vv214: Linear transformations.

Dr.Olga Danilkina

UM-SJTU Joint Institute

May 26, 2020

This week

Today

1. Review: basis and dimension.
2. Linear transformations and linear operators. Linear operators in finite dimensional linear spaces.
3. Linear transformations in 2D and 3D: rotations, reflections, projections.

Next class

1. Composition of linear transformations.
2. Inverse linear transformations.

Later

Spans, image and kernel of a linear transformation.

Last Class

Basis

- Any spanning set of vectors can be reduced to a basis of a linear space.
- Any set of linear independent set of elements can be extended to a basis of a linear space.

Example: $\mathbb{R}^3$: $(2,3,4)$, $(9,6,8)$ are linearly independent. Consider the linearly independent vectors with vectors that span $\mathbb{R}^3$:

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Eliminating linearly dependent vectors from this system, you obtain basis elements:

$$\begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 9 \\ 6 \\ 8 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ is a basis for } \mathbb{R}^3$$

Basis

- If V has a finite basis and U is a linear subspace of V, then there exists a linear subspace W of V such that $V = U \oplus W$
  1. U must have a finite basis $v_1, \ldots, v_m$ as well.
  2. Let $w_1, \ldots, w_n$ span V. Consider the basis of U and the span of V together:
  $$v_1, \ldots, v_m, \ w_1, \ldots, w_n$$
  3. Eliminating linearly dependent elements, we obtain a basis for V:
  $$v_1, \ldots, v_m, \ u_1, \ldots, u_k, \quad u_i = w_j$$
  4. Denote $W = span(u_1, \ldots, u_k) \Rightarrow V = U + W$
  5. It remains to show that $U \cap W = \{0\}$. Let $x \in U \cap W \Rightarrow x \in U$, $x \in W$
  $$x = \alpha_1 v_1 + \ldots + \alpha_m v_m = \beta_1 u_1 + \ldots + \beta_k u_k$$
  $$\Rightarrow \alpha_1 v_1 + \ldots + \alpha_m v_m - \beta_1 u_1 - \ldots - \beta_k u_k = 0$$
  6. $v_1, \ldots, v_m, \ u_1, \ldots, u_k$ is a basis, i.e. linearly independent
  $$\Rightarrow \alpha_1 = \ldots = \alpha_m = \beta_1 = \ldots = \beta_k = 0 \Rightarrow x = 0$$
  7. $V = U + W$, $U \cap W = \{0\} \Rightarrow V = U \oplus W$

Examples

1. $U = span\,(2,3,4),\ (9,6,8))$ is a linear subspace of $\mathbb{R}^3$, and $(2,3,4),\ (9,6,8),\ (0,1,0))$ is a basis for $\mathbb{R}^3$

$$\text{Let } W = span\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \Rightarrow V = U \oplus W$$

2. Let $M = \{p(t) \in P_2(\mathbb{R}):\ p(1) = 0\}$.

$$p(1) = a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 = 0 \Rightarrow a_0 = -a_1 - a_2$$

$M = \{p(t) = a_1(t-1) + a_2(t^2-1)\} \Rightarrow \{t-1,\ t^2-1\}$ is a basis for M

Consider $t-1, t^2-1, 1, t, t^2$ and eliminate linearly dependent elements:

$$t-1, t^2-1, 1 \quad \text{is a basis for } P_2(\mathbb{R})$$

$\Rightarrow W = span(1) \Rightarrow P_2(\mathbb{R}) = M \oplus W$

Dimension

Definition: The number of elements in the basis is called the dimension of a linear space.

Examples:

1. $\dim \mathbb{R}^n = n$
   a. The vectors $\bar{e}_1 = (1, 0, \ldots, 0), \ldots, \bar{e}_n = (0, \ldots, 1) \in \mathbb{R}^n$ are linearly independent:

$$\alpha_1 \bar{e}_1 + \alpha_2 \bar{e}_2 + \ldots + \alpha_n \bar{e}_n = \bar{0}$$

$$\Rightarrow (\alpha_1, \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots, 0) \Rightarrow \alpha_1 = \ldots = \alpha_n = 0$$

$$\dim \mathbb{R}^n \geq n$$

   b. Consider arbitrary $n+1$ vectors in $\mathbb{R}^n$: $\bar{x}^1 = (x_1^1, \ldots, x_n^1)$, $\ldots, \bar{x}^n = (x_1^n \ldots, x_n^n)$, $\bar{x}^{n+1} = (x_1^{n+1} \ldots, x_n^{n+1})$

$$\alpha_1 \bar{x}^1 + \ldots + \alpha_n \bar{x}^n + \alpha_{n+1} \bar{x}^{n+1} = \bar{0}$$

