一. complex number 复数

rectangular form: $x = a + j\cdot b$ (complex conjugate $x^* = a - j\cdot b$) 实矩

polar form: $x = Re^{j\theta} \Rightarrow xy = RSe^{j(\theta+\phi)}$; $\frac{x}{y} = \frac{R}{S}e^{j(\theta-\phi)}$

magnitude: $|x| = \sqrt{a^2+b^2} = R$ — Euler's identity:

phase: $\angle x = \theta = \begin{cases} \tan^{-1}(\frac{b}{a}) & a \ge 0 \\ \tan^{-1}(\frac{b}{a}) + \pi & a < 0 \end{cases}$ — $x = R(\cos\theta + j\cdot\sin\theta)$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$, $\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$

common discrete-time [常见信号]: signal representations

Kronecker delta / unit impulse: $\delta[n] = \begin{cases} 1 & n=0 \\ 0 & n \ne 0 \end{cases}$

unit step: $u[n] = \begin{cases} 1 & n \ge 0 \\ 0 & n < 0 \end{cases}$ — sinusoids: $x[n] = A\sin(w_0 n + \theta)$, $-\infty < n < +\infty$

exponentials: $x[n] = B\alpha^n$, $-\infty < n < +\infty$ — [标记$x[0]$]: $x[n] = \{0, \underline{1}, 2\}$

二. discrete-time systems: T是映射 (画图要画成离散的点) — $x[n] = \{0, 1, 2\}$

$\Rightarrow x[n] \xrightarrow{T} y[n]$ 或 $y[n] = T(x[n])$ — $x[n] = \{0, \underset{\uparrow}{1}, 2\}$

[性质]: ① linear: $T(ax_1[n] + bx_2[n]) = aT(x_1[n]) + bT(x_2[n])$

证法: 让 $z[n] = ax_1[n] + bx_2[n]$, 等式左边是 $T(z[n])$, 看与右边关系

② time-invariant / shift-invariant: $y[n-n_0] = T(x[n-n_0])$

证法: $x[n]$先T映射再 $y[n-n_0]$ 与 $x[n-n_0]$ 输入T映射相比 (相当于 $y[n-n_0]$! ↳只影响n! 不影响n_0)

③ causal: system output not depends on future samples.

④ BIBO stable: for any $x[n]$ that $|x[n]| < \beta$ 对∀n成立, 有 $|T(x[n])| < \alpha$ 对∀n成立, $\alpha < \infty$, $\beta < \infty$. 有限输入, 有限输出

unit pulse response:

1. impulse response: $h[n] = T(\delta[n])$ ←只将 $x[n]$ 改成 $\delta[n]$

⇒用impulse表示signal: $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$

2. system response for LTI system: $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$, $\forall n$

commutative — linear, time-invariant — convolution between $x[n]$, $h[n]$

$\Rightarrow$ convolution: $x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n]$

[步骤]: ① 翻转一个信号 $h[-k]$ ② 平移n格 $h[-(k-n)] = h[n-k]$

③ 相乘 $z_n[k] = x[k]h[n-k]$ ④ 相加 $y[n] = \sum_{k} z_n[k]$

[性质] ① 交换律 ② 结合律 $(x[n]*h_1[n])*h_2[n] = x[n]*(h_1[n]*h_2[n])$

③ 分配律 $x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n] + x[n]*h_2[n]$

④ identity: $x[n]*\delta[n] = x[n]$ ⑥ BIBO stable

⑤ causality: if $h[n] = 0$ 对 $n<0$ 成立; 若 $\sum_{-\infty}^{\infty}|h[n]| < \infty$ 成立.

⑦ $x[n]$起点 n_s, 终点 n_e, 长度N; $h[n]$起点 m_s, 终点 m_e, 长度M, $y[n]$为卷积

$y[n]$起点 $k_s = n_s + m_s$, 终点 $k_e = n_e + m_e$, 长度 $K = N+M-1$

[算法]. $x[n] = \{\underset{\uparrow}{3}, 1, 0, 3, 1, 1\}$ $h[n] = \{2, -1, 1\}$ $y[n] = x[n]*h[n]$

operator matrix: $H = \begin{bmatrix} 2 & & & \\ -1 & 2 & & \\ 1 & -1 & 2 & \\ & 1 & -1 & \ddots \\ & & 1 & \ddots \end{bmatrix}$ 列数: 与$x[n]$相同; 行数: $x[n] + h[n]$ $\Rightarrow y = H\cdot x$

