ECE 310 Fall 2023

Lecture 22

DFT properties

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Learning Objectives

After this lecture, you should be able to:

- Explain and apply DFT properties for analyzing signals in both the time-domain and DFT-domain.
- Understand the circular convolution property of the DFT and perform circular convolution of finite-length sequences.

Recap from previous lecture

We introduced the discrete Fourier transform (DFT) in the previous lecture as a practical realization of the DTFT. It is important to note that the DTFT is still key to this class and in practice. The DFT begins the portion of this course where we start applying many of the topics we have already discussed. We will still rely heavily on the math and theory we have developed, but now we will have an eye towards real-world implementation. This lecture will continue discussion of the DFT by looking at important properties of the DFT, particularly the consequences of periodic extension and circular properties in the DFT.

1 Periodicity of the DFT

The most central property to understanding the DFT is the periodicity we see in both the DFT and inverse DFT equations.

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n},\ 0 \leq k \leq N-1 \tag{1}$$

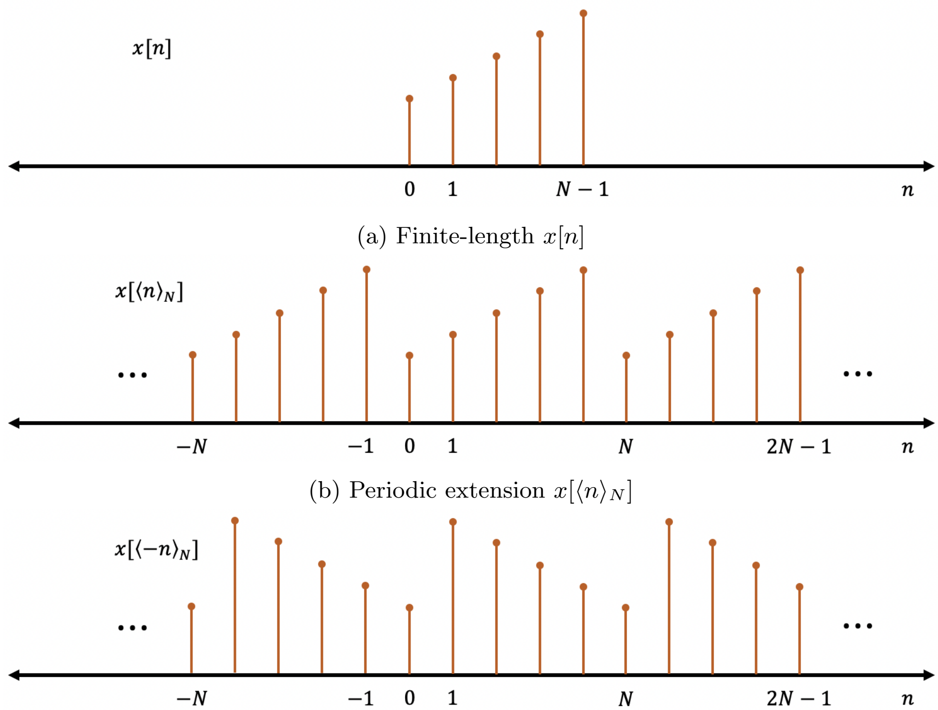
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\frac{2\pi k}{N}n},\ 0 \leq n \leq N-1. \tag{2}$$

We already defined N-periodicity of the DFT in the previous lecture, but it is worth carefully defining again.

N-periodicity in frequency. Let $X[k]$ be the DFT of a length-N signal, $x[n]$. The DFT $X[k]$ will be N-periodic:

$$X[k+mN] = X[k],\ m \in \mathbb{Z}. \tag{3}$$

N-periodicity in time. Let $X[k]$ again be the DFT of a length-N signal, $x[n]$. The DFT $X[k]$ is only computed using the values of $x[n]$ from $n = 0$ to $n = N-1$. Thus, the inverse DFT can only reproduce those values used to compute $X[k]$. In other words, the information in $x[n]$ from $n = 0$ to $n = N-1$ produces



