ECE 310 Fall 2023

Lecture 14

Discrete-time Fourier Transform

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Learning Objectives

After this lecture, you should be able to:

- Compute the discrete-time Fourier series of a periodic discrete-time signal.
- Compute the discrete-time Fourier transform of a discrete-time signal or show that it does not exist.
- Explain what the discrete-time Fourier transform of a signal represents and how it relates to the discrete-time Fourier series.

Recap from previous lecture

We introduced the continuous-time Fourier series (CTFS) in the previous lecture and used the CTFS to informally derive the continuous-time Fourier transform (CTFT). The CTFS and CTFT represent two methods for characterizing the frequency content of periodic and aperiodic continuous-time signals. In this lecture, we will follow a similar process to establish discrete-time equivalents of these two techniques.

1 Discrete-time Fourier series

Recall from the previous lecture that we can define harmonically related periodic signals using a fundamental frequency ω_0 and corresponding fundamental period $N_0 = 2\pi/\omega_0$. For discrete-time complex exponential signals, an example would be:

$$x_k[n] = e^{j\omega_0 kn} \ , 0 \leq k < N_0 - 1. \tag{1}$$

Note that there is no need to let k use all integers due to the 2π periodicity of discrete-time signals. Consider, $k = 0$ and $k = N_0$:

$$k = 0 : x_0[n] = e^{j0} = 1, \ \forall n \tag{2}$$

$$k = N_0 : x_{N_0}[n] = e^{j\omega_0 N_0 n} = e^{j2\pi n} = 1, \ \forall n. \tag{3}$$

Thus, we may use a *finite* set of harmonically related complex exponentials to represent periodic discrete-time signals with period N_0 by

$$x[n] = \sum_{k=0}^{N_0-1} c_k e^{j\frac{2\pi k}{N_0}n}. \tag{4}$$

The above demonstration for $k = 0$ and $k = N_0$ shows that the signal synthesized by Eqn. 4 will be periodic with fundamental period N_0. Following a similar procedure as in lecture 12, we can determine the coefficients c_k for a given periodic signal $x[n]$ by

$$c_k = \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi k}{N_0}n}. \tag{5}$$

Equations 4 and 5 define the *discrete-time Fourier series* (DTFS) of periodic discrete-time signals. Like with the CTFS and CTFT, we may refer to Eqn. 4 as the synthesis equation and Eqn. 5 as the analysis equation. Let's consider an example to make the DTFS more concrete.

Exercise 1: Let $x[n]$ be a periodic rectangular pulse with period N_0 with

$$x[n] = \begin{cases} 1, & 0 \leq n < L \\ 0, & L \leq n < N_0 \end{cases}. \tag{6}$$

Determine the discrete-time Fourier series coefficients c_k.

$$\begin{aligned} c_k &= \frac{1}{N_0} \sum_{n=0}^{N_0-1} x[n] e^{-j\frac{2\pi k}{N_0}n} \\ &= \frac{1}{N_0} \sum_{k=0}^{L-1} e^{-j\frac{2\pi k}{N_0}n} \qquad (7) \\ &= \frac{1}{N_0} \left(\frac{1 - e^{-j\frac{2\pi k}{N_0}L}}{1 - e^{-j\frac{2\pi k}{N_0}}} \right) \qquad (8) \\ &= \frac{1}{N_0} \left(\frac{e^{-j\frac{\pi k}{N_0}L}}{e^{-j\frac{\pi k}{N_0}}} \right) \left(\frac{e^{j\frac{\pi k}{N_0}L} - e^{-j\frac{\pi k}{N_0}L}}{e^{j\frac{\pi k}{N_0}} - e^{-j\frac{\pi k}{N_0}}} \right) \qquad (9) \\ &= \frac{1}{N_0} e^{-j\frac{\pi k}{N_0}(L-1)} \left(\frac{\sin\left(\frac{\pi L}{N_0}k\right)}{\sin\left(\frac{\pi}{N_0}k\right)} \right). \qquad (10) \end{aligned}$$

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We use the finite geometric sum formula in line 8 and a popular trick in line 9 where we factor out complex exponentials in the numerator and denominator in order to invoke Euler's identity in line 10. Our Fourier series coefficients are complex-valued, thus we can write the magnitude and phase for each c_k:

$$|c_k| = \begin{cases} \frac{1}{N_0} \left| \frac{\sin\left(\frac{\pi L}{N_0}k\right)}{\sin\left(\frac{\pi}{N_0}k\right)} \right|, & k \neq 0 \\ \frac{L}{N_0}, & k = 0 \end{cases} \tag{11}$$

$$\angle c_k = -\frac{\pi k}{N_0}(L-1). \tag{12}$$

Above, we use L'Hopital's rule to obtain the magnitude $|c_k|$ for $k = 0$.

