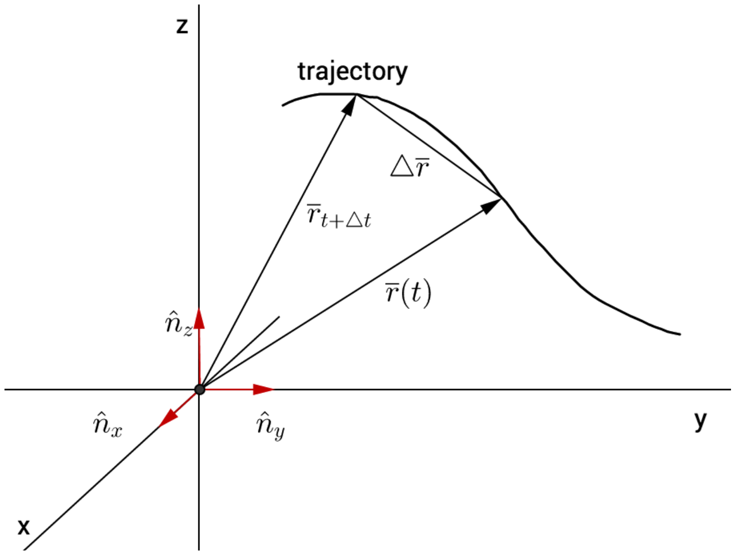
1 Cartesian coordinate system



$\triangle\overline{r}(t)$ - displacement over time interval$(t, t+\triangle t)$

parametric equations of trajectory $\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$

<u>*Position*</u>

$$\overline{r}(t) = x(t)\hat{n}_x + y(t)\hat{n}_y + z(t)\hat{n}_z = (x(t), y(t), z(t)) \tag{1}$$

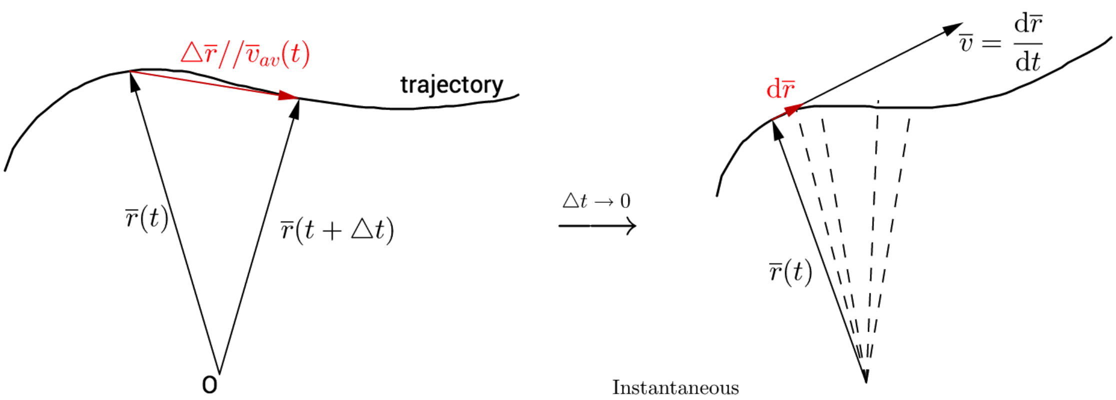
$\hat{n}_x, \hat{n}_y, \hat{n}_z$ - fixed unit vectors (time independent) i.e. $\dot{\hat{n}}_x = \dot{\hat{n}}_y = \dot{\hat{n}}_z = 0$

<u>*Velocity*</u>

$$\overline{v}_{av} = \frac{\triangle\overline{r}}{\triangle t} \qquad \text{(average)} \tag{2}$$

$$\begin{aligned} \overline{v}(t) &= \lim_{\triangle t\to 0}\frac{\triangle\overline{r}}{\triangle t} = \lim_{\triangle t\to 0}\frac{\overline{r}(t+\triangle t)-\overline{r}(t)}{\triangle t} \\ &= \lim_{\triangle t\to 0}\left[\frac{x(t+\triangle t)-x(t)}{\triangle t}\hat{n}_y + \frac{y(t+\triangle t)-y(t)}{\triangle t}\hat{n}_y + \frac{z(t+\triangle t)-z(t)}{\triangle t}\hat{n}_z\right] \\ &= \underbrace{\dot{x}(t)}_{=v_x(t)}\hat{n}_x + \underbrace{\dot{y}(t)}_{=v_y(t)}\hat{n}_y + \underbrace{\dot{z}(t)}_{=v_z(t)}\hat{n}_z = (\dot{x}(t), \dot{y}(t), \dot{z}(t)) \qquad \text{(instantaneous)} \end{aligned} \tag{3}$$

speed $$v(t) = |\overline{v}(t)| = \sqrt{[\dot{x}(t)]^2 + [\dot{y}(t)]^2 + [\dot{z}(t)]^2} \tag{4}$$



