ECE 310 Fall 2023

Lecture 10

Transfer functions and LTI system response: Part 2

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Learning Objectives

After this lecture, you should be able to:

- Work with transfer functions given as improper rational expressions.
- Use z-transform properties and system algebra to compute the response of a larger LTI system composed of multiple LTI systems.

Recap from previous lecture

In the last lecture, we explained how to use transfer functions to compute the response of LTI systems in the z-domain. We showed how the z-transform helps us find the impulse response of an LTI system given as an LCCDE or compute system outputs without the convolution sum. We will continue working with system response and LCCDEs in this lecture by discussing improper rational expressions and system algebra to utilize multiple LTI systems in arbitrary combinations.

1 Improper rational expressions

Thus far, we have only considered transfer functions described by proper rational expressions

$$H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}},\ M \leq N. \tag{1}$$

What if instead $M > N$ and we have an *improper* transfer function? In order to obtain a proper rational expression, we would need to perform polynomial long division to remove these higher degree terms from the numerator. Then, we may decompose them into the summation of a proper rational function and the long division result:

$$H(z) = \sum_{k=0}^{M-N-1} C_k z^{-k} + \frac{\sum_{k=0}^{N-1} d_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}, \tag{2}$$

where C_k gives the coefficients for our polynomial long division. Note that we have substituted new coefficients d_k for the numerator now that we have altered its degree.

Computing the inverse z-transform of such an improper z-transform will give us (1) shifted and scaled impulses for the first summation and (2) a summation of exponential terms from performing partial fraction expansion (PFE). In the spirit of performing PFE for our inverse z-transform, we can also rewrite Eqn. 2 as

$$H(z) = \sum_{k=0}^{M-N-1} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}. \tag{3}$$

To better understand when improper rational functions appear and how to compute our C_k values, let's consider an example.

Exercise 1: Compute the impulse response $h[n]$ for the following LCCDE.

$$y[n] = 2y[n-1] + 3y[n-2] + x[n] - 3x[n-1] + x[n-2] + 4x[n-3] \tag{4}$$

We start by computing $H(z)$:

$$Y(z) = 2z^{-1}Y(z) + 3z^{-2}Y(z) + X(z) - 3z^{-1}X(z) + z^{-2}X(z) + 4z^{-3}X(z) \tag{5}$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1 - 3z^{-1} + z^{-2} + 4z^{-3}}{1 - 2z^{-1} - 3z^{-2}}. \tag{6}$$

Clearly, we have an improper $H(z)$ with $M = 4$ and $N = 2$. Factoring the denominator, we can rewrite $H(z)$ in the form of Eqn. 3:

$$H(z) = C_0 + C_1 z^{-1} + \frac{A_1}{1 - 3z^{-1}} + \frac{A_2}{1 + z^{-1}}. \tag{7}$$

Multiplying both sides by $1 - 2z^{-1} - 3z^{-2}$,

$$C_0(1 - 2z^{-1} - 3z^{-2}) + C_1 z^{-1}(1 - 2z^{-1} - 3z^{-2}) + A_1(1 + z^{-1}) + A_2(1 - 3z^{-1}) = 1 - 3z^{-1} + z^{-2} + 4z^{-3}. \tag{8}$$

We have four unknowns and that may seem like a lot; however, note that our C_0 and C_1 terms are multiplied by polynomials with roots matching our poles. Thus, we can solve for A_1 and A_2 as we did in Lecture 8:

$$z = 3 \implies \frac{4}{3}A_1 = \frac{7}{27} \implies A_1 = \frac{7}{36} \tag{9}$$

$$z = -1 \implies 4A_2 = 1 \implies A_2 = \frac{1}{4}. \tag{10}$$

Plugging in our values of A_1 and A_2, we can simplify line 8 to

$$C_0(1 - 2z^{-1} - 3z^{-2}) + C_1(z^{-1} - 2z^{-2} - 3z^{-3}) = \frac{5}{9} - \frac{22}{9}z^{-1} + z^{-2} + 4z^{-3}. \tag{11}$$

We see by inspection $C_0 = \frac{5}{9}$ and $C_1 = -\frac{4}{3}$ since they exclusively have zero-degree and third-degree terms, respectively. In general, we can solve such an *over-determined* linear system by choosing $M - N$ equations to solve through. For example, we could solve the system given by the coefficients for z^{-1} and z^{-2}:

$$\begin{cases} -2C_0 + C_1 &= -\frac{22}{9} \\ -3C_0 - 2C_1 &= 1 \end{cases}. \tag{12}$$