   This is a homogeneous system of n linear equations in $n+1$ variables $\Rightarrow \exists \alpha_i \neq 0 \Rightarrow$ any $n+1$ vectors are linearly dependent in $\mathbb{R}^n \Rightarrow \dim \mathbb{R}^n < n+1$

   c. $\dim \mathbb{R}^n \geq n$, $\dim \mathbb{R}^n < n+1 \Rightarrow \dim \mathbb{R}^n = n$

Dimension

Examples:

2. $\dim C[a,b] = \infty$
   a. Let $n \in \mathbb{N}$ be arbitrary. The functions $1, x, x^2, \ldots, x^n$ are continuous on any $[a,b] \Rightarrow 1, x, x^2, \ldots, x^n \in C[a,b]$
   b. Check linear dependence/independence of $1, x, x^2, \ldots, x^n$

   $$\alpha_0 \cdot 1 + \alpha_1 x + \ldots + \alpha_n x^n = 0$$

   This equation has n roots $x_1, \ldots, x_n$ for any constants $\alpha_0, \ldots, \alpha_n$. If we want to keep this identity for any x, then $\alpha_0 = \ldots = \alpha_n = 0 \Rightarrow 1, x, x^2, \ldots, x^n$ are linearly independent.
   c. But $n \in \mathbb{N}$ can be any $\Rightarrow$ there is a system of linearly independent elements in $C[a,b]$ which is not finite

   $$\Rightarrow \dim C[a,b] = \infty$$

3. $\dim \mathbb{M}_{2\times 2} = 4$

Dimension

Examples:

4. $\dim P_n(\mathbb{R}) = n+1$
   a. $\forall p(t) \in P_n(\mathbb{R}) \quad p(t) = a_0 \cdot 1 + a_1 t + a_2 t^2 + \dots a_n t^n$
   $$\Rightarrow P_n(\mathbb{R}) = span(1, t, \dots, t^n)$$
   b. The system $1, t, \dots, t^n$ is linearly independent $\Rightarrow \dim P_n(\mathbb{R}) = n+1$
5. $\dim U \oplus W = \dim U + \dim W$
   a. It is enough to prove that
   $$\dim(U+W) = \dim U + \dim W - \dim(U \cap W)$$
   b. Let $u_1, \dots, u_m$ be a basis of $U \cap W \Rightarrow$ we can extend it up to the basis $u_1, \dots, u_m, v_1, \dots, v_j$ of U and up to the basis $u_1, \dots, u_m, w_1, \dots, w_k$ of W.
   c. $\dim U = m + j$, $\dim W = m + k$
   d. Show that $u_1, \dots, u_m, v_1, \dots, v_j, w_1, \dots, w_k$ is the basis for $U + W \Rightarrow \dim(U+W) = m + j + k = (m+j) + (m+k) - m = \dim U + \dim W - \dim(U \cap W)$

Range Subspace of a Matrix

Definition: Let $A \in \mathbb{M}_{m \times n}$. The range subspace of A is the set

$$R(A) = \{Ax, \, x \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

Similarly,

$$R(A^T) = \{A^T y, \, y \in \mathbb{R}^m\} \subset \mathbb{R}^n$$

Example: Let

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}$$

Bases for infinite dimensional linear spaces

Bases

Q: Why do we need to consider different bases in a linear space?

- Is the standard basis $e_i = (0, \dots, \underbrace{1}_{i\ th}, \dots, 0)$ a "good" basis in $\mathbb{R}^n$?
- It gives us only the coordinates of a point. Can we form bases that keep other information?
- Let each coordinate represent brightness of a pixel in an image $\Rightarrow$ the brightness of the whole image is $x_1 + \ldots + x_n$, $x_1 - x_2 + x_3 - \ldots + (-1)^n x_n$ is the "jaggedness" of the image.
- $\mathbb{R}^2$: the vectors $v_1 = (1, 1)$, $v_2 = (1, -1)$ are linearly independent $\Rightarrow \{v_1,\ v_2\}$ is the basis.

$$x = \frac{x_1 + x_2}{2} v_1 + \frac{x_1 - x_2}{2} v_2$$

The coordinates of $x = (x_1, x_2)$ in the basis $\mathfrak{B} = \{v_1,\ v_2\}$ are

$$x_{\mathfrak{B}} = \frac{x_1 + x_2}{2}, \ \frac{x_1 - x_2}{2}$$

Lagrange Interpolation

- You know that p is a polynomial and $deg(p) \leq n-1$. Also $p(\alpha_i) = b_i$, $i = 1, \ldots, n$. Find p.
- The n polynomials

$$g_j = \frac{\prod_{i=1}^{n}(x - \alpha_i)}{x - \alpha_j}$$

  are linearly independent.

  $\Rightarrow g_j$, $j = 1, \ldots, n$ form a basis of $P_{n-1}(\mathbb{R})$.

