NAME: Junsheng Huang

netID: jh103

section:

PROBLEM SET #8

[1]. $X : N(-2,4)$ °F

(a) $P\{X \geq -2\} = P\{\frac{X+2}{2} \geq \frac{-2+2}{2}\} = Q(0) = 1-\Phi(0) = 0.5$

(b) $P\{X \geq -4\} = P\{\frac{X+2}{2} \geq \frac{-4+2}{2}\} = Q(-1) = \Phi(1) = 0.8413$

$P\{X \geq -2 \mid X \geq -4\} = \frac{P\{X \geq -2\}}{P\{X \geq -4\}} = \frac{0.5}{0.8413} = 0.5943$

(c) $Y = \frac{5}{9}(X-32)$

$P\{Y > X\} = P\{\frac{5}{9}X - \frac{160}{9} > X\} = P\{\frac{4}{9}X < -\frac{160}{9}\}$

$= P\{X < -40\} = P\{\frac{X+2}{2} < \frac{-40+2}{2}\} = \Phi(-19) \approx 8.527 \times 10^{-81}$

(d) consider $X = 2Y-2$, Y is $N(0,1)$ distribution.

$E[Y^2] = \int_{-\infty}^{\infty} \frac{u^2}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})\,du = \frac{u}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2})\,du = 1$

$E[(X+2)^2] = E[(2Y-2+2)^2] = 4E[Y^2] = 4$

[2]. $Y = X + Z$, $Z : N(0,1)$ distribution

(a) $X = 1$, the pdf of Y: $p_Y(u) = p_Z(u-1) = \frac{1}{\sqrt{2\pi}} \exp(-\frac{(u-1)^2}{2})$

(b) $X = -1$ or $X = 1$ $\quad Y > 0 \Rightarrow X = 1 \quad Y \leq 0 \Rightarrow X = -1$

$P(\text{error}) = P(X=-1 \mid Y > 0) + P(X=1 \mid Y \leq 0)$

$= \frac{1}{2}[P(Z > 1) + P(Z \leq -1)]$

$= 1 - \Phi(1)$

$= 0.1587$

(c) $Y < -1 \Rightarrow X = -2 \qquad Y > 1 \Rightarrow X = 2 \qquad$ else $\Rightarrow X = 0$

$P(\text{error}) = P(X \neq -2 \mid Y < -1) + P(X \neq 0 \mid -1 \leq Y \leq 1) + P(X \neq 2 \mid Y > 1)$

$= P(X=0, Z<-1) + P(X=2, Z<-3) + P(X=-2, 1 \leq Z \leq 3) + P(X=2, -3 \leq Z \leq -1) + P(X=0, Z>1) + P(X=-2, Z>3)$

$= \frac{1}{3}[P(Z<-1) + P(Z>1) + P(Z<-3) + P(Z>3) + P(1 \leq Z \leq 3) + P(-3 \leq Z \leq -1)]$

$= \frac{1}{3}[2P(Z>1) + P(Z \leq -1) + P(Z \geq 1)]$

$= \frac{4}{3} P(Z>1)$

$= \frac{4}{3} \cdot Q(1)$

$= \frac{4}{3}[1 - \Phi(1)]$

$= 0.2116$

[3]. $n = 10^5$, $p = 10^{-4}$ $\mu_X = np = 10$. $Var_X = np(1-p) = 9.999$

(a) $P\{X = 15\} = \binom{10^5}{15} \cdot (10^{-4})^{15} \cdot (1-10^{-4})^{10^5-15}$

(b) use the Gaussian approximation with continuity correction ($\tilde{X}$)

$P\{X = 15\} \approx P(14.5 \leq \tilde{X} \leq 15.5)$

$\approx P(\frac{14.5-10}{\sqrt{9.999}} \leq \frac{\tilde{X}-10}{\sqrt{9.999}} \leq \frac{15.5-10}{\sqrt{9.999}})$

