Vv156 Lecture 16

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- We are to develop a general method for solving integrals of the form

$$\int f(x)g(x)\,dx$$

- Suppose $G(x)$ an antiderivative of $g(x)$, that is, $G' = g$, and consider

$$\frac{d}{dx}\Big[f(x)G(x)\Big] = f(x)\cdot G'(x) + f'(x)\cdot G(x) = f(x)\cdot g(x) + f'(x)\cdot G(x)$$

- This states that $f(x)G(x)$ is an antiderivative of RHS, in integral notation,

$$f(x)G(x) = \int\Big[f(x)g(x) + f'(x)G(x)\Big]\,dx = \int f(x)g(x)\,dx + \int f'(x)G(x)$$

Theorem

Suppose f and g are continuous, and f' is continuous, then

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx, \qquad \text{where} \quad G'(x) = g(x)$$

- The application of the last theorem is called integration by parts.
- The idea behind integration by parts is to choose one of the factors so that

$$\int f(x)\, g(x)\, dx = f(x)\, G(x) - \int f'(x)\, G(x)\, dx, \quad \text{where} \quad G'(x) = g(x)$$

it becomes "simpler" when differentiated, while antiderivatives of the other factor are readily available.

Exercise

Use integration by parts to find

$$\int x^3 \ln x \, dx$$

- An antiderivative of $\ln x$ can be found using integration by parts.

- Suppose f is a differentiable function, then

$$\begin{aligned}\int f(x)\,dx &= \int f(x)g(x)\,dx, && \text{where} \quad g(x)=1\\ &= f(x)G(x) - \int f'(x)G(x)\,dx, && \text{where} \quad G'(x)=g(x)\\ &= f(x)x - \int x f'(x)\,dx\end{aligned}$$

- Therefore the integral $\int \ln x\,dx$ can be easily determined,

$$\begin{aligned}\int \ln x\,dx &= x\ln x - \int x\frac{1}{x}\,dx\\ &= x\ln x - \int 1\,dx\\ &= x\ln x - x + c = x(\ln x - 1) + c\end{aligned}$$

- The theorem of integration by parts is also true for definite integral, again

$$\frac{d}{dx}\Big(f(x)G(x)\Big) = f(x)\,g(x)\, + f'(x)G(x)$$

means

$$f(x)G(x)$$

is an antiderivative of

$$f(x)\,g(x)\, + f'(x)G(x)$$

- Note the sum is a continuous function under our hypotheses, so FTC states

$$\int_a^x \Big(f(t)\cdot\, g(t)\, + f'(t)\cdot G(t)\Big)\,dt \qquad \text{is an antiderivative as well.}$$

- Hence the two functions are equal up to an additive constant c,

$$f(x)G(x) + c = \int_a^x \Big(f(t)g(t) + f'(t)G(t)\Big)\,dt$$

- For $x = a$, we see that

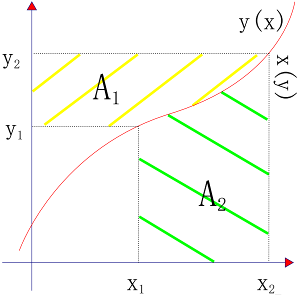
$$\begin{aligned} f(a)G(a) + c &= \int_a^a \Big(f(t)g(t) + f'(t)G(t)\Big)\,dt \\ &= 0 \\ \implies c &= -f(a)G(a) \end{aligned}$$

- Now if $x = b$, and replace the dummy variable t by x,

$$\begin{aligned} \int_a^b f(x)g(x)\,dx + \int_a^b f'(x)G(x)\,dx &= f(b)G(b) - f(a)G(a) \\ &= \Big[f(x)G(x)\Big]_a^b \\ \implies \boxed{\int_a^b f(x)g(x)\,dx} &= \Big[f(x)G(x)\Big]_a^b - \int_a^b f'(x)G(x)\,dx \end{aligned}$$

