vv255_Assignment 6: Triple Integrals.
Due to: 2019-07-15

Problem 1: Calculate the integrals.

a. $\iiint\limits_{\mathcal{R}} xy^2z^3 \; dx \; dy \; dz\,,\quad \mathcal{R}:\; z = xy,\quad y = x,\quad x = 1,\quad z = 0 \qquad Ans: \frac{1}{364}$

b. $\iiint\limits_{\mathcal{R}} \frac{dx \; dy \; dz}{(1+x+y+z)^3},\quad \mathcal{R}:\; x+y+z = 1,\quad x = 0,\quad y = 0,\quad z = 0 \qquad Ans: \frac{1}{2}\ln 2 - \frac{5}{16}$

c. $\iiint\limits_{\mathcal{R}} \sqrt{x^2+y^2} \; dx \; dy \; dz\,,\quad \mathcal{R}:\; x^2+y^2 = z^2,\quad z = 1 \qquad Ans: \frac{\pi}{6}$

Problem 2: Change the order of integration in the following integrals (all possible orders).

a. $\int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{x+y} f(x,y,z)dz$

b. $\int\limits_0^1 dx \int\limits_0^1 dy \int\limits_0^{x^2+y^2} f(x,y,z)dz$

c. $\int\limits_{-1}^1 dx \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy \int\limits_{\sqrt{x^2+y^2}}^{1} f(x,y,z)dz$

Problem 3: Find the volume of the solids bounded by the following surfaces.

a. $z = x^2+y^2,\quad z = 2x^2+2y^2,\quad y = x,\quad y = x^2 \quad Ans: \frac{3}{35}$

b. $z = 6-x^2-y^2,\quad z = \sqrt{x^2+y^2} \qquad Ans: \frac{32\pi}{3}$

c. $x^2+y^2+z^2 = 2az,\quad x^2+y^2 \le z^2,\quad a > 0 \qquad Ans:\; \pi a^3$

d. $x^2+y^2+z^2 = a^2,\quad x^2+y^2+z^2 = b^2,\quad x^2+y^2 = z^2\;\; (z \ge 0) \qquad Ans: \frac{\pi}{3}(\sqrt{2}-2)(b^3-a^3)$

e. $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 + \frac{z^4}{c^4} = 1 \qquad Ans:\; \frac{abc}{3}\sqrt{\frac{\pi}{2}}\Gamma^2(\frac{1}{4})$

f. $(a_1x+b_1y+c_1z)^2+(a_2x+b_2y+c_2z)^2+(a_3x+b_3y+c_3z)^2 = h^2 if$

$$|\Delta| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0 \quad Ans: \frac{4\pi h^3}{2|\Delta|}$$

g. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = \ln\frac{\frac{x}{a}+\frac{y}{b}-\frac{z}{c}}{\frac{x}{a}+\frac{y}{b}},\quad x = 0,\quad z = 0,\quad \frac{y}{b}+\frac{z}{c} = 0,\quad \frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \qquad Ans:\quad 5abc\left(\frac{1}{e}-\frac{1}{3}\right)$

Problem 4: Applications of triple integrals in mechanics

Let $\rho = \rho(x, y, z)$ be the density function of t a solid V. Summing the elements of mass up

$$dm = \rho \ \ dV = \rho \ \ dx \ \ dy \ \ dz,$$

$$m = \iiint\limits_{V} \rho \ \ dx \ \ dy \ \ dz$$

Using elementary moments $dM_{yz} = x \ \ dm = x_\rho dV, \ dM_{zx} = y \ \ dm = y_\rho dV, dM_{xy} = z \ \ dm = z_\rho dV$, we can find the moments

$$M_{yz} = \iiint\limits_{V} x_\rho dV, \qquad M_{zx} = \iiint\limits_{V} y_\rho dV, \qquad M_{xy} = \iiint\limits_{V} z_\rho dV$$

And the coordinates (ξ, η, ζ) of the center of mass

$$\xi = \frac{\iiint_V \ x_\rho dV}{V}, \qquad \eta = \frac{\iiint_V \ y_\rho dV}{V}, \qquad \zeta = \frac{\iiint_V \ z_\rho dV}{V}$$

a. The region V lies between the paraboloiid $z = 24 - x^2 - y^2$ and the cone $z = 2\sqrt{x^2 + y^2}$. Find the centroid of V - the center of mass in the case if the density is constant.

b. Find the center of mass of the solid bounded by the surfaces

$$x^2 + y^2 = 2az, \qquad x^2 + y^2 + z^2 = 3a^2$$

c. Find mass and the coordinates of the center of mass of the sphere $x^2 + y^2 + z^2 \le 2az$ if

$$\rho = \frac{k}{\sqrt{x^2 + y^2 + z^2}}$$