Vv156 Lecture 5

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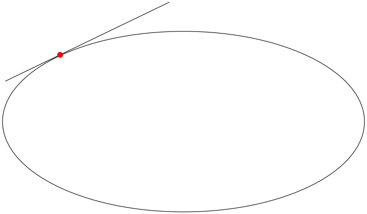
Q: What is a tangent line?

Definition

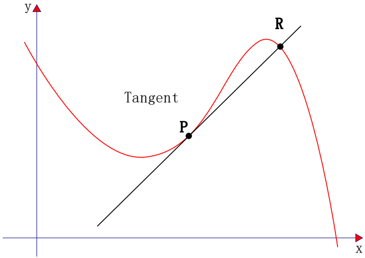
Euclid (300 BC) stated that a line is tangent to a circle if it intersects the curve at one and only one point.

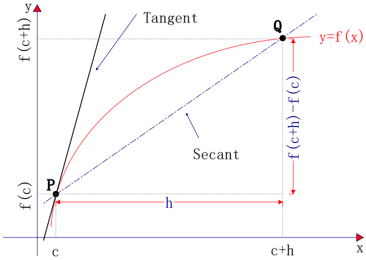
- This definition is also adequate for ellipses, for example,



Q: Is this definition adequate for the function $y = f(x)$ at the point $x = c$?

- Euclid's definition is not applicable to more general curves. For example,





Q: What do we need to define a line? What seems to be the problem?

Q: What is the difference between a secant and the tangent at the point P?

- The slope of the secant is defined to be

$$\frac{f(c+h) - f(c)}{h}$$

also known as the "Difference Quotient" of f at c.

Q: What happens if Q moves towards P?

Definition

Suppose $f(x)$ is defined for $a \le x \le b$, then $f(x)$ is said to be differentiable with the derivative $f'(c)$ at a point c inside the interval if the following limit exists:

$$\lim_{h \to 0} \left[\frac{f(c+h) - f(c)}{h} \right] = f'(c)$$

- The derivative $f'(c)$ is the slope of the tangent line to the graph of $f(x)$ at $x = c$, and it is defined to be the slope of the graph $f(x)$ at $x = c$.
- Alternatively, we can also used the following limit to define the derivative

$$f'(c) = \lim_{x \to c} \left[\frac{f(x) - f(c)}{x - c} \right]$$

since if $x = c + h$, then $0 < |x - c| < \delta$ and $0 < |h| < \delta$ are equivalent.

Exercise

Find the slope of the curve $y = \dfrac{1}{x}$ at $x \neq 0$ using the definition.

Theorem

Let $f(x)$ be defined on $[a, b]$, and suppose $f(x)$ is differentiable at a point c in the interval (a, b), then $f(x)$ is continuous at c.

Proof

- To prove that f is continuous at c, we have to show that

$$\lim_{x \to c} f(x) = f(c)$$

- We start by considering the limit of $f(x)$ and add and subtract $f(c)$,

$$\lim_{x \to c} f(x) = \lim_{x \to c} [f(c) + (f(x) - f(c))] = \underbrace{\lim_{x \to c} f(c)}_{1} + \underbrace{\lim_{x \to c} [f(x) - f(c)]}_{2}$$

- The sum law in the last step is valid because both limits 1 and 2 exist, why?
- For 1, since f is defined on $[a, b]$, thus $f(c)$ is defined and it is a constant,

$$\lim_{x \to c} f(c) = f(c)$$

Proof

- For 2, since $f(x) - f(c) = \dfrac{f(x) - f(c)}{x - c}(x - c)$ when $x \neq c$, thus

$$\begin{aligned}\lim_{x \to c} [f(x) - f(c)] &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c}(x - c) \\ &= \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \lim_{x \to c}(x - c) \\ &= f'(c) \cdot 0 \\ &= 0\end{aligned}$$

- The product law can be used in the above step since both of the limits exist.
- Putting 1 and 2 together

$$\lim_{x \to c} f(x) = \underbrace{\lim_{x \to c} f(c)}_{1} + \underbrace{\lim_{x \to c} [f(x) - f(c)]}_{2} = f(c) + 0 = f(c).$$