[Velocity vector $\overline{v}$ is always tagent to the trajectory!]

Acceleration

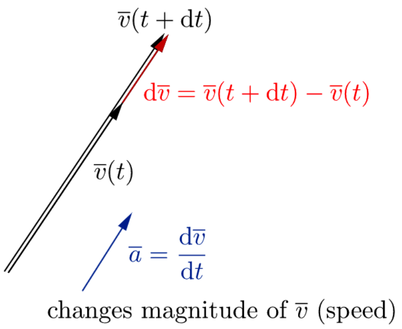
$$\text{analogously} \qquad \overline{a}_{av} = \frac{\triangle\overline{v}}{\triangle t} \qquad \text{(average)}, \qquad \triangle\overline{v} = \overline{v}(t+\triangle t) - \overline{v}(t) \tag{5}$$

$$\begin{aligned} \overline{a}(t) &= \lim_{\triangle t\to 0}\frac{\triangle\overline{v}}{\triangle t} = \dot{v}_x(t)\hat{n}_x + \dot{v}_y(t)\hat{n}_y + \dot{v}_z(t)\hat{n}_z \\ &= \underbrace{\ddot{x}(t)}_{=a_x(t)}\,\hat{n}_x + \underbrace{\ddot{y}(t)}_{=a_y(t)}\,\hat{n}_y + \underbrace{\ddot{z}(t)}_{=a_z(t)}\,\hat{n}_z \qquad \text{(instantaneous)} \end{aligned} \tag{6}$$

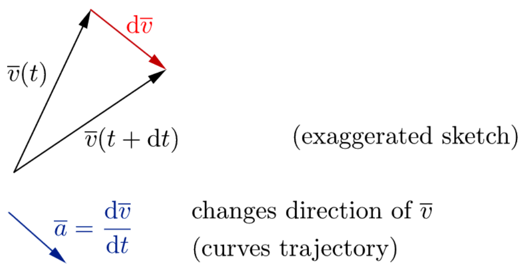
$$\text{magnitude} \qquad a(t) = |\overline{a}(t)| = \sqrt{[\ddot{x}(t)]^2 + [\ddot{y}(t)]^2 + [\ddot{z}(t)]^2} \tag{7}$$

2 Tangential and normal components of acceleration

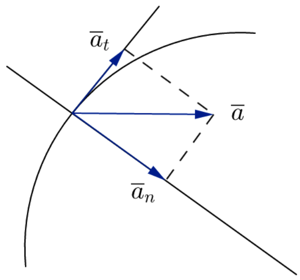
acceleration parallel to velocity
(i.e. tangent to trajectory)



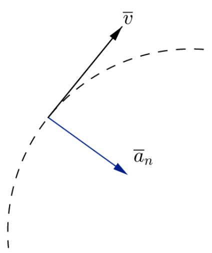
acceleration perpendicular to velocity
(i.e. normal to trajectory)



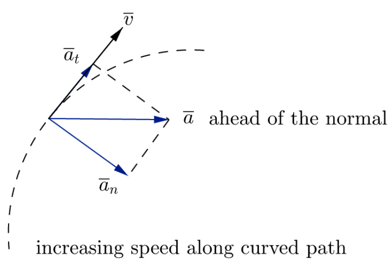
magnitude & direction of $\overline{v}$ both change



