ECE 310 Fall 2023

Lecture 20

Aliasing effect

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Learning Objectives

After this lecture, you should be able to:

- Define and identify the aliasing effect when sampling bandlimited continuous-time signals using Nyquist criterion.
- Demonstrate how sampling below the Nyquist rate aliases sinusoidal signals in the time-domain and frequency-domain.

Recap from previous lecture

We derived the processes for ideal analog-to-digital and digital-to-analog conversion in the previous two lectures. These ideal sampling schemes operate under the assumption that our continous-time signals are bandlimited and we sample above the Nyquist rate. In this lecture, we will explore what happens when we sample below the Nyquist rate and cannot recover the original continuous-time signal we sampled.

1 Nyquist criterion

We begin by briefly reviewing bandlimited continuous-time signals and Nyquist criterion. A continuous-time signal $x(t)$ is bandlimited if

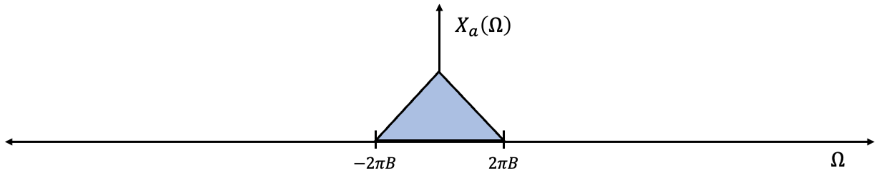
$$X_a(\Omega) = 0,\ |\Omega| > 2\pi B, \tag{1}$$

where $X_a(\Omega)$ is the continuous-time Fourier transform (CTFT) of $x(t)$. The linear frequency B can be referred to as the magnitude of the largest linear frequency present in $x(t)$. Recall that we may relate analog (Ω) and digital (ω) frequencies using the sampling period T:

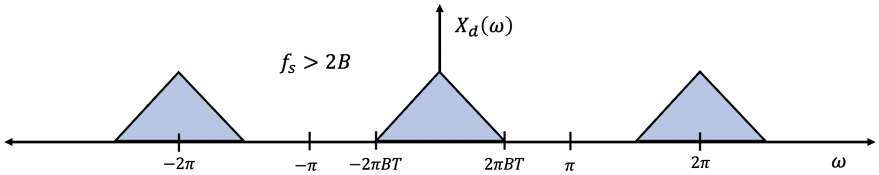
$$\omega = \Omega T. \tag{2}$$

We can use this relationship and the definition of bandlimited signals to derive the Nyquist criterion in an alternative way from lecture 18. Remember that the DTFT of a discrete-time signal is 2π periodic in the frequency-domain. Thus, all unique information present in $x[n]$ and its DTFT $X_d(\omega)$ is contained between $\omega \in [-\pi, \pi]$. Without loss of generality, let's assume that the largest magnitude frequency B in $x(t)$ is positive. We want to make sure after sampling that all frequencies present in $X_a(\Omega)$ are mapped to the $[-\pi, \pi]$ interval for $X_d(\omega)$. This is equivalent to guaranteeing the largest radial frequency, $\Omega_{\max} = 2\pi B$, does not exceed π after converting to a digital frequency. Thus, following Eqn. 2,

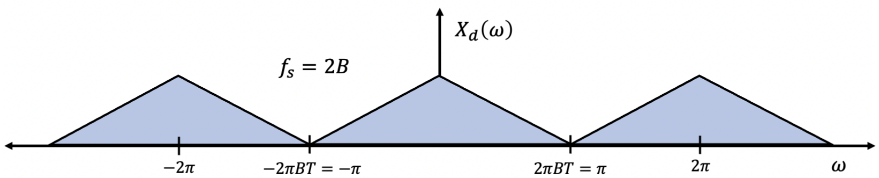
$$\begin{aligned} 2\pi BT &< \pi && (3)\\ T &< \frac{1}{2B} && (4)\\ f_s &> 2B, && (5) \end{aligned}$$



