Midterm Review — Part I

Test functions

smooth functions

$C^k(\Omega)$ — all partial derivatives of φ with order k exist and are continuous
$C^\infty(\Omega)$ — all partial derivatives of φ with any order exist

support

$\text{supp}\ \varphi = \overline{\{x \in \Omega : \varphi(x) \neq 0\}}$ where $\varphi : \Omega \to \mathbb{C}$
$C_0^\infty(\Omega)$ — the set of $\varphi \in C^\infty(\Omega)$ with $\text{supp}\ \varphi \in \Omega$
$C_0^\infty(\mathbb{R}^n) = \mathcal{D}(\mathbb{R}^n)$ — the set of $\varphi \in C^\infty(\mathbb{R}^n)$ with bounded support
Note: in R^n, a bounded support is compact

test functions

test function is a smooth function with compact support
the set of test function is denoted as $\mathcal{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$

bump function — $b(x) = \begin{cases} e^{-\frac{a}{a^2-x^2}} & |x| < a \\ 0 & \text{otherwise} \end{cases}$

smooth step — $B(x) = \displaystyle\int_{-\infty}^{x} b(t)dt$

null sequence

If φ_m is a sequence with $\varphi_m \in \mathcal{D}(\mathbb{R}^n)$

- $\exists R > 0\ \forall m \in \mathbb{N} \to \text{supp}\ \varphi_m \subset B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$
- $\forall \alpha \in \mathbb{N}^n \to \sup\limits_{x \in \mathbb{R}^n} |D^\alpha \varphi_m(x)| \xrightarrow{m\to\infty} 0$

then φ_m is a null sequence in $\mathcal{D}(\mathbb{R}^n)$

Distributions

distributions

a distribution is a **linear continuous functional** $T : \mathcal{D}(\mathbb{R}^n) \to \mathbb{C}$

- linear: for all $\lambda, \gamma \in \mathbb{R} \quad \varphi, \psi \in \mathcal{D}(\mathbb{R}^n)$, then $T(\lambda\varphi + \gamma\psi) = \lambda T\varphi + \gamma T\psi$
- continuous: for all null sequence φ_m, then $T\varphi_m \xrightarrow{m\to\infty} 0$

the set of test function is denoted as $\mathcal{D}'(\mathbb{R}^n)$

locally integrable functions

a function $f : \mathbb{R}^n \to \mathbb{C}$ is called locally integrable if it satisfies

$\displaystyle\int_\Omega |f(x)|dx \leq \infty$ where Ω is an arbitrary bounded set in $\mathbb{R}^n$

the set of locally integrable functions is denoted as $L^1_{\text{loc}}(\mathbb{R}^n)$

regular distributions

a regular distribution can be expressed as the form of

$T\varphi = T_g\varphi = \displaystyle\int_{-\infty}^{+\infty} g(x)\varphi(x)dx$ where $g \in L^1_{\text{loc}}(\mathbb{R}^n)$

for example: $T\varphi = \displaystyle\int_0^\infty \varphi(x)dx$

singular distributions

a singular distribution is a distribution but is not regular
for example: $T\varphi = \varphi(1)$

basic properties

sum — $(T_1+T_2)\varphi = T_1\varphi + T_2\varphi$

scalar multiplication — $(\lambda T)\varphi = \lambda(T\varphi)$

dilation

dilation operator — $D_\alpha : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n) \quad (D_\alpha\varphi)(x) = \alpha^{\frac{n}{2}}\varphi(\alpha x)$

dilation of distribution — $(D_\alpha T)\varphi = T(D_{\frac{1}{\alpha}}\varphi)$

dilation of regular distribution — $(D_\alpha T_g)\varphi = T_g(D_{\frac{1}{\alpha}}\varphi) = T_{D_\alpha g}\varphi$

translation

translation operator — given $y \in \mathbb{R}^n \quad \tau_y : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n) \quad (\tau_y\varphi)(x) = \varphi(x-y)$

translation of distribution — $(\tau_y T)\varphi = T(\tau_{-y}\varphi)$

translation of regular distribution — $(\tau_y T_g)\varphi = T_g(\tau_{-y}\varphi) = T_{\tau_y g}\varphi$

multiplication

multiplication operator — given $h \in C^\infty(\mathbb{R}^n) \quad M_h : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n) \quad (M_h\varphi)(x) = h(x)\varphi(x)$

multiplication of distribution — $(M_h T)\varphi = T(M_h\varphi)$

multiplication of regular distribution — $(M_h T_g)\varphi = T_g(M_h\varphi) = T_{M_h g}\varphi$

derivative

derivative of distribution — $(D^\alpha T)\varphi = (-1)^{|\alpha|}T(D^\alpha\varphi)$

derivative of regular distribution — $(D^\alpha T_g)\varphi = (-1)^{|\alpha|}T_g(D^\alpha\varphi) = T_{D^\alpha g}\varphi$

principle value

$T_{\frac{1}{x}}\varphi = \mathcal{P}\left(\frac{1}{x}\right)(\varphi) = \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon}\frac{\varphi(x)}{x}dx$

