Vv557 Methods of Applied Mathematics II

Green Functions and Boundary Value Problems

Assignment 2

Date Due: 12:55 PM, Wednesday, the 10th of March 2021

Discussion Class Preparation

Please (re-)view Video files 15-18 and/or finish reading the section "The Classical Fourier Transform" and the first half of the section "Tempered Distributions and the Fourier Transform" (up to but excluding the part on convolution for tempered distributions on Slide 162) in the lecture slides. You should be able to answer the following questions:

i) How is the space of Schwartz functions defined?

ii) What is the relationship between the sets of test functions and Schwartz functions?

iii) Give three examples (not trivially equivalent) of Schwartz functions.

iv) How is the Fourier transform defined for Schwartz functions?

v) How is convergence defined in the space of Schwartz functions? What does "continuity of the Fourier transform" mean?

vi) List the basic properties of the Fourier transform and the convolution.

vii) What is a tempered distribution?

viii) How is the Fourier transform defined for tempered distributions?

Exercises (20 Marks)

Exercise 2.1
For $u\colon \mathbb{R}^2 \to \mathbb{R}$ given by[1]

$$u(x,t) = \begin{cases} 1/2 & t - |x| > 0, \\ 0 & \text{otherwise} \end{cases}$$

calculate $u_{tt} - u_{xx}$.
(4 Marks)

Exercise 2.2
Show that

$$\lim_{t\to\infty} \int_{\mathbb{R}} \frac{(1-\cos(tx))\varphi(x)}{x}\, dx = \mathcal{P}\left(\frac{1}{x}\right).$$

(4 Marks)

Exercise 2.3
(Taken from Stakgold/Holst, Exercise 2.2.3)
For fixed $\alpha > 0$, consider the sequence (f_k) of continuous functions

$$f_k\colon \mathbb{R} \to \mathbb{R}, \qquad\qquad f_k(x) = k^\alpha H(x) x e^{-kx}$$

Show that

i) $f_k(x) \to 0$ as $k \to \infty$ for all $x \in \mathbb{R}$ and any value of $\alpha > 0$,

ii) $f_k \to 0$ uniformly on $\mathbb{R}$ as $k \to \infty$ if $\alpha < 1$,

iii) $\int_{\mathbb{R}} |f_k(x)|\, dx \to 0$ as $k \to \infty$ if $\alpha < 2$,

[1] Zuily, C., *Problems in Distributions and Partial Differential Equations*, Exercise 28

iv) $T_{f_k} \to 0$ as $k \to \infty$ in the sense of distributions if $\alpha < 2$,

v) $T_{f_k} \to T_\delta$ as $k \to \infty$ in the sense of distributions if $\alpha = 2$,

vi) T_{f_k} does not converge as $k \to \infty$ in the sense of distributions if $\alpha > 2$.

(12 Marks)