1. Limit of sequence

① as close as you wish 要多近有多近

precise definition

A sequence $\{a_n\}$ has the limit $L \in R$.
if for $\forall \varepsilon > 0$. there is a corresponding integer N such that.
if $n > N$. then $|a_n - L| < \varepsilon$

notation

$$\lim_{n\to\infty} a_n = L. \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

Examples

1. $\lim_{n\to\infty} \frac{1}{n}$
2. $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n}$
3. Q8 in worksheet.
4. Q6
5. $\{a_n\}$. $a_n = n^{\frac{1}{2}} \sin(\frac{n}{2}\pi)$, $\lim_{n\to\infty} a_n$

Solutions

$\lim_{n\to\infty} \frac{1}{n}$

1. ~~for every $\varepsilon > 0$~~.

We need to show. for every $\varepsilon > 0$, there exists an N such that

$$\left|\frac{1}{n} - 0\right| < \varepsilon \quad \text{if } n > N$$

Since

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n}$$

for a given ε, there exists N such that.

$$\left|\frac{1}{n} - 0\right| < \frac{1}{N} < \varepsilon$$

Hence not only such N exists. every natural number $N > \frac{1}{\varepsilon}$ is valid.

2. $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n}$

1° ~~for every $\varepsilon>0$~~,

We need to show that for every $\varepsilon>0$, there exists an N such that

$$\left|\frac{\sqrt{n^2+1}}{n}-1\right|<\varepsilon$$

$1-1<\sqrt{1+\frac{1}{n^2}}-1<\varepsilon$?

what's wrong with

Since

$$\left|\frac{\sqrt{n^2+1}}{n}-1\right|=\sqrt{1+\frac{1}{n^2}}-1<1+\frac{1}{n^2}-1=\frac{1}{n^2}$$

欲证A比B小
先证A比C小
再证C比B小
(valid)
找一个比A小的D
证D比B小
not valid.

for a given ε, there exists such N that

$$\left|\frac{\sqrt{n^2+1}}{n}-1\right|<\frac{1}{N^2}<\varepsilon$$

Hence not only N exists, every natural number $N>\frac{1}{\sqrt{\varepsilon}}$ is valid.

3. $\lim_{k\to\infty} x_{2k-1}=a \qquad \lim_{k\to\infty} x_{2k}=a$

there exists k_1, such that $|x_{2k-1}-a|<\varepsilon$ if $k>k_1$.

exists k_2 such that $|x_{2k}-a|<\varepsilon$ if $k>k_2$

$\therefore$ if we take $N=\max\{2k_1-1, 2k_2\}$

$$|x_n-a|<\varepsilon \text{ for all } n>N.$$

4. Q6. $\forall\varepsilon>0$ (suppose $\varepsilon<1$)

$$|x_n-0|=|q^{n-1}-0|=|q|^{n-1}$$

We need to show that for every $\varepsilon>0$, there exists an N such that

$$|q^{n-1}-0|<\varepsilon$$

Since $|q^{n-1}-0|=|q|^{n-1}<\varepsilon<1$

$$(n-1)\ln|q|<\ln\varepsilon$$

$$n>\frac{\ln\varepsilon}{\ln|q|}+1 \qquad (\ln|q|<0)$$

take $N=\left[\frac{\ln\varepsilon}{\ln|q|}+1\right]$, then we have $|q^{n-1}-0|<\varepsilon$ if $n>N$

terms

convergent / divergent.

$\boxed{\text{Careful}}$: $\lim_{n\to\infty} a_n = \infty$. also divergent.

for $\forall M \in R$
$a_n > M$ for all $n > N$
$\Rightarrow \lim_{n\to\infty} a_n = +\infty$

for $\forall m \in R$
$a_n < m$ for all $n > N$
$\Rightarrow \lim_{n\to\infty} a_n = -\infty$

Note

1. make use of $[\quad]+1$ to represent N.
(比原值大，是整数)
2. 用好 $N = \max\{\cdots,\cdots\}$
3. 放缩的方向
4. a^n 取 ln (自然对数) 注意变号，不等式方向 $\ln a$.

Limit law

条件.
1° 确保 La, Lb 存在.
$\lim_{n\to\infty} a_n = La$
$\lim_{n\to\infty} b_n = Lb$ convergent

1. $\lim_{n\to\infty} a = a$.
2. $\lim_{n\to\infty} a_n \pm b_n = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n = La \pm Lb$
3. $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n = La\,Lb$
4. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} a_n \Big/ \lim_{n\to\infty} b_n = \frac{La}{Lb}$ ($Lb \neq 0$, $b_n \neq 0$)

Examples

1. $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$
2. $\lim_{n\to\infty} \frac{n^2+7n}{n^3}$
3. ~~$\lim_{n\to\infty} \frac{3^n + 5^n}{5^{n+1} - 3^n}$~~

3. Q4. & $\lim_{n\to\infty} \frac{1+n+n^2+\cdots+n^a}{1+n+n^2+\cdots+n^b}$ $(2<a<b)$
4. Q5
5. Q9.

① $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \infty - \infty$

分子有理化(处理根号).

