ECE 310 Fall 2023

Lecture 6

z-transform

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Learning Objectives

After this lecture, you should be able to:

- Explain the motivation of using the z-transform.
- Define the z-transform of a discrete-time signal.
- Explain the significance of the region of convergence (ROC) for a z-transform.
- Compute the z-transform by inspection using a table of transform pairs.

Recap from previous lecture

We continued our discussion of impulse responses, convolution, and LTI systems in the last lecture by introducing linear constant-coefficient difference equations (LCCDEs). We worked with examples for both finite and infinite impulse response systems and their visualization with block diagrams. In this lecture, we will begin working with our first transform of the course: the z-transform.

1 Motivation for the z-transform

In the previous two lectures, we analyzed the system response of LTI systems using the convolution operator and LCCDEs. These are two powerful tools to explain how a given system will act on a given input signal. However, we are still fairly limited in terms of how we can analyze LTI systems. We only know how to demonstrate how one input signal will change in shape after passing through one LTI system. Suppose instead we would like to characterize how a given system $h[n]$ responds to *any* input signal.

How can we do this? We clearly cannot pass any of the infinitely many input signals to $h[n]$! Recall how we defined the impulse response in Lecture 4 and decomposed discrete-time signals into the summation of scaled and shifted unit impulses. The impulse response gave us a building block for how signals pass through LTI systems. However, even a simple impulse input can greatly change shape when acted on since it will take on the same shape as the system's impulse response (identity property of convolution). Let's consider instead a general signal that can pass to any LTI and *not change shape* but perhaps be re-scaled. Thus, our desired behavior is

$$\begin{aligned} y[n] &= x[n] * h[n] && (1) \\ &= A(x[n])x[n]. && (2) \end{aligned}$$

Above, $A(x[n])$ denotes some function that maps $x[n]$ to a complex-valued scalar, i.e. $x[n] \overset{A}{\mapsto} \mathbb{C}$. We can start by looking at our convolution sum

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]. \quad (3)$$

We need to pick a class of signals such that $x[n-k]$ decomposes into two independent parts that are functions of k and n, respectively. This will allow us to separate our convolution sum into one part that is a function of k and then another part that is $x[n]$ itself. An easy choice for this would be complex exponential signals since we can split one exponential into a product of two exponentials, i.e. $z^{n-k} = z^n z^{-k}$. Thus, let $x[n]$ belong to

$$x[n] \in \{z^n : z \in \mathbb{C}, -\infty < n < \infty\}, \tag{4}$$

the class of complex exponential signals. Plugging $x[n]$ into our convolution sum, we see

$$\begin{aligned}
y[n] &= x[n] * h[n] \\
&= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]z^{n-k} \qquad (5)\\
&= \left(\sum_{k=-\infty}^{\infty} h[k]z^{-k}\right) z^n \qquad (6)\\
&= H(z)z^n, \ \forall n. \qquad (7)
\end{aligned}$$

In line 6, we use the product of exponential functions to group our convolution sum into terms that are functions only of k and n respectively. We see that the portion in parentheses is not a function of n, but rather only a function of z, which is just a complex number! This allows us to arrive at our final expression in step 7 that shows our system output will be our original input, $x[n] = z^n$, scaled by the complex-valued summation $H(z) = \sum_{k=-\infty}^{\infty} h[n]z^{-k}$ – *if the summation converges.*

With this result, we have gained a valuable tool in analyzing LTI systems. The sum given by $H(z)$ is known as the *transfer function* of our system $h[n]$ or the *z-transform* of $h[n]$. The transfer function tells us how our system will scale any given complex exponential signal for a given complex base z. Thus, if we express any signal as the summation of complex exponentials, we can immediately describe the system output!

$$x[n] = \sum_{k=1}^{K} b_k z_k^n, \ \forall n \tag{8}$$

$$y[n] = \sum_{k=1}^{K} H(z_k) b_k z_k^n, \ \forall n. \tag{9}$$

In broader mathematical terms, we have demonstrated that complex exponential functions are *eigenfunctions* of LTI systems. Recall from linear algebra that an *eigenvector* or eigenfunction of a system is a function that will maintain the same shape and only be rescaled when acted on by an operator or system. The amount our function is scaled by is known as the *eigenvalue* for that particular eigenfunction. For our complex exponentials passed to an LTI system given by $h[n]$, $H(z)$ is the eigenvalue for $x[n] = z^n$. *It is critical to note that the above results and derivations only hold with the stated condition of "for all n", i.e. infinitely long signals.* This may seem like a strong condition; however, our motivation for using the z-transform remains the same and we will demonstrate its usefulness in the upcoming lectures. Consider for example if $H(z_k) = 0$. This tells us our system removes the exponential component with base z_k and gives us our first intuition for how to design or describe LTI systems as *filtering* or removing certain components from input signals.

