**University of Illinois**
**ECE 310** **Profs. Kamalabadi, Katselis, Liang**

# Midterm Exam II - Solution

8:30-10:00pm, Tuesday, April 12, 2022

**Name:** ________________________________

**Section:** 9:00 AM 12:00 PM 3:00 PM

**NetID:** ____________

**Score:** ____________

| Problem | Pts. | Score |
|---|---|---|
| 1 | 10 | |
| 2 | 8 | |
| 3 | 16 | |
| 4 | 8 | |
| 5 | 8 | |
| 6 | 12 | |
| 7 | 16 | |
| 8 | 10 | |
| 9 | 12 | |
| Total | 100 | |

## Instructions

- You may not use any books, calculators, or notes other than two handwritten two-sided sheets of 8.5" x 11" paper.
- Show all your work to receive full credit for your answers.
- When you are asked to "calculate", "determine", or "find", this means providing closed-form expressions (i.e., without summation or integration signs).
- Neatness counts. If we are unable to read your work, we cannot grade it.
- Turn in your entire booklet once you are finished. No extra booklet or papers will be considered.

**(10 Pts.)**

1. Find a closed-form expression for the frequency response $H_d(\omega)$ of a system with unit pulse response:

$$h[n] = \delta[n] + \delta[n-1] + \delta[n-2].$$

Find $|H_d(0)|$, $|H_d(\frac{\pi}{2})|$, $|H_d(\pi)|$, $\angle H_d(0)$, $\angle H_d(\frac{\pi}{2})$, and $\angle H_d(\pi)$.

**Solution:** By definition of the DTFT:

$$\begin{aligned} H_d(\omega) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = 1 + e^{-j\omega} + e^{-2j\omega} \\ &= e^{-j\omega}(e^{j\omega} + 1 + e^{-j\omega}) = e^{-j\omega}(1 + 2\cos(\omega)) \end{aligned}$$

where the last equality uses the fact that $\cos(\omega) = \frac{e^{-j\omega}+e^{j\omega}}{2}$. Then, the mangnitude and phase of the frequency response is given as

$$|H_d(\omega)| = |1 + 2\cos(\omega)|, \quad \angle H_d(\omega) = \begin{cases} -\omega, & 1 + 2\cos(\omega) \geq 0 \\ -\omega + \pi, & \text{otherwise} \end{cases}$$

By plugging the values of $\omega$ into $|H_d(\omega)|$ and $\angle H_d(\omega)$ we get the following:

| | |
|---|---|
| $H_d(\omega) = e^{-j\omega}(1 + 2\cos(\omega))$ | (4 Pts) |
| $\lvert H_d(0)\rvert = 3$ | (1 Pts) |
| $\lvert H_d(\frac{\pi}{2})\rvert = 1$ | (1 Pts) |
| $\lvert H_d(\pi)\rvert = 1$ | (1 Pts) |
| $\angle H_d(0) = 0$ | (1 Pts) |
| $\angle H_d(\frac{\pi}{2}) = -\pi/2$ | (1 Pts) |
| $\angle H_d(\pi) = 0$ | (1 Pts) |

**(8 Pts.)**

2. Recall that an eigenfunction of a system is an input signal which appears at the output of the system scaled by a complex constant. Which of the following discrete-time signals could be eigenfunctions of any (both real and complex) stable LSI system? (note that $\omega_0$ is a constant; **-2 pt for each incorrect choice; minimum score for the whole problem =0**).

   (a) $5^{-n}u[n]$

   (b) $e^{j2\omega_0 n}$

   (c) $e^{j\omega_0 n} + e^{j2\omega_0 n}$

   (d) $\cos(\omega_0 n)$

   **Solution:** By definition, a sequence $x[n]$ is called an *eigenfunction* of a system if the output of that system when inputting $x[n]$ can be written as

$$y[n] = \lambda \cdot x[n]$$

   for some real or complex-valued constant $\lambda \in \mathbb{C}$, which is called the *eigenvalue* correspondingly. Then we have:

   **(a) is NOT** an eigenfunction for any stable LSI system. **Example**: consider an LSI system with difference equation $y[n] = x[n-1]$. Then the output when inputting $x[n] = 5^{-n}u[n]$ is

$$y[n] = 5^{-n+1}u[n-1] \neq \lambda \cdot 5^{-n}u[n],\ \forall \lambda \in \mathbb{C}$$

   Then $(a)$ is not an eigenfunction in general.

