ECE 313: Problem Set 2: Solutions

1. **[Gambling with Dice]**

   (a) Since the die is fair,

   $$p_X(12) = P\{1\} = 1/6, \quad (1)$$
   $$p_X(6) = P\{2,3\} = 1/3, \quad (2)$$
   $$p_X(-m) = P\{4,5,6\} = 1/2. \quad (3)$$

   (b) From the pmf we found in part (a),

   $$E[X] = \frac{12}{6} + \frac{6}{3} - \frac{m}{2} = 4 - \frac{m}{2}.$$

   Setting

   $$4 - \frac{m}{2} = -1 \quad \Longrightarrow \quad m = 10.$$

   (c) By LOTUS,

   $$E[X^2] = 12^2 \times \frac{1}{6} + 6^2 \times \frac{1}{3} + (-10)^2 \times \frac{1}{2} = 24 + 12 + 50 = 86,$$

   and therefore

   $$\mathrm{Var}(X) = E[X^2] - (E[X])^2 = 86 - (-1)^2 = 85.$$

2. **[Illini T-Shirts]**

   (a) The possible values of X are 0,1, or 2. The event $\{X = 0\}$ corresponds to both t-shirts being orange. Therefore

   $$p_X(0) = \frac{\binom{4}{2}}{\binom{10}{2}} = \frac{4 \cdot 3}{10 \cdot 9} = \frac{12}{90} = \frac{2}{15}.$$

   The event $\{X = 2\}$ corresponds to both t-shirts being blue. Therefore

   $$p_X(2) = \frac{\binom{6}{2}}{\binom{10}{2}} = \frac{6 \cdot 5}{10 \cdot 9} = \frac{30}{90} = \frac{5}{15} = \frac{1}{3}.$$

   And

   $$p_X(1) = 1 - p_X(0) - p_X(2) = \frac{8}{15}.$$

   Alternatively, the event $\{X = 1\}$ corresponds to one of the t-shirts being orange and the other blue. Therefore

   $$p_X(1) = \frac{\binom{4}{1}\binom{6}{1}}{\binom{10}{2}} = \frac{4 \cdot 6 \cdot 2}{10 \cdot 9} = \frac{48}{90} = \frac{8}{15}.$$

   (b) By LOTUS,

   $$E[(X+1)(X+2)] = 2p_X(0) + 6p_X(1) + 12p_X(2) = \frac{4 + 48 + 60}{15} = \frac{112}{15}.$$

3. **[Matching cards to boxes]**

   (a) Here is one possibility. We can take

   $$\Omega = \{y_1y_2y_3 : y_1y_2y_3 \text{ is a permutation of } 123\},$$

   where y_i represents the number on the card placed back in box i, for $1, 2, 3$. For example, 312 indicates that box 1 gets number 3, box 2 gets number 1, and box 3 gets number 2. There are $3! = 6$ elements in Ω, i.e., $|\Omega| = 6$.

   (b) The possible values of X are 0,1, and 3; notice that X cannot take the value 2 since if two of the boxes get back their numbers, so will the third one.
   There is only one outcome that contributes to $\{X = 3\}$, which is 123, so

   $$p_X(3) = \frac{1}{6}.$$

   The outcomes in $\{X = 1\}$ correspond to only one box getting back its number. For each such box i, there is only one outcome such that the other two boxes do not have their numbers. Therefore

   $$\{X = 1\} = \{132, 321, 213\} \implies p_X(1) = \frac{3}{6} = \frac{1}{2}.$$

   Now we can use the fact that the elements of the pmf sum up to one to conclude that

   $$p_X(0) = 1 - \frac{1}{6} - \frac{1}{2} = \frac{1}{3}.$$

   Alternatively, the outcomes in $\{X = 0\}$ correspond to no box getting back its number. Therefore there are 2 choices for y_1, i.e., 2 and 3, and for each of those choices, there is only one choice for of numbers for remaining two boxes. Therefore, $\{X = 0\} = \{231, 312\}$, and $p_X(0) = 1/3$.

   (c) $E[X] = p_X(1) + 3p_X(3) = 1/2 + 1/2 = 1$.

   (d) By the definition of variance and part (c),

   $$\text{Var}(X) = E[(X - 1)^2] = 1^2 p_X(0) + 2^2 p_X(3) = \frac{1}{3} + \frac{4}{6} = 1.$$

   Alternatively, we could first find $E[X^2] = 1^2 p_X(1) + 3^2 p_X(3) = 1/2 + 9/6 = 2$, and then $\text{Var}(X) = E[X^2] - E[X]^2 = 2 - 1 = 1$.

4. **[To diet or not to diet]**

   (a) Define the event H_i = "i-th toss is heads", and T_i = "i-th toss is tails", $i = 1, 2, 3, 4$, and let E = "Cookie Monster eats the cookie".

   $$P(E) = P(H_1) + P(T_1H_2H_3H_4) = \frac{1}{2} + \frac{1}{16} = \frac{9}{16}$$

   because T_1, H_2, H_3, H_4 are mutually independent and each of these events has probability $1/2$.

   (b) Note that if the first toss is heads, then the Cookie Monster eats the cookie, i.e., $H_1 \subset E$. Therefore, $H_1 = H_1E$, and $P(H_1E) = P(H_1) = 1/2$, and

   $$P(H_1|E) = \frac{P(H_1E)}{P(E)} = \frac{P(H_1)}{P(E)} = \frac{1}{2}\frac{16}{9} = \frac{8}{9}.$$

5. **[Faulty solar cells]**

(a) There are $\binom{n}{2}$ possibilities for which two cells fail, and $n-1$ possibilities for two neighboring cells to fail: 1 and 2, 2 and 3, $\ldots$, $n-1$ and n. So the probability is

$$\frac{n-1}{\binom{n}{2}} = \frac{n-1}{\frac{n(n-1)}{2}} = \frac{2}{n}.$$

(b) Let A be the event that at least one of the two failures is among the first four cells, and B be the event that both failures are among the first four cells. Note that $B \subset A$, and therefore $AB = B$.

Now

$$|B| = \binom{4}{2} = 6,$$

because there are $\binom{4}{2}$ ways to select two of the first four cells to fail. And

$$|A| = \binom{4}{2} + 4(n-4) = 4n - 10,$$

because there are $\binom{4}{2}$ ways to select two positions out of the first four, and $4(n-4)$ ways to select a pair of positions with one of those being among the first four and the other among the other $(n-4)$ positions. Thus

$$P(B|A) = \frac{P(AB)}{P(A)} = \frac{|AB|}{|A|} = \frac{|B|}{|A|} = \frac{6}{4n-10} = \frac{3}{2n-5}.$$