vv255: Double Integrals.

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Multivariable integrals

Today

- Review: The Darboux approach to define definite and multivariable integrals.
- Iterated integrals. Fubini's theorem.
- Double integrals over general regions.
- Change of variables. Jacobian.
- Surface area.

Multivariable integrals

- Recall that the Riemann integral is defined as a limit of Riemann finite sums:
$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i)\Delta x_i,\ \xi_i \in \Delta x_i$$

Multivariable integrals

- Recall that the Riemann integral is defined as a limit of Riemann finite sums:

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- The Darboux integral is the common value of the lower and upper Darboux integrals:

  Let $P = a = x_0 < x_1 < x_2 < \ldots < x_n = b$ be a partition of $[a, b]$ and

$$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x),\ M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$$

  Introduce the upper and lower Darboux sums

$$L(f, P) = \sum_{i=1}^n m_i \Delta x_i,\ U(f, P) = \sum_{i=1}^n M_i \Delta x_i$$

$$\int_a^b f(x)\,dx = \underbrace{\sup_{P \text{ of } [a,b]} L(f, P)}_{\underline{\int_a^b} f(x)\,dx} = \underbrace{\inf_{P \text{ of } [a,b]} U(f, P)}_{\overline{\int_a^b} f(x)\,dx}$$

Review: Riemann integration

- The Darboux definition of the integral looks at partitions of $[a, b]$ into finite pieces and find over- and under-approximations of the area between the graph of $y = f(x)$ and the x-axis using rectangles with width described by this partition.

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- A function f is Darboux integrable over $[a, b]$ if $\forall \varepsilon > 0 \, \exists$ partition P on $[a, b]$
$$U(f, P) - L(f, P) < \varepsilon$$

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- A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ iff f is Darboux integrable on $[a, b]$.

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- A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ iff f is Darboux integrable on $[a, b]$.
- (Advanced) Let $f(x)$, $\alpha(x)\colon [a, b] \to \mathbb{R}$ be bounded functions and $\alpha(x)$ be monotone. The Riemann-Stieltjes integral of f with respect to α is defined as

$$\int_a^b f(x)\, d\alpha(x) = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} f(\xi_i) \Delta\alpha_i$$

- Similarly, we can define Darboux-Stieltjes integral of f with respect to α.

Multivariable integrals

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- Instead of finite partitions of intervals, the multivariable Darboux integral instead considers finite partitions of "rectangles"
- The "volume under the surface" is then over- and under-approximated using "boxes" instead of rectangles

Multivariable integrals

Definition

Let $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$ *be such that for all* $1 \leq k \leq n$, $a_k \leq b_k$. *The collection of points in* $\mathbb{R}^n$

$$[a_1, b_1] \times \cdots \times [a_n, b_n] = \{(x_1, \ldots, x_n) \mid a_1 \leq x_1 \leq b_1, \ldots, a_n \leq x_n \leq b_n\}$$

is called a closed rectangle in $\mathbb{R}^n$. *We also write:* $R \subseteq \mathbb{R}^n$ *is a closed rectangle.*

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is called a closed rectangle in $\mathbb{R}^n$. We also write: $R \subseteq \mathbb{R}^n$ is a closed rectangle.

Note that if $a_1 < b_1$ and $a_2 < b_2$, then the closed rectangle $[a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$ looks like a bounded rectangular region that contains all of its boundary points.

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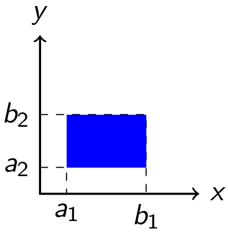

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Definition

Recall that a partition, P, of $[a, b]$ into m pieces is a set of $m + 2$ points $P = \{x_0 < \cdots < x_{m+1}\}$ where $x_0 = a$ and $x_{m+1} = b$. We say that P is partition of the closed rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$ if $P = (P_1, \ldots, P_n)$ where for all $1 \leq k \leq n$, P_k is a partition of $[a_k, b_k]$.

Multivariable integrals

If $P = (P_1, \ldots, P_n)$ is a partition of the closed rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$, then for all $1 \leq k \leq n$, $P_k = \{x_{k,0}, \ldots, x_{k,l_k+1}\}$ where $x_{k,0} = a_k$ and $x_{k,l_k+1} = b_k$. Moreover, every $j_1, \ldots, j_n$ where for all $1 \leq k \leq n$, $1 \leq j_k \leq l_k + 1$, defines a closed rectangle that is contained in R:

$$[x_{1,j_1-1}, x_{1,j_1}] \times \cdots \times [x_{n,j_n-1}, x_{n,j_n}]$$

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Therefore, a partition $P = (P_1, \ldots, P_n)$ of a closed rectangle R in $\mathbb{R}^n$ is a division of R into finitely many closed rectangles R_1, ..., R_N in $\mathbb{R}^n$ such that

$$R = \bigcup_{i=1}^{N} R_i \text{ and } V(R) = \sum_{i=1}^{N} V(R_i)$$

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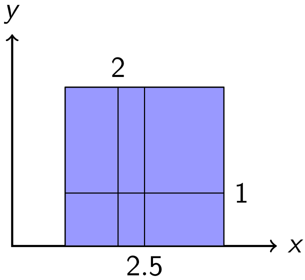
Therefore, it is often easier to specify a partition P of a closed rectangle R in $\mathbb{R}^n$ by the rectangles that it divides R into, i.e. $P = \{R_1, \dots, R_N\}$, rather than the points that define this division.

Multivariable integrals

Example

Consider the closed rectangle $R = [1,4] \times [0,3]$ *in* $\mathbb{R}^2$. *If* $P_1 = \{1, 2, 2.5, 4\}$ *and* $P_2 = \{0, 1, 3\}$, *then* $P = \langle P_1, P_2 \rangle$ *is a partition of* R *into* 6 *rectangles:*



$$[1,2] \times [0,1], [1,2] \times [1,3], [2,2.5] \times [0,1], [2,2.5] \times [1,3],$$

$$[2.5,4] \times [0,1], [2.5,4] \times [1,3]$$

$$V(R) = 3 \cdot 3 = 9$$

Least upper bounds and greatest lower bounds

Definition

We say that $A \subseteq \mathbb{R}$ is bounded above if there exists $b \in \mathbb{R}$ such that for all $x \in A$, $x \leq b$. We say that $A \subseteq \mathbb{R}$ is bounded below if there exists $b \in \mathbb{R}$ such that for all $x \in A$, $b \leq x$. We say that $A \subseteq \mathbb{R}$ is bounded if A is bounded above and bounded below.

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Definition

Let $A \subseteq \mathbb{R}$ be nonempty and bounded below. We use $\inf A$ to denote the greatest lower bound of A in $\mathbb{R}$. Equivalently, $y = \inf A$ if for all $w \in A$, $y \leq w$ and for all $\varepsilon > 0$, there exists $x \in A$ such that $|x - y| < \varepsilon$.

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Note that it is a fundamental property of $\mathbb{R}$ (an axiom) that every nonempty subset that is bound below has an inf, and every nonempty subset that is bounded above has a sup.

Upper and lower Darboux sums

Definition

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle and let $f : R \longrightarrow \mathbb{R}$ be a bounded function. Let $P = \{R_1, \ldots, R_N\}$ be a partition of R into N rectangles. For all $1 \leq i \leq N$, let

$$m_i = \inf\{f(\bar{x}) \mid \bar{x} \in R_i\} \ and\ M_i = \sup\{f(\bar{x}) \mid \bar{x} \in R_i\}$$

The upper Darboux sum of f with respect to P, denote $U(f, P)$, is defined by

$$U(f, P) = \sum_{i=1}^{N} M_i V(R_i)$$

The lower Darboux sum of f with respect to P, denote $L(f, P)$, is defined by

$$L(f, P) = \sum_{i=1}^{N} m_i V(R_i)$$

Upper and lower Darboux sums

Example

Let $R = [1,4] \times [0,3] \subseteq \mathbb{R}^2$*. Consider* $f : R \longrightarrow \mathbb{R}$ *defined by* $f(x,y) = 2y + 3$*. Let* $P = \{R_1, \ldots, R_6\}$ *be the partition of* R *where*

$$\begin{aligned} R_1 &= [1,2] \times [0,1] & R_2 &= [1,2] \times [1,3] \\ R_3 &= [2,2.5] \times [0,1] & R_4 &= [2,2.5] \times [1,3] \\ R_5 &= [2.5,4] \times [0,1] & R_6 &= [2.5,4] \times [1,3] \end{aligned}$$

