ECE 310 Fall 2023

Lecture 29

FIR filter design

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Learning Objectives

After this lecture, you should be able to:

- Explain what defines linear-phase FIR filters.
- Identify the four types of linear-phase filters.
- Describe the difference between linear-phase and generalized linear-phase FIR filters.

Recap from previous lecture

We introduced practical filter design in the previous lecture by discussing the key features of practical filters in the frequency domain including transition bands, stopband attenuation, and passband ripple. In this lecture, we will examine common structures of finite impulse response filters and talk about linear phase filters in particular.

1 Linear-phase FIR filters motivation

Finite-impulse response (FIR) filters have, as the name suggests, a finitely-long impulse response. Referring back to difference equations, an FIR filter only has input terms – $c_j x[n-j]$ terms – and no output/feedback terms. Consider, for example, an ideal low-pass filter with impulse response $h_{\text{lpf}}[n]$ and cutoff frequency ω_c:

$$h_{\text{lpf}}[n] = \frac{\sin(\omega_c n)}{\pi n}, \ -\infty < n < \infty. \tag{1}$$

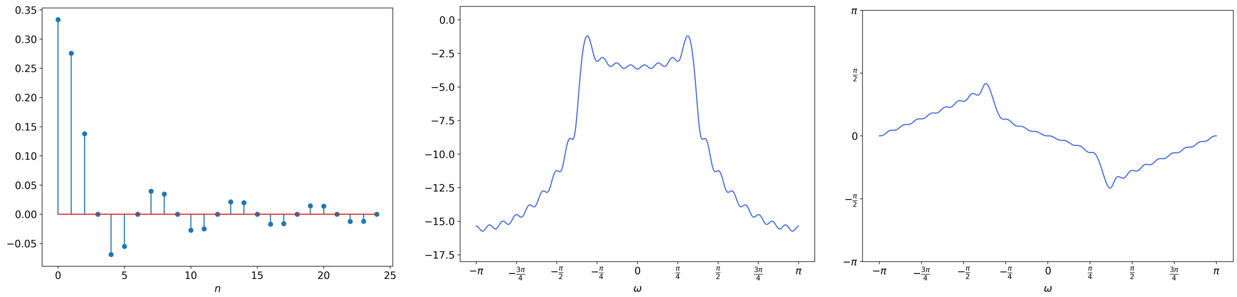
We discussed the issues with this impulse response in the previous lecture including the infinite-length and non-causality of the sequence. A näive approach would be to select the first N causal coefficients to obtain a length-N or $(N-1)$-order FIR filter. This filter would be given by $\tilde{h}_{\text{lpf}}[n]$:

$$\tilde{h}_{\text{lpf}}[n] = \frac{\sin(\omega_c n)}{\pi n}, \ 0 \le n \le N-1. \tag{2}$$

Figure 1 shows this näive impulse response for $N = 25$ and $\omega_c = \pi/3$ along with its magnitude and phase responses. The main concern we have with this figure is the phase response. We see here that $\angle\tilde{H}(\omega)$ is non-linear as we have these ripples through the phase response and multiple changes between roughly linear slopes in the response. Recall from lecture 17 how we defined *group delay*, denoted by τ_{gd}:

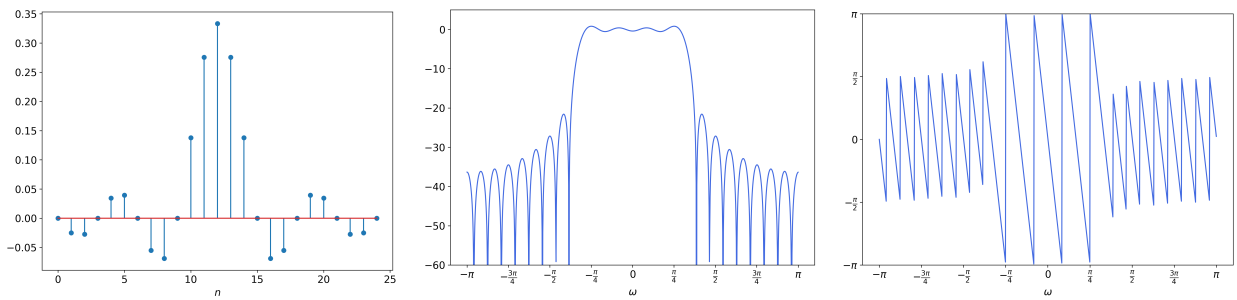
$$\tau_{\text{gd}}(\omega) = -\frac{d\angle H(\omega)}{d\omega}. \tag{3}$$

