ECE 310 Fall 2023

Lecture 24

Spectral analysis and window functions

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Learning Objectives

After this lecture, you should be able to:

- Identify the key challenges in performing spectral analysis including the varying amplitudes and distances between sinusoidal components.
- Describe how the choice of window function can affect spectral analysis.
- Explain the trade-offs between common window functions, e.g. rectangular and Hamming windows.

Recap from previous lecture

In the previous lecture, we introduced a common application of the DFT known as spectral analysis. We established the signal processing foundations for this problem and began identifying the key challenges of spectral analysis. In this lecture, we will further examine these challenges and consider when it is possible to separate sinusoidal components, including how choosing appropriate window functions can help us.

1 Brief overview of spectral analysis problem

We begin by briefly reviewing what we learned about the spectral analysis problem in the previous lecture. Spectral analysis is the task of identifying multiple sinusoidal components present in a finite-length discrete-time signal. Let $x[n]$ be our discrete-time signal:

$$x[n] = \sum_{m=1}^{M} A_m \cos(\omega_m n),\ 0 \leq n \leq N-1. \tag{1}$$

In general, we typically do not know the number of sinusoidal components and certainly do not know their amplitudes or frequencies. The core of the spectral analysis problem is distinguishing one sinusoidal component from another; thus, we may develop our understanding for the problem by examining a signal with only $M = 2$ sinusoidal components, given by $s[n]$:

$$s[n] = A_1 \cos(\omega_1 n) + A_2 \cos(\omega_2 n),\ 0 \leq n \leq N-1. \tag{2}$$

Recall from lecture 23 that the DTFT spectrum of a length-N cosine $x[n] = A\cos(\omega_0 n)$ is given by

$$X(\omega) = \frac{A}{2}C(\omega - \omega_0) + \frac{A}{2}C(\omega + \omega_0), \tag{3}$$

$$C(\omega) = e^{-j\frac{\omega}{2}(N-1)}\left(\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right) = \begin{cases} N, & \omega = 0 \\ e^{-j\frac{\omega}{2}(N-1)}\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}, & \omega \neq 0 \end{cases}. \tag{4}$$

The main source of difficulty in performing spectral analysis comes from the spectral leakage phenomenon due to the finite length of the signal $s[n]$. The resulting challenges we identified in lecture 24 were:

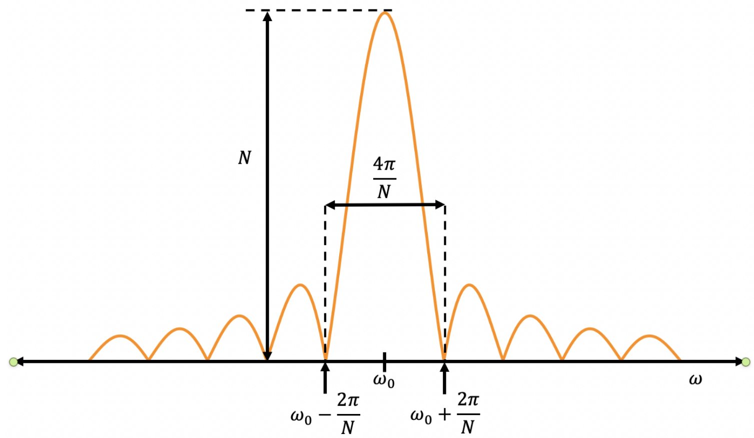


Figure 1: Magnitude spectrum $|C(\omega - \omega_0)|$

1. **The frequencies of components may be too close to separate.** We may not be able to distinguish the main lobes from two components even if they have similar amplitudes should they be too close and form one main lobe. In other words, if $|\omega_1 - \omega_2|$ is too small, the main lobes of each component will look like one combined main lobe.
2. **The amplitudes of components may vary greatly.** Consider if the smaller amplitude A_2 was much smaller than A_1, e.g. $\frac{1}{10}$. In this case, the main lobe of the second component may be covered up by a side lobe of the first component. Thus, we need an effective way to minimize side lobes so that the relevant main lobes are easier to identify.

