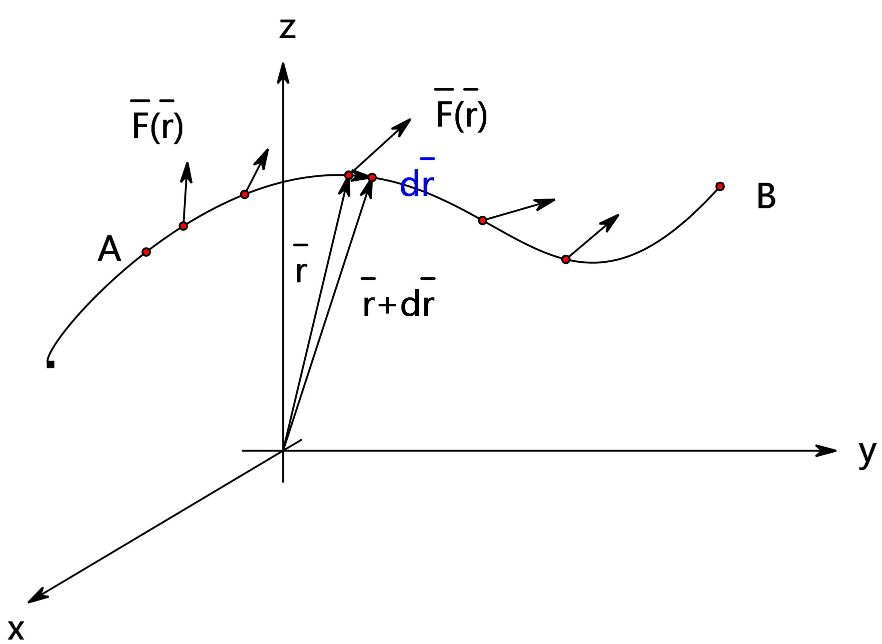
Mechanics:

- Kinetics $(\bar{r}, \bar{v}, \bar{a})$ How?
- Dynamics $(\bar{F}, \bar{a} = \frac{\bar{F}}{m})$ Why?

Scalar quantites:

- Concept of work: how to measure the "effort" put into moving the particle from one place to another?
- Kinetic energy and work-k.e theorem : how this "effort" changes the state of the particle(eg. its velocity)

1 Work



In general, $\bar{F} = \bar{F}(\bar{r})$ (variable force; vector field)
Elementary work done by $\bar{F}$ when the particle moves from $\bar{r}$ to $\bar{r} + d\bar{r}$

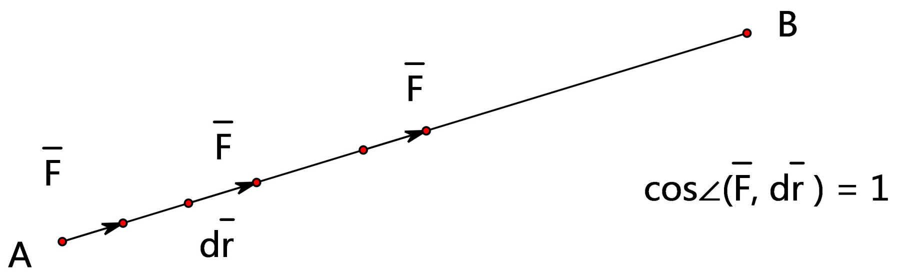
$$\delta W = \bar{F} \circ d\bar{r}$$

Total work done by $\bar{F}$ when the particle moves from A to B along the curve Γ_{AB}–add all infinitesimal contributions

$$W_{AB} = \int_{\Gamma_{AB}} \bar{F} \cdot d\bar{r}$$

How to calculate work in various situations?

