

Chapter 11 – Rigid Body Mechanics (I)

Introduction: Kinematics and Moment of Inertia

UM-SJTU Joint Institute
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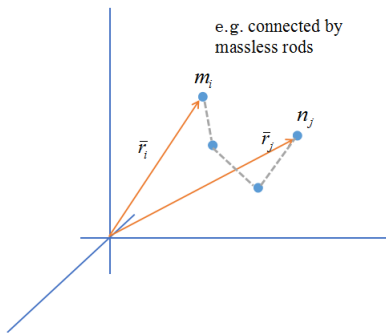
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 - Angular acceleration
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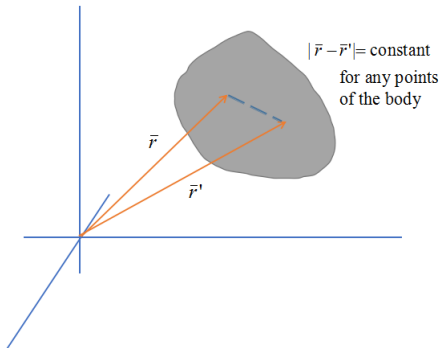
What is a rigid body?

What is a rigid body?

A body is called *rigid* if the distance between any two points of the body remains constant.



Discrete distribution of mass



Continuous distribution of mass

What is a rigid body?

Basic kinematic quantities in rotational motion

Moment of inertia

Gravitational potential energy of a rigid body

Angular velocity

Angular acceleration

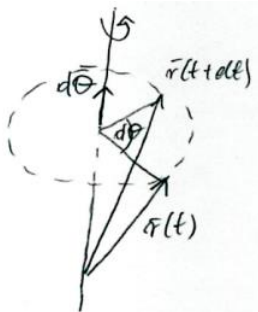
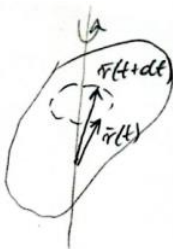
Linear vs. angular quantities in rotational motion

Examples

Basic kinematic quantities in rotational motion

Angular velocity

Angular displacement and angular velocity in rotational motion about a fixed axis



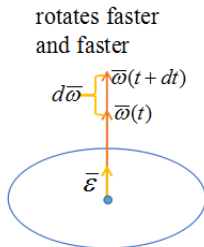
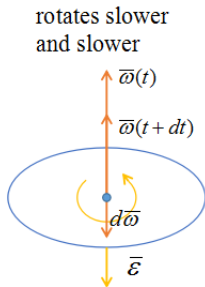
$d\vec{\theta}$ - angular displacement direction: determined by the right hand rule
magnitude: the angle swept by the radius $\bar{\omega} = \frac{d\vec{\theta}}{dt}$ [rad/s] - rate of the change of the angular displacement

Angular acceleration

Consequently, the angular acceleration is defined as $\bar{\varepsilon} = \frac{d\bar{\omega}}{dt}$ (rate of change of the angular velocity; units [rad/s²]).

$$d\bar{\omega} = \bar{\omega}(t + dt) - \bar{\omega}(t), \quad \bar{\varepsilon} = \frac{d\bar{\omega}}{dt}$$

For rotation about a fixed axis $\bar{\varepsilon} = \frac{d\bar{\omega}}{dt} \parallel \bar{\omega}$ ($\bar{\omega}$ does not change the direction (fixed axis), only the magnitude may change



Example

Rotation with constant angular acceleration about a fixed axis (e.g., the z axis) with $\bar{\varepsilon}$ constant (i.e., it has a single non-zero component along the z axis)

Initial conditions: $\omega(0) = \omega_0$, $\theta(0) = \theta_0$

$$\varepsilon = \frac{d\omega}{dt} \Rightarrow d\omega = \varepsilon dt \Rightarrow \int_{\omega_0}^{\omega(t)} d\omega = \int_0^t \varepsilon dt$$

$$\boxed{\omega(t) = \omega_0 + \varepsilon t}$$

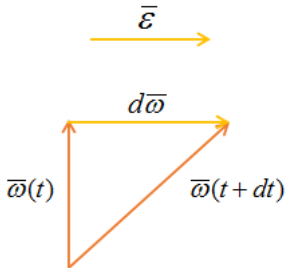
$$\omega = \frac{d\theta}{dt} \Rightarrow d\theta = \omega dt \Rightarrow \int_{\theta_0}^{\theta(t)} d\theta = \int_0^t (\omega_0 + \varepsilon t) dt$$

$$\boxed{\theta(t) = \theta_0 + \omega_0 t + \frac{1}{2}\varepsilon t^2}$$

Compare with $a = \text{const}$, $v(t) = v_0 + at$, $x(t) = x_0 + v_0 t + \frac{1}{2}at^2$

