ECE 313: Problem Set 1: Solutions

1. **[Defining a set of outcomes]**

   (a) A natural choice is $\Omega = \{\{b_1, b_2, b_3\} : 1 \leq b_i \leq 10, b_1, b_2, b_3 \text{ distinct}\}$, where for a given outcome $\{b_1, b_2, b_3\}$, b_i denotes the number on the i^{th} ball drawn from the bag.

   (b) $(10)(9)(8)/(6) = 120$, because there are 10 possible choices for b_1, and given b_1 there are 9 possible choices for b_2, and given b_1 and b_2, there are 8 possible choices for b_3, thus yielding $(10)(9)(8) = 720$. Finally, notice that for a given choice of $\{b_1, b_2, b_3\}$, there are $3! = 6$ permutations that correspond to the same outcome, so we need to divide 720 by 6 which gives the desired result.

2. **[Using set theory to calculate probabilities of events]**

   (a) If $B \subset A$, then $AB = B$ and $P(AB) = P(B) = 1/3$.

   (b) Using relationship:

   $$1 = P(\Omega) = P(A \cup B) = P(A) + P(B) - P(AB) = \frac{3}{4} + \frac{1}{3} - P(AB).$$

   Hence we get $P(AB) = 13/12 - 1 = 1/12$.

3. **[Displaying outcomes in a two event Karnaugh map]**

   (a) First, consider event A. Since there are 3 possible (i.e., $4 = 1 + 3 = 2 + 2 = 3 + 1$) out of total 36 ($= 6 \times 6$) possible summations, $P(A) = 3/36 = 1/12$. For event B, one can readily see that $P(B) = 1/3 + 1/3 - 1/3 \times 1/3 = 5/9$. See below for a Karnaugh map, where $(a, b),\ a, b \in \{1, 2, 3, 4, 5, 6\}$ means that a and b are popped up after rolling the first and the second dices, respectively.

	A	A^c
B	$(2,2)$ $(1,3)$ $(3,1)$	$(1,1)$ $(1,2)$ $(1,4)$ $(1,5)$ $(1,6)$ $(2,1)$ $(2,3)$ $(2,4)$ $(2,5)$ $(2,6)$ $(3,2)$ $(4,1)$ $(4,2)$ $(5,1)$ $(5,2)$ $(6,1)$ $(6,2)$
B^c		$(3,3)$ $(3,4)$ $(3,5)$ $(3,6)$ $(4,3)$ $(4,4)$ $(4,5)$ $(4,6)$ $(5,3)$ $(5,4)$ $(5,5)$ $(5,6)$ $(6,3)$ $(6,4)$ $(6,5)$ $(6,6)$

Figure 1: Karnaugh map for Problem 3.

   (b) $P(AB) = 3/36 = 1/12$.

4. **[A Karnaugh map for three events]**

   Suppose A denotes a set of individuals participating in yoga activities; B denotes a set of individuals participating in running; and C denotes that a set of individuals participating in zumba. Then the final map is shown in the next page. Since there are 100 individuals in

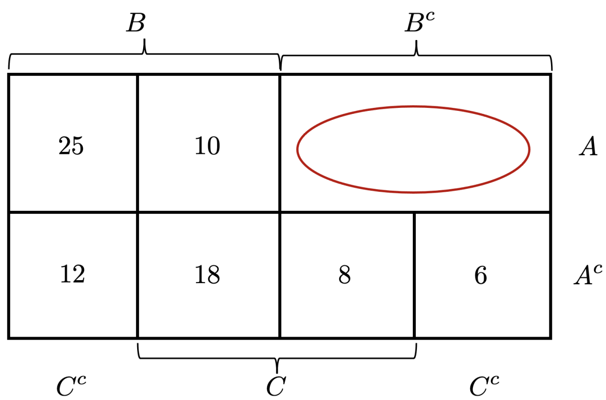

Figure 2: Karnaugh map for Problem 4.

   total, the number of members who participate in yoga but do not run, is $21 = 100 - 65 - 14$ (see red-circled part).

5. **[Selecting socks at random with a twist]**

   (a) One choice is

$$\Omega = \big\{(A_1, A_2, A_3, A_4) : A_i \subset \{BR1, BR2, BG1, BG2, GR1, GR2, GG1, GG2\}, \\ |A_i| = 2, A_iA_j = \emptyset \ \text{ for all } i \neq j\big\},$$

   where an outcome $\{A_1, A_2, A_3, A_4\}$ means person i draws out the two socks in A_i ($1 \leq i \leq 4$).

