ECE 310 Fall 2023

Lecture 11

BIBO stability and causality

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Learning Objectives

After this lecture, you should be able to:

- Explain the necessary and sufficient conditions for an LTI system to be BIBO stable using: the definition of BIBO stability, impulse response, or z-transform.
- Define what makes a causal or non-causal LTI system stable.
- Identify bounded inputs that cause unstable systems to yield unbounded outputs.

Recap from previous lecture

We completed our discussion of transfer functions and LTI system response using the z-transform in the last lecture by covering improper rational functions and LTI system algebra. This lecture will introduce our final method for assessing if an LTI system is BIBO stable. We will demonstrate the necessary and sufficient conditions for a transfer function to represent a BIBO stable system and how to easily check this for causal and non-causal systems.

1 BIBO stability in the z-domain

Thus far, we have stated two conditions for a discrete-time system to be bounded-input bounded-output (BIBO) stable. The first condition says that a system T is BIBO stable if and only if

$$|T(x[n])| < \alpha,\ \forall\ n,\ \text{for any } |x[n]| < \beta,\ \forall\ n,\ \text{where } \alpha, \beta < \infty. \tag{1}$$

This condition simply states the definition of BIBO stability: the output must be finite (bounded) for all values of n for any input signal that is finite-valued (bounded) for all values of n. Note that this condition applies to *any* discrete-time system.

If we know that our system T is LTI, we have a second condition for BIBO stability using its impulse response $h[n]$. For this condition, T is stable if and only if

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{2}$$

We can derive our final condition using Eqn. 2 and the definition of the z-transform. Let $H(z)$ be the z-transform of an LTI system with an ROC that contains the unit-circle $|z| = 1$:

$$|H(z)| = \left| \sum_{n=-\infty}^{\infty} h[n] z^{-n} \right| \tag{3}$$

$$\leq \sum_{n=-\infty}^{\infty} \left|h[n]z^{-n}\right| \tag{4}$$

Above, line 4 follows from triangle inequality. Next, we evaluate at $|z| = 1$. Note that $|z| = e^{j\omega}$ represents any arbitrary point on the unit circle for some choice of ω.

$$\begin{aligned}
\left(\sum_{n=-\infty}^{\infty} \left|h[n]z^{-n}\right|\right)\Bigg|_{|z|=1} &= \sum_{n=-\infty}^{\infty} \left|h[n]e^{-j\omega n}\right| && (5)\\
&= \sum_{n=-\infty}^{\infty} \sqrt{\left(h[n]e^{-j\omega n}\right)\left(h^*[n]e^{j\omega n}\right)} && (6)\\
&= \sum_{n=-\infty}^{\infty} |h[n]| && (7)\\
&< \infty. && (8)
\end{aligned}$$

In the final line, we invoke our initial assumption $|z| = 1$ (the unit-circle in the z-domain) belongs to our ROC.

Recall that the ROC of a z-transform is defined as the values of z for which our z-transform is finite. Thus, line 7 tells us that if an LTI system has a transfer function with an ROC containing $|z| = 1$ then its impulse response is absolutely summable and thus the system is BIBO stable (Eqn. 2). In fact, we can go one step further and say an LTI system with impulse response $h[n]$ is BIBO stable if and only if the ROC of $H(z)$ contains the *unit-circle*, i.e. $|z| = 1$. This is our third necessary and sufficient condition for BIBO stability!

1.1 Identifying causal and non-causal systems

In lecture 4, we stated that an LTI system is causal if

$$h[n] = 0, \; n < 0. \tag{9}$$

Otherwise, our system is non-causal. We now would like to connect our understanding of causality, BIBO stability, and the z-transform.

To do so, we first define a *right-sided* sequence as any signal $x[n]$ where

$$x[n] = 0, \; n < n_0. \tag{10}$$

Similarly, we say a sequence is *left-sided* if

$$x[n] = 0, \; n > n_0. \tag{11}$$

Note that n_0 may be positive or negative for left-sided or right-sided sequences. We also stated in lecture 7 that the ROC of these sequences take the following shapes:

$$\begin{aligned}
\text{ROC } &= |z| > a, \; x[n] \text{ right-sided} && (12)\\
\text{ROC } &= |z| < a, \; x[n] \text{ left-sided.} && (13)
\end{aligned}$$

We can be more specific depending on the sign of n_0 as well. For right-sided signals, if $n_0 < 0$, then the ROC is $a < |z| < \infty$. This is because $n < 0$ gives positive powers in the z-transform which will go to infinity for $|z| = \infty$. Similarly, if $n_0 > 0$ for a left-sided signal, the ROC is $0 < |z| < a$ since $n > 0$ gives negative powers that blow up for $|z| = 0$.

