University of Illinois **Fall 2023**

ECE 313: Problem Set 5

Due: Friday, September 29 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.10 - 2.11

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings. **5 points will be deducted for a submission with incorrectly-assigned page numbers.**

1. **[ML Hypothesis Testing]**
   Consider the hypothesis testing problem in which the pmf's of the observation X under hypotheses H_0 and H_1 are given, respectively, by:

   $$p_0(k) = \frac{1}{4} \text{ for } k = 1, 2, 3, 4.$$

   and

   $$p_1(k) = \frac{k}{10} \text{ for } k = 1, 2, 3, 4.$$

   (a) Find the ML decision rule using the likelihood matrix.
   (b) Confirm that you obtain the same ML rule from the likelihood ratio form.
   (c) Find $p_{\text{false-alarm}}$ and p_{miss} for the ML rule.
   (d) Assuming priors $\pi_0 = \frac{1}{3}$ and $\pi_1 = \frac{2}{3}$, find the average probability of error p_e for the ML rule.

2. **[MAP Hypothesis Testing]**
   Consider the same hypotheses as in Problem 1. Assume priors $\pi_0 = \frac{1}{3}$ and $\pi_1 = \frac{2}{3}$.

   (a) Find the MAP decision rule using the joint probability matrix.
   (b) Confirm that you obtain the same MAP rule using the likelihood ratio form.
   (c) Find the average probability of error p_e for the MAP rule.
   (d) Compare the p_e value for the MAP rule with that of the ML rule for priors $\pi_0 = \frac{1}{3}$ and $\pi_1 = \frac{2}{3}$.

3. **[ML Testing between geometrics]**
Anne has two biased coins which look identical, but with probability of Heads being ρ_0 for one of the coins, and ρ_1 for the other, with $0 < \rho_0 < \rho_1 < 1$. She hands one of these coins to Ben without telling him which one it is. Ben tosses the coin he receives multiple times. Let X denote the number of tosses it takes for Ben to see the first Heads. Let H_i denote the hypothesis that the coin that Ben is tossing is has probability of Heads being ρ_i, $i = 0, 1$. Ben needs to determine which hypothesis is true (i.e., which coin he is tossing) based on the observation X. In this problem Ben uses the ML decision rule since he has no priors on the hypotheses.

   (a) Find the ML decision rule as a function of ρ_0 and ρ_1. Assume that ties are broken in favor of H_1. Recall that $0 < \rho_0 < \rho_1 < 1$, and note that it is easier to use the likelihood ratio test approach.

   (b) Find $p_{\text{false-alarm}}$ and p_{miss} for the ML rule, assuming that $\rho_0 = 0.25$ and $\rho_1 = 0.75$.

4. **[MAP Testing between geometrics]**
Consider the same hypothesis testing problem as in Problem 2. Assume now that Ben knows that Anne is three times as likely to hand him the coin that has probability of Heads being ρ_0, i.e., $\pi_0 = 3\pi_1$.

   (a) Find the MAP decision rule as a function of ρ_0 and ρ_1. Assume that ties are broken in favor of H_1. Again note that it is easier to use the likelihood ratio test approach.

   (b) Find the average probability of error p_e for the MAP rule, assuming that $\rho_0 = 0.25$ and $\rho_1 = 0.75$.