(a) Finite-length $x[n]$

(b) Periodic extension $x[\langle n\rangle_N]$

(c) Circular reversal $x[\langle -n\rangle_N]$

Figure 1: Example illustration of a finite-length signal $x[n]$, it's periodic extension $x[\langle n\rangle_N]$ and the result of reversing the periodically extended signal, $x[\langle -n\rangle_N]$

$X[k]$, and $X[k]$ can only map this information back to the time-domain. The consequence of this is that recovered signals from the inverse DFT formula are N-periodic in the time domain:

$$x[n+mN] = x[n], \;\; m \in \mathbb{Z}. \tag{4}$$

For proofs of either N-periodicity in the time or DFT domains, please refer to lecture 21. The periodic extension of time-domain signals is certainly unintuitive. The important takeaway here is that any DFT corresponds to a finite-length time-domain signal that is periodically extended.

Another way of notating periodic extension is by the modulo operation. For a finite-length signal $x[n]$ and its DFT $X[k]$, we can write

$$x[n] = x[\langle n\rangle_N], \;\; \forall n \tag{5}$$
$$X[k] = X[\langle k\rangle_N], \;\; \forall k, \tag{6}$$

where $\langle n\rangle_N$ denotes the modulo operation that computes the remainder of dividing n by N. For example, $\langle 11\rangle_3 = 2$. This notation will be helpful in deriving and defining properties of the DFT.

Figure 1 provides examples of a finite-length signal and its periodic extension. This figure also includes the visualization for reversing a periodically extended signal. This will be helpful in the next section as we discuss circular convolution and the circular time reversal of signals.

2 Circular convolution

In our previous discussions of the z-transform and the DTFT, we repeatedly used the important convolution property that states

$$x[n] * h[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)H(z) \tag{7}$$

$$x[n] * h[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)H(\omega). \tag{8}$$

In words, we say that convolution in the time-domain is multiplication is the z-domain or frequency-domain. We should check if the same relation holds using the DFT. Let $X[k]$ and $H[k]$ denote the DFTs of two length-N signals, $x[n]$ and $h[n]$, respectively, and $Y[k]$ be the product of these DFTs. We want to see what the resulting $y[n]$ will be:

$$\begin{aligned}
y[n] &= \frac{1}{N}\sum_{k=0}^{N-1} Y[k]e^{j\frac{2\pi k}{N}n} && (9)\\
&= \frac{1}{N}\sum_{k=0}^{N-1} X[k]H[k]e^{j\frac{2\pi k}{N}n} && (10)\\
&= \frac{1}{N}\sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi k}{N}m}\right)\left(\sum_{l=0}^{N-1} h[l]e^{-j\frac{2\pi k}{N}l}\right)e^{j\frac{2\pi k}{N}n} && (11)\\
&= \sum_{m=0}^{N-1}\sum_{l=0}^{N-1} x[m]h[l]\left(\frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi k)\frac{n-m-l}{N}}\right) && (12)
\end{aligned}$$

The last term in line 12 will equal

$$\frac{1}{N}\sum_{k=0}^{N-1} e^{j(2\pi k)\frac{n-m-l}{N}} = \begin{cases} 1, & \langle n-l-m\rangle_N = 0 \\ 0, & \text{otherwise} \end{cases}, \tag{13}$$

due to the orthogonality principle of harmonically related periodic exponential signals we derived in lecture 13. Intuitively, the condition in line 13 for the sum to be non-zero is if $n-m-l$ is a multiple of N. This is also equivalent to if $l = \langle n-m\rangle_N$ or $m = \langle n-l\rangle_N$. Substituting for $l = \langle n-m\rangle_N$, we then find

$$y[n] = \sum_{m=0}^{N-1} x[m]h[\langle n-m\rangle_N]. \tag{14}$$

The result in line 14 is known as the *circular convolution* of two length-N sequences. More clearly, we can state the circular convolution property as follows:

$$x[n] \circledast h[n] \overset{\text{DFT}}{\longleftrightarrow} X[k]H[k], \tag{15}$$

where

$$x[n] \circledast h[n] = \sum_{m=0}^{N-1} x[m]h[\langle n-m\rangle_N] = \sum_{m=0}^{N-1} h[m]x[\langle n-m\rangle_N]. \tag{16}$$

Note also that the circular convolution of two length-N sequences will always result in another length-N signal. Thus, circular convolution should be between equal-length signals. If, however, two signals are different lengths, we may zero-pad the shorter signal to match the length of the longer signal.