   (b) $|\Omega| = 8!/2^4 = 2,520$ because there are $8!$ orders that the socks could be drawn out one at a time, but this over counts by a factor of $(2)^4$ because the order each person draws two socks doesn't matter. Another way to get this answer is to note that there are $\binom{8}{2}$ possible choices for A_1, then $\binom{6}{2}$ possible choices for A_2, then $\binom{4}{2}$ choices for A_3, then A_4 is determined as well. So $|\Omega| = \binom{8}{2}\binom{6}{2}\binom{4}{2} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{2 \cdot 2 \cdot 2}$

   (c) $|M| = 2 \cdot 2 = 4$. Notice that one specific possible outcome would be $(\{BR1, BR2\}, \{BG1, BG2\}, \{GR1, GR2\}, \{GG1, GG2\})$ in M. Other outcomes in M can be obtained if the boys exchanged their socks with each other and so did the girls leading to $2 \cdot 2 =$ permutations.

   (d) $P(M) = \frac{|M|}{|\Omega|} = \frac{2^4 \cdot 4}{8!} = \frac{1}{7 \cdot 6 \cdot 5 \cdot 3} = \frac{1}{630}$.

   (e) Suppose the two boys go first followed by the two girls in drawing socks. The first boy has a 1/2 chance of drawing the sock of the correct gender, then in the second draw has a 1/7 chance of drawing the sock which matches the color and gender of the first sock. The second boy has 1/3 chance of drawing the sock of correct gender, and in the second draw, has a 1/5 chance of drawing the sock that matches the color and gender of the first sock. The first girl draws a sock and then has a 1/3 chance of drawing

a sock of the right color. The remaining two socks are left for the second girl. So $P(M) = (1/2)(1/7)(1/3)(1/5)(1/3) = \frac{1}{630}$.
If instead the boys and girls took turns, i.e., first boy, first girl, second boy, and the second girl, to draw then the first boy has a 1/2 chance of drawing the sock of the correct gender, then in the second draw has a 1/7 chance of drawing the sock which matches the color and gender of the first sock. The first girl has 2/3 chance of drawing the sock of correct gender, and in the second draw, has a 1/5 chance of drawing the sock that matches the color and gender of the first sock. The second boy has a 1/2 chance of drawing a sock of the right gender and in the second draw a 1/3 chance of getting the matching sock. The remaining two socks are left for the second girl. Thus, $P(M) = (1/2)(1/7)(2/3)(1/5)(1/2)(1/3) = \frac{1}{630}$. Therefore, it does not matter in what gender sequence the individuals take turns to draw.

6. **[Two more poker hands]**

   (a) There are $\binom{13}{5}$ ways to select the numbers for the five cards, then 4 ways to choose the suit. Thus,

$$\begin{aligned} P(FLUSH) &= \frac{\binom{13}{5} \cdot 4}{\binom{52}{5}} \\ &= \frac{13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 4}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} \approx 0.00198 \end{aligned}$$

   (b) Notice that there are 16 cards in total, which are one of A, K, Q, J. Hence,

$$\begin{aligned} P(SPECIAL) &= \frac{\binom{16}{5}}{\binom{52}{5}} \\ &= \frac{1}{595} \approx 0.00168 \end{aligned}$$

7. **[Fishing]**

   (a) There are $\binom{10}{5}$ ways to catch 5 fish. Of these, there are $\binom{6}{3} \cdot \binom{4}{2}$ ways of catching 3 goldfish and 2 catfish. Thus,

$$\begin{aligned} P(G) &= \frac{\binom{6}{3} \cdot \binom{4}{2}}{\binom{10}{5}} \\ &= \frac{10}{21} \approx 0.476. \end{aligned}$$

   (b) There are $\binom{10}{3}$ ways to catch 3 fish. Of these, there are $\binom{3}{2}$ ways of catching exactly two goldfish that were previously caught. Notice that there are $\binom{7}{1}$ ways of catching the third fish since the third fish should not be the remaining one goldfish caught in the first place. This gives:

$$\begin{aligned} P(A) &= \frac{\binom{3}{2} \cdot \binom{7}{1}}{\binom{10}{3}} \\ &= \frac{7}{40} = 0.175. \end{aligned}$$