ECE 310 Fall 2023

Lecture 13

Fourier analysis

Corey Snyder

Learning Objectives

After this lecture, you should be able to:

- Explain what defines a periodic function, including continuous-time and discrete-time complex exponentials.
- Define and compute the continuous time Fourier series of a periodic continuous-time signal.
- Understand how we may derive the continuous-time Fourier transform of aperiodic signals from the continuous-time Fourier series.

Recap from previous lecture

We concluded our discussion of the z-transform in the previous lecture with the BIBO stability and causality of transfer functions. We will make use of the z-transform again soon, but for this lecture we will cover some important mathematical preliminaries for harmonic analysis. We will review periodic signals in both continuous-time and discrete-time then explore the continuous-time Fourier series and Fourier transform.

1 Periodic signals

1.1 Continuous-time signals

Continuous-time periodic signals are defined by their period T_0 such that

$$x(t + kT_0) = x(t), \ \forall k \in \mathbb{Z}. \tag{1}$$

Above, the symbol $\mathbb{Z}$ denotes "all integers". The smallest T_0 for which Eqn. 1 holds is referred to as the *fundamental period*. Each period corresponds to a *linear frequency* F_0 according to

$$F_0 = \frac{1}{T_0}. \tag{2}$$

Furthermore, we define the *radial frequency* Ω_0 as

$$\Omega_0 = 2\pi F_0. \tag{3}$$

When describing each of these quantities, linear frequency is measured in cycles per second or *Hertz* (Hz), radial frequency in radians per second, and period in seconds.

Arguably the most fundamental periodic signal in signal processing is given by complex exponentials of the form:

$$x(t) = e^{j\Omega_0 t}. \tag{4}$$

We also commonly refer to *harmonically related* periodic signals. Two periodic signals are harmonically related if one signal has a frequency that is an integer multiple of the other. For example, harmonically related complex exponentials are given by

$$x_k(t) = e^{j\Omega_0 kt}, \; k \in \mathbb{Z}. \tag{5}$$

The frequency Ω_0 above is referred to as the *fundamental frequency.* To derive one key property of harmonically related complex exponential signals, consider the following where k and l are both integers:

$$\begin{aligned}
\int_{t=0}^{T_0} x_k(t)x_l^*(t)dt &= \int_{t=0}^{T_0} e^{j\Omega_0 kt}e^{-j\Omega_0 lt}dt && (6)\\
&= \int_{t=0}^{T_0} e^{j\Omega_0 (k-l)t}dt && (7)\\
&= \frac{e^{j\Omega_0(k-l)t}}{j\Omega_0(k-l)}\Bigg|_0^{T_0} && (8)\\
&= \frac{e^{j\Omega_0 T_0(k-l)} - 1}{j\Omega_0(k-l)} && (9)\\
&= \frac{e^{j2\pi(k-l)} - 1}{j\Omega_0(k-l)} && (10)\\
&= \begin{cases} 0, & k \neq l \\ T_0, & k = l \end{cases}. && (11)
\end{aligned}$$

We obtain the final line by applying L'Hopital's rule at $k = l$. We see that this integral is only non-zero if $k = l$. In this case, the complex exponential evaluates to one and the integral gives us T_0. This result is known as the *orthogonality property* and will be useful to us soon and throughout the remainder of this course. Note that the above result holds for integration over any contiguous period of the harmonically related exponentials, i.e. t_0 to $t_0 + T_0$

1.2 Discrete-time signals

Similar to continuous-time signals, discrete-time periodic signals are defined by their period N_0 such that

$$x[n + kN_0] = x[n], \; \forall k \in \mathbb{Z}. \tag{12}$$

The period N_0 corresponds to a linear frequency f_0 and radial frequency ω_0 where the following relations from continuous-time signals still hold in discrete-time:

$$\begin{aligned}
\omega_0 &= 2\pi f_0 && (13)\\
f_0 &= \frac{1}{N_0}. && (14)
\end{aligned}$$

For discrete-time signals, we describe linear frequency in cycles per sample, radial frequency in radians per sample, and the period in samples.

In discrete-time, complex exponential signals are now written instead as

$$x[n] = e^{j\omega_0 n}. \tag{15}$$

One major difference between our continuous-time and discrete-time periodic signals is the range of unique frequencies. For continuous-time periodic signals, our radial frequency Ω_0 can be *any* real-valued number. However, discrete-time signals can only have unique frequencies between zero and 2π, i.e. $\omega_0 \in [0, 2\pi)$. We can prove this in just a couple lines:

$$\begin{aligned}
e^{j(\omega_0 + 2\pi k)n} &= e^{j\omega_0 n}e^{j2\pi kn} && (16)\\
&= e^{j\omega_0 n}e^{j2\pi m}, \; m \in \mathbb{Z} && (17)
\end{aligned}$$

$$= e^{j\omega_0 n}. \tag{18}$$

In line 17, we show the explicit step of how the product of k and n will always yield some integer m. Thus, $e^{j2\pi m}$ will always evaluate to one and the result holds. This counter-intuitive result shows that any periodic discrete-time signal with radial frequency beyond 2π is equivalent to the same signal with a frequency modulated down to fit between zero and 2π. For example, $\cos\left(\frac{17\pi}{3}n\right) \equiv \cos\left(\frac{5\pi}{3}n\right)$.