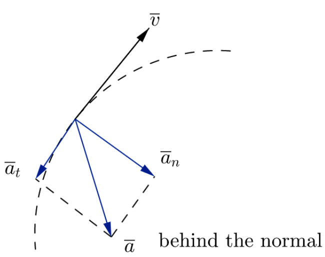
special cases



constant speed along curved path

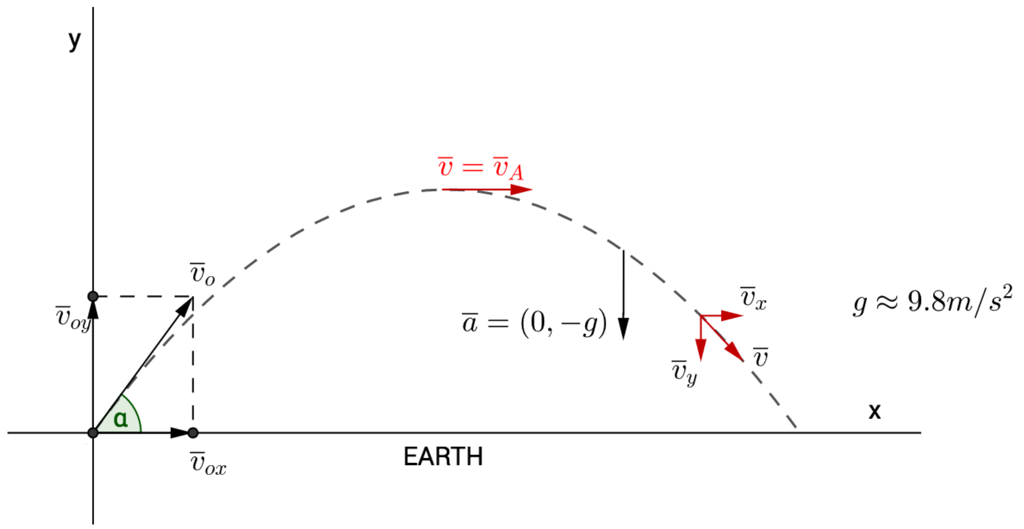


increasing speed along curved path



decreasing speed along curved path

Example (2D) projectile motion



Initial conditions (t=0)

$$\begin{cases} x(0) = 0 \\ y(0) = 0 \end{cases} \qquad \begin{cases} v_x(0) = v_{0x} = v_0 \cos\alpha \\ v_y(0) = v_{0y} = v_0 \sin\alpha \end{cases}$$

Accleration

$$\begin{aligned} a_x(t) &\equiv 0 \\ a_y(t) &= -g \end{aligned} \qquad \text{that is} \qquad \begin{aligned} \frac{\mathrm{d}v_x}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}v_y}{\mathrm{d}t} &= -g \end{aligned}$$

Velocity

$$\int_{v_{0x}}^{v_x(t)} \mathrm{d}v_x = \int_0^t 0\mathrm{d}t = 0 \qquad \Rightarrow \qquad v_x(t) = \text{const} = v_0 \cos\alpha \tag{8}$$

$$\int_{v_0 \sin\alpha}^{v_y(t)} \mathrm{d}v_y = -\int_0^t g\mathrm{d}t \qquad \Rightarrow \qquad v_y(t) = v_0 \sin\alpha - gt \tag{9}$$

Position

$$v_x(t) = \frac{\mathrm{d}x}{\mathrm{d}t} = v_0 \cos\alpha \quad \Rightarrow \quad \int_0^{x(t)} \mathrm{d}x = \int_0^t v_0 \cos\alpha \mathrm{d}t \quad \Rightarrow \quad x(t) = v_0 t \cos\alpha \tag{10}$$

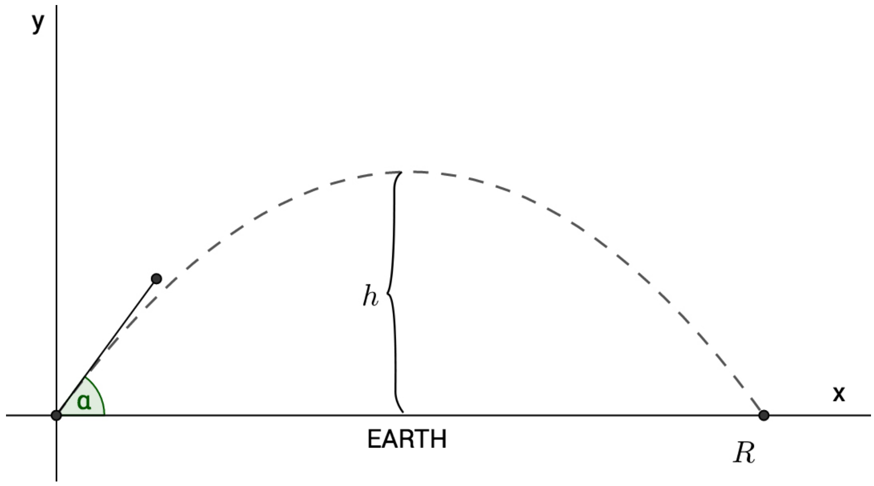
$$v_y(t) = \frac{\mathrm{d}y}{\mathrm{d}t} = v_0 \sin\alpha - gt \quad \Rightarrow \quad \int_0^{y(t)} \mathrm{d}y = \int_0^t [v_0 \sin\alpha - gt]\mathrm{d}t \quad \Rightarrow \quad y(t) = v_0 t \sin\alpha - \frac{1}{2}gt^2 \tag{11}$$

