Vev557 Methods of Applied Mathematics II

Sample Exercises for the Midterm Exam

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The following exercises are sample exercises of a difficulty comparable to those found the actual first midterm exam. The exam will usually include of 5 to 8 such exercises to be completed in 100 minutes.

Definitions and Concepts

Some questions will test your understanding of basic definitions and concepts. The answers will involve either multiple choice selections or ask you to write a sentence or two explaining the concept.

Exercise 1 Multiple Choice

In the following exercises, mark the boxes corresponding to true statements with a cross (☒). In each case, it is possible that none of the statements are true or that more than one statement is true.

i) Which of the following statements are correct

- ☐ All distributions have a Fourier transform.
- ☒ All tempered distributions have a Fourier transform.
- ☒ All distributions have a derivative.
- ☒ All tempered distributions have a derivative.

ii) ‘ If the Fourier transform of $f(x)$ is $\hat{f}(\xi)$, what is the Fourier transform of $f(ax+b)$?

- ☐ $e^{-ib\xi}\hat{f}\left(\dfrac{\xi}{a}\right)$
- ☐ $\dfrac{e^{-ib\xi}}{|a|}\hat{f}\left(\dfrac{\xi}{a}\right)$
- ☐ $e^{ib\xi/a}\hat{f}\left(\dfrac{\xi}{a}\right)$
- ☒ $\dfrac{e^{ib\xi/a}}{|a|}\hat{f}\left(\dfrac{\xi}{a}\right)$

iii) Given a test function $\varphi \in \mathcal{D}(\mathbb{R})$, the sequence $\{\varphi(mx)/m\}$ is NOT a null sequence because

- ☐ the support of $\{\varphi(mx)/m\}$ is not bounded as $m \to \infty$;
- ☐ the support of the derivatives of $\{\varphi(mx)/m\}$ is not bounded as $m \to \infty$;
- ☒ $\{\varphi(mx)/m\}$ does not converge to 0 uniformly as $m \to \infty$;
- ☒ the derivatives of $\{\varphi(mx)/m\}$ do not converge to 0 uniformly as $m \to \infty$;

iv) Which of the following functions of real numbers are distributions?

- ☒ $f(x) = e^{x^4}$
- ☒ $f(x) = e^{-x^2}$
- ☐ $f(x) = e^{-x^2}/x$
- ☒ $f(x) = e^{-x^2}/\sqrt{x}$

(8 Marks)

Distributions

You should be familar with basic properties of distributions, such as how they are defined and what a regular distribution is. It is also important to know how to differentiate distributions and perform other basic operations.

Exercise 2 Straightforward Derivatives

Differentiate the following elements of $\mathcal{D}'(\mathbb{R})$:

i) $\delta(x-2) + 2H(x-2)$,

ii) $x^2\delta(x-1)$,

(4 Marks)

Proof.

i) $\delta'(x-2) + 2\delta(x-2)$

ii) $2x\delta(x-1) + x^2\delta'(x-1)$

□

Exercise 3 Derivative of a Function with a Jump Discontinuity

Prove the following statement: Let $I \subset \mathbb{R}$ be an open interval and $f \in L^1_{\text{loc}}(I) \cap C^1(I \setminus \{\xi\})$ such that the left- and right-hand limits of f and f' at ξ exist. Denote

$$[f]_\xi := \lim_{\varepsilon \to 0} \big(f(\xi+\varepsilon) - f(\xi-\varepsilon)\big).$$

Then

$$(T_f)' = [f]_\xi \cdot \delta(x-\xi) + T_{f'}$$

where we define $f'(\xi)$ to have any value we like.

