ECE 310 Fall 2023

Lecture 7

z-transform properties

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Learning Objectives

After this lecture, you should be able to:

- Understand key properties of the z-transform and the ROC.
- Apply properties of the z-transform to derive the z-transform for discrete-time signals.

Recap from previous lecture

We motivated and introduced the z-transform in the previous lecture. We saw that the z-transform is a powerful tool for determining the response of LTI systems to complex exponential inputs. We will continue in this lecture by further defining the z-transform, region of convergence, and their key properties.

1 Overview of the z-transform

We define the z-transform of a discrete time signal $x[n]$ as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \tag{1}$$

This sum is not guaranteed to converge for all values of z. Thus, we must specify a region of convergence (ROC) to identify where $X(z)$ is well-defined. Moreover, we saw in the previous lecture that the ROC is necessary to make each $X(z)$ unique. It is important to note that $X(z)$ is not defined for values of z that lie outside the ROC *even if the expression for $X(z)$ computes a finite value.* Consider, for example,

$$x[n] = \left(\frac{1}{2}\right)^n u[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}},\ |z| > \frac{1}{2}. \tag{2}$$

The above $X(z)$ evaluates to -1 for $z = \frac{1}{4}$. However, let's check the original z-transform sum for this signal $x[n]$:

$$X\left(\frac{1}{4}\right) = \left(\sum_{n=-\infty}^{\infty} x[n]z^{-n}\right)\Bigg|_{z=\frac{1}{4}} \tag{3}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{1}{4}\right)^{-n} \tag{4}$$

$$= \sum_{n=0}^{\infty} 2^n. \tag{5}$$

Clearly, the z-transform sum does not converge for $z = \frac{1}{4}$, which is consistent with the given ROC.

Let's summarize some key properties that define the z-transform and its ROC:

1. Values of z where $X(z) = 0$ are known as zeros while values where $X(z) \to \infty$ are referred to as poles.
2. The ROC defines values of z for which the z-transform sum converges. Thus, the ROC cannot include any poles.
3. The ROC is a connected region, i.e. you can connect any two points in the ROC with a (curvy) line without exiting the ROC.
4. For infinite-length signals, the ROC can have one the following shapes:
   - Right-sided sequence, i.e. $x[n] = 0,\ n < n_0$: ROC=$|z| > a$.
   - Left-sided sequence, i.e. $x[n] = 0,\ n > n_0$: ROC=$|z| < a$.
   - Two-sided: ROC=$a < |z| < b$.
5. For finite-length signals, the ROC is the entire complex z-plane, except perhaps $z = 0$ for causal sequences or $z = \infty$ for non-causal sequences. For example, the causal sequence $x[n] = \delta[n] + \delta[n-1]$ has $X(z) = 1 + z^{-1}$ where the ROC is $|z| > 0$. Likewise, the anti-causal sequence $x[n] = \delta[n+1] + \delta[n]$ has $X(z) = z + 1$ and the ROC of $|z| < \infty$. If we add these two sequences, we would have an ROC of $0 < |z| < \infty$.

2 Key properties of the z-transform

We have computed the z-transform thus far using the z-transform sum or by inspection using a table of transform pairs. This is of course quite limiting. We would like to quickly compute the new z-transform of a signal if we scale it, shift it, reverse it, add to it, and so on. In this section, we will list several important z-transform properties along with a few proofs while further derivations and examples are reserved for lecture. For the below properties, let R_x denote the ROC for signal $x[n]$.

Time-shifting.

$$\begin{aligned}&x[n-k] \overset{\mathcal{Z}}{\Longleftrightarrow} z^{-k}X(z) \qquad (6)\\ &\text{ROC} = R_x \text{ (except possibly } z = 0 \text{ or } z = \infty).\end{aligned}$$

To derive the time-shifting property, let $y[n] = x[n-k]$.

$$\begin{aligned}Y(z) &= \sum_{n=-\infty}^{\infty} y[n]z^{-n} && (7)\\ &= \sum_{n=-\infty}^{\infty} x[n-k]z^{-n} && (8)\\ &= \sum_{m=-\infty}^{\infty} x[m]z^{-(m+k)} && (9)\\ &= z^{-k}\sum_{m=-\infty}^{\infty} x[m]z^{-m} && (10)\\ &= z^{-k}X(z). && (11)\end{aligned}$$

Above, line 9 follows from substituting $n = m + k$.