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Therefore $V(R_1) = 1$, $V(R_2) = 2$, $V(R_3) = 0.5$, $V(R_4) = 1$, $V(R_5) = 1.5$ *and* $V(R_6) = 3$, *and*

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$$\begin{aligned} m_1 &= 3 & m_2 &= 5 & m_3 &= 3 \\ m_4 &= 5 & m_5 &= 3 & m_6 &= 5 \\ M_1 &= 5 & M_2 &= 9 & M_3 &= 5 \\ M_4 &= 9 & M_5 &= 5 & M_6 &= 9 \end{aligned}$$

$$L(f,P) = 39 \qquad U(f,P) = 69$$

Upper and lower Darboux sums

Lemma

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle and let $f : R \longrightarrow \mathbb{R}$ be a bounded function. If P is a partition of R, then

$$V(R) \inf\{f(\bar{x}) \mid \bar{x} \in R\} \leq L(f, P) \leq U(f, P) \leq V(R) \sup\{f(\bar{x}) \mid \bar{x} \in R\}$$

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Definition

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. Let $P = \langle P_1, \ldots, P_n \rangle$ and $Q = (Q_1, \ldots, Q_n)$ be partitions of R. We say that Q is a refinement of P if for all $1 \leq k \leq n$, $P_k \subseteq Q_k$. I.e. for all $1 \leq k \leq n$, the points that are in the partition P_k are also in the partition Q_k.

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We can show the following key combinatorial lemma:

Lemma

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. If P and Q are partitions of R, then there exists a partition S of R that is a refinement of both P and Q.

Upper and lower Darboux integrals

Definition

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. Let $f : R \longrightarrow \mathbb{R}$ be a bounded function. The upper Darboux integral of f over R is defined by

$$\overline{\int_R} f = \inf\{U(f, P) \mid P \textit{ is a partition of } R\}$$

The lower Darboux integral of f over R is defined by

$$\underline{\int_R} f = \sup\{L(f, P) \mid P \textit{ is a partition of } R\}$$

We say that f is Darboux integrable or just integrable over R if $\overline{\int_R} f = \underline{\int_R} f$, and if this is the case then we use

$$\int_R f \textit{ or } \int_R f \, dV$$

to denote this common value, called the integral of f over R.

The definite multivariable integral

Note that if $f : R \longrightarrow \mathbb{R}$ is an integrable function where $R \subseteq \mathbb{R}^2$ is a close rectangle, then the integral of f over R is also sometimes denoted

$$\iint_R f \ dA$$

to highlight the fact that $R \subseteq \mathbb{R}^2$.

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Lemma

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. Let $f : R \longrightarrow \mathbb{R}$ be a bounded function. Let P and Q be partitions of R. If Q is a refinement of P, then $U(f, Q) \leq U(f, P)$ and $L(f, P) \leq L(f, Q)$.

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Lemma

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. Let $f : R \longrightarrow \mathbb{R}$ be a bounded function. Let P and Q be partitions of R. If Q is a refinement of P, then $U(f, Q) \leq U(f, P)$ and $L(f, P) \leq L(f, Q)$.

Lemma

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. Let $f : R \longrightarrow \mathbb{R}$ be a function with $m, M \in \mathbb{R}$ such that for all $\bar{x} \in R$, $m \leq f(\bar{x}) \leq M$. Then

$$mV(R) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq MV(R)$$

The definite multivariable integral

This shows that if $R \subseteq \mathbb{R}^n$ is a closed rectangle, P is a partition of R and $f : R \longrightarrow \mathbb{R}$ is an integrable function, then

$$L(f, P) \leq \int_R f \; dV \leq U(f, P)$$

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Example

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle and let $c \in \mathbb{R}$. The function $f : R \longrightarrow \mathbb{R}^n$ defined by: for all $\bar{x} \in R$, $f(\bar{x}) = c$, is integrable because

$$cV(R) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq cV(R)$$

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This shows that if $R \subseteq \mathbb{R}^n$ is a closed rectangle, P is a partition of R and $f : R \longrightarrow \mathbb{R}$ is an integrable function, then

$$L(f, P) \leq \int_R f \; dV \leq U(f, P)$$

Example

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle and let $c \in \mathbb{R}$. The function $f : R \longrightarrow \mathbb{R}^n$ defined by: for all $\bar{x} \in R$, $f(\bar{x}) = c$, is integrable because

$$cV(R) \leq \underline{\int_R} f \leq \overline{\int_R} f \leq cV(R)$$

The multivariable integral has many of the nice properties that hold for the integral of functions of a single variable. The behaviour of partitions (they act like a lattice) gives us Cauchy's Criterion:

Cauchy's Criterion

Theorem

(Cauchy's Criterion) Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. Let $f : R \longrightarrow \mathbb{R}$ be a bounded function. Then f is integrable if and only if for all $\varepsilon > 0$, there exists a partition P of R such that

$$U(f, P) - L(f, P) < \varepsilon$$

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$$U(f, P) - L(f, P) < \varepsilon$$

Example

Let $R = [1, 4] \times [0, 1] \subseteq \mathbb{R}^2$. Consider $f : R \longrightarrow \mathbb{R}$ defined by $f(x, y) = 2y + 5$. For all $N \in \mathbb{N}$ with $N \geq 1$, define $P_N = \{R_1, \ldots, R_N\}$ to be the partition of R into N rectangles where for all $1 \leq k \leq N$,

$$R_k = [1, 4] \times \left[\frac{(k-1)}{N}, \frac{k}{N}\right]$$

Cauchy's Criterion

Example

(Continued.) So, given $N \in \mathbb{N}$ with $N \geq 1$, we have for all $1 \leq k \leq N$,

$$m_k = \frac{2(k-1)}{N} + 5 \text{ and } M_k = \frac{2k}{N} + 5$$

and $V(R_k) = \frac{3}{N}$.

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$$U(f, P_N) = 15 + \frac{6}{N^2} \sum_{k=1}^{N} k \text{ and } L(f, P_N) = 15 + \frac{6}{N^2} \sum_{k=1}^{N-1} k$$

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So,

$$U(f, P_N) - L(f, P_N) = \frac{6}{N}$$

Therefore, for any $\varepsilon > 0$, we can choose $N \in \mathbb{N}$ with $\frac{6}{N} < \varepsilon$ and this gives

$$U(f, P_N) - L(f, P_N) < \varepsilon, \text{ so } f \text{ is integrable.}$$

The definite multivariable integral

Theorem

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. Let $f : R \longrightarrow \mathbb{R}$ and $g : R \longrightarrow \mathbb{R}$ be integrable functions. If for all $\bar{x} \in R$, $f(\bar{x}) \leq g(\bar{x})$, then

$$\int_R f \; dV \leq \int_R g \; dV$$

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Theorem

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. Let $f : R \longrightarrow \mathbb{R}$ and $g : R \longrightarrow \mathbb{R}$ be integrable functions, and let $\alpha \in \mathbb{R}$. Then

(i) *αf is integrable and*

$$\int_R \alpha f \; dV = \alpha \int_R f \; dV$$

(ii) *$f + g$ is integrable and*

$$\int_R (f+g) \; dV = \int_R f \; dV + \int_R g \; dV$$

The definite multivariable integral

We are also able to generalise the fact that every continuous function is integrable to the multivariable integral. The following is straightforward adaption of the corresponding result for functions of a single variable:

Theorem

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. If $f : R \longrightarrow \mathbb{R}$ is continuous, then f is integrable.

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Theorem

Let $R \subseteq \mathbb{R}^n$ be a closed rectangle. If $f : R \longrightarrow \mathbb{R}$ is continuous, then f is integrable.

It should be clear that if $R \subseteq \mathbb{R}^2$ and $f(x, y)$ is an integrable function of two variables, then

$$\int_R f \; dV$$

is the volume between the rectangle R lying on the xy-plane and the surface described by the graph $z = f(x, y)$ with a sign to tell you whether this surface is above or below the xy-plane.

Multivariable integrals

Example

Let $R = [-1, 1] \times [-2, 2]$ *and consider* $f : R \longrightarrow \mathbb{R}$ *defined by* $f(x, y) = \sqrt{1 - x^2}$*. The surface described by the* $z = f(x, y)$ *is a half cylinder with radius* 1 *sitting on* R *and running parallel to the* y*-axis. By recognising the volume enclosed by this surface and* R *we see that*

$$\iint_R f(x, y)\, dV = \iint_R \sqrt{1 - x^2}\, dV = 2\pi$$

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- Computing multivariable integrals directly from the definition of the Darboux integral is going to get hideously complicated very quickly.
- We will now turn to developing some techniques that allow us to compute integrals of functions of more than one variable over regions.
- Our focus, to begin with, will be functions of two variable. This means that the integrals that we will be computing correspond to volumes in 3D space.

