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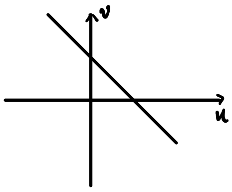
section:

PROBLEM SET #9

[1]

$$F_{X,Y}(u,v) = \begin{cases} 0 & u+v \le 1 \\ 1 & u+v > 1 \end{cases}$$



① $0 \le F(u,v) \le 1$ for all $(u,v) \in \mathbb{R}^2$: true

② $F(u,v)$ is nondecreasing on u/v: true

③ $F(u,v)$ is right-continuous in u/v:

consider $u+v=1$, $\lim_{\substack{v \to 1-u \\ v > 1-u}} F(u,v) = 1$

$F(u,v) = 0$, so $F(u,v) \neq F(u,v^+)$,

thus not right-continuous

so $F(u,v)$ is not a valid joint CDF.

④ $P\{0.5 < X \le 1, 0.5 < Y \le 1\}$

$= F_{X,Y}(1,1) - F_{X,Y}(0.5,1) - F_{X,Y}(1,0.5) + F_{X,Y}(0.5,0.5)$

$= -1$, but probability should be $[0,1]$.

[2].

	$u=0$	$u=1$	$u=2$	$u=3$
$v=4$	0	0.1	0.1	0.2
$v=5$	0.2	0	0	0
$v=6$	0	0.2	0.1	0.1

(a) marginal pmf: $p_X(0) = 0.2$, $p_X(1) = 0.3$, $p_X(2) = 0.2$, $p_X(3) = 0.3$

$p_Y(4) = 0.4$ $p_Y(5) = 0.2$, $p_Y(6) = 0.4$

0.08 0.06

(b) support of Z is $\{4,5,6,7,8,9\}$, $Z = X+Y$ 0.04 0.12 0.08 0.06 0.08

0.12 0.04 0.12

$p_Z(4) = p_{X,Y}(0,4) = 0$

$p_Z(5) = p_{X,Y}(0,5) + p_{X,Y}(1,4) = 0.3$

$p_Z(6) = p_{X,Y}(0,6) + p_{X,Y}(1,5) + p_{X,Y}(2,4) = 0.1$

$p_Z(7) = p_{X,Y}(1,6) + p_{X,Y}(2,5) + p_{X,Y}(3,4) = 0.4$

$p_Z(8) = p_{X,Y}(2,6) + p_{X,Y}(3,5) = 0.1$

$p_Z(9) = p_{X,Y}(3,6) = 0.1$

z	4	5	6	7	8	9
$p(z)$	0	0.3	0.1	0.4	0.1	0.1

(c) $p_{Y|X}(v|3) = P(Y=v|X=3) = \frac{p_{X,Y}(3,v)}{p_X(3)} = \frac{p_{X,Y}(3,v)}{0.3}$

for all v: $p_{Y|X}(4|3) = \frac{0.2}{0.3} = \frac{2}{3}$

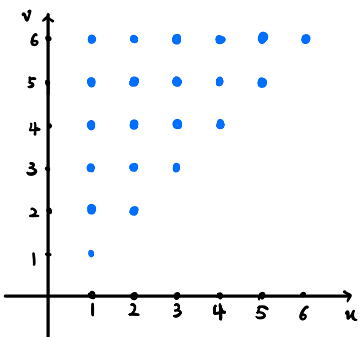
$p_{Y|X}(5|3) = \frac{0}{0.3} = 0$

$p_{Y|X}(6|3) = \frac{0.1}{0.3} = \frac{1}{3}$

$E[Y|X=3] = 4 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} = \frac{14}{3}$

[3]. $Z_1, Z_2 \in \{1,2,3,4,5,6\}$, $X = \min\{Z_1, Z_2\}$ $Y = \max\{Z_1, Z_2\}$

(a)



for the support of u,v:

$u \le v$, $u,v \in \{1,2,3,4,5,6\}$

(b)

v \ u	1	2	3	4	5	6
1	$\frac{1}{36}$	×	×	×	×	×
2	$\frac{1}{18}$	$\frac{1}{36}$	×	×	×	×
3	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	×	×	×
4	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	×	×
5	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$	×
6	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{18}$	$\frac{1}{36}$

(c) marginal pmf of Y:

$P_Y(1) = \frac{1}{36}$

$P_Y(2) = \frac{1}{18} + \frac{1}{36} = \frac{3}{36} = \frac{1}{12}$

$P_Y(3) = \frac{1}{18} + \frac{1}{18} + \frac{1}{36} = \frac{5}{36}$

$P_Y(4) = \frac{1}{18} \times 3 + \frac{1}{36} = \frac{7}{36}$

$P_Y(5) = \frac{1}{18} \times 4 + \frac{1}{36} = \frac{9}{36} = \frac{1}{4}$

$P_Y(6) = \frac{1}{18} \times 5 + \frac{1}{36} = \frac{11}{36}$

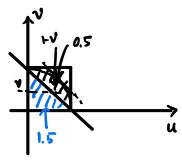
(d) $E[Y-X] = \frac{1}{18}[(2-1)+(3-2)+(3-1)+\cdots+(6-5)+(6-4)+\cdots+(6-1)]$