(3 Marks)

Proof. We have

$$\begin{aligned}
(T_f)'\varphi &= -T_f\varphi' = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx \\
&= -\int_{-\infty}^{\xi} f(x)\varphi'(x)\,dx - \int_{\xi}^{\infty} f(x)\varphi'(x)\,dx \\
&= \lim_{\varepsilon\to 0}\Big(f(x)\varphi(x)\big|_{-\infty}^{\xi-\varepsilon} - f(x)\varphi(x)\big|_{\xi+\varepsilon}^{\infty}\Big) + \int_{\mathbb{R}} f'(x)\varphi(x)\,dx \\
&= \lim_{\varepsilon\to 0}\big(f(\xi-\varepsilon)\varphi(\xi-\varepsilon) - f(\xi+\varepsilon)\varphi(\xi+\varepsilon)\big) + T_{f'}\varphi \\
&= T_{f'}\varphi + \lim_{\varepsilon\to 0}\big(f(\xi-\varepsilon)\varphi(\xi) - f(\xi+\varepsilon)\varphi(\xi)\big) \\
&\quad + \lim_{\varepsilon\to 0}\big(f(\xi-\varepsilon)(\varphi(\xi-\varepsilon)-\varphi(\xi)) - f(\xi+\varepsilon)(\varphi(\xi+\varepsilon)-\varphi(\xi))\big) \\
&= [f]_\xi\varphi(\xi) + T_{f'}\varphi \\
&\quad + \underbrace{\lim_{\varepsilon\to 0} f(\xi-\varepsilon)}_{\text{exists}} \cdot \underbrace{\lim_{\varepsilon\to 0}(\varphi(\xi-\varepsilon)-\varphi(\xi))}_{=0} - \underbrace{\lim_{\varepsilon\to 0} f(\xi+\varepsilon)}_{\text{exists}} \cdot \underbrace{\lim_{\varepsilon\to 0}(\varphi(\xi+\varepsilon)-\varphi(\xi)))}_{=0} \\
&= \big([f]_\xi \cdot \delta(x-\xi) + T_{f'}\big)\varphi
\end{aligned}$$

□

Families of Distributions

Finding limits of families (sequences) of distributions is also an important part of the exam.

Exercise 4 Dirac Comb
For $N \in \mathbb{N}$, define the distribution $T_N \in \mathcal{D}'(\mathbb{R})$ by

$$T_N = \sum_{k=-N}^{N} \delta(x-k), \qquad \text{where } \delta(x-k)\varphi = \varphi(k).$$

Show that $T := \lim_{N\to\infty} T_N$ exists in the sense of distributions. Is T a tempered distribution?
(6 Marks)

Proof. We define $T \in \mathcal{D}(\mathbb{R})$ by

$$T\varphi := \sum_{k=-\infty}^{\infty} \varphi(k) \qquad \text{for } \varphi \in C_0^\infty(\mathbb{R}).$$

For any $\varphi \in \mathcal{D}(\mathbb{R})$ the support of ph is compact, so that the sum is actually finite, hence $T\varphi \in \mathbb{R}$ is well-defined. Furthermore, T is linear, as follows: let $\varphi, \psi \in \mathcal{D}(\mathbb{R})$ and choose $N \in \mathbb{N}$ such that $\operatorname{supp}\varphi \cup \operatorname{supp}\psi \subset [-N, N]$. Then for $\lambda, \mu \in \mathbb{C}$

$$\begin{aligned}
T(\lambda\varphi + \mu\psi) &= \sum_{k=-\infty}^{\infty} \big(\lambda\varphi + \mu\psi\big)(k) \\
&= \sum_{k=-N}^{N} \big(\lambda\varphi(k) + \mu\psi(k)\big) \\
&= \lambda \sum_{k=-N}^{N} \varphi(k) + \mu \sum_{k=-N}^{N} \psi(k) \\
&= \lambda T\varphi + \mu T\psi.
\end{aligned}$$

Lastly, T is continuous: let (φ_n) be a null sequence in $\mathcal{D}(\mathbb{R})$. Then $\bigcup_n \operatorname{supp}\varphi_n \subset [-N, N]$ for some $N \in \mathbb{N}$ and

$$\begin{aligned}
\lim_{n\to\infty} T\varphi_n &= \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \varphi_n(k) \\
&= \sum_{k=-N}^{N} \lim_{n\to\infty} \varphi_n(k) \\
&= 0
\end{aligned}$$

where we have used that each $\varphi_n(k) \to 0$ as $n \to \infty$. Hence, T is a distribution. We now show that T is the limit of the T_N: for any $\varphi \in \mathcal{D}(\mathbb{R})$ we can find $M > 0$ such that have $\operatorname{supp}\varphi \subset [-M, M]$ and then

$$(T_N - T)\varphi = \sum_{|k|>N} \varphi(k) = 0 \qquad \text{for } N > M.$$

Hence, $T := \lim_{N\to\infty} T_N$.