   **(b) is** an eigenfunction for any stable LSI system, since by eigenfunction property

$$y[n] = H_d(2\omega_0) \cdot x[n]$$

   where $\lambda = H_d(2\omega_0)$ is the corresponding eigenvalue and $H_d(\cdot)$ is the frequency response of this system. To prove this, let $h[n]$ be the impulse response of the system, then the output $y[n]$ when inputting $x[n] = e^{j2\omega_0 n}$ is

$$\begin{aligned} y[n] = x[n] * h[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k] \cdot e^{j2\omega_0(n-k)} \\ &= \underbrace{e^{j2\omega_0 n}}_{=x[n]} \cdot \underbrace{\sum_{k=-\infty}^{\infty} h[k]e^{-j2\omega_0 k}}_{=H_d(2\omega_0)} \\ &= H_d(2\omega_0) \cdot x[n] \end{aligned}$$

   which holds for any sequence $h[n]$ as long as the DTFT $H_d(\cdot)$ exists (which is guaranteed by the fact that the system is stable).

   **(c) is NOT** an eigenfunction for any stable LSI system. Since from part (b), in this case the output is

$$y[n] = H_d(\omega_0)e^{j\omega_0 n} + H_d(2\omega_0)e^{j2\omega_0 n} \neq \lambda \cdot (e^{j\omega_0 n} + e^{j2\omega_0 n})$$

for some $\lambda \in \mathbb{C}$, where the last inequality holds as long as $H_d(\omega_0) \neq H_d(2\omega_0)$. Therefore, $e^{j\omega_0 n} + e^{j2\omega_0 n}$ is not an eigenfunction in general.

**(d) is NOT** an eigenfunction for any stable LSI system. This is because, similar to part (c), in this case the input is

$$x[n] = \cos(\omega_0 n) = \frac{e^{j\omega n} + e^{-j\omega n}}{2}.$$

Then the output is

$$y[n] = \frac{1}{2}H_d(\omega_0)e^{j\omega n} + \frac{1}{2}H_d(-\omega_0)e^{-j\omega n} \neq \lambda \cdot \frac{e^{j\omega n} + e^{-j\omega n}}{2}$$

for some $\lambda \in \mathbb{C}$, where the last inequality holds as long as $H_d(-\omega_0) \neq H_d(\omega_0)$. Therefore, $\cos(\omega_0 n)$ is not an eigenfunction in general.

**(16 Pts.)**

3. An ideal low-pass filter has unit pulse response $h_{\text{lp}}[n]$ and frequency response:

$$H_{\text{lp}}(\omega) = \begin{cases} 1, & |\omega| < \frac{\pi}{5} \\ 0, & \frac{\pi}{5} \leq |\omega| \leq \pi \end{cases}$$

   (a) Determine the frequency response, $H_1(\omega)$, of a filter whose unit pulse response is $h_1[n] = (-1)^n h_{\text{lp}}[n]$, and plot it for $|\omega| \leq \pi$. What type of filter is this?

   (b) Determine the frequency response, $H_2(\omega)$, of a filter whose unit pulse response is $h_2[n] = 2h_{\text{lp}}[n]\cos(\frac{\pi}{2}n)$, and plot it for $|\omega| \leq \pi$. What type of filter is this?

**Solutions**

**(a)** We do not need to find $h_{\text{lp}}[n]$ to complete this problem if we artfully use pairs and properties of the DTFT. In particular, we note that, given $e^{j\pi n} \leftrightarrow 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \pi - 2\pi l)$,

$$\begin{aligned} h_1[n] &= (-1)^n h_{\text{lp}}[n] = e^{j\pi n} h_{\text{lp}}[n] \implies \\ H_1(\omega) &= \frac{1}{2\pi} \int_0^{2\pi} H_{\text{lp}}(\theta) 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \theta - \pi - 2\pi l) d\theta \\ &= \frac{2\pi}{2\pi} \int_0^{2\pi} H_{\text{lp}}(\theta) \delta((\omega - \pi) - \theta) d\theta \\ &= H_{\text{lp}}(\omega - \pi). \end{aligned}$$

