ECE 310 Fall 2023

Lecture 5

Difference equations and block diagrams

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Learning Objectives

After this lecture, you should be able to:

- Interpret LTI systems defined by a Linear Constant-Coefficient Difference Equation (LCCDE).
- Identify whether a given LCCDE will yield a finite or infinite impulse response.
- Translate between LCCDEs and their corresponding block diagrams

Recap from previous lecture

We covered a lot of ground in the previous lecture to introduce impulse responses, convolution, and their relationship with LTI systems. We will continue discussing LTI systems in this lecture as we define linear constant-coefficient difference equations and their graphical representations.

1 Linear constant-coefficient difference equations

A popular representation of LTI systems is given by *linear constant-coefficient difference equations* known as LCCDEs. An LCCDE is given by a linear combination of system inputs and system outputs to generate our output $y[n]$

$$y[n] = \sum_{i=1}^{K} b_i y[n-i] + \sum_{j=0}^{M-1} c_j x[n-j],\ 0 \leq K < \infty,\ 1 \leq M < \infty. \tag{1}$$

This definition is quite flexible so let's unpack everything. We have two summations that gather terms corresponding to the input and output sequences. We have K output or *feedback* terms with constant coefficients $\{b_i\}_{i=1}^{K}$ at shift locations $\{i\}_{i=1}^{K}$. We then have M input terms with coefficients $\{c_j\}_{j=0}^{M-1}$ at shift locations $\{j\}_{j=0}^{M-1}$. Note that we assume K and M are finite. We will also assume for this class that we have zero initial conditions, meaning our system is "at rest" when it receives our input $x[n]$. This is easily realizable by zeroing out memory locations before receiving an input signal. From the general definition in Eqn. 1, we can describe some important properties of our LTI system. If all shifts by i and j are non-negative, then our system is causal because we will use no future inputs or outputs to compute $y[n]$. The form shown in Eqn. 1 depicts the system as being causal; however, we could also have non-causal terms that look like $x[n+2]$ or $y[n+1]$, for example.

1.1 Infinite impulse response systems

If $K > 0$, we say our system has an *infinite impulse response* or is IIR. This is because the feedback terms allow the output to continue even with a finitely long input that goes to zero. Consider the following example input and system:

$$x[n] = \delta[n] \tag{2}$$

$$y[n] = \frac{1}{2}y[n-1] + x[n]. \tag{3}$$

Since we assume zero initial conditions, i.e. $y[n] = 0, n < 0$, our output sequence will proceed as

$$y[n] = \left\{ \underset{\uparrow}{1}, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots \right\}. \tag{4}$$

Our input signal only has one non-zero sample yet the response will still go infinitely long following the shown geometric sequence. Clearly, our $y[n]$ in Eqn. 4 represents the impulse response $h[n]$ of our system since our input is $\delta[n]$ (the impulse response of this system is given by $h[n] = \left(\frac{1}{2}\right)^n u[n]$). Here, we arrive at an important feature of LCCDEs. We cannot use the impulse response for the system in Eqn. 4 in practice since it would require an infinitely long convolution sum. However, we can still easily implement this system in computer code or hardware via the recursive structure described by its LCCDE.

1.2 Finite impulse response systems

If $K = 0$, we say our system has a *finite impulse response* or is FIR. We can verify this by computing the impulse response of our system given in Eqn. 1 when $K = 0$

$$h[n] = \sum_{j=0}^{M-1} c_j \delta[n-j], 1 \leq M < \infty. \tag{5}$$

This impulse response is finitely long and easily utilized via convolution. One common FIR system is known as the *moving average filter* given below for length L:

$$y[n] = \frac{1}{L} \sum_{i=0}^{L-1} x[n-i]. \tag{6}$$

As the name suggests, a length-L moving average filter computes the average of the present sample and the $L-1$ previous samples. This system has a finite impulse response with L entries with value $\frac{1}{L}$, i.e. $h[n] = (1/L)(u[n] - u[n-L])$, thus we may implement the system via the convolution sum. However, we can also use a more compact representation with a feedback term

$$\begin{aligned} y[n] &= \frac{1}{L} \sum_{i=0}^{L-1} x[n-i] \\ &= y[n-1] + \frac{1}{L}(x[n] - x[n-L]). \end{aligned} \tag{7}$$

The alternate form in Eqn. 7 utilizes the computation of the sum at $n-1$, which contains $L-2$ identical terms as the sum at n. Thus, using this LCCDE only requires two sum operations at each index n while the convolution sum would require L sum operations!