☆遇到无限, 不能列向量: $x[n] = u[n]$, $h[n] = (-\frac{3}{4})^n u[n]$

$y[n] = x[n]*h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty}(-\frac{3}{4})^k u(k)u(n-k)$ ☆

$= \sum_{k=0}^{\infty}(-\frac{3}{4})^k$

3. LCCDE: linear constant-coefficient difference equations.

$y[n] = \sum_{i=1}^{K} b_i y[n-i] + \sum_{j=0}^{M-1} c_j x[n-j]$

① K>0: IIR: infinite impulse response, impulse response:

② K=0: FIR: finite impulse response, $h[n] = \sum_{j=0}^{M-1} c_j\delta[n-j]$, $1 \le M < \infty$

[eg]. moving average filter: $y[n] = \frac{1}{L}\sum_{i=0}^{L-1} x[n-i]$ $= y[n-1] + \frac{1}{L}(x[n] - x[n-L])$

4. Block diagrams:

① delay block: $x[n-k] \rightarrow \boxed{z^{-1}} \rightarrow x[n-k-1]$

② coefficient / gain block: $x[n] \rightarrow \triangleright_c \rightarrow cx[n]$

③ adder block: $x[n] \rightarrow \oplus \rightarrow x[n] + z[n]$ ($z[n]$)

三. z-transform: discrete-time signal $x[n]$ 的 z-transform:

$X(z) = \sum_{n=-\infty}^{\infty} x[n]\cdot z^{-n}$, $x[n] \xrightarrow{z} X(z)$

ROC: region of convergence 收敛区域

1. 常见z-transform对:

$\delta[n] \xrightarrow{z} 1$, ROC: all z — $u[n] \xrightarrow{z} \frac{1}{1-z^{-1}}$ ROC: $|z| > 1$

$a^n u[n] \xrightarrow{z} \frac{1}{1-az^{-1}}$, ROC: $|z| > |a|$ — $-a^n u[-(n+1)] \xrightarrow{z} \frac{1}{1-az^{-1}}$, ROC: $|z| < |a|$

$na^n u[n] \xrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^2}$ ROC: $|z| > |a|$ — $-na^n u[-(n+1)] \xrightarrow{z} \frac{az^{-1}}{(1-az^{-1})^2}$ $|z| < |a|$

$\cos(w_0 n)u[n] \xrightarrow{z} \frac{1-\cos(w_0)\cdot z^{-1}}{1-2\cos(w_0)\cdot z^{-1} + z^{-2}}$ ROC: $|z| > 1$

$\sin(w_0 n)u[n] \xrightarrow{z} \frac{\sin(w_0)\cdot z^{-1}}{1-2\cos(w_0)z^{-1} + z^{-2}}$ ROC: $|z| > 1$

$a^n\cos(w_0 n)u[n] \xrightarrow{z} \frac{1-a\cos(w_0)z^{-1}}{1-2a\cos(w_0)z^{-1} + a^2z^{-2}}$ ROC: $|z| > |a|$

$a^n\sin(w_0 n)u[n] \xrightarrow{z} \frac{a\sin(w_0)\cdot z^{-1}}{1-2a\cos(w_0)z^{-1} + a^2z^{-2}}$ ROC: $|z| > |a|$

2. 一些性质: ① $X(z) = 0$: zeros $X(z) \to \infty$: poles

② ROC是connected region, 所有属于的z都使z变换converge

③ 对于finite-length signal, ROC是整个z平面, casual: z=0除外 (non-casual: $z=\infty$ 除外)

① time-shifting: $x[n-k] \xleftrightarrow{z} z^{-k}X(z)$, ROC: R_x (除了 $z=0$, $z=\infty$)

② linearity: $ax_1[n] + bx_2[n] \xleftrightarrow{z} aX_1(z) + bX_2(z)$ ROC $\supseteq R_{x_1} \cap R_{x_2}$

③ convolution: $x_1[n] * x_2[n] \xleftrightarrow{z} X_1(z)\cdot X_2(z)$ ROC $\supseteq R_{x_1} \cap R_{x_2}$

④ differentiation: $nx[n] \xleftrightarrow{z} -z\cdot\frac{dX(z)}{dz}$, ROC $= R_x$