□

- The last theorem essentially states

$$\text{Differentiability} \Rightarrow \text{ Continuity}$$

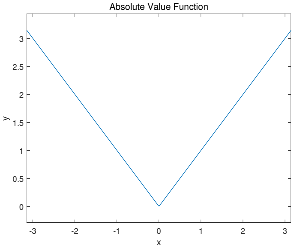
- The contrapositive of the last theorem is surely true; that is

$$\text{Not continuous } \Rightarrow \text{ Not differentiable}$$

- However, the converse of the last theorem is NOT true; that is,

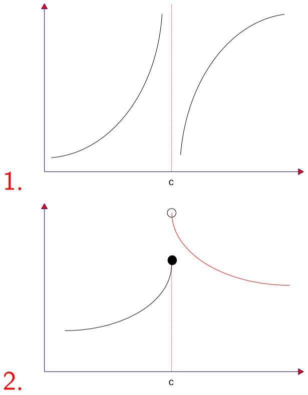
$$\text{Continuity } \not\Rightarrow \text{ Differentiability}$$

Q: Can you think of a counterexample?



- There are a few ways that a function can be non-differentiable at a point c:

1. The function is not continuous at c.



2. The function is continuous at c, but the graph of f has a corner at c,

   e.g. $f(x) = |x|$ at $x = 0$ belongs to this category.

- To understand 2. formally instead relying on intuition, we define one-sided derivative using one-sided limit,

Definition

The function f has a right-hand derivative at c if the right-hand limit exists,

$$f'(c^+) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

and a left-hand derivative at c if the left-hand limit exists,

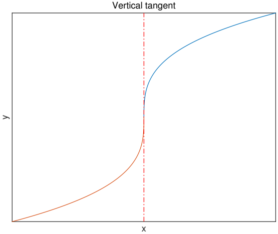
$$f'(c^-) = \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

- Having a corner at c is simply a result of the right-hand derivative being NOT equal to the left-hand derivative at c, i.e.,

$$f(x) = |x| \implies f'(0^+) = 1 \qquad \text{and} \qquad f'(0^-) = -1$$

Q: Is there a third way of not having a well defined slope for $f(x)$ at $x = c$?

3. A third possibility is that the curve has a vertical tangent line at c;



that is, f is continuous at c but the difference quotient is approaching ∞

$$\lim_{h\to 0}\left|\frac{f(c+h)-f(c)}{h}\right| = \infty$$

- The tangent lines become steeper and steeper as $x \to c$.
- A function is differentiable at a point if and only if it is differentiable from the left and right side and these derivatives coincide.

- When the derivative function is given, we can detect a vertical tangent using

$$\lim_{x \to c} |f'(x)|$$

- If the above limit is not finite, then f has a vertical tangent at c.

Exercise

(a) Show the following function is continuous and has a vertical tangent at $x = 2$

$$f(x) = \sqrt[5]{2-x}$$

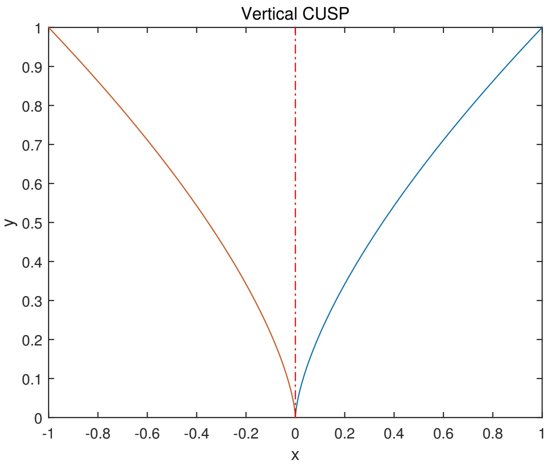
- We can further categorise 2. and 3.

Definition

A vertical tangent is also known as a vertical cusp if the one-sided derivatives are both infinite, but one is positive and the other is negative.

Exercise

(b) Show $g(x) = \sqrt[3]{x^2}$ has a vertical cusp at $x = 0$.



Matlab

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>> x = [0:0.0001:3]; plot(x,x.^(2/3)); hold on; plot(-x,x.^(2/3)); obj = line([0,0],[0,1]);
>> set(obj, 'color','red'); set(obj, 'LineStyle', '-.'); clear obj; hold off; axis([-1,1,0,1]);
>> xlabel('x'); ylabel('y'); title('Vertical CUSP');
```

Q: Is there a function that is continuous everywhere but nowhere differentiable?

$$W(x) = \sum_{k=0}^{\infty} a^k \cos\left(b^k 2\pi x\right)$$

where $a \in (0,1)$ and b is an positive integer such that $ab \geq 1$.