Linearity.

$$\begin{aligned}&ax_1[n] + bx_2[n] \overset{\mathcal{Z}}{\Longleftrightarrow} aX_1(z) + bX_2(z) \qquad (12)\\ &\text{ROC} \supseteq R_{x_1} \cap R_{x_2}.\end{aligned}$$

This simply states that the z-transform is a linear transform. The superset symbol $\supseteq$ indicates that the ROC contains at least the intersection of each signal's ROCs. This makes intuitive sense since we only need one z-transform sum to diverge in order to make the entire z-transform diverge. The statement of "at least" refers to exceptions where we have certain pole-zero interactions between the two signals. Most typically, the ROC is simply the intersection of the two ROCs. More on this in later lectures.

Convolution.

$$x_1[n] * x_2[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X_1(z)X_2(z) \tag{13}$$
$$\text{ROC} \supseteq R_{x_1} \cap R_{x_2}.$$

To derive the convolution property, let $y[n] = x_1[n] * x_2[n]$.

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y[n]z^{-n} && (14)\\
&= \sum_{n=-\infty}^{\infty} \left(\sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right) z^{-n} && (15)\\
&= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]z^{-k}z^{-(n-k)} && (16)\\
&= \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k]x_2[m]z^{-k}z^{-m} && (17)\\
&= \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \sum_{k=-\infty}^{\infty} x_1[k]z^{-k} && (18)\\
&= X_1(z)X_2(z). && (19)
\end{aligned}$$

Above, line 17 follows from the substitution $n = m + k$.

This is a remarkable property! We now can say that convolution in the time-domain corresponds to multiplication in the z-domain. This property will be key when we further discuss transfer functions and the characterization of LTI systems using the z-transform.

Lecture exercise: Prove the time shifting property using the convolution property.

Differentiation.

$$nx[n] \overset{\mathcal{Z}}{\Longleftrightarrow} -z\frac{dX(z)}{dz} \tag{20}$$
$$\text{ROC} = R_x.$$

The differentiation property will be useful to us when we discuss double-poles and stability of LTI systems in upcoming lectures.

Conjugation.

$$x^*[n] \overset{\mathcal{Z}}{\Longleftrightarrow} X^*(z^*) \tag{21}$$
$$\text{ROC} = R_x.$$

Time reversal.

$$x[-n] \overset{\mathcal{Z}}{\Longleftrightarrow} X\left(z^{-1}\right) \tag{22}$$
$$\text{ROC} = \frac{1}{R_x}.$$

Table 1: Useful z-transform properties.

Property	Signal	z-transform	ROC
Time-shifting	$x[n-k]$	$z^{-k}X(z)$	R_x except $z=0$ or $z=\infty$
Linearity	$ax_1[n]+bx_2[n]$	$aX_1(z)+bX_2(z)$	At least $R_{x_1} \cap R_{x_2}$
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_{x_1} \cap R_{x_2}$
Differentiation	$nx[n]$	$-z\frac{dX(z)}{dz}$	R_x
Conjugation	$x^*[n]$	$X^*(z^*)$	R_x
Time reversal	$x[-n]$	$X\left(z^{-1}\right)$	$1/R_x$
Scaling	$a^n x[n]$	$X\left(z/a\right)$	$\|a\|R_x$
Real-part	$\text{Re}\{x[n]\}$	$(1/2)[X(z)+X^*(z^*)]$	At least R_x
Imaginary-part	$\text{Im}\{x[n]\}$	$(1/2)[X(z)-X^*(z^*)]$	At least R_x

Lecture exercise: Prove the time reversal property.

For convenience, we summarize the above properties and a couple additional properties in Table 1, which corresponds to Table 3.2 in the *Manolakis and Ingle* textbook.