ECE 310 Fall 2023

Lecture 16

Frequency response

Corey Snyder

Learning Objectives

After this lecture, you should be able to:

- Compute the response of LTI systems to sinusoidal input signals.
- Describe and evaluate the response of LTI systems to aperiodic input signals using the DTFT.

Recap from previous lecture

We continued defining the discrete-time Fourier transform (DTFT) in the previous lecture by explaining the connections between the z-transform and DTFT. We also reviewed important DTFT transform pairs and properties that are key to applying the DTFT in practice. In this lecture, we will build on these tools to explore the response of LTI systems using the DTFT.

1 Frequency response

Recall that we can represent LTI systems by their impulse response $h[n]$. When discussing the z-transform, we defined the *transfer function* $H(z)$ as the z-transform of $h[n]$:

$$\text{Impulse response } h[n] \overset{\mathcal{Z}}{\longleftrightarrow} H(z) \text{ Transfer function.} \tag{1}$$

Similarly, we refer to the DTFT of an LTI system's impulse response as the *frequency response* of that system:

$$\text{Impulse response } h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(\omega) \text{ Frequency response.} \tag{2}$$

In the following sections, we will demonstrate how we can use the frequency response of LTI systems to compute system outputs to input signals.

1.1 LTI response to periodic exponential signals

We proved in lecture 6 that complex exponential signals are *eigenfunctions* of LTI systems. Recall that an eigenfunction or eigenvector of a system or operator is an input that will only be rescaled after being acted on by the system. For LTI systems, this means

$$y[n] = x[n] * h[n],\ x[n] = Az^n,\ A, z \in \mathbb{C} \tag{3}$$

$$\implies y[n] = H(z)Az^n. \tag{4}$$

Thus, $H(z)$ gives the *eigenvalue* of the LTI system for the eigenfunction of z^n. Note that $H(z)$ may be complex-valued, thus the rescaling includes changes in magnitude and/or phase. Looking to LTI systems with a given frequency response $H(\omega)$, we are guaranteed the same relationship for periodic complex exponentials, i.e. $z^n = e^{j\omega_0 n}$:

$$y[n] = x[n] * h[n],\ x[n] = Ae^{j\omega_0 n},\ A \in \mathbb{C}, \omega_0 \in \mathbb{R} \tag{5}$$
$$\implies y[n] = H(\omega_0)Ae^{j\omega_0 n}. \tag{6}$$

We can take Eqn. 6 one step further by acknowledging

$$H(\omega_0) = |H(\omega_0)|e^{j\angle H(\omega_0)}, \tag{7}$$

where $|H(\omega_0)|$ and $\angle H(\omega_0)$ give the magnitude and phase, respectively, of the complex number $H(\omega_0)$.

$$H(\omega_0)Ae^{j\omega_0 n} = |H(\omega_0)|Ae^{j(\omega_0 n + \angle H(\omega_0))} \tag{8}$$

1.2 LTI response to sinusoids

With the response of LTI systems to periodic complex exponential signals in hand, we can look to how LTI systems act on sinusoidal signals. Consider the following general cosine signal:

$$x[n] = A\cos(\omega_0 n + \theta). \tag{9}$$

Using Euler's identity for cosine, we may also write $x[n]$ as

$$x[n] = \frac{A}{2}\left(e^{j(\omega_0 n+\theta)} + e^{-j(\omega_0 n+\theta)}\right). \tag{10}$$

We can then compute the response of an LTI system with frequency response $H(\omega)$ to this cosine using Eqn. 8:

$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \frac{A}{2}\left(|H(\omega_0)|e^{j(\omega_0 n+\theta+\angle H(\omega_0))} + |H(-\omega_0)|e^{-j(\omega_0 n+\theta-\angle H(-\omega_0))}\right) \end{aligned} \tag{11}$$

Suppose now we know that our LTI system has a real-valued impulse response. We proved in the previous lecture that a real-valued signal will have a Hermitian symmetric DTFT. Thus, if $h[n]$ is real-valued,

$$H^*(\omega) = H(-\omega) \tag{12}$$
$$\implies |H(\omega)| = |H(-\omega)|,\ \angle H(\omega) = -\angle H(-\omega). \tag{13}$$