  $\Rightarrow \forall p \in P_{n-1}(\mathbb{R}) \quad \exists c_j \colon p = \sum_j c_j g_j$
- The coefficients c_j equal

$$c_i = \frac{p(\alpha_i)}{(\alpha_i - \alpha_1)\ldots(\alpha_i - \alpha_{i-1})(\alpha_i - \alpha_{i+1})\ldots(\alpha_i - \alpha_n)}$$

Lagrange Interpolation

- You want to keep your special code safe and you know 5 reliable friends. Ensure that you need only 3 people to recover your code.
- Consider a polynomial $p = code + p_1 x + p_2 x^2$.
- Choose a_1, a_2, a_3, a_4, a_5 and set $b_i = p(a_i)$.
- Give (a_i, b_i) to your ith friend.

Simplest Coding-Decoding Transformations

- Your location at the JI ($\approx x_1 = 121$, $x_2 = 31$) is sent to the central admin. The coordinates are encoded with the code

$$y_1 = x_1 - x_2$$
$$y_2 = -5x_1 + x_2$$

- The received coordinates are

$$\bar{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 121 - 31 \\ -5 \cdot 121 + 31 \end{pmatrix} = \begin{pmatrix} 90 \\ -574 \end{pmatrix}$$

- The coding transformation is

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_2 \\ -5x_1 + x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & -1 \\ -5 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- A transformation of the form $\bar{y} = A\bar{x}$ is called a linear transformation.

Simplest Coding-Decoding Transformations

- ⋆ How can one find your actual location?
- ⋆ One has to solve the system

$$\begin{cases} x_1 - x_2 = y_1 \\ -5x_1 + x_2 = y_2 \end{cases}$$

- ⋆ $\bar{y} \to \bar{x}$ is the decoding transformation.
- ⋆

$$\begin{cases} x_1 = -\frac{1}{4}y_1 - \frac{1}{4}y_2 \\ x_2 = -\frac{5}{4}y_1 - \frac{1}{4}y_2 \end{cases}$$

- ⋆ The inverse (decoding) transformation is $\bar{x} = B\bar{y}$

$$B = \begin{pmatrix} -\frac{1}{4} & -\frac{1}{4} \\ -\frac{5}{4} & -\frac{1}{4} \end{pmatrix}$$

- ⋆ B is the coefficient matrix of the inverse transformation.

Simplest Coding-Decoding Transformations

Q: Is it possible to find the inverse transformation for any linear transformation?

- Consider a linear transformation defined by

$$\begin{cases} y_1 = & x_1 - x_2 \\ y_2 = & -2x_1 + 2x_2 \end{cases}$$

- Multiply the first equation by 2 and add to the second one:

$$\begin{cases} x_1 - x_2 = & y_1 \\ 0 = & 2y_1 + y_2 \end{cases}$$

- The system does not have a solution unless $y_2 = -2y_1$, and it gives infinitely many solutions.
- The inverse transformation does not exist.
- What do you notice about the coefficient matrix

$$A = \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}?$$

Linear Transformations

Definition: A function $T\colon \mathbb{R}^m \to \mathbb{R}^n$ is called a linear transformation if there exists an $n \times m$ matrix A such that

$$T\bar{x} = A\bar{x} \quad \forall \bar{x} \in \mathbb{R}^m.$$

Remark: A linear transformation is a special case of a linear operator:
Let V, U be linear spaces over $\mathbb{K}$. A map $T\colon V \to U$ is a linear operator if

1. $T(v_1 + v_2) = Tv_1 + Tv_2 \quad \forall\, v_1,\, v_2 \in V$
2. $T(\alpha v) = \alpha Tv \quad \forall\, \alpha \in \mathbb{K}, \forall v \in V$

Remark: Any linear operator defined on finite dimensional linear spaces is represented by a matrix.

Linear Operators

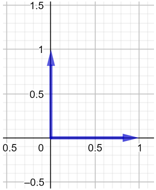
Linear Transformations

Examples of linear transformations:

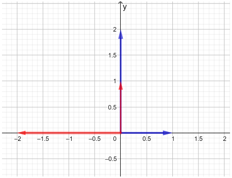
1. $T : \mathbb{R}^3 \to \mathbb{R}^2$, $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$
2. The identity transformation

$$I : \mathbb{R}^n \to \mathbb{R}^n,\ I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

Linear Transformations



3 Let $T\bar{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \bar{x}$



$$\begin{pmatrix} 0 \\ 2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The rotation through $\frac{\pi}{2}$ in the counterclockwise direction.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$\Rightarrow \sqrt{x_1^2 + x_2^2} = \sqrt{(-x_1)^2 + x_2^2}$$

Linear Transformations

For a linear transformation with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, find the inverse linear transformation.