- Consider the area under a smooth 1-to-1 curve

$$A_1 = \int_{y_1}^{y_2} x(y)\,dy; \quad A_2 = \int_{x_1}^{x_2} y(x)\,dx$$



- The area is also given by

$$A_1 + A_2 = x_2 y_2 - x_1 y_1 = \Big[x \cdot y(x)\Big]_{x_1}^{x_2}$$

$$\int_{y_1}^{y_2} x(y)\,dy + \int_{x_1}^{x_2} y(x)\,dx = \Big[x \cdot y(x)\Big]_{x_1}^{x_2}$$

- If $y = f(x) \implies \int_{y_1}^{y_2} x(y)\,dy = \int_{x_1}^{x_2} x f'(x)\,dx$ by substitution.
- And if we introduce $G(x) = x$ and $g(x) = G'(x) = 1$, then

$$\int_{x_1}^{x_2} f(x)g(x)\,dx = \Big[f(x)G(x)\Big]_{x_1}^{x_2} - \int_{x_1}^{x_2} f'(x)G(x)\,dx$$

- One strategy often works for choosing $f(x)$ and $g(x)$ is known as LIATE.

  choose $f(x)$ to be the function whose category occurs earlier in the following list and take $g(x)$ to be the remaining factor.

  Logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential

Exercise

Use integration by parts to evaluate

(a) $\displaystyle\int x\cos x\,dx$.

(b) $\displaystyle\int x^2e^{-x}\,dx$.

(c) $\displaystyle\int e^x\cos x\,dx$.

- Suppose we would like to apply integration by parts to

$$\begin{aligned}
\int \frac{1}{x}\,dx &= \int 1\cdot\frac{1}{x}\,dx && (1)\\
&= x\cdot\frac{1}{x} + \int x\frac{1}{x^2}\,dx && (2)\\
&= 1 + \int \frac{1}{x}\,dx && (3)\\
\Longrightarrow\ {\color{red}0 = 1} & && (4)
\end{aligned}$$

- Oops! What was the error? Everything is correct until the last step
- The mistake is in assuming $\int f(x)\,dx - \int f(x)\,dx = 0$.
- The correct way to manipulate it is

$$\int f(x)\,dx - \int f(x)\,dx = \int \Big(f(x) - f(x)\Big)\,dx = \int 0\,dx = c$$

Reduction formula for powers of sine and cosine

- For positive integers n, we have

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1}x\cos x + \frac{n-1}{n}\int \sin^{n-2}x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1}x\sin x + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

Proof

- Split a copy of $\cos x$,

$$\int \cos^n x\,dx = \int \cos^{n-1}x\cos x\,dx,$$

- Apply integration by parts

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$$

Proof

- If we let,

$$f(x) = \cos^{n-1} x; \qquad g(x) = \cos x$$
$$f'(x) = -(n-1)\cos^{n-2} x \sin x; \qquad G(x) = \sin x$$

- then

$$\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \sin x \sin x\,dx$$

- With the identity $\sin^2\theta + \cos^2\theta = 1$, we have

$$\boxed{\int \cos^n x\,dx} = \cos^{n-1} x \sin x + (n-1)\left[\int \cos^{n-2} x\,dx - \boxed{\int \cos^n x\,dx}\right]$$
$$\boxed{\int \cos^n x\,dx} = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx \quad \square$$

Odd Power

If both m and n are positive integers, and suppose either m or n is odd, then

$$\int \sin^m x \cos^n x \, dx$$

can be evaluated by

1. splitting off a factor of sine or cosine whichever has the odd power
2. using substitution with the other factor being g, and use

$$\sin^2 \theta + \cos^2 \theta = 1$$

Exercise

Evaluate

$$\int \sin^4 x \cos^5 x \, dx$$

Even Power

If both m and n are even positive integers, then

$$\int \sin^m x \cos^n x \, dx$$

can be evaluated by using

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

1. to eliminate sine
2. then apply the reduction formula for powers of sine and cosine

Exercise

Evaluate

$$\int \sin^2 x \cos^4 x \, dx$$

- Integrals of products of sines and cosines of the form

$$\int \sin mx \cos nx \, dx \qquad \int \sin mx \sin nx \, dx \qquad \int \cos mx \cos nx \, dx$$

can be found by using the trigonometric identities

$$\begin{aligned}
\sin\alpha\cos\beta &= \frac{1}{2}\Big[\sin(\alpha-\beta)+\sin(\alpha+\beta)\Big]\\
\sin\alpha\sin\beta &= \frac{1}{2}\Big[\cos(\alpha-\beta)-\cos(\alpha+\beta)\Big]\\
\cos\alpha\cos\beta &= \frac{1}{2}\Big[\cos(\alpha-\beta)+\cos(\alpha+\beta)\Big]
\end{aligned}$$

Exercise

Evaluate

$$\int \sin 3x \cos 5x \, dx$$

- Integrating tangent and secant closely parallel those for sine and cosine.