We show that T is a tempered distribution. Let $\varphi \in \mathcal{S}(\mathbb{R})$, then for some $C > 0$ we have $|\varphi(x)| \le C/(1+x^2)$ and so

$$\sum_{k=-\infty}^{\infty} |\varphi(k)| \le \sum_{k=-\infty}^{\infty} \frac{C}{1+k^2} < \infty$$

and this implies that

$$T\varphi = \sum_{k=-\infty}^{\infty} \varphi(k)$$

converges. Hence T is well-defined. Furthermore, T is linear, since

$$\begin{aligned}
T(\lambda\varphi + \mu\psi) &= \sum_{k=-\infty}^{\infty} \big(\lambda\varphi + \mu\psi\big)(k) \\
&= \sum_{k=-\infty}^{\infty} \big(\lambda\varphi(k) + \mu\psi(k)\big) \\
&= \lambda \sum_{k=-\infty}^{\infty} \varphi(k) + \mu \sum_{k=-N}^{N} \psi(k) \\
&= \lambda T\varphi + \mu T\psi.
\end{aligned}$$

where we are able to add the series individually because they converge absolutely. Lastly, T is continuous: let (φ_n) be a null sequence in $\mathcal{S}(\mathbb{R})$. Then $\sup_{x\in\mathbb{R}}|(1+x^2)\varphi_n| \to 0$, so $\varphi_n(x) \le C_n/(1+x^2)$, where the coefficients C_n satisfy $C_n \to 0$.

$$\begin{aligned}
\lim_{n\to\infty} T\varphi_n &= \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \varphi_n(k) \\
&= \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \frac{C_n}{1+k^2} \\
&= \lim_{n\to\infty} C_n \sum_{k=-\infty}^{\infty} \frac{1}{1+k^2} = 0.
\end{aligned}$$

Hence T is a tempered distribution. □

Exercise 5 An Oscillating Delta Family

Calculate the limit

$$\lim_{t\to\infty} e^{ixt}\mathcal{P}\left(\frac{1}{x}\right)$$

in the sense of distributions.

(5 Marks)

Proof. We note that

$$\begin{aligned}
\left[e^{ixt}\mathcal{P}\left(\frac{1}{x}\right)\right]\varphi &= \mathcal{P}\left(\frac{1}{x}\right)(e^{ixt}\varphi) \\
&= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{e^{itx}\varphi(x)}{x}\,dx \\
&= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\cos(tx)\varphi(x)}{x}\,dx + i\int_{\mathbb{R}} \frac{\sin(tx)}{x}\varphi(x)\,dx \\
&= \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{(\cos(tx)-1)\varphi(x)}{x}\,dx + \lim_{\varepsilon\to 0}\int_{|x|>\varepsilon} \frac{\varphi(x)}{x}\,dx + i\int_{\mathbb{R}} \frac{\sin(tx)}{x}\varphi(x)\,dx \\
&= \int_{\mathbb{R}} \frac{(\cos(tx)-1)\varphi(x)}{x}\,dx + \mathcal{P}\left(\frac{1}{x}\right)\varphi + i\int_{\mathbb{R}} \frac{\sin(tx)}{x}\varphi(x)\,dx
\end{aligned}$$

Recall that

$$\lim_{t\to\infty}\int_{\mathbb{R}} \frac{\sin(tx)}{x}\varphi(x)\,dx = \pi\cdot\varphi(0), \qquad \lim_{t\to\infty}\int_{\mathbb{R}} \frac{(1-\cos(tx))\varphi(x)}{x}\,dx = \mathcal{P}\left(\frac{1}{x}\right),$$

so

$$\lim_{t\to\infty} e^{ixt}\mathcal{P}\left(\frac{1}{x}\right) = i\pi\delta(x).$$

□

Fourier Transform and Tempered Distributions

You need to be able to calculate trigonometric Fourier series. Straightforward calculations like thse should not present any serious problems.

