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section:

PROBLEM SET #13

[1]. fair die n times, $S_n = X_1 + \cdots + X_n$

(a) $P\{0.9\mu_x \le \frac{S_n}{n} \le 1.1\mu_x\} \ge 0.92$,

$E[\frac{S_n}{n}] = E[X_1] = \frac{7}{2}$, $Var[\frac{S_n}{n}] = Var(X_1) = \frac{35}{12}$

$P\{|\frac{S_n}{n} - \mu| \ge \delta\} \le \frac{\sigma^2}{n \cdot \delta^2}$,

$P\{|\frac{S_n}{n} - \mu| \le 0.1\mu\} \le 0.08$, $\delta = 0.1\mu = \frac{7}{20}$

$\Rightarrow \frac{Var(\frac{S_n}{n})}{n \cdot \delta^2} \le 0.08$, $n \ge \frac{1}{0.08 \cdot (0.1)^2} \cdot \frac{4}{49} \cdot \frac{35}{12} = 297.6$

since n is an integer, $n \ge 298$

(b) $E[X_1] = \frac{7}{2}$. $Var(X_1) = \frac{35}{12}$

$\frac{S_n - nE[X_1]}{\sqrt{n\sigma^2}}$ is Gaussian r.v. consider n rolls,

S_n has mean $\overset{\mu}{\frac{7}{2}n}$. Var $\frac{35}{12} \cdot n$

so $P\{|S_n - \mu| \le \frac{7}{20}n\} = P\{|\frac{S_n - \mu}{\sqrt{\frac{35}{12}n}}| \le \frac{\frac{7}{20}n}{\sqrt{\frac{35}{12}n}}\} = 1 - 2Q(\frac{\frac{7}{20}n}{\sqrt{\frac{35}{12}n}}) = 2\Phi(\frac{\frac{7}{20}n}{\sqrt{\frac{35}{12}n}}) - 1 \ge 0.92$

$\therefore \Phi(\frac{\frac{7}{20}n}{\sqrt{\frac{35}{12}n}}) \ge 0.96$, $\frac{\frac{7}{20}n}{\sqrt{\frac{35}{12}n}} \ge 1.75$, $n > 72.9 \Rightarrow n \ge 73$

[2].

(a) $\int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} f_{X,Y}(u,v) dv = \int_0^{\frac{\pi}{4}} d\theta \cdot \int_0^{10} \frac{k \cdot r^2 \sin\theta\cos\theta}{10000} \cdot r dr$

$= \frac{k}{20000} \cdot \int_0^{\frac{\pi}{4}} \int_0^{10} r^3 \sin 2\theta \, dr \, d\theta$

$= \frac{k}{20000} \cdot [\frac{1}{4} r^4]_0^{10} \cdot [-\frac{1}{2}\cos 2\theta]_0^{\frac{\pi}{4}}$

$= \frac{k}{20000} \cdot \frac{1}{4} \cdot 10^4$

$= 1$

$\Rightarrow k = 8$

(b) $f_Y(v) = \int_0^{\sqrt{100-v^2}} \frac{8v}{10000} \cdot u \, du = \frac{4v}{10000} \cdot [u^2]_0^{\sqrt{100-v^2}}$

$= \frac{v}{2500} \cdot (100 - v^2)$

$E[Y] = \int_0^{10} \frac{v^2}{2500} \cdot (100 - v^2) dv = \int_0^{10} (\frac{v^2}{25} - \frac{v^4}{2500}) dv$

$= [\frac{1}{75} v^3 - \frac{1}{12500} v^5]_0^{10}$

$= \frac{40}{3} - 8$

$= \frac{16}{3}$

$E[Y^2] = \int_0^{10} \frac{v^3}{2500} (100 - v^2) dv = \frac{100}{3}$

MSE: $Var(Y) = E[Y^2] - E[Y]^2 = \frac{44}{9}$

(c) $g^*(x) = \int_{-\infty}^{+\infty} v \cdot f_{Y|X}(v|u) dv$,

$f_{Y|X}(v|u_0) = \frac{f_{X,Y}(u_0, v)}{f_X(u_0)} = \frac{\frac{8 \cdot u_0 \cdot v}{10000}}{\frac{u_0}{2500}(100 - u_0^2)} = \frac{2v}{100 - u_0^2}$

$\therefore g^*(x) = E[Y | X = u_0] = \int_0^{\sqrt{100-u_0^2}} \frac{2v^2}{100 - u_0^2} dv$

