ECE 310 Fall 2023

Lecture 19

Ideal digital-to-analog conversion

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Learning Objectives

After this lecture, you should be able to:

- Explain ideal digital-to-analog conversion using sinc interpolation.
- Describe ideal digital-to-analog conversion in the frequency domain via ideal low-pass filtering.

Recap from previous lecture

In the previous lecture, we began our discussion of sampling with ideal analog-to-digital conversion. We worked through this ideal process to examine the resulting frequency spectra after sampling with an impulse train and conversion to a digital sequence. We saw that sampling analog signals leads to spectral copies in the frequency domain and this motivated our derivation of the Nyquist criterion for sampling bandlimited signals. This lecture will cover the ideal digital-to-analog process where we will derive the necessary math and intuition for the ideal reconstruction of analog signals.

1 Ideal reconstruction of bandlimited signals

In the previous lecture, we sampled continuous-time bandlimited signals denoted by $x(t)$ with continuous-time Fourier transform (CTFT) $X_a(\Omega)$. We achieved instantaneous sampling by multiplying with an impulse train to obtain $x_{\text{sampled}}(t)$

$$x_{\text{sampled}}(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \tag{1}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \tag{2}$$

where the CTFT of $x_{\text{sampled}}(t)$, $X_s(\Omega)$, is given by

$$X_s(\Omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\Omega - k\frac{2\pi}{T}\right) \tag{3}$$

for sampling period T. This sampled continuous-time signal $x_{\text{sampled}}(t)$ is then stored digitally to form our discrete-time signal $x[n]$. Our goal in this lecture is to recover bandlimited signal $x(t)$ from its digital samples $x[n]$.

Figure 1 illustrates the ideal digital-to-analog procedure we would like to derive. We have discrete-time samples that can be re-associated in time with sampling period T (most likely the same as we used to sample the original analog signal). The main problem then is how to "connect the dots" or fill in the signal values

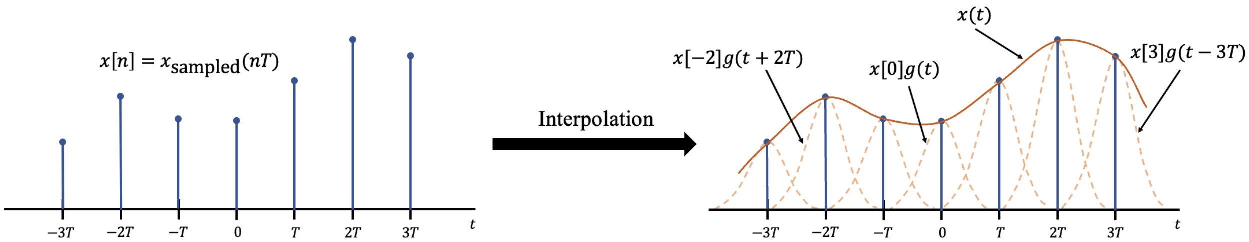

Figure 1: Depiction of interpolating discrete-time values into a continuous-time signal. Note how we sum up all the scaled and shifted interpolation functions $g(t)$ to create the reconstructed signal $x(t)$.

between these samples. We refer to this process as *interpolation* and it can be generally expressed by the following *reconstruction formula*:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]g(t - nT). \tag{4}$$

Equation 4 reconstructs a continuous-time signal by summing shifted copies of an *interpolation function* $g(t)$. These shifted copies are separated by integer multiples of the sampling period and scaled by the discrete-time samples $x[n]$. Thus, we must find an appropriate interpolation function that will exactly recover any bandlimited signal $x(t)$.

After re-associating each sample from $x[n]$ at instantaneous location $t = nT$, we again have

$$x[n] = x_{\text{sampled}}(nT) \tag{5}$$

and thus

$$x(t) = \sum_{n=-\infty}^{\infty} x_{\text{sampled}}(nT)g(t - nT). \tag{6}$$

The summation in Eqn. 6 is then equivalent to

$$x(t) = \int_{-\infty}^{\infty} x_{\text{sampled}}(\tau)g(t - \tau)d\tau \tag{7}$$

because $x_{\text{sampled}}(\tau)$ is only non-zero when $\tau = nT$. Equation 7 is also known at the *continuous-time convolution* between two continuous-time signals. Note how this integral resembles our discrete-time convolution where now we integrate instead of summing, but we still flip-and-shift one of the two signals: $g(t)$ in this case. By the convolution property of the Fourier transform, we can now say

$$X_a(\Omega) = X_s(\Omega)G_a(\Omega) \tag{8}$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_a\left(\Omega - k\frac{2\pi}{T}\right)G_a(\Omega). \tag{9}$$

Now we are able to solve for our ideal interpolation function $g(t)$ via its CTFT $G_a(\Omega)$:

$$G_a(\Omega) = \begin{cases} T, & |\Omega| \leq \frac{\pi}{T} \\ 0, & |\Omega| > \frac{\pi}{T} \end{cases}. \tag{10}$$