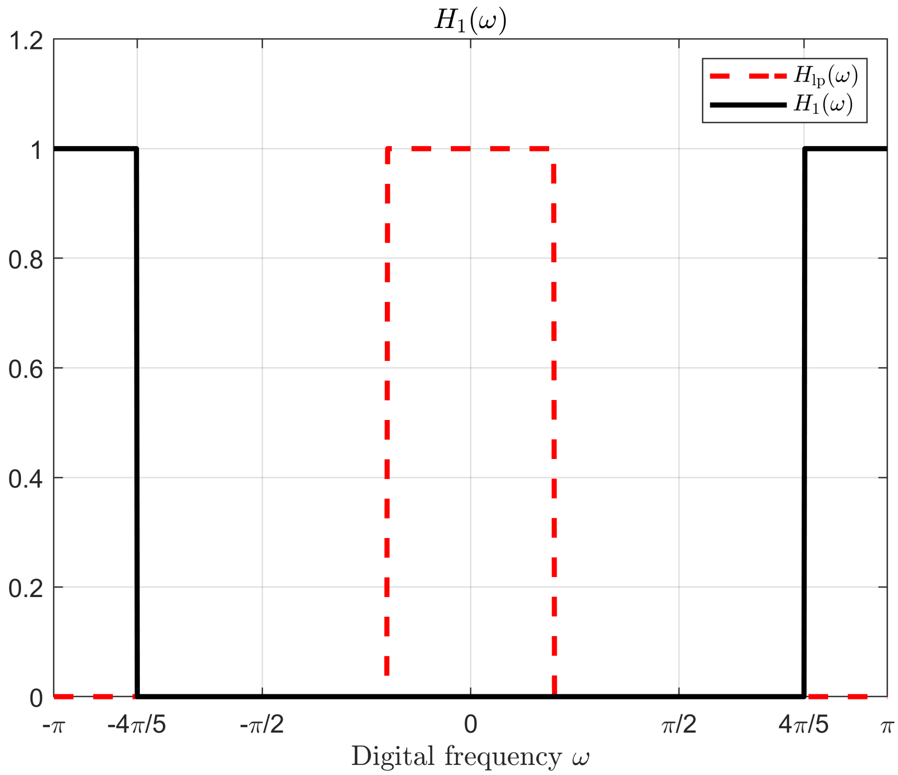
This can be seen more readily by the frequency-shifting property. Effectively, the multiplication of $(-1)^n$ ends up shifting $H_{lp}(\omega)$ to the right by $\pi$, with no additional scaling. This ends up forming a **high-pass filter**, as shown in the figure below. It is high pass because, on the interval of $[-\pi, \pi]$, it only passes "high frequencies", ones above a particular cutoff $|\omega| > \omega_c = 4\pi/5$.

**(b)** The same process can be followed, noting that $2\cos\left(\frac{\pi}{2}n\right) = e^{j\frac{\pi}{2}n} + e^{-j\frac{\pi}{2}n}$. As such,

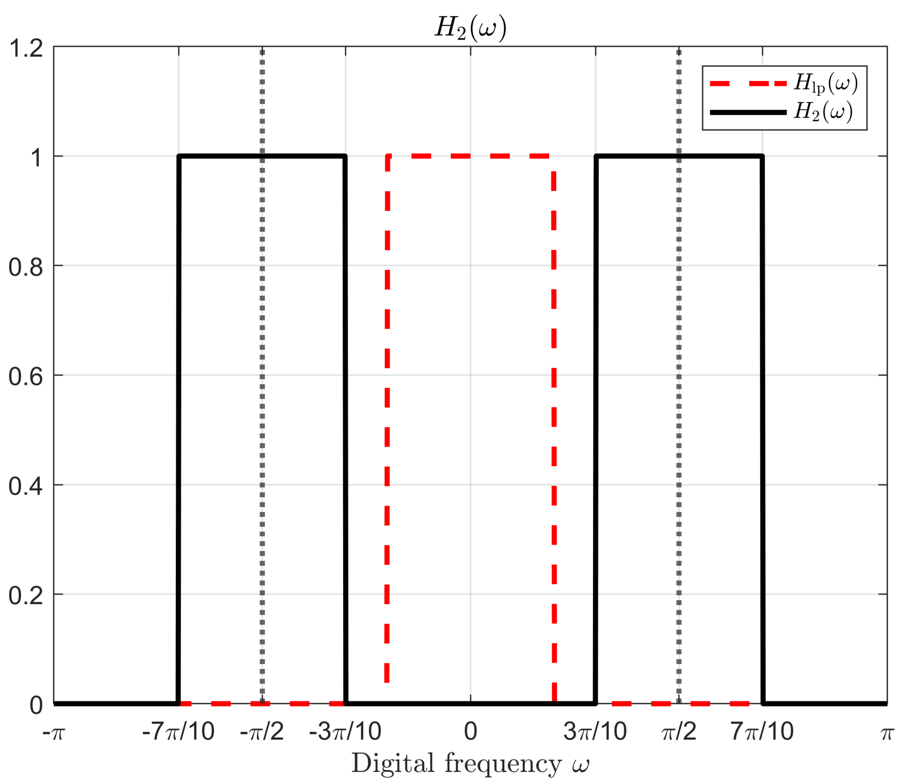
$$\begin{aligned} h_2[n] &= 2\cos\left(\frac{\pi}{2}n\right) h_{\text{lp}}[n] = e^{j\frac{\pi}{2}n} h_{\text{lp}}[n] + e^{-j\frac{\pi}{2}n} h_{\text{lp}}[n] \implies \\ H_2(\omega) &= H_{\text{lp}}\left(\omega - \frac{\pi}{2}\right) + H_{\text{lp}}\left(\omega + \frac{\pi}{2}\right). \end{aligned}$$

This results in a **band-pass filter**, where only frequencies in a particular pair of non-touching bands on the interval $\omega \in [-\pi, \pi]$ are allowed to pass, the rest being rejected.