$\lim_{\omega\to\infty}\int_{\mathbb{R}}\frac{(1-\cos(\omega x))\varphi(x)}{x}dx = \mathcal{P}\left(\frac{1}{x}\right)(\varphi)$

$\Delta\left(\frac{1}{|x|}\right) = -4\pi\delta(x)$ in $\mathbb{R}^3$ distributionally

Families of distributions

convergence

For the following conditions

- $I \subset \mathbb{R}$
- $\{T_\alpha\}_{\alpha\in I} \quad T_\alpha \in \mathcal{D}'(\mathbb{R}^n)$
- $\alpha_0 \in \overline{I}$
- $T \in \mathcal{D}'(\mathbb{R}^n)$
- $\lim_{\alpha\to\alpha_0} T_\alpha\varphi = T\varphi$

We say $\lim_{\alpha\to\alpha_0} T_\alpha = T$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$

uniformly convergence $\Rightarrow$ pointwise convergence

uniformly convergence $\Rightarrow$ distributionally convergence

delta families

Given the following conditions

- $I \subset \mathbb{R}$
- $\{f_\alpha\}_{\alpha\in I}$ is a families of functions and $f_\alpha \in L^1_{\text{loc}}(\mathbb{R}^n)$
- $\lim_{\alpha\to\alpha_0} f_\alpha = \delta$

$\{f_\alpha\}_{\alpha\in I}$ is called a delta family

if $I = \mathbb{N}, \alpha_0 = \infty$, it is also called a delta sequence

The following procedures construct a delta family $\{f_\alpha\}_{\alpha\in(0,+\infty)}$ as $\alpha \searrow 0$

- $\forall x\ f(x) \geq 0$
- $\int_{\mathbb{R}^n} f(x)dx = 1$

- $f_\alpha = \frac{1}{\alpha^n} f\left(\frac{x}{\alpha}\right)$ with $\alpha > 0$

delta families examples

$f(x) = \frac{1}{\pi(x^2+1)}$ — $f_y(x) = \frac{1}{y} f\left(\frac{x}{y}\right) = \frac{1}{\pi(x^2+y^2)}$ $(y > 0)$ as $y \searrow 0$

$f(x) = \frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}$ — $f_t(x) = \frac{1}{\sqrt{t}} f\left(\frac{x}{\sqrt{t}}\right) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$ $(t > 0)$ as $t \searrow 0$

$f(x) = H(x)xe^{-x}$ — $f_k(x) = kf(kx) = k^2 H(x)xe^{-kx}$ $(k \in \mathbb{N})$ as $k \to +\infty$

Poisson kernel — $f_r(\theta) = \begin{cases} \frac{1-r^2}{2\pi(1+r^2-2r\cos\theta)} & |\theta| < \pi \\ 0 & |\theta| > \pi \end{cases}$ $(0 \le r < 1)$ as $r \nearrow 1$

Dirichlet kernel — $f_R(x) = \frac{1}{2\pi}\int_{-R}^{R} e^{i\omega x} d\omega = \frac{\sin(Rx)}{\pi x}$ $(R > 0)$ as $R \to +\infty$

Fourier Transform

functions of rapid decrease

a function of rapid decrease φ satisfies $\varphi \in C^\infty(\mathbb{R}^n)$ and $\forall \alpha, \beta \in \mathbb{N}^n \ \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi(x)| < \infty$

the set of functions of rapid decrease is denoted as $\mathcal{S}(\mathbb{R}^n)$

a test function is also a function of rapid decrease $\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n)$

Fourier transform

Fourier transform on $\varphi \in \mathcal{S}(\mathbb{R}^n)$ — $(\mathcal{F}\varphi)(\xi) = \widehat{\varphi}(\xi) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} \varphi(x) dx$

inversion Fourier transform on $\varphi \in \mathcal{S}(\mathbb{R}^n)$ — $\varphi(x) = \frac{1}{(\sqrt{2\pi})^n} \int_{\mathbb{R}^n} e^{-i\langle x, \xi \rangle} \widehat{\varphi}(\xi) d\xi$