The DTFS of a periodic discrete-time signal is guaranteed to exist for bounded signals since the analysis and synthesis equations are over finite summations. Moreover, we can guarantee that the synthesis equation will exactly recover our original signal and the corresponding analysis coefficients give a unique DTFS for each periodic discrete-time signal.

2 Discrete-time Fourier transform

Like the CTFS, the DTFS also expresses how much each harmonic frequency is present in a given periodic signal. Thus, the DTFS decomposes a discrete-time periodic signal into a complex-weighted sum of harmonically related complex exponential signals.

We again would like to expand our analysis beyond periodic signals to include aperiodic signals as well. We can exploit the same intuitive trick as in lecture 12 by acknowledging that aperiodic signals are described by the limit as $N_0 \to \infty$ and $\omega_0 \to 0$. Informally, we will obtain the following new analysis and synthesis equations:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{13}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega. \tag{14}$$

The analysis equation given by Eqn. 13 is known as the *discrete-time Fourier transform* (DTFT) and the synthesis equation shown in Eqn. 14 is referred to as the inverse discrete-time Fourier transform (inverse DTFT). We use $X(\omega)$ to denote the DTFT or *frequency spectrum* of $x[n]$ (note that *Manolakis and Ingle* use $X(e^{j\omega})$ instead). We can guarantee the existence of the DTFT if the analysis sum converges. Thus, the DTFT exists if

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty. \tag{15}$$

Finally, just like the CTFS, CTFT, and DTFS, the DTFT of a discrete-time signal is guaranteed to be unique if it exists. Let's revisit the rectangular signal from Exercise 1 by now assuming it is aperiodic.

Exercise 2: Compute the DTFT $X(\omega)$ of the rectangular pulse given by

$$x[n] = \begin{cases} 1, & 0 \leq n < L \\ 0, & \text{otherwise} \end{cases}. \tag{16}$$

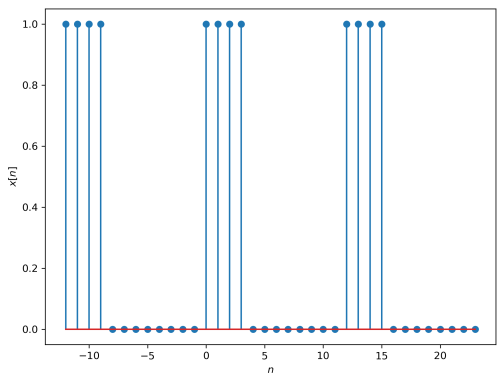
$$\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\
&= \sum_{n=0}^{L-1} e^{-j\omega n} && (17) \\
&= \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} && (18) \\
&= \left(\frac{e^{-j\frac{\omega L}{2}}}{e^{-j\frac{\omega}{2}}}\right)\left(\frac{e^{j\frac{\omega L}{2}} - e^{j\frac{\omega L}{2}}}{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}\right) && (19) \\
&= e^{-j\frac{\omega}{2}(L-1)}\left(\frac{\sin\left(\frac{\omega L}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}\right). && (20)
\end{aligned}$$

The above result is remarkably close to the DTFS coefficients obtained in Exercise 1. In fact, if we let $\omega_k = \frac{2\pi k}{N_0}$, these coefficients become

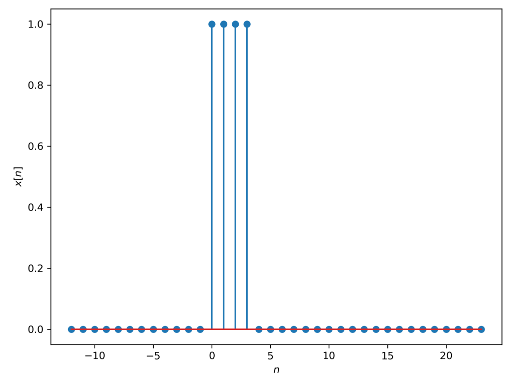
$$c_k = \frac{1}{N_0}e^{-j\frac{\omega_k}{2}(L-1)}\left(\frac{\sin\left(\frac{\omega_k L}{2}\right)}{\sin\left(\frac{\omega_k}{2}\right)}\right). \tag{21}$$

We see then that the DTFS coefficients actually sample the DTFT values at N_0 evenly spaced frequencies! The only difference then is the scaling factor by $\frac{1}{N_0}$.