Lecture exercise: Prove the orthogonality property for discrete-time harmonically related complex exponentials.

2 Continuous-time Fourier series

Suppose we have a set of harmonically related complex exponentials with fundamental frequency Ω_0 as shown in Eqn. 5. We can synthesize a new signal $x(t)$ as a linear combination of these signals using

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\Omega_0 kt}, \tag{19}$$

where each c_k gives the coefficient associated with the exponential of frequency $k\Omega_0$. If this summation converges for all t, we will have a result that is the sum of periodic signals all with common period $T_0 = \frac{2\pi}{\Omega_0}$. Thus, the resulting $x(t)$ will also be periodic with period T_0. Suppose we would like to find the corresponding coefficients c_k for some periodic signal $x(t)$ with period T_0. We can solve for these coefficients as follows:

$$\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j\Omega_0 kt} \\
\int_{T_0} x(t)e^{-j\Omega_0 lt}dt &= \int_{T_0}\left(\sum_{k=-\infty}^{\infty} c_k e^{j\Omega_0 kt}\right)e^{-j\Omega_0 lt}dt && (20)\\
&= \sum_{k=-\infty}^{\infty} c_k \int_{T_0} e^{j\Omega_0 kt}e^{-j\Omega_0 lt}dt && (21)\\
&= \begin{cases} 0, & k \neq l \\ c_k T_0, & k = l \end{cases} && (22)\\
c_k &= \frac{1}{T_0}\int_{T_0} x(t)e^{-j\Omega_0 kt}dt && (23)
\end{aligned}$$

Above, $\int_{T_0}$ is shorthand for integration over one contiguous period of the signal. We use the orthogonality principle in line 22 to obtain the final result in Eqn. 23. Together, Eqns. 19 and 23 describe the *continuous-time Fourier series* of the periodic signal $x(t)$. Equation 19 is known as the *synthesis equation* and Eqn. 23 is referred to as the *analysis equation*. We can guarantee the existence of a periodic signal's Fourier series if the following sufficient conditions are satisfied:

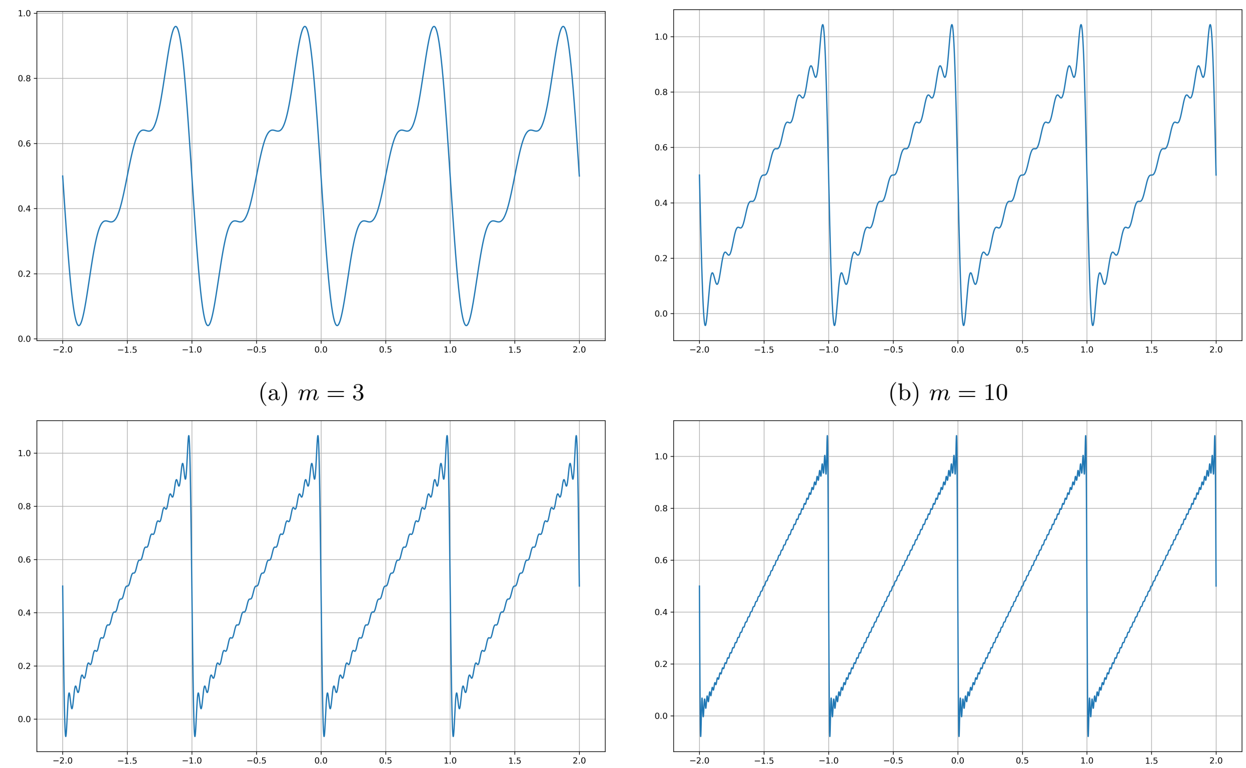
1. The periodic signal $x(t)$ is absolutely integrable over one period:

$$\int_{T_0} |x(t)|dt < \infty \tag{24}$$

2. The signal has a finite number of minima, maxima, and finite-size discontinuities per period.

These conditions are known as the *Dirichlet conditions* and will guarantee that the Fourier series analysis and synthesis equations will converge. We can also guarantee another form of convergence if

$$\int_{T_0} |x(t)|^2 dt < \infty. \tag{25}$$



(a) $m = 3$ (b) $m = 10$

(c) $m = 20$ (d) $m = 50$

Figure 1: Partial sums for the Fourier series of a sawtooth wave $x(t) = t$ with period $T_0 = 1$. The Fourier series is given by $x(t) = \sum_{k=-m}^{m} c_k e^{j2\pi kt}$ with $c_k = \frac{j}{\pi k}$ for $k \neq 0$ and $c_0 = \frac{1}{2}$.

The Fourier series of a periodic signal tells us how similar the signal is to each harmonic frequency. This means the magnitude of each c_k value is directly proportional to the amount of energy that harmonic frequency $k\Omega_0$ contributes to the signal. This is similar to how we described the z-transform as the response to complex discrete-time exponentials of any magnitude. Like the z-transform, the Fourier series of a periodic signal is guaranteed to be unique if it exists.

Figure 1 provides an example of the Fourier series of a given periodic function. Here, we see how the series sum converges to the specified sawtooth waveform as the limits of the summation increase.

2.1 Continuous-time Fourier transform

The continuous-time Fourier series is a useful tool for representing any periodic continuous-time signal as the synthesis of one type of periodic function. However, we would like to be able to decompose *any* continuous-time signal – periodic or aperiodic – in a similar manner. Consider that aperiodic signals can be described as the limit of periodic signals as the period $T_0 \to \infty$. This means our fundamental frequency $\Omega_0 \to 0$ and the spacing between our harmonically related frequencies will go to zero. Informally, this will give us the following new analysis and synthesis equations from our Fourier series equations:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \tag{26}$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\Omega) e^{j\Omega t} d\Omega. \tag{27}$$

Let's briefly recap the changes seen above. For our new analysis equation given by Eqn. 26:

1. The limits of integration are now for all time since the period extends to infinity.

2. We have replaced $k\Omega_0$ with the continuous variable Ω since $\Omega_0 \to 0$ and we evaluate at infinitely many values of k.

3. The scaling factor $1/T_0$ is gone now.

Meanwhile, the new synthesis equation is shown in Eqn. 27:

1. We now integrate instead of summing because we need to evaluate over all continuous frequency values for $\Omega \in (-\infty, \infty)$.

2. We again replace $k\Omega_0$ with Ω

Together, Eqns. 26 and 27 describe the *continuous-time Fourier transform*, often simply called the Fourier transform, and its inverse transform counterpart. We use the symbol $X(\Omega)$ to denote the Fourier transform of $x(t)$ as well as

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(\Omega). \tag{28}$$

The existence of the Fourier transform is similarly guaranteed by the Dirichlet conditions as mentioned above for the Fourier series. The only difference is that the second condition is over any finite interval instead of one period. The Fourier transform of a given signal is also guaranteed to be unique if it exists, like the Fourier series of a periodic signal.

The Fourier transform is a central result and tool of signal processing. Intuitively, the Fourier transform works the same as the Fourier series. Where the Fourier series tells us how much energy is allocated to each harmonic of a fundamental frequency, the Fourier transform gives us how much *every* possible frequency is represented in a given periodic or aperiodic signal. This is consequential not only because analyzing the frequency content of a signal is useful, but because it demonstrates a general method used throughout signal processing. We often find it useful to transform signals into a new space that is more meaningful for a given task, more compact in representation, or better fits a prior model for some phenomenon we measure. In upcoming lectures, we will introduce and more fully explore the discrete-time version of the Fourier transform.