ECE 310 Fall 2023

Lecture 17

Magnitude and phase response

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Learning Objectives

After this lecture, you should be able to:

- Compute and plot the magnitude response of an LTI system.
- Compute and plot the phase response of an LTI system.
- Define and explain the significance of group delay using the phase response.

Recap from previous lecture

The previous lecture introduced us to the frequency response of LTI systems and how to apply the DTFT to compute system outputs. In this lecture, we will further explore frequency response by carefully looking at the magnitude response and phase response.

1 Complex numbers review

Before discussing the magnitude and phase response of LTI systems, it will be helpful to review important complex numbers preliminaries. We can represent a complex number $z \in \mathbb{C}$ in *rectangular form*:

$$z = a + jb, \tag{1}$$

or *polar form*:

$$z = Re^{j\theta}. \tag{2}$$

Looking at polar form, R denotes the magnitude of z and θ gives us the phase of z. We can calculate the magnitude and phase of a complex number in a couple equivalent ways:

$$|z| = R = \sqrt{a^2 + b^2} \tag{3}$$

$$|z| = R = \sqrt{zz^*} \tag{4}$$

$$\angle z = \theta = \tan^{-1}\left(\frac{\text{Im}\{z\}}{\text{Re}\{z\}}\right) \tag{5}$$

Above, $z^* = a - jb$ is the complex conjugate of z. By convention, we represent phase by its *principle angle* that lies in $[-\pi, \pi]$. Thus we modulate any $|\theta| > \pi$ to lie in this range. It will also be important to keep in mind that

$$-1 = e^{\pm j\pi}. \tag{6}$$

For example, let

$$z = -2e^{j\frac{\pi}{4}}. \tag{7}$$

The proper polar representation of z will replace the negative sign with $e^{-j\pi}$:

$$z = 2e^{-j\frac{3\pi}{4}}. \tag{8}$$

We choose $-\pi$ in this case to ensure the phase is in the $[-\pi, \pi]$ range.

2 Magnitude and phase response

2.1 Magnitude response

Let $H(\omega)$ denote the frequency response of an LTI system. We define the *magnitude response* as the magnitude of the frequency response:

$$|H(\omega)| = \sqrt{H(\omega)H^*(\omega)}. \tag{9}$$

When plotting the magnitude response of an LTI system, we may do so on a linear or *decibel* (dB) scale. A linear scale simply takes the values of $|H(\omega)|$ as is. In other words, the y-axis scales evenly or linearly. A decibel scale is logarithmic, i.e. non-linear, and we express the magnitude response on a decibel scale as

$$|H(\omega)|_{\text{db}} = 20\log_{10}|H(\omega)|. \tag{10}$$

We commonly use decibel scaling when the magnitude response of a system covers multiple orders of magnitude. For example, we will refer to plots on a decibel scale later in the course when we cover filter design.

2.2 Phase response

We define the *phase response* of an LTI system as the angle or phase of the frequency response:

$$\angle H(\omega) = \tan^{-1}\left(\frac{\text{Im}\{H(\omega)\}}{\text{Re}\{H(\omega)\}}\right). \tag{11}$$

When plotting the phase response, it is convention to keep the y-axis range between $[-\pi, \pi]$ in accordance with our principle angle values. Thus, when a phase response goes above π or below $-\pi$ we will modulate the plot by 2π to remain within this $[-\pi, \pi]$ range. Altogether, we can express the frequency response using the magnitude and phase responses as

$$H(\omega) = |H(\omega)|e^{j\angle H(\omega)}. \tag{12}$$

Let's take a look at a couple examples to practice working with magnitude and phase responses.

Exercise 1: Compute the frequency response, magnitude response, and phase response of the following LTI system with impulse response