Now that we know the z-transform is an important tool for explaining the response of LTI systems to input signals, let's formalize our definition and consider examples of the transform.

2 The z-transform

The z-transform of a discrete-time signal $x[n]$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{10}$$

$$x[n] \overset{\mathcal{Z}}{\mapsto} X(z). \tag{11}$$

Recall from our derivation in the previous section that we noted the z-transform sum may not converge. Thus, it is important for us to note the values of z for which the z-transform is defined. We define the *region of convergence* (ROC) as the continuous region in the complex z-plane for which the z-transform is well-defined. We refer to values of z for which $H(z)$ is infinite as *poles* and values for which $H(z) = 0$ as *zeros*. Let's get a better sense of the ROC using a couple examples.

Exercise 1: Let $x[n] = u[n]$. What is the corresponding $X(z)$ and ROC?

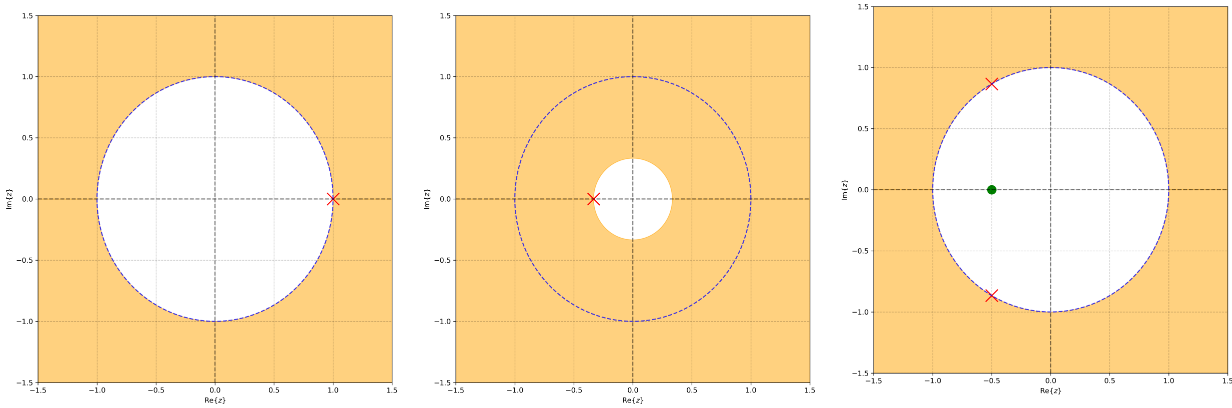
$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} u[n]z^{-n} && (12) \\ &= \sum_{n=0}^{\infty} z^{-n} && (13) \\ &= \frac{1}{1-z^{-1}}. && (14) \end{aligned}$$

Above, line 14 follows from the formula for sum of an infinite-length geometric sequence, i.e. $\sum_{n=0}^{\infty} ab^n = \frac{a}{1-b}$. This sum, however, only converges if $|b| < 1$. For our example we have $b = z^{-1}$; thus, we have an ROC of $|z| > 1$. We also have one pole for $X(z)$ at $z = 1$ which coincides nicely with our ROC.

Above, we have computed the z-transform of the *right-sided* unit-step function. What if we instead compute the z-transform for the (negated) *left-sided* unit-step function: $-u[-(n + 1)]$?

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=-\infty}^{\infty} -u[-(n+1)]z^{-n} && (15) \\ &= -\sum_{n=-\infty}^{-1} z^{-n} && (16) \\ &= -\sum_{m=0}^{\infty} z^{m+1} && (17) \\ &= -z\sum_{m=0}^{\infty} z^{m} && (18) \\ &= \frac{-z}{1-z} \cdot \frac{-z^{-1}}{-z^{-1}} && (19) \\ &= \frac{1}{1-z^{-1}}. && (20) \end{aligned}$$

Above, line 17 follows from substituting $m = -n - 1$. We see the resulting z-transform is the same as for $u[n]$! However, we still need to find the ROC. The geometric sequence sum in line 18 converges if $|z| < 1$. Thus, we have the same expression for $X(z)$ with two different signals, yet they will have different ROCs. We have demonstrated a critical property of the z-transform: *the z-transform of a signal is unique if and only if we specify the region of convergence.* Always remember to state the ROC of your z-transforms!