1.1 Constant force acting in the direction of the displacement; displacement along a stright line



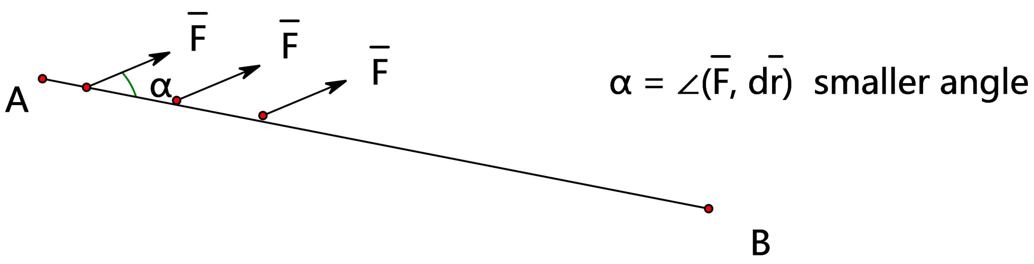
$$\delta W = \bar{F} \cdot d\bar{r} = |\bar{F}| \cdot |d\bar{r}|$$

$$W_{AB} = \int_{\Gamma_{AB}} \bar{F} \cdot d\bar{r} = \int_{\Gamma_{AB}} |\bar{F}||d\bar{r}| = |\bar{F}| \underbrace{\int_{\Gamma_{AB}} |d\bar{r}|}_{S_{AB}-\text{the length of AB}} = |\bar{F}|S_{AB}$$

In this case

$$W_{AB} = |\bar{F}|S_{AB}$$

1.2 Constant force acting at an angle to the direction of stright-line displacement



$$\delta W = \bar{F} \cdot d\bar{r} = |\bar{F}| \cdot |d\bar{r}|cos\alpha$$

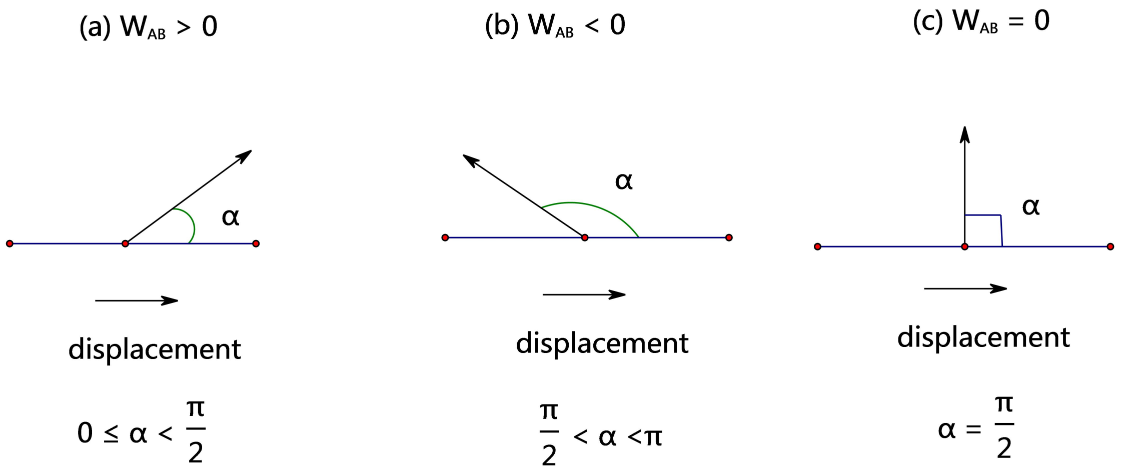
$$W_{AB} = \int_{\Gamma_{AB}} \bar{F} \cdot d\bar{r} = \int_{\Gamma_{AB}} |\bar{F}| \cdot |d\bar{r}|cos\alpha = |\bar{F}|cos\alpha \underbrace{\int_{\Gamma_{AB}} |d\bar{r}|}_{S_{AB}-\text{length of AB}} = |\bar{F}|S_{AB}cos\alpha$$

