ECE 310 Fall 2023

Lecture 8

Inverse z-transform

Corey Snyder

Learning Objectives

After this lecture, you should be able to:

- Write the common forms for the z-transform of finite-length and infinite-length signals.
- Compute the inverse z-transform of a given rational expression in the z-domain using partial fraction expansion and inspection.

Recap from previous lecture

We have spent the previous two lectures introducing and defining the z-transform and its key properties. We know how to compute the z-transform of finite-length and infinite-length sequences, and how to manipulate these signals in both the time-domain and z-domain using z-transform properties. This lecture will introduce the inverse z-transform and popular techniques for converting a z-transform to its time-domain representation.

1 Common forms of the z-transform

Before defining and exploring the inverse z-transform, it will be helpful to establish some general notation for the z-transform of any finite-length or infinite-length signal.

1.1 Finite-length signals

For finite-length signals, we need only refer to our decomposition of signals as a summation of unit impulse functions we used in Lecture 4:

$$x[n] = \sum_{k=k_s}^{k_e} x[k]\delta[n-k], \tag{1}$$

where k_s and k_e indicate the start and end indices of our finite-length sequence, e.g. a causal length-N signal starting at $n = 0$ has $k_s = 0$ and $k_e = N - 1$. The resulting z-transform is then

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x[n]z^{-n} && (2)\\ &= \sum_{n=-\infty}^{\infty} \left(\sum_{k=k_s}^{k_e} x[k]\delta[n-k]\right) z^{-n} && (3)\\ &= \sum_{k=k_s}^{k_e} x[k] \sum_{n=-\infty}^{\infty} \delta[n-k]z^{-n} && (4) \end{aligned}$$

$$= \sum_{k=k_s}^{k_e} x[k]z^{-k}. \tag{5}$$

Above, line 4 follows from exchanging our sums for n and k and line 5 from the z-transform of $\delta[n]$ and time shifting property.

1.2 Infinite-length signals

For infinite-length signals, we note from the common z-transform pairs from Lecture 6 that these transforms are given by *rational expressions*. More precisely, they are the ratio between two polynomials in z. We say these rational expressions are *proper* if the degree of the numerator is less than the degree of the denominator. We may generalize these proper rational expressions in a couple ways:

$$X(z) = \frac{\sum_{k=0}^{N-1} b_k z^{-k}}{1 + \sum_{k=1}^{M} a_k z^{-k}} \tag{6}$$

$$X(z) = \frac{\prod_{k=1}^{N-1}(1 - q_k z^{-1})}{\prod_{k=1}^{M}(1 - p_k z^{-1})}. \tag{7}$$

The form given in Eqn. 6 simply expresses the polynomials of the numerator and denominator. The product form shown in Eqn. 7 factorizes the numerator and denominator to make clear the locations of the zeros and poles for $X(z)$ at q_k and p_k, respectively. Note that the $\prod$ symbol denotes the product operator which computes the product of the elements it iterates over (product version of the sum operator). In both forms, we have N input terms and M output/feedback terms.

2 The inverse z-transform

The inverse z-transform is formally defined as

$$x[n] = \frac{1}{j2\pi} \oint_C X(z) z^{n-1} dz. \tag{8}$$

This formula requires us to perform a closed line integral over the complex z-domain. Fortunately, we can avoid performing this integral for most signals we are interested in using a couple different techniques. We will entirely avoid this integral in this course.

2.1 Finite-length signals

The inverse z-transform of finite-length signals is easy to define. In fact, we have already done it! Equations 1 and 5 describe our z-transform pair for any finite-length signal:

$$x[n] = \sum_{k=k_s}^{k_e} x[k]\delta[n-k] \overset{\mathcal{Z}}{\longleftrightarrow} \sum_{k=k_s}^{k_e} x[k]z^{-k}. \tag{9}$$

We of course may have finite-length signals not expressed as a summation of scaled and shifted impulses that we can identify by inspection. Still, the transform pair expressed in Eqn. 9 is perfectly valid.

