_{o'} + \bar{a}' + 2\bar{\omega} \times \bar{v}' + \frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' + \bar{\omega} \times (\bar{\omega} \times \bar{r}')}$$

Multiply by m and rearrange (leave $m\bar{a}'$ on one side, all other terms move to the other side)

$$m\bar{a}' = m\bar{a} - m\bar{a}_{o'} - m\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' - 2m(\bar{\omega} \times \bar{v}') - m\bar{\omega} \times (\bar{\omega} \times \bar{r}')$$

But $m\bar{a} = \bar{F}$ because xyz is an inertial FoR (can use Newton's laws here)

$\downarrow$

real force (i.e. of material origin)

Equation of motion in a non-inertial frame of reference

$$m\bar{a}' = \bar{F} \underbrace{- m\bar{a}_{o'} - m\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' - 2m(\bar{\omega} \times \bar{v}') - m\bar{\omega} \times (\bar{\omega} \times \bar{r}')}$$

pseudo forces (also called fictitious forces or forces of inertia) — kinematic corrections (have units of $[N]$) that are due to the fact that we describe dynamics in a non-inertial FoR. These "forces" never appear in inertial FoRs

$$\begin{cases} - m\bar{a}_{o'} & \text{d}'\text{Alembert "force"} \\ - m\dfrac{\mathrm{d}\omega}{\mathrm{d}t} \times \bar{r}' & \mathcal{E}\text{uler "force"} \\ - 2m\bar{\omega} \times \bar{v}' & \text{Coriolis "force"} \\ - m\bar{\omega} \times (\bar{\omega} \times \bar{r}') & \text{centrifugal "force"} \end{cases}$$

always proportional to the mass "m" of a particle

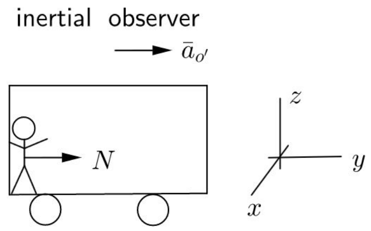
Note. If you discuss dynamics in a non-inertial frame of reference and include these "forces" in your free body diagrams, please make sure you remember that these are pseudo forces (and distinguish them from any real forces)

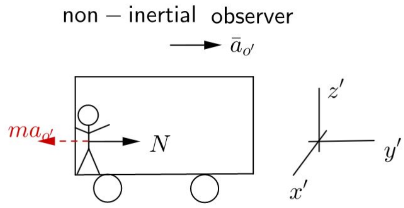
Note. You cannot have the centripetal and the centrifugal forces on the same free-body diagram!

6 Examples

$$m\bar{a}' = \bar{F} - m\bar{a}_{o'} - m\frac{\mathrm{d}\bar{\omega}}{\mathrm{d}t} \times \bar{r}' - 2m(\bar{\omega} \times \bar{v}') - m\bar{\omega} \times (\bar{\omega} \times \bar{r}')$$

(1) $\bar{a}_{o'} \neq 0,\ \bar{\omega} = 0, \bar{v}' = 0$; e.g. acclerating car moving along a straight line





$$ma_{o'} = N$$

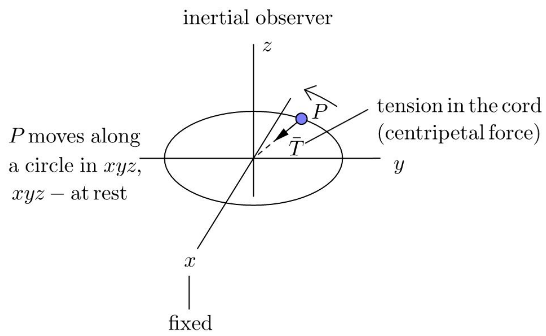
"He moves with acceleration $\bar{a}_{o'}$, because there is a force (normal force $\bar{N}$ due to the wall) acting upon him"

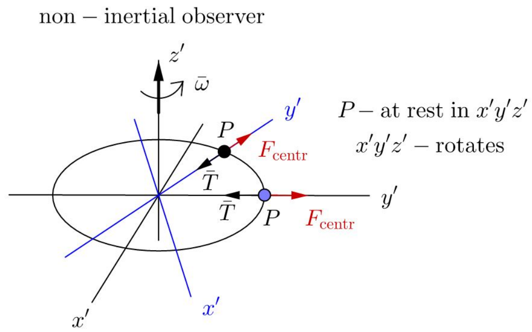
$$0 = N - ma_{o'}$$

"He is at rest in my FoR because the normal force is balanced by the d'Alembert 'force' "

mathematically we have identical equations but their physical interpretation is different

