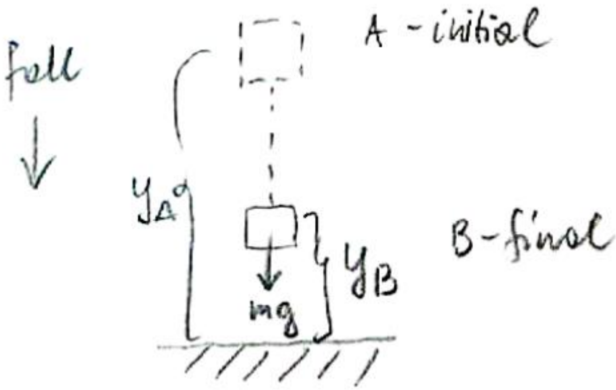
CHAPTER 10

Potential Energy

Gravitational Force & Gravitaitonal Potential Energy



Work: (Recall the definition)

$$W_{grav} = mg\,(y_A - y_B) = mg\,y_A - mg\,y_B$$

Define:

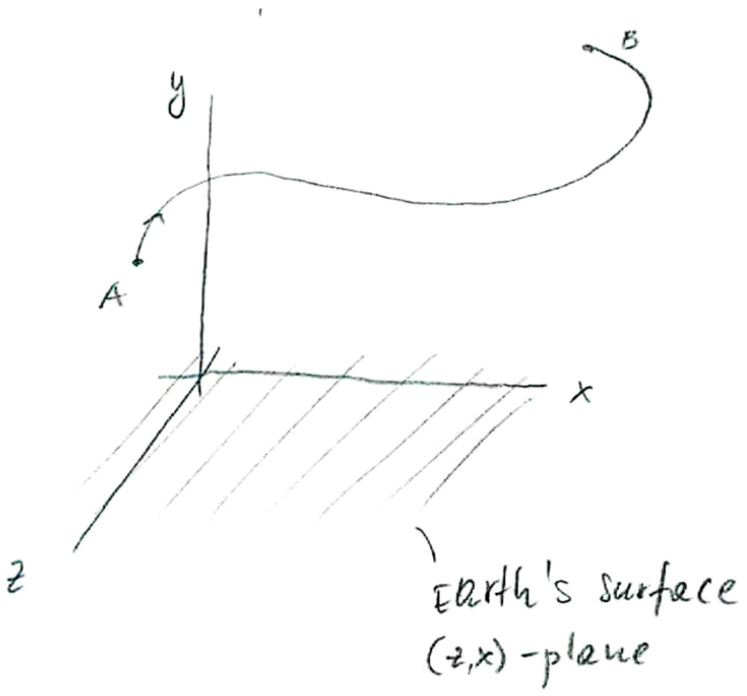
$$\boxed{U_{grav}(y) = mgy}$$

Final - Initial

$$W_{grav} = U_{grav,A} - U_{grav,B} = -\left(U_{grav,B} - U_{grav,A}\right) = -\Delta U_{grav}$$