$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$

$= \frac{\lim_{n\to\infty} 1}{\lim_{n\to\infty}\sqrt{n+1} + \lim_{n\to\infty}\sqrt{n}} = 0$.

② $\lim\limits_{n\to\infty} \dfrac{n^2+7n}{n^3 \cancel{+7n}}$

$= \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}+\frac{7}{n^2}}{1}$

$= 0$

Note: 数列中， $\lim\limits_{n\to\infty} \dfrac{a_1 n^k + b_1 n^{k-1} + \dots + c_1 n}{a_2 n^m + b_2 n^{m-1} + \dots + c_2 n}$

$(a_1, a_2 \neq 0)$

$k > m \Rightarrow \pm\infty \begin{cases} +\infty & a_1 \cdot a_2 > 0 \\ -\infty & a_1 \cdot a_2 < 0 \end{cases}$

$k < m \Rightarrow 0$

$k = m \Rightarrow \dfrac{a_1}{a_2}$

③ $\lim\limits_{n\to\infty} \dfrac{(-2)^n + 3^n}{(-2)^{n+1} + 3^{n+1}}$

$= \lim\limits_{n\to\infty} \dfrac{\frac{1}{3}(-\frac{2}{3})^n + \frac{1}{3}}{(-\frac{2}{3})^{n+1} + 1}$

$= \dfrac{1}{3}$

Note. distingush $\underline{\underline{a^n \ \& \ n^k}}$.

See the next case.

④ $\lim\limits_{n\to\infty} \dfrac{1+a+a^2+\dots+a^n}{1+b+b^2+\dots+b^n}$ & $\lim\limits_{n\to\infty} \dfrac{1+n+n^2+\dots+n^a}{1+n+n^2+\dots+n^b}$ $2<a<b$ (~~a=b>2~~)

$= \lim\limits_{n\to\infty} \dfrac{\frac{1-a^{n+1}}{1-a}}{\frac{(1-b^{n+1})}{1-b}}$

$= \dfrac{1-b}{1-a}$

divided by n^b

$= \lim\limits_{n\to\infty} \dfrac{0+0+\dots+0}{0+0+\dots+1} = 0$

⑤. $\lim\limits_{n\to\infty} \dfrac{\frac{1}{3}n(2n+1)(2n-1)}{2n^3} = \dfrac{2}{3}$

how to derive?

1° $1^2+2^2+\dots+n^2 = \frac{1}{6}n(2n+1)(n+1) = S_1$

2° $1^2+2^2+\dots+(2n)^2 = \frac{1}{6}(2n)(4n+1)(2n+1) = S_2$

连偶^2 $2^2+4^2+\dots+(2n)^2 = 4S_1 = \frac{2}{3}n(2n+1)(n+1) = S_3$

Squeeze theorem

suppose $\{a_n\}$ and $\{b_n\}$ are convergent.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$$

if for some $N \in \mathbb{N}$. $a_n \le c_n \le b_n$ for all $n > N$.

then $\lim_{n\to\infty} c_n = L$.

Examples

1. $\lim_{n\to\infty} \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n}$

2. $\lim_{n\to\infty} \frac{\sqrt{n^2+1}}{n}$

2. $\frac{\sqrt{n^2+1}}{n} = \sqrt{1+\frac{1}{n^2}}$

$$1 \le \sqrt{1+\frac{1}{n^2}} \le 1+\frac{1}{n^2}$$

$\lim_{n\to\infty} 1 = 1$

$\lim_{n\to\infty} 1+\frac{1}{n^2} = 1$

$\therefore \lim_{n\to\infty} \sqrt{1+\frac{1}{n^2}} = 1$

$$0 < \frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{2n+1}}.$$

Mathematical Induction.

when $n=1$ $0<\frac{1}{2}<\frac{1}{\sqrt{3}}$. valid.

suppose when $n=k$ $\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2k-1}{2k} < \frac{1}{\sqrt{2k+1}}$

then when $n=k+1$

$$\frac{1}{2}\cdots\frac{2k+1}{2k+2} < \frac{1}{\sqrt{2k+1}}\cdot\frac{2k+1}{2k+2} < \frac{1}{\sqrt{2k+3}}$$

since

$$\sqrt{(2k+1)(2k+3)} < \sqrt{(2k+2)^2}.$$

then $\lim_{n\to\infty} 0 = 0$

$$\lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = \frac{1}{\infty} = 0$$

Monotonic sequence theorem

A monotonic ① sequence converges if and only if it is bounded ②

① $a_{n+1} \geq a_n$ or $b_{n+1} \leq b_n$ for all n

② If an unempty set has a lower bound, it also has an infimum

③ infimum. for set A

1° $\inf(A) \leq a_n$ for all $a_n \in A$.