Trajectory

$$\text{parmetric form} \quad \begin{cases} x(t) = v_0 t \cos\alpha & \Rightarrow t = \dfrac{x}{v_0 \cos\alpha} \\ y(t) = v_0 t \sin\alpha - \dfrac{1}{2} g t^2 & \end{cases} \tag{12}$$

$$\Rightarrow \text{trajectory (implicit form)} \qquad y(x) = x \tan\alpha - \frac{1}{2} \frac{g}{v_0^2 \cos^2\alpha} x^2 \qquad \text{(parabola)} \tag{13}$$

Height and range



Height

$$\text{At the highest point} \qquad v_y(t_h) = 0 \Rightarrow t_h = \frac{v_0 \sin\alpha}{g} \tag{14}$$

$$\text{and} \quad y(t_h) = \frac{v_0^2 \sin^2\alpha}{2g} = h \qquad \text{use (11)} \tag{15}$$

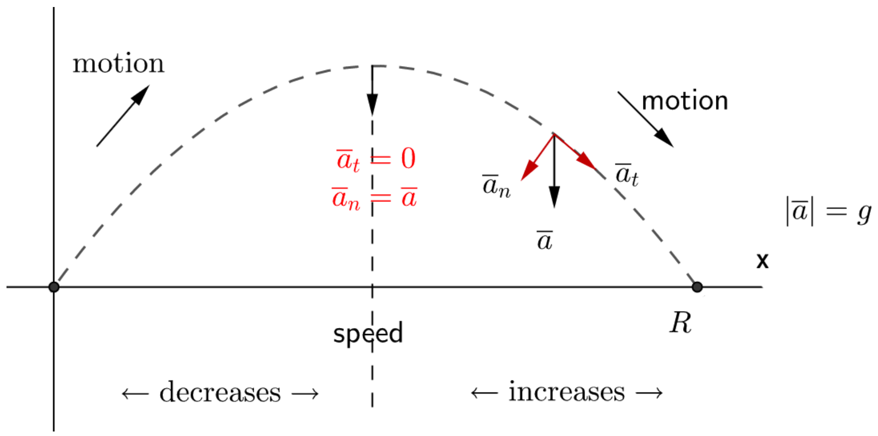
Range

$$y(x_R) = 0 \quad \Rightarrow \quad x_R \tan\alpha - \frac{1}{2} \frac{g}{v_0^2 \cos^2\alpha} x_R^2 = 0 \qquad \text{use (13)}$$

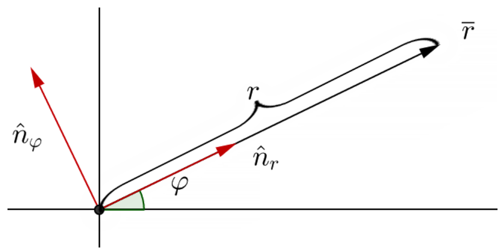
$$\Rightarrow \quad x_R = 0 \quad \text{(starting point)} \quad \text{or} \quad x_R = \frac{v_0^2 \sin 2\alpha}{g} \tag{16}$$

What angle gives maximum (a)height? $\alpha = 90°$
(b)range? $\alpha = 45°$

Acceleration in projectile motion



3 Polar coordinates (2D kinematics)



Position vector $\qquad \overline{r} = r\hat{n}_r$

Trajectory

$$\text{parametric form} \quad \begin{cases} r = r(t) \\ \varphi = \varphi(t) \end{cases} \tag{17}$$