2.1 Computing circular convolution

Calculating the ordinary convolution between two discrete-time sequences is hard enough, so it may feel unfair that we need to consider a new wrinkle with circular convolution. In this section, we will look at one example of circular convolution with two methods of solving: a modified table method from regular convolution and using a matrix operator. For this example, we will circularly convolve the following two sequences:

$$x[n] = \begin{bmatrix} 3 & 1 & 0 & 3 & 1 & 1 \end{bmatrix} \tag{17}$$

$$h[n] = \begin{bmatrix} -4 & -1 & -7 & 5 & 5 & 1 \end{bmatrix}. \tag{18}$$

Before moving on, it will be helpful to define the *circular time reversal* of a finite-length signal:

$$x[\langle -n \rangle_N] = \begin{cases} x[0], & n = 0 \\ x[N-n], & 1 \le n \le N-1 \end{cases}. \tag{19}$$

Figure 1c depicts an example of circular time reversal.

2.1.1 Modified table method

Recall from computing regular convolution in lecture 4 how we used the table method to calculate the convolution of two discrete-time sequences. We flipped-and-shifted one sequence while the other sequence stayed stationary. If the moving sequence had partial overlap with the stationary sequence, we assumed zero-extension, i.e. the ends of each signal were simply zero. Now with circular convolution, we must reconsider the table method with the modulo operator in the convolution sum. We can modify our existing table method by periodically extending the sequence we flip and shift along.

m	0	1	2	3	4	5	
$x[m]$	3	1	0	3	1	1	
$y[0] = \sum_m x[m]h[\langle -m \rangle_N]$	-4	1	5	5	-7	-1	-4
$y[1] = \sum_m x[m]h[\langle 1-m \rangle_N]$	-1	-4	1	5	5	-7	6
$y[2] = \sum_m x[m]h[\langle 2-m \rangle_N]$	-7	-1	-4	1	5	5	-9
$y[3] = \sum_m x[m]h[\langle 3-m \rangle_N]$	5	-7	-1	-4	1	5	2
$y[4] = \sum_m x[m]h[\langle 4-m \rangle_N]$	5	5	-7	-1	-4	1	14
$y[5] = \sum_m x[m]h[\langle 5-m \rangle_N]$	1	5	5	-5	-1	-4	-18

$$y[n] = \begin{bmatrix} -4 & 6 & -9 & 2 & 14 & -18 \end{bmatrix} \tag{20}$$

2.1.2 Using a matrix operator

Like with ordinary convolution, we can unpack the table method into a matrix-vector multiplication to compute circular convolution. Let x, h, and y denote the length-N vector representations of $x[n]$, $h[n]$, and $y[n]$, respectively. We may calculate y as

$$y = \tilde{\mathbf{H}}x = \tilde{\mathbf{X}}h, \tag{21}$$

where

$$\tilde{\mathbf{H}} = \begin{bmatrix} -4 & 1 & 5 & 5 & -7 & -1 \\ -1 & -4 & 1 & 5 & 5 & -7 \\ -7 & -1 & -4 & 1 & 5 & 5 \\ 5 & -7 & -1 & -4 & 1 & 5 \\ 5 & 5 & -7 & -1 & -4 & 1 \\ 1 & 5 & 5 & -7 & -1 & -4 \end{bmatrix} \tag{22}$$

$$\tilde{\mathbf{X}} = \begin{bmatrix} 3 & 1 & 1 & 3 & 0 & 1 \\ 1 & 3 & 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 1 & 1 & 3 \\ 3 & 0 & 1 & 3 & 1 & 1 \\ 1 & 3 & 0 & 1 & 3 & 1 \\ 1 & 1 & 3 & 0 & 1 & 3 \end{bmatrix}. \tag{23}$$

