University of Illinois **Fall 2023**

ECE 313: Problem Set 13

Due: Friday, December 1 at 7:00:00 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.9–4.11

Note on reading: For most sections of the course notes there are short answer questions at the end of the chapter. We recommend that after reading each section you try answering the short answer questions. Do not hand in; answers to the short answer questions are provided in the appendix of the notes.

Note on turning in homework: Homework is assigned on a weekly basis on Fridays. You must upload handwritten homework to Gradescope. Alternatively, you can typeset the homework in LaTeX. However, no additional credit will be awarded to typeset submissions. No late homework will be accepted. Please write on the top right corner of the first page:
NAME AS IT APPEARS ON Canvas
NETID
SECTION
PROBLEM SET #
Page numbers are encouraged but not required. Five points will be deducted for improper headings.
5 points will be deducted for a submission with incorrectly-assigned page numbers.

1. [**Law of Large Numbers and Central Limit Theorem**]
A fair die is rolled n times. Let $S_n = X_1 + X_2 + \ldots + X_n$, where X_i is the number showing on the ith roll. Determine a condition on n such that the probability the sample average $\frac{S_n}{n}$ is within 10% of the mean μ_X is greater than 0.92.

   (a) Solve the problem using the form of the law of large numbers based on the Chebychev inequality.

   (b) Solve the problem using the Gaussian approximation for S_n, which is suggested by the CLT. If you need to find $\mathcal{Q}(x)$ or $\Phi(x)$, round x to the nearest hundredth. (**Note**: Do not use the continuity correction for this question, because, unless $3.5n \pm (0.1)n\mu_X$ are integers, inserting the term 0.5 is not applicable).

2. [**An estimation problem**]
Suppose X and Y have the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} \frac{kuv}{10000} & u \geq 0, v \geq 0, u^2 + v^2 \leq 100 \\ 0 & \text{else} \end{cases}$$

   (a) Find k. (You will use this value for the following questions.)

   (b) Find the constant estimator, δ^*, of Y, with the smallest mean square error (MSE), and find the MSE.

   (c) Find the unconstrained estimator, $g^*(X)$, of Y based on observing X, with the smallest MSE, and find the MSE.

   (d) Find the linear estimator, $L^*(X)$, of Y based on observing X, with the smallest MSE, and find the MSE. (**Hint**: You may use the fact $E[XY] = \frac{25\pi}{3}$, which can be derived using integration in polar coordinates.)

3. **[Jointly Gaussian Random Variables]**
Suppose X and Y are jointly Gaussian with $\mu_X = 2$, $\mu_Y = 3$, $\sigma_X^2 = 16$, $\sigma_Y^2 = 25$, and correlation coefficient $\rho_{XY} = 0.6$.

   (a) Find the marginal pdf of X, i.e. $f_X(u)$.

   (b) Find the pdf of Y given $X = 6$, i.e. $f_{Y|X}(v|6)$.

   (c) Find the numerical value of $P(Y \geq 2|X = 6)$. If you need to find $Q(x)$ or $\Phi(x)$, round x to the nearest hundredth.

   (d) Find the numerical value of $E[Y^2|X = 6]$.

4. **[Minimum MSE Linear Estimators]**
Let $(N_t, t \geq 0)$ be a Poisson process with rate $\lambda > 0$, and let a, b be positive integers. Suppose it is known that $N_a = k$.

   (a) Suppose $a > b$. Find the linear minimum mean square error (LMMSE) estimator of N_b given N_a and the corresponding MSE. (**Hint**: Recall that Poisson processes have independent increments.)

   (b) Find the LMMSE estimator and corresponding MSE of N_b given N_a, supposing this time that $a < b$.