(2) $\bar{a}_{o'} = 0$; $\bar{\omega} = \text{const}$; $\bar{v}' = 0$; e.g. uniform circular motion





$$m\bar{a} = \bar{T}$$

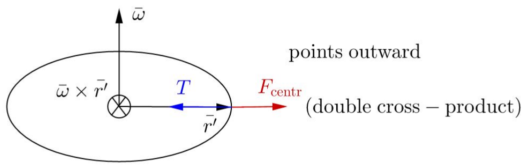
$$-m\omega^2 R\hat{n}_r = \bar{T}$$

"He moves along a circle with centripetal acceleration $-\omega^2 R\hat{n}_r$, because there is tension in the cord that plays a role of a centripetal force"

$$0 = \bar{F}_{\mathrm{centr}} + \boxed{\bar{T}}$$

$\downarrow$

cannot call it centripetal



Since $\bar{r}' \perp \bar{\omega}$ and $\bar{\omega} \perp (\bar{\omega} \times \bar{r}')$ then

$$\begin{aligned}|\bar{F}_{\text{centr}}| &= |-m\bar{\omega} \times (\bar{\omega} \times \bar{r}')| \\ &= m\omega^2 r' \\ &= m\omega^2 R\end{aligned}$$

Or use the formula for the double cross product

$$\bar{\omega} \times (\bar{\omega} \times \bar{r}') = \bar{\omega}(\underbrace{\bar{\omega} \circ \bar{r}'}_{=0}) - \bar{r}'(\underbrace{\omega \circ \bar{\omega}}_{=\omega^2})$$

$$\bar{F}_{\text{centr}} = m\omega^2 \bar{r}' = m\omega^2 R\hat{n}_r$$

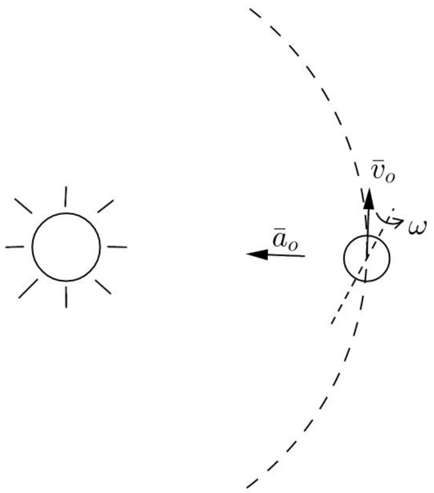
So eventually

$$0 = m\omega^2 R\hat{n}_r + \bar{T}$$

"He rests in my frame of reference, because the tension in the cord is balanced by the 'centrifugal force' "

Again both equations are algebraically identical but they are formulated by the both observers using different language

7 The Earth as a frame of reference



orbital motion + rotational motion

$$\left.\begin{aligned} v_o &\sim 30 \text{ km/s} \\ \omega = \frac{2\pi}{T} &= \frac{2\pi}{24h} \sim 7 \cdot 10^{-5} \ \frac{1}{s} \end{aligned}\right\} \Rightarrow \text{non} - \text{inertial FoR}$$

For an (non-inertial) observer on the Earth ($\bar{r'}$, $\bar{v'}$ are in the Earth's FoR)

$$m\bar{a'} = \boxed{\bar{F}} - m\boxed{\bar{a}_o} - m\bar{\omega} \times (\bar{\omega} \times \bar{r'}) - 2m(\bar{\omega} \times \bar{v'})$$

$\downarrow$ material force (gravitation) $\bar{F}_{\text{Earth}} + \bar{F}_{Sun}$

$\searrow$ $-\bar{a}_o$ "centrifugal" acceleration

In the non-inertial FoR associated with the Earth $\bar{F}_{\text{Sun}} - m\bar{a}_o \cong 0$

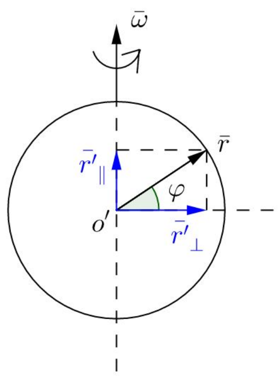
$\downarrow$ -$\bar{a}_o$, acceleration due to orbital motion

$$\boxed{m\bar{a'} = \underbrace{\bar{F}_{\text{Earth}}}_{\text{weight}} - m\bar{\omega} \times (\bar{\omega} \times \bar{r'}) - 2m(\bar{\omega} \times \bar{v'})}$$