Iterated single variable integrals

Recall, that First Fundamental Theorem of Calculus provides a tool that allows us to algebraically evaluate definite integrals. We will now see that multivariable integrals can be reduced and computed as iterations of integrals of functions of a single variable.

Iterated single variable integrals

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Definition

Let $R = [a, b] \times [c, d]$. Let $f : R \longrightarrow \mathbb{R}$ be integrable. We define the partial integral of f with respect to x between a and b, denoted

$$\int_a^b f(x, y)\ dx,$$

to be the function of y obtained by holding y constant and evaluating the integral of $f(x, y)$ with respect to x between a and b.

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to be the function of x obtained by holding x constant and evaluating the integral of $f(x, y)$ with respect to y between c and d. The iterated integral

$$\int_a^b \int_c^d f(x,y)\ dy\ dx = \int_a^b \left[\int_c^d f(x,y)\ dy \right] dx$$

is the integral of $\int_c^d f(x,y)\ dy$ between a and b, and

$$\int_c^d \int_a^b f(x,y)\ dx\ dy = \int_c^d \left[\int_a^b f(x,y)\ dx \right] dy \ \text{ is } \ \int_a^b f(x,y)\ dx$$

integrated between c and d.

Iterated single variable integrals

Example

Consider

$$\int_1^4 \int_0^2 (6x^2y - 2x)\ dy\ dx$$

and

$$\int_0^2 \int_1^4 (6x^2y - 2x)\ dx\ dy$$

The fact that these two iterated integrals evaluate to the same value is no accident.

Fubini's Theorem

One can compute the volume of a solid object by integrating over the area of its cross-section.

- Let $R = [a, b] \times [c, d]$ and let $f(x, y)$ be an integrable function defined on R and $\mathcal{S}$ be the solid corresponding to the volume being computed by $\iint_R f \ dA$. I.e. the volume lying between R on the xy-plane and the surface described by the graph $z = f(x, y)$.

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- Recall, that we can compute volumes as integrals of the areas of cross-sections:

$$\iint_R f \; dA = \int_a^b A(x) \; dx$$

  where $A(x)$ is the the area of the cross-section of $\mathcal{S}$ parallel to the yz-plane at point x on the x-axis.

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$$A(x) = \int_c^d f(x, y) \; dy \text{ and } \iint_R f \; dA = \int_a^b \int_c^d f(x, y) \; dy \; dx$$

Fubini's Theorem

Moreover, the same argument appears to work if we compute the volume of $\mathcal{S}$ by integrating over cross-sections that are parallel to the xz-plane instead of the yz-plane. This would yield:

$$\iint_R f \; dA = \int_c^d \int_a^b f(x, y) \; dx \; dy$$

Formalizing the ideas in the preceding slide into a rigorous mathematical proof, which we will not do in this course, yields Fubini's Theorem:

Theorem

(Fubini's Theorem) Let $R = [a, b] \times [c, d]$. If $f : R \longrightarrow \mathbb{R}$ is continuous on R, then

$$\iint_R f \; dA = \int_a^b \int_c^d f(x, y) \; dy \; dx = \int_c^d \int_a^b f(x, y) \; dx \; dy$$

The assumption that f is continuous can be replaced by the assumption that f is bounded on R, discontinuous only on a finite number of smooth curves, and that the iterated integrals involved in Fubini's Theorem exist.

Fubini's Theorem

Example

Let $R = [1,2] \times [0,\pi]$. *Compute*

$$\iint_R y \sin(xy)\ dA$$

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Let $R = [1, 2] \times [0, \pi]$. *Compute*

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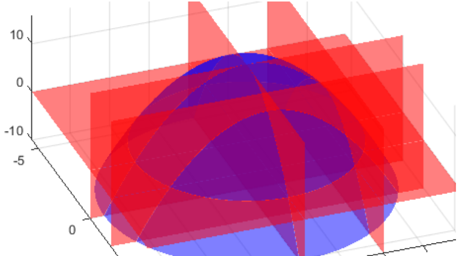
Example

Let $\mathcal{S}$ *be the solid that is bounded by the elliptic paraboloid* $x^2 + 2y^2 + z = 16$, *the planes* $x = 2$ *and* $y = 2$, *and the xz-plane, the xy-plane and the yz-plane. Find the volume of* $\mathcal{S}$.



Fubini's Theorem

Fubini's Theorem: if a function of two variables can be factored into two functions of single variable, then its multivariable integral is particularly easy to compute.
Let $R = [a, b] \times [c, d]$. If $f : R \longrightarrow \mathbb{R}$ is continuous on R and $f(x, y) = g(x)h(y)$, then

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Example

Compute

$$\int_0^2 \int_0^\pi r \sin^2(\theta) \, d\theta \, dr$$

Integrals of functions of two variables over regions

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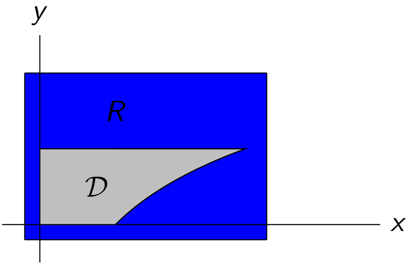
Let $\mathcal{D}$ be a bounded region in $\mathbb{R}^2$ that contains all of its boundary points. Let $f : \mathcal{D} \longrightarrow \mathbb{R}$ be a function. Since $\mathcal{D}$ is bounded, $\mathcal{D}$ can be completely enclosed in a closed rectangle R.



Integrals of functions of two variables over regions

This allows us to define $\tilde{f} : R \longrightarrow \mathbb{R}$ by

$$\tilde{f}(\bar{x}) = \begin{cases} f(\bar{x}) & \text{if } \bar{x} \in \mathcal{D} \\ 0 & \text{if } \bar{x} \in R \text{ and } \bar{x} \notin \mathcal{D} \end{cases}$$

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It should be clear that he choice of the closed rectangle R that encloses $\mathcal{D}$ does not change the value of the integral.

Unfortunately, discussing the exact conditions that ensure that $\tilde{f}$ is integrable is outside the scope of this course. However, if f is continuous on $\mathcal{D}$ and $\mathcal{D}$ is a bounded region that contains all of its boundary points, then $\tilde{f}$ defined above will be integrable.

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Obviously, if we want to compute the integral of a function of two variables over a bounded region $\mathcal{R}$ in $\mathbb{R}^2$, then $\mathcal{R}$ needs to have some kind of nice description.

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Let $g_1 : [a, b] \longrightarrow \mathbb{R}$ and $g_2 : [a, b] \longrightarrow \mathbb{R}$ be continuous functions. Let $\mathcal{R}$ be a region that lies between the graphs $y = g_1(x)$ and $y = g_2(x)$ on $[a, b]$. I.e.

$$\mathcal{R} = \{(x, y) \mid a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}$$

A region of this form is said to be a type I bounded region.

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Integrals of functions of two variables over regions

Example

Find the volume of the solid that lies under the graph $z = x^2 + y^2$ *and above the region* $\mathcal{D}$ *in the* xy*-plane that is bounded by* $y = 2x$ *and* $y = x^2$.

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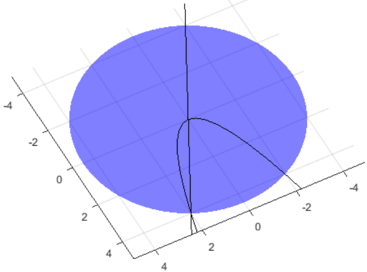

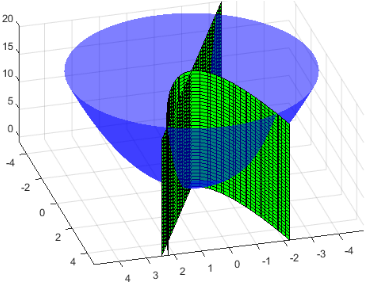

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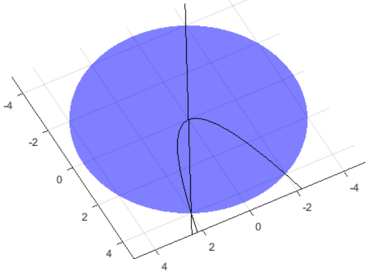
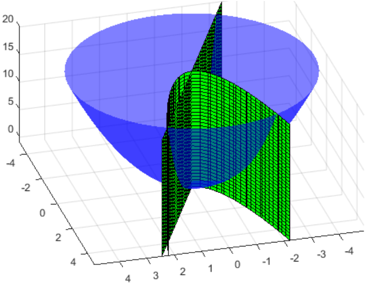
Find the volume of the solid that lies under the graph $z = x^2 + y^2$ and above the region $\mathcal{D}$ in the xy-plane that is bounded by $y = 2x$ and $y = x^2$.