Exercise 6

Calculate the (distributional) Fourier transform of the functions $f, g\colon \mathbb{R} \to \mathbb{R}$, given by

$$f(x) = \begin{cases} 0 & x < 0, \\ x & x \geq 0, \end{cases} \qquad \text{and} \qquad g(x) = \sin^2(x).$$

(4 Marks)

Proof. We have that $f(x) = (-i)(ix)H(x)$ and $\widehat{(ix)H(x)}(\xi) = \widehat{H}'(\xi)$, so

$$\hat{f}(\xi) = (-i)\widehat{H}'(\xi) = -i\frac{d}{dx}\left(\frac{i}{\sqrt{2\pi}}\mathcal{P}\left(\frac{1}{x}\right) + \sqrt{\frac{\pi}{2}}\delta(\xi)\right)$$

Noting that

$$\sin^2(x) = \frac{1 - \cos(2x)}{2} = \frac{2 - e^{2ix} - e^{-2ix}}{4}$$

we have

$$\hat{g}(\xi) = \sqrt{\frac{\pi}{2}}\delta(\xi) - \sqrt{\frac{\pi}{8}}\delta(\xi - 2) - \sqrt{\frac{\pi}{8}}\delta(\xi + 2)$$

□

Exercise 7 Continuity of the Fourier Transform

i) Show that the Fourier transform is continuous as a map $\mathcal{F}\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

ii) Show that the Fourier transform is well-defined on $\mathcal{S}'(\mathbb{R})$: if $T \in \mathcal{S}'(\mathbb{R})$, then $\mathcal{F}T \in \mathcal{S}'(\mathbb{R})$.

iii) Show that the Fourier transform is continuous as a map $\mathcal{F}\colon \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$.

(6 Marks)

Proof.

i) We note that

$$\begin{aligned} \sup_{\xi\in\mathbb{R}}|\hat{f}(\xi)| &\leq \frac{1}{\sqrt{2\pi}} \sup_{\xi\in\mathbb{R}} \int_{\mathbb{R}} |f(x)e^{-ix\xi}|\, dx \\ &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{1}{1+x^2}\, dx \cdot \sup_{x\in\mathbb{R}}|(1+x^2)f(x)| \end{aligned}$$

so that if (f_n) is a null sequence, then $(\widehat{f_n})$ converges to zero uniformly. Since the Fourier transforms takes derivatives into multiplications and vice-versa, we see that in fact all derivatives and products with monomials of $(\widehat{f_n})$ also converge to zero uniformly, hence $(\widehat{f_n})$ is a null sequence if (f_n) is. This shows that the Fourier transform is continuous as a map $\mathcal{F}\colon \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$.

ii) We define $\widehat{T}\varphi := T\hat{\varphi}$, which exists, since $\hat{\varphi} \in \mathcal{S}$ if φ is. We need top show that $\widehat{T}$ is linear and continuous. But because both T and $\mathcal{F}$ are linear and continuous, the same is true for their composition.

iii) Suppose that (T_n) is a null sequence in $\mathcal{S}'$. Then we claim that $\widehat{T_n}$ is also a null sequence (which shows the continuity of $\mathcal{F}$). We need to establish that $\widehat{T_n}\varphi \to 0$ for any $\varphi \in \mathcal{S}$. However, $\widehat{T_n}\varphi = T_n\hat{\varphi}$ and since (T_n) is a null sequence and $\hat{\varphi} \in \mathcal{S}$, $T_n\hat{\varphi} \to 0$.

