NAME: Junsheng Huang

netID: jh103

section:

PROBLEM SET # 9

[1]. X uniform $\mu_X = 2$, $Var(X) = \frac{1}{3}$, $Y = X^2+2X+3$

(a) consider $X \in [a,b]$, $\mu_X = \frac{a+b}{2} = 2$, $Var(X) = \frac{(a-b)^2}{12} = \frac{1}{3}$

$\therefore a+b=4,\ b-a=2 \Rightarrow a=1,\ b=3$

Since $X \in [1,3]$, $Y = X^2+2X+3 = (X+1)^2+2$

$\therefore$ the set of possible values of Y is $[6,18]$

(b) $F_Y(c) = P(Y \le c) = P((X+1)^2 \le c-2) = P(X \le \sqrt{c-2}-1)$

$= \frac{\sqrt{c-2}-1-1}{2} = \frac{\sqrt{c-2}-2}{2}$

$f_Y(c) = \frac{dF_Y(c)}{dc} = \frac{1}{4\sqrt{c-2}}$

so the pdf of Y is $f_Y(c) = \begin{cases} \frac{1}{4\sqrt{c-2}} & c \in [6,18] \\ 0 & \text{else.} \end{cases}$

(c)

$E[Y] = E[X^2+2X+3] = E[X^2]+2E[X]+3$

$= Var(X) + (E[X])^2 + 2E[X] + 3$

$= \frac{1}{3} + 2^2 + 2\times 2 + 3$

$= \frac{34}{3}$

[2]. $f_X(u) = \frac{1}{\pi(1+u^2)}$ $-\infty < u < \infty$ $\quad Y = \frac{1}{X}$,

$F_X(c) = \int_{-\infty}^{c} f_X(u)du$

$= \frac{1}{\pi} \cdot \arctan(u) \Big|_{-\infty}^{c}$

$= \frac{1}{\pi} \cdot [\arctan(c) + \frac{\pi}{2}]$

$= \frac{1}{\pi}\arctan(c) + \frac{1}{2}$

for $c > 0$:

$F_Y(c) = P(\frac{1}{X} \le c) = P(X \ge \frac{1}{c}) = 1 - P(0 < X \le \frac{1}{c}) = 1 - F(X \le \frac{1}{c}) + F(X \le 0)$

$= 1 - \frac{1}{\pi}\arctan(\frac{1}{c}) - \frac{1}{2} + \frac{1}{2}$

$= 1 - \frac{1}{\pi}\arctan(\frac{1}{c})$

$f_Y(c) = \frac{dF_Y(c)}{dc} = -\frac{1}{c^2} \cdot \frac{-\frac{1}{\pi}}{1+\frac{1}{c^2}} = \frac{1}{\pi} \cdot \frac{1}{1+c^2} = \frac{1}{\pi(1+c^2)}$

for $c < 0$:

$F_Y(c) = F(\frac{1}{X} \le c) = F(0 > X \ge \frac{1}{c}) = F(X < 0) - F(X \le \frac{1}{c})$

$= \frac{1}{2} - \frac{1}{\pi}\arctan(\frac{1}{c}) - \frac{1}{2}$

$= -\frac{1}{\pi}\arctan(\frac{1}{c})$

$f_Y(c) = \frac{dF_Y(c)}{dc} = \frac{1}{\pi(1+c^2)}$

$\therefore$ pdf of Y is $f_Y(c) = \frac{1}{\pi(1+c^2)}$, $-\infty < c < +\infty$

[3]. U uniformly distributed in $[0,1]$, $X = g(U)$, Gaussian $N(2,2)$

consider X, Y that satisfy Gaussian $N(2,2)$

$f_X(u) = f_Y(u) = \frac{1}{2\sqrt{\pi}} \exp[-\frac{(u-2)^2}{4}]$

$f(u,v) = \frac{1}{2\sqrt{\pi}} \cdot \exp[-\frac{(u-2)^2}{4}] \cdot \frac{1}{2\sqrt{\pi}} \cdot \exp[-\frac{(v-2)^2}{4}]$ $\quad \begin{cases} (u-2)^2+(v-2)^2 = r^2 \\ \tan\theta = \frac{u-2}{v-2} \end{cases}$