Example

Compute

$$\int_0^1 \int_x^1 \sin(y^2)\; dy\; dx$$

Double Integrals: Properties

We are to consider properties of double integrals using the Riemann definition of the double integral. Recall that definition here:

- Let a bounded function $f(x, y)$ be defined on $\mathcal{R}$.
- We divide $\mathcal{R}$ into sub-domains $\mathcal{R}_1, \ldots, \mathcal{R}_n$ with the areas $A_1, \ldots, A_n$.
- In each elementary domain $\mathcal{R}_i$, we choose an arbitrary point (ξ_i, η_i), find the value of f at that point $f(\xi_i, \eta_i)$, and approximate the volume of the solid bounded above by $f(x, y)$ defined in the sub-domain $\mathcal{R}_i$ by $f(\xi_i, \eta_i)A_i$.

$$\sigma = \sum_{i=1}^{n} f(\xi_i, \eta_i) A_i$$

- Let $\lambda = \max A_i$.
- The finite limit of the integral sum σ as $\lambda \to 0$ is called the double integral of the function $f(x, y)$ over the region $\mathcal{P}$.

$$\sigma \to \iint_{\mathcal{R}} f(x, y)\, dx\, dy$$

Double Integrals: Properties

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$$\iint_{\mathcal{R}} f \; dA = \iint_{\mathcal{R}_1} f \; dA + \iint_{\mathcal{R}_2} f \; dA$$

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8. (The mean value theorem) If $f(x,y)$ is continuous for all $(x,y) \in A$, then
$$\exists (x^*, y^*) \in A\colon \iint_{\mathcal{R}} f \; dA = f(x^*, y^*) A_{\mathcal{R}}$$

Change of variables: Example 1

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Consider the functions

$$x(r,\theta) = r\cos(\theta) \text{ and } y(r,\theta) = r\sin(\theta)$$

The functions $x(r,\theta)$ *and* $y(r,\theta)$ *map the closed rectangle*

$$R = \{(r,\theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$$

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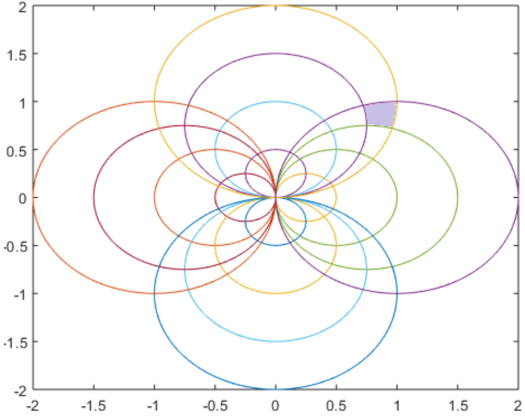
So, we need some way of integrating functions that are obtained composing a function with a maps that transform one region in $\mathbb{R}^2$ into another region in $\mathbb{R}^2$.

Change of variables: Example 2

Consider a one-to-one and onto map from uv plane to xy plane defined by

$$x = \frac{u}{u^2+v^2}, \quad y = \frac{v}{u^2+v^2}$$

(u, v do not vanish simultaneously)



Change of variables: Example 2

This map is invertible:

$$u = \frac{x}{x^2 + y^2}, \quad v = \frac{y}{x^2 + y^2}$$

Any (u, v) defines the point (x, y) and vice-versa. The values u, v are called coordinates of the point (x, y). A curve that consists of points in xy- plane for which one of the corresponding coordinates u, v is a constant, is called the coordinate curve (coordinate line).
The coordinate lines in xy plane are the circles

$$x^2 + y^2 - \frac{1}{u_0}x = 0 \quad x^2 + y^2 - \frac{1}{v_0}y = 0$$

centered at points on x and y axes and passing through the origin.
For example, the square region $[\frac{1}{2}, 1] \times [\frac{1}{2}, 1] \in uv$ is mapped onto the dashed region in xy-plane.
u, v are "curved" coordinates.

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Functions of a single real variable:

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Definition

Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and let $f_1, \ldots, f_n$ be such that for all $1 \leq k \leq n$, $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$. We say that $f_1, \ldots, f_n$ are the components of F if for all $\bar{x} \in \mathbb{R}^n$,

$$F(\bar{x}) = \langle f_1(\bar{x}), \ldots, f_n(\bar{x}) \rangle$$

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Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ have components $f_1, \ldots, f_n$ where for all $1 \leq k \leq n$, $f_k : \mathbb{R}^n \longrightarrow \mathbb{R}$ has independent variables $x_1, \ldots, x_n$, and let $\bar{a} \in \mathbb{R}^n$. If for all $1 \leq k \leq n$, f_k is differentiable at $\bar{a}$, then we define the Jacobian of F at $\bar{a}$, by $J_F(\bar{a}) = \det(A)$ where $A = (a_{ij})$ is the $n \times n$ matrix with entries

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Example

Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ *be defined by*

$$F(r, \theta) = \langle r\cos(\theta), r\sin(\theta) \rangle$$

$$\mathit{Then}\ J_F(r, \theta) = \begin{vmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{vmatrix} = r$$

Change of variables

In order to present a "change of variables" result for multivariable integrals, we will also require that the functions that map a region $\mathcal{R}_1$ to a region $\mathcal{R}_2$ that we want to integrate over never map two distinct points in $\mathcal{R}_1$ to the same point in $\mathcal{R}_2$.

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Theorem

Let $\mathcal{R}$ and $\mathcal{S}$ be bounded regions $\mathbb{R}^2$ that contain all of their boundary points. Let $T : D \longrightarrow \mathbb{R}^2$ where $D \subseteq \mathbb{R}^2$ be an injective onto map that maps $\mathcal{S}$ in the uv-plane to $\mathcal{R}$ in the xy-plane. If $f(x, y)$ is continuous on $\mathcal{R}$, all of the components of T have continuous partial derivative and $J_T(u, v)$ is never 0 on $\mathcal{S}$, then

$$\iint_{\mathcal{R}} f(x, y)\ dx\ dy = \iint_{\mathcal{S}} f(x(u, v), y(u, v))|J_T(u, v)|\ du\ dv$$

Change of Variables: Proof of the Theorem

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Let $s = S_1S_2S_3S_4$ be an infinitely small rectangular in uv plane with sides du, dv parallel to the coordinate axes u and v.
Then the image of S in xy plane is a figure $p = P_1P_2P_3P_4$ with 4 vertices :

$$S_1(u, v) \to P_1(x(u, v), y(u, v)),\ S_2(u+du, v) \to P_2(x(u+du, v), y(u+du, v))$$

$$S_3(u + du, v + dv) \to P_3(x(u + du, v + dv), y(u + du, v + dv)),$$

$$S_4(u, v + dv) \to P_4(x(u, v + dv), y(u, v + dv))$$

$$\Rightarrow P_1(x, y),\ P_2(x + \frac{\partial x}{\partial u}du, y + \frac{\partial y}{\partial u}du)$$

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The segments P_1P_2, P_3P_4 have equal projections onto both axes
$\Rightarrow p = P_1P_2P_3P_4$ is a parallelogram.

Change of Variables: Proof of the Theorem

$$A_p = 2A_{P_1P_2P_3} = abs\begin{vmatrix} \frac{\partial x}{\partial u}du & \frac{\partial x}{\partial v}dv \\ \frac{\partial y}{\partial u}du & \frac{\partial y}{\partial v}dv \end{vmatrix} = |J_T(u,v)|dudv$$

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Dividing the whole region S in uv plane into sub-regions (rectangles) by the lines parallel to coordinate axes, we produce a partition of the region P in xy plane into "curved" rectangles with the areas $|J_T(u,v)|dudv$. Summing them up, we obtain

$$A_P = \iint_S |J_T(u,v)|dudv$$

By the mean value theorem,

$$A_P = |J_T(u^*,v^*)|A_S$$

$\Rightarrow$ "shrinking" the region S into the point (u,v), $\quad |J_T(u,v)| = \lim \dfrac{A_P}{A_S}$

The absolute value of the Jacobian is the distortion coefficient (for the transformation of the plane uv onto xy)

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Applying the mean value theorem for each R_i, i.e $A_{R_i} = |J(u^*, v^*)| A_{S_i}$, we obtain

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Attention: we can't choose the point (u^*, v^*) but the point (x_i, y_i) is arbitrary. We make use of it $x_i = x_i(u^*, v^*)$, $y_i = y_i(u^*, v^*)$

$$\sigma = \sum_{i=1}^{n} f(x(u^*, v^*), y_i(u^*, v^*)) |J(u^*, v^*)| A_{S_i}$$

Change of Variables: Proof of the Theorem

It is the integral sum for the integral $\iint_S f(u,v)|J(u,v)|dudv$.

