Vv156 Lecture 4

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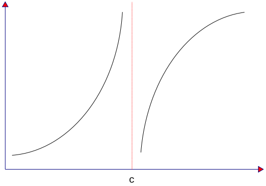
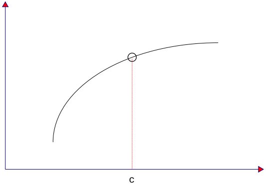
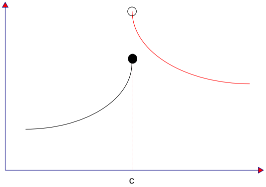
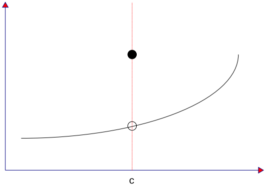
Q: What does a curve, that is not continuous, look like?

1. 

1. 

2. 

3. 

Definition

Let f be a function defined on some *open* interval that contains the number c, then f is said to be continuous at $x = c$ if the following conditions are satisfied:

1. $f(c)$ is defined; 2. $\lim\limits_{x\to c} f(x)$ exists ; 3. $\lim\limits_{x\to c} f(x) = f(c)$

Exercise

(a) Find K which makes the following function continuous at $x = 1$.

$$f(x) = \begin{cases} x^2 - 2 & \text{if} \quad x < 1, \\ Kx - 4 & \text{if} \quad 1 \le x. \end{cases}$$

(b) Show the sine function is continuous at every point $c \in (-\infty, \infty)$.

(c) Determine whether the following function is continuous at every point $c \in \mathbb{R}$.

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & \text{if} \quad x \neq 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

(d) Show the function $f(x) = \begin{cases} 1 & \text{if} \quad x \in \mathbb{Q}, \\ 0 & \text{if} \quad x \notin \mathbb{Q} \end{cases}$ is nowhere continuous.

- Discontinuities can be further classified according to their nature.

Defintion

- Removable discontinuity: Both $f(c)$ and $\lim_{x \to c} f(x) = L$ exist, but

$$f(c) \neq L$$

  in which case we can make f continuous at c by redefining $f(c) = L$.

- Jump discontinuity: Both of the one-sided limits exist, but

$$\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$$

- Essential discontinuity: At least one of the one-sided limits does not exist.

Q: Can we have an essential discontinuity where the function f is bounded?

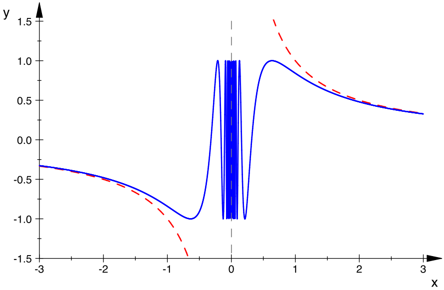
$$f(x) = \sin\left(\frac{1}{x}\right) \qquad \text{at} \qquad x = 0$$

