University of Illinois Fall 2020

ECE 313: Midterm Exam II

Monday, November 09, 2020

The exam consists of **6** problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. SHOW YOUR WORK. Answers without appropriate justification will receive very little credit. Reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

The solutions must be handwritten on pieces of paper. Please scan your solutions and upload them onto Gradescope before the deadline.

1. **[20 points]** Let X be a random variable with CDF given by

$$F_X(u) = \begin{cases} C_1, & c \leq 0 \\ a + bu^2, & 0 < c \leq 2 \\ C_2, & c > 2. \end{cases}$$

   (a) [**5 points**] Obtain the values of constants C_1, C_2, a, and b.
   Solution: By properties of the CDF, $\lim_{u\to-\infty} F(u) = 0$, which implies that $C_1 = 0$. Similarly, $\lim_{u\to\infty} F(u) = 1$, which implies that $C_2 = 1$. Also, $F_X(u)$ must be non-decreasing and right continuous, which implies that $F_X(0+) = F_X(0) = 0$ and $F_X(2) = F_X(2+) = 1$. Therefore, $a = 0$ and $b = \frac{1}{4}$.

   (b) [**5 points**] Obtain $f_X(u)$, the pdf of X, for all u.
   Solution: The pdf of X is the derivative of its CDF, hence,

$$f_X(u) = \frac{d}{du}F_X(u) = \begin{cases} \frac{1}{2}u, & 0 \leq u \leq 2 \\ 0, & \text{else} \end{cases}$$

   (c) [**5 points**] Obtain $P\{X \geq 1\}$.
   Solution: $P\{X \geq 1\} = 1 - F_X(1) = 1 - \frac{1}{4} = \frac{3}{4}$.

   (d) [**5 points**] Obtain the mean of X.
   Solution: $\mathbb{E}[X] = \int_0^2 u f_X(u)\,du = \int_0^2 \frac{1}{2}u^2\,du = \frac{1}{6}\,u^3\Big|_0^2 = \frac{4}{3}$.

2. **[20 points]** Buses arrive at a station starting at time zero according to a Poisson process $(N_t : t \geq 0)$ with rate λ, where N_t denotes the number of buses arriving up to time t.

   (a) [**8 points**] Given that two buses arrive during the interval $[0, 3]$, find the probability that exactly one bus arrives before $t = 1$.
   Solution: Let event A be exactly one bus arriving during the interval $[0, 1]$. B is the event that two buses arrive during the interval $[0, 3]$. Then AB is the event that exactly one bus arrives during the interval $[0, 1]$ and one bus arrives during the interval $[1, 3]$. $P\{AB\} = \frac{\lambda\cdot e^{-\lambda}}{1!} \times \frac{(2\lambda)\cdot e^{-2\lambda}}{1!} = 2\lambda^2 e^{-3\lambda}$. $P\{B\} = \frac{(3\lambda)^2\cdot e^{-3\lambda}}{2!} = \frac{9}{2}\lambda^2 e^{-3\lambda}$. Therefore, the conditional probability is,

$$P\{A|B\} = \frac{P\{AB\}}{P\{B\}} = \frac{4}{9}.$$

(b) [**8 points**] Let T be the arrival time of the third bus. What is $P\{T > 3\}$? What is the mean of T?
Solution: Let N be the number of buses that arrive during the interval $[0, 3]$, which is a Poisson random variable with parameter 3λ. $P\{T > 3\} = P\{N \leq 2\} = \sum_{k=0}^{2} \frac{(3\lambda)^k e^{-3\lambda}}{k!} = (1 + 3\lambda + \frac{9}{2}\lambda^2)e^{-3\lambda}$. T follows the Erlang distribution with $r = 3$ and the rate is λ. $\mathbb{E}(T) = \frac{r}{\lambda} = \frac{3}{\lambda}$.

(c) [**4 points**] What is the probability that there is a bus arriving exactly at time $t = 1$?
Solution: The probability is zero.

3. [**15 points**] Let X be a random variable uniformly distributed in $[0, 1]$ and let Θ be a random variable uniformly distributed in $[-\pi/2, \pi/2]$.

(a) [**10 points**] Find the CDF, pdf and expected value of the random variable $Y = X^n$.
Solution: We start with the CDF:

$$F_Y(y) = P\{Y \leq y\} = P\{X^n \leq y\} = P\{X \leq y^{1/n}\} = y^{1/n}.$$

By differentiating the CDF, we obtain the pdf $f_Y(y) = \frac{1}{n} y^{1/n-1}$, valid for $y \in [0, 1]$, and equal to 0 otherwise. We can find the expected value of Y using the LOTUS formula:

$$\int_0^1 x^n \, dx = \frac{1}{n+1}.$$

(b) [**5 points**] Find the mean of $Y = \cos(\Theta)$.
Solution: Using LOTUS, we have

$$\mathbb{E}[T] = \int_{-\pi/2}^{\pi/2} \cos(\theta) \frac{1}{\pi} \, d\theta = \sin(\theta)|_{-\pi/2}^{\pi/2} = \frac{2}{\pi}$$

4. [**10 points**] Let X be the number of times that a fair coin, flipped 40 times, lands heads. Find the probability that $X = 20$ using the normal approximation for a binomial distribution with continuity correction. Please use the Normal tables in the course notes to simplify your final solution.

