# Vv156 Lecture 10

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- Recall the sequence $\{a_n\}$ is said to be increasing if

$$a_{n+1} \geq a_n \qquad \text{for all } n.$$

and it is said to be decreasing if

$$a_{n+1} \leq a_n \qquad \text{for all } n.$$

Q: Let $\mathcal{I}$ be an interval, How to define the notion of increasing/decreasing for

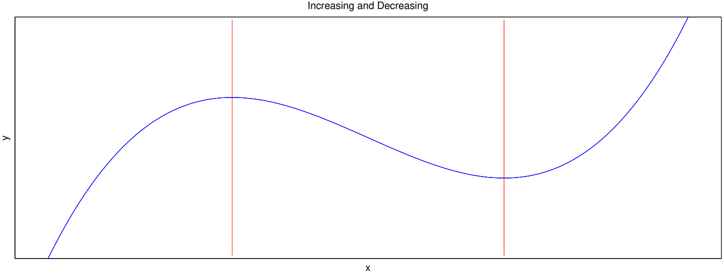
$$\text{a function} \qquad f(x) \qquad \text{where} \qquad x \in \mathcal{I}$$

### Definition

Suppose $f$ is defined on an interval $\mathcal{I}$, and $x_1$ and $x_2$ denote points in $\mathcal{I}$, then

1. $f$ is increasing on the interval if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2$.
2. $f$ is decreasing on the interval if $f(x_1) \geq f(x_2)$ whenever $x_1 < x_2$.

Q: Can you think of any connection between increasing/decreasing and $f'(x)$?




### Theorem

Suppose $f(x)$ is continuous on an interval $\mathcal{I}$, and differentiable on its interior.

1. If $f'(x) > 0$ for every interior point of $\mathcal{I}$, then $f$ is increasing on $\mathcal{I}$.
2. If $f'(x) < 0$ for every interior point of $\mathcal{I}$, then $f$ is decreasing on $\mathcal{I}$.

## Proof

- Consider an interior point $c$ of $\mathcal{I}$, if $f'(x) > 0$ for every interior point $\mathcal{I}$, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L > 0$$

- By definition, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon \qquad \text{if} \qquad 0 < |x - c| < \delta$$

- Expanding the left, we have

$$-\epsilon + L < \frac{f(x) - f(c)}{x - c} < \epsilon + L$$

- For $x$ sufficiently close to $c$ but greater than $c$, we have

$$x - c > 0$$

## Proof

- So we can rearrange the last inequality

$$\boxed{(L-\epsilon)(x-c) < f(x) - f(c)} < (L+\epsilon)(x-c)$$

- If we look at the lower bound provided of $f(x) - f(c)$ by the last inequality

$$(L-\epsilon)(x-c) < f(x) - f(c)$$

- Since $L > 0$, there is always some $0 < \epsilon < L$, such that

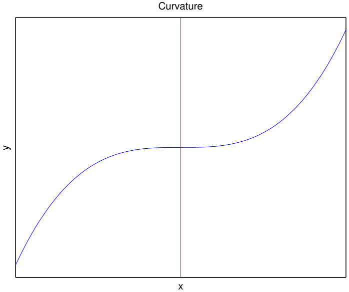
$$f(x) - f(c) > 0 \qquad \text{for} \qquad x - c > 0.$$

- So there is an open interval extending right from $c$ such that the function is

increasing

- Since $c$ is arbitrary, this shows that $f$ is increasing on the entire interval $\mathcal{I}$.
- This proves the first part, the second part is true for a similar reason. □

- The sign of the derivative of f reveals where the graph of f is increasing and where it is decreasing, but it does not reveal the direction of curvature, i.e.




### Definition

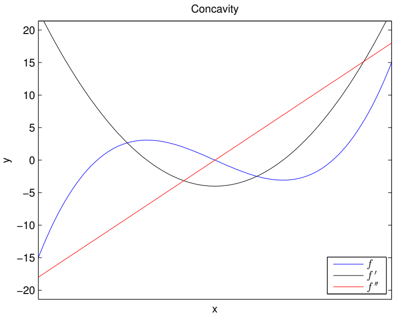
Let $f(x)$ be differentiable on an interval $\mathcal{I}$. The graph of $f(x)$ is said to be

1. concave up on $\mathcal{I}$ if and only if $f'(x)$ is increasing on $\mathcal{I}$.
2. concave down on $\mathcal{I}$ if and only if $f'(x)$ is decreasing on $\mathcal{I}$.

## Theorem

Suppose $f(x)$ is twice differentiable on an interval $\mathcal{I}$.