Right-sided systems. From Eqn. 9 and our understanding of right-sided signals, we can say that a given $H(z)$ represents a causal LTI system if and only if its ROC is the exterior of a circle that goes to infinity. If such an ROC does not include infinity, we instead of have a non-causal, right-sided impulse response.

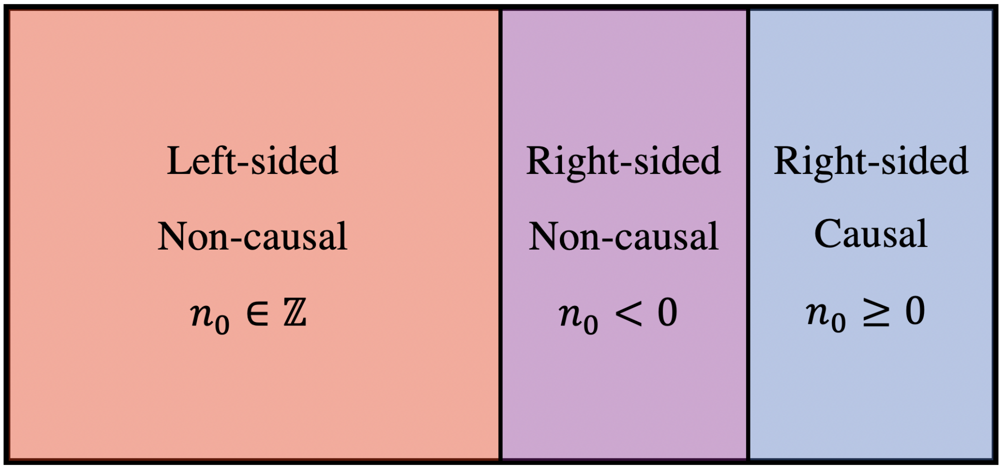


Figure 1: Illustration of when right-sided and left-sided signals are causal or non-causal.

We can also describe the ROC of causal systems simply using their poles. Because the ROC cannot include any values for which $|H(z)|$ goes to infinity and the ROC is the exterior of a circle for causal systems, we have that

$$\text{ROC } = |z| > |p_{\max}| \tag{14}$$

where $p_{\max}$ denotes the largest magnitude pole of our system. This means then that a causal system is BIBO stable if and only if the largest pole of its transfer function lies inside the unit-circle.

Left-sided systems. We can similarly describe the ROC of non-causal, left-sided signals as

$$\text{ROC } = |z| < |p_{\min}|. \tag{15}$$

Here, we acknowledge that the ROC of left-sided signals must be the interior of a circle. Thus, the ROC must be inside the smallest magnitude pole, $p_{\min}$, to avoid including any other poles. Therefore, an LTI system with a left-sided impulse response is BIBO stable if and only if the smallest pole of its transfer function lies outside the unit-circle. Remember that the ROC will also exclude $|z| = 0$ if there are any values of $n > 0$ where the sequence is non-zero. Figure 1 depicts the distinction between right-sided, left-sided, causal, and non-causal sequences.

Two-sided systems. Finally, we can characterize infinite-length, two-sided systems. For such systems, we can write the impulse response $h[n]$ as