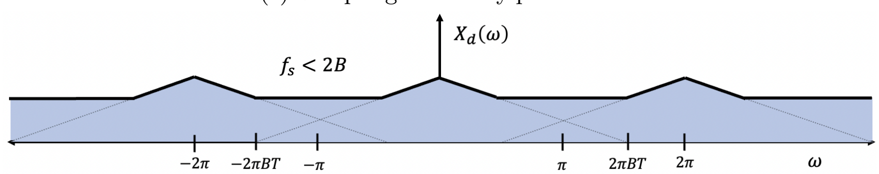
(a) Original continuous-time spectrum.



(b) Sampling above the Nyquist rate.



(c) Sampling at the Nyquist rate.



(d) Sampling below the Nyquist rate.

Figure 1: Frequency spectrum of a bandlimited continuous-time signal $X_a(\Omega)$. The following three graphs show the DTFT $X_d(\omega)$ after sampling $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X_a(\Omega)$ with different choices of sampling rate above, at, and below the Nyquist rate. Note the aliasing when sampling below the Nyquist rate results in overlap between spectral copies and a change in the shape of each spectral copy.

where $f_s = 1/T$ is the sampling frequency. Equations 4 and 5 are equivalent forms of the *Nyquist criterion* for sampling bandlimited continuous-time signals. The quantity $2B$ is known as the *Nyquist rate*.

2 Aliasing effect

In lectures 18 and 19, we considered sampling above the Nyquist rate of bandlimited signals. When $f_s > 2B$, we have no overlap in our spectral copies after sampling as in Fig. 1b. We may even be able to sample at the Nyquist rate when $X_d(\pm\pi) = 0$ as in Fig. 1c. We now want to examine the scenario of Fig. 1d where we sample below the Nyquist rate and the spectral copies overlap.

When the spectral copies overlap, the shape of the original spectrum $X_a(\Omega)$ is no longer accurately represented between $[-\pi, \pi]$ in $X_d(\omega)$. Thus, if we performed ideal digital-to-analog conversion, we would not be able to recover the original continuous-time signal $x(t)$ that gives us $X_a(\Omega)$. We instead would recover a corrupted version of the continuous-time signal that has had its frequencies and information distorted. *Aliasing* is the phenomenon whereby a signal is undersampled and its sampled representation no longer accurately expresses the original signal. Instead, the sampled signal takes on a different appearance or *alias*. Once a signal is aliased, we cannot recover its original representation. In general, aliasing is difficult to quantify and compute. We can, however, look at the simpler scenario of sampling sinusoidal signals where aliasing is actually manageable to interpret and fully explain.

2.1 Aliasing of sinusoidal signals

Suppose we have a continuous-time signal $x(t)$ given by a sinuosoid with radial frequency Ω_0:

$$x(t) = \sin(\Omega_0 t). \tag{6}$$

We then sample this sinusoid with sampling period T to obtain discrete-time signal $x[n] = x(nT)$:

$$\begin{align} x[n] &= \sin(\Omega_0 nT) \tag{7} \\ &= \sin(\omega_0 n), \tag{8} \end{align}$$

where $\omega_0 = \Omega_0 T$. Clearly, $x(t)$ is a bandlimited signal with $B = \Omega_0/2\pi$. Referring back to Nyquist criterion, we will have aliasing if $T > 1/2B = \pi/\Omega_0$. For example, let $T = 1/2B = \pi/\Omega_0$. This will give us a sampled $x[n]$ of

$$\begin{align} x[n] &= \sin\left(\Omega_0 n \frac{\pi}{\Omega_0}\right) \tag{9} \\ &= \sin(\pi n) \tag{10} \\ &= 0, \text{ for all n.} \tag{11} \end{align}$$

This is clearly aliasing because we have all zeros instead of a discrete-time sinusoid! There would be no way of knowing what continuous-time signal was sampled to yield this sequence of zeros.