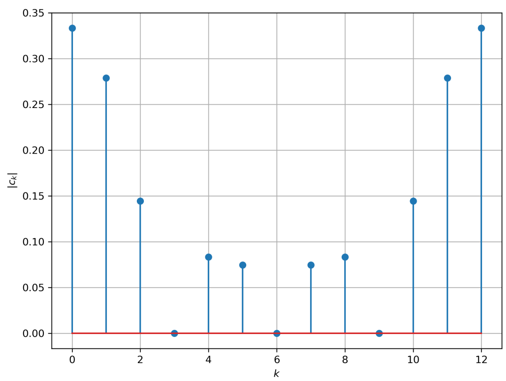
The DTFT of a discrete-time signal is quite similar to the CTFT of a continuous-time signal. Like the CTFT, the DTFT tells us how much each frequency is present in our signal. Both the CTFT and DTFT are complex-valued, continuous functions of frequency Ω and ω, respectively. Remember, however, that the DTFT is 2π-periodic over ω while the CTFT is uniquely defined for all Ω. In addition to the frequency spectrum $X(\omega)$, we also commonly refer to the *magnitude spectrum* $|X(\omega)|$ and *phase spectrum* $\angle X(\omega)$. Figure 1 depicts the DTFS coefficients and DTFT of an example rectangular pulse. We will further explore the magnitude and phase spectra of signals and systems as well as key properties of the DTFT and its relation to the z-transform in upcoming lectures.



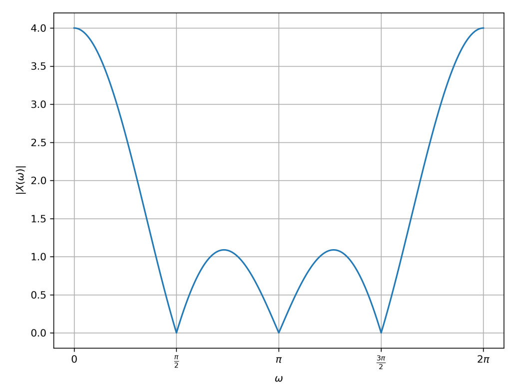
(a) Periodic rectangular pulse



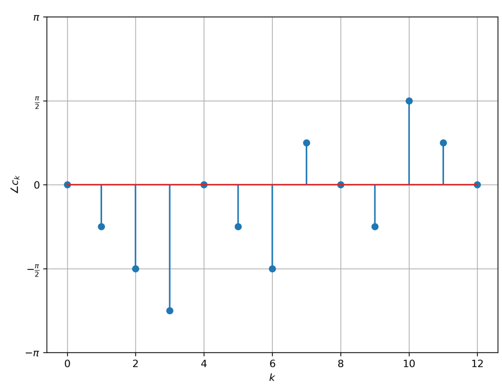
(b) Aperiodic rectangular pulse



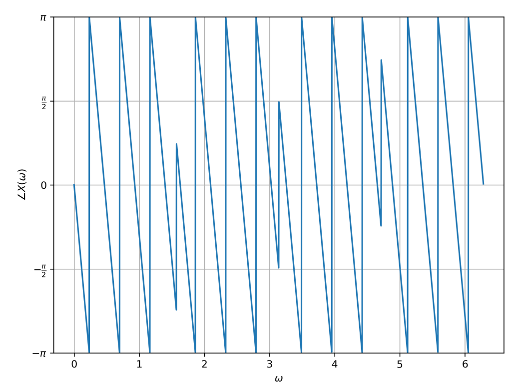
(c) Magnitude of DTFS coefficients $|c_k|$



(d) Magnitude spectrum $|X(\omega)|$



(e) Phase of DTFS coefficients $\angle c_k$



(f) Phase spectrum $\angle X(\omega)$

Figure 1: Depiction of the DTFS and DTFT of a rectangular pulse signal with $N_0 = 12$ and $L = 4$. The left column shows the periodic rectangular pulse and corresponding DTFS coefficients. We include c_0 and c_{12} to demonstrate the N_0 periodicity of the DTFS. The right column illustrates the aperiodic pulse signal and corresponding DTFT. Note how the DTFS "samples" the DTFT spectra with the scaling difference of $1/N_0$ for the magnitude coefficients.