- ⋆ $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \bar{y} = A\bar{x} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{pmatrix}$
- ⋆ Solve the sytem

$$\begin{cases} ax_1 + bx_2 = & y_1 \\ cx_1 + dx_2 = & y_2 \end{cases}$$

$$\begin{cases} x_1 = & \dfrac{1}{ad - cb}(dy_1 - by_2) \\ x_2 = & \dfrac{1}{ad - cb}(ay_2 - cy_1) \end{cases} \Rightarrow B = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\bar{x} = B\bar{y}$$

Linear Transformations

Definition: For a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the quantity $ad - bc$ is called the determinant of A:

$$\det A = |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

If $\det A \neq 0$, then the inverse linear transformation exists and

$$B = A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Lemma: Let $A = (\bar{a}_1 \quad \bar{a}_2)$ be a non-zero matrix. Then

1. $\det A = |\bar{a}_1| \sin\theta |\bar{a}_2|$ where θ is oriented from $\bar{a}_1$ to $\bar{a}_2$, $-\pi < \theta < \pi$
2. The area of the parallelogram spanned by $\bar{a}_1$, $\bar{a}_2$ is $\det A$.
3. $\det A = 0 \Rightarrow \bar{A}_1 || \bar{a}_2$

Problems

2. The cross product of two vectors

$$\bar{x} = (x_1, x_2, x_3),\ \bar{y} = (y_1, y_2, y_3) \in \mathbb{R}^3$$

is given by

$$\bar{x} \times \bar{y} = (x_2 y_3 - x_3 y_2, -x_1 y_3 + x_3 y_1, x_1 y_2 - x_2 y_1)$$

Let $\bar{v} = (v_1, v_2, v_3)$ be fixed and $T\bar{x} = \bar{v} \times \bar{x}$.
Is T a linear transformation?

Problems

3. Consider an arbitrary vector $\bar{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$.
Is the transformation $T\bar{x} = (\bar{x},\ \bar{v})$ linear? $(\cdot, \cdot)$ denotes the dot product. If so, find the matrix of T. Show that the converse is also true: for a linear transformation $T \colon \mathbb{R}^3 \to \mathbb{R}$, there exists $\bar{v} \in \mathbb{R}^3$ such that $T\bar{x} = (\bar{x},\ \bar{v})$.

Problems

4. Find a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\bar{x}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \bar{y}_1 = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

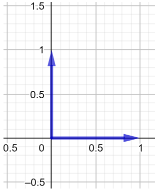
$$\bar{x}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \to \bar{y}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

What is the matrix A^{-1} of the inverse transformation T^{-1}?
How can one use the representation $A^{-1} = (A^{-1}\bar{e}_1 \quad A^{-1}\bar{e}_2)$ to find the matrix of T^{-1}?

Problems

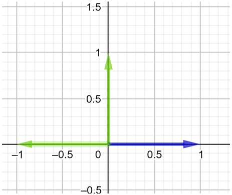
Examples of linear transformations in $\mathbb{R}^n$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



the identity transformation

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$



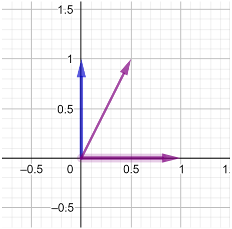
the reflection about the vertical axis

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$ the clockwise rotation through $\dfrac{\pi}{2}$

Examples of linear transformations in $\mathbb{R}^n$

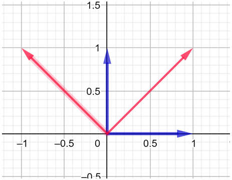
$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ the orthogonal projection onto the horizontal axis

$A = \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{pmatrix}$



the horizontal shear

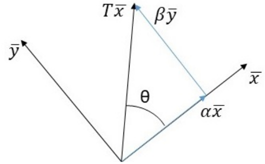
$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$



the scaling by the factor $\sqrt{2}$ and the rotation through $\frac{\pi}{4}$

Rotations

Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that rotates any vector $\bar{x}$ through a fixed angle θ in the counterclockwise direction.



$$||T\bar{x}|| = ||\bar{x}|| = ||\bar{y}|| \text{ and } \quad \bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \perp \bar{y} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$T\bar{x} = \alpha\bar{x} + \beta\bar{y} = \cos\theta\bar{x} + \sin\theta\bar{y}$$

$$\cos\theta = \frac{||\alpha\bar{x}||}{||T\bar{x}||} = \alpha, \ \sin\theta = \frac{||\beta\bar{y}||}{||T\bar{y}||} = \beta$$

$$T\bar{x} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Rightarrow T \text{ is linear}$$

Rotations

The rotation matrix has the form $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, $a^2 + b^2 = 1$.

Example: The matrix of the counterclockwise rotation through $\dfrac{\pi}{4}$ is

$$A = \begin{pmatrix} \dfrac{\sqrt{2}}{2} & -\dfrac{\sqrt{2}}{2} \\ \dfrac{\sqrt{2}}{2} & \dfrac{\sqrt{2}}{2} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Scaling

For any positive scalar k the matrix

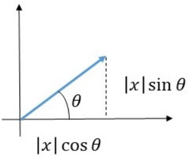
$$A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

defines a scaling by k.