To make things more concrete, let's examine a sinusoid of a known frequency and see what happens as we sample above the Nyquist rate and lower the sampling rate until we are well below the Nyquist rate. Let $x(t)$ be

$$x(t) = \sin(100\pi t). \tag{12}$$

We have $x(t)$ is bandlimited with $B = 50$ Hz and maximum radial frequency $\Omega_0 = 100\pi$. Thus, the Nyquist criterion for sampling $x(t)$ and avoiding aliasing would be $f_s > 100$ Hz. Using Eqn. 2, we can quickly examine what will happen when we try the following samplings scenarios: (1) sampling well above, (2) sampling a little above, (3) sampling a little below, and (4) sampling well below the Nyquist rate. For each scenario, $x_k[n]$ denotes the sampled signal for the given sampling rate, e.g. $x_2[n]$ for scenario 2. We will show the resulting sampled radial frequency ω_k, discrete-time signal $x_k[n]$, and recovered continuous-time radial frequency $\hat{\Omega}_k$ if we use the same sampling rate/period for ideal digital-to-analog conversion. Recall also that

$$\sin(\omega_0 n) \stackrel{\mathcal{F}}{\longleftrightarrow} -j\pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0). \tag{13}$$

Scenario 1: $f_s = 160$ **Hz.** (i.e. $T_1 = 1/160$ seconds)

$$\omega_1 = \frac{5\pi}{8},\ x_1[n] = \sin\left(\frac{5\pi}{8}n\right),\ \hat{\Omega}_1 = \omega_1/T_1 = 100\pi \tag{14}$$

Scenario 2: $f_s = 120$ **Hz.**

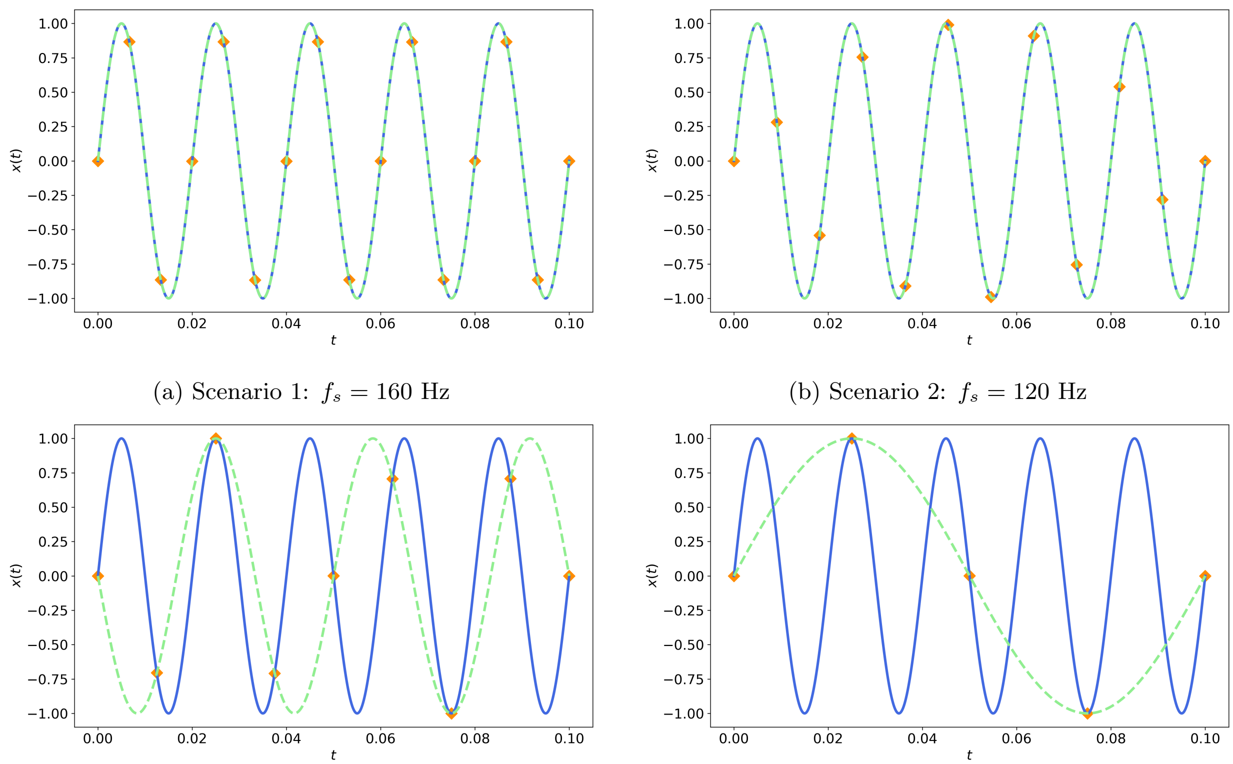
$$\omega_2 = \frac{5\pi}{6},\ x_2[n] = \sin\left(\frac{5\pi}{6}n\right),\ \hat{\Omega}_2 = \omega_2/T_2 = 100\pi \tag{15}$$