2.2 Infinite-length signals

Computing the inverse z-transform of infinite-length sequences is more challenging; however, our rational expression forms of Eqns. 6 and 7 give us a helpful starting point.

First, We can separate our product form from Eqn. 7 into a sum of rational expressions:

$$\frac{\prod_{k=1}^{N-1}(1 - q_k z^{-1})}{\prod_{k=1}^{M}(1 - p_k z^{-1})} = \sum_{k=1}^{M} \frac{A_k}{1 - p_k z^{-1}}. \tag{10}$$

The numerators represented by each A_k are the necessary quantities such that combining each rational expression into a common denominator will yield the numerator given in Eqn. 6.

Next, let's consider an example of what each individual term of this sum represents. For concreteness, let $A_1 = 3$, $p_1 = \frac{1}{2}$, $A_2 = -1$, and $p_2 = -2$ and assume the signal is causal:

$$\sum_{k=1}^{2} \frac{A_k}{1 - p_k z^{-1}} = \frac{3}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 + 2z^{-1}} \tag{11}$$

$$\frac{3}{1 - \frac{1}{2}z^{-1}} \overset{\mathcal{Z}^{-1}}{\mapsto} 3\left(\frac{1}{2}\right)^n u[n] \tag{12}$$

$$-\frac{1}{1 + 2z^{-1}} \overset{\mathcal{Z}^{-1}}{\mapsto} -(-2)^n u[n] \tag{13}$$

$$\frac{3}{1 - \frac{1}{2}z^{-1}} - \frac{1}{1 + 2z^{-1}} \overset{\mathcal{Z}^{-1}}{\mapsto} 3\left(\frac{1}{2}\right)^n u[n] - (-2)^n u[n]. \tag{14}$$

Above, we use $\overset{\mathcal{Z}^{-1}}{\mapsto}$ to denote the inverse z-transform. We obtain our inverse z-transform in lines 12 and 13 by *inspection* using our z-transform pairs table. Finally, we obtain our corresponding $x[n]$ by combining these two terms using the linearity of the z-transform.

We have shown that the summation in Eqn. 10 decomposes our z-transform into a form that corresponds to a sum of exponentials in the time-domain. This allows us to compute our inverse z-transform by inspection using table look-ups and common z-transform properties so long as we have our A_k quantities. Let's look at an example where we must solve for these A_k values.

Exercise 1: Compute the inverse z-transform for the following transfer function:

$$H(z) = \frac{2 - z^{-1}}{1 - \frac{7}{3}z^{-1} + \frac{2}{3}z^{-2}}. \tag{15}$$

First, we must factor our denominator into a product of binomial terms.

$$1 - \frac{7}{3}z^{-1} + \frac{2}{3}z^{-2} = \left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 2z^{-1}\right) \tag{16}$$

Now, we have

$$H(z) = \frac{A_1}{1 - \frac{1}{3}z^{-1}} + \frac{A_2}{1 - 2z^{-1}}. \tag{17}$$

Multiplying both sides by $\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - 2z^{-1}\right)$ then gives us

$$2 - z^{-1} = A_1\left(1 - 2z^{-1}\right) + A_2\left(1 - \frac{1}{3}z^{-1}\right). \tag{18}$$

We can solve for A_1 and A_2 by setting $z = \frac{1}{3}$ and $z = 2$, respectively. Doing so will give us $A_1 = \frac{1}{5}$ and $A_2 = \frac{9}{5}$. Using our z-transform pairs, we then arrive at our final inverse z-transform:

$$h[n] = \frac{1}{5}\left(\frac{1}{3}\right)^n u[n] + \frac{9}{5}(2)^n u[n], \;\; h[n] \text{ right-sided} \tag{19}$$

$$h[n] = -\frac{1}{5}\left(\frac{1}{3}\right)^n u[-n-1] - \frac{9}{5}(2)^n u[-n-1], \;\; h[n] \text{ left-sided.} \tag{20}$$

Note that we did not specify our ROC for $H(z)$ (to not reveal the poles when factorizing); thus, we have provided both solutions depending on if our signals are both right-sided or left-sided. Also note that it could be possible for one to be right-sided and the other left-sided! We must check the ROC to determine this.

