University of Illinois **Fall 2023**

ECE 313: Problem Set 8

Due: Friday, October 20 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.6.2, 3.6.3, and 3.7

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays, and is due by 7 p.m. on the following Friday. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings.
5 points will be deducted for a submission with incorrectly-assigned page numbers.

1. **[Gaussian Distribution]**
   Suppose X is a $N(-2, 4)$ random variable that models the winter temperature on a very cold day, as measured in Fahrenheit degrees. Compute the following quantities.

   (a) $P\{X \geq -2\}$.
   (b) $P(X \geq -2 | X \geq -4)$.
   (c) Let Y be the temperature in Celsius. $P\{Y > X\}$.
   (d) $E[(X + 2)^2]$.

2. **[Communication in Gaussian Noise]**
   A wireless communication system consists of a transmitter and a receiver. The transmitter sends a signal x, and the receiver observes

   $$Y = x + Z,$$

   where Z is a noise term, modeled as a Gaussian random variable with mean $\mu_Z = 0$ and variance $\sigma_Z^2 = 1$.

   (a) Suppose the transmitted signal is $x = 1$. What is the pdf of the received signal Y?
   (b) Now suppose the transmitted signal can be either $x = -1$ or $x = 1$. The receiver uses the following decoding rule: if $Y > 0$, it declares that $x = 1$; if $Y \leq 0$, it declares that $x = -1$. Assuming that the transmitter sends -1 or $+1$ with probability $1/2$ each, what is the receiver's error probability?
   (c) Now suppose the transmitted signal x can be chosen from three possible values: $x = -2$, $x = 0$ and $x = 2$. The receiver now uses the following decoding rule: if $Y < -1$, it declares $x = -2$, if $Y > 1$, it declares $x = 2$, and otherwise it declares $x = 0$. Assuming the transmitter sends each possible symbol with probability $1/3$, what is the receiver's error probability?

3. **[Gaussian versus Poisson Approximation for Binomial Distribution]**
A communication receiver recovers a block of $n = 10^5$ bits. It is known that each bit in the block can be in error with probability 10^{-4}, independently of whether other bits are in error. Let X be the number of bit errors.

   (a) Write down an exact expression for $P\{X = 15\}$. You do not need to compute a numerical value for this probability.

   (b) Determine an approximate value for $P\{X = 15\}$ via the Gaussian approximation with continuity correction.

   (c) Solve part (b) using the Poisson approximation of a binomial distribution.

4. **[Gaussian Approximation]**
You go to a carnival and decide to play a game in which, with probability 0.4, you win 1 rubber duck, and with probability 0.6, you lose 1 rubber duck. You decide to play this same game repeatedly 100 times. Let $X_i \in \{-1, 1\}$ represent your earned rubber ducks at the ith game, for $i = 1, \ldots, 100$. Assume that $X_1, X_2, ..., X_{100}$ are all independent. Let $X = \sum_{i=1}^{n} X_i$ be your total earnings of rubber ducks (which may be negative).

   (a) Let $Z_i = (X_i + 1)/2$, for $i = 1, \ldots, 100$ and $Z = \sum_{i=1}^{100} Z_i$. Notice that Z_i is a binary indicator of whether the ith game was won. What is the distribution of Z?

   (b) Express the event $\{X \geq 10\}$ in terms of Z, and use the Gaussian approximation with continuity correction to compute $P\{X \geq 10\}$.

   (c) Express the event $\{X = 0\}$ in terms of Z and use the Gaussian approximation with continuity correction to compute $P\{X = 0\}$.

   (d) Solve part (c) using the Poisson approximation of a binomial distribution instead.

5. **[Maximum Likelihood Estimation]**
Suppose that X has the following pdf:

$$f_\theta(u) = \begin{cases} \left(\frac{u}{\theta}\right) e^{-\frac{u^2}{2\theta}} & u \geq 0 \\ 0 & u < 0 \end{cases}$$

with parameter $\theta > 0$. Suppose the value of θ is unknown and it is observed that $X = 10$. Find the maximum likelihood estimate, $\widehat{\theta}_{ML}$, of θ.