Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 8

Date Due: 12:55 PM, Wednesday, the 21st of April 2021

Exercises (13 Marks)

Exercise 8.1

Consider the half-disk in $\mathbb{R}^2$,

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 \colon |x|^2 < 1,\ x_2 > 0\}.$$

In the following, polar coordinates (r, θ) given by $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ will be used. You may use that the Laplace operator in these coordinates is given by

$$\Delta_{(r,\theta)} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

which simply means that

$$\Delta_{(r,\theta)} u(r\cos\theta, r\sin\theta) = \Delta u(x_1, x_2)\big|_{(x_1,x_2)=(r\cos\theta, r\sin\theta)}$$

if u is a twice-dfferentiable function on $\mathbb{R}^2$. For short we will write $u(r, \theta)$ to refer to u on Ω and omit the polar coordinate composition.

i) In euclidean coordinates in $\mathbb{R}^2$, it is easy to see that we can symbolically write

$$\delta(x - \xi) = \delta\big((x_1, x_2) - (\xi_1, \xi_2)\big) = \delta(x_1 - \xi_1)\delta(x_2 - \xi_2).$$

Explain why in polar coordinates this becomes

$$\delta(x - \xi) = \frac{1}{r}\delta(r - \varrho)\delta(\theta - \vartheta)$$

if $x = (r\cos\theta, r\sin\theta)$ and $\xi = (\varrho\cos\vartheta, \varrho\sin\vartheta)$.

Green's function for the Dirichlet problem on Ω satisfies

$$-\Delta g(x; \xi) = \delta(x, \xi), \qquad x, \xi \in \operatorname{int}\Omega, \qquad\qquad g(\,\cdot\,, \xi)\big|_{\partial\Omega} = 0.$$

in euclidean coordinates, or

$$-\Delta_{(r,\theta)} g(r, \theta; \varrho, \vartheta) = \frac{1}{r}\delta(r - \varrho)\delta(\theta - \vartheta), \qquad 0 < r, \varrho < 1,\ 0 < \theta, \vartheta < \pi$$

and

$$g(1, \theta; \varrho, \vartheta) = 0, \qquad 0 < \varrho < 1,\ 0 < \theta, \vartheta < \pi, \quad g(r, 0; \varrho, \vartheta) = g(r, \pi; \varrho, \vartheta) = 0, \qquad 0 < r, \varrho < 1,\ 0 < \vartheta < \pi,$$

in polar coordinates.

ii) Separate variables in the Dirichlet problem

$$\Delta_{(r,\theta)} u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

and find the angular eigenfunctions and the eigenvalues.

iii) Give the formal partial eigenfunction expansion for Green's function in terms of suitable θ eigenfunctions (use the boundary conditions for g). Do not yet determine the coefficient functions.

iv) Determine the one-dimensional Green's function problem that the coefficients must satisfy.

v) Solve the Green's function problem and find the coefficients, giving the partial eigenfunction expansion.

vi) Use the Green's function expansion to find the solution to the Dirichlet problem in i) above.

vii) Plot (e.g., using Mathematica)) the first few terms of the solution.

(13 Marks)