With the assumption that $h[n]$ is real-valued, we may then re-write Eqn. 11 as

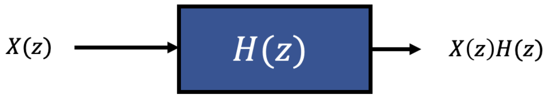
$$\begin{aligned} y[n] &= x[n] * h[n] \\ &= \frac{A}{2}\left(|H(\omega_0)|e^{j(\omega_0 n+\theta+\angle H(\omega_0))} + |H(-\omega_0)|e^{-j(\omega_0 n+\theta-\angle H(-\omega_0))}\right) \\ &= \frac{A}{2}\left(|H(\omega_0)|e^{j(\omega_0 n+\theta+\angle H(\omega_0))} + |H(\omega_0)|e^{-j(\omega_0 n+\theta+\angle H(\omega_0))}\right) && (14) \\ &= |H(\omega_0)|A\cos(\omega_0 n + \theta + \angle H(\omega_0)). && (15) \end{aligned}$$

The above result holds the same for sine instead of cosine. Thus, Eqn. 15 describe the sinusoidal response of *real-valued* LTI systems with frequency response $H(\omega)$.

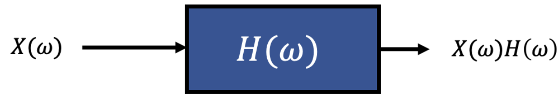
1.3 LTI response to aperiodic inputs

We may describe the response of an LTI system to any aperiodic input using the DTFT and its convolution property:

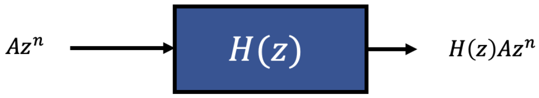
$$y[n] = x[n] * h[n]$$



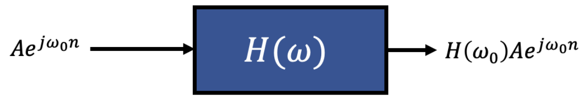
(a) System response using transfer function.



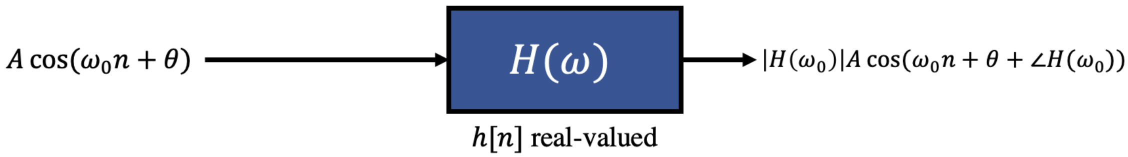
(b) System response using frequency response.



(c) Eigenfunction of transfer functions.



(d) Eigenfunction of frequency responses.



(e) Sinusoidal response of real-valued LTI systems.

Figure 1: Summary of the response of LTI systems using the z-transform and DTFT.

$$Y(\omega) = X(\omega)H(\omega). \tag{16}$$

Just like with the z-transform, we see that convolution in the time-domain corresponds to multiplication in the transform domain. Thus, we may compute the response of an LTI system to any aperiodic input by multiplying their respective DTFTs, assuming they exist, and taking the inverse DTFT of the result.

For example, using our DTFT pairs from the previous lecture, we can quickly derive the result from Eqn. 6.

$$\begin{aligned}
x[n] &= Ae^{j\omega_0 n} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi A\delta(\omega - \omega_0) && (17)\\
Y(\omega) &= X(\omega)H(\omega) \\
&= 2\pi A\delta(\omega - \omega_0)H(\omega) && (18)\\
&= 2\pi H(\omega_0)A\delta(\omega - \omega_0) && (19)\\
y[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(\omega)e^{j\omega n}d\omega && (20)\\
&= H(\omega_0)A\int_{-\pi}^{\pi} \delta(\omega - \omega_0)e^{j\omega n}d\omega && (21)\\
&= H(\omega_0)Ae^{j\omega_0 n}. && (22)
\end{aligned}$$

The frequency response of LTI systems is of fundamental importance because we may use it to

1. Compute the response to any signal with a defined DTFT.
2. Design and analyze systems that pass or stop certain signal components according to their frequency contents.

We will explore these two elements in the upcoming lectures as we further discuss frequency responses using their magnitude and phase responses and ideal LTI filter representations. Figure 1 provides a summary of what we have covered thus far in the course regarding the response of LTI systems to input signals using the z-domain or frequency-domain.