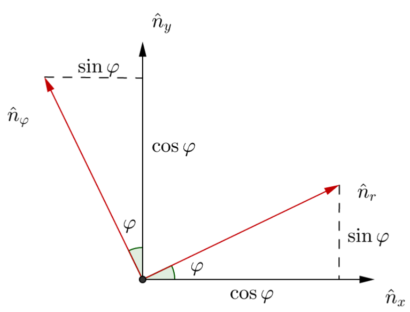
$$\text{or} \quad \text{implicit form} \quad r = r(\varphi) \text{ or } \varphi = \varphi(r) \tag{18}$$

Velocity

$$\overline{v} = \dot{\overline{r}} = \dot{r}\hat{n}_r + r\dot{\hat{n}}_r \tag{19}$$

Note that $\dot{\hat{n}}_r$ is not zero, unlike in Cartesian coordinates, here $\hat{n}_r, \hat{n}_\varphi$ are not fixed.

How to find $\dot{\hat{n}}_r$ and $\dot{\hat{n}}_\varphi$?



$$
\begin{cases} \hat{n}_r = \cos\varphi \hat{n}_x + \sin\varphi \hat{n}_y \\ \hat{n}_\varphi = -\sin\varphi \hat{n}_x + \cos\varphi \hat{n}_y \end{cases} \tag{20}
$$

Now using (20)

$$
\dot{\hat{n}}_r = -\dot{\varphi}\sin\varphi \hat{n}_x + \dot{\varphi}\cos\varphi \hat{n}_y = \dot{\varphi}(-\sin\varphi \hat{n}_x + \cos\varphi \hat{n}_y) = \dot{\varphi}\hat{n}_\varphi \tag{21}
$$

$$
\dot{\hat{n}}_\varphi = -\dot{\varphi}\cos\varphi \hat{n}_x - \dot{\varphi}\sin\varphi \hat{n}_y = -\dot{\varphi}(\cos\varphi \hat{n}_x + \sin\varphi \hat{n}_y) = -\dot{\varphi}\hat{n}_r \tag{22}
$$

So velocity

$$
\boxed{\overline{v} = \dot{r}\hat{n}_r + r\dot{\hat{n}}_r = \underbrace{\dot{r}\hat{n}_r}_{\text{radial component}} + \underbrace{r\dot{\varphi}\hat{n}_\varphi}_{\text{transversal component}}} \tag{23}
$$

$$
\text{speed} \qquad v = |\overline{v}| = \sqrt{(\dot{r})^2 + (r\dot{\varphi})^2} \tag{24}
$$

Accelearation

$$
\begin{aligned}
\overline{a} = \dot{\overline{v}} &= \ddot{r}\hat{n}_r + \dot{r}\dot{\hat{n}}_r + \dot{r}\dot{\varphi}\hat{n}_\varphi + r\ddot{\varphi}\hat{n}_\varphi + r\dot{\varphi}\dot{\hat{n}}_\varphi \\
&= \ddot{r}\hat{n}_r + \dot{r}\dot{\varphi}\hat{n}_\varphi + \dot{r}\dot{\varphi}\hat{n}_\varphi + r\ddot{\varphi}\hat{n}_\varphi - r(\dot{\varphi})^2\dot{\hat{n}}_r \\
&= (\ddot{r} - r\dot{\varphi}^2)\hat{n}_r + (r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi
\end{aligned}
$$

$$
\boxed{\overline{a} = \underbrace{(\ddot{r} - r\dot{\varphi}^2)\hat{n}_r}_{\text{radial component}} + \underbrace{(r\ddot{\varphi} + 2\dot{r}\dot{\varphi})\hat{n}_\varphi}_{\text{transversal component}}} \tag{25}
$$

CAUTION! in general, radial $\neq$ normal transversal $\neq$ tangential

Example (kinematics in the polar coordinates)

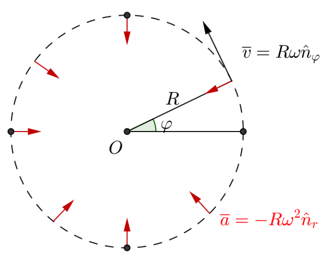
Circular motion: $r = R = \text{const} \quad \Rightarrow \quad \dot{r} = \ddot{r} = 0$