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} kx_1 \\ kx_2 \end{pmatrix} = k \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- $k > 1$ enlargement
- $0 < k < 1$ shrinking
- $k = -1$ the rotation through π
- $-1 < k < 0$ shrinking and rotation through π
- $k < -1$ enlargement and rotation through π

Rotation combined with scaling



$$\bar{x} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} ||\bar{x}|| \cos\theta \\ ||\bar{x}|| \sin\theta \end{pmatrix}$$

The matrix

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} ||\bar{x}|| \cos\theta & -||\bar{x}|| \sin\theta \\ ||\bar{x}|| \sin\theta & ||\bar{x}|| \cos\theta \end{pmatrix}$$

$$= ||\bar{x}|| \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

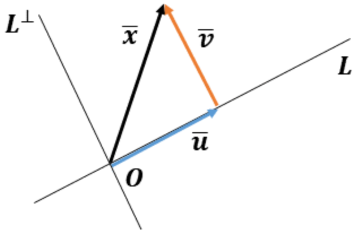
represents the rotation combined with scaling.

Orthogonal Projections in $\mathbb{R}^2$

Let L be a line running through the origin

$$\bar{x} = \bar{u} + \bar{v}, \quad \bar{u}||L, \quad \bar{v} \perp L$$

$$T\bar{x} = proj_L \bar{x} = \bar{u}, \quad \bar{v} = proj_{L^\perp} \bar{x}$$



$$\bar{w} \neq \bar{0},\ \bar{w}||L \Rightarrow \bar{u} = \alpha \bar{w} \quad \text{and}\, (\bar{v}, \bar{w}) = 0 \Rightarrow (\bar{x} - \alpha \bar{w}, \bar{w}) = 0$$

$$\bar{u} = proj_L \bar{x} = \frac{(\bar{x}, \bar{w})}{(\bar{w}, \bar{w})} \bar{w}, \quad A = \frac{1}{w_1^2 + w_2^2} \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix}$$

Orthogonal Projections in $\mathbb{R}^2$

If $||\bar{w}|| = 1$, then $(\bar{w}, \bar{w}) = 1$ and the matrix of the orthogonal projection becomes

$$A = \begin{pmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{pmatrix}$$

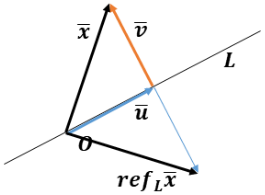
Example: The orthogonal projection onto the line $L = span\{(-1, 3)\}$ is given by the matrix

$$A = \frac{1}{(-1)^2 + 3^2} \begin{pmatrix} (-1)^2 & -1 \cdot 3 \\ -1 \cdot 3 & 3^2 \end{pmatrix} = \begin{pmatrix} 0.1 & -0.3 \\ -0.3 & 0.9 \end{pmatrix}$$

Reflections in $\mathbb{R}^2$

Let L be a line in $\mathbb{R}^2$ running through the origin

$$\bar{x} = \bar{u} + \bar{v}, \quad \bar{u} || L, \quad \bar{v} \perp L, \quad proj_L \bar{x} = \bar{u}$$



The linear transformation $ref_L \bar{x} = \bar{u} - \bar{v}$ is called the reflection of $\bar{x}$ about L.

$$ref_L \bar{x} = \bar{u} - \bar{v} = \bar{u} - (\bar{x} - \bar{u}) = 2\bar{u} - \bar{x} = 2proj_L \bar{x} - \bar{x}$$

$$A = \begin{pmatrix} 2w_1^2 - 1 & 2w_1 w_2 \\ 2w_1 w_2 & 2w_2^2 - 1 \end{pmatrix}, \; ||\bar{w}|| = 1, \; \bar{w} || L$$

Reflections in $\mathbb{R}^2$

$$(2w_1^2-1)+(2w_2^2-1) = 2(w_1^2+w_2^2)-2 = 0 \Rightarrow 2w_1^2-1 = -(2w_2^2-1)$$

$$\text{Also,} \quad 2w_2^2 = 2(1-w_1^2)$$

$$(2w_1^2-1)^2 + (2w_1w_2)^2 = 4w_1^4 - 4w_1^2 + 1 + 4w_1^2w_2^2$$

$$= 4w_1^4 - 4w_1^2 + 1 + 4w_1^2(1-w_1^2) = 1$$

The reflection matrix is of the form

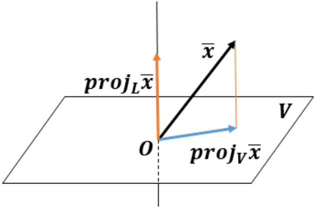
$$A = \begin{pmatrix} a & b \\ b & -a \end{pmatrix},\ a^2 + b^2 = 1$$

Orthogonal Projections and reflections in $\mathbb{R}^3$

Let L be a line in $\mathbb{R}^3$ running through the origin.