this minus sign is important

Along a Curved Path



$$W_{grav,A\to B} = \int_{\Gamma_{AB}} \bar{F}_{grav} \circ d\bar{r}$$

$$\bar{F}_{grav} = (0, -mg, 0)$$

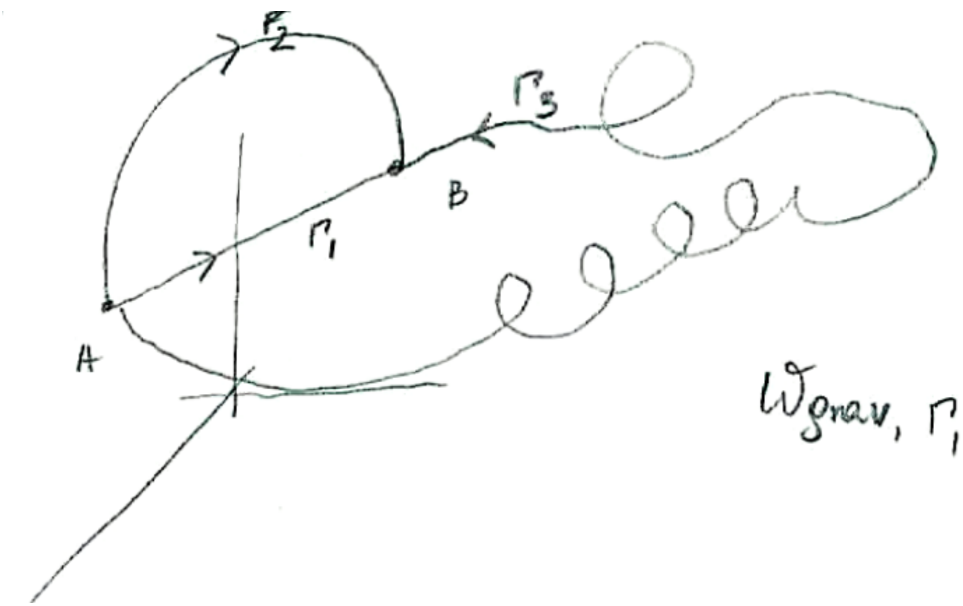
$$d\bar{r} = (dx, dy, dz)$$

$$W_{grav,A\to B} = \int_{\Gamma_{AB}} \bar{F}_{grav} \circ d\bar{r} = \int_{y_A}^{y_B} (-mg)\, dy =$$

$$= mg\,y_A - mg\,y_B = U_{grav,A} - U_{grav,B}$$

Conclusion: $W_{grav, A \to B} = U_{grav, A} - U_{grav, B}$

- Work done by gravitational force does not depend on path!
- Depends only on the initial and final positions of the particle.



$$W_{grav, \Gamma_1} = W_{grav, \Gamma_2} = W_{grav, \Gamma_3}$$

General: Conservative force

- 1) According to VP140:
- A force with the property that work done by it does not depend on path along which the particles is moved, but only on initial and final positions of the particle is called a potential (or conservative) force.

$$W_{loop} = \oint_{\Gamma} \bar{F} \cdot d\bar{r} = 0$$

- 2) According to Vv285:

3.1.14. Definition. Let $\Omega \subset \mathbb{R}^n$ be open and $F: \Omega \to \mathbb{R}^n$ a vector field. If the integral along any open curve C^* depends only on the initial and final points or, equivalently,

$$\oint_{\mathcal{C}} F\, d\vec{s} = 0 \qquad \text{for any closed curve } \mathcal{C},$$

then F is called *conservative*.

General: Potential Force (Field)

3.1.11. Definition. Let $\Omega \subset \mathbb{R}^n$ be an open set. A vector field $F\colon \Omega \to \mathbb{R}^n$ is said to be a *potential field* if there exists a differentiable potential function $U\colon \Omega \to \mathbb{R}$ such that

$$F(x) = \nabla U(x).$$

E.g.: Gravitational Force, Elastic Force

See: Problem Set 7 Q1

Potential vs Conservative

Potential Fields are Conservative

In physical terms, a conservative force field has the property that the work required to move a particle from one point to another does not depend on the path taken. Therefore, energy is conserved.

3.1.15. Remark. We note explicitly that every potential field is a conservative field.

In fact, under certain conditions a conservative field is also a potential field.

3.1.16. Definition. Let $\Omega \subset \mathbb{R}^n$. Then Ω is said to be *(pathwise) connected* if for any two points in Ω there exists an open curve within Ω joining the two points.

