University of Illinois Fall 2021

ECE 313: Midterm Exam II

Monday, November 8, 2021
8:45 p.m. — 10:00 p.m.

1. **[10+6 points]** The two parts of the problem are unrelated.

   (a) Assume that the lifetime of an electronic device is a random variable that follows the **uniform distribution** with a mean of 3 years and a variance of 3 years^2. Suppose the device has been working for 1 year. What is the conditional probability that it will still work for at least another year?
   Solution: Lifetime T is uniformly distributed over the interval $[a, b]$.
   $E[T] = (a+b)/2 = 3 \Rightarrow a + b = 6$
   $\mathrm{Var}(T) = (a-b)^2/12 = 3 \Rightarrow b - a = 6$
   Therefore, $a = 0, b = 6$.
   $P\{T \geq 1{+}1 | T \geq 1\} = \dfrac{P\{T \geq 2, T \geq 1\}}{P\{T \geq 1\}} = \dfrac{P\{T \geq 2\}}{P\{T \geq 1\}} = \dfrac{1 - P\{T < 2\}}{1 - P\{T < 1\}} = \dfrac{1 - 2/6}{1 - 1/6} = \dfrac{4}{5}.$

   (b) Assume that the lifetime of an electronic device is a random variable that follows the **exponential distribution** with a mean of 3 years. Suppose the device has been working for 1 year. What is the conditional probability that it will still work for at least another year? Express your answer in terms of e.
   Solution: Lifetime T has the exponential distribution with parameter λ.
   $E[T] = 1/\lambda = 3 \Rightarrow \lambda = 1/3$
   According to the memoryless property of exponential distribution,
   $P\{T \geq 1 + 1 | T \geq 1\} = P\{T \geq 1\} = e^{-\lambda} = e^{-1/3}$.

2. **[10+14 points]** The two parts of the problem are unrelated.

   (a) Let X be a $N(2, 4)$ Gaussian random variable, and let $Y = 2X$. Find the following two probabilities $P\{Y \geq X + 2\}$ and $P\{Y \geq 2\}$. In the latter case, express your answer in two ways, using the Φ function in one and the Q function in the other.
   Solution: Clearly, $P\{Y \geq X + 2\} = P\{X \geq 2\} = 1/2$, due to the fact that X has mean 2. For the other probability, observe that Y is Gaussian $N(4, 16)$, so that

   $$P\{Y \geq 2\} = P\{\frac{Y-4}{4} \geq \frac{2-4}{4}\} = Q(-0.5) = 1 - \Phi(-0.5).$$

   Alternatively, you could have just written

   $$P\{Y \geq 2\} = P\{X \geq 1\} = P\{\frac{X-2}{2} \geq \frac{1-2}{2}\} = Q(-0.5) = 1 - \Phi(-0.5).$$

   (b) Suppose that T is uniformly distributed in $[0, \pi/2]$. Find the pdf of the random variable $C = \cos(T)$.
   Solution: Over the interval $[0, \pi/2]$, the cos function takes values in $[0, 1]$. Hence, C is supported on $[0, 1]$ and the cos function is monotonically decreasing on the interval $[0, \pi/2]$. We have

   $$F_C(c) = P\{C \leq c\} = P\{\cos(T) \leq c\} = P\{\cos^{-1}(c) \leq T \leq \pi/2\} = 1 - \frac{2\cos^{-1}(c)}{\pi},$$

   provided that $c \in [0, 1]$. For $c < 0$, $F_C(c) = 0$ and for $c > 1$, $F_C(c) = 1$. The pdf hence equals $f_Y(y) = \frac{2}{\pi\sqrt{1-y^2}}$ for $y \in [0, 1]$ and is 0 elsewhere.

3. **[14+6 points]** Assume that in a binary hypothesis testing problem, the continuous random variable X has a uniform pdf $f_0(u)$ for $u \in [0,1]$ under hypothesis H_0 and a "triangle" pdf over $[0,1]$ under hypothesis H_1, i.e.,

$$f_1(u) = \begin{cases} 4u, \text{ for } u \in [0,1/2], \\ 4-4u, \text{ for } u \in [1/2,1], \\ 0, \text{ otherwise.} \end{cases}$$

(a) State the MAP rule for the two hypothesis, given that $\pi_1 = \pi_0$.
Solution: The likelihood function is of the form

$$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \begin{cases} 4u, \text{ for } u \in [0,1/2], \\ 4-4u, \text{ for } u \in [1/2,1], \\ 0, \text{ otherwise.} \end{cases}$$

The MAP rule for $\pi_1 = \pi_0$ reduces to the ML rule, which relies on using the threshold $\tau = 1$, i.e.,

$$\begin{cases} \Lambda(u) > 1: \text{ decide in favor of } H_1, \\ \Lambda(u) < 1: \text{ decide in favor of } H_0. \end{cases}$$

This results in the following decision rule

$$\begin{cases} H_0 : 0 < X < 1/4 \text{ or } 3/4 < X < 1, \\ H_1 : 1/4 < X < 3/4. \end{cases}$$

(b) Find the probability of false alarm.
Solution: We have

$$P_{fa} = P\{\text{Decide } H_1 | H_0\} = P\{1/4 < X < 3/4 | H_0\} = 1/2.$$

4. **[10+10 points]** The two parts of the problem are unrelated.

(a) If the joint CDF of random variables X, Y is given by

$$F_{X,Y}(u,v) = u^2 v^2 \text{ for } 0 \le u \le 1, 0 \le v \le 1,$$

find $F_{X,Y}(1/3,3), F_{X,Y}(2,1/2)$.
Solution: As $F_{X,Y}(1,1) = 1$, both X, Y are at most one with probability 1. So, $F_{X,Y}(1/3,3) = \mathbb{P}(X \le 1/3, Y \le 3) = \mathbb{P}(X \le 1/3, Y \le 1) = F_{X,Y}(1/3,1) = 1/9$. Similarly, $F_{X,Y}(2,1/2) = 1/4$.

(b) Let X be uniform on $[0,1]$, $Y = 1 - X$ and $F_{X,Y}$ be the joint CDF of these random variables. Find $F_{X,Y}(2/3,3/4)$.
Solution: By definition, $F(2/3,3/4) = \mathbb{P}(X \le 2/3, 1 - X \le 3/4) = \mathbb{P}(1/4 \le X \le 2/3) = 5/12$.

5. **[10+10 points]** Let U, V and W be independent, and U, V taking values ± 1 with probability $1/2$. Consider two new random variables, $X = UW, Y = VW$.

(a) If W takes the values 1 and -1 with probabilities p and $(1-p)$, respectively (here $0 < p < 1$), are X and Y independent?
Solution: It is easy to compute the pmf for X, Y, and see that it has masses $1/4$ at the fours points $(\pm 1, \pm 1)$, so that X, Y are obviously independent.

(b) If W takes the values 0 and 1 with probabilities p and $(1-p)$, respectively (here $0 < p < 1$), are X and Y independent?
Solution: The pair (X, Y) takes values $(0,0)$ and $(1,1)$ with nonzero probabilities (p and $(1-p)/4$, respectively), but $\mathbb{P}((X,Y) = (0,1)) = 0$, so X and Y are not independent because the product criterion of independence fails.