Fourier transform on $\varphi \in \mathbb{R}$ — $\widehat{\varphi}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\xi} \varphi(x) dx$

inversion Fourier transform on $\varphi \in \mathbb{R}$ — $\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\xi} \widehat{\varphi}(\xi) d\xi$

basic properties

$\mathcal{F}[D^\alpha((-ix)^\beta)\varphi(x)](\xi) = (i\xi)^\alpha D^\beta(\mathcal{F}\varphi)(\xi)$

- $\alpha = 0, \beta = 1 \quad \widehat{-ix\varphi}(\xi) = \widehat{\varphi}'(\xi)$
- $\alpha = 1, \beta = 0 \quad \widehat{\varphi'}(\xi) = i\xi\widehat{\varphi}(\xi)$
- $\alpha = 1, \beta = 1 \quad -i\widehat{\varphi - ix\varphi'}(\xi) = i\xi\widehat{\varphi}'(\xi)$
- $\vdots$

$\hat{\hat{\varphi}}(x) = \varphi(-x)$

null sequence

if φ_m is a sequence with $\varphi_m \in \mathcal{S}(\mathbb{R}^n)$

- $\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi_m(x)| \xrightarrow{m \to \infty} 0$

then φ_m is called a null sequence in $\mathcal{S}(\mathbb{R}^n)$

Tempered distributions

tempered distributions

a linear continuous functional T on $\mathcal{S}(\mathbb{R}^n)$

- linear — for all $\lambda, \gamma \in \mathbb{R} \quad \varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$, then $T(\lambda\varphi + \gamma\psi) = \lambda T\varphi + \gamma T\psi$
- continuous — for all null sequence $\varphi_m T\varphi_m \xrightarrow{m \to \infty} 0$

is called a tempered distribution

the set of tempered distributions is denoted as $\mathcal{S}'(\mathbb{R}^n)$

$\mathcal{D}(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$

Fourier transform

Fourier transform $\mathcal{F}$ is a tempered distribution

Fourier transform on tempered distribution — $\widehat{T}\varphi = T\widehat{\varphi}$

Fourier transform on regular tempered distribution — $\widehat{T_g}\varphi = T_g\widehat{\varphi} = T_{\hat{g}}\varphi$

inversion Fourier transform

inversion Fourier transform on tempered distribution — $(\mathcal{F}^{-1}T)\varphi = T(\mathcal{F}^{-1}\varphi)$

inversion Fourier transform on regular tempered distribution — $(\mathcal{F}^{-1}T_g)\varphi = T_g(\mathcal{F}^{-1}\varphi) = T_{\mathcal{F}^{-1}g}\varphi$

convolution

convolution on functions — $(f * g)(x) = \int_{\mathbb{R}^n} f(x-\tau)g(\tau)d\tau = \int_{\mathbb{R}^n} f(\tau)g(\tau)d\tau$

convolution on distributions — $(T * \psi)(\varphi) = T(\widetilde{\psi} * \varphi)$ with $\widetilde{\psi}(x) = \psi(-x)$

convolution on regular distributions — $(T_g * \psi)(\varphi) = T_g(\widetilde{\psi} * \varphi) = T_{g*\psi}\varphi$

Fourier transform pair

$\varphi(x)$	$\widehat{\varphi}(\xi)$
$\delta(x)$	$\frac{1}{\sqrt{2\pi}}$
1	$\sqrt{2\pi}\delta(\xi)$
$H(x)$	$-\frac{i}{\sqrt{2\pi}}\mathcal{P}\left(\frac{1}{x}\right) + \sqrt{\frac{\pi}{2}}\delta(\xi)$
$\mathrm{sgn}(x)$	$-i\sqrt{\frac{2}{\pi}}\mathcal{P}\left(\frac{1}{x}\right)$
$\frac{1}{x}$ or $\mathcal{P}\left(\frac{1}{x}\right)$	$-i\sqrt{\frac{\pi}{2}}\mathrm{sgn}(x)$
$\frac{1}{a^2+x^2}$	$\frac{\sqrt{2\pi}}{2a}e^{-a\lvert\xi\rvert}$
$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{a}{a^2+\xi^2}$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}e^{-\frac{\xi^2}{4a}}$

