Question1 (1 points)
Find the interval of convergence for the power series

$$\sum_{n=10}^{\infty} \frac{(3x+2)^n}{n^2}$$

Solution:

1M Ratio test

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n^2}{(n+1)^2}|3x+2| \to |3x+2| \qquad \text{as} \qquad n \to \infty$$

thus the series is convergent for

$$-1 < x < -\frac{1}{3}$$

and diverges for

$$x < -1 \qquad \text{or} \qquad x > -\frac{1}{3}$$

Checking the endpoints, we have

$$\sum_{n=10}^{\infty} \frac{(3x+2)^n}{n^2} = \sum_{n=10}^{\infty} \frac{1}{n^2} \qquad \text{when} \qquad x = -\frac{1}{3}$$

which is convergent since it is the p-series with $p = 2 > 1$. When $x = -1$, we have

$$\sum_{n=10}^{\infty} \frac{(3x+2)^n}{n^2} = \sum_{n=10}^{\infty} \frac{(-1)^n}{n^2}$$

which converges absolutely thus is convergent according to the case when $x = -\frac{1}{3}$. Therefore, the interval of convergence is

$$\left[-1, -\frac{1}{3}\right]$$

Question2 (1 points)
Find the Taylor series of the following function around $x = 0$

$$f(x) = \frac{2}{3x-5}$$

and determine the radius of convergence.

Solution:

1M If a function has a power series representation centred at $x = a$, then Taylor series is the power representation centred at $x = a$. Thus there is an easy way in this case,

$$f(x) = \frac{2}{3x-5} = -\frac{2}{5}\left(\frac{1}{1-\dfrac{3x}{5}}\right) = -\frac{2}{5}\sum_{n=0}^{\infty}\left(\frac{3x}{5}\right)^n = -\frac{2}{5}\sum_{n=0}^{\infty}\underbrace{\left(\frac{3}{5}\right)^n}_{c_n} x^n$$

the radius of convergence is given by

$$-1 < \frac{3x}{5} < 1 \implies -\frac{5}{3} < x < \frac{5}{x} \implies |x| < \frac{5}{3} \implies R = \frac{5}{3}$$

Question3 (1 points)
Find the Taylor series of

$$f(x) = \cos x$$

and show it converges to $f(x)$ for all $x \in \mathbb{R}$.

Solution:

1M Consider an arbitrary centre $x_0 \in \mathbb{R}$, the derivatives are

$$\begin{aligned}
f'(x_0) &= -\sin x_0 = \cos\left(x_0 + \frac{\pi}{2}\right)\\
f''(x_0) &= -\cos x_0 = \cos\left(x_0 + \frac{2\pi}{2}\right)\\
f'''(x_0) &= \quad \sin x_0 = -\cos\left(x_0 + \frac{\pi}{2}\right) = \cos\left(x_0 + \frac{3\pi}{2}\right)\\
f^{(4)}(x_0) &= \quad \cos x_0 = \cos\left(x_0 + \frac{4\pi}{2}\right)
\end{aligned}$$

since $f^{(4)}(x_0) = \cos x_0 = f(x_0)$, this patten will repeat. In general, we have

$$f^{(n)}(x_0) = \cos\left(x_0 + \frac{n\cdot\pi}{2}\right) \qquad \text{for} \quad n \in \mathbb{N}_0$$

Thus the Taylor series for $f(x)$ around $x = x_0$ is given by

$$\sum_{n=0}^{\infty} \frac{\cos\left(x_0 + \frac{n\cdot\pi}{2}\right)}{n!}(x - x_0)^n$$

Since cosine is bounded between -1 and 1,

$$\left|f^{(n+1)}(x)\right| \leq 1$$

then the remainder $R_n(x)$ of the Taylor series satisfies the following

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}\right| \leq \frac{1}{(n+1)!}|x-x_0|^{n+1}, \qquad \text{for all } x \text{ and } n.$$

Hence by the Taylor's inequality theorem, it is converges to $f(x)$ for all x, that is,

$$f(x) = \sum_{n=0}^{\infty} \frac{\cos\left(x_0 + \frac{n \cdot \pi}{2}\right)}{n!}(x - x_0)^n \qquad \text{for all } x \in \mathbb{R}.$$

Question4 (1 points)
Find the value of the following series if it is convergent. If not, justify why it is divergent.

