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section:

PROBLEM SET #11

[1].

(a) $f_{X,Y}(u,v) = \begin{cases} C(u^4+v^4), & 0\le u\le 1 \text{ and } 0\le v\le 1 \\ 0 & \text{otherwise} \end{cases}$

assume it is independent: $f_{X,Y}(u,v)=f_X(u)\cdot f_Y(v)$

consider $0\le u_1<u_2\le 1$, $0\le v_1<v_2\le 1$

$f_X(u_1)\cdot f_Y(v_1)\cdot f_X(u_2)\cdot f_Y(v_2) = f_X(u_1)\cdot f_Y(v_2)\cdot f_X(u_2)\cdot f_Y(v_1)$

$\Rightarrow (u_1^4+v_1^4)(u_2^4+v_2^4) = (u_1^4+v_2^4)(u_2^4+v_1^4)$

$\Rightarrow u_1^4v_2^4+u_2^4v_1^4 = u_1^4v_1^4+u_2^4v_2^4 \Rightarrow (v_1^4-v_2^4)\cdot(u_1^4-u_2^4)=0$

since $u_1\ne u_2$, $v_1\ne v_2$ and all are non-negative, contradiction.

so X, Y are not independent

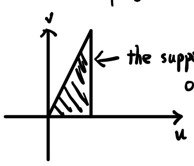
(b) $f_{X,Y}(u,v)=\begin{cases} C\cdot u\cdot v\cdot e^{-\frac{u^2}{2}} & 0\le u\le 1,\ 1\le v\le 2 \\ 0 & \text{otherwise} \end{cases}$

the support is product set, consider:

$f_X(u)=u\cdot e^{-\frac{u^2}{2}}$, $0\le u\le 1$, $f_Y(v)=C\cdot v$, $1\le v\le 2$

X, Y are independent

(c) $f_{X,Y}(u,v)=\begin{cases} C\cdot \exp(v-u) & 0\le u\le 1,\ 0\le v\le 2u \\ 0 & \text{otherwise} \end{cases}$



← the support: consider (0,0) and (1,2), obviously $(0,2)\notin$ the support

so the support is not product sets

X, Y are not independent

[2]. $Z=X+Y$; $X=1,2,3$; $Y=0,1,2,3$

so the support of Z is 1, 2, 3, 4, 5, 6

$Z=1$: $(X,Y)=(1,0)$: $P_Z(1)=\frac{1}{3}\cdot\frac{1}{2}=\frac{1}{6}$

$Z=2$: $(X,Y)=(1,1),(2,0)$ $P_Z(2)=\frac{1}{3}\cdot(\frac{1}{2}+\frac{1}{4})=\frac{1}{4}$

$Z=3$: $(X,Y)=(1,2),(2,1),(3,0)$ $P_Z(3)=\frac{1}{3}\cdot(\frac{1}{2}+\frac{1}{4}+\frac{1}{8})=\frac{7}{24}$

$Z=4$: $(X,Y)=(1,3),(2,2),(1,3)$ $P_Z(4)=\frac{1}{3}\cdot(\frac{1}{4}+\frac{1}{8}+\frac{1}{8})=\frac{1}{6}$

$Z=5$: $(X,Y)=(2,3),(3,2)$ $P_Z(5)=\frac{1}{3}\cdot(\frac{1}{8}+\frac{1}{8})=\frac{1}{12}$

$Z=6$: $(X,Y)=(3,3)$ $P_Z(6)=\frac{1}{3}\cdot\frac{1}{8}=\frac{1}{24}$

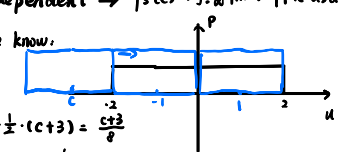
Z	1	2	3	4	5	6
P	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{7}{24}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{24}$

[3]

(a) X: uniformly distributed in [-2, 2] Y: uniformly distributed in [-1, 1]