$\varphi = \varphi(t)$ - in general: any function of time

(a) uniform circular motion (particle travels at constant speed, assume counter-clockwise)

Velocity

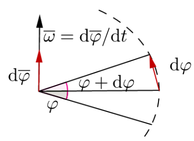
$$
\overline{v} = \underset{=0}{\dot{r}}\hat{n}_r + r\dot{\varphi}\hat{n}_\varphi = R\dot{\varphi}\hat{n}_\varphi \tag{26}
$$



uniform motion, so $|\overline{v}| = |v_\varphi| = v = \text{const}$

$$
R\dot{\varphi} = v \quad \Rightarrow \quad \frac{\mathrm{d}\varphi}{t} = \frac{v}{R} \quad \Rightarrow \quad \int_0^{\varphi(t)} \mathrm{d}\varphi = \int_0^t \frac{v}{R}\mathrm{d}t
$$

Hence



$$
\varphi = \varphi(t) = \frac{v}{R}t = \omega t \tag{27}
$$

Note ω is angular velocity (here constant)   Angular velocity is a vector, in general, $\overline{\omega} = \frac{\mathrm{d}\overline{\varphi}}{\mathrm{d}t}$

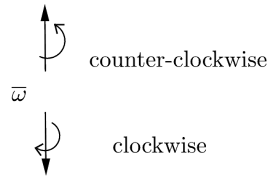
Acceleration

$$\overline{a} = (\underset{=0}{\ddot{r}} - r\dot{\varphi}^2)\hat{n}_r + (r\underset{=0}{\ddot{\varphi}} + 2\underset{=0}{\dot{r}}\dot{\varphi})\hat{n}_\varphi = -R\omega^2\hat{n}_r \tag{28}$$

Summary $\boxed{\begin{aligned}\overline{v} &= R\omega\hat{n}_\varphi \\ \overline{a} &= -R\omega^2\hat{n}_\varphi\end{aligned}}$

Here: $\hat{n}_\varphi$ || tangential direction
$\hat{n}_r$ || normal direction
both $|\overline{v}|$ and $|\overline{a}|$ are constant in time

(b) non-uniform circular motion: $\varphi = \varphi(t)$ - arbitrary function of time
Now

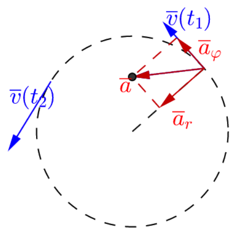


$\dot{\varphi} = \dot{\varphi}(t) = \omega(t)$ angular velocity

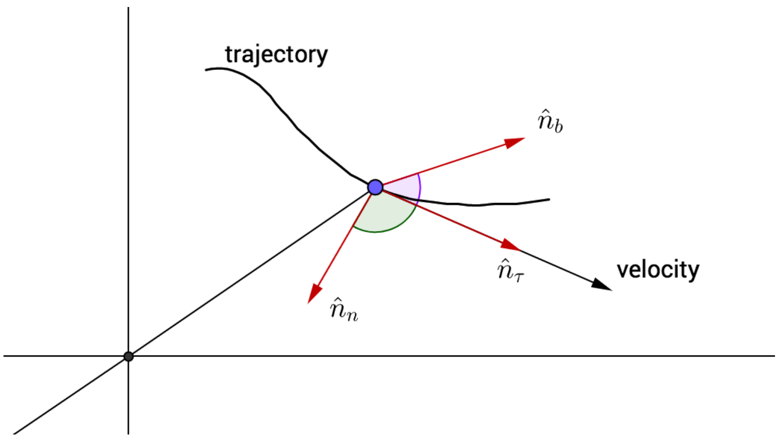
$\ddot{\varphi} = \ddot{\varphi}(t) = \dot{\omega}(t) = \varepsilon(t)$ angular acceleration (in general) $\overline{\varepsilon} = \frac{\mathrm{d}^2\overline{\varphi}}{\mathrm{d}t^2}$

$$\overline{v} = R\omega(t)\hat{n}_\varphi \tag{29}$$

$$\overline{a} = \underbrace{-R\omega^2(t)\hat{n}_r}_{\text{curves the trajectory}} + \underbrace{R\varepsilon(t)\hat{n}_\varphi}_{\substack{\text{changes the magnitude of } \overline{v} \\ \text{(i.e. } v = R|\omega(t)| \neq \text{const)}}} \tag{30}$$