We demonstrated in lecture 17 that a linear phase response yields a uniform group delay while a non-linear phase response gives a non-uniform group delay. The significance of group delay lies in how an LTI system



(a) $\tilde{h}_{\mathrm{lpf}[n]}$ (b) $|\tilde{H}(\omega)|_{\mathrm{dB}}$ (c) $\angle\tilde{H}(\omega)$

Figure 1: Impulse response, magnitude response (on dB scale), and phase response for näive impulse response from Eqn. 2 with $N = 25$ and $\omega_c = \pi/3$.



(a) $\hat{h}_{\mathrm{lpf}[n]}$ (b) $|\hat{H}(\omega)|_{\mathrm{dB}}$ (c) $\angle\hat{H}(\omega)$

Figure 2: Impulse response, magnitude response (on dB scale), and phase response for shifted impulse response from Eqn. 4 with $N = 25$, $\alpha = 12$, and $\omega_c = \pi/3$.

will potentially shift or delay an input signal. Uniform group delay means all elements in the time-domain are shifted uniformly or identically regardless of what frequency content they correspond to. On the other hand, non-uniform group delay means time-domain elements will be shifted by non-uniform or inconsistent amounts based on the frequency content they correspond to. These non-uniform shifts can make the result of applying a filter look or sound incoherent; thus, we would like to maintain linear phase and uniform group delay in our FIR filters.

We can modify the example from Fig. 1 to achieve linear phase by shifting the ideal impulse response $h_{\mathrm{lpf}}[n]$ by $\alpha = (N-1)/2 = 12$ samples to obtain $\hat{h}_{\mathrm{lpf}}[n]$:

$$\hat{h}_{\mathrm{lpf}}[n] = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)},\ 0 \le n \le N-1. \tag{4}$$

Figure 2 shows the resulting impulse response, magnitude response, and phase response of this new FIR filter. We note a few interesting things here. First, the impulse response is now symmetric about $n = 12$. Second, the stopband attenuation is better with this shifted filter. And lastly, we do now have effectively linear phase. Although the phase response $\angle\hat{H}(\omega)$ may look quite crowded, it is in fact linear almost everywhere ignoring the finitely-many π-jumps in phase. Altogether, this motivates the design of symmetric FIR filters since they will guarantee we have linear phase.

2 Linear-phase filter types

There are four types of linear-phase (LP) FIR filters we can define. These four types can be separated based on whether they have even or odd symmetry about their center point and whether they are even or odd length. Next, we will define each of the four filter types.

2.1 Type-I linear-phase filter

The type-I LP FIR filter has **even symmetry and odd length** N. For impulse response $h_1[n]$, we can then say

$$h_1[n] = h_1[N-n-1]. \tag{5}$$

Note that when we say even or odd symmetry when referring to these linear phase filters, it is symmetry about their midpoint given by $n = \alpha = (N-1)/2$. The frequency response of any type-I filter follows a common general form. We can get a sense of that form by looking at a simple example. Let $h_1[n]$ be a type-I filter with impulse response $h_1[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2] + 2\delta[n-3] + \delta[n-4]$. The resulting frequency response is then

$$\begin{aligned}
H_1(\omega) &= \sum_{n=-\infty}^{\infty} h_1[n]e^{-j\omega n} && (6)\\
&= 1 + 2e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega} + e^{-j4\omega} && (7)\\
&= [3 + 2(e^{j\omega} + e^{-j\omega}) + (e^{j2\omega} + e^{-j2\omega})]e^{-j2\omega} && (8)\\
&= [3 + 4\cos(\omega) + 2\cos(2\omega)]\, e^{-j2\omega}. && (9)
\end{aligned}$$