$S=X+Y$, X, Y independent $\Rightarrow f_S(c)=\int_{-\infty}^{\infty} f_X(u)\cdot f_Y(c-u)du$

from the figure we know:



$c<-3$: $f_S(c)=0$

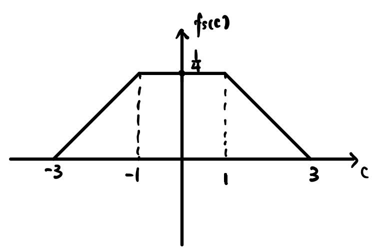
$-3\le c\le -1$: $f_S(c)=\frac{1}{4}\cdot\frac{1}{2}\cdot(c+3)=\frac{c+3}{8}$

$-1<c<1$: $f_S(c)=\frac{1}{4}\cdot\frac{1}{2}\cdot 2=\frac{1}{4}$

$1\le c\le 3$: $f_S(c)=\frac{1}{4}\cdot\frac{1}{2}\cdot(3-c)=\frac{3-c}{8}$

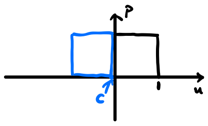
$c>3$: $f_S(c)=0$

$f_S(c)=\begin{cases} 0 & c<-3 \\ \frac{c+3}{8} & -3\le c<-1 \\ \frac{1}{4} & -1\le c<1 \\ \frac{3-c}{8} & 1\le c\le 3 \\ 0 & c>3 \end{cases}$



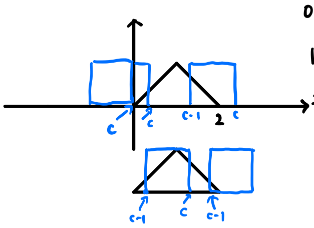
(b) $W = X + Y + Z = S + Z$, $S = X + Y$; X, Y, Z: uniformly distributed over $[0,1]$

Consider $S = X + Y$ first. from the figure:



$$f_S(u) = \begin{cases} u & 0 < u \le 1 \\ 2-u & 1 < u \le 2 \\ 0 & \text{else} \end{cases}$$

consider $W = S + Z$:



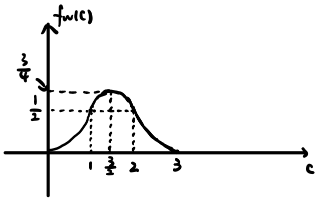
$c < 0$: $f_W(c) = 0$

$0 < c \le 1$: $f_W(c) = \frac{c^2}{2}$

$1 < c < 2$: $f_W(c) = 1 - \frac{(2-c)^2}{2} - \frac{(c-1)^2}{2} = \frac{2-(4-4c+c^2+c^2-2c+1)}{2} = \frac{-2c^2+6c-3}{2} = -c^2+3c-\frac{3}{2}$

$2 \le c \le 3$: $f_W(c) = \frac{(2-c+1)^2}{2} = \frac{(3-c)^2}{2} = \frac{c^2-6c+9}{2} = \frac{1}{2}c^2 - 3c + \frac{9}{2}$

$c > 3$: $f_W(c) = 0$



$$\Rightarrow f_W(c) = \begin{cases} \frac{c^2}{2} & 0 < c \le 1 \\ -c^2+3c-\frac{3}{2} & 1 < c < 2 \\ \frac{1}{2}c^2-3c+\frac{9}{2} & 2 \le c \le 3 \\ 0 & \text{otherwise} \end{cases}$$

[4].

(a) X, Y: independent exponential r.v. with parameter λ.

