Question 1.1
Prove or disprove:
If A is an invertible 3×3 matrix, then its rows form a basis of $\mathbb{R}^3$.
(6 points)

$\dim \mathbb{R}^3 = 3$, any 3 lin. ind. vectors form a basis

Question 1.2
Prove or disprove:

$$\dim Im\, A = \dim Im\, rref\, A = \text{\# lin. ind. columns}$$

(6 points)

$\parallel$ # lin. independent columns $=$ equal

Question 1.3
Prove or disprove:
There exists an invertible 3×3 matrix, for which 7 of the 9 entries are the same.
(6 points)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Question 1.4
Prove or disprove:
There is a 3×3 matrix $A: \quad Im\, A = Ker\, A$
(6 points)

If $\dim Im A = \dim Ker A$

$A: \mathbb{R}^3 \to \mathbb{R}^3 \qquad 3 = \underbrace{2 \dim Ker A}_{\notin \mathbb{N}} \leftarrow$ not possible

Question 1.5
Prove or disprove:
There is a 6×6 matrix A : $\quad Im\, A = Ker\, A$.
(6 points)

$6 = 2 \dim Ker A \Rightarrow \dim Ker A = 3$

$$A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Question 1.6
Prove or disprove:

$$a.\{(1,1,1)\} \quad b.\{(1,2,1),(2,0,-1),(4,4,1)\}$$

are the spanning sets of $\mathbb{R}^3$.
(6 points)

no

$1 < \dim \mathbb{R}^3$

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 \\ 0 & -4 & -4 \\ 0 & -3 & -3 \end{pmatrix} \Rightarrow \text{rank} < 3$$

$\Rightarrow$ (b) is not lin. ind.

Question 1.7
Prove or disprove:

$$a.\{(1,2,1),(2,0,-1),(4,4,0)\}. \quad b.\{(1,0,0),(0,1,0),(0,0,1),(1,1,1)\}$$

are the spanning sets of $\mathbb{R}^3$.
(6 points)

NO $\quad 4 > \dim \mathbb{R}^3$

$$\begin{pmatrix} 1 & 2 & 4 \\ 2 & 0 & 4 \\ 1 & -1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 \\ 0 & -4 & -4 \\ 0 & -3 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$\Rightarrow \text{rank} = 3$
$\Rightarrow$ (b) is lin. independent

Question 1.8
Prove or disprove:
The plane $x + 3y - 5z = 1$ is a subspace of $\mathbb{R}^3$.
(6 points)

$(0,0,0) \notin P: x+3y-5z=1$

Question 1.9
Prove or disprove:

$$A = (\bar{a}_1 \dots \bar{a}_m), \quad \exists A^{-1}_{m\times m} \Rightarrow \mathbb{R}^m = span(\bar{a}_1, \dots, \bar{a}_m)$$

(6 points)

$\exists A^{-1} \Rightarrow$ ref $A = I_n \Rightarrow$ m lin. ind. columns in A

Question 1.10
Prove or disprove:

$$A, B \in \mathbb{M}_{n\times n}(\mathbb{R}): \quad AB = 0 \Rightarrow A = 0 \text{ or } B = 0$$

(6 points)

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Question 1.11
Prove or disprove:
If a linear map is one-to-one and onto, then its inverse is also one-to-one and onto.
(6 points)

Question 1.12
Prove or disprove: If $\{x_1, x_2, x_3\}$ is the basis of a linear space X, then $\{x_1, x_1 - x_2, x_1 - x_2 - x_3\}$ is also a basis of X.
(6 points)

$\alpha x_1 + \beta(x_1 - x_2) + \gamma(x_1 - x_2 - x_3) = 0$

$(\alpha + \beta + \gamma)x_1 + (-\beta - \gamma)x_2 - \gamma x_3 = 0$

$\Rightarrow \alpha + \beta + \gamma = 0, \; -\beta + \gamma = 0, \; \gamma = 0 \Rightarrow \alpha = \beta = \gamma = 0$

Question 1.13
Prove or disprove:

If there exists a solution to $A^2\bar{x} = \bar{b}$ then the system $A\bar{x} = \bar{b}$ is consistent. (6 points)

$$\exists \bar{x}_0 : A^2\bar{x}_0 = \bar{b} \Rightarrow A\underbrace{(A\bar{x}_0)}_{\bar{y}} = \bar{b} \Rightarrow \exists \bar{y} : A\bar{y} = \bar{b}$$

