Vv156 Lecture 18

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- Recall the area under a continuous curve $y = f(x)$ over $[a, b]$ is defined as

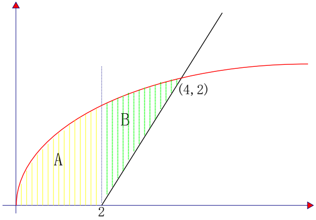
$$\int_a^b f(x)\, dx$$

Exercise

Find the area of the region in the first quadrant that is bounded above by

$$y = \sqrt{x}$$

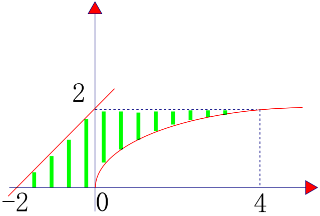
and below by the x-axis and the line $y = x - 2$.



Exercise

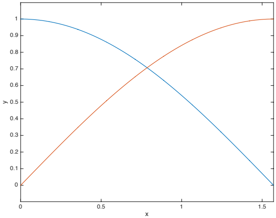
(a) Find the area of the region enclosed by the curves.

$$y = x + 2; \quad y = \sqrt{x}; \quad y = 2; \quad y = 0.$$



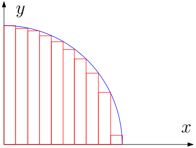
(b) Find the area of the region enclosed by the curves.

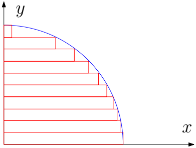
$$y = \sin x; \quad y = \cos x; \quad x = \frac{\pi}{2}; \quad x = 0.$$



- Recall the underlying principle for defining and finding the area is to:

1. Divide the region into thin strips,





2. Approximate the area of each strip by the area of a rectangle,

$$\text{small} \approx \text{Height} \times \text{Width}$$

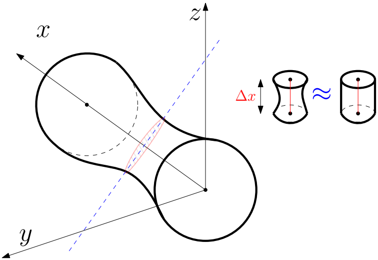
3. Add the approximations to form a Riemann sum,

$$\sum f(x_k^*)\Delta x_k \qquad \text{or} \qquad \sum g(y_k^*)\Delta y_k$$

4. Take the limit of the Riemann sum

$$\int_a^b \Big(\text{Height}\Big)\, dx \qquad \text{or} \qquad \int_c^d \Big(\text{Width}\Big)\, dy$$

- The same idea can be used to define and find the volume of a solid object.



1. Divide the solid into thin slices,
2. Approximate the volume of each slice by the volume of a cylinder,

$$\text{small} \approx \text{Base} \times \text{Height}$$

3. Add the approximations to form a Riemann sum,

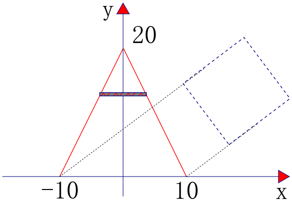
$$\sum A(x_k^*)\Delta x_k$$

4. Take the limit of the Riemann sum to find the volume.

$$\int_a^b \Big(Area\Big)\,dx$$

Exercise

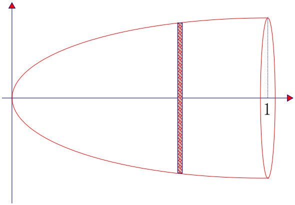
(a) Find the volume of a pyramid with a square base that is 20 meters tall and 20 meters on a side at the base.



(b) Find the volume of the solid obtained by rotating the region under the curve

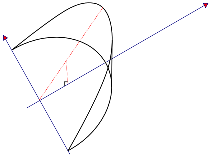
$$y = \sqrt{x}$$

about the x-axis from 0 to 1.



Exercise

(a) A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the centre of the cylinder.



Find the volume of the wedge.

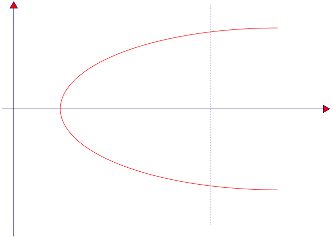
(b) Find the volume of the solid generated by revolving about the line $x = 3$

the region between the parabola

$$x = y^2 + 1$$

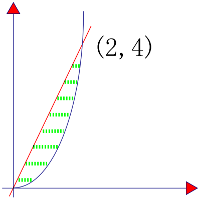
and the line

$$x = 3$$



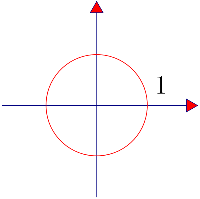
Exercise

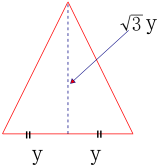
(a) The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid.