(a) $H(z) = \frac{1}{1-z^{-1}}$ (b) $H(z) = \frac{1}{1+\frac{1}{3}z^{-1}}$ (c) $H(z)\frac{1+\frac{1}{2}z^{-1}}{1+z^{-1}+z^{-2}}$

Figure 1: Example pole-zero plots for (a) $x[n] = u[n]$, (b) $x[n] = \left(-\frac{1}{3}\right)^n u[n]$, and (c) $\cos\left(\frac{2}{3}\pi n\right) u[n]$. The dashed blue circle represents the unit-circle.

Exercise 2: Compute the z-transform for right-sided exponential signals $x[n] = a^n u[n],\ a \in \mathbb{R}$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n && (21) \\ &= \frac{1}{1-az^{-1}}. && (22) \end{aligned}$$

Line 22 again follows from our geometric sequence sum. We know that this sum will only converge if $|a/z| = |a|/|z| < 1$. Thus, we have the ROC of $|z| > |a|$ with one pole at $z = a$.

We commonly choose to visualize our z-transforms on the complex z-plane using a *pole-zero* plot. A pole-zero plot demonstrates each zero location with a "$\circ$" symbol, each pole with a "$\times$", and shades in the region of convergence. For convenience, we provide a table of common z-transform pairs in Table 1 that corresponds to Table 3.1 from the *Manolakis and Ingle* textbook.

Table 1: Common z-transform pairs.

$x[n]$	$X(z)$	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-(n+1)]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-(n+1)]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(\omega_0 n)u[n]$	$\frac{\sin(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$a^n \cos(\omega_0 n)u[n]$	$\frac{1-a\cos(\omega_0)z^{-1}}{1-2a\cos(\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n \sin(\omega_0 n)u[n]$	$\frac{a\sin(\omega_0)z^{-1}}{1-2a\cos(\omega_0)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

2.1 Linear algebra intuition for z-transform (optional reading if interested!)

We will conclude this lecture with a brief look at the intuition for what the z-transform computes for a given signal. To do so, we will use some basic linear algebra knowledge. Recall that the *dot product* between two length-N real-valued vectors u and v is given by

$$v^\top u = \sum_{n=1}^{N} u_n v_n,\ \ u, v \in \mathbb{R}^N \tag{23}$$

where $v^\top$ denotes the *transpose* of u. Furthermore, if our sequences are complex-valued, our dot product is expressed as

$$v^H u = \sum_{n=1}^{N} u_n v_n^*,\ \ u, v \in \mathbb{C}^N \tag{24}$$

where v^H is the *Hermitian transpose* or *conjugate transpose* of v. At a high-level, the dot product tells us how similar or well-aligned two vectors are. For two unit-length vectors u and v, we know that

$$v^H u = \begin{cases} 1, & \text{if and only if } u = v \\ 0, & \text{if and only if } u \perp v \\ -1, & \text{if and only if } u = -v \end{cases}. \tag{25}$$

Thus, the magnitude of the dot product is maximized when two sequences are perfectly aligned and minimized when they are orthogonal to one another.

Returning to the z-transform, we can also view it as a dot product! Our two vectors or sequences here are $u = x[n]$ and $v^H = z^{-n}$. This means $X(z)$ gives us an expression for the dot product between $x[n]$ with any possible complex exponential sequence z^{-n}. Thus, each value of $X(z)$ tells us how similar $x[n]$ is to the particular exponential defined by z with Hermitian transpose z^{-n}. The larger the magnitude, the more similar; the closer to zero, the less similar. The z-transform in effect decomposes our signal into its similarity to all complex exponentials!

Looking ahead to future lectures, we will work with the z-transform to describe the response of LTI systems to input signals. Referring back to Eqn. 9, we know how our LTI system will respond to a linear combination of complex exponentials. Thus, if the $X(z)$ tells us how much of each exponential is represented in $x[n]$, we may find a convenient way of computing system responses in the complex z domain. More on this in upcoming lectures!