NAME: Junsheng Huang

netID: jh103

section:

PROBLEM SET # 7

[1]. X exponential distribution λ

(a) $E[X^n] = \int_0^\infty t^n \cdot \lambda e^{-\lambda t} dt$

$= -t^n e^{-\lambda t}\Big|_0^\infty + \int_0^\infty n \cdot t^{n-1} e^{-\lambda t} dt$

$= \frac{n}{\lambda} E[X^{n-1}]$,

$E[X] = \frac{1}{\lambda} \Rightarrow E[X^2] = \frac{2}{\lambda^2}$

(b) $P\{[X^2] = 4\} = P\{4 \le X^2 < 5\}$ this is because for exponential distribution, $f_X(t) = 0$ for $t<0$, no need to consider negative part.

since X is continuous-type random variable,

$P\{4 \le X^2 < 5\} = P\{4 \le X^2 \le 5\} = P\{2 \le X \le \sqrt{5}\} = \int_2^{\sqrt{5}} \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_2^{\sqrt{5}} = e^{-2\lambda} - e^{-\sqrt{5}\lambda}$

[2]. 2 fishes per hour, 6.AM start

(a) for poisson process with $\lambda = 2$:

$P(N_{0.5} - N_0 = 1) = \frac{[2 \cdot 0.5]^1}{1!} \cdot e^{-2 \cdot 0.5} = e^{-1}$

(b) the fish between 6.AM and 6:30AM: X_1

total fish by 8.AM: X

$P(X_1 = 1, X = 4) = P(N_{0.5} - N_0 = 1, N_4 - N_{0.5} = 3)$ independent!

$= P(N_{0.5} - N_0 = 1) \cdot P(N_4 - N_{0.5} = 3)$

$= e^{-1} \cdot \frac{[2 \cdot 2.5]^3}{3!} \cdot e^{-2 \cdot 2.5}$

$= e^{-1} \cdot \frac{5^3}{6} \cdot e^{-5}$

$= \frac{125}{6} e^{-6}$

(c) $P\{N_6 \ge 5\} = 1 - P(N_6 = 0) - P(N_6 = 1) - P(N_6 = 2) - P(N_6 = 3) - P(N_6 = 4)$

$= 1 - e^{-12} - \frac{12}{1!} \cdot e^{-12} - \frac{12^2}{2!} e^{-12} - \frac{12^3}{3!} e^{-12} - \frac{12^4}{4!} e^{-12}$

$= 1 - 1237 e^{-12}$

(d) the fish from 6AM till noon: X

the fish from 6AM to 6:30AM: X_1

the probability is: $P(X_1 \ge 5 \mid X = 6) = \frac{P(X_1 = 5, X = 6) + P(X_1 = 6, X = 6)}{P(X = 6)} = \frac{\frac{1}{5!} \cdot e^{-1} \cdot 11 \cdot e^{-11} + \frac{1}{6!} e^{-1} \cdot e^{-11}}{\frac{12^6}{6!} e^{-12}} = \frac{67}{12^6} = \frac{67}{2985984}$

[3]. 4 light bulbs, burnout rate: 0.1 per day.

the burn out number of light bulbs should be a Poisson process with $\lambda = 0.1 \cdot 4 = 0.4$

(a) without going to store $\Rightarrow$ burn out number ≤ 3

$P(N_7 \le 3) = P(N_7 = 0) + P(N_7 = 1) + P(N_7 = 2) + P(N_7 = 3)$

$= e^{-2.8} + \frac{2.8^1}{1!} e^{-2.8} + \frac{2.8^2}{2!} e^{-2.8} + \frac{2.8^3}{3!} e^{-2.8}$

$= 0.6919$

(b) consider each day as X_i, $i = 1, 2 \cdots 7$, each is independent with each other

Since each day at most 4 burnout, X_i follows binomial distribution

$P(X_i = 0) = (0.9)^4$, $P(X_i \ne 0) = 1 - 0.9^4$

$\therefore$ the probability is $P = [(0.9)^4]^6 \cdot (1 - 0.9^4) \cdot C_7^1 + [0.9^4]^7$

$= 0.244$

[4]. $X \in [-3, 5]$, uniformly distributed

$$f_X(u) = \begin{cases} \frac{1}{8} & -3 \le u \le 5 \\ 0 & \text{otherwise} \end{cases}$$

to obtain Y that uniformly distributed over $[0,1]$,

$Y = aX + b$, $f_Y(u) = f_X(\frac{u-b}{a}) \cdot \frac{1}{a}$

and $$f_Y(u) = \begin{cases} 1 & 0 \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

so $a = \frac{1}{8}$, $-3 \le \frac{u-b}{8} \le 5 \Rightarrow -\frac{3}{8} \le u-b \le \frac{5}{8}$, since $0 \le u \le 1$, $b = \frac{3}{8}$

so $Y = \frac{1}{8}X + \frac{3}{8}$