University of Illinois **Fall 2023**

ECE 313: Problem Set 3: Solutions

Due: Friday, September 15 at 07:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.4 - 2.7

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings.
5 points will be deducted for a submission with incorrectly-assigned page numbers.

1. [**Karnaugh Puzzle**]

   (a) First note that $P(A) = P(AB) + P(AB^c) = 0.5$. Then by independence of A and B, we can conclude that $P(B) = P(AB)/P(A) = 0.5$. Furthermore, $P(A^c) = 1 - P(A) = 0.5$. Similarly, $P(B^c) = 0.5$. Also, by the independence of A and B, $P(A^cB) = P(A^c)P(B) = 0.25$ and $P(A^cB^c) = P(A^c)P(B^c) = 0.25$. So the Karnaugh map is as shown in the next page.

	A	A^c
B	$P(AB) = 0.25$	$P(A^cB) = 0.25$
B^c	$P(AB^c) = 0.25$	$P(A^cB^c) = 0.25$

Figure 1: Karnaugh map for Problem 6 (a).

   (b) First note that since $A^cC = \emptyset$, $P(A^cBC) = P(A^cB^cC) = 0$. This will allow us to fill in the bottom row of the Karnaugh map shown below, since

$$P(A^cBC^c) = P(A^cB) - P(A^cBC) = P(A^cB) = 0.25.$$

and similarly $P(A^cB^cC^c) = 0.25$. Now

$$P(ABC^c) = P(AB) - P(ABC) = 0.25 - 0.1 = 0.15.$$

Also, we get:

$$\begin{aligned} P(AB^cC) =& P(C) - P(A^cC) - P(ABC) \\ =& P(C) - (P(A^cBC) + P(A^cB^cC)) - P(ABC) \\ =& 0.2 - 0.1 = 0.1. \end{aligned}$$

Therefore $P(AB^cC^c) = P(AB^c) - P(AB^cC) = 0.25 - 0.1 = 0.15$. This completes the top row of the Karnaugh map below.

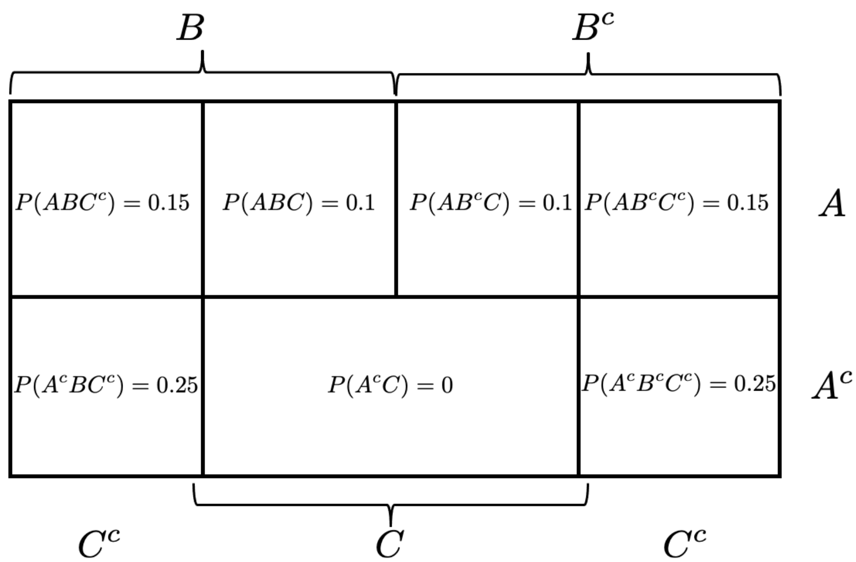

Figure 2: Karnaugh map for Problem 6 (b).

(c) The easiest way is to use De Morgan's Law and the Karnaugh map from part (b) as follows:

$$P(A \cup B \cup C) = 1 - P(A^cB^cC^c) = 1 - 0.25 = 0.75.$$

The "brute-force" way is to use Property P.9 on page 9 of your class notes, and the Karnaugh map as follows:

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC) \\ &= 0.5 + 0.5 + 0.2 - 0.25 - 0.1 - 0.2 + 0.1 = 0.75. \end{aligned}$$

Alternatively, since A^c and C are mutually exclusive, it follows that $C \subset A$. Therefore, $P(A \cup B \cup C) = P(A \cup B) = P(A) + P(B) - P(AB) = 0.75$.

2. **[Triple your money in five weeks?]**

(a) Notice first that we are interested in the value in dollars of your investment at the end of the fifth week and the initial investment was 32\$. Let k be the number of weeks that the value is doubled and $(5-k)$ be the number of weeks that the value decreases by half. Then the final value is

$$X = 32 \cdot 2^k \cdot \frac{1}{2}^{(5-k)} = 2^{2k},$$

where $k \in \{0, 1, 2, 3, 4, 5\}$. Hence one can readily see that the possible values of X are $\{1, 4, 16, 64, 256, 1024\}$.

(b) Since both doubling probability and halving probability of the money for each week are 0.5, we see that the pmf of X follows Binomial distribution, i.e.,

$$\Pr(X = 2^{2k}) = \binom{5}{k} \cdot \left(\frac{1}{2}\right)^5, \quad k \in \{0, 1, 2, 3, 4, 5\}$$

(c)

$$\begin{aligned}
\mathbb{E}[X] :=& \sum_x \Pr(X = x) \\
=& 1 \cdot \frac{1}{32} + 4 \cdot \frac{5}{32} + 16 \cdot \frac{10}{32} + 64 \cdot \frac{10}{32} + 256 \cdot \frac{5}{32} + 1024 \cdot \frac{1}{32} \\
=& \frac{1}{32}(1 + 20 + 160 + 640 + 1280 + 1024) \\
=& \frac{3125}{32} \approx 97.656 \\
>& 32 \times 3 = 96,
\end{aligned}$$

where the last inequality implies that the brochure is accurate (as claimed in the problem statement).