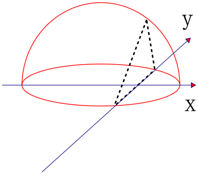


Find the volume of the solid.

(b) Find the volume of a solid with a circular base of radius one unit.







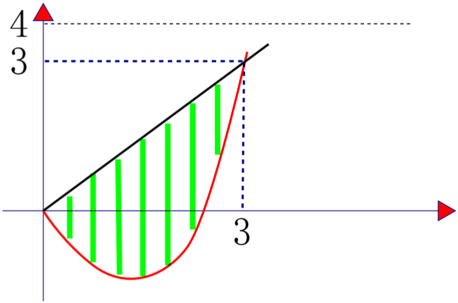
The Parallel cross-sections perpendicular to its base are equilateral triangles.

Exercise

Find the volume of the solid obtained by rotating the region bounded by

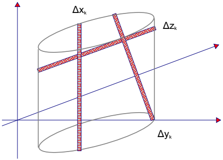
$$y = x^2 - 2x$$

and $y = x$ about the line $y = 4$.



Cross-sections

1. Divide the solid into thin slices,



2. Approximate the volume of each slice by the volume of a cylinder,

$$\text{small} \approx \text{Base} \times \text{Height}$$

3. Add the approximations to form a Riemann sum,

$$\sum A(x_k^*)\Delta x_k = \sum B(y_k^*)\Delta y_k = \sum C(z_k^*)\Delta z_k$$

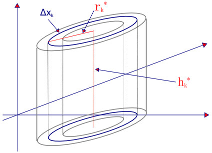
4. Take the limit of the Riemann sum to find the volume.

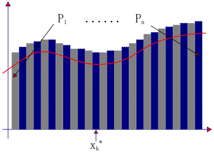
$$\int_a^b \Big(Area\Big)\,dx = \int_c^d \Big(Area\Big)\,dy = \int_g^h \Big(Area\Big)\,dz$$

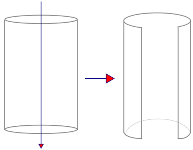
Cylindrical shells

- It is sometimes easier to find the volume by dividing in a special way:

1. Divide the solid into hollow cylinders known as cylindrical shells,







2. Approximate the volume of each shells by the volume of a box,

$$\text{small} \approx \text{circumference} \times \text{height} \times \text{thickness}$$

3. Add the approximations to form a Riemann sum,

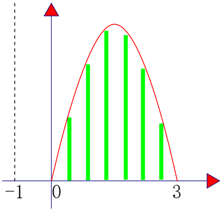
$$\sum A(x_k^*)\Delta x_k$$

4. take the limit of the Riemann sums to find the volume.

$$\int_a^b 2\pi \begin{pmatrix}\text{Shell}\\ \text{Radius}\end{pmatrix} \begin{pmatrix}\text{Shell}\\ \text{Height}\end{pmatrix} dx$$

Exercise

(a) The region in the first quadrant enclosed by the x-axis and the parabola



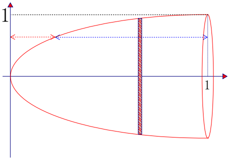
$$y = 3x - x^2$$

is revolved about the vertical line

$$x = -1$$

Find the volume of the solid.

(b) Find the volume of the solid obtained by revolving the region under



$$y = \sqrt{x}$$

about the x-axis from 0 to 1.

Exercise

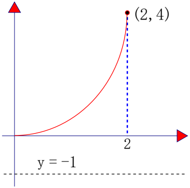
(a) A solid is generated when the region under

$$y = x^2$$

over the interval $[0, 2]$ is revolved about the line

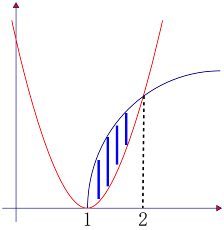
$$y = -1$$

Use cylindrical shells to find the volume of the solid.



(b) A solid is formed by rotating the finite region bounded by the graphs of

$$y = \sqrt{x-1} \quad \text{and} \quad y = (x-1)^2$$



about the y-axis. Find the volume of the solid.