- The first graph is a plot of

$$y = \sin\left(\frac{1}{x}\right)$$

together with

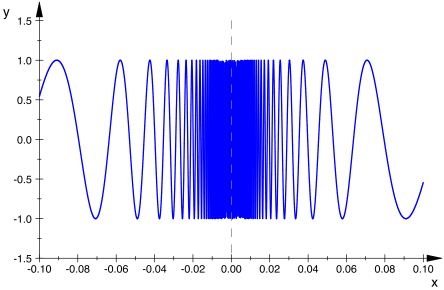
$$y = \frac{1}{x}$$



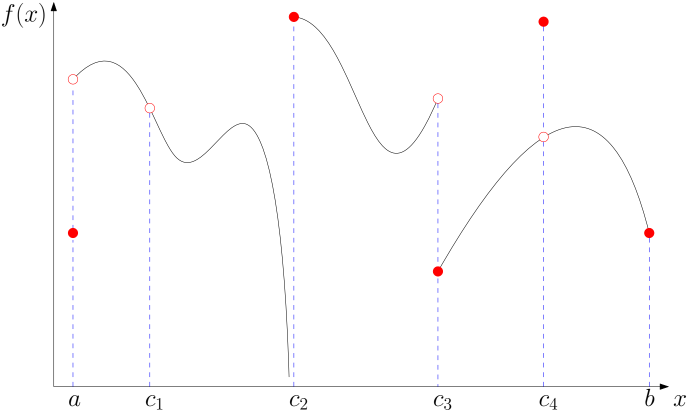
- The second is a closer look at

$$y = \sin\left(\frac{1}{x}\right)$$

near $x = 0$.



Q: What types of discontinuities does the following function on $[a, b]$ has?



- Note that our definition of continuity at the moment only concerns interior points of the domain of the function $y = f(x)$, we need to extend it before we can discuss the continuity of f at $x = a$ or $x = b$.

Definition

Let f be a function defined on a *closed* interval $[a, b]$, then f is

continuous at $x = a$

if the following conditions are satisfied:

1. $f(a)$ is defined; 2. $\lim_{x \to a^+} f(x)$ exists ; 3. $\lim_{x \to a^+} f(x) = f(a)$

continuous at $x = b$

if the following conditions are satisfied:

1. $f(b)$ is defined; 2. $\lim_{x \to b^-} f(x)$ exists ; 3. $\lim_{x \to b^-} f(x) = f(b)$

Definition

A function is said to be continuous on a *set* if it is continuous at every point in it, and simply continuous if it is continuous at everywhere in its domain.

Theorem

If the functions f and g have the same domain and are continuous at c, then

1. The scalar multiple, kf, is also continuous at c, for a constant k.
2. The sum or the difference, $f \pm g$, is also continuous at c.
3. The product, fg, is also continuous at c.
4. The quotient, $\dfrac{f}{g}$, is continuous at c if $g(c) \neq 0$.

Q: Why this theorem is obviously true?

- For the same reason, the following is true.

Theorem

Every polynomial function or rational function is continuous on its domain.

Q: How can we argue $y = \exp(\sin x) = e^{\sin x}$ is continuous ?

Defintion

Suppose $\mathcal{A}$ is the domain of f and $\mathcal{B}$ is the domain of g, where

$$g(\mathcal{B}) \subset \mathcal{A}$$

that is, the domain of f contains the range of g, then the composition

$$(f \circ g)(x) = f\Big(g(x)\Big)$$

is a function of x.

Q: What is the natural domain of

$$f \circ g, \qquad \text{where } f(x) = \sqrt{x-1} \text{ and } g(x) = (x-1)^{-1}$$

Theorem

Let f and g be two continuous functions in their domains. If c is in the domain of g and $g(c)$ is in the domain of f, then the composite function

$$f \circ g$$

is also continuous at c.

Proof

- For $\epsilon > 0$, since f is continuous at $g(c)$, there exists δ_1 such that

$$|y - g(c)| < \delta_1 \implies |f(y) - f(g(c))| < \epsilon.$$

- Next, since g is continuous at c, there exists $\delta > 0$ such that

$$|x - c| < \delta \implies |g(x) - g(c)| < \delta_1.$$

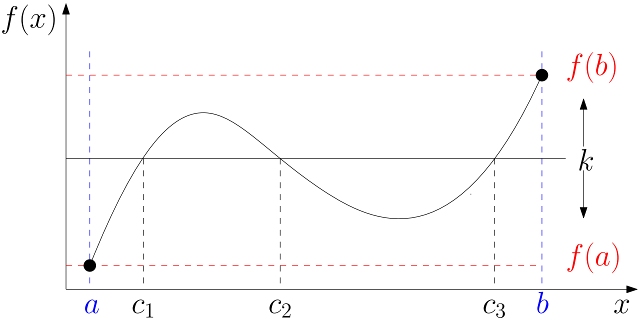
- Combine these inequalities shows that the composite function is continuous.

$$|x - c| < \delta \implies |f(g(x)) - f(g(c))| < \epsilon \quad \square$$

Q: A Tibetan monk leaves the monastery at 7:00 AM and takes his usual path to the top of the mountain, arriving at 7:00 PM. The following morning, he starts at 7:00 AM at the top and takes the same path back, arriving at the monastery at 7:00 PM. Is there always a point on the path that the monk will cross at exactly the same time of day on both days?