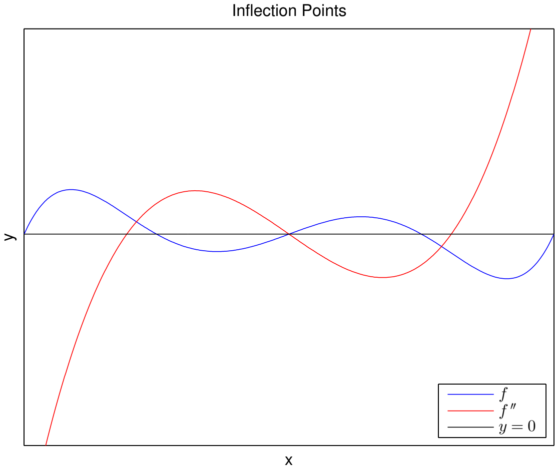
1. If $f''(x) > 0$ for all $x$ in $\mathcal{I}$, then $f$ is concave up on $I$.
2. If $f''(x) < 0$ for all $x$ in $\mathcal{I}$, then $f$ is concave down on $I$.




- This theorem follows directly from the last theorem P3.

## Definition

If $f$ changes the direction of concavity at the point $(x_0, f(x_0))$, then we say that $f$ has an inflection point at $x_0$.



## Exercise

(a) Find the intervals on which

$$f(x) = x + \sin x$$

is increasing or decreasing.

(b) Use the first and second derivatives of the function

$$f(x) = x^3 - 3x^2 + 1$$

to determine the intervals on which $f(x)$ is increasing, decreasing, concave up, and concave down. Identify all inflection points, if any.

(c) Describe the concavity of the graph of

$$f(x) = x^4$$

## Definition

Let $c$ be a point in the domain $\mathcal{D}$ of a function $y = f(x)$. Then $f(c)$ is a

- *global/absolute* maximum of $f$ for a set $\mathcal{I} \subset \mathcal{D}$ if

$$f(c) \geq f(x) \qquad \text{for all } x \in \mathcal{I}.$$

- *global/absolute* minimum of $f$ for a set $\mathcal{I} \subset \mathcal{D}$ if

$$f(c) \leq f(x) \qquad \text{for all } x \in \mathcal{I}.$$

- local/relative maximum of $f$ if there is a neighborhood $\mathcal{U} \subset \mathcal{D}$ of $c$ such that

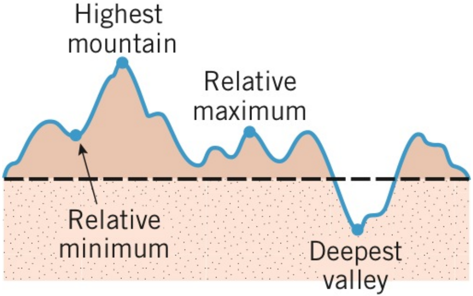
$$f(c) \geq f(x) \qquad \text{for all } x \in \mathcal{U}.$$

- local/relative minimum of $f$ if there is a neighborhood $\mathcal{U} \subset \mathcal{D}$ of $c$ such that

$$f(c) \leq f(x) \qquad \text{for all } x \in \mathcal{U}.$$

- We say $f$ has an extremum at $c$ if $f$ has a maximum or a minimum at $c$.

- If we imagine the graph of a function $f(x)$ to be a two-dimensional mountain range with hills and valleys,




- Relative maxima or local maxima are the tops of the hills.
- Relative minima or local maxima are the bottoms of the valleys.
- The relative extrema are the high or low points in their immediate vicinity

Q: Find the relative extrema, if any, for the following functions

1. $f(x) = x^2$:

   Relative minimum at $x = 0$.

2. $f(x) = x^3$:

   No relative extremum.

3. $f(x) = \cos x$:

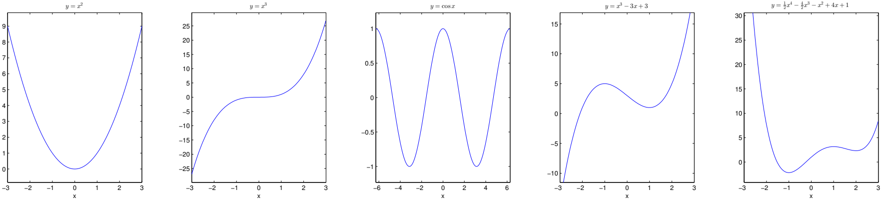
   Relative maxima at even $\pi$; Relative minima at odd $\pi$.

4. $f(x) = x^3 - 3x + 3$:

   Relative maximum at $x = -1$; Relative minimum at $x = 1$.