(d) The probability that we will lose money is nothing but $\Pr(X < 32)$, which can be obtained by the following decomposition:

$$\begin{aligned}
\Pr(X < 32) =& \Pr(X = 1) + \Pr(X = 4) + \Pr(X = 16) \\
=& \frac{1}{32} + \frac{5}{32} + \frac{10}{32} \\
=& \frac{16}{32} = \frac{1}{2}.
\end{aligned}$$

3. **[Geometric Random Variable]**

(a)

$$p_Y(k) = p(1-p)^{k-1} \text{ for } k \geq 1.$$

(b)

$$E[Y] = \frac{1}{p} = \begin{cases} 1.0526 & \text{A student} \\ 6.6667 & \text{C student.} \end{cases}$$

(c) P(an A student does not get an invitation in 5 trials)

$$\begin{aligned}
=& \sum_{k=6}^{\infty} p(1-p)^{k-1} \\
=& (1-p)^5 \sum_{k'=0}^{\infty} p(1-p)^{k'} \\
=& (1-p)^5 = (1 - 0.95)^5 = 3.125 \times 10^{-7}.
\end{aligned}$$

It can also be observed directly that P(an A student does not get an invitation in 5 trials) $= (1-p)^5$ by the independence of each trial.
P(C gets an invitation in 5 trials) =

$$1 - (1-p)^5 = 1 - (1 - 0.15)^5 = 0.5563.$$

(d) The A student, by the memoryless property of geometric random variable, the booths the students have already visited and the interviews they have already received has no influence on the future booth visits.

4. **[Rolling dice]**

(a) First observe that $\Pr(X = 0) = \Pr(X = 1) = \Pr(X = 2) = 0$. And for $X > 3$, we see that X follows Negative binomial distribution with probability $1/6$ (since the event of interest is getting 4, getting other numbers can be thought of as one case).

$$\Pr(X = k) = \begin{cases} \binom{k-1}{2}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^{k-3}, & k \geq 3; \\ 0, & k = \{0, 1, 2\}. \end{cases}$$

Referring to Section 2.6 in the course note, we see that

$$\mathbb{E}[X] = \frac{3}{1/6} = 18,$$
$$\mathrm{Var}(X) = \frac{3 \cdot 5/6}{(1/6)^2} = 90.$$

(b) With a similar argument (except the probability of interest becomes $1/3$, instead of $1/6$), we get:

$$\Pr(X = k) = \begin{cases} \binom{k-1}{3}\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^{k-4}, & k \geq 4; \\ 0, & k = \{0, 1, 2, 3\}. \end{cases}$$

As described in Section 2.6 in the course note, we see that

$$\mathbb{E}[X] = \frac{4}{1/3} = 12,$$
$$\mathrm{Var}(X) = \frac{4 \cdot 2/3}{(1/3)^2} = 24.$$

5. **[Customer support center]**

(a) As described in the course note, the event of interest follows Poisson distribution with mean $\lambda \times 3 = 4 \times 3 = 12$, and thus we get:

$$\Pr(X = 2) = \frac{12^2}{2} e^{-12} \approx 0.00044.$$

(b)

$$\begin{aligned} \Pr(X \geq 3) =& 1 - (\Pr(X = 0) + \Pr(X = 1) + \Pr(X = 2)) \\ =& 1 - \left(e^{-4} + 4e^{-4} + \frac{4^2}{2} e^{-4}\right) \\ =& 1 - 13e^{-4} \approx 0.762. \end{aligned}$$

6. **[Overbooked flights]**

(a) Let X be the number of passengers showed up at the gate.

$$\begin{aligned}\Pr(X \leq 100) =& 1 - \sum_{k=101}^{105} \Pr(X = k) \\ =& 1 - \sum_{k=101}^{105} \binom{105}{k} \cdot 0.8^k \cdot 0.2^{105-k} \\ \approx& 0.9999985.\end{aligned}$$

(b)

$$\mathbb{E}[X] = 105 \times 0.8 = 84.$$

(c) According to the problem statement, we saw that the number of show-ups can be modeled as a binomial random variable X with $n = 105$ and the showing-up probability 0.8. Now the key observation is that

$$\begin{aligned}\Pr(Y = k) =& \Pr(X = 105 - k) \\ =& \binom{105}{105-k} (0.8)^{105-k} (0.2)^k \\ =& \binom{105}{k} (0.2)^k (0.8)^{105-k},\end{aligned}$$

where the last equality holds since $\binom{105}{k} = \binom{105}{105-k}$. Hence we see that Y follows Binomial distribution with $n = 105$ and the no-show-up probability $p = 0.2$.

(d) If we use Poisson approximation with $\lambda \approx np = 105 \cdot 0.2 = 21$, we get:

$$\begin{aligned}\Pr(Y \geq 5) =& 1 - \sum_{k=0}^{4} \Pr(Y = k) \\ =& 1 - e^{-21} \left(\frac{21^0}{0!} + \frac{21^1}{1!} + \frac{21^2}{2!} + \frac{21^3}{3!} + \frac{21^4}{4!} \right) \\ \approx& 0.9999925.\end{aligned}$$

Notice that the final answer we get here is similar to the one we got in (a).