The Intermediate-Value theorem

If f is continuous on a closed interval $[a, b]$ and
k is any number strictly between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$,
then there exits
a number c in the interval (a, b) such that $f(c) = k$.



Proof

- Let us assume

$$f(a) < k < f(b)$$

- Under this assumption, suppose $g(x) = f(x) - k$, then

$$g(a) < 0 \qquad \text{and} \qquad g(b) > 0$$

- To show there exists $a < c < b$ such that

$$f(c) = k,$$

  we need to show that there exist c such that

$$g(c) = 0 \qquad \text{for some} \quad a < c < b$$

- Let

$$\mathcal{S} = \Big\{ x \in [a, b] \mid g(x) < 0 \Big\}$$

Proof

- Note the set $\mathcal{S}$ is nonempty since $a \in \mathcal{S}$ and $\mathcal{S}$ is bounded from above by b.
- There is no gap in $\mathbb{R}$, we take as given that $\sup(\mathcal{S})$ exist, and claim

$$c = \sup(\mathcal{S})$$

- To show this claim is correct, we need to show

$$g(x = c) = 0, \qquad \text{where} \quad c = \sup(\mathcal{S})$$

- Suppose $g(c) \neq 0$, since f and thus g is continuous at c, there exists $\delta > 0$

$$|g(x) - g(c)| < \frac{1}{2}|g(c)| \qquad \text{when} \quad |x - c| < \delta$$

- Now if $g(c) < 0$, then $c \neq b$ and for all x such that $|x - c| < \delta$.

$$g(c) + g(x) - g(c) < g(c) - \frac{1}{2}g(c) \iff g(x) < \frac{1}{2}g(c) < 0$$

Proof

- So the following is true, which means there are points $x \in \mathcal{S}$ bigger than c,

$$g(x) < 0 \qquad \text{for all } x \text{ such that} \quad |x - c| < \delta.$$

- That leads us to a contradiction of c being an upper bound of $\mathcal{S}$.
- Similarly, if $g(c) > 0$, then $c \neq a$ and for all x such that $|x - c| < \delta$

$$g(c) + g(x) - g(c) > g(c) - \frac{1}{2}g(c) \iff g(x) > \frac{1}{2}g(c) > 0$$

- Thus

$$g(x) > 0 \qquad \text{for all } x \text{ such that } |x - c| < \delta.$$

- It follows that there exists $\mu > 0$ such that $c - \mu \geq a$ and

$$g(x) > 0 \qquad \text{for} \quad c - \mu \leq x \leq c$$

- So $x = c - \mu$, which is less than c, is also an upper bound of $\mathcal{S}$, this leads to a contradiction of c being the least upper bound.

Proof

- The above forces us to conclude that

$$g(c) = 0$$

- Finally, $c \neq a$ and $c \neq b$ since g is nonzero at the endpoints

$$a < c < b$$

- If the original assumption is not true, that is, $f(b) < k < f(a)$ instead, then we apply the argument to the reflect of $y = f(x)$ about x-axis, and

$$-f(c) = -k \quad \square$$

Q: Can you, in theory, slice a rain drop on a windscreen exactly in half no matter how irregular its shape with a single straight-line cut?

Q: In theory, can you always simultaneously slice two rain drops each exactly in half with a single straight-line cut, no matter the shapes of the rain drops nor their location on the windscreen.