$= \frac{1}{18}[1+(1+2)+\cdots+(1+2+\cdots+5)]$

$= \frac{1}{18}[1+3+6+10+15]$

$= \frac{35}{18}$

[4] $f_{X,Y}(u,v) = \begin{cases} 1.5 & 0\le u<1,\ 0\le v<1,\ 0\le u+v<1 \\ 0.5 & 0\le u<1\ \ 0\le v<1,\ 1\le u+v<2 \end{cases}$



(a) the marginal pdf of Y:

$f_Y(v) = \int_0^{1-v} 1.5\,du + \int_{1-v}^{1} 0.5\,du$

$= \frac{3}{2}u\Big|_0^{1-v} + \frac{1}{2}u\Big|_{1-v}^{1}$

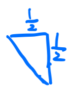
$= \frac{3}{2} - \frac{3}{2}v + \frac{1}{2} - \frac{1}{2} + \frac{1}{2}v$

$= \frac{3}{2} - v$

$\therefore f_Y(u) = \begin{cases} \frac{3}{2} - v & 0\le v\le 1 \\ 0 & \text{else} \end{cases}$

(b) $P\{X+Y\ge \frac{3}{2}\} = P\{u+v \ge \frac{3}{2}\} = 0.5\times0.5\times0.5\div2 = \frac{1}{16}$

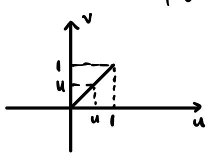
↑from the figure, use the area to calculate

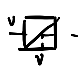


(c) $P\{X^2+Y^2\le1\} = 0.5\cdot(\frac{\pi}{4}-\frac{1}{2}) + 1.5\cdot\frac{1}{2} = \frac{\pi}{8}+\frac{1}{2}$

still from the figure

[5]. $f_{X,Y}(u,v) = \begin{cases} A(1-|u-v|) & 0<u<1,\ 0<v<1 \\ 0 & \text{else} \end{cases}$





(a) from the figure, we know $F_{X,Y}(1,1)=1$

$F_{X,Y}(1,1) = \int_0^1 du\cdot[\int_0^u A(1-(u-v))\,dv + \int_u^1 A(1-(v-u))\,dv]$

$= \int_0^1 A\,du\cdot[\int_0^u (1-u+v)\,dv + \int_u^1 (1-v+u)\,dv]$

$= \int_0^1 A\,du\cdot[(v-uv+\frac{1}{2}v^2)\Big|_0^u + (v-\frac{1}{2}v^2+uv)\Big|_u^1]$

$= \int_0^1 A\,du\cdot[u-u^2+\frac{1}{2}u^2+(1-\frac{1}{2}+u)-(u-\frac{1}{2}u^2+u^2)]$

$= \int_0^1 A\cdot[u-\frac{1}{2}u^2+\frac{1}{2}+u-u-\frac{1}{2}u^2]\,du$

$= \int_0^1 A\cdot[-u^2+u+\frac{1}{2}]\,du$

$= A\cdot(-\frac{1}{3}u^3+\frac{1}{2}u^2+\frac{1}{2}\ \ )\Big|_0^1$

$= A\cdot(\frac{1}{2}+\frac{1}{2}-\frac{1}{3})$

$= \frac{2}{3}A = 1 \Rightarrow A = \frac{3}{2}$

(b) marginal pdf for X:

$f_X(u) = \int_0^u A(1-(u-v))\,dv + \int_u^1 A(1-(v-u))\,dv$

$= \frac{3}{2}[-u^2+u+\frac{1}{2}]$

$= -\frac{3}{2}u^2+\frac{3}{2}u+\frac{3}{4}$

marginal pdf for Y:

$f_Y(v) = \int_0^v A(1-(v-u))\,du + \int_v^1 A(1-(u-v))\,du$

$= \frac{3}{2}[\int_0^v (1-v+u)\,du + \int_v^1 (1-u+v)\,du]$

$= \frac{3}{2}[(u-vu+\frac{1}{2}u^2)\Big|_0^v + (u-\frac{1}{2}u^2+vu)\Big|_v^1]$

$= \frac{3}{2}[v-v^2+\frac{1}{2}v^2+(1-\frac{1}{2}+v)-(v-\frac{1}{2}v^2+v^2)]$

$= \frac{3}{2}[v-\frac{1}{2}v^2+\frac{1}{2}+v-v-\frac{1}{2}v^2]$

$= \frac{3}{2}[-v^2+v+\frac{1}{2}]$

$= -\frac{3}{2}v^2+\frac{3}{2}v+\frac{3}{4}$

(c) $P\{X>Y\} = P\{u>v\} = \int_0^1 du\cdot\int_0^u A(1-(u-v))\,dv$

$= \frac{3}{2}\int_0^1 du\cdot\int_0^u (1-u+v)\,dv$

$= \frac{3}{2}\int_0^1 du\cdot(v-uv+\frac{1}{2}v^2)\Big|_0^u$

$= \frac{3}{2}\int_0^1 du\cdot(u-u^2+\frac{1}{2}u^2)$

$= \frac{3}{2}\int_0^1 (u-\frac{1}{2}u^2)\,du$

$= \frac{3}{2}(\frac{1}{2}u^2-\frac{1}{6}u^3)\Big|_0^1$

$= \frac{3}{2}\cdot\frac{1}{3} = \frac{1}{2}$

(d) $P\{X+Y<1 \mid X>\frac{1}{2}\}$

$$= \frac{P\{X+Y<1,\ X>\frac{1}{2}\}}{P(X>\frac{1}{2})}$$

$$P\{X>\tfrac{1}{2}\} = \int_{\frac{1}{2}}^{1} f_X(u)\,du = \int_{\frac{1}{2}}^{1}\left(-\tfrac{3}{2}u^2+\tfrac{3}{2}u+\tfrac{3}{4}\right)du$$

$$= \left(-\tfrac{1}{2}u^3+\tfrac{3}{4}u^2+\tfrac{3}{4}u\right)\Big|_{\frac{1}{2}}^{1}$$

$$= -\tfrac{1}{2}+\tfrac{3}{4}+\tfrac{3}{4}-\left(-\tfrac{1}{2}\cdot(\tfrac{1}{2})^3+\tfrac{3}{4}\cdot(\tfrac{1}{2})^2+\tfrac{3}{4}\cdot\tfrac{1}{2}\right)$$

$$= 1-\left[-\tfrac{1}{16}+\tfrac{3}{16}+\tfrac{3}{8}\right]$$

$$= 1-\tfrac{1}{2}$$

$$= \tfrac{1}{2}$$

$$P\{X+Y<1,\ X>\tfrac{1}{2}\}$$

$$= P\{X>\tfrac{1}{2},\ Y<1-X\}$$

$$= \int_{\frac{1}{2}}^{1} du \cdot \int_{0}^{1-u} f_{X,Y}(u,v)\,dv$$

$$= \int_{\frac{1}{2}}^{1} A\,du \cdot \int_{0}^{1-u} (1-(u-v))\,dv$$

$$= \tfrac{3}{2}\int_{\frac{1}{2}}^{1} du \cdot \int_{0}^{1-u} (1-u+v)\,dv$$

$$= \tfrac{3}{2}\int_{\frac{1}{2}}^{1} du \cdot \left(v-uv+\tfrac{1}{2}v^2\right)\Big|_{0}^{1-u}$$

$$= \tfrac{3}{2}\int_{\frac{1}{2}}^{1} du \left[(1-u)-u(1-u)+\tfrac{1}{2}(1-u)^2\right]$$

$$= \tfrac{3}{2}\int_{\frac{1}{2}}^{1}\left(\tfrac{1}{2}u^2-u+\tfrac{1}{2}+u^2-u+1-u\right)du$$

$$= \tfrac{3}{2}\int_{\frac{1}{2}}^{1}\left(\tfrac{3}{2}u^2-3u+\tfrac{3}{2}\right)du$$

$$= \tfrac{3}{2}\left(\tfrac{1}{2}u^3-\tfrac{3}{2}u^2+\tfrac{3}{2}u\right)\Big|_{\frac{1}{2}}^{1}$$

$$= \tfrac{3}{2}\left[\left(\tfrac{1}{2}-\tfrac{3}{2}+\tfrac{3}{2}\right)-\left(\tfrac{1}{2}\cdot(\tfrac{1}{2})^3-\tfrac{3}{2}\cdot(\tfrac{1}{2})^2+\tfrac{3}{2}\cdot\tfrac{1}{2}\right)\right]$$

$$= \tfrac{3}{2}\left[\tfrac{1}{2}-\left(\tfrac{1}{16}-\tfrac{3}{8}+\tfrac{3}{4}\right)\right] = \tfrac{3}{32}$$

$$\therefore P\{X+Y<1 \mid X>\tfrac{1}{2}\} = \frac{\frac{3}{32}}{\frac{1}{2}} = \frac{3}{16}$$