In this case

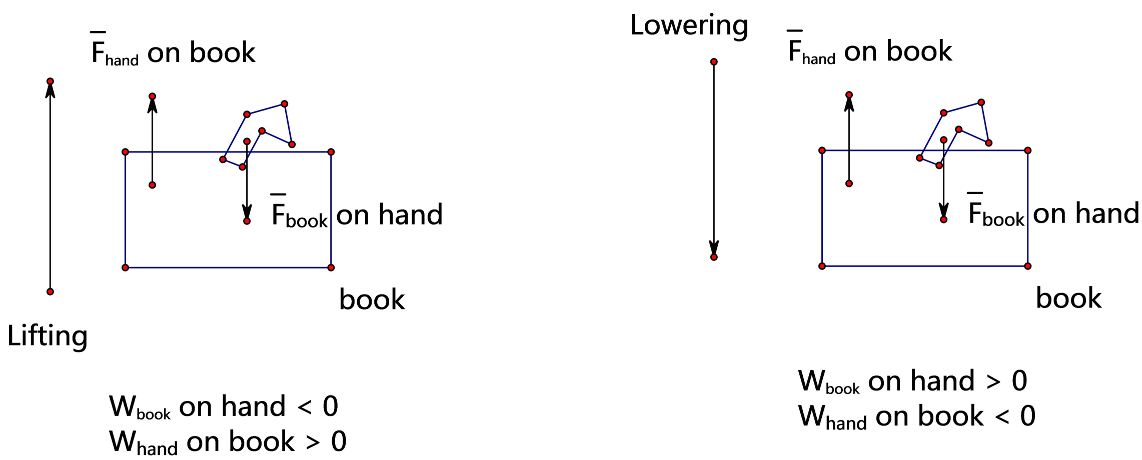
$$W_{AB} = |\bar{F}|S_{AB}cos\alpha$$

Conclusion:

$$work \quad can \quad be \begin{cases} < 0 \\ = 0 \\ > 0 \end{cases}$$



Example: positive/negetive work



Remember what does work on what!

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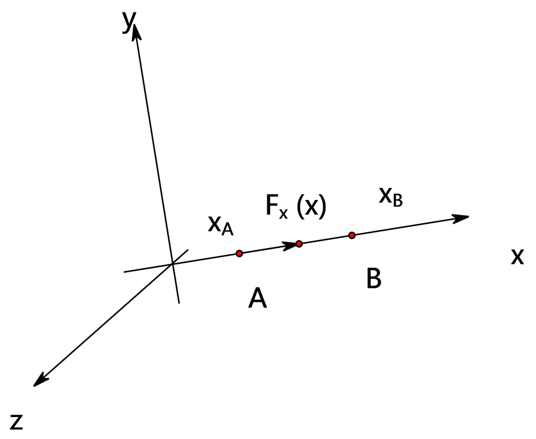
Work done in the case when many forces act on a particle

$$\begin{aligned} \delta W &= (\bar{F}_1 + \bar{F}_2 + ... + \bar{F}_N) \cdot d\bar{r} = \bar{F}_1 \cdot d\bar{r} + \bar{F}_2 \cdot d\bar{r} + ... + \bar{F}_N \circ d\bar{r} \\ &= \delta W_1 + \delta W_2 + ... + \delta W_N \end{aligned}$$

or, equivalently,

$$\delta W = \bar{F}_{net} \cdot d\bar{r}$$

1.3 Force with varing magnitude acting along a stright-line path



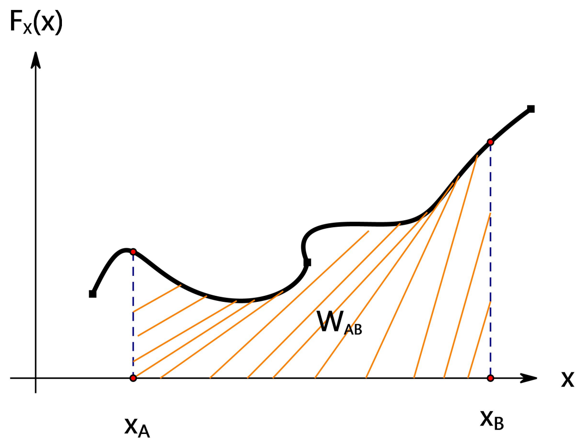
$$\bar{F} = (\underbrace{F_x(x)}_{\text{depends on x only}}, 0, 0)$$