Fourier transform properties

$\varphi(x)$	$\widehat{\varphi}(\xi)$
$f(ax)$	$\frac{1}{\lvert a\rvert}\widehat{f}\left(\frac{x}{a}\right)$
$f(x+a)$	$e^{iax}\widehat{f}(\xi)$
$e^{iax}f(x)$	$\widehat{f}(\xi-a)$
$\cos(\omega x)f(x)$	$\frac{1}{2}\left(\widehat{f}(\xi+\omega)+\widehat{f}(\xi-\omega)\right)$
$\sin(\omega x)f(x)$	$\frac{i}{2}\left(\widehat{f}(\xi+\omega)-\widehat{f}(\xi-\omega)\right)$
$D^a f(x)$	$(i\xi)^a\widehat{f}(\xi)$
$x^a f(x)$	$i^a D^a\widehat{f}(\xi)$
$f(x)g(x)$	$\frac{1}{\sqrt{2\pi}}\left(\widehat{f} * \hat{g}\right)(\xi)$
$(f * g)(x)$	$\sqrt{2\pi}\widehat{f}(\xi)\hat{g}(\xi)$
$\int_{\infty}^{x} f(t)dt = (f * H)(x)$	$-i\widehat{f}(\xi)\mathcal{P}\left(\frac{1}{\xi}\right) + \pi\widehat{f}(0)\delta(\xi)$
$\widehat{f}(x)$	$f(-x)$

heat kernel

The heat equation of $u(x,t)$ — $\frac{\partial u}{\partial t} - \Delta u = 0$ with $x,t \in \mathbb{R}^n \times R_+$

with the initial condition — $u(x,0) = f(x) \in \mathcal{S}'(\mathbb{R}^n)$

has the unique solution $u(\cdot,t) \in \mathcal{S}'(\mathbb{R}^n)$ — $u(x,t) = (f * p)(x,t)$ for $t > 0$

where $p(x,t)$ is the heat kernel $p(x,t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$

Differential Operators

linear ordinary differential operators

an ordinary differential operator L of order p

$L = \sum_{k=0}^{p} a_k(x) \frac{d^k}{dx^k}$ with $a_k \in C^\infty((\mathrm{a},\mathrm{b}),\mathbb{R})$

formal adjoint

The formal adjoint L^* is defined as $(LT)\varphi = T(L^*\varphi)$

calculate the formal adjoint with $D^n(aT)\varphi = (-1)^n T(D^n(a\varphi))$

second order —

$L = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x) \quad \Rightarrow \quad L^* = a_2(x)\frac{d^2}{dx^2} + (2a_2'(x) - a_1(x))\frac{d}{dx} + (a_2''(x) - a_1'(x) + a_0(x))$

conjunct

inner product — $\langle \varphi, \psi \rangle_{L^2([a,b])} = \int_a^b \varphi(x)\psi(x)dx$

conjunct J by Green's formula — $J(\varphi,\psi)|_a^b = \langle \psi, L\varphi \rangle_{L^2([a,b])} - \langle L^*\psi, \varphi \rangle_{L^2([a,b])}$

$J(\varphi,\psi) = \sum_{k=1}^{p} \sum_{i+j=k-1} (-1)^i D^i(a_k(x)\psi(x)) D^j \varphi(x)$

second order — $L = a_2(x)\frac{d^2}{dx^2} + a_1(x)\frac{d}{dx} + a_0(x) \quad \Rightarrow \quad J(\varphi,\psi) = a_2(x)(\psi\varphi' - \varphi\psi') + (a_1(x) - a_2'(x))\varphi\psi$

Lagrange's identity — $\psi L\varphi - \varphi L^*\psi = \frac{d}{dx} J(\varphi,\psi)$

classical solutions

the differential equation $Lu = f$ on Ω satisfies

- L is an ordinary or partial differential operator
- Ω is a domain in $\mathbb{R}^n$
- f is a continuous function on Ω

A classical solution is a function $u \in C^p(\Omega)$ satisfying $Lu = f$ on Ω

weak solutions

the differential equation $Lu = f$ on Ω satisfies

- L is an ordinary or partial differential operator
- Ω is a domain in $\mathbb{R}^n$
- $f \in L^1_{\mathrm{loc}}(\Omega)$ is a locally integrable function on Ω

A weak solution is a function $u \in L^1_{\mathrm{loc}}(\Omega)$ satisfying $(LT_u) = T_f\varphi$ on Ω with $\operatorname{supp}\varphi \subset \Omega$

distributional solutions

the differential equation $Lu = f$ on Ω satisfies

- L is an ordinary or partial differential operator
- Ω is a domain in $\mathbb{R}^n$
- $S \in \mathcal{D}'(\mathbb{R}^n)$

