Chapter 1 – Physical Quantities, Scalars, and Vectors

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Physics I (Summer 2019)
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Agenda

1. Scalar Quantities
2. Vector Quantities
   - Basic Vector Operations
   - Scalar (Dot) Product
   - Vector (Cross) Product
3. Position in Space. Vectors in the Cartesian Coordinate System
   - Position Vector
   - Cartesian Coordinate System

Scalar Quantities

Scalar Quantities

Scalar quantities – defined by a single number

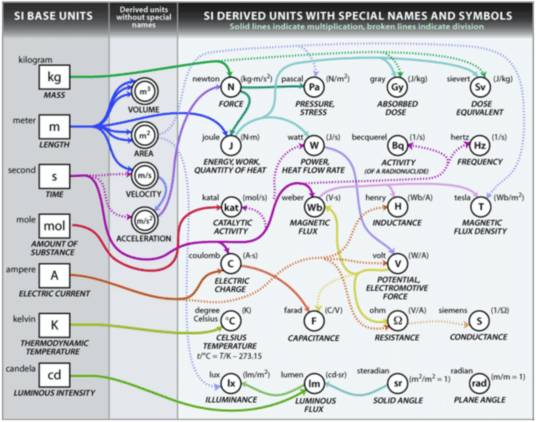
Examples: time, mass, length, volume, density of matter, electric charge, potential energy, pressure, kinetic/potential energy,. . .

In general physics, many scalar quantities are defined operationally (contrary to theoretical definitions). That is, they are obtained as a result of a sequence of measurement operations. A typical example is density.

$$\text{density of (bulk) matter} = \frac{\text{mass}}{\text{volume}}$$

Physical quantities all have corresponding units. There are different kinds of unit systems in use around the world and in various fields of science. In experimental physics and engineering, the most commonly used one is the metric system or SI (from French *Système international d'unités*).

SI Units



Factor	Name	Symbol	Factor	Name	Symbol
10^{24}	yotta	Y	10^{-1}	deci	d
10^{21}	zetta	Z	10^{-2}	centi	c
10^{18}	exa	E	10^{-3}	milli	m
10^{15}	peta	P	10^{-6}	micro	μ
10^{12}	tera	T	10^{-9}	nano	n
10^{9}	giga	G	10^{-12}	pico	p
10^{6}	mega	M	10^{-15}	femto	f
10^{3}	kilo	k	10^{-18}	atto	a
10^{2}	hecto	h	10^{-21}	zepto	z
10^{1}	deka	da	10^{-24}	yocto	y

Source: physics.nist.gov

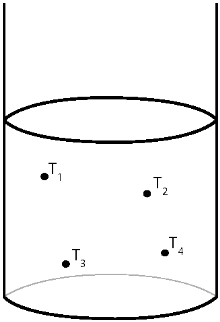
Operations on Scalar Quantities

Different mathematical operations performed on physical quantities have their limitations regarding the units of operation arguments.

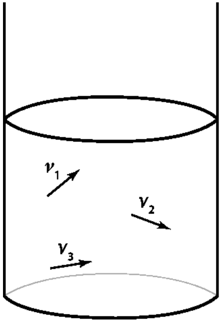
$+$ or $-$	only compatible units allowed as arguments
$*$ or $/$	may involve quantities with different units
sin(...), ln(...), exp(...), ...	only dimensionless arguments allowed

Vector Quantities

Scalar vs. Vector Quantities



Local temperature in a liquid is a scalar (a single number is enough to define it at a given point)



Velocities of objects floating in a liquid are vectors (both a number and a direction needed)

Vector Quantities

Vector quantities – have both a *magnitude* and a *direction*

Examples: velocity, force, linear momentum, angular velocity, angular momentum, electric/magnetic field, electric current density,. . .

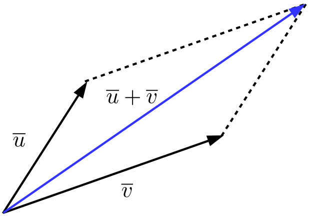
Notation for vectors: $\vec{u}$ or $\overline{u}$ or $\underline{u}$ or $\mathbf{u}$ (boldface, typical in printed material),. . .