4 Natural coordinate system



unit vectors (versors)
$\hat{n}_\tau$ - tagent (along $\overline{v}$)
$\hat{n}_n$ - normal
$\hat{n}_b$ - binormal

Velocity $\overline{v}(t) = v\hat{n}_\tau$ or $\hat{n}_\tau = \frac{\overset{\text{velocity(vector)}}{\overline{v}}}{\underset{\text{speed(scalar)}}{v}} = \frac{\dot{\overline{r}}}{|\dot{\overline{r}}|}$

Note: Normal unit vector - perpendicular (orthogonal) to $\hat{n}_\tau$ - many choices possible ! (in 3D)

Unique choice:

$$\hat{n}_n = \frac{\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}}{\left|\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}\right|} \qquad \text{because } \frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t} \text{ in general is not a unit}$$

vector.

Is $\hat{n}_n \perp \hat{n}_\tau$? Yes!

$$\hat{n}_\tau \cdot \hat{n}_\tau = 1 \qquad \underset{\text{with respect to } t}{\overset{\text{differentiate}}{\Longrightarrow}} \qquad \frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t} \cdot \hat{n}_\tau + \hat{n}_\tau \cdot \frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t} = 0$$

$$\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t} \cdot \hat{n}_\tau = 0 \qquad \Longrightarrow \qquad \frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t} \perp \hat{n}_\tau \quad \text{and} \quad \hat{n}_n \perp \hat{n}_\tau$$

The normal versor $\hat{n}_n$ points along the radius of curvature.

The binormal unit vector determined by

$$\hat{n}_b = \hat{n}_\tau \times \hat{n}_n \qquad \text{(right-handed system)}$$

$\hat{n}_\tau$, $\hat{n}_n$ and $\hat{n}_b$ are the three unit vectors of the natural coordinate system, "sliding" along the particle's trajectory.

Velocity

$$\overline{v} = v\hat{n}_\tau \tag{31}$$

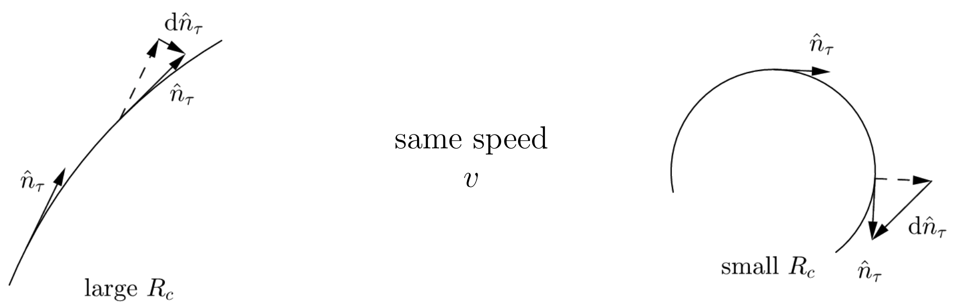
Acceleration

$$\overline{a} = \dot{\overline{v}} = \dot{v}\hat{n}_\tau + v\dot{\hat{n}}_\tau \overset{\text{use definition of } \hat{n}_n}{=} \dot{v}\hat{n}_\tau + v\left|\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}\right|\hat{n}_n$$

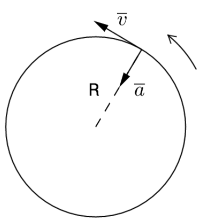
Define the (instantaneous) radius of trajectory's curvature $R_c \overset{\text{def}}{=} \dfrac{v}{\left|\frac{\mathrm{d}\hat{n}_\tau}{\mathrm{d}t}\right|}$. Then

$$\overline{a} = \underbrace{\underbrace{\dot{v}\hat{n}_\tau}_{\text{tangential component } \overline{a}_\tau} + \underbrace{\frac{v^2}{R_C}\hat{n}_n}_{\text{normal component } \overline{a}_n}}_{\text{mutually perpendicular } (|\overline{a}|=\sqrt{a_\tau^2+a_n^2})} \tag{32}$$

Note Interpretation of R_C



Example Uniform circular motion ($|\overline{v}| = \text{const}$)



$R_C = R$ (exercise)