Conservative Forces & Energy Conservation

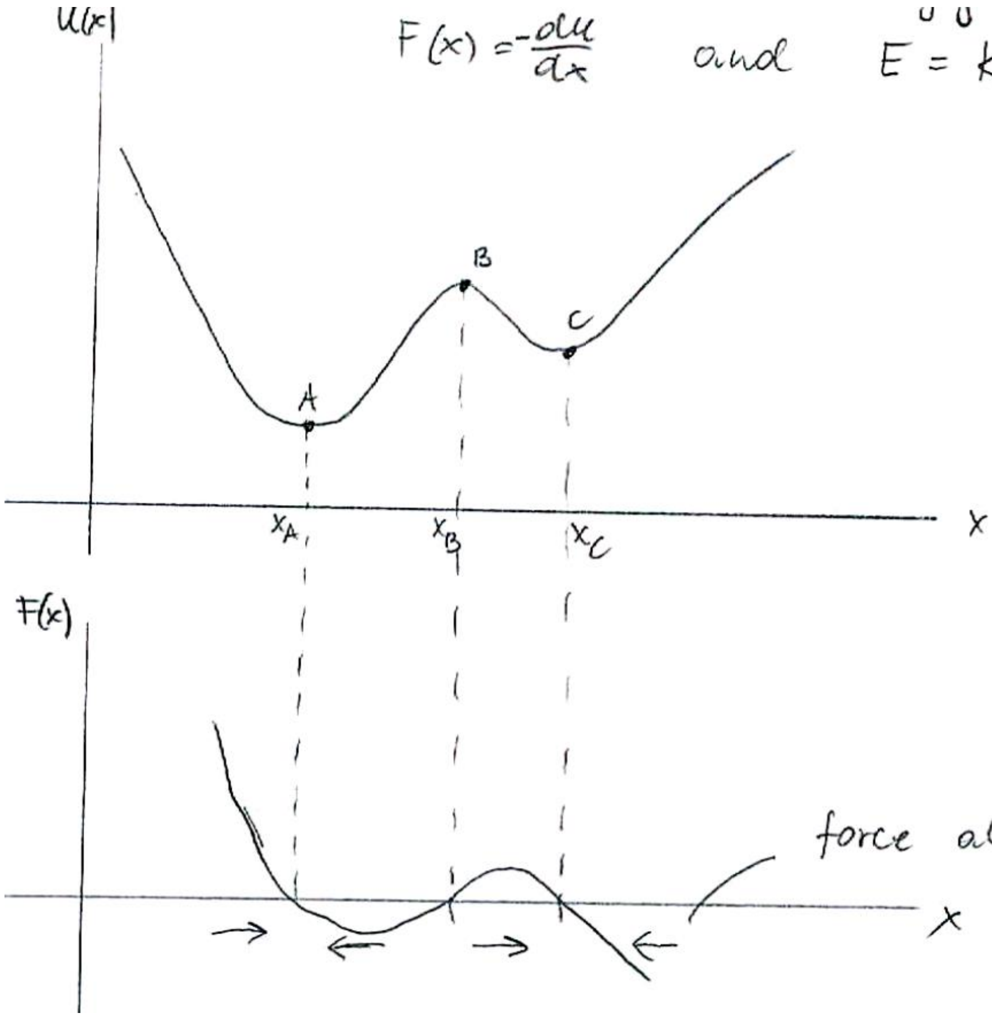
$$\boxed{F_x = -\frac{dU}{dx}}$$

$$W_{A\to B} = U(A) - U(B) \qquad W_{loop} = \oint_{\Gamma} \bar{F}\cdot d\bar{r} = 0$$