Magnitude – the length of a vector (denoted as $|\overline{u}|$ or simply u)

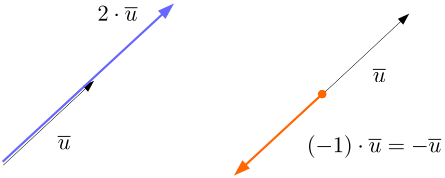
Basic Operations on Vector Quantities

Mathematical operations on vector quantities are more complicated than those on scalars.

Vector addition (parallelogram rule)

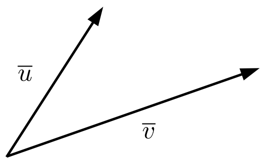


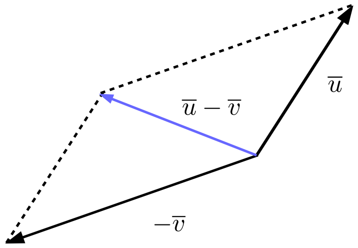
Multiplication by a scalar



Basic Operations on Vector Quantities

Vector subtraction





Vector Multiplication. Scalar Product

The result of multiplying two vectors can be defined in two ways: either as the *scalar product* or the *vector product*.

The **scalar (dot) product** of two vectors $\overline{u}$ and $\overline{w}$ is defined as

$$\boxed{\overline{u} \circ \overline{w} \overset{def}{=} |\overline{u}||\overline{w}| \cos \alpha} \qquad \text{(scalar!)},$$

where $\alpha = \measuredangle(\overline{u}, \overline{w})$ is the smaller angle between vectors $\overline{u}$ and $\overline{w}$.

<u>Observation:</u> The dot product of two non-zero vectors is zero, if and only if the vectors are perpendicular,

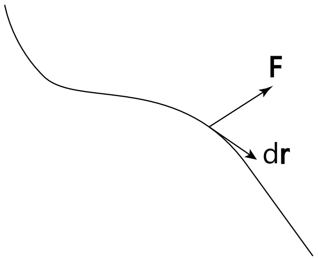
$$\overline{u} \circ \overline{w} = 0 \Longleftrightarrow \overline{u} \perp \overline{w}.$$

Scalar (Dot) Product. Example

Elementary work

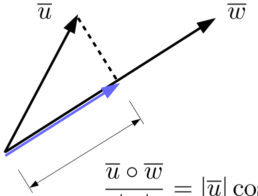
$$\delta W \stackrel{def}{=} \overline{F} \circ \mathrm{d}\overline{r},$$

where $\overline{F}$ is the force and $d\overline{r}$ is the elementary displacement.



<u>Observation:</u> Elementary work is zero, if the force acts perpendicular to the elementary displacement.

Scalar (Dot) Product. Example: Orthogonal Projection



$$\frac{\overline{u} \circ \overline{w}}{|\overline{w}|} = |\overline{u}| \cos \measuredangle(\overline{u}, \overline{w})$$

The component vector of $\overline{u}$ pointing in the direction of $\overline{w}$ is therefore given by

$$\frac{\overline{u} \circ \overline{w}}{|\overline{w}|} \cdot \frac{\overline{w}}{|\overline{w}|}$$

It is called the orthogonal projection vector of the vector $\overline{u}$ onto the vector $\overline{w}$. Note that, the vector $\frac{\overline{w}}{|\overline{w}|}$ is of unit length and points in the direction of vector $\overline{w}$.

Vector Multiplication. Vector Product

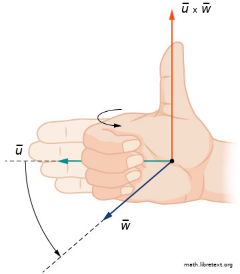
The **vector (cross) product** of two vectors $\overline{u}$ and $\overline{w}$ is denoted as

$$\boxed{\overline{b} \overset{def}{=} \overline{u} \times \overline{w}} \qquad \text{(vector!)}$$

The length (magnitude) of $\overline{b}$ is defined as

$$\boxed{|\overline{b}| = |\overline{u}||\overline{w}| \sin\alpha,}$$

where $\alpha = \measuredangle(\overline{u}, \overline{w})$ is the smaller angle between vectors $\overline{u}$ and $\overline{w}$. The vector $\overline{b}$ is perpendicular to both vectors $\overline{u}$ and $\overline{w}$, and its direction is defined by the right-hand rule.