$$\bar{x} = proj_L \bar{x} + \bar{v},\ \bar{v} \perp L$$

If $V = L^{\perp}$ is a plane through the origin perpendicular to L then



$$proj_V \bar{x} = \bar{x} - proj_L \bar{x} = \bar{x} - (\bar{x}, \bar{w})\bar{w},\ \bar{w}||L,\ ||\bar{w}|| = 1$$

$$ref_L \bar{x} = proj_L \bar{x} - proj_V \bar{x} = 2(\bar{x}, \bar{w})\bar{w} - \bar{x}$$

$$ref_V \bar{x} = proj_V \bar{x} - proj_L \bar{x} = \bar{x} - 2(\bar{x}, \bar{w})\bar{w}$$

Examples

1. Let L be the line in $\mathbb{R}^3$ spanned by the vector $\bar{y} = (2, 1, 2)$. Find the orthogonal projection of the vector $(1, 1, 1)$ onto L.
Solution:

$$proj_L \bar{x} = (\bar{x}, \bar{w})\bar{w},\ \bar{w} || L,\ ||\bar{w}|| = 1$$

$$\bar{y} = (2, 1, 2) || L \Rightarrow \bar{w} = \frac{\bar{y}}{||\bar{y}||} = \left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

$$proj_L \bar{x} = \left(\frac{2}{3}x_1 + \frac{1}{3}x_2 + \frac{2}{3}x_3\right)\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$$

$$= \left(\frac{4}{9}x_1 + \frac{2}{9}x_2 + \frac{4}{9}x_3, \frac{2}{9}x_1 + \frac{1}{9}x_2 + \frac{2}{9}x_3, \frac{4}{9}x_1 + \frac{2}{9}x_2 + \frac{4}{9}x_3\right)$$

$$= \begin{pmatrix} \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \\ \frac{2}{9} & \frac{1}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{2}{9} & \frac{4}{9} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Examples

$$= \begin{pmatrix} w_1^2 & w_1 w_2 & w_1 w_3 \\ w_1 w_2 & w_2^2 & w_2 w_3 \\ w_1 w_3 & w_2 w_3 & w_3^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \left(\frac{10}{9}, \frac{5}{9}, \frac{10}{9} \right)$$

2. Find the matrices of the following linear transformations $\mathbb{R}^3 \to \mathbb{R}^3$:

a. the orthogonal projection onto xy plane.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

b. the reflection about xz plane.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Examples

c. the rotation about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

d. the rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.

$$\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

e. the rotation about the z-axis through $\pi/4$ turning the positive x-axis towards the positive y-axis

f. the orthogonal projection onto the line $y = x$ on the xy-plane

Composition of Linear Transformations

Any matrix defines a linear transformation $\Rightarrow$ a matrix product defines a composition of linear transformations.

- $D_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$ defines a counterclockwise rotation through α.
- $D_\alpha D_\beta = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$
  $= \begin{pmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{pmatrix}$
- $D_\alpha D_\beta = D_\beta D_\alpha$

Composition of Linear Transformations

$$\bar{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \bar{y}$$

The reflection about the line L with the direction vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$$\bar{x}_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \to \underbrace{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}_{B} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix} = \bar{y}$$

The reflection about the line L with the direction vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$BA = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ the rotation through} \frac{\pi}{2}$$

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ the rotation through} -\frac{\pi}{2}$$

Image and Kernel of a Linear Transformation

Definition: The kernel and image of a linear operator $T\colon V \to W$ are defined by

$$Ker\ T = \{v \in V\colon\ Tv = 0\} \quad Im\ T = \{w \in W\colon\ w = Tv,\ v \in V\}$$

Examples: 1. $T\colon \mathbb{R} \to \mathbb{R},\ Tx = x^2$ (not linear)

$$Ker\ T = \{0\},\ Im\ T = \mathbb{R}_+ \cup \{0\}$$

2. $T\colon \mathbb{R} \to \mathbb{R}^2,\ T(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$ (not linear)

$$Ker\ T = \varnothing,\ Im\ T = \text{unit circle}$$

3. $T\colon \mathbb{R}^3 \to \mathbb{R}^3,\ T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$

$$Ker\ T = k\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ k = const,\ Im\ T = xy \text{ plane}$$

Image and Kernel of a Linear Transformation

4. $T\colon \mathbb{R}^2 \to \mathbb{R}^2,\ T\bar{x} = A\bar{x},\ A = \begin{pmatrix} 2 & 3 \\ 6 & 9 \end{pmatrix}$

$$Ker\ A = t\begin{pmatrix} -3/2 \\ 1 \end{pmatrix},\ Im\ A = span\begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

5. $T\colon P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ Tp(t) = p'(t)$

$$p(t) = a_0 + a_1 t + a_2 t^2 \Rightarrow Tp(t) = p'(t) = a_1 + 2a_2 t$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$Ker\ T = \{p(t)\colon\ Tp = 0\} = \{a_0\} = span(1),\ Im\ T = span(1, t)$$

Image and Kernel of a Linear Transformation

Lemma 1: Let $T: \mathbb{R}^m \to \mathbb{R}^n$ be defined by the matrix $A_{n\times m}$. The columns of the matrix A are linearly independent iff

$$Ker\, A_{n\times m} = \{\bar{0}\} \iff rank\, A = m \Rightarrow m \leq n$$

Lemma 2: Let $T: V \to W$ be a linear operator. $Im\, T$ and $Ker\, T$ are linear subspaces of V and W $\Rightarrow$ there exist bases of the kernel and the image of a linear transformation.