Scenario 3: $f_s = 80$ **Hz.**

$$\omega_3 = \frac{5\pi}{4},\ x_3[n] = \sin\left(\frac{5\pi}{4}n\right) \equiv \sin\left(-\frac{3\pi}{4}n\right),\ \hat{\Omega}_3 = \omega_3/T_3 = -60\pi \tag{16}$$

Scenario 4: $f_s = 40$ **Hz.**

$$\omega_4 = \frac{5\pi}{2},\ x_4[n] = \sin\left(\frac{5\pi}{2}n\right) \equiv \sin\left(\frac{\pi}{2}n\right),\ \hat{\Omega}_4 = \omega_4/T_4 = 20\pi \tag{17}$$



(a) Scenario 1: $f_s = 160$ Hz

(b) Scenario 2: $f_s = 120$ Hz

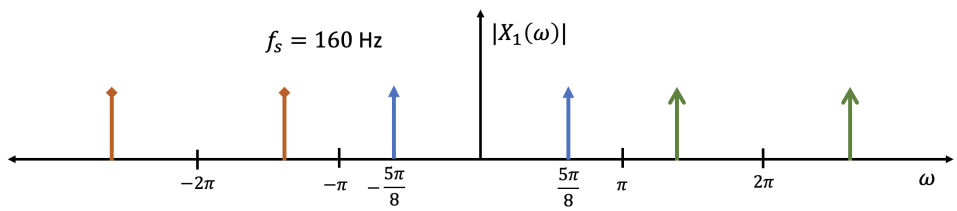
(c) Scenario 3: $f_s = 80$ Hz

(d) Scenario 4: $f_s = 40$ Hz

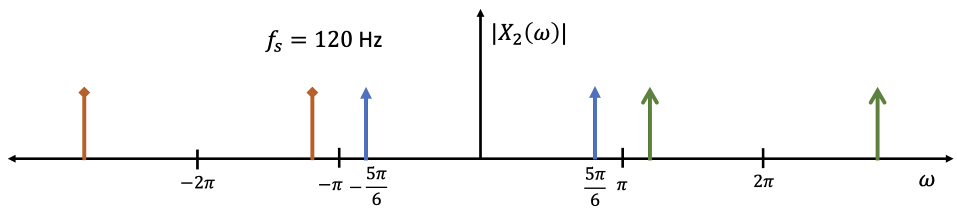
Figure 2: Illustration of the recovered sinusoid in the time-domain after sampling $x(t) = \sin(100\pi t)$ at varying sampling frequencies f_s. In each figure, the blue line is the original $x(t)$, the orange diamonds identify the points we would sample at the given sampling frequency, and the green dashed line is the signal $\hat{x}(t)$ we would recover from ideal digital-to-analog conversion of the given samples.