The map T of uv onto xy is continuous $\Rightarrow$ if the regions S_i are shrinking to infinitely small regions, it happens with the regions R_i as well. And then, σ is the integral sum for the integral $\iint_R f(x,y)\,dx\,dy$

$$\Rightarrow \iint_R f(x,y)\,dx\,dy = \iint_S f(u,v)|J(u,v)|du\,dv$$

Polar coordinates

Example

Perhaps the most commonly used change of variables is the transformation to polar coordinates. Consider $f(x, y) = 3x + 4y^2$ *and suppose that we want to compute*

$$\iint_{\mathcal{R}} f(x, y)\, dA$$

where $\mathcal{R}$ *is the region in the upper half-plane bounded by the circles* $x^2 + y^2 = 1$ *and* $x^2 + y^2 = 4$.

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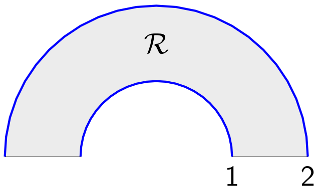

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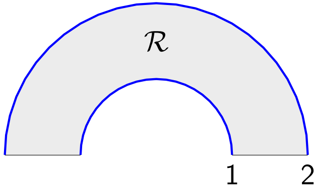

Let $D = \{(r, \theta) \in \mathbb{R}^2 \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$*.*

Polar coordinates

Example

(Continued.) The map $T : D \longrightarrow \mathbb{R}^2$ *defined by*

$$T(r, \theta) = \langle r\cos(\theta), r\sin(\theta) \rangle$$

is an injective onto map that maps the closed rectangle D *in the* $r\theta$*-plane to the region* $\mathcal{R}$ *in the* xy*-plane.*

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$$\begin{aligned} \iint_{\mathcal{R}} f \; dA &= \int_0^{\pi} \int_1^2 f(x(r,\theta), y(r,\theta)) r \; dr \; d\theta \\ &= \int_0^{\pi} \int_1^2 (3r\cos(\theta) + 4r^2\sin^2(\theta)) r \; dr \; d\theta \end{aligned}$$

Polar coordinates

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Now,

$$\int_1^2 (3r^2\cos(\theta) + 4r^3\sin^2(\theta)) \; dr = \Big[r^3\cos(\theta) + r^4\sin^2(\theta)\Big]_1^2$$

Polar coordinates

Example

(Continued.)

$$= 7\cos(\theta) + 15\sin^2(\theta)$$

Polar coordinates

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Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ *and above the* xy*-plane, and inside the cylinder* $x^2 + y^2 = 2x$.

Polar coordinates

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(Continued.)

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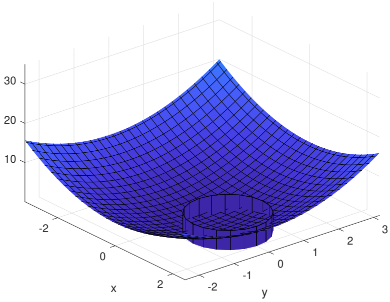
Example

Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

After completing the square we see that the cylinder is described by $(x-1)^2 + y^2 = 1$. Therefore, it is a cylinder of radius 1 running parallel to the z-axis and centred at $\langle 1, 0\rangle$ on the xy-plane.

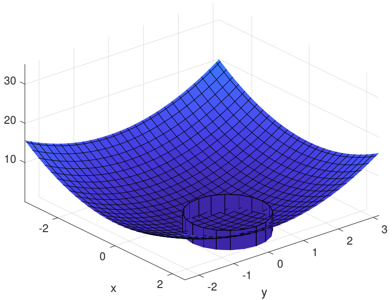
Change of variables

Example



Change of variables

Example



So, the volume of the solid is given by

$$\iint_{\mathcal{R}} (x^2 + y^2)\ dx\ dy$$

where $\mathcal{R}$ is the unit circle centered at $(1, 0)$ on the xy-plane.

Change of variables

Example

(Continued.) The translation of the map used in the previous example: $T : D \longrightarrow \mathcal{R}$ *defined by*

$$T(r, \theta) = \langle r\cos(\theta) + 1, r\sin(\theta)\rangle,$$

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where $D = \{\langle r,\theta\rangle \in \mathbb{R}^2 \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$, *is an injective map from* D *to* $\mathcal{R}$. *As with the previous example, we see that* $J_T(r,\theta) = r$. *After changing variable, we see that*

$$\iint_{\mathcal{R}} (x^2 + y^2)\ dx\ dy = \int_0^{2\pi}\int_0^1 (r^2 + 2r\cos(\theta) + 1)r\, dr\, d\theta = \frac{3\pi}{2}$$

Change of variables

Example

Compute

$$\iint_{\mathcal{R}} \left(\frac{x-y}{x+y+2}\right)^2 \, dA$$

where $\mathcal{R}$ *is the region that contains all of its boundary points and is bounded by the lines* $x+y=\pm 1$ *and* $x-y=\pm 1$.

Area of the Region: Example 1

Find the area of the region bounded by the curves

$$(x^2 + y^2)^2 = 2a^2(x^2 - y^2), \quad x^2 + y^2 = a^2 \quad (x^2 + y^2 \geq a^2)$$

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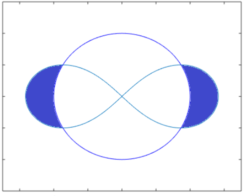

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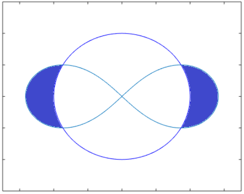

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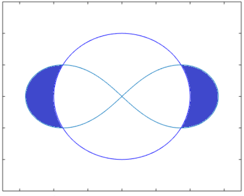

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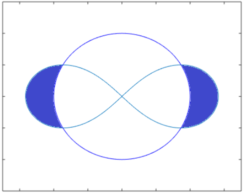

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$$A = 4\int_0^{\frac{\pi}{6}} \int_a^{a\sqrt{2\cos 2\theta}} r\,dr\,d\theta = 2a^2 \int_0^{\frac{\pi}{6}} (2\cos 2\theta - 1)\,d\theta = \frac{3\sqrt{3}-\pi}{3} a^2$$

Area of the Region: Example 2

Find the area of the region bounded by the curves

$$\frac{x^3}{a^3} + \frac{y^3}{b^3} = \frac{x^2}{k^2} + \frac{y^2}{h^2} \quad x = 0,\ y = 0 \quad (a,\ b,\ k,\ h > 0)$$

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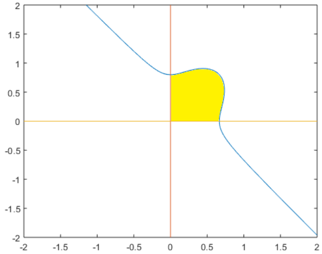

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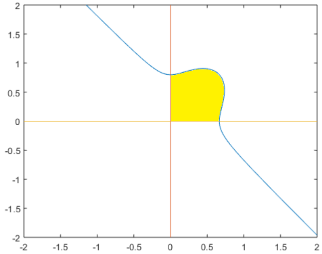

We define generalized polar coordinates:

$$x = ar\cos^{\frac{2}{3}}\theta,\ y = br\sin^{\frac{2}{3}}\theta,\ 0 \le \theta \le \frac{\pi}{2}$$

Area of the Region: Example 2

In generalized polar coordinates, the region is defined by

$$r = \frac{a^2}{k^2}\cos^{\frac{4}{3}}\theta + \frac{b^2}{h^2}\sin^{\frac{4}{3}}\theta\,(0 < \theta < \frac{\pi}{2}),\ \theta = 0,\ \theta = \frac{\pi}{2}$$

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$$= \frac{ab}{3}\left(\frac{a^2b^2}{k^2h^2} + \frac{2\pi}{3\sqrt{3}}\left(\frac{a^4}{k^4} + \frac{b^4}{h^4}\right)\right)$$

Volume: Example 1

Find the volume of a solid body bounded by the surfaces

$$z = x^2 + y^2, \quad y = x^2, \quad z = 0, \quad y = 1$$

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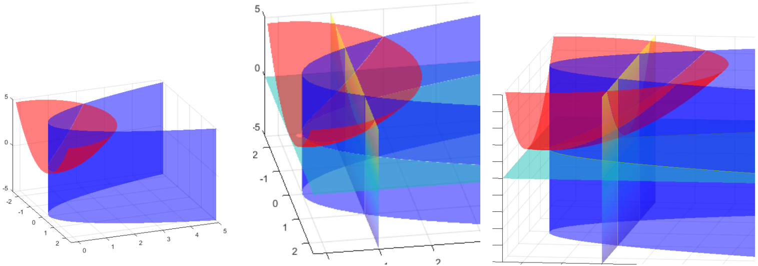