Reduction formulae for tangent and secant function

For integer powers $n \geq 2$ of tangent and secant, we use the reduction formulae

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx,$$
$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$$

- When n is even, it is solved by successive applications of the formula.
- When n is odd, the exponent can be reduced to 1, and we then need to solve

$$\int \tan x \, dx \qquad \text{or} \qquad \int \sec x \, dx$$

- Neither $\left(\int \tan x\, dx\right)$ nor $\left(\int \sec x\, dx\right)$ are usually in the derivative table,

$$\int \tan x\, dx = \int \frac{\sin x}{\cos x}\, dx$$

$$u = g(x) = \cos x \implies g' = -\sin x$$

$$\int \tan x\, dx = -\int u^{-1}\, du$$

$$= -\ln|\cos x| + C$$

$$= \ln|\sec x| + C$$

$$\int \sec x\, dx = \int \sec x \frac{\tan x + \sec x}{\tan x + \sec x}\, dx$$

$$= \int \frac{\sec x \tan x + \sec^2 x}{\tan x + \sec x}\, dx$$

$$u = g(x) = \tan x + \sec x$$

$$\implies g' = (\sec^2 x + \sec x \tan x)\, dx$$

$$\int \sec x\, dx = \int u^{-1}\, du$$

$$= \ln|\tan x + \sec x| + C$$

- The following integrals are common, and worth to commit to memory

$$\int \tan^2 x\, dx = \tan x - x + C; \qquad \int \sec^2 x\, dx = \tan x + C$$

Exercise

Find the following integrals

(a)

$$\int \tan^2 x \sec^4 x \, dx$$

(b)

$$\int \tan^3 x \sec^3 x \, dx$$

(c)

$$\int \tan^2 x \sec x \, dx$$

Integrating products of tangents and secants

- If both m and n are even positive integers, then the integral

$$\int \tan^m x \sec^n x \, dx$$

can be evaluated by using the following identity and the reduction formulae

$$\sec^2 x = \tan^2 x + 1$$

- Alternatively,

1. If we have even powers of $\sec x$, then consider the substitution $u = \tan x$
2. If we have odd powers of $\tan x$, then consider the substitution $u = \sec x$
3. If we have even powers of $\tan x$ and odd powers of $\sec x$, then

   use the above identity to reduce the integrand to powers of $\sec x$ alone.

- Trigonometric substitutions is a method in which we replace the variable of integration by a trigonometric function. If x is the variable of integration,

$$x = g(\theta), \qquad \text{where } g(\theta) \text{ is some trigonometric function.}$$

- Notice it is different from the usual substitution

$$u = g(x) \qquad \text{where } g(x) \text{ is a part of the integrand.}$$

- This is useful when one of the followings is a part of the integrand.

$$1. \quad \sqrt{a^2 - x^2}; \qquad 2. \quad \sqrt{a^2 + x^2}; \qquad 3. \quad \sqrt{x^2 - a^2}$$

  where a is a positive constant.

- The corresponding substitutions are

$$1. \quad x = a\sin\theta; \qquad 2. \quad x = a\tan\theta; \qquad 3. \quad x = a\sec\theta$$

- The basic idea for making such a substitution is to eliminate the radical.

Exercise

(a) Find
$$\int \sqrt{4-x^2}\,dx$$

(b) Find
$$\int \frac{dx}{\sqrt{4+x^2}}$$

(c) Evaluate
$$\int_0^{3\sqrt{3}/2} \frac{x^3}{(4x^2+9)^{3/2}}\,dx$$

(d) Find the area of the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$