$$d\bar{r} = (dx, 0, 0)$$

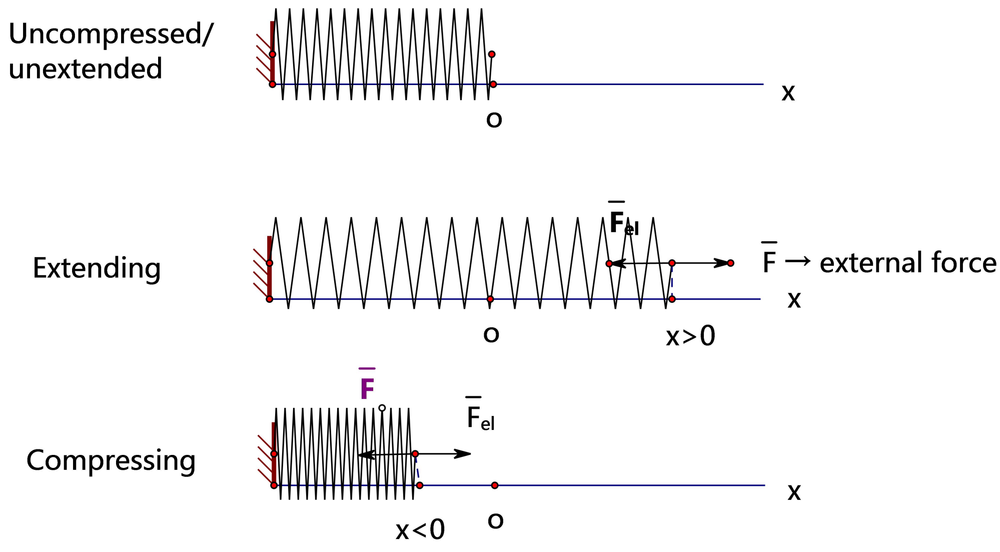
$$\delta W = \bar{F} \circ d\bar{r} = F_x(x)dx \quad \text{(takes care of the sign)}$$

$$W_{AB} = \int_{\Gamma_{AB}} \delta W = \int_{\Gamma_{AB}} F_x(x)dx = \int_{x_B}^{x_A} F_x(x)dx = \int_{x_B}^{x_A} F_x(x)dx$$

Interpretation:



Example: Work doen on/by a spring

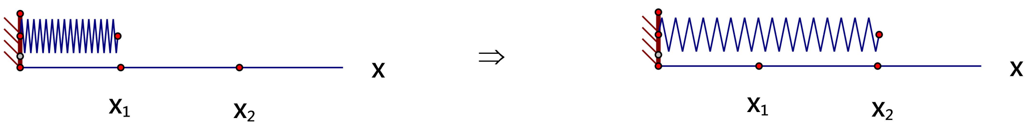


$$\underbrace{\bar{F}_{el}}_{\text{elastic (restoring force)}} = (-kx, 0, 0)$$

$$\bar{F} = (kx, 0, 0)$$

$$d\bar{r} = (dx, 0, 0)$$

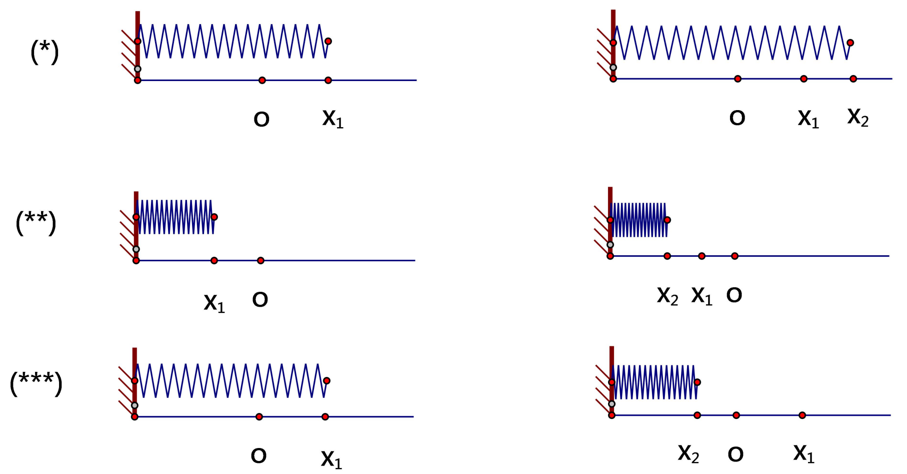
If the position of the right (free) end ofthe spring changes from x_1 to x_2



Work done by

- $\rightarrow$ external force (hand) on spring $\quad W^{ext}_{1\rightarrow 2} = \int_{x_1}^{x_2} kxdx = \frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2$
- $\rightarrow$ elastic force (spring) on hand $\quad W^{ext}_{1\rightarrow 2} = \int_{x_1}^{x_2} -kxdx = -(\frac{1}{2}kx_2^2 - \frac{1}{2}kx_1^2) = -W^{ext}_{1\rightarrow 2}$