2 Block diagrams

We commonly visualize our LTI systems using *block diagrams* that illustrate the corresponding LCCDE. We will come back to these diagrams later in the course when we discuss FIR and IIR filter design; however, we can still demonstrate the important building blocks at this stage. Looking at Eqn. 1, we see that we need ways of expressing scaling coefficients and delays for inputs and outputs. Figure 1 shows a *delay block* that delays an input signal by one sample. The z^{-1} notation will make more sense in Lecture 6 when we begin discussing the z-transform. Figure 2 depicts how we scale an input signal or apply a *gain* of c. Finally, we have our *adder block* given in Fig. 3 that computes the sum of two input signals.

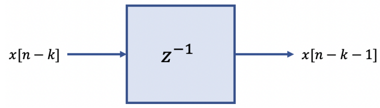

Figure 1: Delay block for LCCDE block diagram.

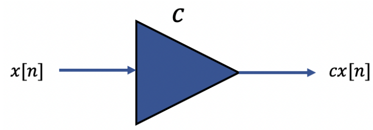

Figure 2: Coefficient/gain block for LCCDE block diagram.

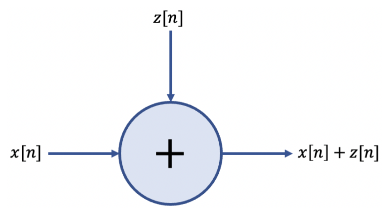

Figure 3: Adder block for LCCDE block diagram.

We can connect these blocks together to represent any LCCDE. Let's look at an example now:

$$y[n] = \frac{1}{2}y[n-1] + \frac{1}{2}y[n-2] + 3x[n] - 2x[n-1] + x[n-2]. \tag{8}$$

Figure 4 illustrates the block diagram for this system. There are multiple choices for translating an LCCDE into a block diagram and the example in Fig. 4 is known as the *direct form I* representation of the system. We will mention other block diagram forms later in this course.

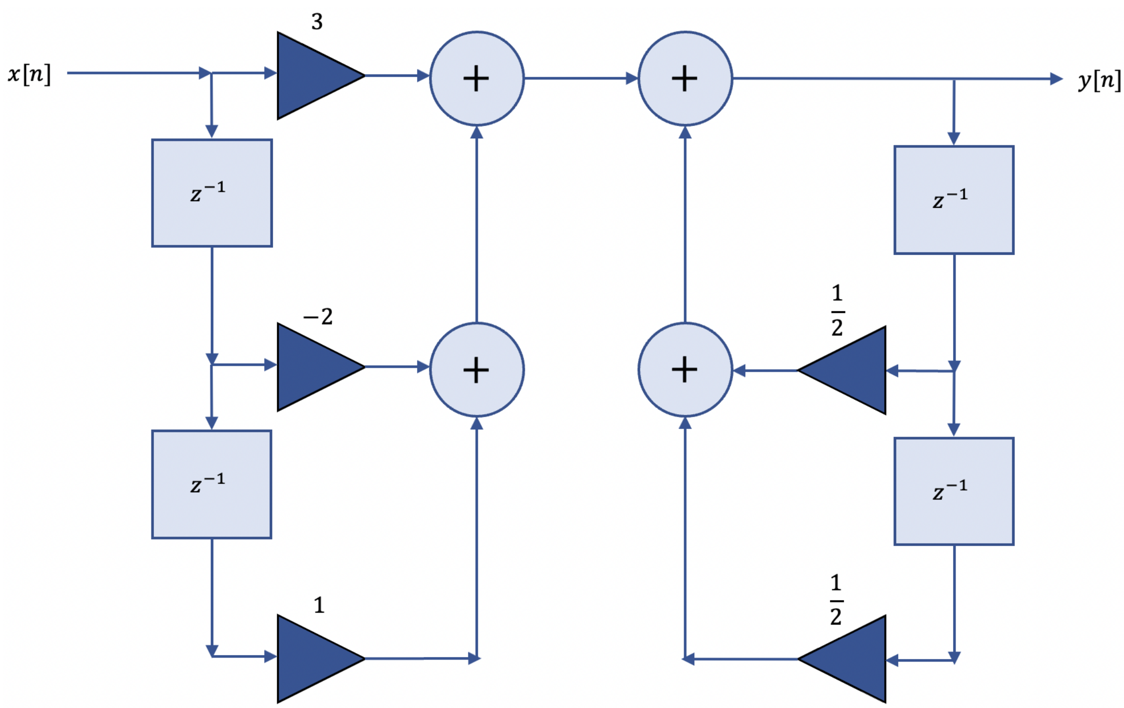

Figure 4: Direct form I representation of system in Eqn. 8.