University of Illinois **Fall 2023**

ECE 313: Problem Set 11

Due: Friday, November 10 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.4–4.6

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings.
5 points will be deducted for a submission with incorrectly-assigned page numbers.

1. [**Independent or not?**]
   Decide whether X and Y are independent for each of the following pdfs. The constant C in each case represents the value making the pdf integrate to one. Justify your answer.

   (a)
   $$f_{X,Y}(u,v) = \begin{cases} C(u^4+v^4), & \text{if } 0 \le u \le 1 \text{ and } 0 \le v \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

   (b)
   $$f_{X,Y}(u,v) = \begin{cases} Cuv\, e^{-\frac{u^2}{2}}, & \text{if } 0 \le u \le 1 \text{ and } 1 \le v \le 2; \\ 0, & \text{otherwise.} \end{cases}$$

   (c)
   $$f_{X,Y}(u,v) = \begin{cases} C\exp(v-u), & \text{if } 0 \le u \le 1 \text{ and } 0 \le v \le 2u; \\ 0, & \text{otherwise.} \end{cases}$$

2. [**Sums of integer-valued random variables**]
   Let X and Y be independent random variables with PMFs:
   $$p_X(u) = \begin{cases} \frac{1}{3}, & \text{if } u = 1,2,3; \\ 0, & \text{otherwise.} \end{cases}$$
   $$p_Y(v) = \begin{cases} \frac{1}{2}, & \text{if } v = 0; \\ \frac{1}{4}, & \text{if } v = 1; \\ \frac{1}{8}, & \text{if } v = 2; \\ \frac{1}{8}, & \text{if } v = 3; \\ 0, & \text{otherwise.} \end{cases}$$
   Find the PMF of $Z = X + Y$.

3. **[Sums of jointly continuous-type random variables]**
Let the random variables X, Y, and Z are mutually independent.

   (a) Suppose that X is uniformly distributed over the interval [-2, 2] and Y is uniformly distributed over the interval [-1, 1]. Find and sketch the pdf of $S = X + Y$.

   (b) Suppose now that X, Y, and Z is uniformly distributed over the interval [0, 1]. Find and sketch the pdf of $W = X+Y+Z$. (Hint: First consider the random variable $S = X+Y$, and then consider $W = S + Z$.)

4. **[Sums of jointly continuous-type random variables]**
Suppose that X and Y are independent exponential random variables with parameter λ.

   (a) Find the PDF of $Z = X + Y$.

   (b) Suppose now that the random variables X and Y are independent and exponentially distributed with different parameters λ and μ, respectively. Find the PDF of $S = X - Y$.

5. **[Additional examples using joint distributions]**
The lifetimes of three light bulbs are modeled as mutually independent and exponential random variables X_1, X_2, and X_3 with rates λ.

   (a) The time at which a light bulb first burns out is denoted by a random variable Z_1 which is given by:

   $$Z_1 = \min\{X_1, X_2, X_3\}.$$

   Find the pdf of Z_1.

   (b) Let us consider the first time at which two of the three light bulbs burn out. We denote this time by a random variable Z_2. Find the expected value of Z_2, i.e., $E[Z_2]$. (Hint: Can you find the expected value of Z_2 without finding the pdf of Z_2? First consider the time that the first light bulb burn out. Then, consider the additional time required for the second bulb (from the remaining two bulbs) to burn out.)

   (c) Suppose that we have n light bulbs and their lifetimes are modeled by mutually independent exponential random variables with rate λ denoted by X_i for $i = 1, \ldots, n$. Let us denote the first time that k of the n light bulbs burn out as $Z_{k:n}$, where $1 \leq k \leq n$. What is the expected value of $Z_{k:n}$, i.e., $E[Z_{k:n}]$?