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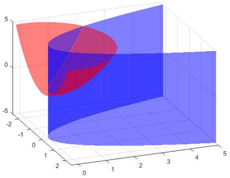

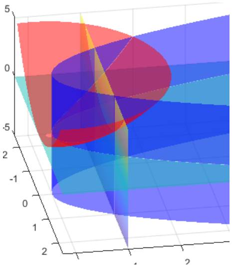

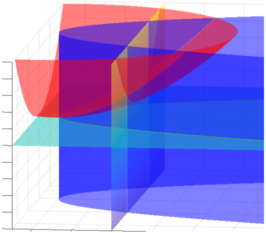

$$V = \iint_D (x^2 + y^2)\, dx\, dy, \quad D = \{(x, y) \in \mathbb{R}^2 \colon |x| \le 1,\, x^2 \le y \le 1\}$$

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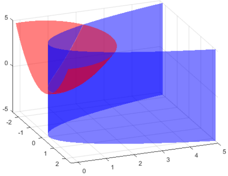
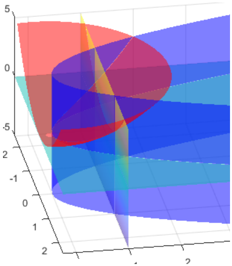
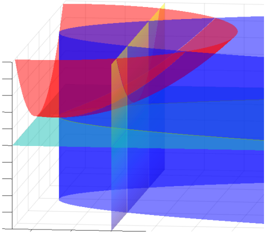

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$$V = \int_{-1}^{1} \int_{x^2}^{1} (x^2 + y^2)\, dy\, dx = \int_{-1}^{1} (x^2 - x^4 + \frac{1}{3} - \frac{x^6}{3})\, dx = \frac{88}{105}$$

Volume: Example 2

Example

Find the volume of a solid body bounded by the surfaces

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$z > 0, \quad a, b, c > 0$$

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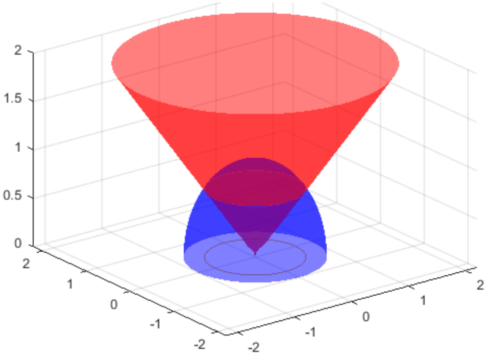

Volume: Example 2

Example

$$V = \iint_D (z_1 - z_2)\, dx\, dy, \quad \mathcal{D} = \{(x, y) \in \mathbb{R}^2 \colon \frac{x^2}{a^2} + \frac{y^2}{b^2} \le \frac{1}{2}\}$$

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$$z_1(x, y) = c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}},\; z_2(x, y) = c\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2}}$$

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$$V = abc\int_0^{2\pi} d\theta \int_0^{\frac{1}{\sqrt{2}}} (r\sqrt{1 - r^2} - r^2) dr$$

$$= \frac{2}{3}\pi abc((1 - r^2)^{3/2} + r^3)|_{1/\sqrt{2}}^{0} = \frac{\pi}{3} abc(2 - \sqrt{2})$$

Surface area

The definite integral of functions of two variables can be used to compute the surface area of a surface described by a graph $z = f(x, y)$, where f has continuous partial derivatives, in the same way as the definite integral of functions of a single variable could be used to compute the arc length of smooth lines.

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Surface area

The definite integral of functions of two variables can be used to compute the surface area of a surface described by a graph $z = f(x, y)$, where f has continuous partial derivatives, in the same way as the definite integral of functions of a single variable could be used to compute the arc length of smooth lines.

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$$\bar{a} = h\bar{i} + f_x(x_0, y_0)h\bar{k} \text{ and } \bar{b} = k\bar{j} + f_y(x_0, y_0)k\bar{k}$$

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Surface area

$$\bar{a} \times \bar{b} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ h & 0 & f_x(x_0, y_0)h \\ 0 & k & f_y(x_0, y_0)k \end{vmatrix} = -f_x(x_0, y_0)hk\bar{i} - f_y(x_0, y_0)kh\bar{j} + hk\bar{k}$$

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It follows that the surface area of the tangent plane approximation of $z = f(x, y)$ above the rectangle R is given by

$$hk\sqrt{(f_x(x_0, y_0))^2 + (f_y(x_0, y_0))^2 + 1}$$

$$= V(R)\sqrt{(f_x(x_0, y_0))^2 + (f_y(x_0, y_0))^2 + 1}$$

Surface area

Generalizing and formalizing this argument shows that if $f : D \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^2$, has continuous partial derivatives and $\mathcal{R}$ is a region contained in D that contains all of its boundary points, then the surface area, A, of the surface $z = f(x, y)$ is given by

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Example

Find the surface area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Triple integrals

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Definition

Let $R = [a,b] \times [c,d] \times [r,s]$. Let $f : R \longrightarrow \mathbb{R}$ be integrable. We define the partial integral of f with respect to x between a and b, denoted

$$\int_a^b f(x,y,z)\;dx,$$

to be the function of z and y obtained by holding z and y constant and evaluating the integral of $f(x,y,z)$ with respect to x between a and b. The partial integral of f with respect to y between c and d and the partial integral of f with respect to z between r and s are then defined similarly.

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and the other iterated integrals are defined similarly.

If $f : D \longrightarrow \mathbb{R}$, where $D \subseteq \mathbb{R}^3$ is a closed rectangle and $\mathcal{R}$ is a region contained in D, then define

$$\tilde{f}(\bar{x}) = \begin{cases} f(\bar{x}) & \bar{x} \in \mathcal{R} \\ 0 & \bar{x} \notin \mathcal{R} \end{cases}$$

Triple integrals

If $\tilde{f}$ is integrable over D, then the integral of f over $\mathcal{R}$, denoted

$$\iiint_{\mathcal{R}} f \; dV,$$

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Again, I do not want to go into the details about when $\tilde{f}$ is integrable, but it is safe to assume that if $\mathcal{R}$ is defined by continuous functions, is bounded, contains all of its boundary points and f is continuous on $\mathcal{R}$, then $\tilde{f}$ is integrable.

Triple integrals

Theorem

(Fubini's Theorem II) Let $R = [a, b] \times [c, d] \times [r, s]$. If $f : R \longrightarrow \mathbb{R}$ is continuous on R, then

$$\iiint_R f \; dV = \int_a^b \int_c^d \int_r^s f(x, y, z) \; dz \; dy \; dx$$

Moreover, this equation holds for any rearrangement of the order of the iterated integral on the right-hand side.

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(Fubini's Theorem II) Let $R = [a, b] \times [c, d] \times [r, s]$*. If* $f : R \longrightarrow \mathbb{R}$ *is continuous on* R*, then*

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As was the case with double integrals, Fubini's Theorem allows us to compute triple integrals over regions (solids) that are defined by continuous functions.

Let $R = [a, b] \times [c, d]$ and let $u_1 : R \longrightarrow \mathbb{R}$ and $u_2 : R \longrightarrow \mathbb{R}$ be continuous functions. Let D be a region in $\mathbb{R}^2$ that is contained in R (i.e. $D \subseteq R$). Let $\mathcal{R}$ be the solid region whose projection onto the xy-plane is D and that is bounded on the z-axis by the continuous functions $u_1(x, y)$ and $u_2(x, y)$.

Triple integrals

I.e.

$$\mathcal{R} = \{\langle x, y, z\rangle \in \mathbb{R}^3 \mid \langle x, y\rangle \in D \text{ and } u_1(x, y) \leq z \leq u_2(x, y)\}$$

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A region of this form is said to be a type I solid region. If $f : \mathcal{R} \longrightarrow \mathbb{R}$ is a function such that $\tilde{f}$ is integrable on any closed rectangle containing $\mathcal{R}$, then

$$\iiint_{\mathcal{R}} f \; dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z) \; dz \right] dA$$

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Note that this uses Fubini's Theorem to reduce a triple integral to a double integral of a partial integral. The bounded region D in $\mathbb{R}^2$ over which the double integral is taken could be either a type I or type II bounded region in $\mathbb{R}^2$.