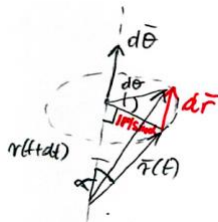
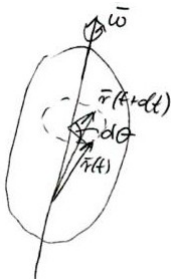
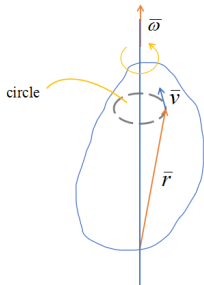
What if the axis of rotation is not fixed? (i.e. changes its orientation)

The direction of $\bar{\omega}$ changes and $\bar{\epsilon} \nparallel \bar{\omega}$



Conclusion: $\bar{\epsilon}$ is not parallel to the axis of rotation.

Linear vs. angular quantities in rotational motion: velocity (rotation about a fixed axis)



$$|d\vec{r}| = |\vec{r}| \sin \alpha d\theta$$

[α – angle between $\vec{r}(t)$ and axis of rotation, i.e. $d\vec{\theta}$]

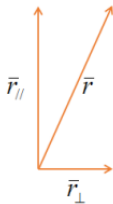
Introducing vector $d\vec{\theta}$ as before (right hand rule), we have

$$d\vec{r} = d\vec{\theta} \times \vec{r} \quad \implies \quad \frac{d\vec{r}}{dt} = \frac{d\vec{\theta}}{dt} \times \vec{r}$$

$$\boxed{\vec{v} = \vec{\omega} \times \vec{r}}$$

Decomposing \vec{r} as

$$\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$$



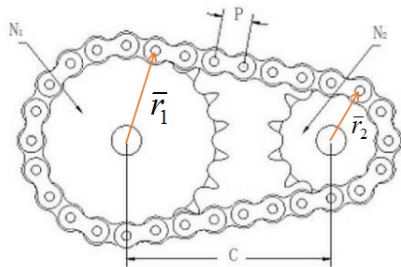
we have

$$\vec{v} = \vec{\omega} \times (\vec{r}_{\parallel} + \vec{r}_{\perp}) = \vec{\omega} \times \vec{r}_{\parallel} + \vec{\omega} \times \vec{r}_{\perp} = \vec{\omega} \times \vec{r}_{\perp}$$

The magnitude

$$|\vec{v}| = |\vec{\omega} \times \vec{r}_{\perp}| = |\vec{\omega}| \cdot |\vec{r}_{\perp}|$$

Example. Bicycle gears



$$v = r_1\omega_1 = r_2\omega_2$$



$$\frac{\omega_1}{\omega_2} = \frac{r_2}{r_1}$$

sprocket of larger
radius rotates
slower

$p = \frac{2\pi r_1}{N_1} = \frac{2\pi r_2}{N_2}$, so that $\frac{\omega_1}{\omega_2} = \frac{N_2}{N_1}$ or $\frac{\omega_2}{\omega_1} = \frac{N_1}{N_2}$ (N_1, N_2 are numbers of teeth on sprockets)

Conclusion

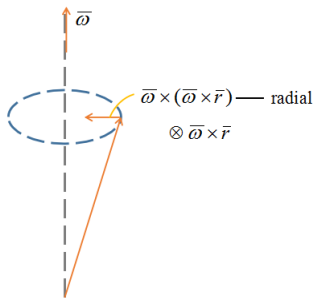
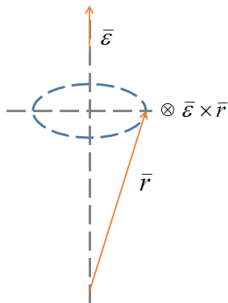
Given constant pedalling rate, the rear wheel rotates with maximum angular speed if N_1 is largest, and N_2 – smallest.