We see that the resulting frequency response does have linear phase present in the $e^{-j2\omega}$ term. The central sample ($h_1[2] = 3$ in this case) provides a constant offset for the frequency response while the matched symmetric samples pair up to form cosines using Euler's identity. Let $\alpha = (N-1)/2$ be the center sample of a type-I LP FIR filter. Tthe frequency response of a length-N type-I filter is then given by

$$H_1(\omega) = \left(h_1[\alpha] + \sum_{k=1}^{(N-1)/2} 2h_1[\alpha + k]\cos(k\omega) \right) e^{-j\alpha\omega\alpha} = A_1(\omega)e^{-j\alpha\omega}. \tag{10}$$

2.2 Type-II linear-phase filter

A type-II linear-phase filter is defined by having **even symmetry and even length**. We may generally define the frequency response of any type-II filter as we did for type-I

$$H_2(\omega) = \left(\sum_{k=1}^{N/2} 2h_2\left[\alpha + k - \frac{1}{2}\right] \cos\left(\left(k - \frac{1}{2}\right)\omega\right) \right) e^{-j\alpha\omega} = A_2(\omega)e^{-j\alpha\omega}, \tag{11}$$

where we still have $\alpha = (N-1)/2$.

2.3 Type-III linear-phase filter

Type-III linear-phase filters are characterized by **odd symmetry and odd length**. Thus, we have

$$h_3[n] = -h_3[N-n-1]. \tag{12}$$

Note that because we have odd symmetry for type-III, the center point at $n = (N-1)/2$ must be zero! The resulting general frequency response for type-III filters is given by

$$H_3(\omega) = \left(\sum_{k=1}^{(N-1)/2} 2h_3[\alpha + k]\sin(k\omega) \right) je^{-j\alpha\omega} = A_3(\omega)je^{-j\alpha\omega}. \tag{13}$$

We still have $\alpha = (N-1)/2$ and notice that $H_3(\omega)$ looks close to $H_1(\omega)$. Here, we replace the cosines with sines due to the odd symmetry leading to the "sine" Euler's identity which also provides the j factor next to the complex phase term. We no longer have the constant offset of the center sample since it must be zero to guarantee the odd symmetry of the impulse response about the center point.

Table 1: Summary of the four types of linear-phase FIR filters. We also note whether the frequency response is necessarily zero at $\omega = 0$ or $\omega = \pi$. These conditions may prohibit one of the four canonical filter types – low-pass (LP), high-pass (HP), band-pass (BP), band-stop (BS) – from being possible for a given filter type. For example, type-II filters cannot represent high-pass filters since $H(\pi)$ will always be zero for a type-II filter.

Filter type	Symmetry	Length	$H_d(0)$	$H_d(\pi)$	Possible canonical filter types
Type-I	Even	Odd	May be non-zero	May be non-zero	LP, HP, BP, BS
Type-II	Even	Even	May be non-zero	Always zero	LP, BP
Type-III	Odd	Odd	Always zero	Always zero	BP
Type-IV	Odd	Even	Always zero	May be non-zero	BP, HP

2.4 Type-IV linear-phase filter

Finally, type-IV linear phase filters have **odd symmetry and even length**. Their frequency response is similar to the type-II frequency response except we now have sine instead of cosine in the summation along with the added j factor like with type-III filters:

$$H_4(\omega) = \left(\sum_{k=1}^{N/2} 2h_4\left[\alpha + k - \frac{1}{2}\right] \sin\left(\left(k - \frac{1}{2}\right)\omega\right)\right) je^{-j\alpha\omega} = A_4(\omega)je^{-j\alpha\omega}. \tag{14}$$

Table 1 provides a summary regarding the symmetry type, length, and possible filter structures for each of the four linear-phase filter types.