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Similarly, let $\mathcal{R}$ be the solid region whose projection onto the yz-plane is D and that is bounded on the x-axis by the continuous functions $u_1(y, z)$ and $u_2(y, z)$. I.e.

$$\mathcal{R} = \{(x, y, z) \in \mathbb{R}^3 \mid (y, z) \in D \text{ and } u_1(y, z) \leq x \leq u_2(y, z)\}$$

Triple integrals

A region of this form is said to be a type II solid region. If $f : \mathcal{R} \longrightarrow \mathbb{R}$ is a function such that $\tilde{f}$ is integrable on any closed rectangle containing $\mathcal{R}$, then

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A region of this form is said to be type III solid region. If $f : \mathcal{R} \longrightarrow \mathbb{R}$ is a function such that $\tilde{f}$ is integrable on any closed rectangle containing $\mathcal{R}$, then

$$\iiint_{\mathcal{R}} f \; dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z) \; dy \right] dA$$

Triple integrals

Just as computing the double integral of the function that is contantly 1 over a region gives the area of that region, computing the triple integral of the function that is contantly 1 over a solid region yields the volume of that region. That is, if $\mathcal{R}$ is a solid region in $\mathbb{R}^3$ such that the function that is constantly 1 is integrable over $\mathcal{R}$, then the volume of $\mathcal{R}$ is given by

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Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$.

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Example

Find the volume of the tetrahedron enclosed by the coordinate planes and the plane $2x + y + z = 4$.

Example

Compute

$$\iiint_{\mathcal{R}} \sqrt{x^2 + z^2}\, dV$$

where $\mathcal{R}$ is the solid region bounded by $y = x^2 + z^2$ and $y = 4$.

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Theorem

Let $\mathcal{R}$ and $\mathcal{S}$ be bounded regions $\mathbb{R}^3$ that contain all of their boundary points. Let $T : D \longrightarrow \mathbb{R}^2$ where $D \subseteq \mathbb{R}^3$ be an injective onto map that maps $\mathcal{S}$ in the uvw-plane to $\mathcal{R}$ in the xyz-plane. If $f(x, y, z)$ is continuous on $\mathcal{R}$, all of the components of T have continuous partial derivative and $J_T(u, v, w)$ is never 0 on $\mathcal{S}$, then

$$\iiint_{\mathcal{R}} f(x, y, z)\ dx\ dy\ dz = \iiint_{\mathcal{S}} f(T(u, v, w))|J_T(u, v, w)|\ du\ dv\ dw$$

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We now turn to discussing two particularly useful "change of variables" for triple integrals: Cylindrical Coordinates and Spherical Coordinates

Cylindrical Coordinates

Let $R = [0, a] \times [0, 2\pi] \times [c, d]$. Consider $x : R \longrightarrow \mathbb{R}$, $y : R \longrightarrow \mathbb{R}$ and $z : R \longrightarrow \mathbb{R}$ defined by

$$x(r, \theta, s) = r\cos\theta \qquad y(r, \theta, s) = r\sin\theta \qquad z(r, \theta, s) = s$$

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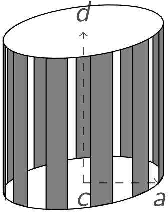
The map F is onto injective and maps the closed rectangle R in $\mathbb{R}^3$ into the cylinder of radius a centred around the z-axis that runs between c and d.



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So, given a description of a region $\mathcal{R}$ in cylindrical coordinates:

$$\mathcal{R} = \{(r,\theta,s) \in \mathbb{R}^3 \mid (r,\theta) \in D \text{ and } u_1(r,\theta) \leq s \leq u_2(r,\theta)\}$$

where

$$D = \{\langle r,\theta\rangle \in \mathbb{R}^2 \mid \alpha \leq \theta \leq \beta, h_1(\theta) \leq r \leq h_2(\theta)\}$$

Then the integral of a function $f(x,y,z)$ over the region $\mathcal{S}$ that F maps $\mathcal{R}$ to is given by

$$\iiint_{\mathcal{S}} f(x,y,z)\ dV = \int_{\alpha}^{\beta} \int_{h_1(\alpha)}^{h_2(\beta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, s) r\ ds\ dr\ d\theta$$

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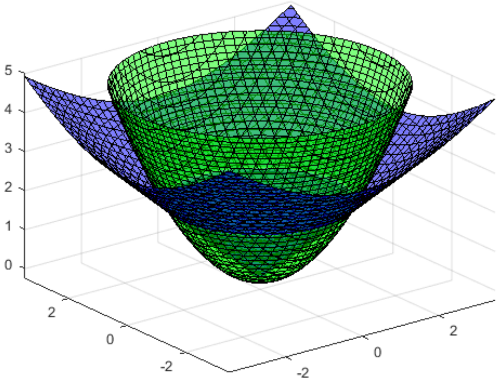

Cylindrical Coordinates: Example

$$\Big\{(x, y, z) \in \mathbb{R}^3 \colon -1 \leq x \leq a, -\sqrt{a^2 - x^2} \leq y \leq \sqrt{a^2 - x^2},$$
$$\frac{x^2 + y^2}{a} \leq z \leq \sqrt{x^2 + y^2}\Big\}$$

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$$V=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{\frac{x^2+y^2}{a}}^{\sqrt{x^2+y^2}} dz\,dy\,dx$$

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$$V=4\int_0^{\frac{\pi}{2}} d\theta\int_0^a r\,dr\int_{\frac{r^2}{a}}^{r} dz=2\pi\int_0^a r(r-\frac{r^2}{a})\,dr=$$

Cylindrical Coordinates: Example

$$\Big\{(x,y,z)\in\mathbb{R}^3:\ -1\le x\le a,\ -\sqrt{a^2-x^2}\le y\le\ \sqrt{a^2-x^2},$$
$$\frac{x^2+y^2}{a}\le z\le\ \sqrt{x^2+y^2}\Big\}$$

$$V=\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_{\frac{x^2+y^2}{a}}^{\sqrt{x^2+y^2}} dz\,dy\,dx$$

Use the cylindrical coordinates:

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Cylindrical Coordinates: Example

Find the volume of the solid bounded by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}, \quad a, b, c > 0$$

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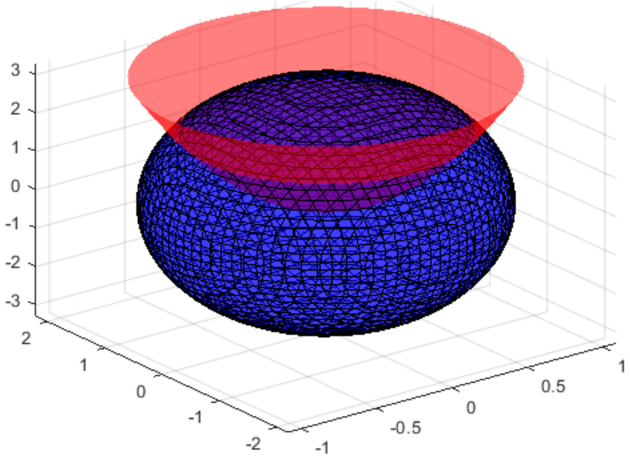

Cylindrical Coordinates: Example

Find the projection of the intersection curve onto xy plane:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = 1 \Rightarrow$$

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Use the generalized cylindrical coordinates:

$$x = a\sqrt{\frac{\sqrt{5}-1}{2}}\, r\cos\theta, \quad y = b\sqrt{\frac{\sqrt{5}-1}{2}}\, r\sin\theta, \quad z = z \quad 0 \le \theta \le \frac{\pi}{2}$$

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$$J(\rho, \theta, z) = \frac{ab}{2}(\sqrt{5}-1)r$$

Cylindrical Coordinates: Example

$$V = 4\frac{ab}{2}(\sqrt{5}-1)\int_0^{\frac{\pi}{2}} d\theta \int_0^1 r\,dr \int_{cr^2\frac{\sqrt{5}-1}{2}}^{c\sqrt{1-\frac{r^2(\sqrt{5}-1)}{2}}} dz$$

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$$= \frac{5}{12}\pi abc(3-\sqrt{5})$$

Cylindrical Coordinates: Example

Find the volume of the solid bounded by

$$az = a^2 - x^2 - y^2, \quad z = a - x - y, \quad y = 0, \quad x = 0, \quad z = 0$$