Illustration:

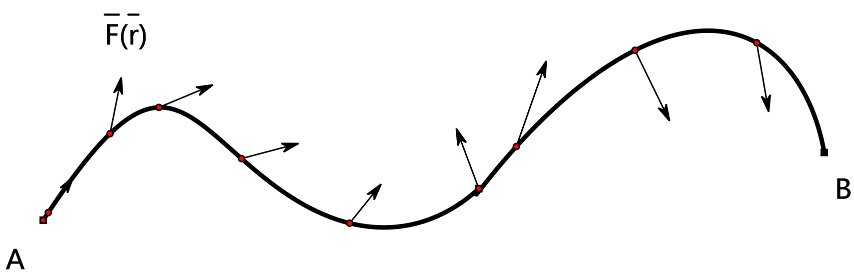


(*) $W_{1\to2}^{ext} > 0$, $W_{1\to2}^{el} < 0$

(**) $W_{1\to2}^{ext} > 0$, $W_{1\to2}^{el} < 0$

(***) $W_{1\to2}^{ext} < 0$, $W_{1\to2}^{el} > 0$

1.4 the most general case - variable force along a curve

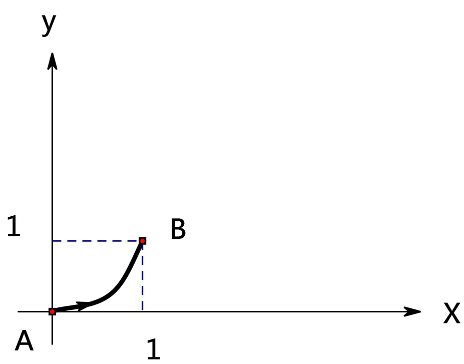


$$\delta W = \bar{F} \circ d\bar{r} \qquad \bar{F} = \bar{F}(\bar{r}) \neq const$$

$$W_{AB} = \int_{\Gamma_{AB}} \bar{F} \cdot d\bar{r}$$

Example:

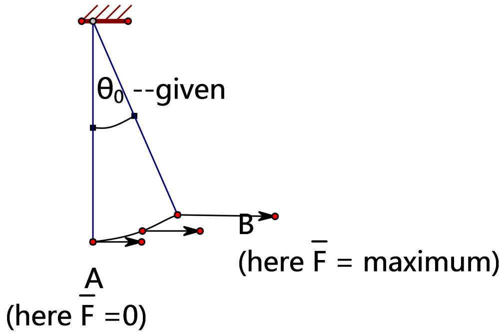
a) $\bar{F}(\bar{r}) = (x^2 + y^2)\hat{n}_x + x\hat{n}_y$ [N] from (0,0) to (1,1) along a segment of parebola $y = x^2$



$$\Gamma_{AB} \begin{cases} y = & x^2 \\ 0 \leq x & \leq 1 \end{cases}$$

$$W_{AB} = \int_{\Gamma_{AB}} \bar{F} \cdot d\bar{r} = \int_{\Gamma_{AB}} F_x dx + F_y dy = \int_{\Gamma_{AB}} (x^2 + y^2) dx + x dy$$
$$= \int_0^1 [(x^2 + x^4) + (x \cdot 2x)] dx = \frac{1}{3} + \frac{1}{5} + \frac{2}{3} = \frac{18}{15} \quad [J]$$

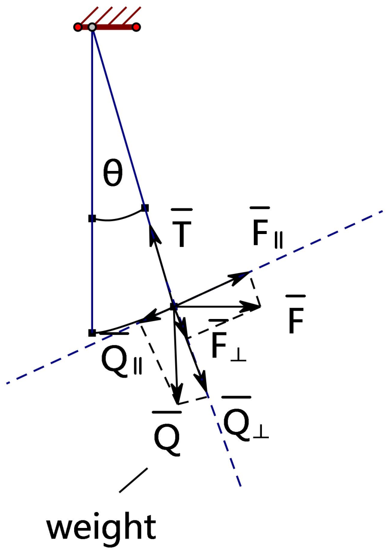
b) Less abstract example: find minimum work needed to move an object of weight Q, suspended on a cord, from A to B, acting with a horizontal force