Linear vs. angular quantities in rotational motion: acceleration (rotation about a fixed axis)

Recall that $\vec{v} = \vec{\omega} \times \vec{r}$. Consequently, the acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega} \times \vec{r}) = \frac{d\vec{\omega}}{dt} \times \vec{r} + \vec{\omega} \times \frac{d\vec{r}}{dt} = \underbrace{\vec{\varepsilon} \times \vec{r}}_{\vec{a}_{\text{tan}}} + \underbrace{\vec{\omega} \times (\vec{\omega} \times \vec{r})}_{\vec{a}_{\text{centripetal}}}$$

What is the direction of these terms?



Decomposing once again $\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$,

$$\vec{a} = \vec{\epsilon} \times (\vec{r}_{\parallel} + \vec{r}_{\perp}) + \vec{\omega} \times [\vec{\omega} \times (\vec{r}_{\parallel} + \vec{r}_{\perp})] = \vec{\epsilon} \times \vec{r}_{\perp} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{\perp})$$

The magnitudes

$$|\vec{a}_{\text{tan}}| = |\vec{\epsilon}| \cdot |\vec{r}_{\perp}|$$

$$|\vec{a}_{\text{centr}}| = |\vec{\omega} \times (\vec{\omega} \times \vec{r}_{\perp})| = |\vec{\omega}| |\vec{\omega} \times \vec{r}_{\perp}| = \omega^2 |\vec{r}_{\perp}|$$

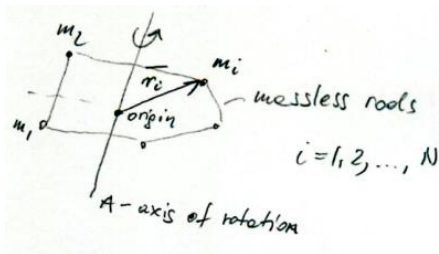
Moment of inertia

Kinetic energy of a rotating rigid body

Discrete distribution of mass

Total kinetic energy

$$K = \sum_{i=1}^N K_i = \sum_{i=1}^N \frac{1}{2} m_i v_i^2$$



But $\vec{v}_i = \vec{\omega} \times \vec{r}_i$ and $|\vec{v}_i| = |\vec{\omega}| |\vec{r}_{i\perp}|$. Hence

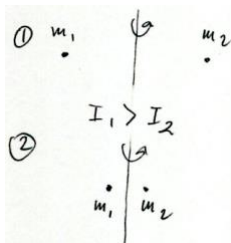
$$\boxed{K} = \sum_{i=1}^N \frac{1}{2} m_i \omega^2 r_{i\perp}^2 = \frac{1}{2} \left(\sum_{i=1}^N m_i r_{i\perp}^2 \right) \omega^2 = \boxed{\frac{1}{2} I \omega^2},$$

where $I = \sum_{i=1}^N m_i r_{i\perp}^2$ is the **moment of inertia** about the (fixed) axis of rotation A.

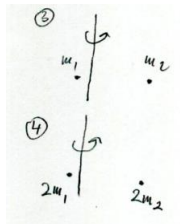
Moment of inertia. Illustration

$I = \sum_{i=1}^N m_i r_{i\perp}^2$ - depends on the distribution (arrangement) of mass

(a) (b)

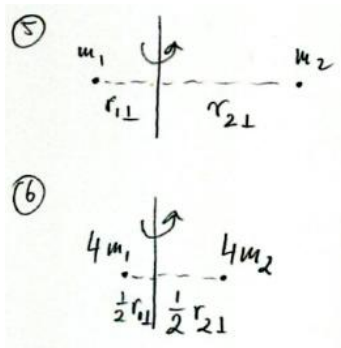


$$I_1 > I_2$$



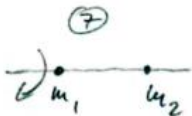
$$I_3 < I_4 = 2I_3$$

(c)



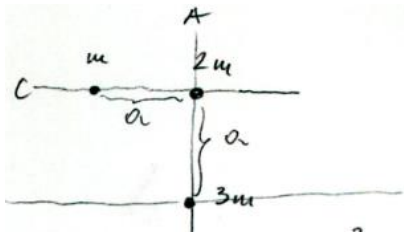
$$I_5 = m_1 r_{1\perp}^2 + m_2 r_{2\perp}^2 \quad \text{and} \quad I_6 = 4m_1 \left(\frac{1}{2} r_{1\perp} \right)^2 + 4m_2 \left(\frac{1}{2} r_{2\perp} \right)^2 = I_5$$