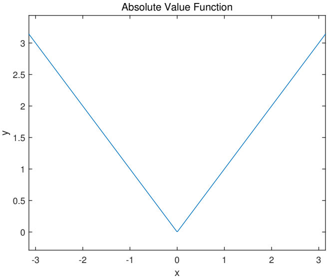
5. $f(x) = \frac{1}{2}x^4 - \frac{4}{3}x^3 - x^2 + 4x + 1$

Q: How to find relative extrema for a differentiable function except possibly for finite number of points?




Q: What do you notice regarding extrema and the slope at the extremum?

Q: Is there any other way to have a relative extreme?



### Definition

We define a critical point for $f$ to be a point in the domain of $f$ at which either

1. The graph of $f$ has a horizontal tangent line.

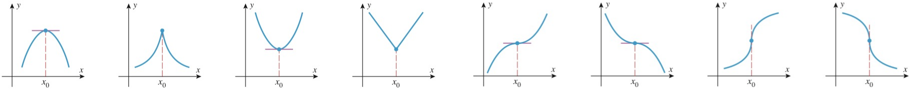
$$f' = 0$$

2. The derivative function $f'$ does not exist.

To distinguish between the two types of critical points we call
point $c$ a stationary point of $f$ if $f'(c)$ is defined.

Q: Which of the followings $x_0$ is a critical point/stationary point?

(a) (b) (c) (d) (e) (f) (g) (h)



Q: What will ensure that a critical point is a relative extrema?

## The first derivative test

Suppose $c$ is a critical point for $f(x)$.

1. If $f'$ changes from positive to negative at $c$, then

   $$f \text{ has a relative maximum at } c.$$

2. If $f'$ changes from negative to positive at $c$, then

   $$f \text{ has a relative minimum at } c.$$

3. If $f'$ does not change sign at $c$, then

   $$f \text{ has no local maximum or minimum at } c.$$

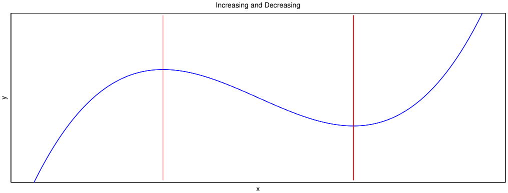
## Exercise

Find all critical points of

$$f(x) = 3x^5 - 5x^3$$

and then determine their nature by using the first derivative test.

Q: Is there any connection between the relative extrema of a twice differentiable function $f(x)$ and the concavity of $f(x)$?




### The second derivative test

Suppose that $f''$ exists at the point $c$.

1. If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a relative minimum at $c$.
2. If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a relative maximum at $c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test is inconclusive; that is,

   $f$ may have a relative maximum, a relative minimum, or neither at $c$.

- The second derivative test is more convenient than the first derivative test.

**Exercise**

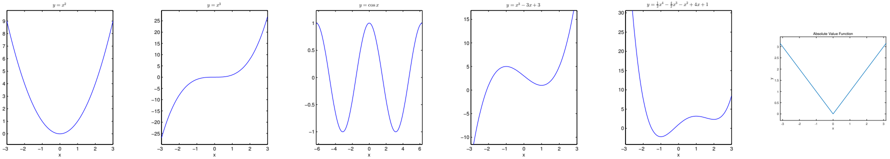
Find all critical points of

$$f(x) = 3x^5 - 5x^3$$

and then determine their nature using the second derivative test.

- Neither the first nor the second derivative test gives us a procedure directly to find relative extrema, they are merely tests for points in the domain of $f$.

Q: How can we narrow it down to a finite number of points?




- The next theorem proves our previous formally.

## Theorem

If $f(x)$ is differentiable at $x = c$ and $f(c)$ is a relative extremum, then the point $c$ is a stationary point

$$f'(c) = 0$$

## Proof

- If $f$ has a relative maximum at $c$, then

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in a } \delta\text{-neighbourhood of } c$$

so

$$\frac{f(c+h) - f(c)}{h} \leq 0 \qquad \text{for all } 0 < h < \delta,$$

which implies that

$$f'(c) = \lim_{h \to 0^+} \left[ \frac{f(c+h) - f(c)}{h} \right] \leq 0.$$

## Proof

- Moreover,

$$\frac{f(c+h)-f(c)}{h} \geq 0 \qquad \text{for all } -\delta < h < 0,$$

which implies that

$$f'(c) = \lim_{h\to 0^-}\left[\frac{f(c+h)-f(c)}{h}\right] \geq 0.$$

- If follows that $f'(c) = 0$ in order to have no contradiction of differentiability.
- If $f$ has a relative minimum at $c$, the argument is similar. The difference is the signs in these inequalities are reversed and the conclusion remains to be

$$f'(c) = 0 \quad \square$$

- This limits the search for an extremum on the domain to critical points.
- However, the converse of this theorem is not true.

Q: Is there an example in which a critical point is not a relative extremum?