A distributional solution is a distribution $T \in \mathcal{D}'(\mathbb{R}^n)$ satisfying $(LT) = S\varphi$ on Ω with $\operatorname{supp}\varphi \subset \Omega$

fundamental solutions

Let $\xi \in \mathbb{R}^n$ be fixed. A fundamental solution for L with pole at ξ is $E(\cdot, \xi) \in \mathcal{D}'(\mathbb{R}^n)$ satisfying

$LE(x, \xi) = \delta(x - \xi)$

if operator L has constant coefficients, then $E(x, \xi) = E(x - \xi, 0)$

causal fundamental solutions — $E(x, \xi) = 0$ for $x < \xi$

types of solutions

u is a weak solution $+$ $u \in C^p(\Omega) \Rightarrow u$ is a classical solution

u is a classical solution $+$ $u \in L^1_{\text{loc}}(\Omega) \Rightarrow u$ is a weak solution

u is a distributional solutions $+$ (S is regular $\Leftrightarrow$ T is regular) $\Rightarrow u$ is a weak regular

E is a fundamental solutions $\Rightarrow E$ is a distributional solutions

Heuristic construction

Construct $Lu_\xi = 0$, given initial condition $u_\xi(\xi) = u'_\xi(\xi) = \cdots = u_\xi^{(p-2)}(\xi) = 0, u_\xi^{(p-1)}(\xi) = \dfrac{1}{a_p(\xi)}$ with its solution $u_\xi(x)$

then a causal fundamental solution for $LE(x, \xi) = \delta(x, \xi)$ is given by $E(x, \xi) = H(x - \xi)u_\xi(x)$

Initial Value Problems

ordinary differential equations

Consider an ordinary differential equation $Lu = f$ of order p on an open interval $I \subset \mathbb{R}$ satisfying

- L is a linear ordinary differential operator
- f is a piecewise continuous on $\overline{I}$
- function coefficients of L satisfy $a_p, \cdots, a_0 \in C(\overline{I})$
- $a_p(x) \neq 0$ for $x \in I$

initial value problem

an initial value problem follows the patterns below

- ordinary differential equation — $Lu = f$ on I
- initial conditions — $u(x_0) = \gamma_1, u'(x_0) = \gamma_2, \cdots, u^{(p-1)}(x_0) = \gamma_p$ with $x_0 \in \overline{I}, \gamma_1, \gamma_2, \cdots, \gamma_p \in \mathbb{R}$

which is denoted as $\{f; \gamma_1, \gamma_2, \cdots, \gamma_p\}_{x_0}$

IVP has a classical unique solution on $\overline{I}$

Abel's formula for the Wronskian

Suppose $u_1, u_2, \cdots, u_p$ is p solutions for $Lu = 0$ on $I \subset \mathbb{R}$

then $W(u_1, u_2, \cdots, u_p; x) = Ce^{-m(x)}$ where

- W indicates the Wronskian
- C is a constant value
- m is a particular solution for $m'(x) = \dfrac{a_{p-1}(x)}{a_p(x)}$

indicating — $W(u_1, u_2, \cdots, u_p; x) = 0$ $(\forall x \in I) \Leftrightarrow W(u_1, u_2, \cdots, u_p; x_0) = 0$ $(\exists x_0 \in I)$

indicating — $u_1, u_2, \cdots, u_p$ is dependent $\Leftrightarrow W(u_1, u_2, \cdots, u_p; x_0) = 0$ $(\exists x_0 \in I)$

basis of solutions

$\{u_1, u_2, \cdots, u_p\}$ forms a basis of solutions for L on I given by

- u_1 solves $\{0; 1, 0, \cdots, 0\}_{x_0}$
- u_2 solves $\{0; 0, 1, \cdots, 0\}_{x_0}$
- $\vdots$
- u_p solves $\{0; 0, 0, \cdots, 1\}_{x_0}$

and the solution for $Lu = 0$ with $\{0; \gamma_1, \gamma_2, \cdots, \gamma_p\}$ is given by $u(x) = \gamma_1 u_1(x) + \gamma_2 u_2(x) + \cdots + \gamma_p u_p(x)$

particular solution

we consider the differential equation $Lu = f$ with $\{f; 0, 0, \cdots, 0\}_{x_0}$

the solution is given by $u(x) = \int_{x_0}^{x} u_\xi(x) f(\xi) d\xi$

where u_ξ is the solution for $\left\{0; 0, 0, \cdots, 0, \frac{1}{a_p(\xi)}\right\}_\xi$

inhomogeneous equation

the solution for $Lu = f$ with $\{f; \gamma_1, \gamma_2, \cdots, \gamma_p\}_{x_0}$ is given by

$$u(x) = \underbrace{\int_{x_0}^{x} u_\xi(x) f(\xi) d\xi}_{\text{particular solution}} + \underbrace{\sum_{i=1}^{p} \gamma_i u_i(x)}_{\text{general solution}}$$