(d)



$$I_7 = 0$$

(e)



$$I_A = ma^2, \quad I_B = 2ma^2 + ma^2, \quad I_C = 3ma^2$$

Moment of inertia about an axis for a continuous distribution of mass



Contribution to the kinetic energy due to the element of mass dm

$$dK = \frac{1}{2}(dm)v^2 = \frac{1}{2}\omega^2 r_{\perp}^2 dm$$

Total kinetic energy (added all contributions)

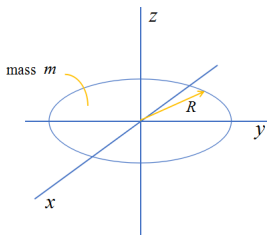
$$K = \int_{\text{object}} dK = \int_{\text{object}} \frac{1}{2}\omega^2 r_{\perp}^2 dm = \frac{1}{2} \left(\int_{\text{object}} r_{\perp}^2 dm \right) \omega^2$$

I_A — moment of inertia of the *object* about the axis A

$$I_A = \int_{\text{object}} r_{\perp}^2 dm$$

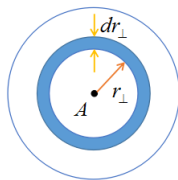
Example (a): uniform disk

Uniform disk (mass m , radius R) about the axis of symmetry perpendicular to the plane of the disk



$\sigma = \frac{m}{\pi R^2}$ – surface
density of mass

$$I_A = \int_{\text{object}} r_{\perp}^2 dm$$



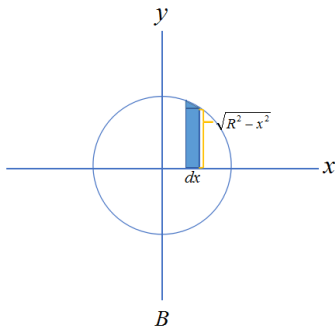
$$dm = 2\pi r_{\perp} \sigma dr_{\perp}$$

$$dI_A = r_{\perp}^2 dm \Rightarrow I_A = \int_{\text{object}} r_{\perp}^2 dm$$

$$\boxed{I_A} = \int_0^R 2\pi \sigma r_{\perp}^3 dr_{\perp} = \sigma 2\pi \frac{R^4}{4} = \frac{1}{2} \sigma \pi R^2 = \boxed{\frac{1}{2} m R^2}$$

Example (b): the same disk, different axis

The same disk about the axis contained in the plane of the disk.



$$I_B = 4 \int_{\text{quarter of disk}} x^2 dm$$

$$= 4\sigma \int_0^R x^2 \sqrt{R^2 - x^2} dx$$

Change of variables: $x = R \sin u$, so that $dx = R \cos u du$.

$$\begin{aligned} I_B &= 4\sigma R^4 \int_0^{\pi/2} \sin^2 u \cos^2 u du = 4\sigma R^4 \int_0^{\pi/2} \left(\frac{1}{2} \sin 2u\right)^2 du \\ &= \sigma R^4 \int_0^{\pi/2} \frac{1}{2} [1 - \cos 4u] du = \sigma R^4 \frac{\pi}{4} = \frac{1}{4} m R^2 \end{aligned}$$

$$I_B = \frac{1}{4} m R^2$$

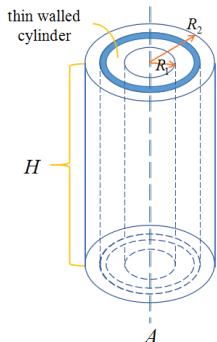
Remark 1. Because of the symmetry, the moments of inertia about axes y and x are both equal to $\frac{1}{4}mR^2$.

Remark 2. It turns out that $I_z = I_x + I_y$.

Is it a general rule for planar objects? [\rightarrow see problem set]

Example (c): uniform hollow cylinder

Mass m , outer radius R_2 , inner radius R_1 ; about the vertical axis of symmetry



$dm = \rho 2\pi r_{\perp} H dr_{\perp}$, where ρ – density of mass

$$\begin{aligned} I_A &= \int_{\text{cylinder}} r_{\perp}^2 dm = \int_{R_1}^{R_2} r_{\perp}^2 \rho 2\pi r_{\perp} H dr_{\perp} = \\ &= \frac{2\pi\rho H}{4} (R_2^4 - R_1^4) \end{aligned}$$

Express in terms of mass and radii $m = \rho V = \rho\pi(R_2^2 - R_1^2)H$.