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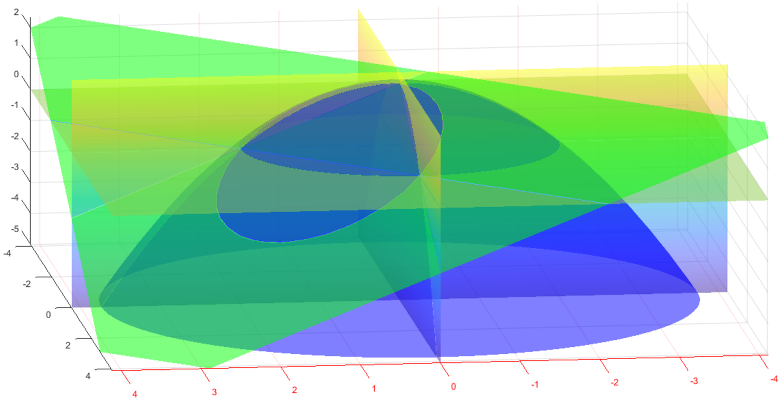


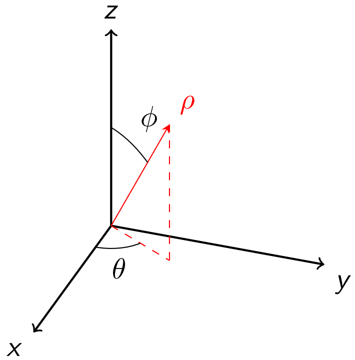
Spherical Coordinates

Let $R = [0, a] \times [0, 2\pi] \times [0, \pi]$. Consider $x : R \longrightarrow \mathbb{R}$, $y : R \longrightarrow \mathbb{R}$ and $z : R \longrightarrow \mathbb{R}$ defined by

$$x(\rho, \theta, \phi) = \rho \sin\phi \cos\theta \qquad y(\rho, \theta, \phi) = \rho \sin\phi \sin\theta$$

$$z(\rho, \theta, \phi) = \rho \cos\phi$$

$$\rho \geq 0, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi$$



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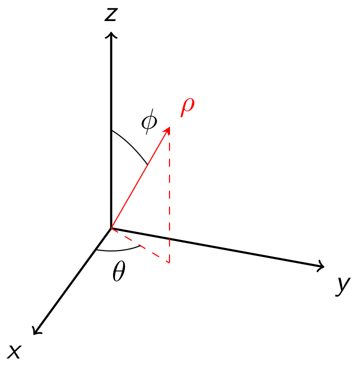

Observe that $\rho^2 = x^2 + y^2 + z^2$

Spherical Coordinates

The functions x, y and z form the components of a map $F : R \longrightarrow \mathbb{R}^3$ defined by:

$$F(\rho, \theta, \phi) = (\rho \sin \phi \cos \theta, \rho \sin \phi \sin \theta, \rho \cos \phi)$$

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$$\begin{aligned} x_\rho &= \sin\phi\cos\theta & x_\theta &= -\rho\sin\phi\sin\theta & x_\phi &= \rho\cos\phi\cos\theta \\ y_\rho &= \sin\phi\sin\theta & y_\theta &= \rho\sin\phi\cos\theta & y_\phi &= \rho\cos\phi\sin\theta \\ z_\rho &= \cos\phi & z_\theta &= 0 & z_\phi &= -\rho\sin\phi \end{aligned}$$

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And, so, the Jacobian is

$$J_F = \begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

Spherical Coordinates

So

$$J_F = \cos\phi(-\rho^2 \sin^2\theta \sin\phi\cos\phi - \rho\cos^2\theta\sin\phi\cos\phi)$$
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And, since $0 \leq \phi \leq \pi$,

$$|J_F(\rho, \theta, \phi)| = \rho^2 \sin\phi$$

Spherical Coordinates: Example

Compute

$$\iiint_{\mathcal{R}} (x^2 + y^2)\ dV$$

where $\mathcal{R}$ is the solid region that lies between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$.

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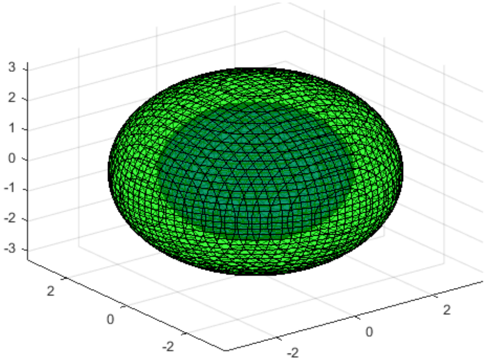

Spherical Coordinates: Example

Use the spherical coordinates:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$

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$$= 2\pi \frac{3^5 - 2^5}{5} \int_0^{\pi} -(1 - \cos^2 \phi)\, d(\cos \phi)$$

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$$= 2\pi \frac{211}{5}\ (-\cos\phi + \frac{\cos^3\phi}{3})\Big|_0^{\pi} =$$

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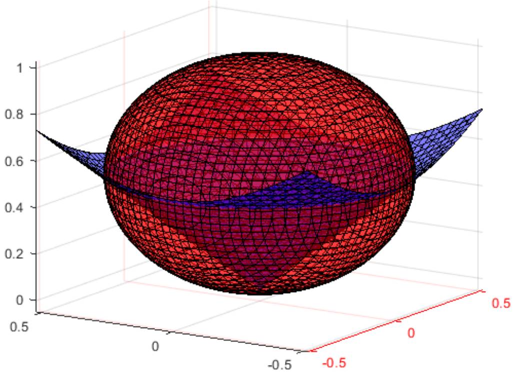
$$= 2\pi \frac{211}{5}\ (-\cos\phi + \frac{\cos^3\phi}{3})\Big|_0^{\pi} = \frac{1688\pi}{15}$$

Spherical Coordinates: Example

Find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.



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Spherical Coordinates: Example

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$$\sqrt{x^2 + y^2} = \rho \sin\phi = z = \rho \cos\phi \Rightarrow \phi = \frac{\pi}{4}$$

$$x^2 + y^2 + z^2 = \rho^2 = z = \rho \cos\phi \Rightarrow$$

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$$\mathcal{R} = \{(\rho, \theta, \phi) \colon 0 \le \rho \le \cos\phi,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4}\}$$

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$$V = 4V = 4 \int_0^{\cos\phi} \rho^2 \, d\rho \int_0^{\frac{\pi}{2}} \, d\theta \int_0^{\frac{\pi}{4}} \sin\phi \, d\phi$$

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$$= \frac{2\pi}{3}\left(\frac{1}{4} - \frac{1}{16}\right) = \frac{\pi}{8}$$

Spherical Coordinates: Example

Find the volume of the solid bounded by

$$(x^2 + y^2 + z^2)^2 = a^2(x^2 + y^2 - z^2)$$

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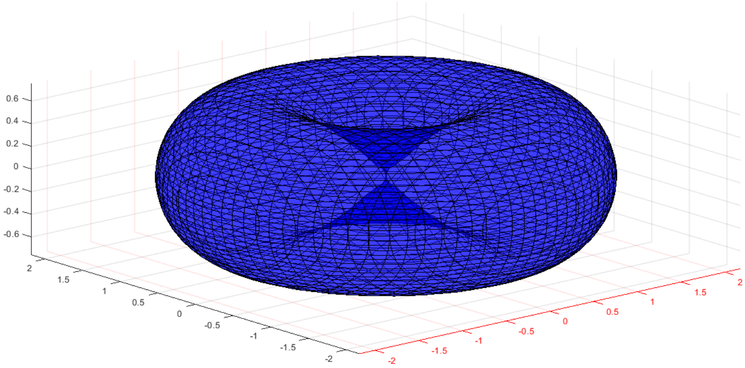

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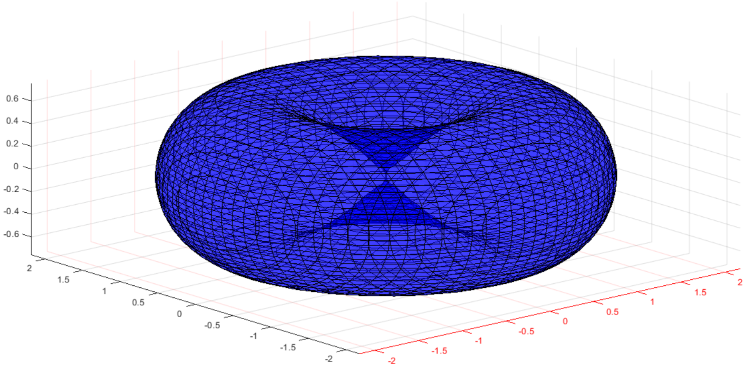

The solid is symmetric w.r.t all coordinate planes $\Rightarrow \frac{1}{8}$ of the body is in the 1st octane.

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Use the spherical coordinates:

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Let $\frac{\pi}{2} - \phi = t$.

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Triple Integrals: Change of Varibles

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