$F_{X,Y}(u,v) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \exp[-\frac{(u-2)^2+(v-2)^2}{4}] = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\infty} \exp[-\frac{r^2}{4}] \cdot r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2\pi} d\theta \int_0^{\infty} \frac{1}{2} \exp[-\frac{r^2}{4}] r\,dr$

$F_\Theta(\theta) = \frac{\theta}{2\pi}$ $\quad \theta \in [0, 2\pi]$ $\quad F_R(r) = 1 - \exp(-\frac{r^2}{4})$, $r \in [0,\infty)$

$\frac{\theta}{2\pi} = U_1$ $\quad \theta = 2\pi U_1$ $\quad 1 - e^{-\frac{R^2}{4}} = U_2$, $\quad R = \sqrt{-4\ln(1-U_2)}$

and $X = R \cdot \cos\theta$, $Y = R\sin\theta$

$\Rightarrow \begin{cases} X = \sqrt{-4\ln(1-U_2)} \cdot \cos(2\pi U_1) \\ Y = \sqrt{-4\ln(1-U_2)} \cdot \sin(2\pi U_1) \end{cases}$ (Consider $1-U_2$ is also uniformly distributed $\Rightarrow \begin{cases} X = \sqrt{-4\ln U_2}\cos(2\pi U_1) \\ Y = \sqrt{-4\ln U_2}\sin(2\pi U_1) \end{cases}$

another method for [3]. ← using $\Phi^{-1}(u)$

$F(c) = \Phi(\frac{c-2}{\sqrt{2}})$, $\quad F(c) = u \Rightarrow \Phi(\frac{c-2}{\sqrt{2}}) = u$

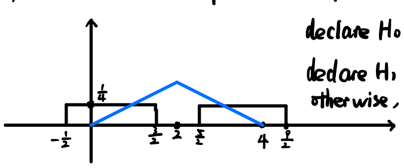
$\frac{c-2}{\sqrt{2}} = \Phi^{-1}(u)$, $\quad c = \sqrt{2}\,\Phi^{-1}(u) + 2$

so $g(u) = \sqrt{2}\,\Phi^{-1}(u) + 2$

[4]. $f_0(u) = \begin{cases} \frac{1}{4} & u \in [-\frac{1}{2}, \frac{3}{2}] \cup [\frac{5}{2}, \frac{9}{2}] \\ 0 & \text{else} \end{cases}$ $\quad f_1(u) = \begin{cases} \frac{1}{4}u & u \in [0,2] \\ -\frac{1}{4}u+1 & u \in [2,4] \\ 0 & \text{else} \end{cases}$

$4\pi_0 = \pi_1 \Rightarrow \pi_0 = \frac{1}{5},\ \pi_1 = \frac{4}{5}$

(a) for the ML rule: always choose the largest entry



declare H_0 is true when $u \in [-\frac{1}{2}, 1] \cup [3, \frac{9}{2}]$ $\quad \to u \in (-\infty, 1] \cup [3, +\infty)$ ← put the equal part for declaring H_0 is true.

declare H_1 is true when $u \in (1,3)$

otherwise, $f_1(u) = f_0(u)$, declare H_0 / H_1 depends on whatever you like

(b) $p_{\text{false alarm}} = \int_1^{\frac{3}{2}} \frac{1}{4}\,du + \int_{\frac{5}{2}}^{3} \frac{1}{4}\,du = \frac{1}{4}$

$p_{\text{miss}} = \int_0^1 \frac{1}{4}u\,du + \int_3^4 \cdot (-\frac{1}{4}u+1)\,du = \frac{1}{8}u^2\Big|_0^1 + [(-\frac{1}{8})u^2+u]\Big|_3^4$