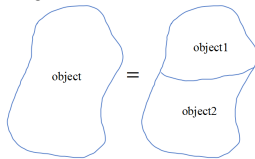
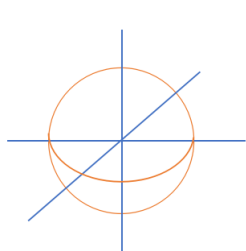
$$\text{Hence } \boxed{I_A} = \frac{\pi\rho H}{2} (R_2^2 - R_1^2)(R_2^2 + R_1^2) = \boxed{\frac{1}{2}m(R_2^2 + R_1^2)}.$$

Note. In particular for a full cylinder ($R_1 = 0$) with the same mass

$$I_A = \frac{1}{2}mR_2^2$$

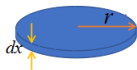
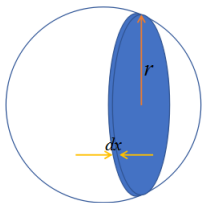
Example (d): uniform ball

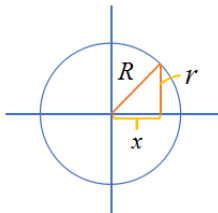
Mass m , radius R (hence $\rho = m/\frac{4}{3}\pi R^3$); any axis of symmetry



$$\int_{\text{object}} r_{\perp}^2 dm = \int_{\text{object 1}} r_{\perp}^2 dm + \int_{\text{object 2}} r_{\perp}^2 dm$$

Idea: Cut the ball into slices (cylinders of infinitesimal height), add the contributions.



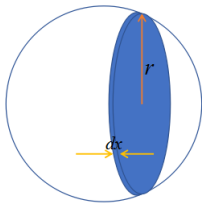


$$r = \sqrt{R^2 - x^2}$$

$$dm = \rho \pi r^2 dx = \rho \pi (R^2 - x^2) dx$$

$$dI_A = \frac{1}{2} r^2 dm$$

Adding all contributions due to single slices



$$I_A = 2 \cdot \int_0^R \frac{1}{2} (R^2 - x^2) \rho \pi (R^2 - x^2) dx$$

$$= \pi \rho \int_0^R (R^4 - 2R^2 x^2 + x^4) dx$$

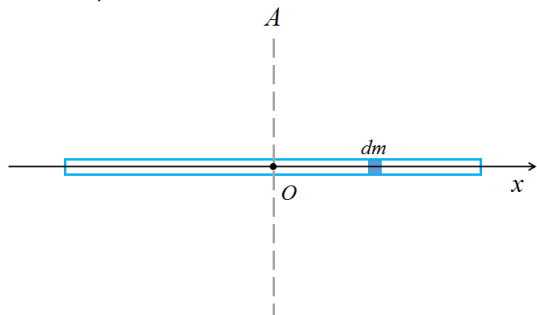
$$= \pi \rho \left(R^5 - \frac{2}{3} R^2 R^3 + \frac{1}{5} R^5 \right)$$

$$= \pi \rho \frac{15 - 10 + 3}{15} R^5 = \frac{8}{15} R^5 \pi \rho$$

$$I_A = \frac{2}{5} m R^2$$

Example (e): uniform slender rod

Mass m and length l about the axis of symmetry (perpendicular to the rod)



linear density of mass

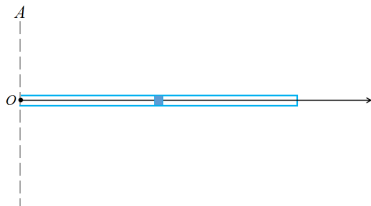
$$\lambda = \frac{m}{l}$$

$$dm = \lambda dx$$

$$\begin{aligned} \boxed{I_A} &= \int_{\text{rod}} r_{\perp}^2 dm = \int_{-\frac{l}{2}}^{\frac{l}{2}} x^2 \lambda dx = \frac{1}{3} \lambda l^3 \left(\frac{1}{8} + \frac{1}{8} \right) \\ &= \frac{1}{12} \lambda l^3 = \boxed{\frac{1}{12} m l^2} \end{aligned}$$

Example (f): same rod, different axis

Sometimes we may need to know the moment of inertia about an axis A' (parallel to A), e.g. through one of ends of the rod.