⑤ conjugation: $x^*[n] \xleftrightarrow{z} X^*(z^*)$, ROC $= R_x$

⑥ time reversal: $x[-n] \xleftrightarrow{z} X(z^{-1})$, ROC $= \frac{1}{R_x}$

⑦ scaling: $a^n x[n] \xleftrightarrow{z} X(\frac{z}{a})$, ROC $= |a|\cdot R_x$

⑧ real-part: $Re\{x[n]\} \xleftrightarrow{z} \frac{1}{2}[X(z) + X^*(z^*)]$, ROC: at least R_x

⑨ imaginary-part: $Im\{x[n]\} \xleftrightarrow{z} \frac{1}{2}[X(z) - X^*(z^*)]$, ROC: at least R_x

right-sided: 右边有界 — $x[n] = 0$, $n < n_0$ ROC: $|z| > a$

left-sided: 左边有界 — $x[n] = 0$, $n > n_0$ ROC: $|z| < a$

two-sided: ROC = $a < |z| < |b|$

3. the inverse z-transform: $x[n] = \frac{1}{j\cdot 2\pi}\oint_C X(z)\cdot z^{n-1}dz$

① finite-length signal: $x[n] = \sum_{k=k_s}^{k_e} x[k]\delta[n-k] \xleftrightarrow{z} \sum_{k=k_s}^{k_e} x[k]z^{-k}$

② infinite-length signal: partial fraction expansion

[eg]. $H(z) = \frac{2-z^{-1}}{1-\frac{7}{3}z^{-1}+\frac{2}{3}z^{-2}} = \frac{A_1}{1-\frac{1}{3}z^{-1}} + \frac{A_2}{1-2z^{-1}}$ ←先因式分解

$2 - z^{-1} = A_1(1-2z^{-1}) + A_2(1-\frac{1}{3}z^{-1})$ ←解 A_1, A_2

最后再查表对应回去, 有 $h[n] = \frac{1}{5}(\frac{1}{3})^n u[n] + \frac{9}{5}(2)^n u[n]$ 注意ROC! $h[n]$ right-sided

[eg]. $H(z) = \frac{3-\frac{3}{2}z^{-1}}{1-z^{-1}+z^{-2}}$, 分母分解出来有负数: $p_k = \frac{1\pm\sqrt{1-4}}{2} = e^{\pm j\cdot\frac{\pi}{3}}$

$\Rightarrow H(z) = \frac{A_1}{1-e^{j\cdot\frac{\pi}{3}}z^{-1}} + \frac{A_2}{1-e^{-j\cdot\frac{\pi}{3}}z^{-1}}$ ←一样写 A_1, A_2 解!

$3 - \frac{3}{2}z^{-1} = A_1(1-e^{-j\cdot\frac{\pi}{3}}z^{-1}) + A_2(1-e^{j\cdot\frac{\pi}{3}}z^{-1}) \Rightarrow A_1 = A_2 = \frac{3}{2}$

$\Rightarrow h[n] = \frac{3}{2}(e^{j\frac{\pi}{3}n} + e^{-j\cdot\frac{\pi}{3}n})u[n] = 3\cos(\frac{\pi}{3}n)u[n]$.

四. transfer function 对于 $y[n] = x[n]*h[n]$, transfer function

$H(z) = \frac{Y(z)}{X(z)}$

1. LCCDEs: $y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$ (feedback) N个pole, M个zeros.

$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$ [eg]. $y[n] = \frac{1}{2}y[n-1] + x[n]$

两边进行z变换 — 算H(z) — 进行z逆变换

$\Rightarrow Y(z) = \frac{1}{2}\cdot z^{-1}Y(z) + X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{2}z^{-1}} \Rightarrow h[n] = (\frac{1}{2})^n u[n]$

有feedback项吗 —NO→ FIR: finite-length impulse response

↓yes

有at least one pole 没有与zero抵消吗 —Yes→ IIR: infinite-length impulse response

2. improper rational expression $M > N$ 的情况, 可以分解出整项

$H(z) = \sum_{k=0}^{M-N-1} C_k z^{-k} + \sum_{k=1}^{N} \frac{A_k}{1-p_k z^{-1}}$

3. system algebra: (注意ROC有无延伸! pole可能被zero抵消!)