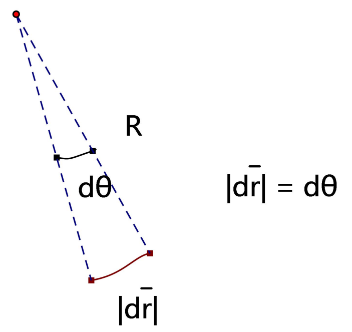
$$W_{A \to B} = \int_{\Gamma_{AB}} \bar{F} \cdot d\bar{r}$$

1st method

$\delta\omega = \bar{F} = (\bar{F}_{\parallel} + \bar{F}_{\perp}) \cdot d\bar{r} = \underbrace{\bar{F}_{\parallel}}_{\text{varies}} \cdot d\bar{r}$ and express in terms of θ



Infinitesimal arc length



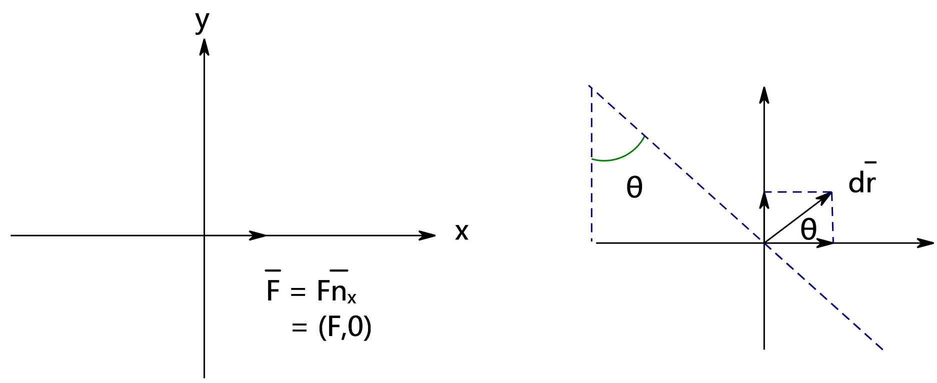
So

$$\delta W = F_{\|}|d\bar{r}| = Qsin\theta Rd\theta$$

work done by $\bar{F}$ $(A \to B)$

$$W_{A\to B} = \int_{\Gamma_{AB}} \bar{F}\cdot d\bar{r} = \int_0^{\theta_0} \underbrace{Qsin\theta}_{F_u} Rd\theta = QR\int_0^{\theta_0} sin\theta d\theta$$
$$= QR[1 - cos\theta_0]$$

2nd method: use cartesian coordinates



$d\bar{r} = |d\bar{r}|cos\theta\hat{n}_x + |d\bar{r}|sin\theta\hat{n}_y = (|d\bar{r}|cos\theta, |d\bar{r}|sin\theta)$
$\delta W = \bar{F}\circ d\bar{r} = Fcos\theta \underbrace{|d\bar{r}|}_{Rd\theta} = Fcos\theta Rd\theta$
and $Fcos\theta = Qsin\theta$
$\delta W = Qsin\theta Rd\theta$
and $W_{A\to B} = \int_0^{\theta_0} Qsin\theta Rd\theta = QR[1 - cos\theta_0]$

2 Work and motion, kinetic energy, and work-kinetic energy THM

Obdervation:

$$\begin{array}{lcr} W > 0 & \rightarrow & \text{particle speeds up} \\ W < 0 & \rightarrow & \text{particle slows down} \\ W = 0 & \rightarrow & \text{no change in speed} \end{array}$$

Recall

$$\delta W = \bar{F} \cdot d\bar{r}$$

so

$$\underbrace{\frac{\delta W}{dt}}_{\text{rate of work done on a particle}} = F \cdot \frac{d\bar{r}}{dt} = \bar{F} \cdot \bar{r} = m\bar{a} \cdot \bar{v} = m\dot{\bar{v}} \cdot \bar{v}$$

But $v^2 = \bar{v} \cdot \bar{v}$, so $\frac{d}{dt}(v^2) = \frac{d}{dt}(\bar{v} \cdot \bar{v}) = \dot{\bar{v}} \cdot \bar{v} + \bar{v} \cdot \dot{\bar{v}} = 2\bar{v} \cdot \dot{\bar{v}}$

$$\frac{\delta W}{dt} = \frac{d}{dt}(\underbrace{\frac{1}{2}mv^2}_{K \quad \text{Kinetic energy}})$$