$a_\tau = 0$

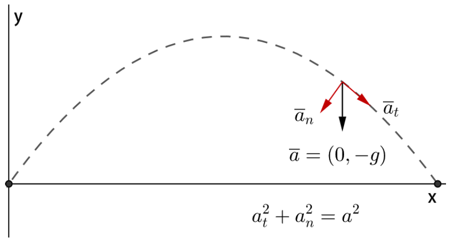
$a_n = v^2/R$

Again:

$$a_\tau = \dot{v} \qquad \text{changes the } \underline{\text{magnitude}} \text{ of } \overline{v}$$
$$a_n = v^2/R_C \qquad \text{changes the } \underline{\text{direction}} \text{ of } \overline{v}$$

Example Projectile motion (also see hm2)

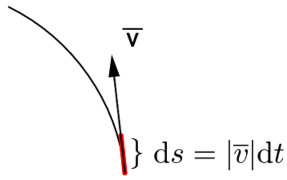
$$\overline{v} = (v_0 \cos\alpha)\hat{n}_x + (v_0 \sin\alpha - gt)\hat{n}_y$$
$$|\overline{v}| = v = \sqrt{[v_x(t)]^2 + [v_y(t)]^2}$$
$$\text{and} \quad a_t = \dot{v}$$
$$a_n = \sqrt{g^2 - (\dot{v})^2}$$



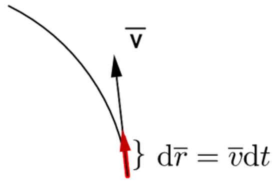
5 Concluding remarks

5.1 average speed vs average time

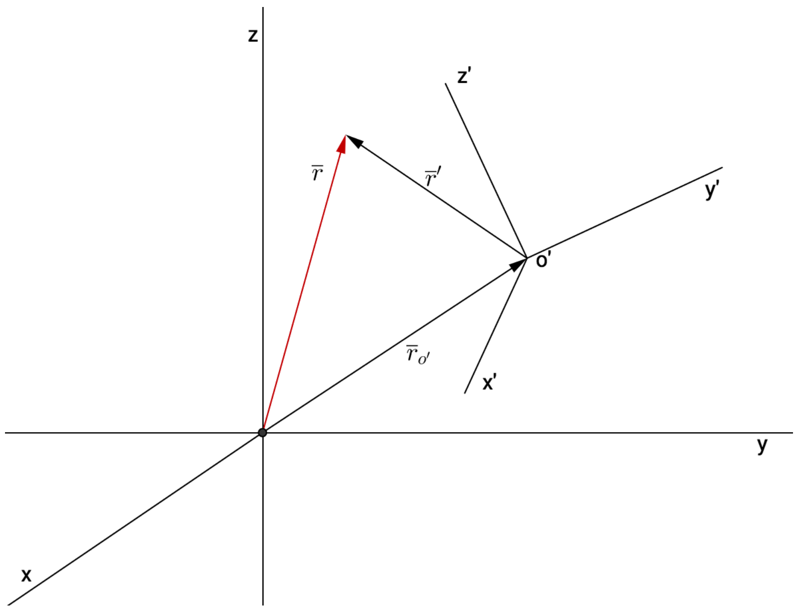
$$\underset{\text{(number)}}{\text{average speed}} = \frac{\overset{\text{distance traveled over time interval } (t_1, t_2)}{\int_{t_1}^{t_2} |\overline{v}(t)|\mathrm{d}t}}{t_2 - t_1}$$



$$\underset{\text{(vector)}}{\text{average velocity}} = \frac{\overset{\text{net displacement over time interval } (t_1, t_2)}{\int_{t_1}^{t_2} \overline{v}(t)\mathrm{d}t}}{t_2 - t_1}$$



5.2 relative motion and Galilean transformation



$$\overline{r} = \overline{r}_{o'} + r'$$

Assume $\dot{\overline{r}}_{o'} = \overline{v}_{o'} =$ const, that is $x'y'z'$ moves along a straight line (no rotations, either)

$$\Rightarrow \overline{v} = \overline{v}_{o'} + \overline{v}'$$

Use $\overline{r}_{o'} = \overline{v}_{o'}t + \overline{r}_{o',init}$ (choose to be 0) (i.e. $O = O'$ at $t = 0$)

$$\Rightarrow \boxed{\overline{r} = \overline{v}_{o'}t + \overline{r}'} \qquad \text{Galilean transformation} \tag{33}$$