$Z = X + Y$, support of Z is $[0, +\infty)$, $f_X(u) = f_Y(u) = \lambda e^{-\lambda u}$

$f_Z(c) = \int_{-\infty}^{\infty} f_X(u) \cdot f_Y(c-u)\,du = \int_0^c \lambda \cdot e^{-\lambda u} \cdot \lambda \cdot e^{-\lambda(c-u)}\,du$

$= \int_0^c \lambda^2 \cdot e^{-\lambda \cdot c}\,du = \lambda^2 \cdot c \cdot e^{-\lambda \cdot c}$

(b) X, Y: independent exponential r.v. with parameter λ, μ

$Z = X - Y$, $f_X(u) = \lambda \cdot e^{-\lambda u}$, $f_Y(u) = \mu \cdot e^{-\mu u}$

$c \ge 0$:

$F_Z(c) = P\{X - Y < c\} = P\{X < Y + c\}$

$= \int_0^{\infty} dy \cdot \int_0^{y+c} \lambda \cdot e^{-\lambda x} \cdot \mu \cdot e^{-\mu y}\,dx$

$= \mu \cdot \int_0^{\infty} e^{-\mu y}\,dy \cdot (-e^{-\lambda x})\Big|_0^{y+c}$

$= \mu \cdot \int_0^{\infty} -e^{-\mu y} \cdot [e^{-\lambda(y+c)} + 1]\,dy$

$= 1 - \frac{\mu}{\lambda+\mu} e^{-\lambda c}$

$f_Z(c) = \frac{dF_Z(c)}{dc} = \frac{\lambda \cdot \mu}{\mu+\lambda} \cdot e^{-\lambda c}$

for $c < 0$, the same procedure:

$F_Z(c) = \int_{-c}^{\infty} \int_0^{y+c} \lambda \mu e^{-\lambda x} e^{-\mu y}\,dx\,dy$

$= e^{cu} - \frac{\mu}{\lambda+\mu} \cdot e^{cu}$

$= \frac{\lambda}{\lambda+\mu} e^{cu}$

$f_Z(c) = \frac{dF_Z(c)}{dc} = \frac{\lambda\mu}{\lambda+\mu} \cdot e^{cu}$

$$\therefore f_Z(c) = \begin{cases} \frac{\lambda\mu}{\lambda+\mu} e^{-\lambda c} & c \ge 0 \\ \frac{\lambda\mu}{\lambda+\mu} e^{\mu c} & c < 0 \end{cases}$$

[5]. X_1, X_2, X_3 mutually independent, exponentially with rate λ

(a) $Z_1 = \min\{X_1, X_2, X_3\}$

$1 - F_{Z_1}(c) = P(Z_1 > c) = P(X_1 > c) \cdot P(X_2 > c) \cdot P(X_3 > c)$

$= e^{-\lambda c} \cdot e^{-\lambda c} \cdot e^{-\lambda c} = e^{-3\lambda c}$

$\therefore F_{Z_1}(c) = 1 - e^{-3\lambda c}$, $f_{Z_1}(c) = \frac{dF_{Z_1}(c)}{dc} = 3\lambda \cdot e^{-3\lambda c}$

$$\text{so } f_{Z_1}(c) = \begin{cases} 3\lambda \cdot e^{-3\lambda c} & c \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(b) Z_2: first time two light bulbs burn out

consider Z_3: after one light bulbs burn out, the additional time for a light bulb burn out

$Z_2 = Z_1 + Z_3$, $E[Z_2] = E[Z_1] + E[Z_3]$,

from (a), Z_1 is exponentially distributed with rate 3λ., $E[Z_1] = \frac{1}{3\lambda}$

consider Z_3: $1 - F_{Z_3}(c) = P(X_1 > c)\, P(X_2 > c)$ (we don't care what light bulb left. so use X_1, X_2 for notation)

the same, $F_{Z_3}(c) = 1 - e^{-2\lambda c}$, $f_{Z_3}(c) = \begin{cases} 2\lambda \cdot e^{-2\lambda c} & c \ge 0 \\ 0 & \text{otherwise} \end{cases}$ $E[Z_3] = \frac{1}{2\lambda}$

so $E[Z_2] = \frac{1}{3\lambda} + \frac{1}{2\lambda} = \frac{5}{6\lambda}$

(c) n light bulbs, from b, we can add the time each light bulb burn out:

$$E[Z_{k:n}] = \sum_{i=0}^{k-1} \frac{1}{\lambda \cdot (n-i)}$$