Hence,

$$\frac{\delta W}{dt} = \frac{dK}{dt}$$

or

$$\delta W = dK$$

Work-Kinetic energy theorem:

The work done by the net force on a particle is equal to the change in the particle's kinetic energy

For finite increments

$$\underbrace{W}_{\text{work done by the net force}} = \Delta K$$

$$\begin{array}{llr} W > 0 \implies \Delta K > 0 & & \implies \Delta v > 0 \\ W < 0 \implies \Delta K < 0 & & \implies \Delta v < 0 \\ W = 0 \implies K = const & & \implies v = const \end{array}$$

Notes:

1. derived in a general case

2. used Newton's second law; can use only in inertial FoRs

3. K has the units of work $[J = N \cdot M = kg\frac{m^2}{s^2}]$

Power - characteries "how fast" is work being done

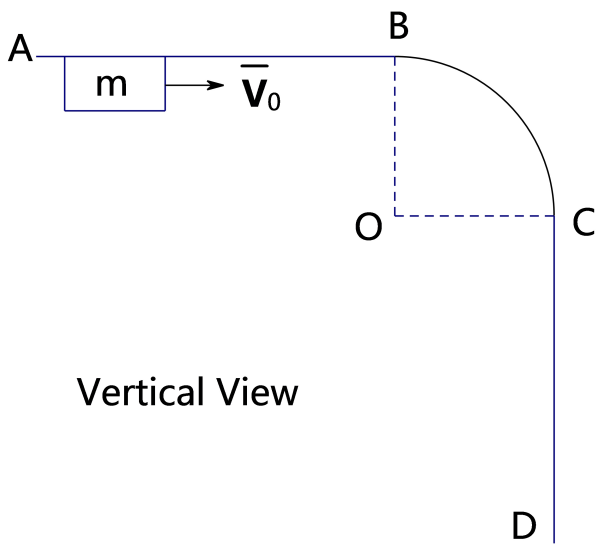
$$\underbrace{\frac{\delta W}{dt}}_{\text{rate of work done}} = \bar{F} \circ \bar{v} = \underbrace{P}_{\text{instantaneous power}}$$

$$\frac{W}{\Delta} = P_{av}$$

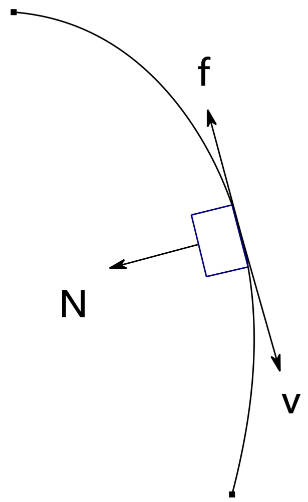
other units: $\underbrace{hp}_{\text{horse power}} = 746w = 0.746kw$

3 Exercise 1

There is an object with mass m, is moving on a frictionless plane. Its motion is subjected to a fixed baffle on the plane. The baffle 0, consists of two straight board and an arc board with radius R. At t=0, the object is moving along AB with speed v_0. The friction coefficient is μ. Calculate the work done by firction when the object is moving from A to D.



Solution:



Firstly, we try to calculate the speed at D.
when the object is moving along line segment AB, the is no interaction between the baffle and the object. When the object is moving through the arc BC, its interaction with the baffle is shown above. Accordignly, we can have the following relation:

$$-f = m\frac{dv}{dt} \tag{1}$$

$$N = m\frac{v^2}{R} \tag{2}$$

What's more,

$$f = \mu N, \tag{3}$$

By the equation above, we have

$$m\frac{dv}{dt} = -\mu m\frac{v^2}{R}$$

by substitution,

$$\frac{dv}{dt} = \frac{dv}{ds}\frac{ds}{dt} = \frac{dv}{ds}v,$$

which implies:

$$\frac{dv}{ds} = \mu\frac{v}{R},$$

saparate variable:

$$\frac{dv}{v} = -\frac{\mu}{R}ds,$$

integral both sides:

$$\int_{v_0}^{v}\frac{dv}{v} = \int_0^{\frac{\pi R}{2}} -\frac{\mu}{R}ds,$$

Hence,

$$\ln \frac{v}{v_0} = \mu \frac{\pi}{2},$$

and v is the speed when the object is moving to C. Since the object is moving in 1D in CD line segemnt, its sped remain constant.