In the next two sections, we will demonstrate how we can resolve nearby frequencies or two components with different amplitudes. Note that we will use the DTFT for our analysis since the notation is easier to read; however, all the same criteria and design trade-offs may be found exactly the same using the DFT.

2 Separating main lobes of components

We first consider how we may resolve two sinusoidal components with relatively equal amplitudes that are close in frequency. For simplicity, we assume $A_1 = A_2 = 2$. We will then find the resulting spectrum to be

$$S(\omega) = C(\omega - \omega_1) + C(\omega + \omega_1) + C(\omega - \omega_2) + C(\omega + \omega_2). \tag{5}$$

Looking at the positive frequencies, $\omega \in [0, \pi]$, we see then that the periodic sinc functions centered at ω_1 and ω_2 may distort one another. If N is sufficiently large, we may assume these interactions are negligible and then the resulting magnitude spectrum is approximately:

$$|S(\omega)| \approx |C(\omega - \omega_1)| + |C(\omega + \omega_1)| + |C(\omega - \omega_2)| + |C(\omega + \omega_2)|, \ (N \text{ large}). \tag{6}$$

This assumption for N is easy to satisfy in practice and typically not a concern for real-world signals. In order to differentiate two similar frequencies, we must be able to distinguish the respective main lobes of their DTFTs. Figure 1 shows that the main lobe of $C(\omega)$ has width $\frac{4\pi}{N}$. We may use this main lobe width to derive criteria for separating two main lobes.

Full-lobe separation. Suppose we would like to have no overlap between the main lobes of the two sinusoidal components. We call this the condition for full-lobe separation. Without loss of generality, assume that $\omega_1 < \omega_2$. Full-lobe separation tells us that we want the right edge of the main lobe of ω_1 to be less than the left edge of the main lobe of ω_2

$$\omega_1 + \frac{2\pi}{N} < \omega_2 - \frac{2\pi}{N}. \tag{7}$$

Rearranging, we obtain a condition for the minimum difference in frequency we can resolve for a signal of length N:

$$\omega_2 - \omega_1 > \frac{4\pi}{N}. \tag{8}$$

This full-lobe separation criterion tells us that as we increase the length of our signal $s[n]$, we may distinguish frequencies that are closer to one another. *It is important to note that zero-padding will not help us with lobe separation because zero-padding does not change the DTFT of a finite-length signal!*

Half-lobe separation. In practice, we may often relax the full-lobe separation criterion to allow up to half of each main lobe to overlap. This half-lobe separation criterion may be derived similarly to obtain:

$$\omega_2 - \omega_1 > \frac{2\pi}{N}. \tag{9}$$

Exercise 1: Suppose we want to perform spectral analysis on a recording of a guitar in standard tuning playing a musical chord, i.e. a simultaneous combination of multiple notes. The music is recorded at 8 kHz and we may assume all musical notes present have similar amplitudes. The lowest note in standard tuning is $E_2 \approx 82$ Hz and the next lowest note on the guitar is $F_2 \approx 87$ Hz. What is the minimum number of samples we need, and thus how long we need to record, to guarantee we can separate all recorded notes with full-lobe separation?

The spacing between musical notes in Western music are in *semitone intervals* such that each note is $2^{1/12} \approx 1.059$ times higher in frequency than the previous note. Thus, notes become further apart as we move up in pitch or frequency. This means the two closest, and hardest to resolve, notes on a guitar are the lowest notes E_2 and F_2 as stated above. We have

$$f_s = \frac{1}{T} = 8 \text{ kHz}. \tag{10}$$

Thus,

$$\omega_1 = \Omega_1 T = \frac{2\pi(82)}{8000} \text{ rad/sample} \tag{11}$$

$$\omega_2 = \Omega_2 T = \frac{2\pi(87)}{8000} \text{ rad/sample}. \tag{12}$$

According to full-lobe separation, we will need

$$\omega_2 - \omega_1 > \frac{4\pi}{N} \tag{13}$$

$$N > \frac{4\pi}{\frac{2\pi}{8000}(87-82)} = \frac{16000}{5} = 3{,}200 \text{ samples}. \tag{14}$$

These 3,200 samples amounts less than 0.5 seconds of recording with an 8 kHz sampling rate! This is a fairly easy requirement to meet in practice as the guitarist needs to hold the chord for less than a second. (It is fair to point out that E_2 and F_2 are impossible to play simultaneously in standard tuning! But let's assume we may want to differentiate the spectra of multiple recorded chords, e.g. E major and F major.)