① 并联 parallel

$H(z) = H_1(z) + H_2(z)$

$x[n] \rightarrow \boxed{H_1(z) + H_2(z)} \rightarrow y[n]$

② 串联 series (cascade)

$x[n] \rightarrow \boxed{H_1(z)} \rightarrow \boxed{H_2(z)} \rightarrow y[n]$

$x[n] \rightarrow \boxed{H_1(z)\cdot H_2(z)} \rightarrow y[n]$

$H(z) = H_1(z)\cdot H_2(z)$

impulse response $h[n]$:

并: $h_1[n] + h_2[n]$

串: $h_1[n] * h_2[n]$

4. BIBO stable (前提, LTI系统)

① $\sum_{n=-\infty}^{\infty}|h[n]| < \infty$ ② transfer function H(z)的ROC contains unit circle ($|z| = 1$)

5. causal (前提, LTI系统)

left-sided	right-sided	right-sided
non-causal	non-causal	causal
$n_0 \in Z$	$n_0 < 0$	$n_0 \ge 0$

对于unBIBO的, 构造input出unbounded output?

① $h[n]$是unbounded sequence $\Rightarrow x[n] = \delta[n]$

② 构造: 考虑cancel the pole 出来是bounded output!

ROC of H(z)	causal		condition of stability
$\|z\| > p_{max}$	✓	right-sided	$p_{max} < 1$
$\|z\| < p_{min}$	×	left-sided	$p_{min} > 1$
$p_{max} < \|z\| < \infty$	×	right-sided	$p_{max} < 1$
$a < \|z\| < b$	×	both-sided	$p_{max} = a < 1$ for $h_r[n]$, $p_{min} = b > 1$ for $h_l[n]$

(p_{min}, p_{max} 是最小/最大的pole)

6. marginal stability 边缘稳定: ROC包括部分但不是全部 unit circle ($|z| = 1$).

① only unstable to periodic input that oscillate at the same frequency.

↳ resonate with our LTI system to create an unbounded output.

input signal match at least one of poles → 成为second-order pole (double pole)

[eg]. $h[n]=e^{jw_1 n}u[n]$, $x[n]=e^{jw_2 n}u[n]$,

$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}e^{jw_1 k}u[k]e^{jw_2(n-k)}u[n-k]=e^{jw_2 n}\sum_{k=0}^{n}e^{jk(w_1-w_2)}$

如果求 real-value bounded input, 可以考虑写成 LCCDE 式子然后分析.

五. Fourier transform 傅立叶变换

1. periodic signals:

① continuous-time periodic signals (period T_0): $x(t+kT_0)=x(t), \forall k\in\mathbb{Z}$

最小的 T_0: fundamental period linear frequency $F_0=\frac{1}{T_0}$ (cycle per sec; Hz)

radial frequency $\Omega_0=2\pi F_0$ (radians per sec; period in sec)

信号还可以写成 $x(t)=e^{j\Omega_0 t}$ ⇒ harmonically related periodic signal: frequency 是别的整数倍, $x_k(t)=e^{j\Omega_0\cdot kt}$, $k\in\mathbb{Z}$

☆ orthogonality property $\int_{t=0}^{T_0}x_k(t)\cdot x_l^*(t)\,dt=\frac{e^{j2\pi(k-l)}-1}{j\Omega_0(k-l)}=\begin{cases}0 & k\neq l\\ T_0 & k=l\end{cases}$

② discrete-time signals (period N_0): $x[n+kN_0]=x[n], \forall k\in\mathbb{Z}$

linear frequency $f_0=\frac{1}{N_0}$ (cycle per sample); radius frequency $w_0=2\pi f_0$ (radius per sample / the periods in sample)

也可以写成 $x[n]=e^{jw_0 n}$ $w_0\in[0,2\pi)$!

2. continuous-time Fourier Series.

$x(t)=\sum_{k=-\infty}^{\infty}c_k\cdot e^{j\Omega_0 kt}$, $c_k=\frac{1}{T_0}\int_{T_0}x(t)e^{-j\Omega_0 kt}dt$

synthesis equation analysis equation

[条件] Dirichlet conditions 也可以用 $\int_{T_0}|x(t)|^2dt<\infty$

I. $\int_{T_0}|x(t)|dt<\infty$ II. 有finite数量的极小, 极大, 断点在每个周期中

① continuous-time Fourier transform: $x(t)\xleftrightarrow{F}X(\Omega)$

$X(\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt$; $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\Omega)e^{j\Omega t}d\Omega$