$$h[n] = \delta[n] - \delta[n-2]. \tag{13}$$

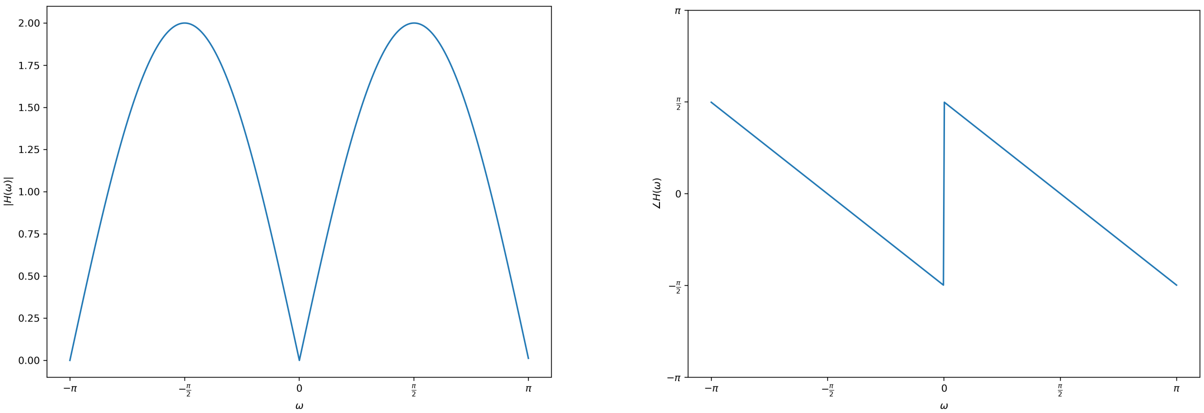
We begin by finding the frequency response $H(\omega)$:

$$\begin{aligned}
H(\omega) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} && (14)\\
&= 1 - e^{-j2\omega} && (15)\\
&= e^{-j\omega}\left(e^{j\omega} - e^{-j\omega}\right) && (16)\\
&= 2je^{-j\omega}\sin(\omega) && (17)\\
&= 2e^{j\left(\frac{\pi}{2}-\omega\right)}\sin(\omega). && (18)
\end{aligned}$$

We use a common trick in line 16 by removing a complex exponential from two terms in order to invoke one of Euler's identities. Of course, the form of $H(\omega)$ shown in line 15 is a valid expression; however, the final result we show in line 18 makes it much easier for us to find the magnitude and phase responses.

$$\begin{aligned}
|H(\omega)| &= 2\,|\sin(\omega)| && (19)\\
\angle H(\omega) &= \frac{\pi}{2} - \omega + \angle\sin(\omega). && (20)
\end{aligned}$$

Above, we use $\angle\sin(\omega)$ to account for $\pm\pi$ phase jumps that can occur due to sin becoming negative. Figure 1 plots these magnitude and phase responses. Note the π phase jump at $\omega = 0$ since sine is negative for $\omega \in (-\pi, 0)$.



(a) $|H(\omega)|$ (b) $\angle H(\omega)$

Figure 1: Magnitude and phase responses of $H(\omega)$ for Exercise 1.

We are able to neatly separate the magnitude and phase responses from the full frequency response in Exercise 1. Next, we will try one more example where we have to rely more on the definitions of magnitude and phase response and the result is less neat.

Exercise 2: Compute the frequency response, magnitude response, and phase response of the following LTI system with impulse response

$$h[n] = 3\delta[n] + 2\delta[n-1] + \delta[n-2]. \tag{21}$$

First, we compute the frequency response $H(\omega)$:

$$\begin{aligned} H(\omega) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \\ &= 3 + 2e^{-j\omega} + e^{-j2\omega}. \end{aligned} \tag{22}$$