Final Review — Part II

Second-Order Boundary Value Problems

General problem

Consider the ODE

$Lu = f \quad \Rightarrow \quad a_2(x)u''(x) + a_1(x)u'(x) + a_0(x)u(x) = f(x)$

on $(a, b) \in \mathbb{R}$ with boundary conditions

$B_1 u = \alpha_{11} u(a) + \alpha_{12} u'(a) + \beta_{11} u(b) + \beta_{12} u'(b) = \gamma_1$

$B_2 u = \alpha_{21} u(a) + \alpha_{22} u'(a) + \beta_{21} u(b) + \beta_{22} u'(b) = \gamma_2$

$\{f; \gamma_1, \gamma_2\}$ is the data for the problem $\{L, B_1, B_2\}$

Fundamental solution

a fundamental solution $E(x; \xi)$ for L with pole at $\xi \in [a, b]$ satisfies

$LE = \delta(x - \xi)$

Method 1

$LE = 0$ for $x \in (a, \xi) \cup (\xi, b)$

E is continuous on (a, b)

$$\left.\frac{dE}{dx}\right|_{x=\xi^+} - \left.\frac{dE}{dx}\right|_{x=\xi^-} = \frac{1}{a_2(x)}$$

Method 2

Heuristic Construction before mid-term to find a causal fundamental solution

Green's function

the Green's function $g(x; \xi)$ for $\{L, B_1, B_2\}$ satisfies

$Lg = \delta(x - \xi) \quad B_1 g = 0 \quad B_2 g = 0$

Basic functions

two basic functions u_1, u_2 are independent solutions satisfying

$Lu_1 = Lu_2 = 0 \quad B_1 u_1 = 0 \quad B_2 u_2 = 0$

Unmixed boundary conditions

Let the boundary condition for a second-order ODE

$B_1 u = \alpha_1 u(a) + \alpha_2 u'(a)$

$B_2 u = \beta_1 u(b) + \beta_2 u'(b)$

We first find two basic functions u_1, u_2

The Green's function can be derived as

$g(x; \xi) = \begin{cases} c_1 u_1(x) & x < \xi \\ c_2 u_2(x) & x > \xi \end{cases}$ where c_1, c_2 are constants to be specified by the following equations

$$\begin{cases} c_1 u_1(\xi) - c_2 u_2(\xi) = 0 \\ -c_1 u_1'(\xi) + c_2 u_2'(\xi) = \dfrac{1}{a_2(\xi)} \end{cases} \Rightarrow \begin{cases} c_1 = \dfrac{u_2(\xi)}{a_2(\xi)\Big[(u_1(\xi)u_2'(\xi) - u_1'(\xi)u_2(\xi)\Big]} \\ c_2 = \dfrac{u_1(\xi)}{a_2(\xi)\Big[(u_1(\xi)u_2'(\xi) - u_1'(\xi)u_2(\xi)\Big]} \end{cases}$$

General boundary conditions

In general boundary conditions B_1 and B_2

we find the causal fundamental solution $E(x;\xi) = H(x-\xi)u_\xi(x)$ first

then determine two basic functions u_1, u_2

$g(x;\xi) = E(x;\xi) + c_1 u_1(x) + c_2 u_2(x)$ satisfying the boundary conditions $B_1 g = B_2 g = 0$

Solution formula

Given the second-order ODE problem $\{L, B_1, B_2\}$ with data $\{f; \gamma_1, \gamma_2\}$ on (a,b)

we can find two basic functions u_1, u_2 and the Green's function $g(x;\xi)$

then $u(x) = \displaystyle\int_a^b g(x;\xi) f(x) dx + \frac{\gamma_2}{B_2 u_1} u_1(x) + \frac{\gamma_1}{B_1 u_2} u_2(x)$

Adjoint BVPs and Higher-Order Equations

Find adjoint problem

Given an ODE problem $Lu = f$ with boundary operators $\{B_k\}$ $(k = 1, 2, \cdots, p)$