$$\begin{aligned}\boxed{I_{A'}} &= \int_{\text{rod}} r_{\perp}^2 dm \\ &= \int_0^l x^2 \lambda dx \\ &= \frac{1}{3} \lambda l^3 = \boxed{\frac{1}{3} m l^2}\end{aligned}$$

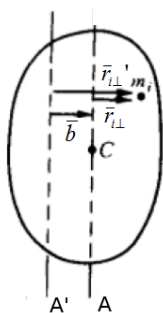
Compare

$$I_{A'} - I_A = \left(\frac{1}{3} - \frac{1}{12} \right) m l^2 = \frac{1}{4} m l^2 = m \left(\frac{l}{2} \right)^2 \Rightarrow \boxed{I_{A'} = I_A + m \left(\frac{l}{2} \right)^2}$$

Is there a universal relation between I_A and $I_{A'}$?

Yes! *Steiner's theorem* (also known as the *parallel axis theorem*)

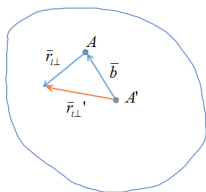
Parallel axis theorem (Steiner's theorem)



A – axis **through the center of mass**

A' – any axis parallel to A

view from
the top



For an element of mass m_i

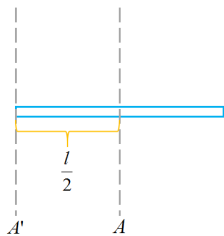
$$\bar{r}'_{i\perp} = \bar{r}_{i\perp} + \bar{b}$$

$$\begin{aligned} I_{A'} &= \sum_{i=1}^N m_i r_{i\perp}'^2 = \sum_{i=1}^N m_i (\bar{r}_{i\perp} + \bar{b})^2 = \sum_{i=1}^N m_i (r_{i\perp}^2 + 2\bar{r}_{i\perp} \circ \bar{b} + b^2) \\ &= \sum_{i=1}^N m_i r_{i\perp}^2 + 2\bar{b} \circ \sum_{i=1}^N m_i \bar{r}_{i\perp} + b^2 \sum_{i=1}^N m_i = I_A + 2\bar{b} \circ m\bar{r}_{cm\perp} + mb^2 \end{aligned}$$

Recall that A passes through the center of mass; hence $|\vec{r}_{cm\perp}| = 0$ (c.m. lies on that axis). Eventually, $I_{A'} = I_A + mb^2$, where b is the distance between the axes A and A'.

Examples

(a) slender uniform rod (mass m , length l)

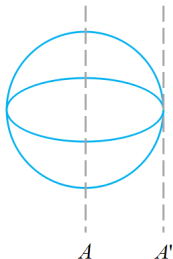


$$I_{A'} = I_A + m \left(\frac{l}{2} \right)^2 = I_A + \frac{ml^2}{4}$$

$$I_{A'} = \frac{1}{3}ml^2$$

$$I_A = I_{A'} - \frac{ml^2}{4} = \frac{1}{3}ml^2 - \frac{1}{4}ml^2 = \frac{1}{12}ml^2$$

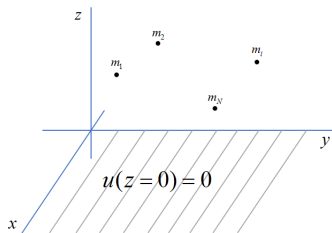
(b) uniform ball (mass m , radius R)



$$I_{A'} = I_A + mR^2 = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2$$

Gravitational potential energy of a rigid body

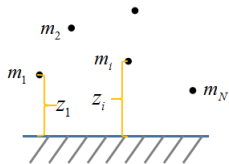
Gravitational potential energy of a rigid body



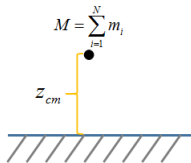
$$U_{\text{grav}} = \sum_{i=1}^N m_i g z_i = g \underbrace{\sum_{i=1}^N m_i z_i}_{z_{\text{cm}} \cdot \sum_{i=1}^N m_i}$$

$$\text{Recall that } \bar{r}_{\text{cm}} = \frac{\sum_{i=1}^N m_i \bar{r}_i}{\sum_{i=1}^N m_i}$$

Hence $U_{\text{grav}} = M g z_{\text{cm}}$ (where $M = \sum_{i=1}^N m_i$).



Both have the same gravitational potential energy.



Note. Also applies to any continuous distribution of mass.