2^{nd} law of dynamics on the Earth (including kinematic corrections due to fact that the Earth is a non-inertial FoR)

8 Estimation of "fictitious" forces

8.1 centrifugal "force"



φ – latitude
($\varphi > 0$ for the N hemisphere
$\varphi < 0$ for the S hemisphere)

$$\bar{r'} = \bar{r'}_{\perp} + \bar{r'}_{\parallel}$$

$\swarrow$ perpendicular to the axis of rotation

$\searrow$ parallel to the axis of rotation

$$\begin{aligned} \bar{F}_{\text{centr}} &= -m\bar{\omega} \times (\bar{\omega} \times (\bar{r'}_{\perp} + \bar{r'}_{\parallel})) \\ &= -m\bar{\omega} \times (\bar{\omega} \times \bar{r'}_{\perp} + \underbrace{\bar{\omega} \times \bar{r'}_{\parallel}}_{=0}) \\ &= -m\bar{\omega} \times (\bar{\omega} \times \bar{r'}_{\perp}) \end{aligned}$$

The magnitude

$$F_{\text{centr}} = m\omega^2 r'_{\perp} \qquad \text{Because } \bar{\omega} \perp \bar{r'}_{\perp} \text{ and } \bar{\omega} \perp (\bar{\omega} \times \bar{r'}_{\perp})$$

But

$$|\bar{r'}_{\perp}| = |\bar{r}| \cos\varphi = \boxed{R} \cos\varphi$$

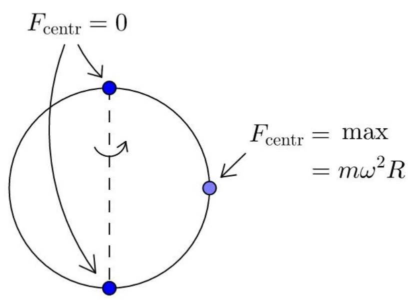
$\downarrow$

Earth's radius $\sim$ 6357 km to 6378 km

$\swarrow$ pde $\searrow$ the Equator

So

$$\boxed{F_{\text{centr}} = m\omega^2 R \cos\varphi}$$

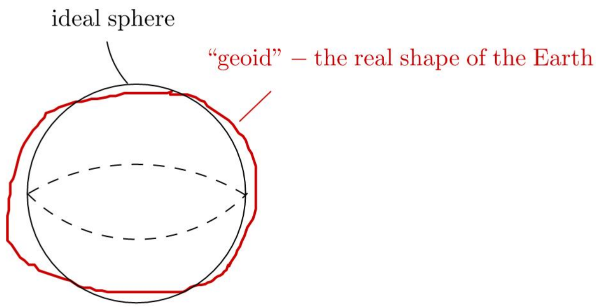


Estimation of the contribution of this "force" to non-inertiality

$$\frac{(F_{\text{centr}})_{\max}}{mg} = \frac{\omega^2 R}{g} \approx \underline{0.003} \qquad \underline{\text{very small effect}}$$

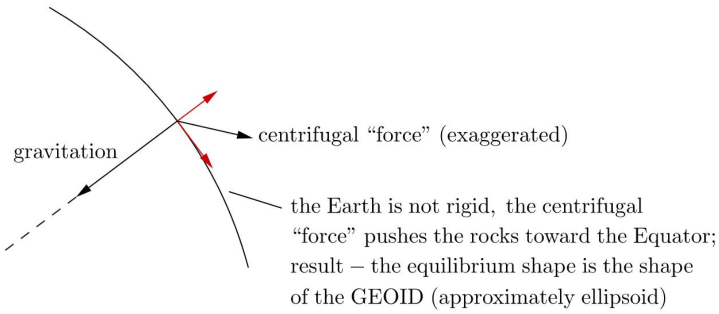
Note. To make $(F_{\text{centr}})_{\max} \sim mg$, the angular velocity should increase 18-fold. Then the day would last only 1h 19mins!