Intuitively, $G_a(\Omega)$ is an ideal low-pass filter that extracts only the central copy of the sampled spectra $X_s(\Omega)$, i.e. at $k = 0$, and corrects the scaling by $1/T$. Figure 2 shows how $G_a(\Omega)$ overlaps with $X_s(\Omega)$ to produce the desired $X(\Omega)$. The inverse CTFT of $G_a(\Omega)$ is then

$$g(t) = \frac{\sin\left(\frac{\pi t}{T}\right)}{\frac{\pi t}{T}} \tag{11}$$

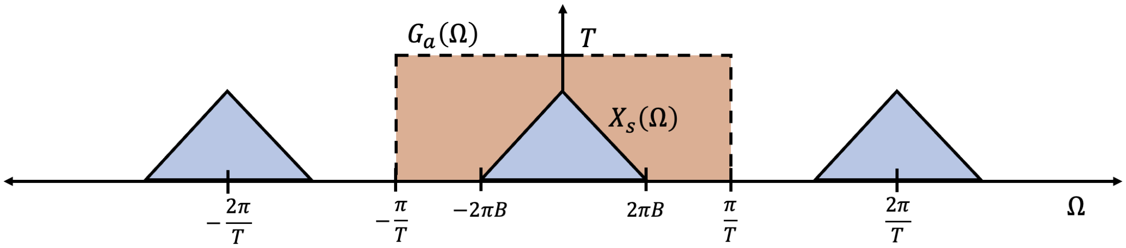

Figure 2: Illustration of how interpolation function $g(t) \xleftrightarrow{\mathcal{F}} G_a(\Omega)$ extracts only the central spectral copy from $X_s(\Omega)$ in the frequency domain. The resulting output of $G_a(\Omega)X_s(\Omega)$ gives the original $X_a(\Omega)$ spectrum.

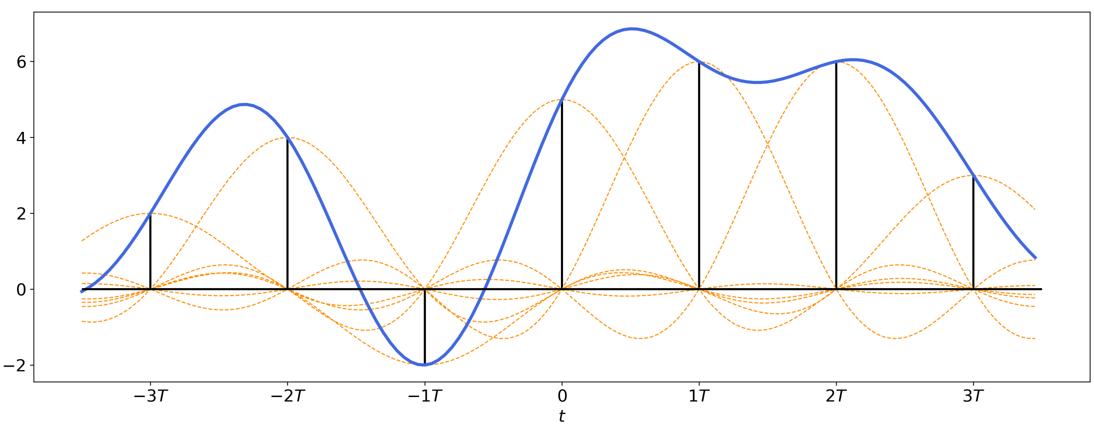

Figure 3: Example of recovering a bandlimited continuous-time signal by applying sinc interpolation to a discrete-time sequence. Each dashed orange line is a shifted and scaled sinc interpolation function, each black line is a discrete-time signal value $x[n]$, and the solid blue line is the final result of sinc interpolation.

$$= \text{sinc}\left(\frac{\pi t}{T}\right). \tag{12}$$

We simplify $g(t)$ using the sinc function in Eqn. 12 where

$$\text{sinc}(t) = \frac{\sin t}{t} = \begin{cases} 1, & t = 0 \\ \frac{\sin t}{t}, & t \neq 0 \end{cases}. \tag{13}$$

Finally, now that we have solved for the interpolation function $g(t)$, we are able to state the ideal digital-to-analog reconstruction formula:

$$x(t) = \sum_{n=-\infty}^{\infty} x[n]\text{sinc}\left(\frac{\pi(t - nT)}{T}\right). \tag{14}$$

This reconstruction formula is also commonly referred to as the *sinc interpolation formula*. Figure 3 provides an example of the sinc interpolation formula in action. Note how each individual sinc function has the same height as the sample $x[n]$ it interpolates over. Furthermore, each sinc function has nulls (goes to 0) at every multiple of T to the left and right of its center point at nT.

We have made an important assumption in this lecture: the bandlimited analog signal $x(t)$ is recoverable. We are able to perfectly reconstruct an analog signal if the central spectral copy is separable from the other spectral copies like we see in Fig. 2. We stated in lecture 18 that we will not have overlap if we sample fast enough according to the Nyquist criterion. In the next lecture, we will explore what happens when we sample below the Nyquist rate and interpret the resulting phenomenon known as aliasing.