□

Exercise 8 Wave Equation

Consider the wave equation problem for a function $u\colon \mathbb{R}^2 \to \mathbb{R}$,

$$u_{tt} - u_{xx} = 0, \qquad u(x,0) = f(x), \qquad u_t(x,0) = g(x).$$

Take the Fourier transform of the equation with respect to the x-variable to obtain an ODE in the t-variable and solve the ODE to obtain

$$\widehat{u}(\xi,t) = \hat{f}(\xi)\cos(\xi t) + \frac{\hat{g}(\xi)}{\xi}\sin(\xi t).$$

Then calculate the inverse Fourier transform (in the distributional sense) to obtain a solution formula for $u(x,t)$. **(5 Marks)**

Proof. Taking the Fourier transform, we obtain

$$\widehat{u}_{tt} + \xi^2\widehat{u} = 0, \qquad \widehat{u}(\xi,0) = \widehat{f}(\xi), \qquad \widehat{u}_t(\xi,0) = \widehat{g}(\xi).$$

(1/2 Mark) The ODE has solution

$$\widehat{u}(\xi,t) = c_1\cos(\xi t) + c_2\sin(\xi t), \qquad c_1, c_2 \in \mathbb{R}.$$

(1/2 Mark) Then the initial conditions give

$$\widehat{u}(\xi,0) = c_1 = \widehat{f}(\xi), \qquad \widehat{u}_t(\xi,0) = c_2\xi = \widehat{g}(\xi),$$

(1/2 Mark) so

$$\begin{aligned}
\widehat{u}(\xi,t) &= \hat{f}(\xi)\cos(\xi t) + \frac{\hat{g}(\xi)}{\xi}\sin(\xi t) \\
&= \frac{1}{2}e^{i\xi t}\hat{f}(\xi) + \frac{1}{2}e^{-i\xi t}\hat{f}(\xi) + \frac{\hat{g}(\xi)}{\xi}\sin(\xi t).
\end{aligned}$$

(1/2 Mark) We now use that $e^{i\xi t}\hat{f}(\xi) = \mathcal{F}(f(\,\cdot - t))$ to see that

$$u(x,t) = \frac{f(x-t)+f(x+t)}{2} + t\mathcal{F}^{-1}_{\xi\to x}\left(\frac{\sin(\xi t)}{\xi t}\hat{g}(\xi)\right).$$

Using $t\mathcal{F}(f(t\cdot(\,\cdot\,))) = (\mathcal{F}f)(\,\cdot/t)$ and

$$\widehat{\Pi}(\xi) = \frac{2}{\sqrt{2\pi}}\sin(\xi)/\xi, \qquad \Pi(x) = \begin{cases} 1 & |x| < 1, \\ 0 & \text{otherwise} \end{cases}$$

we see that

$$\begin{aligned}
u(x,t) &= \frac{f(x-t)+f(x+t)}{2} + t\mathcal{F}^{-1}_{\xi\to x}\left(\frac{\sin(\xi t)}{\xi t}\hat{g}(\xi)\right) \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{\sqrt{2\pi}}{2}\mathcal{F}^{-1}_{\xi\to x}\left(\widehat{\Pi}(\xi/t)\hat{g}(\xi)\right) \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}(\Pi(\,\cdot/t) * g)(x) \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{-\infty}^{\infty}\Pi(y/t)g(x-y)\,dy \\
&= \frac{f(x-t)+f(x+t)}{2} + \frac{1}{2}\int_{-t}^{t} g(x-y)\,dy \\
&= \underbrace{\frac{f(x-t)+f(x+t)}{2}}_{\textbf{(1 Mark)}} + \underbrace{\frac{1}{2}\int_{x-t}^{x+t} g(y)\,dy}_{\textbf{(2 Marks)}}
\end{aligned}$$

□

Fundamental Solutions and Initial Value problems

You should know what classical, weak and distributional solutions of differential solutions are, what a fundamental solution is and how to find causal fundamental solutions for ordinary differential equations. You should be familiar with the solution formula for initial value problems.