Solution: We seek to find $P\{X = 20\}$ by approximating the Binomial(40,1/2) distribution of X with a Gaussian distribution, and by adjusting the boundaries of integration to include the continuity correction. In this case,

$$P\{X = 20\} = P\{20 - 1/2 < X < 20 + 1/2\} = P\{\frac{19.5 - 20}{\sqrt{10}} < \frac{X - 20}{\sqrt{10}} < \frac{20.5 - 20}{\sqrt{10}}\}.$$

The last probability can be expressed as $\Phi(0.16) - \Phi(-0.16)$ which equals 0.1272.

5. [**20 points**] In a communication system, a bit $X \in \{0, 1\}$ is transmitted. The received signal $Y = X + W$ where the random noise W is standard normal (i.e., Gaussian distribution with mean 0 and variance 1). The goal is to make a decision on whether X is 0 or 1 based on the received signal Y.

(a) [**10 points**] Suppose X is equally likely to be 0 or 1. Find the MAP decision rule to decide if X is 0 or 1.
Solution: If $X = 0$ then $Y \sim N(0, 1)$ and if $X = 1$ then $Y \sim N(1, 1)$. Since the prior probabilities are equal, the MAP rule and ML rule are identical. Suppose we observe $Y = v$ The ML decision rule decides $X = 0$ when the likelihood (pdf value) of $Y = v$

conditioned on $X = 0$ is larger than the likelihood (pdf value) of $Y = v$ conditioned on $X = 1$. We do this by comparing the two conditional pdfs: decide $X = 0$ if

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} &> \frac{1}{\sqrt{2\pi}} e^{-\frac{(v-1)^2}{2}} && (1) \\
-\frac{v^2}{2} &> -\frac{(v-1)^2}{2} && (2) \\
v &< \frac{1}{2}. && (3)
\end{aligned}
$$

The MAP rule is intuitive: decide $X = 0$ if $Y < \frac{1}{2}$ and $X = 1$ otherwise.

(b) [**10 points**] Now suppose that $P[X = 0] = 0.3$ and $P[X = 1] = 0.7$. Find the MAP decision rule to decide if X is 0 or 1. You can use the approximations $\log_e 0.3 = -1.2$ and $\log_e 0.7 = -0.4$.

Solution: The MAP decision rule decides $X = 0$ when the weighted likelihood (pdf value) of $Y = v$ conditioned on $X = 0$ is larger than the weighted likelihood (pdf value) of $Y = v$ conditioned on $X = 1$; the weighting of the likelihoods is proportional to the prior probabilities of the two values of X. We discover the MAP rule by comparing the two weighted conditional pdfs: decide $X = 0$ if

$$
\begin{aligned}
0.3\frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}} &> 0.7\frac{1}{\sqrt{2\pi}} e^{-\frac{(v-1)^2}{2}} && (4) \\
-1.2 - \frac{v^2}{2} &> -0.4 - \frac{(v-1)^2}{2} && (5) \\
v &< 1.3. && (6)
\end{aligned}
$$

The MAP rule is thus: decide $X = 0$ if $Y < 1.3$ and $X = 1$ otherwise.

6. [**15 points**] X and Y are jointly distributed discrete random variables. The joint pmf table is below.

	X = 0	X=1	X=2
Y=0	0.1	0.1	0.2
Y=1	0.1	0.1	0.1
Y=2	0.1	0.1	?

Table 1: Joint PMF table of X and Y.

(a) [**3 points**] Find the missing entry ($P[\{X = 2\} \cap \{Y = 2\}]$) in the joint pmf table.

Solution: The total probability has to add to 1. So the missing entry is 0.1.

(b) [**6 points**] Are X and Y independent? Explain.

Solution: No, X and Y are *not* independent. This is seen by the fact that $P[\{X = 2\} \cap \{Y = 2\}] \neq P(X = 2) \cdot P(Y = 2)$ because $0.1 \neq 0.3 \cdot 0.4 = 0.12$.

(c) [**6 points**] Find $E[X|Y = 0]$.

Solution: The pmf of X conditioned on $Y = 0$ is $[0.25, 0.25, 0.5]$ centered around the values $0, 1, 2$ respectively. So the corresponding conditional expectation of X is $0 \cdot 0.25 + 1 \cdot 0.25 + 2 \cdot 0.5 = 0.75$