$$v = v_0 e^{-\frac{\mu\pi}{2}}.$$

And only when the object is moving in CD line segment, the friction will do work, by work-energy theorem we have

$$W_f = \int \mathbf{f} \cdot d\mathbf{r} = \Delta E_k = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_0^2, \tag{4}$$

when we plug in v_f, we have

$$W_f = \frac{1}{2} m v_0^2 (e^{-\mu\pi} - 1)$$

Remark: since $e^{-\mu\pi} < 1$, $W_f < 0$, which suggests the friction does negatie work.

4 Exercise 2

Find work done by the force $\mathbf{F} = (2xy + y)\hat{n}_x + (x^2 + 1)\hat{n}_y$ if a particle is being moved from (-1,0) to (0,1) along

(a) the straight line connecting these points,

(b) the (shorter) arc of the circle $x^2 + y^2 = 1$,

(c) the axes of the Cartesian coordinate system: first from (-1,0) to (0,0) along the x axis, then from (0,0) to (0,1) along the y axis.

Solution: $\delta W = \mathbf{F} \cdot d\mathbf{r} = F_x dx + F_y dy + F_z dz$
for (a), the path is a line signment $y = x + 1$

$$\begin{aligned}
\delta W &= \mathbf{F} \cdot d\mathbf{r} \\
&= F_x dx + F_y dy + F_z dz \\
&= (2x(x+1) + x + 1)dx + ((y-1)^2 + 1)dy \\
&= (2x^2 + 3x + 1)dx + (y^2 - 2y + 2)dy
\end{aligned}$$

$$
\begin{aligned}
W &= \int \delta W \\
&= \int_{-1}^{0} (2x^2 + 3x + 1)dx + \int_{0}^{1} (y^2 - 2y + 2)dy \\
&= (2/3 - 3/2 + 1) + (1/3 - 1 + 2) \\
&= 3/2
\end{aligned}
$$

for (b), the path is a curve, we need to consider substitution.

let $x = -\cos t (0 \le t \le \pi/2)$ and $y = \sin t (0 \le t \le \pi/2)$

Then

$$
dx = \sin t dt
$$

$$
dy = \cos t dt
$$

$$
\begin{aligned}
\delta W &= \mathbf{F} \cdot d\mathbf{r} \\
&= F_x dx + F_y dy + F_z dz \\
&= (2(-\cos t)\sin t + \sin t)\sin t dt + ((\cos t)^2 + 1)\cos t dt \\
&= -2\cos t(1 - \cos^2 t) + \sin^2 t + \cos^3 t + \cos t \\
&= (3\cos^3 t + \sin^2 t - \cos t)dt
\end{aligned}
$$

$$
\boxed{\int_0^{\pi/2} \sin^n x = \int_0^{\pi/2} \cos^n x = \begin{cases} \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \dfrac{2}{3} \cdot 1 & \text{n is odd} \\ \dfrac{n-1}{n} \cdot \dfrac{n-3}{n-2} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2} & \text{n is even} \end{cases}}
$$

$$
\begin{aligned}
W &= \int \delta W \\
&= \int_0^{\pi/2} (3\cos^3 t + \sin^2 t - \cos t)dt \\
&= 3 \cdot 2/3 + 1/2 \cdot \pi/2 - 1 \\
&= 1 + \pi/4
\end{aligned}
$$

for (c), the path is two line segment: $L_1 : y = 0 \ \ (-1 \le x \le 0)$ and $L_2 : x = 0 \ \ (0 \le y \le 1)$

$$
\begin{aligned}
\delta W &= \mathbf{F} \cdot d\mathbf{r}_1 + \mathbf{F} \cdot d\mathbf{r}_2 \\
&= F_x dx + F_y dy \\
&= 0dx + dy \\
&= dy
\end{aligned}
$$

$$
\begin{aligned}
W &= \int \delta W \\
&= \int_0^1 dy \\
&= 1
\end{aligned}
$$