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

Solution:

1M The given series can be thought as the following power series evaluated at the $x = \dfrac{1}{3}$

$$\sum_{n=0}^{\infty} nx^n$$

for which a closed form can be obtained using the following approach

$$f(x) = \sum_{n=0}^{\infty} nx^n \implies \frac{f(x)}{x} = \sum_{n=0}^{\infty} nx^{n-1}$$

Considering the antiderivatives for both sides, we have

$$\int \frac{f(x)}{x}\, dx = C + \sum_{n=0}^{\infty} x^n = C + \frac{1}{1-x} \qquad \text{for} \quad |x| < 1$$

Now consider the derivatives,

$$\frac{d}{dx}\int \frac{f(x)}{x}\, dx = \frac{d}{dx}\left(C + \frac{1}{1-x}\right) \implies \frac{f(x)}{x} = \frac{1}{(1-x)^2}$$

from which, we see the close form is given by

$$f(x) = \frac{x}{(1-x)^2} \implies \sum_{n=1}^{\infty} \frac{n}{3^n} = f(1/3) = \frac{1/3}{(2/3)^2} = \frac{3}{4}$$

Question5 (1 points)
Let $f \in \mathcal{C}^3[a,b]$, that is, $f^{(3)}$ is continuous on $[a,b]$. Show there exists $c \in (a,b)$ such that

$$f(b) = f(a) + (b-a)f'\left(\frac{a+b}{2}\right) + \frac{1}{24}(b-a)^3 f^{(3)}(c)$$

Solution:

1M Applying Taylor's formula for $x^* = \dfrac{a+b}{2} \in (a,b)$

$$f(x) = f(x^*) + (x-x^*)f'(x^*) + \frac{(x-x^*)^2}{2!}f''(x^*) + \frac{(x-x^*)^3}{3!}f^{(3)}(c)$$

where c is a number between x and x^*. When $x = a$, we have $c_a \in (a, x^*)$ such that

$$\begin{aligned} f(a) &= f(x^*) + (a-x^*)f'(x^*) + \frac{(a-x^*)^2}{2!}f''(x^*) + \frac{(a-x^*)^3}{3!}f^{(3)}(c_a) \\ &= f(x^*) + \left(\frac{a-b}{2}\right)f'(x^*) + \frac{(a-b)^2}{8}f''(x^*) + \frac{(a-b)^3}{48}f^{(3)}(c_a) \end{aligned}$$

Similarly, when $x = b$, we have $c_b \in (x^*, b)$ such that

$$\begin{aligned} f(b) &= f(x^*) + (b-x^*)f'(x^*) + \frac{(b-x^*)^2}{2!}f''(x^*) + \frac{(b-x^*)^3}{3!}f^{(3)}(c_b) \\ &= f(x^*) + \left(\frac{b-a}{2}\right)f'(x^*) + \frac{(b-a)^2}{8}f''(x^*) + \frac{(b-a)^3}{48}f^{(3)}(c_b) \end{aligned}$$

Consider the difference $f(b) - f(a)$, we have

$$f(b) - f(a) = (b-a)f'(x^*) + \frac{(b-a)^3}{24}\frac{f^{(3)}(c_b) + f^{(3)}(c_a)}{2}$$

Since $f^{(3)}$ is continuous on $[a,b]$, thus continuous on $[c_a, c_b]$. IVP states there is $c \in (c_a, c_b) \subset (a,b)$ such that

$$f^{(3)}(c) = \frac{f^{(3)}(c_b) + f^{(3)}(c_a)}{2}$$

thus

$$f(b) = f(a) + (b-a)f'\left(\frac{a+b}{2}\right) + \frac{1}{24}(b-a)^3 f^{(3)}(c)$$