Exercise 9 Equilibrium Diffusion

The equilibrium concentration u of a substance diffusing in a homogeneous, absorbing, infinite, one-dimensional medium (such as an infinite tube) is given by

$$Lu = -\frac{d^2u}{dx^2} + q^2 u = f(x), \qquad x \in \mathbb{R},$$

where f is the source density of the substance and $q > 0$ is a positive constant.

i) Let $\xi \in \mathbb{R}$ be fixed. Use the Fourier transform to find a fundamental solution $E(x;\xi)$ of L satisfying

$$LE(x;\xi) = \delta(x-\xi), \qquad \lim_{|x|\to\infty} E(x,\xi) = 0. \tag{1}$$

Is this a causal fundamental solution? Why or why not?

ii) Verify that the candidate function found satisfies (1) distributionally.

(8 Marks)

Proof.

i) Since the differential equation has constant coefficients, we can assume $\xi = 0$. Taking the Fourier transform of the equation

$$-\frac{d^2u}{dx^2} + q^2 u = \delta(x)$$

we obtain

$$|k|^2 \hat{u}(k) + q^2 \hat{u}(k) = \frac{1}{\sqrt{2\pi}}.$$

or

$$\hat{u}(k) = \frac{1}{\sqrt{2\pi}} \frac{1}{k^2+q^2}.$$

The inverse Fourier transform is

$$u(x) = \frac{1}{2q} e^{-q|x|}$$

and the solution is

$$E(x;\xi) = \frac{1}{2q} e^{-q|x-\xi|}$$

The solution is not causal, since it does not vanish for $x < \xi$.

ii) We apply the left-hand side to a test function $\varphi \in \mathcal{D}(\mathbb{R})$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \big(-E(x;\xi)\varphi''(x) + q^2 E(x;\xi)\varphi(x)\big)\, dx \\
&= \frac{1}{2q}\int_{-\infty}^{\infty} \big(-e^{-q|x-\xi|}\varphi''(x) + q^2 e^{-q|x-\xi|}\varphi(x)\big)\, dx \\
&= \frac{1}{2q}\int_{-\infty}^{\infty} \big(-e^{-q|x-\xi|}\varphi''(x) + q^2 e^{-q|x-\xi|}\varphi(x)\big)\, dx \\
&= -\frac{1}{2q}\int_{\xi}^{\infty} e^{-q(x-\xi)}\varphi''(x)\, dx - \frac{1}{2q}\int_{-\infty}^{\xi} e^{q(x-\xi)}\varphi''(x)\, dx + \frac{1}{2q}\int_{-\infty}^{\infty} q^2 e^{-q|x-\xi|}\varphi(x)\, dx \\
&= \underbrace{-\frac{1}{2q}e^{-q(x-\xi)}\varphi'(x)\big|_{\xi}^{\infty}}_{=\varphi'(\xi)/(2q)} - \frac{1}{2}\int_{\xi}^{\infty} e^{-q(x-\xi)}\varphi'(x)\, dx \underbrace{-\frac{1}{2q}e^{q(x-\xi)}\varphi'(x)\big|_{-\infty}^{\xi}}_{=-\varphi'(\xi)/(2q)} + \frac{1}{2}\int_{-\infty}^{\xi} e^{q(x-\xi)}\varphi'(x)\, dx \\
&\quad + \frac{q}{2}\int_{-\infty}^{\infty} e^{-q|x-\xi|}\varphi(x)\, dx \\
&= \underbrace{-\frac{1}{2}e^{-q(x-\xi)}\varphi(x)\big|_{\xi}^{\infty}}_{=\varphi(\xi)/2} - \frac{q}{2}\int_{\xi}^{\infty} e^{-q(x-\xi)}\varphi(x)\, dx + \underbrace{\frac{1}{2}e^{q(x-\xi)}\varphi(x)\big|_{-\infty}^{\xi}}_{=\varphi(\xi)/2} - \frac{q}{2}\int_{-\infty}^{\xi} e^{q(x-\xi)}\varphi(x)\, dx \\
&\quad + \frac{q}{2}\int_{-\infty}^{\infty} e^{-q|x-\xi|}\varphi(x)\, dx \\
&= \varphi(\xi) = \int_{\mathbb{R}} \delta(x-\xi)\varphi(x)\, dx.
\end{aligned}$$

□