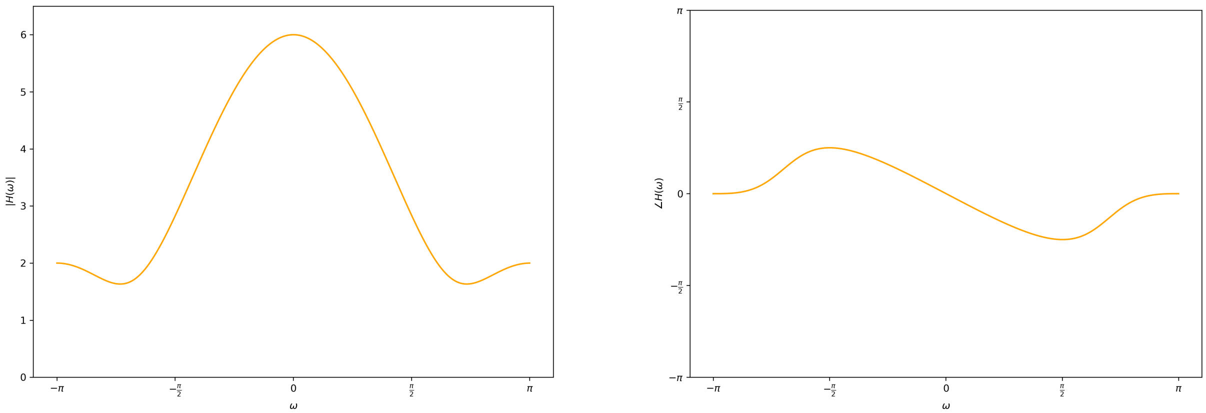
Next, using Euler's identity of

$$e^{j\theta} = \cos(\theta) + j\sin(\theta) \tag{23}$$

we can compute the magnitude and phase responses:

$$|H(\omega)| = \sqrt{H(\omega)H^*(\omega)}$$
$$= \sqrt{\left(3 + 2^{-j\omega} + e^{-j2\omega}\right)\left(3 + 2^{j\omega} + e^{j2\omega}\right)} \tag{24}$$
$$= \sqrt{14 + 8e^{j\omega} + 8e^{-j\omega} + 3e^{j2\omega} + 3e^{-j2\omega}} \tag{25}$$
$$= \sqrt{|14 + 16\cos(\omega) + 6\cos(2\omega)|} \tag{26}$$
$$\angle H(\omega) = \tan^{-1}\left(\frac{\text{Im}\{H(\omega)\}}{\text{Re}\{H(\omega)\}}\right)$$
$$= \tan^{-1}\left(\frac{2\sin(-\omega) + \sin(-2\omega)}{3 + 2\cos(-\omega) + \cos(-2\omega)}\right) \tag{27}$$

We place absolute value signs within the square-root in line 26 as a convention to make sure the magnitude response remains real-valued and positive. In this case, the expression inside the square root happens to remain positive without the absolute value. The corresponding plots for the magnitude and phase responses can be seen in Fig. 2. Note how the phase response has no $\pm\pi$ jumps since the expression inside the absolute value signs never goes negative.



(a) $|H(\omega)|$

(b) $\angle H(\omega)$

Figure 2: Magnitude and phase responses of $H(\omega)$ for Exercise 2.

2.2.1 Group delay

Consider the delay or time-shifting property of the DTFT:

$$x[n-k] \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)e^{-j\omega k}. \tag{28}$$

One way of describing the delay property is that a shift in the time-domain causes the addition of a linear phase in the frequency-domain. In Eqn. 28, the linear phase can be written as the function

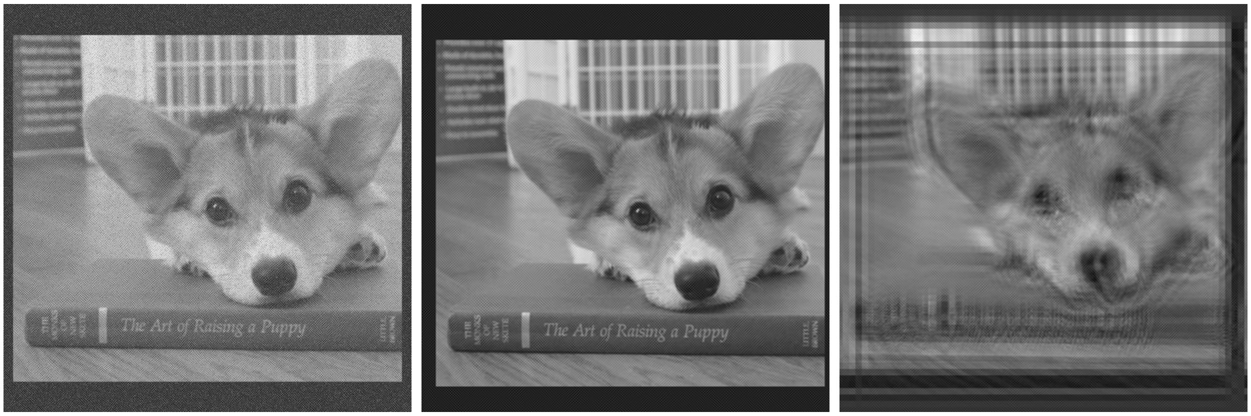
$$f(\omega) = -k\omega. \tag{29}$$

Thus, the shift amount k determines the slope of this linear phase. This is the intuition for the *group delay* τ_{gd} of the frequency response of an LTI system:

