University of Illinois **Fall 2023**

ECE 313: Problem Set 12

Due: Friday, November 17 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.7–4.8

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings.
5 points will be deducted for a submission with incorrectly-assigned page numbers.

1. **[Joint pdf of functions of random variables]**
   Let X and Y be random variables, and define two other random variables as $W = \frac{1}{\sqrt{2}}(X-Y)$ and $Z = \frac{1}{\sqrt{2}}(X+Y)$

   (a) If X and Y are *independent* $N(0,1)$ random variables, then show that W and Z are also *independent* $N(0,1)$ random variables.

   (b) Now consider the case where X and Y are still independent, but $X \sim N(0,2)$ whereas $Y \sim N(0,4)$. Find the joint distribution of W and Z. Are W and Z independent?

2. **[Covariance I]**
   Suppose X and Y are random variables on some probability space.

   (a) If $\mathrm{Var}(X+2Y) = \mathrm{Var}(X-2Y)$, are X and Y uncorrelated? If they are uncorrelated, briefly show/explain why. If they can be correlated, give an example.

   (b) If $\mathrm{Var}(X) = \mathrm{Var}(Y)$, are X and Y uncorrelated? If they are uncorrelated, briefly show why. If they can be correlated, give an example.

   (c) Rewrite $\mathrm{Cov}(5X+3, 2X+3Y-1)$ in terms of $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$ and $\mathrm{Cov}(X,Y)$.

3. **[Covariance II]**
   Consider a scenario where a distant point source in space transmits a message signal, denoted as M. This signal is intercepted by two observatories on Earth, leading to the reception of signals M_1 and M_2 at each observatory. Due to the vast distance the signal travels, both received signals have been corrupted by noise. Assume that they have the same mean μ and variance σ^2. The *signal-to-noise-ratio (SNR)* of the observation M_1 or M_2 is defined as the ratio $SNR_M = \frac{\mu}{\sigma^2}$. Motivated to enhance the reliability of their observations, the observatories decide to collaborate and jointly analyze their received signals. They opt for an averaging strategy, constructing a new random variable S defined as $S = \frac{M_1+M_2}{2}$.

(a) Show that the SNR of S is twice that of the individual observations, if M_1 and M_2 are uncorrelated.

(b) Let us assume that M_1 and M_2 are correlated with ρ_{M_1,M_2} as their correlation coefficient. Express the ratio $\frac{SNR_S}{SNR_M}$ as a function of ρ_{M_1,M_2} and plot the function.

(c) It is noticed that the averaging strategy gives $SNR_S = (1.5)SNR_M$. Determine the value of the correlation coefficient ρ_{M_1,M_2}. Under what condition on ρ_{M_1,M_2} can the averaging strategy result in an SNR_S that is arbitrarily high? Is the condition practical?

4. **[Weak law of large numbers for slightly correlated variables]**
Suppose $X_1, \cdots, X_{50}$ are random variables, each with mean $\mu = 10$ and variance $\sigma^2 = 1$. Suppose also that $|\mathrm{Cov}(X_i, X_j)| \leq 0.05$ if $|i - j| \leq 2$ and $i \neq j$, and $|\mathrm{Cov}(X_i, X_j)| = 0$ if $|i - j| > 2$. Let $S_{50} = X_1 + \cdots + X_{50}$.

   (a) Compute a tight upper bound for $\mathrm{Var}(S_{50})$.

   (b) Use the upper bound you get in part (a) and Chebychev's inequality to find an upper bound on $P(|\frac{S_{50}}{50} - 10| \geq 0.5.)$