$= \frac{1}{8} + [4-2-3+\frac{9}{8}]$

$= \frac{1}{4}$

$p_e = \pi_0 p_{\text{false alarm}} + \pi_1 p_{\text{miss}} = \frac{1}{5}\cdot\frac{1}{4} + \frac{4}{5}\cdot\frac{1}{4} = \frac{5}{20} = \frac{1}{4}$

(c) MAP rules: $\tau = \frac{\pi_0}{\pi_1} = \frac{1}{4}$

for $u \in [0, \frac{3}{2}]$: $\Lambda(X) = \frac{f_1(u)}{f_0(u)} = \frac{\frac{1}{4}u}{\frac{1}{4}} = u > \frac{1}{4}$ when $u > \frac{1}{4}$

$\Rightarrow$ declare H_1 is true when $u > \frac{1}{4}$

for $u \in [\frac{5}{2}, 4]$: $\Lambda(X) = \frac{f_1(u)}{f_0(u)} = \frac{-\frac{1}{4}u+1}{\frac{1}{4}} = -u+4 > \frac{1}{4}$ when $u < \frac{15}{4}$

so declare H_0 is true when $u \in (-\infty, \frac{1}{4}] \cup [\frac{15}{4}, +\infty)$

declare H_1 is true when $u \in (\frac{1}{4}, \frac{15}{4})$

(d) $p_{\text{false alarm}} = \int_{\frac{1}{4}}^{\frac{3}{2}} \frac{1}{4}\,du + \int_{\frac{5}{2}}^{\frac{15}{4}} \frac{1}{4}\,du = \frac{1}{4}(\frac{3}{2} + \frac{15}{4} - \frac{1}{4} - \frac{5}{2}) = \frac{5}{8}$

$p_{\text{miss}} = \int_0^{\frac{1}{4}} \frac{1}{4}u\,du + \int_{\frac{15}{4}}^{4} (-\frac{1}{4}u+1)\,du = (\frac{1}{8}u^2)\Big|_0^{\frac{1}{4}} + (-\frac{1}{8}u^2+u)\Big|_{\frac{15}{4}}^{4} = \frac{1}{8}\cdot(\frac{1}{4})^2 + \frac{1}{4} - 2 + \frac{1}{8}\cdot(\frac{15}{4})^2 = \frac{1}{64}$

$p_e = \pi_0 p_{\text{false alarm}} + \pi_1 p_{\text{miss}} = \frac{1}{5}\cdot\frac{5}{8} + \frac{4}{5}\cdot\frac{1}{64} = \frac{1}{8} + \frac{1}{80} = \frac{11}{80}$

[5]. X Gaussian distribution, observed $X=1$

(a) $Var(X) = 1 \Rightarrow f_X(u) = \frac{1}{\sqrt{2\pi}} \exp[-\frac{(u-\mu)^2}{2}]$

$\frac{df_X(1)}{d\mu} = \frac{1}{\sqrt{2\pi}} \cdot [-(\mu-1)] \exp[-\frac{(\mu-1)^2}{2}]$

$\Rightarrow \mu = 1$ so the ML estimate of the mean of X is 1.

(b) $\mu_X = 1$, consider $Var(X) = a$, $f_X(u) = \frac{1}{\sqrt{2\pi a}} \cdot \exp[-\frac{u^2}{2a}]$

$\frac{df_X(1)}{da} = \frac{1}{\sqrt{2\pi}} \cdot (-\frac{1}{2}) \cdot a^{-\frac{3}{2}} \cdot \exp[-\frac{1}{2a}] + \frac{1}{\sqrt{2\pi a}} \cdot \frac{1}{2a^2} \cdot \exp[-\frac{1}{2a}]$

$= \frac{1}{\sqrt{2\pi}} \cdot \exp[-\frac{1}{2a}] \cdot [-\frac{1}{2}a^{-\frac{3}{2}} + \frac{1}{2}a^{-\frac{5}{2}}]$

$\Rightarrow a = 1$ so the ML estimate of the variance of X is 1.