Question 1.14
Prove or disprove:
If $dimX = n$ and S is a linearly independent set in X, then S is the basis in X.
(6 points)

dim S must be n

Question 1.15
Prove or disprove:
The linear space of all polynomials is infinite dimensional.
(6 points)

$1, t, \dots, t^n, \dots$ are lin. independent

Question 1.16
Prove or disprove:
If A is a matrix of a linear operator $T : V \to W$ then A^T is the matrix of T^{-1}.
(6 points)

$A^T : W' \to V'$

Question 1.17
Prove or disprove:

$$\exists A^{-1}, B^{-1} \Rightarrow \exists (A + B)^{-1}$$

(6 points)

$$\nexists (A+B)^{-1}$$
$$\Uparrow$$
$$A+B = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Rightarrow \exists A^{-1}; \quad B = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} \Rightarrow \exists B^{-1}$$

Question 1.18
Prove or disprove:

$\forall \dim X, \dim Y < +\infty$

$$\dim L(X, Y) = \dim X \cdot \dim Y$$

(6 points)

$L(X,Y) \cong \mathbb{M}_{m\times n} \Rightarrow \dim L(X,Y) = \dim \mathbb{M}_{m\times n} = m\cdot n$

$\dim X = n \quad \dim Y = m$ $\qquad = \dim X \cdot \dim Y$

Question 1.19
Prove or disprove:

$$rank \begin{array}{c} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 1 \\ 3 & 6 & 1 \end{pmatrix} \\ \begin{array}{ccc} \bar{a}_1 & \bar{a}_2 & \bar{a}_3 \end{array} \end{array} = 3$$

(6 points)

No as $\bar{a}_1$ and $\bar{a}_2$ are lin. dependent

Question 1.20
Prove or disprove:

$$rref\ A \cdot B = (rref\ A) \cdot (rref\ B)$$

(6 points)

Ex: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = rref\ A$; $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow rref\ B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$AB = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = rref(AB) \neq (rref\ A)(rref\ B) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Question 1.21
Prove or disprove:

$$\exists (AB)^{-1} \Rightarrow \exists A^{-1}$$

(6 points)

$(AB)(AB)^{-1} = J \Rightarrow A(B(AB)^{-1}) = J \Rightarrow \exists X : AX = J \Rightarrow X = A^{-1}$

$X = B(AB)^{-1}$

Question 2A

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that:

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} \quad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix} \quad T\begin{pmatrix}2\\-1\\2\end{pmatrix} = \begin{pmatrix}1+4c\\2+2c\end{pmatrix}$$

1. Find the matrix representation of T.
2. Is T one-to-one and onto?

(20 marks)

$A = (T\bar{e}_1 \quad T\bar{e}_2 \quad T\bar{e}_3)$

$(2,-1,2) = 2\bar{e}_1 - \bar{e}_2 + 2\bar{e}_3 \Rightarrow \underbrace{T\begin{pmatrix}2\\-1\\2\end{pmatrix}}_{\begin{pmatrix}1+4c\\2+2c\end{pmatrix}} = 2T\bar{e}_1 - T\bar{e}_2 + 2T\bar{e}_3 = \begin{pmatrix}4\\8\end{pmatrix} - \begin{pmatrix}3\\6\end{pmatrix} + 2T\bar{e}_3$

$\Rightarrow T\bar{e}_3 = \begin{pmatrix}2c\\c\end{pmatrix}$

$A = \begin{pmatrix}2&3&2c\\4&6&c\end{pmatrix} \sim \begin{pmatrix}1&3/2&0\\0&0&1\end{pmatrix} \Rightarrow r(A) = \dim \operatorname{Im} A = 2$

$\dim \mathbb{R}^3 = 3 = \dim \operatorname{Ker} A + \dim \operatorname{Im} A$

$\Rightarrow \dim \operatorname{Ker} A = 1 \Rightarrow \operatorname{Ker}\{A\} \neq \{0\} \Rightarrow T$ is not injective

$\forall \bar{y} = \begin{pmatrix}y_1\\y_2\end{pmatrix} \in \mathbb{R}^2 \;\; \exists x \in \mathbb{R}^3 : A\bar{x} = \bar{y} \Rightarrow \begin{matrix}2x_1+3x_2+2cx_3=y_1\\4x_1+6x_2+cx_3=y_2\end{matrix}$ if $c=0$ then $y_2-2y_1=0 \Rightarrow T$ is not onto