2.5 Amplitude response and linear phase

For all four linear phase FIR filter types above, we found a frequency response of the general form

$$H(\omega) = A(\omega)e^{j\Psi(\omega)}. \tag{15}$$

We refer to $A(\omega)$ as the real-valued *amplitude response* of an LTI system and $\Psi(\omega)$ is the remaining complex phase term. For type-I and type-II filters, we have

$$\Psi_{1/2}(\omega) = -\alpha\omega, \tag{16}$$

and for type-III or type-IV filters we have

$$\Psi_{3/4}(\omega) = -\alpha\omega + \frac{\pi}{2}, \tag{17}$$

where the additional $\frac{\pi}{2}$ term comes from the j in their respective frequency responses.

We may translate the factorization in Eqn. 15 into a more familiar form that specifies the magnitude and phase responses:

$$H(\omega) = |A(\omega)|e^{j(\Psi(\omega) + \angle A(\omega))}. \tag{18}$$

Recall that we can always write the frequency response of a system in terms of its magnitude and phase response as follows:

$$H(\omega) = |H(\omega)|e^{j\angle H(\omega)}. \tag{19}$$

Thus, for each linear phase filter type we have the following expressions for their magnitude and phase responses, respectively.

$$|H(\omega)| = |A(\omega)| \tag{20}$$

$$\angle H(\omega) = \Psi(\omega) + \angle A(\omega). \tag{21}$$

Recall that the $\angle A(\omega)$ is accounted for by including a $\pm\pi$ jump in the phase whenever $A(\omega)$ becomes negative. The key takeaway here is that all four linear phase filter types have a phase response that is linear in ω – with slope $-\alpha = (N-1)/2$ – up to potential constant offsets like $\frac{\pi}{2}$ or $\pm\pi$ jumps in phase. Thus, the phase response of each filter type is linear almost everywhere for $\omega \in [-\pi, \pi]$ and locally linear wherever it is differentiable with the same slope. Therefore, these filters will have uniform group delay for almost every frequency.

3 Linear phase vs. generalized linear phase

We have shown how the phase of the four linear-phase FIR filters types are locally linear in ω. Thus, we know they will locally have uniform group delay. This linearity is potentially broken by any constant offsets in phase. For example, type-III and type-IV filters have an additional $\pi/2$ phase. Also, we may have $\pm\pi$ jumps in phase due to the amplitude response changing sign.

Thus, we would like to differentiate between linear phase responses with no jumps and phase responses that are locally linear with additional jumps in phase.

$$\text{Linear phase} : \angle H(\omega) = -\alpha\omega \tag{22}$$

$$\text{Generalized linear phase} : \angle H(\omega) = -\alpha\omega + \beta(\omega). \tag{23}$$

A linear phase response is stricly linear in ω, is differentiable everywhere, and has a constant derivative, e.g. $\angle H(\omega) = -3\omega$. A *generalized linear phase* (GLP) response is linear and differentiable almost everywhere except where we have constant offsets in phase. For example, we may have $\beta(\omega) = \angle\cos(k\omega)$ like when we account for sign changes that give $\pm\pi$ jumps in phase. Thus, linear phase filters are a subset of GLP filters, i.e. linear phase filters are the case when $\beta(\omega) = 0$. This means that type-III and type-IV linear phase filters can never have true linear phase due to the additional $\frac{\pi}{2}$ in their phase response. They may only achieve GLP.

While we do differentiate between linear phase and GLP filters, it is important to note that we can safely use GLP filters in practice and achieve effectively uniform group delay. This is because the passband of any useful filter will not go to zero anywhere. Thus, we do not have sign changes in the passband of an amplitude response and will have truly uniform group delay amongst the frequencies passed through in a signal. We will have some minor phase distortions in the stopband; however, we are already strongly attenuating these frequencies, so the impact is minimal.