As expected, we see that the sampled frequencies in scenarios 1 and 2 lie within the $[-\pi, \pi]$ limits of the central copy of the DTFT. We then obtain the original radial frequency Ω_0 when converting back to continuous-time. Moving to the last two scenarios, we obtain sampled frequencies greater than π: $\frac{5\pi}{4}$ and $\frac{5\pi}{2}$, respectively. These larger frequencies alias down to smaller frequencies by the 2π periodicity of the DTFT (and discrete-time signals in general). In scenario 3, $\omega_3 = \frac{5\pi}{4}$ aliases to $\frac{-3\pi}{4}$ while ω_4 in scenario 4 aliases to $\frac{\pi}{2}$. Both scenarios 3 and 4 give us recovered frequencies $\hat{\Omega}_3 = \omega_3/T_3$ and $\hat{\Omega}_4 = \omega_4/T_4$ that do not match the original radial frequency Ω_0.

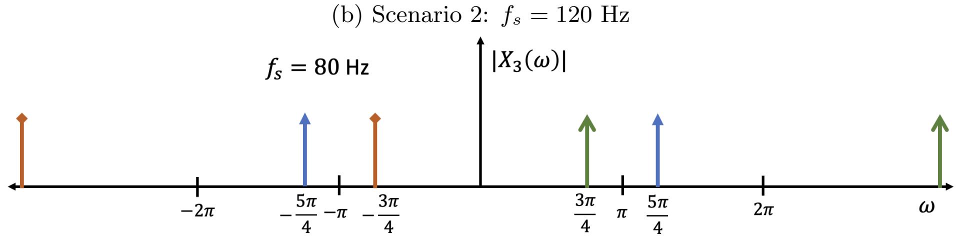
A helpful way to understand this phenomenon is to look at the DTFTs for each scenario to point out how crossover in the spectral copies yields these aliased frequencies. Figure 3 illustrates the magnitude spectra for the DTFT in each of these four sampling scenarios. Note how the blue arrows representing the central copy of the DTFT (centered at $\omega = 0$) spread out further from the center as we sample slower. Once these blue arrows extend past $\pm\pi$ and aliasing occurs, we see adjacent spectral copies, e.g. centered at $\omega = \pm 2\pi$, appear within our central $[-\pi, \pi]$ domain. Finally, Fig. 2 gives the time-domain depiction of each scenario. We see in scenarios 3 and 4 how the recovered sinusoid from the given samples oscillates at a different frequency from the original continuous-time signal.



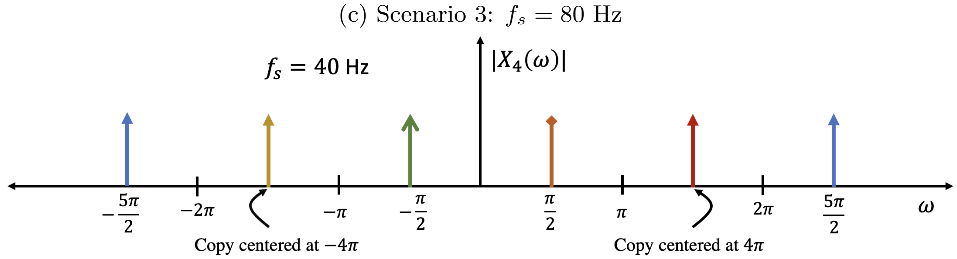
(a) Scenario 1: $f_s = 160$ Hz



(b) Scenario 2: $f_s = 120$ Hz



(c) Scenario 3: $f_s = 80$ Hz



(d) Scenario 4: $f_s = 40$ Hz

Figure 3: Illustration of the DTFT after sampling $x(t) = \sin(100\pi t)$ for varying choices of sampling frequency f_s. Blue arrows denote the middle spectral copy centered at $\omega = 0$. The orange and green arrows identify the spectral copies centered at -2π and 2π, respectively. In scenario 4, we have yet another two copies appear in the figure with the gold and red arrows representing spectral copies centered at -4π and 4π. We also differentiate spectral copies with different arrowhead markers in case the colors are not easy to recognize.