First determine the adjoint operator L^*

Wisely choose $\{B_k\}$ $(k = p+1, p+2, \cdots, 2p)$

so that $\{B_k\}$ $(k = 1, 2, \cdots, 2p)$ are all independent (and easy to calculate)

then calculate and rewrite the conjunct into the following form

$J(u,v)\Big|_a^b = \displaystyle\sum_{k=1}^{2p} \Big[B_k u \cdot B^*_{2p-k+1} v\Big]$

Then we get the adjoint boundary operator $\{B_k^*\}$ $(k = 1, 2, \cdots, 2p)$ (Note: not unique)

Adjoint Green's function

the original Green's function $g(x;\xi)$ satisfies

$Lg = \delta(x-\xi) \quad B_k g = 0 \ (k = 1, 2, \cdots, p)$

the adjoint Green's function $g^*(x;\xi)$ satisfies

$L^* g^* = \delta(x-\xi) \quad B_k^* g = 0 \ (k = 1, 2, \cdots, p)$

Solution formula

$$u(x) = \int_a^b g(x;\xi) f(\xi) d\xi - J\Big(u, g(x;\cdot)\Big)\Big|_a^b$$

Modified Green's Functions

Solvability conditions

Given the ODE problem of order p on (a,b)

$Lu = f \quad x \in (a,b)$ with boundary conditions $B_k u = \gamma_k$ $(k = 1, 2, \cdots, p)$

Then we find the adjoint operators L^* and adjoint boundary operators $\{B_k^*\}$ $(k = 1, 2, \cdots, 2p)$

We find all m independent non-trivial solutions $\{v_k\}$ for the completely homogeneous adjoint problem

$L^* v = 0 \quad B_k^* = 0 \ (k = 1, 2, \cdots, p)$

Then the solvability conditions can be expressed as

$\displaystyle\int_a^b f(x) v_k(x) dx = \sum_{i=1}^{p} \Big[\gamma_i \cdot (B^*_{2p-i+1} v_k)\Big] \quad k = 1, 2, \cdots, m$

(Note: there will be undetermined coefficients in $v_k(x)$, but they should be cancelled later)

Modified Green's functions

We can see that Green's function doesn't exist generally

by plugging $f(x) = \delta(x-\xi)$ into the solvability conditions

We change to find the modified Green's function $g_M(x;\xi)$

Step 1: Find one fundamental solution $E(x;\xi)$

Step 2: Transform m independent non-trivial solutions $\{v_k\}$ for the completely homogeneous adjoint problem (in solvability conditions step) into m orthonormal solutions $\{w_k\}$ $(k=1,2,\cdots,m)$

Step 3: Find m solutions $\{z_k\}$ $(k=1,2,\cdots,m)$ satisfying the differential equations $Lz_k = w_k$ $(k=1,2,\cdots,m)$

Step 4: Find p independent solutions u_k $(k=1,2,\cdots,p)$ for the homogeneous equation $Lu=0$

Step 5: $g_M(x;\xi) = E(x;\xi) - \sum_{k=1}^{m}\Big[v_k(\xi)z_k(x)\Big] + \sum_{k=1}^{p}\Big[c_k u_k(x)\Big]$

Step 6: Use $B_k g_m = 0$ $(k=1,2,\cdots,p)$ to determine the coefficients $\{c_k\}$ $(k=1,2,\cdots,p)$

BVP for PDE

General problem

$Lu = -\nabla\cdot(p\nabla u) + q \quad x\in\Omega$ where

- $p(x) > 0$
- $q(x) \geq 0$
- $\Omega \subset \mathbb{R}^2$ or $\Omega \subset \mathbb{R}^3$

Second-order equations

Elliptic equation: $Lu = \rho(x)F(x)$

Parabolic equation: $\rho(x)\dfrac{\partial u}{\partial t} + Lu = \rho(x)F(x,t)$

Hyperbolic equation: $\rho(x)\dfrac{\partial^2 u}{\partial t^2} + Lu = \rho(x)F(x,t)$

Boundary conditions

boundary operator: $Bu = \alpha(x)u + \beta(x)\dfrac{\partial u}{\partial n}\Big|_{\partial\Omega}$

$\alpha(x)\geq 0 \quad \beta(x)\geq 0 \quad \alpha(x)+\beta(x) > 0$

Separate $\partial\Omega = S_1\cup S_2\cup S_3$ where

- S_1 — $\beta(x) = 0$
- S_2 — $\alpha(x) = 0$
- S_3 — $\alpha(x)\neq 0$ and $\beta(x)\neq 0$