We now have our fully decomposed $H(z)$ into

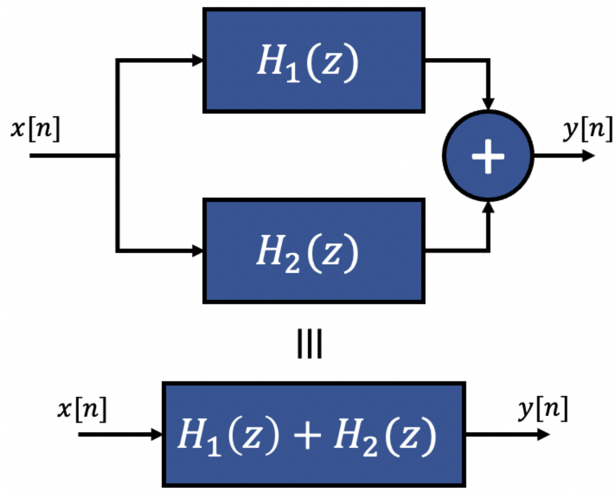
$$H(z) = \frac{5}{9} - \frac{4}{3}z^{-1} + \frac{\frac{7}{36}}{1 - 3z^{-1}} + \frac{\frac{1}{4}}{1 + z^{-1}}. \tag{13}$$

Finally, by inspection and recognizing our LCCDE is causal:

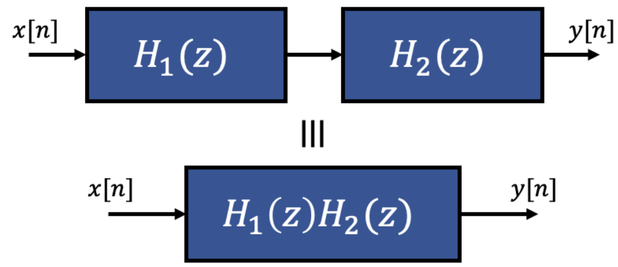
$$h[n] = \frac{5}{9}\delta[n] - \frac{4}{3}\delta[n-1] + \frac{7}{36}3^n u[n] + \frac{1}{4}(-1)^n u[n]. \tag{14}$$

The above exercise demonstrates that we can have improper rational expressions when we have more input terms than output/feedback terms. Specifically, if we have at least one input term that is shifted by at least as much as the most shifted feedback term, we will have an improper rational expression. In the above example, we have an input term three samples away at $x[n-3]$ while our furthest feedback term is at $y[n-2]$.

We also saw that PFE can be performed the same as with a proper rational expression. Solving for



(a) Parallel connection of LTI systems.



(b) Series connection of LTI systems.

Figure 1: Pictorial representation of system algebra for parallel and series connection of LTI systems. The top and bottom systems in each figure are equivalent.

Table 1: Summary of system algebra for parallel and series connections of LTI systems.

	Impulse response $h[n]$	Transfer function $H(z)$
Parallel	$h_1[n] + h_2[n]$	$H_1(z) + H_2(z)$
Series	$h_1[n] * h_2[n]$	$H_1(z)H_2(z)$

the A_k coefficients reduces the computation of C_k values to solving a consistent and over-determined linear system of equations. We can still use polynomial long division, while the above method works just as well with our preferred PFE technique.

2 System algebra

The z-transform and analysis of transfer functions also enables us to efficiently work with more complex LTI systems that are the composition of multiple LTI systems. Consider, for example, if we have two systems $h_1[n]$ and $h_2[n]$ connected in series or parallel. Figure 1 depicts these two scenarios.

Two systems in parallel will give us the result

$$y[n] = x[n] * h_1[n] + x[n] * h_2[n]. \tag{15}$$

By the distributive property of convolution, we can rewrite this as

$$y[n] = x_n * (h_1[n] + h_2[n]). \tag{16}$$

This implies we have a larger system given by

$$h[n] = h_1[n] + h_2[n] \overset{\mathcal{Z}}{\longleftrightarrow} H_1(z) + H_2(z) = H(z). \tag{17}$$

Two systems in series will give us the result

$$y[n] = x[n] * h_1[n] * h_2[n]. \tag{18}$$

By the associative property of convolution, we can also write this as

$$y[n] = x[n] * (h_1[n] * h_2[n]). \tag{19}$$

Thus, we have a larger LTI system represented by

$$h[n] = h_1[n] * h_2[n] \stackrel{\mathcal{Z}}{\longleftrightarrow} H_1(z)H_2(z) = H(z). \tag{20}$$

Table 1 summarizes how parallel and series connections combine to form a larger LTI system in the time-domain and z-domain.

We see that the combination of any number of LTI systems using series or parallel connections results in another LTI system that can be compactly represented by one impulse response $h[n]$ and its corresponding transfer function $H(z)$. These multiple systems "collapse" into one LTI system.