$$h[n] = h_l[n] + h_r[n], \tag{16}$$

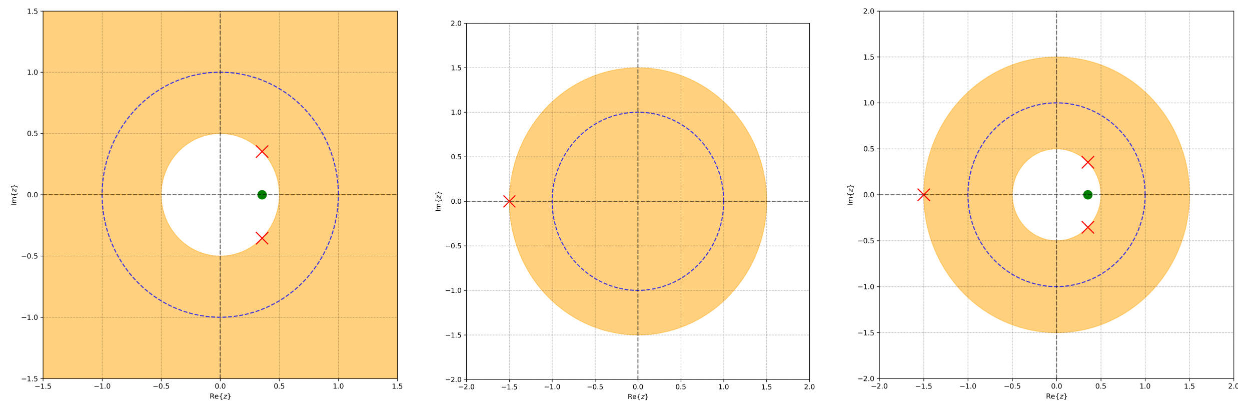
where

$$h_r[n] = \begin{cases} h[n], & n \geq 0 \\ 0, & n < 0 \end{cases} \tag{17}$$

$$h_l[n] = \begin{cases} 0, & n \geq 0 \\ h[n], & n < 0 \end{cases}. \tag{18}$$

We have decomposed our two-sided impulse response into the summation of a right-sided and left-sided component. From our z-transform properties we know that the resulting ROC will be at least the intersection of the respective ROCs of $h_r[n]$ and $h_l[n]$. We know from our previous discussion of right-sided and left-sided signals that this ROC will be

$$\text{ROC} = \text{ at least } (|z| > p_{\max}) \cap (|z| < p_{\min}), \tag{19}$$



(a) $h_r[n] = \left(\frac{1}{2}\right)^n \cos\left(\frac{\pi}{2}\right) u[n]$ (b) $h_l[n] = \left(-\frac{3}{2}\right)^n u[-n-1]$ (c) $h[n] = h_r[n] + h_l[n]$

Figure 2: Regions of convergence for (a) causal, right-sided sequence, (b) left-sided sequence, (c) two-sided sequence. The dashed circle represents the unit-circle.

Table 1: Summary of system properties for different shapes of the ROC of $H(z)$.

ROC shape of $H(z)$	Causal	Right-sided	Left-sided	Condition for stability
$\lvert z\rvert > p_{\max}$	✓	✓	✗	$p_{\max} < 1$
$\lvert z\rvert < p_{\min}$	✗	✗	✓	$p_{\min} > 1$
$p_{\max} < \lvert z\rvert < \infty$	✗	✓	✗	$p_{\max} < 1$
$a < \lvert z\rvert < b$	✗	✓	✓	$p_{\max} = a < 1$ for $h_r[n]$, $p_{\min} = b > 1$ for $h_l[n]$

where $p_{\max}$ and $p_{\min}$ give the largest and smallest poles of $h_r[n]$ and $h_l[n]$, respectively. The ROC is strictly equal to the intersection of these two sets if no zeros from $H_l(z)$ or $H_r(z)$ cancel out $p_{\max}$ or $p_{\min}$. Figure 2 gives an example of the ROC for a two sided sequence we construct from combining a right-sided and left-sided signal. Table 1 summarizes the system properties for different shapes of the ROC of a given transfer function.