2° for every $\varepsilon > 0$. there exists a^*

$a^* < \inf(A) + \varepsilon$

Proof. decreasing + bounded from below $\Longrightarrow$ convergent.

$\inf\{x_n\} = L$

$\therefore$ for any ε. there exists x_N

$L \leq x_N < L + \varepsilon$

and since x_n is decreasing & bounded from below

$\therefore L \leq x_n \leq x_N < L + \varepsilon$ for all $n > N$

$\therefore 0 \leq x_n - L < \varepsilon$

$|x_n - L| < \varepsilon$ for all $n > N$.

$\therefore \lim_{n\to\infty} x_n = L$.

~~Other properties~~

[redacted]

Q3. sub-sequence.

在数列中抽取无限多项并保持其原有顺序

monotonic & converges $\Rightarrow$ bounded + monotonic $\Longrightarrow$ bounded

(sub [redacted]). (原数列).

假设 increasing + 有上界.

$x_1, x_2 \dots x_i, x_{i+1}, \dots x_n \dots x_k \dots$

对原数列任一项，都之后都有一项 x_k 在子数列，设子数列上界为 L

$x_i \leq x_k \leq L$ ∴原数列 bounded.

Q3 sub-sequence

从序数列中选取无限多项按原顺序组成子数列

子数列

monotonic + convergent $\Longrightarrow$ bounded + monotonic $\overset{?}{\Longrightarrow}$ bounded

子数列 原数列 原数列

+ monotonic $\Rightarrow$ convergent.

原数列.

? set L as the lower bound of $\{x_n\}$ and suppose it is decreasing (subsequence)

$L \le x_n$ for all n

the original sequence $\{a_n\}$

for any $a_n \in \{a_n\}$ there is at ~~example~~ least one $\underbrace{a_i \in \{x_n\}}_{①}$ $\underbrace{\text{behind it}}_{②}$

~~$L \le a_n \le a_i$~~

$L \overset{①}{\le} a_i \overset{②}{\le} a_n$

Other properties

1. Show that if a sequence is convergent, it is bounded.

~~Since $\{x_n\}$ is convergent.~~
~~Let $\{x_n\}$ converges to L.~~

1. Let $\{x_n\}$ denotes a sequence that converges to L. Then there exits $N \in \mathbb{N}$ such that.

$$|x_n - L| < 1$$

for all $n > N$. By triangle inequality

$$|x_n| - |L| \le |x_n - L| < 1$$

Therefore when $n > N$.

$$|x_n| < 1 + |L|$$

which shows

$$M = \max\{|x_1|, |x_2|, \ldots, |x_N|, 1+|L|\}$$

can be used as a bound for $|x_n|$ and $\{x_n\}$ is bounded

2. Let $\lim_{n\to\infty} e_n = L$

Since $e_n \to L$ as $n \to \infty$. there exists N such that.

$$n > N \Rightarrow |e_n - L| < \varepsilon$$

By the reverse triangle inequality, we have

$$n > N \Rightarrow \big||e_n| - |L|\big| \le |e_n - L| < \varepsilon.$$

$$\therefore \lim_{n\to\infty} |e_n| = |\lim_{n\to\infty} e_n| = |L|$$

$|a| - |b| \le |a-b|$

Note

$\big||a| - |b|\big| \le |a-b|$

3. Let $\{x_n\}$ denotes a sequence that converges

suppose there exists two limits L_a and L_b

and $L_a < L_b$

for $\varepsilon = \frac{L_b - L_a}{2}$ since $\lim_{n\to\infty} x_n = L_a$ there exists $N_a \in \mathbb{N}$ such that.

$$n > N_a \Rightarrow |x_n - L_a| < \frac{L_b - L_a}{2} \Rightarrow x_n < \frac{L_a + L_b}{2}$$

Similarly. there exists N_b. such that

$$n > N_b \Rightarrow |x_n - L_b| < \frac{L_b - L_a}{2} \Rightarrow x_n > \frac{L_a + L_b}{2}.$$

By setting $N = \max\{N_a, N_b\}$ we reach a contradiction that.

$$x_n < \frac{L_a + L_b}{2} \quad \text{and} \quad x_n > \frac{L_a + L_b}{2}$$

Therefore if a sequence is convergent. its limit is unique.

$\varepsilon = -\frac{a}{2}$ $\quad \exists N \quad (a<0)$.

$n>N \Rightarrow |x_n - a| < -\frac{a}{2}$

$x_n < a - \frac{a}{2} = \frac{a}{2} < 0$