The full-lobe and half-lobe separation criteria are helpful in designing the necessary specifications of a spectral analysis system or algorithm. However, we have assumed the amplitudes of each sinusoidal component are roughly equal and thus equally visible in a given DFT spectrum. Now, we must consider how we can separate spectral components with quite different amplitudes.

3 Window functions

Suppose now our two sinusoidal components have amplitudes A_1 and A_2 such that $A_1 \gg A_2$, i.e. A_1 is much greater than A_2. Even if ω_1 and ω_2 are adequately separated by our criteria from the previous section, we

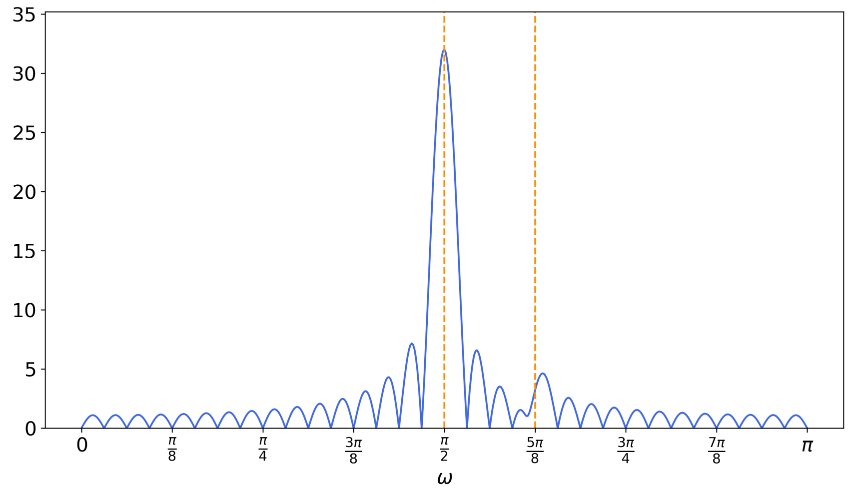


Figure 2: Magnitude spectrum $|S_1(\omega)|$. The orange dashed lines identify ω_1 and ω_2, respectively, in $s_1[n]$.

may find that the main lobe for the small sinusoid blends into the side lobes of the larger sinusoid. To make this all concrete, we will consider the following finite-length signal $s_1[n]$:

$$s_1[n] = \cos(0.5\pi n) + \frac{1}{10}\cos(0.625\pi n),\ 0 \le n \le 63. \tag{15}$$

We see by full-lobe or half-lobe separation criteria $\omega_1 = 0.5\pi$ and $\omega_2 = 0.625\pi$ are easily separable for this length-64 signal. However, Fig. 2 shows us that the second smaller component is hard to identify in the spectrum! The larger first component has side lobes that interfere substantially with the second main lobe. The result is the second main lobe is somewhat masked and distorted by these side lobes.

To address this problem, we need to consider how we mathematically represent finite-length signals. Thus far, we have only indicated the support of the signal, e.g. $0 \le n \le 63$ for $s_1[n]$. We may also write $s_1[n]$ as

$$s_1[n] = (u[n] - u[n-64])\left[\cos(0.5\pi n) + \frac{1}{10}\cos(0.625\pi n)\right],\quad -\infty < n < \infty. \tag{16}$$

This representation shows us that $s_1[n]$ is the product of two infinitely long signals. The first is the combination of unit-step functions in parentheses that form a rectangular function, and the second is the original combination of sinusoids in square brackets. Recall the windowing property of the DTFT:

$$x[n]w[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}X(\omega) * W(\omega). \tag{17}$$

We see then that $S_1(\omega)$ is the convolution of the spectra for two infinitely long cosines and the DTFT of a rectangular function. Since we know

$$\cos(\omega_0 n) \overset{\mathcal{F}}{\longleftrightarrow} \pi(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)), \tag{18}$$

the DTFT spectrum of $S_1(\omega)$ will be shifted and scaled copies of the window function's spectrum by the identify property of convolution! *This statement is true for any signal we consider for spectral analysis and any choice of window function.* This all should motivate us to consider the DTFT of a rectangular function and observe what we may desire in the spectrum of a window function.