Digression: the shape of the Earth and the centrifugal "force"

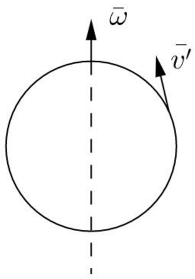


$R_{\text{pole}} \approx 6357$ km

$R_{\text{Equator}} \approx 6378$ km



8.2 the Coriolis' force



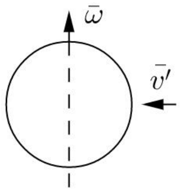
$$\bar{F}_C = -2m(\bar{\omega} \times \bar{v'})$$

Its magnitude

$$F_C = |\bar{F}_C| = 2m|\bar{\omega}||\bar{v}|\sin\angle(\bar{\omega}, \bar{v'})$$

$$\downarrow$$

maximum if $\bar{\omega} \perp \bar{v'}$
(e.g. free fall at the Equator)



<u>Estimation of its contribution</u> to the corrections to Newton's 2^{nd} law

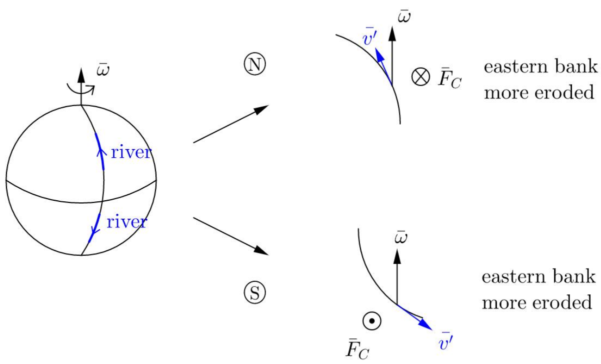
$$\frac{(F_C)_{\max}}{mg} = \frac{2\omega}{g}v' \approx 1.50 \times 10^{-5}\ [\frac{s}{m}]v'$$

To make this ratio ~ 1 we should have our object moving at speed 66 km/s, i.e. 2.2 times faster than the Earth moves in its orbital motion

<u>Conclusion:</u> The Earth can be treated, with a good accuracy, as an inertial frame of reference.

Digression: role of the Coriolis' force in the nature

(1) erosion of river banks



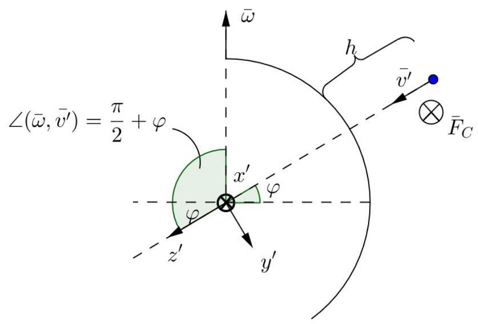
General observation: (1) on the N hemisphere objects are deflected to the right (with respect to the direction of motion)

(2) on the S hemisphere objects are deflected to the left

(2) winds: the prevailing westerlies, cyclones and anticyclones → RECITATION CLASS

Example: Free fall with the Coriolis' force (simplified/approximate discussion)

Observation: Free-falling objects are deflected toward east



($\boxed{\text{sketch exaggerated !}}$: we still assume constant gravitational force close to Earth's surface)

φ—latitude

The Coriolis force:

$$\bar{F}_C = -2m(\bar{\omega} \times \bar{v}')$$

$$F_C = |\bar{F}_C| = 2m\omega v' \sin\angle(\bar{\omega}, \bar{v}')$$

But $\angle(\bar{\omega}, \bar{v}') = \frac{\pi}{2} + \varphi$

$$F_C = 2m\omega v' \cos\varphi \quad (!)$$

In free fall $v' = gt$, so

$$F_C = 2mgt\omega\cos\varphi$$

The acceleration due to this force (directed to the east)

$$a_{x'} = \frac{F}{m} = 2g\omega t\cos\varphi$$

So the velocity toward the east:

$$v_{x'} = \int_0^t a_C \mathrm{d}t = g\omega t^2 \cos\varphi$$

and deflection

$$x' = \int_0^t v_{x'} \mathrm{d}t = \frac{1}{3} g\omega t^3 \cos\varphi$$

But $h = \frac{gt_f^2}{2} \Rightarrow t_f = \sqrt{\frac{2h}{g}} \quad \Rightarrow \quad x_{\text{defl}} = x'(t_f) = \boxed{\frac{1}{3} g\omega (\frac{2h}{g})^{\frac{3}{2}} \cos\varphi}$ (E.g. h=642m, x_{defl}=22cm)

Discussion:

→ a pole (N or S), the $\varphi = \frac{\pi}{2}$ — no deflection

→ the equator, then $\varphi = 0$ — maximum deflection

$$(x_{\text{defl}})_{\max} = \frac{1}{3} g\omega (\frac{2h}{g})^{\frac{3}{2}}$$

Note !!! This is an approximate analysis. In fact $\bar{v} = \bar{v}_{z'} + \bar{v}_{x'}$ and this total velocity should be used in

the formula for the Coriolis' force.

We have only included the $z'-$component (see step (!)). The full analysis of the problem is

complicated.