Vector (Cross) Product. Properties and Examples

Observations:

- The cross product is anticommutative,

$$\overline{u} \times \overline{w} = -\overline{w} \times \overline{u}.$$

- The cross product is a zero vector, if and only if the two non-zero vectors are parallel (or antiparallel),

$$\overline{u} \times \overline{w} = \overline{0} \quad \Longleftrightarrow \quad \overline{u} \parallel \overline{w}.$$

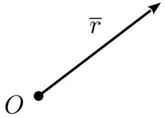
Cross products usually appear whenever there is some kind of rotational motion involved. Typical examples of physical quantities defined through a cross product include the torque and the angular momentum.

$\overline{r} \times \overline{F} \overset{def}{=} \overline{\tau}$ (torque) $\qquad$ $\overline{r} \times \overline{p} \overset{def}{=} \overline{L}$ (angular momentum)

Position in Space. Vectors in the Cartesian Coordinate System

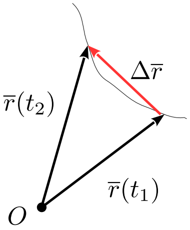
Position Vector and Displacement Vector

To uniquely describe an object's position in space, with respect to a certain point O (called the origin, or the reference point), we use the **position** vector.



The position vector is generally time-dependent $\overline{r} = \overline{r}(t)$, and we may define the displacement vector between two instants of time t_1 and t_2 as

$$\Delta\overline{r} = \overline{r}(t_2) - \overline{r}(t_1)$$



Coordinate Systems

Any vector, including the position vector and the displacement vector, can be represented in the so-called coordinate system, with the origin at point O.

The most commonly used coordinate system in three dimensions is the Cartesian coordinate system.

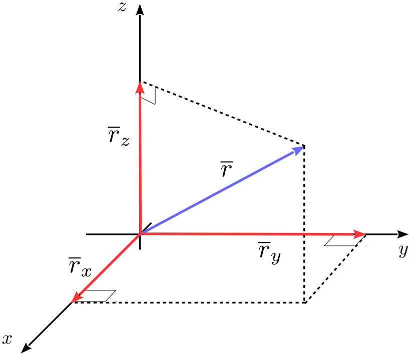
Cartesian Coordinate System

Cartesian Coordinate System

The Cartesian coordinate system is defined by three mutually perpendicular axes, traditionally labelled as x, y, and z, intersecting at the point O called the **origin**.

Every vector in the Cartesian coordinate system can be represented as a sum of three *component vectors* (orthogonal projections) along the three axes. In particular, for the position vector

$$\overline{r} = \overline{r}_x + \overline{r}_y + \overline{r}_z.$$

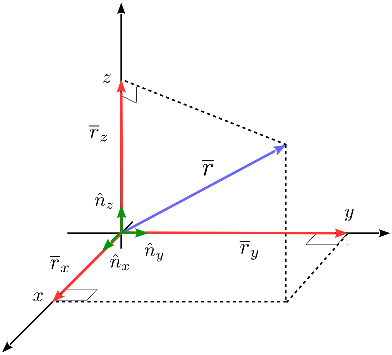


Cartesian Coordinate System. Coordinates and Unit Vectors

Defining <u>fixed</u> **unit vectors** (unit = of unit length) $\hat{n}_x$, $\hat{n}_y$, and $\hat{n}_z$ along the axes x, y, z, the position vector (or any other vector) can be represented as

$$\overline{r} = x\hat{n}_x + y\hat{n}_y + z\hat{n}_z.$$

The numbers x, y, z are called the Cartesian coordinates of the position vector.



Cartesian Coordinate System. Coordinates and Unit Vectors

In the matrix notation, vectors in the Cartesian coordinate system are sometimes denoted as column vectors $\bar{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ or row vectors $\bar{r} = (x, y, z)$. *Note.* The Cartesian unit vectors are also denoted as $\hat{i}$, $\hat{j}$, $\hat{k}$ or $\bar{e}_x$, $\bar{e}_y$, $\bar{e}_z$.