1.2 Unstable inputs and marginal stability

For systems that are not BIBO stable, we would like to answer the question: for what bounded inputs does our system yield an unbounded output? The answer to this question is simple for most unstable systems. We can choose $x[n] = \delta[n]$ if our impulse response $h[n]$ represents an unbounded sequence. For example, consider $h[n] = 3^n u[n]$. Since we can decompose any discrete-time signal into a sum of scaled and shifted impulse, we will have infinitely many bounded inputs that give unbounded outputs! The one exception would be if we can cancel the pole of this system. Our corresponding $H(z)$ is given by

$$H(z) = \frac{1}{1 - 3z^{-1}}, \ |z| > 3. \tag{20}$$

We have a pole at $z = 3$, so we need a signal with a zero at $z = 3$. One simple choice is the $X(z)$ which matches the denominator:

$$X(z) = 1 - 3z^{-1} \overset{\mathcal{Z}}{\longleftrightarrow} \delta[n] - 3\delta[n-1] = x[n]. \tag{21}$$

This input signal will cancel the pole in $h[n]$ and simply yield $y[n] = \delta[n]$ as the output. Note that we can choose any scaled or shifted version of the above $x[n]$ and the output will still be bounded.

1.2.1 Marginal stability

Let's now consider a more interesting class of unstable systems. We know that an LTI system is stable if and only if the ROC of its transfer function contains the unit circle. However, what happens if our ROC touches, but does not include the unit-circle? For causal signals, this means the ROC is $|z| > 1$. Such systems are referred to as *marginally stable*. To be clear, these systems are still unstable! However, unlike normal unstable systems, there exist only finitely many bounded inputs that give an unbounded output (up to scaling and shifting transformations).

Without loss of generality, we will consider causal marginally stable systems. Such systems will have their largest pole lying on the unit-circle. Ignoring smaller poles, these systems will contain at least one term of the form

$$e^{j\omega n}u[n] \overset{\mathcal{Z}}{\longleftrightarrow} \frac{1}{1-e^{j\omega}z^{-1}}. \tag{22}$$

Recall that if these poles occur in a conjugate pair, we will see a cosine or sine in the time-domain. This all means marginally stable systems are described by periodic impulse responses in the time-domain.

To identify bounded inputs that yield unbounded outputs, first consider the marginally stable system where $\omega = 0$. This will give us $h[n] = u[n]$. For any bounded exponential input $x[n]$, we will either have the magnitude decreasing or the magnitude constant at one. First, consider the output $y[n] = x[n] * h[n]$ for decreasing exponentials, i.e. $|a| < 1$:

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{\infty} a^k u[k]u[n-k] && (23) \\ &= \sum_{k=0}^{n} a^k. && (24) \end{aligned}$$

This sum converges for all n and all a with magnitude less than one. However, if $a = 1$, our output will be $y[n] = (n+1)u[n]$, which goes to infinity as $n \to \infty$. Thus, we may have an unbounded output if our bounded exponential has magnitude one. Setting the magnitude to one will yield exponential signals of the general form $e^{j\omega n}$ for some real-valued ω.

Let's see in general which bounded inputs will give an unbounded output for our marginally stable system from Eqn. 22. Let ω_1 and ω_2 be the frequencies of our impulse response $h[n]$ and input signal $x[n]$, respectively.

$$\begin{aligned} y[n] &= x[n] * h[n] && (25) \\ &= \sum_{k=-\infty}^{\infty} e^{j\omega_1 k}u[k]e^{j\omega_2(n-k)}u[n-k] && (26) \\ &= e^{j\omega_2 n}\sum_{k=0}^{n} e^{jk(\omega_1-\omega_2)} && (27) \\ & && (28) \end{aligned}$$

The above result gives us the original input signal $x[n] = e^{j\omega_2 n}$ scaled by a summation of complex exponentials for all n. This yields a bounded output for all ω_1 and ω_2 except if $\omega_1 = \omega_2$. In this case, our output will be

$$y[n] = e^{j\omega_2 n}(n+1), \tag{29}$$

which is a complex exponential with linearly increasing magnitude. Thus, it is an unbounded output! The same result can be proven for cosine and sine since they are the sum of complex exponentials by Euler's identities.

We have arrived at an interesting result. Our marginally stable systems are only unstable to periodic inputs that oscillate at the same frequency. In this sense, we can say that such inputs *resonate* with our LTI system to create an unbounded output. Looking to the z-domain, our marginally stable systems are unstable when an input signal matches at least one of its poles on the unit-circle. We say then that the input signal creates a *second-order pole* or *double pole*. This happens if the input signal has a frequency matching the angle of at least one such pole on the unit circle in the complex z-domain.

Lecture exercise: Demonstrate how double poles on the unit circle translate to unbounded outputs for marginally stable systems.