Rectangular widow. We define the rectangular window function as follows:

$$r[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}. \tag{19}$$

Figures 3a and 3b show the corresponding time-domain and magnitude spectrum representations of the rectangular window $r[n]$. From here, we see the corresponding half-width of the main lobe is $\frac{2\pi}{N} = \frac{\pi}{32}$ and the height of the largest side lobe is about -13 dB.

Hamming window. We would like to lower these side lobe heights to prevent the interference we saw in Fig. 2. One popular window function with better *side lobe attenuation* is the *Hamming window* $h[n]$:

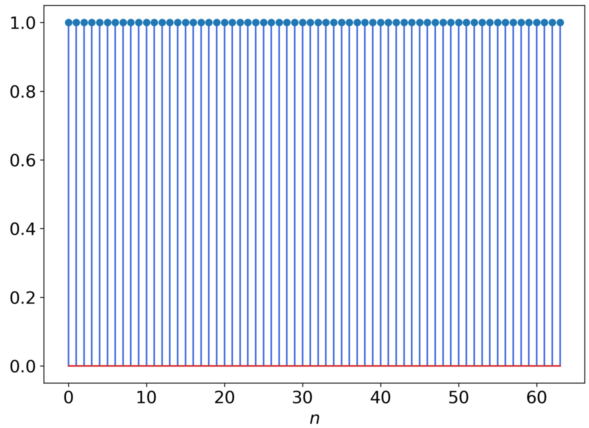
$$h[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}. \tag{20}$$

Figures 3c and 3d demonstrate the Hamming window in the time-domain and its magnitude spectrum, respectively. The Hamming window has the appearance of a "raised cosine" function and its magnitude spectrum also resembles a sinc function. The key differences in the Hamming window's DTFT is that its main lobe has twice the width ($\frac{8\pi}{N}$) – the half-lobe width is $\frac{4\pi}{N} = \frac{\pi}{16}$ in Fig. 3d – and the side lobe attenuation is much stronger at about -42 dB. If we apply the Hamming window to the signal $s_1[n]$, we will obtain the windowed spectrum shown in Fig. 4. We see now that the Hamming window greatly lowers the sidelobes while widening each of the main lobes. Each of the spectral components are now clearly visible!

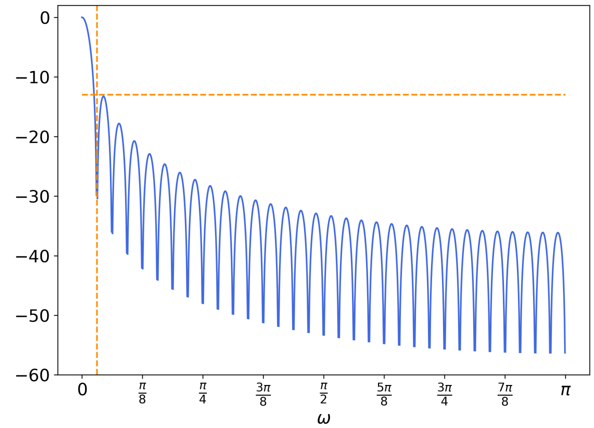
3.1 Window function design tradeoff

The primary tradeoff when selecting a window function for spectral analysis is the main lobe width and side lobe attenuation. Ideally, we want narrow main lobes and small side lobes. However, as we see in Fig 3, we cannot have both. Window functions like the rectangular window have a narrow main lobe with high side lobes, while the Hamming window has a wider main lobe with lower side lobes. Other window functions include the Hann window, Bartlett/triangular window, and Kaiser window which give greater control over this tradeoff. Still, we primarily consider the rectangular and Hamming windows in practice since one of them is often sufficient and they are relatively simple. *Please also do not forget that windowing is not the same as convolution! Windowing is multiplication in time, which corresponds to convolution in the frequency domain. Conversely, convolution in time corresponds to multiplication in the frequency domain. Thus, windowing does not represent an LTI system!*