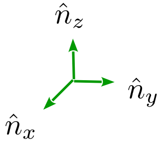
This approach can be generalized to any vector represented in the Cartesian coordinate system

$$\overline{w} = w_x \hat{n}_x + w_y \hat{n}_y + w_z \hat{n}_z.$$

The length (magnitude) of vector $\overline{w}$ can be found as (use Pythagorean theorem twice)

$$\boxed{|\overline{w}| = w = \sqrt{w_x^2 + w_y^2 + w_z^2}.}$$

Properties of Cartesian Unit Vectors



Properties of Cartesian unit vectors

- mutually perpendicular
  $\hat{n}_x \circ \hat{n}_y = \hat{n}_y \circ \hat{n}_z = \hat{n}_z \circ \hat{n}_x = 0$

- unit length $|\hat{n}_x| = |\hat{n}_y| = |\hat{n}_z| = 1$, or equivalently
  $\hat{n}_x \circ \hat{n}_x = \hat{n}_y \circ \hat{n}_y = \hat{n}_z \circ \hat{n}_z = 1$

- For a right-handed system (note the cyclic permutation)

$$
\begin{aligned}
\hat{n}_x \times \hat{n}_y &= \hat{n}_z \\
\hat{n}_y \times \hat{n}_z &= \hat{n}_x \\
\hat{n}_z \times \hat{n}_x &= \hat{n}_y
\end{aligned}
$$

Time-dependent Vectors in Cartesian Coordinates. Differentiation and Integration

In general, vectors can be time-dependent.
For a vector $\overline{u} = \overline{u}(t)$, its derivative with respect to (w.r.t.) time, that is the time-rate of change, is

$$\begin{aligned}
\boxed{\frac{\mathrm{d}\overline{u}}{\mathrm{d}t}} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(u_x(t)\hat{n}_x + u_y(t)\hat{n}_y + u_z(t)\hat{n}_z\right) \\
&= \boxed{\dot{u}_x(t)\hat{n}_x + \dot{u}_y(t)\hat{n}_y + \dot{u}_z(t)\hat{n}_z}
\end{aligned}$$

Note that $\frac{\mathrm{d}}{\mathrm{d}t}\hat{n}_x = \frac{\mathrm{d}}{\mathrm{d}t}\hat{n}_y = \frac{\mathrm{d}}{\mathrm{d}t}\hat{n}_z = 0$ because they are fixed vectors.

Consequently,

$$\int_{t_0}^{t_1} \overline{u}\mathrm{d}t = \left(\int_{t_0}^{t_1} u_x(t)\mathrm{d}t\right)\hat{n}_x + \left(\int_{t_0}^{t_1} u_y(t)\mathrm{d}t\right)\hat{n}_y + \left(\int_{t_0}^{t_1} u_z(t)\mathrm{d}t\right)\hat{n}_z.$$

Scalar Product in the Cartesian Coordinate System

For $\overline{u} = (u_x, u_y, u_z)$ and $\overline{w} = (w_x, w_y, w_z)$, the dot product

$$\boxed{\overline{u} \circ \overline{w} = u_x w_x + u_y w_y + u_z w_z}$$

Hint. Substitute the representations of vectors in terms of unit vectors, multiply out and notice that the six terms involving dot products of different unit vectors give zero (unit vectors are mutually orthogonal).

Note. $\overline{u} \circ \overline{u} = u_x^2 + u_y^2 + u_z^2 = |\overline{u}|^2$.

Vector Product in the Cartesian Coordinate System

Vector product of $\overline{u} = (u_x, u_y, u_z)$ and $\overline{w} = (w_x, w_y, w_z)$ in the Cartesian coordinate system is found from the formula

$$\begin{aligned} \overline{u} \times \overline{w} &= (u_y w_z - u_z w_y)\hat{n}_x + (u_z w_x - u_x w_z)\hat{n}_y + (u_x w_y - u_y w_x)\hat{n}_z \\ &= \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ u_x & u_y & u_z \\ w_x & w_y & w_z \end{vmatrix}. \end{aligned}$$

Hint. Again, start out with $\overline{u}$, $\overline{w}$ written as sums of components; then use the 3^{rd} property of unit vectors ($\hat{n}_x \times \hat{n}_y = \hat{n}_z$, ...) and $\hat{n}_x \times \hat{n}_x = 0, \ldots$