$$\boxed{E = U + K = const}$$

Energy Diagrams

$F(x) = -\frac{du}{dx}$ and $E = K + u = \text{const}$



force always in the direction of decreasing u

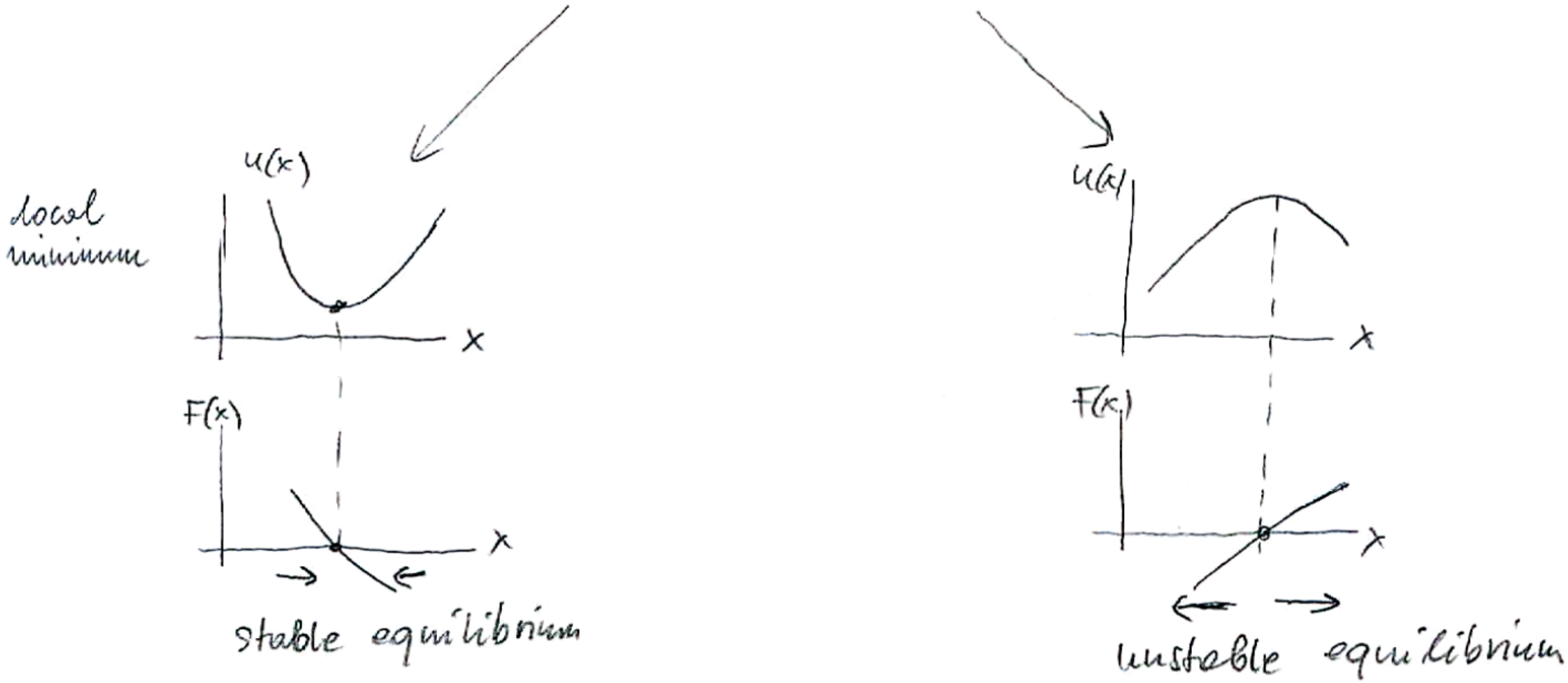
Given u(x),
how to determine F(x)?