Image and Kernel of a Linear Transformation

Definition: A map $T\colon V \to W$ is called injective if $Tu = Tv$ implies $u = v$.
"distinct inputs to distinct outputs"
Lemma: A linear operator $T\colon V \to W$ is injective iff $Ker\ T = \{0\}$.

- Let T be injective. As $\{0\} \subset Ker\ T$, so we need to show that $Ker\ T \subset \{0\}$.
  Let $v \in Ker\ T \Rightarrow Tv = 0 = T(0) \Rightarrow v = 0 \Rightarrow Ker\ T \subset \{0\}$.
- Let $Ker\ T = \{0\}$. If $Tu = Tv$, then
  $T(u - v) = 0 \Rightarrow u - v \in Ker\ T \Rightarrow u - v = 0 \Rightarrow u = v$

Example: $T\colon \mathbb{R}^2 \to \mathbb{R}^3,\ T(x, y) = (2x, 3y, x + 2y)$

Definition: A map $T\colon V \to W$ is called surjective if $Im\ T = W$.

Example: $T : P_5(\mathbb{R}) \to P_5(\mathbb{R}),\ Tp(t) = p'(t)$ is not surjective.

Image and Kernel of a Linear Transformation: Example

$$T\colon \mathbb{R}^6 \to \mathbb{R}^4,\ A = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow Ker\ A = \{\bar{x} \in \mathbb{R}^6 \colon A\bar{x} = 0\}$$

$$\Rightarrow \begin{cases} x_2 + 2x_3 + 3x_6 = 0 & \Rightarrow x_2 = -2x_3 - 3x_6 \\ x_4 + 4x_6 = 0 & \Rightarrow x_4 = -4x_6 \\ x_5 + 5x_6 = 0 & \Rightarrow x_5 = -5x_6 \end{cases}$$

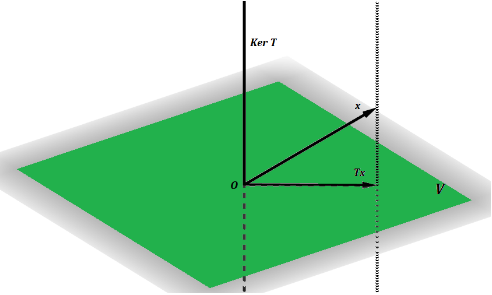
$$\bar{x} = r \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -3 \\ 0 \\ -4 \\ -5 \\ 1 \end{pmatrix} \Rightarrow \dim Ker\ A = 3$$

Rank-Nullity Theorem

$$\boxed{\dim Ker\ T + \dim Im\ T = \dim V}$$

Example: $T : \mathbb{R}^3 \to \mathbb{R}^3,\ T\bar{x} = proj_V \bar{x},\ V \subset \mathbb{R}^3$

$$Ker\ T = \{\bar{x} \in \mathbb{R}^3 : proj_V \bar{x} = \bar{0}\},\ Im\ T = V$$



$$Ker\ T = \text{line orthogonal to } V$$

$$\underbrace{m}_{3} - \underbrace{\dim(Ker\ T)}_{1} = \underbrace{\dim Im\ T}_{2}$$

Rank-Nullity Theorem: Proof

Let $\dim(Ker\ T) = n$ and $\dim Ker\ T = k \Rightarrow k \leq n$.

$\Rightarrow$ there exists a basis $v_1, \ldots, v_k$, of $Ker\ T$. Complete this basis up to the basis of V: $v_1, \ldots, v_k,\ v_{k+1}, \ldots, v_n$

We are to prove that $Tv_{k+1}, \ldots,\ Tv_n$ form the basis for $Im\ T$:

1. $Tv_{k+1}, \ldots,\ Tv_n$ are linearly independent:

   $$\alpha_1 Tv_{k+1} + \ldots + \alpha_{n-k} Tv_n = 0 \Rightarrow T(\alpha_1 v_{k+1} + \ldots + \alpha_{n-k} v_n) = 0$$

   $\Rightarrow \alpha_1 v_{k+1} + \ldots + \alpha_{n-k} v_n \in Ker\ T$

   $\Rightarrow \alpha_1 v_{k+1} + \ldots + \alpha_{n-k} v_n \in span(v_1, \ldots, v_k)$