Other Operations on Vectors in Cartesian Coordinates

- addition/subtraction

$$\begin{aligned} \overline{u} \pm \overline{w} &= (u_x \hat{n}_x + u_y \hat{n}_y + u_z \hat{n}_z) \pm (w_x \hat{n}_x + w_y \hat{n}_y + w_z \hat{n}_z) \\ &= (u_x \pm w_x)\hat{n}_x + (u_y \pm w_y)\hat{n}_y + (u_z \pm w_z)\hat{n}_z \\ &= (u_x \pm w_x, u_y \pm w_y, u_z \pm w_z) \end{aligned}$$

- multiplication by a scalar α

$$\begin{aligned} \alpha\overline{u} &= \alpha(u_x \hat{n}_x + u_y \hat{n}_y + u_z \hat{n}_z) \\ &= (\alpha u_x, \alpha u_y, \alpha u_z) \end{aligned}$$

Example: Basic Vector Operations

Given two vectors in the Cartesian coordinate system $\overline{u} = (1, 2, 3)$ and $\overline{w} = (2, -1, -1)$ answer the following questions:

- Are $\overline{u}$ and $\overline{w}$ parallel to each other?

  No. For $\overline{u}$ and $\overline{w}$ to be parallel (or antiparallel), the Cartesian coordinates of $\overline{u}$ should be proportional to the corresponding Cartesian coordinates of $\overline{w}$. This is clearly not the case here. [Another method (lengthy one): check that $\overline{u} \times \overline{w} \neq 0$.]

- Does $\overline{u}$ have a component along $\overline{w}$? If so, what is its magnitude (length)? Does the component point in the direction of $\overline{w}$ or opposite to it?

  To answer the first question, we need to check if the orthogonal projection of $\overline{u}$ onto $\overline{w}$ is non-zero. That is, we need to find out

$$\overline{u} \circ \overline{w} = (1, 2, 3) \circ (2, -1, -1) = 2 - 2 - 3 = -3 \neq 0.$$

  Hence $\overline{u}$ has a component along $\overline{w}$.

Example: Basic Vector Operations (contd.)

Its magnitude is $u_{\|} = |\overline{u} \circ \overline{w}| / |\overline{w}| = 3/\sqrt{6}$.
Finally, since $\overline{u} \circ \overline{w} < 0$, the angle between the vectors $\overline{u}$ and $\overline{w}$ must be greater that $\pi/2$. Therefore the component of $\overline{u}$ along $\overline{w}$ points in the direction opposite to $\overline{w}$.

- What is the magnitude of the component of $\overline{u}$ perpendicular to $\overline{w}$?
  From the Pythagorean theorem
  $$u_{\perp} = \sqrt{u^2 - u_{\|}^2} = \sqrt{14 - 3/2} = 5/\sqrt{2}.$$
- Find a vector that is perpendicular to the plane spanned by $\overline{u}$ and $\overline{w}$.
  Recalling that the cross-product of two vectors is a vector perpendicular to both these vectors (hence to the plane spanned by them), it is enough to find
  $$\overline{u} \times \overline{w} = \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ u_x & u_y & u_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} \hat{n}_x & \hat{n}_y & \hat{n}_z \\ 1 & 2 & 3 \\ 2 & -1 & -1 \end{vmatrix} = (1, 7, -5)$$

Example: Time-dependent Vectors

Given a time-dependent vector $\overline{u} = (\cos At, Bt, C)$, where A, B, C are non-zero constants with appropriate units, find $\dot{\overline{u}}$ and check whether $\overline{u}$ is perpendicular to $\dot{\overline{u}}$ for any instant t.

First find

$$\dot{\overline{u}} = (-A\sin At, B, 0).$$

Then check

$$\overline{u} \circ \dot{\overline{u}} = (\cos At, Bt, C) \circ (-A\sin At, B, 0) = -A\cos At \sin At + B^2 t.$$

Hence, the two vectors are not perpendicular to each other, except at the instant $t = 0s$ (in general, at the instants, when $A\cos At \sin At = B^2 t$).

Question: How does the answer change if $\overline{u}$ has constant magnitude for all instants t? [See Problem Set 1]