Solution formula for elliptic equation

$Lu = -\nabla\cdot(p\nabla u) + q \qquad Lu = \rho(x)F(x)$

$Bu = \alpha(x)u + \beta(x)\dfrac{\partial u}{\partial n}\Big|_{\partial\Omega} = \gamma$

Since elliptic equation is self-adjoint, then

$$u(\xi) = \int_{\Omega} g(x;\xi)\rho(x)F(x)dx - \int_{S_1}\frac{p\gamma}{\alpha}\frac{\partial g(\cdot;\xi)}{\partial n}d\sigma - \int_{S_2\cup S_3}\frac{p\gamma}{\beta}g(\cdot;\xi)d\sigma$$

Solution formula for parabolic equation

$Lu = -\nabla\cdot(p\nabla u) + q \qquad \rho(x)\dfrac{\partial u}{\partial t} + Lu = \rho(x)F(x,t)$

$\widetilde{L} = \rho(x)\dfrac{\partial}{\partial t} + L \qquad \widetilde{L}^* = -\rho(x)\dfrac{\partial}{\partial t} + L$

$$\partial V = \underbrace{\Big(\Omega\times\{0\}\Big)}_{\text{bottom}} \cup \underbrace{\Big(\partial\Omega\times[0,T]\Big)}_{\text{mantle}} \cup (\underbrace{\Big(\Omega\times\{T\}\Big)}_{\text{top}}$$

$Bu = \alpha(x)\cdot u\Big|_{\partial\Omega\times[0,T]} + \beta(x)\cdot\dfrac{\partial u}{\partial n}\Big|_{\partial\Omega\times[0,T]} = \gamma(x,t) \qquad \widetilde{B_1}u = u\Big|_{\Omega\times\{0\}} = u(x,0) = f(x)$

$B^*v = Bv = \alpha(x)\cdot v\Big|_{\partial\Omega\times[0,T]} + \beta(x)\cdot\dfrac{\partial v}{\partial n}\Big|_{\partial\Omega\times[0,T]} \qquad \widetilde{B_1}^* v = v\Big|_{\Omega\times\{T\}} = v(x,T)$

The adjoint green's function $g^*(x,t;\xi,\tau)$ satisfies

$$\widetilde{L}^* g^* = \delta(x,t;\xi,\tau) \quad B^* g^* = 0 \quad \widetilde{B_1}^* g^* = 0 \quad \Rightarrow \quad \widetilde{L}^* g^* = \delta(x,t;\xi,\tau) \quad Bg^* = 0 \quad g^*(x,T;\xi,\tau) = 0$$

$$\begin{aligned} u(\xi,\tau) = & \int_V \rho(x)F(x,t)g^*(x,t;\xi,\tau)d(x,t) + \int_\Omega \rho(x)f(x)g^*(x,0;\xi,\tau)dx \\ & - \int_{\widetilde{S_1}} \frac{p\gamma}{\alpha}\frac{\partial g^*(\cdot;\xi,\tau)}{\partial n_x}d\sigma + \int_{\widetilde{S_2}\cup\widetilde{S_3}} \frac{p\gamma}{\beta}g^*(\cdot;\xi,\tau)d\sigma \end{aligned}$$

where $\widetilde{S_k} = S_k \times [0,T]$ $(k = 1,2,3)$

Find Green's Function

Partial eigenfunction expansion

Step 1: Separate variables of x_1 and x_2

Step 2: Choose one variable (x_1 here as example) and calculate its eigenvalue λ_n with orthonormal eigenfunction $\varphi_n(x_1)$

Step 3: Write $g(x;\xi) = \sum_{n=1}^{\infty} g_n(x_2;\xi_2)\varphi_n(x_1)$ and $g_n(x_2;\xi_2) = \left\langle g(x;\xi), \varphi_n(x_1) \right\rangle$

Step 4: Combine the original PDE and find the ODE for $g_n(x_2;\xi_2)$

Step 5: Solve the ODE for $g_n(x_2;\xi_2)$ and derive the solution

Step 6: Find $g(x;\xi)$ by plugging into the solution of $g_n(x_2;\xi_2)$

Step 7: Use solution formula to find u by the Green's function g

The method of images

Step 1: Find one fundamental solution $E(x;\xi)$

Step 2: Use models to find $v(x;\xi)$ so that $g(x;\xi) = E(x;\xi) + v(x;\xi)$ and $Bg = BE = \delta(x-\xi)$ and $Bg = 0$

Step 3: Use derived $g(x;\xi)$ and solution formula to find the solution u