   But $v_1, \ldots, v_k,\ v_{k+1}, \ldots, v_n$ are linearly independent

   $\Rightarrow \alpha_1 = \ldots = \alpha_{n-k} = 0$
2. $span(Tv_{k+1}, \ldots,\ Tv_n) = Im\ T$
   - A $w \in Im\ T \Rightarrow \exists v \in V:\ Tv = w \Rightarrow T(\beta_1 w_1 + \ldots + \beta_n v_n) = w$

     $$w = \beta_1 \underbrace{Tv_1}_{=0} + \ldots + \beta_k \underbrace{Tv_k}_{=0} + \beta_{k+1} Tv_{k+1} + \ldots + \beta_n Tv_n$$

     $$w \in span(Tv_{k+1}, \ldots,\ Tv_n) \Rightarrow Im\ T \subset span(Tv_{k+1}, \ldots,\ Tv_n)$$
   - B $w \in span(Tv_{k+1}, \ldots,\ Tv_n) \Rightarrow w = \alpha_{k+1} Tv_{k+1} + \ldots + \alpha_{n-k} Tv_n$

     $w = T(\alpha_{k+1} v_{k+1} + \ldots + \alpha_{n-k} v_n) \Rightarrow w \in Im\ T$

Inverse Linear Transformations

Definition: Let V, W be linear spaces.
A linear operator $T\colon V \to W$ is called invertible if there exists a linear operator $S\colon W \to V$ such that ST equals the identity map on V and TS equals the identity map on W.

A linear operator $S\colon W \to V$ satisfying $ST = I$ and $TS = I$ is called an inverse of T.
Here the first I is the identity map on V and the second I is the identity map on W. We shall denote the inverse linear operator by T^{-1}.

$$T^{-1}(Tv) = v \quad \text{and} \quad T(T^{-1}w) = w \quad \forall v \in V \, \forall w \in W$$

Inverse Linear Transformations

Lemma: A linear operator is invertible iff it is one-to-one (injective) and onto (surjective).

- Let T^{-1} exists.
  - A Let $u, v \in V$ and $Tu = Tv$

    $$u = T^{-1}(Tu) = T^{-1}(Tv) = v \Rightarrow \; T \text{ is injective}$$

  - B Let $w \in W \Rightarrow w = T(T^{-1}w) \Rightarrow w \in Im\,T \Rightarrow W \subset Im\,T$
    As also $Im\,T \subset W$, so $W = Im\,T$
- Let T be injective and surjective. For any $w \in W$, define Sw be a unique element of V such that $T(Sw) = w$. This element exists since T is one-to-one and onto.
  - A From the definition, $TS = I$. Also

    $$T((ST)v) = (TS)(Tv) = ITv = Tv \Rightarrow STv = v \Rightarrow ST = I$$

  - B S is linear:

    $$w_1,\, w_2 \in W \Rightarrow T(Sw_1 + Sw_2) = TSw_1 + TSw_2 = w_1 + w_2$$

    Apply the definition of $S \Rightarrow S(w_1 + w_2) = Sw_1 + Sw_2$
    Similarly, $S(\alpha w) = \alpha Sw \, \forall w \in W \, \forall \alpha \in \mathbb{K}$

Inverse Linear Transformations

Remarks:

1. $(T^{-1})^{-1} = T$
2. Let $V, W = \mathbb{R}^n$. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible if the system $A\bar{x} = \bar{y}$ has a unique solution

$$\iff \textit{rank}\, A = n \iff \textit{rref}\, A = I_n$$

Definition: A square matrix A is invertible if the linear transformation $T\bar{x} = A\bar{x}$ is invertible.

Inverse Linear Transformations

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 3 & 2 \end{pmatrix} \Rightarrow A\bar{x} = I\bar{y}$$

$$\left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ -1 & 3 & 2 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 3 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & -5 & 7 & -3 & 1 \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{array}\right) \sim \left(\begin{array}{cccccc} 1 & 0 & 0 & -\frac{2}{5} & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & 0 & \frac{4}{5} & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{7}{5} & \frac{3}{5} & -\frac{1}{5} \end{array}\right)$$

$$I\bar{y} = A^{-1}\bar{x} \Rightarrow A^{-1} = \frac{1}{5}\begin{pmatrix} -2 & 3 & -1 \\ 4 & -1 & 2 \\ -7 & 3 & -1 \end{pmatrix}$$

Inverse Linear Transformations

1. Let $A_{n\times n}$. If A^{-1} exists, then the system $A\bar{x} = \bar{0}$ has a unique solution

   $$\Rightarrow \textit{rank}\, A = n \Rightarrow \text{ columns of } A \text{ are linearly independent.}$$

2. If A^{-1} exists, then $A^{-1}A = AA^{-1} = I$.
3. $(AB)^{-1} = B^{-1}A^{-1}$