Stable & Unstable Equilibrium

At A, B, C $\frac{du}{dx} = 0 \Rightarrow F = -\frac{du}{dx} = 0$ there

$\Downarrow$

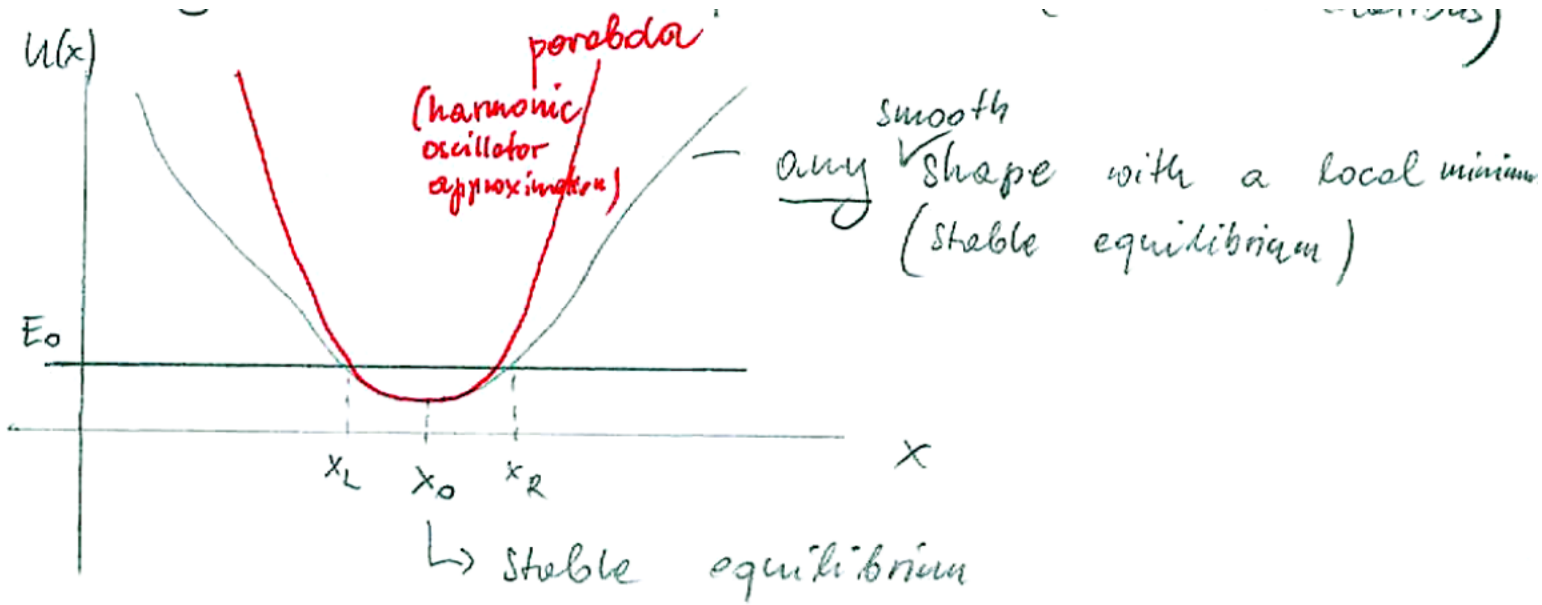
equilibrium points



See: Q2

Harmonic Oscillator Approximation

- Consider motion in the vicinity of a stable equilibrium (**small oscillations**) ⟸ How small?



Taylor expansion

$$U(x) = U(x_0) + \frac{1}{1!}U'(x_0)(x-x_0) + \frac{1}{2!}U''(x_0)(x-x_0)^2 + \cancel{\frac{1}{3!}U'''(x_0)(x-x_0)^3 + \ldots}$$

neglect these terms
$(x-x_0)$ - small

$$U(x) \approx U(x_0) + U'(x_0)(x-x_0) + \frac{1}{2}U''(x_0)(x-x_0)^2$$

But x_0 - Equilibrium, so $F(x_0) = 0 = -\left.\frac{dU}{dx}\right|_{x=x_0} \Rightarrow U'(x_0) = 0$, and

$$\boxed{U(x) \approx U(x_0) + \frac{1}{2}U''(x_0)(x-x_0)^2}$$

harmonic approximation

Corresponding force: $F(x) = -\frac{dU}{dx} = -k(x-x_0)$ where $k = U''(x_0)$

↳ harmonic oscillator

CONCLUSION

A particle moving in a potential well $U = U(x)$ of any shape, not too far away from a stable equilibrium, moves as if it was in harmonic motion with the natural frequency $\boxed{\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{U''(x_0)}{m}}}$

Hence:

$$\boxed{x(t) = x_0 + A\cos(\omega_0 t + \varphi)}$$

See: Q3, Q4