$= \Phi(\frac{15.5-10}{\sqrt{9.999}}) - \Phi(\frac{14.5-10}{\sqrt{9.999}})$

$= \Phi(1.74) - \Phi(1.42)$

$= 0.0389$

(c) consider each bit a time tick, $\lambda = p = 10^{-4}$

$P\{N_n = 15\} = \frac{[np]^{15}}{15!} \cdot e^{-np} = \frac{10^{15}}{15!} e^{-10} = 0.0347$

[4]. $p=0.4$ win $\quad p=0.6$ lose $\quad x_i \in \{-1, 1\}$

$$X = \sum_{i=1}^{n} x_i$$

(a) $z_i = \frac{x_i+1}{2} \in \{0, 1\}$, $z = \sum_{i=1}^{100} z_i \quad z_i$ is 0-1 distribution, $\begin{cases} 0.4, & 1 \\ 0.6 : & 0 \end{cases}$

and $z_1 \sim z_{100}$ is independent, so z is Binomial distribution

(b) $z = \sum_{i=1}^{100} z_i = \frac{1}{2}\sum_{i=1}^{100} x_i + 100 \cdot \frac{1}{2} = 50 + \frac{1}{2}X$; $X \geq 10 \Rightarrow z \geq 55$

event $\{X \geq 10\}$ is event $\{z \geq 55\}$

consider $\tilde{z}$ as Gaussian distribution, then $\mu = [0.4 \cdot 1 + 0.6 \cdot 0] \cdot 100 = 40$, $var = 100 \cdot (0.4 \cdot 0.6^2 + 0.6 \cdot 0.4^2) = 24$

$$P\{X \geq 10\} = P\{z \geq 55\} = P\{\tilde{z} \geq 54.5\} = P\left\{\frac{\tilde{z}-40}{\sqrt{24}} \geq \frac{54.5-40}{\sqrt{24}}\right\}$$

$$= Q\left(\frac{14.5}{\sqrt{24}}\right) = 1 - \Phi(2.96) = 0.0015$$

(c) $X=0 \Rightarrow z = 50$

$$P\{X=0\} = P\{z=50\} = P\{49.5 \leq \tilde{z} \leq 50.5\} = P\left\{\frac{49.5-40}{\sqrt{24}} \leq \frac{\tilde{z}-40}{\sqrt{24}} \leq \frac{50.5-40}{\sqrt{24}}\right\}$$

$$= \Phi\left(\frac{10.5}{\sqrt{24}}\right) - \Phi\left(\frac{9.5}{\sqrt{24}}\right) = \Phi(2.14) - \Phi(1.94) = 0.9838 - 0.9738 = 0.01$$

(d) Poisson approximation: $\lambda = np = 100 \cdot 0.4 = 40$

$$P(X=0) = P\{z=50\} = \frac{40^{50}}{50!}e^{-40} = 0.0177$$

[5]. $f_\theta(u) = \begin{cases} \left(\frac{u}{\theta}\right)\exp\left(-\frac{u^2}{2\theta}\right) & u \geq 0 \\ 0 & u < 0 \end{cases} \quad \theta > 0 \quad$ observed: $X = 10 \quad \underline{\hat{\theta}_{ML}}$

$$f_\theta(10) = \left(\frac{10}{\theta}\right) \cdot \exp\left(-\frac{10^2}{2\theta}\right) = \frac{10}{\theta} \cdot \exp\left(-\frac{50}{\theta}\right)$$

$$\frac{df_\theta(10)}{d\theta} = -\frac{10}{\theta^2}\exp\left(-\frac{50}{\theta}\right) + \frac{10}{\theta} \cdot \left(\frac{50}{\theta^2}\right) \cdot \exp\left(-\frac{50}{\theta}\right)$$

$$= \exp\left(-\frac{50}{\theta}\right) \cdot \left[\frac{500}{\theta^3} - \frac{10}{\theta^2}\right] = \exp\left(-\frac{50}{\theta}\right)$$

$$= \exp\left(-\frac{50}{\theta}\right) \cdot \left[\frac{500-10\theta}{\theta^3}\right]$$

when $\theta = 50$, $f_\theta(10)$ is largest $\Rightarrow \hat{\theta}_{ML} = 50$