For example,

$$y = \begin{bmatrix} 3 & 1 & 1 & 3 & 0 & 1 \\ 1 & 3 & 1 & 1 & 3 & 0 \\ 0 & 1 & 3 & 1 & 1 & 3 \\ 3 & 0 & 1 & 3 & 1 & 1 \\ 1 & 3 & 0 & 1 & 3 & 1 \\ 1 & 1 & 3 & 0 & 1 & 3 \end{bmatrix} \begin{bmatrix} -4 \\ -1 \\ -7 \\ 5 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ 6 \\ -9 \\ 2 \\ 14 \\ -18 \end{bmatrix}. \tag{24}$$

The matrix operators $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{X}}$ have a special structure known as a *circulant matrix*. We construct $\tilde{\mathbf{H}}$ and $\tilde{\mathbf{X}}$ by circularly reversing $x[n]$ and $h[n]$ to form the first row (like in the first line of the table method above). Each subsequent row is the result of circularly shifting forward one place where the last sample wraps around to the front of the row. Note also how the diagonals of circulant matrices are constant.

3 More DFT properties

Additional properties of the DFT and some brief explanations are as follows.

Circular time shifting.

$$x[\langle n-m\rangle_N] \overset{\text{DFT}}{\longleftrightarrow} e^{-j\frac{2\pi km}{N}} X[k] \tag{25}$$

Similar to the time-shifting property of the DTFT, adding a linear phase in the DFT-domain, i.e. linear in k with slope $-2\pi m/M$, induces a shift in the time-domain. The key difference here is that our shift is now a *circular shift* due to the periodicity of the DFT in both the time and frequency domains.

Circular frequency shifting.

$$e^{j\frac{2\pi mn}{N}} x[n] \overset{\text{DFT}}{\longleftrightarrow} X[\langle k-m\rangle_N] \tag{26}$$

We have the same duality between time and frequency shifting with the DFT as we did with the DTFT. Here, we see that multiplying by a linear phase in the time-domain, i.e. linear in n now, gives a circular shift in the DFT domain.

Circular modulation. Extending the circular frequency shifting property, we obtain the circular modulation property:

$$\cos\left(\frac{2\pi m}{N} n\right) x[n] \overset{\text{DFT}}{\longleftrightarrow} \frac{1}{2} X[\langle k+m\rangle_N] + \frac{1}{2} X[\langle k-m\rangle_N]. \tag{27}$$

Circular time reversal.

$$x[\langle -n\rangle_N] \overset{\text{DFT}}{\longleftrightarrow} X[\langle -k\rangle_N]. \tag{28}$$

Conjugation.

$$x^*[n] \overset{\text{DFT}}{\longleftrightarrow} X^*[\langle -k\rangle_N]. \tag{29}$$

Table 1 summarizes the above DFT properties and a few additional properties as well.

Table 1: Summary of DFT properties

Property	Length-N $x[n]$	DFT $X[k]$
Linearity	$ax_1[n] + bx_2[n]$	$aX_1[k] + bX_2[k]$
Circular time shifting	$x[\langle n-m \rangle_N]$	$e^{-j\frac{2\pi mk}{N}} X[k]$
Circular frequency shifting	$e^{j\frac{2\pi mn}{N}} x[n]$	$X[\langle k-m \rangle_N]$
Circular modulation	$\cos\left(\frac{2\pi m}{N} n\right) x[n]$	$\frac{1}{2} X[\langle k+m \rangle_N] + \frac{1}{2} X[\langle k-m \rangle_N]$
Circular time reversal	$x[\langle -n \rangle_N]$	$X[\langle -k \rangle_N]$
Conjugation	$x^*[n]$	$X^*[\langle -k \rangle_N]$
Duality	$X[n]$	$Nx[\langle -k \rangle_N]$
Circular convolution	$x[n] \circledast h[n]$	$X[k]H[k]$
Windowing	$x[n]w[n]$	$\frac{1}{N} X[k] \circledast W[k]$
Parseval's theorem	$\sum_{n=0}^{N-1} x[n]y^*[n]$	$\frac{1}{N} \sum_{k=0}^{N-1} X[k]Y^*[k]$
Parseval's relation	$\sum_{n=0}^{N-1} \lvert x[n] \rvert^2$	$\frac{1}{N} \sum_{k=0}^{N-1} \lvert X[k] \rvert^2$