The procedure we followed in this exercise to solve for each A_k is known as *partial fraction expansion* or *partial fraction decomposition.* In brief, this procedure is as follows:

1. Factorize denominator of $H(z)$ to obtain the summation form given in Eqn. 10.
2. Multiply both sides – $H(z)$ and your new factorized form – by their shared denominator.
3. Solve for each A_k by setting $z = p_k$.

Finally, let's consider one more useful example.

Exercise 2: Compute inverse z-transform for the following transfer function of a causal signal:

$$H(z) = \frac{3 - \frac{3}{2}z^{-1}}{1 - z^{-1} + z^{-2}}. \tag{21}$$

By the discriminant rule, we can see that our denominator will not have real roots. We can solve for these roots using the quadratic formula:

$$\begin{aligned} p_k &= \frac{1 \pm \sqrt{1 - 4(1)(1)}}{2} && (22)\\ &= \frac{1}{2} \pm j\frac{\sqrt{3}}{2} && (23)\\ &= e^{\pm j\frac{\pi}{3}}. && (24) \end{aligned}$$

Now we can write

$$H(z) = \frac{A_1}{1 - e^{j\frac{\pi}{3}}z^{-1}} + \frac{A_2}{1 - e^{-j\frac{\pi}{3}}z^{-1}}. \tag{25}$$

Multiplying both sides by $1 - z^{-1} + z^{-2}$,

$$3 - \frac{3}{2}z^{-1} = A_1\left(1 - e^{-j\frac{\pi}{3}}z^{-1}\right) + A_2\left(1 - e^{j\frac{\pi}{3}}z^{-1}\right). \tag{26}$$

Setting $z = e^{j\frac{\pi}{3}}$, we obtain

$$\begin{aligned} 3 - \frac{3}{2}e^{-j\frac{\pi}{3}} &= A_1\left(1 - e^{-j\frac{2\pi}{3}}\right) && (27)\\ \frac{9}{4} + j\frac{3\sqrt{3}}{4} &= A_1\left(\frac{3}{2} + j\frac{\sqrt{3}}{2}\right) && (28)\\ A_1 &= \frac{3}{2}, && (29) \end{aligned}$$

where line 29 follows from multiplying both sides by the conjugate $\frac{3}{2} - j\frac{\sqrt{3}}{2}$. We can similarly set $z = e^{-j\frac{\pi}{3}}$ to find $A_2 = \frac{3}{2}$. Using our transform pairs, we will find

$$\begin{aligned} h[n] &= \frac{3}{2}\left(e^{j\frac{\pi}{3}n} + e^{-j\frac{\pi}{3}n}\right)u[n] && (30)\\ &= 3\cos\left(\frac{\pi}{3}n\right)u[n]. && (31) \end{aligned}$$

In the final line, we simply invoke Euler's identity for cosine.

This exercise helps us identify some special rules for our inverse z-transform. If we have a pair of conjugate poles, we will be able to use Euler's identity to find cosine or sine in the time-domain. For a given pair of poles expressed by $z = ae^{\pm j\omega}$, we will obtain $a^n\cos(\omega n)u[n]$ or $a^n\sin(\omega n)u[n]$ (for causal signals) in the time-domain by following Euler's identities. If $a = 1$, then we have poles on the unit-circle in the z-domain and will have a truly periodic cosine or sine appear in our inverse z-transform. If $a < 1$, we have a decaying sinusoid and for $a > 1$, we have a growing sinusoid.