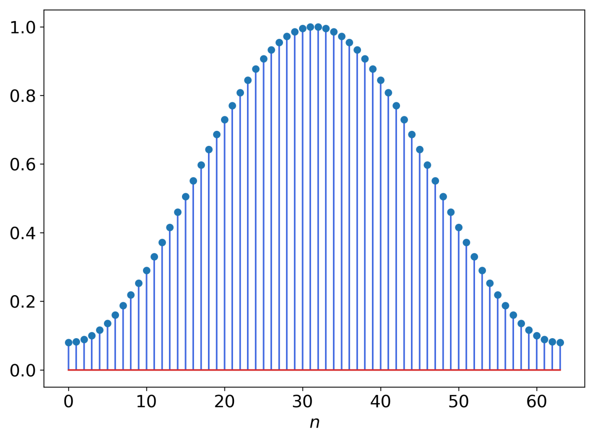
Intuitively, we may see window functions as applying a "weight" to each sample they pull out of a given signal. A rectangular window applies an equal weight of one to all samples. Conversely, a Hamming window tapers down as we move from the center of the window. Thus, the Hamming window gives less weight or importance to samples near the end of the window. Doing so is potentially helpful because of the periodic discontinuities introduced by the DFT. When we periodically extend our finite-length signal in the time-domain, these discontinuities introduce spurious frequencies, i.e. spectral leakage. Down-weighting samples near the edges of the window, as the Hamming window does, makes these samples influence the frequency content of the finite-length signal less and therefore lowers the impact of the spectral leakage phenomenon.



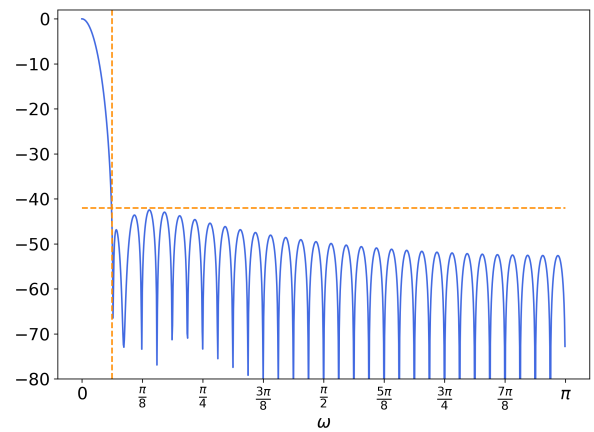
(a) Rectangular window $r[n]$



(b) DTFT $R(\omega)$



(c) Hamming window $h[n]$



(d) DTFT $H(\omega)$

Figure 3: Representation of rectangular and Hamming window functions in time and frequency domains.

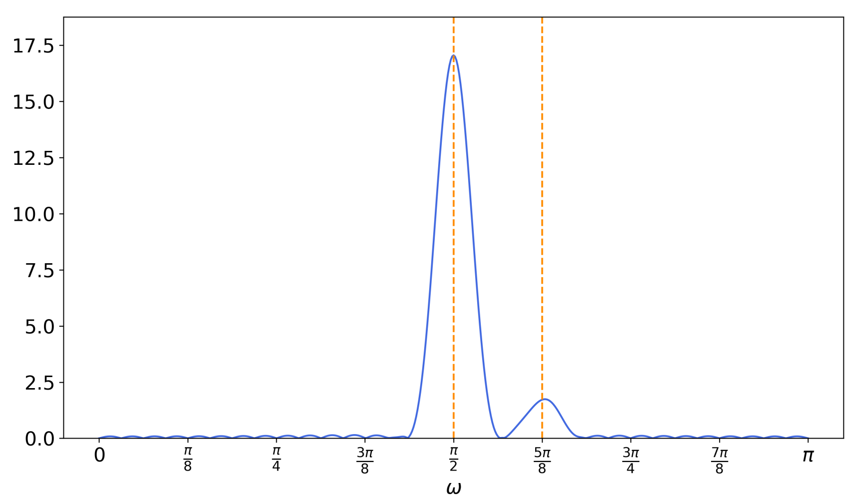


Figure 4: Magnitude spectrum for $s_1[n]$ after applying Hamming window.