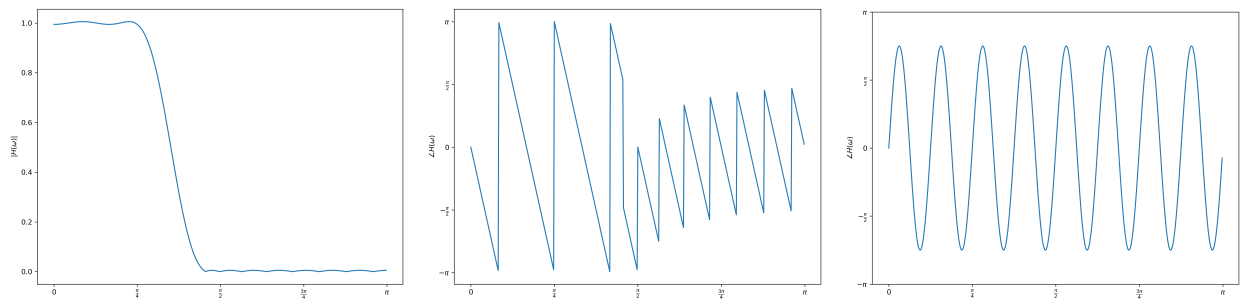
$$\tau_{\text{gd}}(\omega) = -\frac{d\angle H(\omega)}{d\omega}. \tag{30}$$

The phase responses we obtained earlier in Exercises 1 and 2 yield very different group delays. For clarity, let $\angle H_1(\omega)$ and $\angle H_2(\omega)$ be the phase responses for the systems in Exercises 1 and 2, respectively. First, $\angle H_1(\omega)$ is not differentiable since we have a $\pm\pi$ jump in phase at $\omega = 0$. However, $\angle H_1(\omega)$ is still differentiable everywhere except $\omega = 0$. The derivative $d\angle H_1(\omega)/d\omega$ is then -1 almost everywhere with a finite number of discontinuities along the continuous domain $[-\pi, \pi]$. Thus, the group delay of $H_1(\omega)$ will be a constant function of ω, $\tau_{\text{gd}}(\omega) = 1$, and all frequencies present in a signal passed through this system will shift uniformly by one sample. The phase response of $H_2(\omega)$ is differentiable everywhere, but is clearly non-linear. Thus, the group delay for $H_2(\omega)$ will be a non-constant function of ω. We say that $H_1(\omega)$ has *uniform group delay* while $H_2(\omega)$ has *non-uniform group delay.* The consequence of non-uniform group delay is that different frequencies will be shifted by different amounts in the time-domain after passing through such an LTI system.

The concept of group delay can be challenging to understand and visualize. Figure 3 depicts the effects of uniform and non-uniform group delays. Here, we have an image that is corrupted by some amount of noise. We apply two different LTI systems in order to denoise this image. The systems share a common magnitude response (Fig. 3d) but have different phase responses. The first system has uniform group delay (Fig. 3e) while the second system has a non-uniform group delay arising from a sinusoidal phase response (Fig. 3f). Both systems are calibrated to appropriately de-noise the image due to their shared magnitude responses; however, their outputs are remarkably different! The uniform group delay output is a clean image that has been slightly shifted in the x and y-dimensions due to the uniform group delay of the corresponding LTI system. The non-uniform group delay output, on the other hand, contains strong artifacts. These shaking or ringing effects are caused by the non-uniform group delay of the system which acted on it. The



(a) Noisy image (b) Uniform group delay output (c) Non-uniform group delay output



(d) Shared magnitude response (e) Uniform group delay (f) Non-uniform group delay

Figure 3: Effects of uniform vs. non-uniform group delay in image de-noising. The noisy image (a) is acted on by two LTI systems with shared magnitude response (d). The result in (b) corresponds to the system with uniform group delay in its phase response (e). The result in (c) corresponds to the system with non-uniform group delay in its phase response (f).

different frequencies of the image are shifted by different amounts after passing through its LTI system and thus the image lacks the coherence it originally had. Since we are working with an image, these are spatial frequencies, i.e. cycles per pixel. We will return to group delay in future lectures when we discuss practical digital filter design methods.