$= \frac{2}{100 - u_0^2} \cdot [\frac{1}{3} v^3]_0^{\sqrt{100-u_0^2}}$

$= \frac{2}{3}\sqrt{100 - u_0^2} = \frac{2}{3}\sqrt{100 - x^2}$

$E[E[Y|X]^2] = E[\frac{4}{9}(100 - x^2)] = \frac{4}{9}(100 - \frac{100}{3}) = \frac{800}{27}$

MSE: $E[Y^2] - E[E[Y|X]^2] = \frac{100}{3} - \frac{800}{27} = \frac{100}{27}$

(d) $Cov(X,Y) = E[XY] - E[X]E[Y] = \frac{25}{3}\pi - \frac{256}{9}$

$L^*(x) = ax + b$, $a = \frac{Cov(Y,X)}{Var X} = \frac{\frac{25}{3}\pi - \frac{256}{9}}{\frac{44}{9}} = \frac{75\pi - 256}{44}$

$b = E[Y] - aE[X] = \frac{16}{3} \cdot (1 - \frac{75\pi - 256}{44}) = \frac{16}{3} - \frac{4(75\pi - 256)}{33}$

$\therefore L^*(x) = \frac{16}{3} + \frac{75\pi - 256}{44} \cdot (x - \frac{16}{3})$

$MSE = \sigma_Y^2 - \frac{Cov(Y,X)^2}{Var X} = \frac{44}{9} - (\frac{25}{3}\pi - \frac{256}{9})^2 \cdot \frac{9}{44}$

[3]: $\mu_X = 2,\ \mu_Y = 3 \quad \sigma_X^2 = 16,\ \sigma_Y^2 = 25,\ \rho_{X,Y} = 0.6$

(a) $f_X(u) = \frac{1}{\sqrt{32\pi}} \exp\left(-\frac{(u-2)^2}{32}\right)$

(b) $X=6$: $\hat{E}[Y \mid X=6] = \mu_Y + \left(\frac{Cov(Y,X)}{Var\,X}\right)(6-\mu_X)$

$$= 3 + \left(\frac{0.6 \cdot 4 \cdot 5}{16}\right)(6-2)$$

$$= 6$$

MSE: $Var(Y)(1-\rho^2) = 25(1-0.36) = 16$

$f_{Y|X}(u|6) = \frac{1}{\sqrt{32\pi}} \exp\left(-\frac{(u-6)^2}{32}\right)$

(c) $P(Y \geq 2 \mid X=6) = P\left\{\frac{Y-6}{\sqrt{16}} \geq \frac{2-6}{\sqrt{16}} \mid X=6\right\} = Q(-1) = \Phi(1) \approx 0.8413$

(d) $E[Y^2 \mid X=6] = Var(Y \mid X=6) + E[Y \mid X=6]^2$

$$= 16 + 36$$

$$= 52$$

[4]. Poisson process, $\lambda > 0$ $\quad E = \lambda \cdot t$, $Var = \lambda t$

variable $N \Rightarrow Y, N$ independent

(a) given N_a and MSE. $a > b$. get LMMSE: consider $N_b = Y$, $N_a = X$, $X-Y$ follows Poission process, distribution: $Poi(\lambda(a-b))$

estimator: $L^*(X) = mX + n$, $m = \frac{Cov(Y,X)}{Var\,X}$,

$Cov(Y,X) = Cov(Y, Y+N) = Var(Y)$, $m = \frac{Var\,Y}{Var\,X} = \frac{\lambda b}{\lambda a} = \frac{b}{a}$

$\therefore L^*(X) = \lambda b + \frac{b}{a}(X - \lambda a)$, $X = k \Rightarrow L^*(k) = \frac{b}{a}k$

MSE $= \sigma_Y^2 - \frac{Cov(Y,X)^2}{Var(X)} = \lambda b - \frac{\lambda^2 b^2}{\lambda a} = \lambda b \cdot \left(\frac{a-b}{a}\right)$

(b) consider $N_b = Y$, $N_a = X$, $N = Y - X$, N, X independent

estimator: $L^*(X) = mX + n$, $m = \frac{Cov(Y,X)}{Var\,X}$.

$Cov(Y,X) = Cov(X+N, X) = Var\,X$, $m = \frac{Var\,X}{Var\,X} = 1$

$\therefore L^*(X) = \lambda b + x - \lambda a \Rightarrow L^*(k) = \lambda(b-a) + k$

MSE $= \sigma_Y^2 - \frac{Cov(Y,X)^2}{Var\,X} = Var\,Y - Var\,X = \lambda(b-a)$