Question 2B

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that:

$$T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}2\\4\end{pmatrix} \quad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\6\end{pmatrix} \quad T\begin{pmatrix}2\\-1\\2\end{pmatrix} = \begin{pmatrix}1+4c\\2+2c\end{pmatrix}$$

1. Find the matrix representation of T.
2. Is T one-to-one and onto?

(20 marks)

$A = (T\bar{e}_1 \quad T\bar{e}_2 \quad T\bar{e}_3)$

$(2,-1,2) = 2\bar{e}_1 - \bar{e}_2 + 2\bar{e}_3 \Rightarrow T\begin{pmatrix}2\\-1\\2\end{pmatrix} = 2T\bar{e}_1 - T\bar{e}_2 + 2T\bar{e}_3$

$\begin{pmatrix}1+4c\\2+2c\end{pmatrix} = 2T\bar{e}_1 - \begin{pmatrix}2\\4\end{pmatrix} + \begin{pmatrix}6\\12\end{pmatrix} \Rightarrow T\bar{e}_1 = \begin{pmatrix}2c-3/2\\c-3\end{pmatrix}$

$A = \begin{pmatrix}2c-3/2&2&3\\c-3&4&6\end{pmatrix}$ $r(A) = 2 = \dim \operatorname{Im} A$

$\Rightarrow \dim \mathbb{R}^3 = 3 = \dim \operatorname{Ker} A + \dim \operatorname{Im} A \Rightarrow \dim \operatorname{Ker} A = 1 \Rightarrow \operatorname{Ker}\{T\} \neq \{0\}$

T is not injective

$\forall y \in \mathbb{R}^2 \; \exists \bar{x} \in \mathbb{R}^3 : A\bar{x} = \bar{y}$?

$\left(\begin{array}{ccc|c}2c-3/2&2&3&y_1\\c-3&4&6&y_2\end{array}\right) \sim \left(\begin{array}{ccc|c}2c-3/2&2&3&y_1\\-3c&0&0&y_2-2y_1\end{array}\right) \Rightarrow$ if $c=0$ then $y_2-2y_1=0 \Rightarrow \bar{y}$ is not any

$\Rightarrow T$ is not onto

Question 3A

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x,y,z) = (2x - 4y + 3z + \alpha, 7x + \beta xyz).$$

Is T linear? (15 marks)

Iff $\alpha = \beta = 0$

Question 3B

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by

$$T(x, y, z) = (3x - 5y + 2z + \beta, 4x + \alpha xyz).$$

Is T linear? (15 marks)

Iff $\alpha = \beta = 0$

Question 4

The matrix $AB + B^T A^T$ is always symmetric.
(15 marks)

A is symmetric $\iff A = A^T$

$(AB + B^T A^T)^T = (AB)^T + (B^T A^T)^T = B^T A^T + (A^T)^T (B^T)^T$
$= B^T A^T + AB = AB + B^T A^T$

Question 5

Can $L(V, W)$ be finite dimensional if $0 < \dim V < \infty$ and $\dim W = \infty$ (20 points)

No

$L(V, W) = \{T : V \to W, T \text{ is linear}\}$

Let $\dim V = n$, $\{v_1, \ldots, v_n\} \subset V$ be a basis

and let $w_1, \ldots, w_n, \ldots$ be independent in W
(not finite)

Let $T_i : V \to W$: $T_i(v_j) = \begin{cases} w_i & j = 1 \\ 0 & j \neq 1 \end{cases}$

If $\alpha_1 T_1 + \cdots + \alpha_N T_N = 0 \Rightarrow \alpha_1 T_1(v_1) + \alpha_2 T_2(v_1) + \cdots + \alpha_N T_N(v_1) = 0$
$\alpha_1 w_1 + \alpha_2 w_2 + \cdots + \alpha_N w_N = 0$

$\{w_1, \ldots, w_n, \ldots\}$ lin. independent $\Rightarrow \forall N$ $\{w_1, \ldots, w_N\}$ is lin. indep.
$T_1, \ldots, T_N$ is lin. ind. $\forall N \Leftarrow$ $\Rightarrow \alpha_1 = \cdots = \alpha_N = 0$
$\Rightarrow \dim L(V, W) = \infty$