Points are assigned as follows: +2 for each correct filter interpretation, +2 for each correct plot and +2 for each correct closed-form $H(\omega)$. If a student had a correct plot with preliminary work but no closed-form $H(\omega)$, points were awarded for $H(\omega)$. One point was deducted for each minor error, such as mislabelling an axis or having an incorrect scale factor.



$H_1(\omega)$
$H_{\mathrm{lp}}(\omega)$
$H_1(\omega)$
1.2
1
0.8
0.6
0.4
0.2
0
-π
-4π/5
-π/2
0
π/2
4π/5
π
Digital frequency $\omega$

$H_2(\omega)$
$H_{\mathrm{lp}}(\omega)$
$H_2(\omega)$
1.2
1
0.8
0.6
0.4
0.2
0
-π
-7π/10
-π/2
-3π/10
0
3π/10
π/2
7π/10
π
Digital frequency $\omega$

**(8 Pts.)**

4. Let $(X[m])_{m=0}^{99}$ be the 100−point **DFT** of a **real-valued** sequence $(x[n])_{n=0}^{99}$ and $X_d(\omega)$ be the **DTFT** of $x[n]$ zero-padded to infinite length. Choose all the correct answers (**-2 pt for each incorrect choice; minimum score for the whole problem =0**).

   (a) $X[70] = X_d\left(-\frac{6\pi}{10}\right)$
   (b) $X[70] = X_d\left(\frac{70\pi}{50}\right)$
   (c) $|X[70]| = |X_d\left(\frac{70\pi}{100}\right)|$
   (d) $\angle X[70] = -\angle X_d\left(\frac{3\pi}{5}\right)$

   **Solution**

   We know that by definition of the DFT,

$$X[k] = \sum_{n=0}^{99} x[n]e^{-j\frac{2\pi}{100}nk},$$

   and by definition of the DTFT of the zero-padded version,

$$X_d(\omega) = \sum_{n=-\infty}^{\infty} x_{zp}[n]e^{-j\omega n} = \sum_{n=0}^{99} x[n]e^{-j\omega n},$$

   as each term in $x_{zp}$ outside the range $n \in \{0, 1, 2, ..., 99\}$ is zero. As such, we can write

$$X[k] = X_d\left(\omega_k = \frac{2\pi k}{100}\right).$$

   This relationship will form the basis of how to analyze these four options.

   **(a)** Notice that, from the above analysis, $X[70] = X_d\left(\frac{2\pi\times 70}{100}\right) = X_d\left(\frac{140\pi}{100}\right) = X_d\left(\frac{14\pi}{10}\right)$. Using the fact that any DTFT is $2\pi$-periodic, we know that for any integer $l$,

$$X_d\left(\frac{14\pi}{10}\right) = X_d\left(\frac{14\pi}{10} + 2\pi l\right) = X_d\left(\frac{14\pi - 20\pi l}{10}\right).$$

   If we let $l = 1$, this shows

$$X_d\left(\frac{14\pi}{10}\right) = X_d\left(\frac{14\pi - 20\pi}{10}\right) = X_d\left(\frac{-6\pi}{10}\right).$$

   Therefore, $X[70] = X_d\left(\frac{6\pi}{10}\right)$, so **(a) is true**.

   **(b)** Similar analysis shows that $X_d\left(\frac{70\pi}{50}\right) = X_d\left(\frac{140\pi}{100}\right)$, which is equal to $X[70]$ as proved in (a). Thus, **(b) is true**.

   **(c)** Similar analysis shows that $|X_d\left(\frac{70\pi}{100}\right)| = |X_d\left(\frac{70\pi}{100} + 2\pi l\right)| = |X_d\left(\frac{70\pi+200\pi l}{100}\right)|$, and given that $x[n]$ is real, $|X_d(\omega)| = |X_d(-\omega)|$. However, no manipulation of the argument can show equality to $|X[70]| = |X_d\left(\frac{70\pi}{50}\right)|$, so **(c) is false**.

   **(d)** Converting $X[70] = X_d\left(\frac{140\pi}{100}\right)$, we can use conjugate symmetry to know that given $x[n]$ is real, $\angle X_d(\omega) = -\angle X_d(-\omega)$. Upon similar manipulations as above,

$$\angle X_d\left(\frac{140\pi}{100}\right) = -\angle X_d\left(-\frac{140\pi}{100}\right) = -\angle X_d\left(-\frac{7\pi}{5}\right) = -\angle X_d\left(-\frac{7\pi}{5} + 2\pi\right) = -\angle X_d\left(\frac{3\pi}{5}\right),$$

   so **(d) is true**.

**(8 Pts.)**

5. The following linear convolution

$$\{x_n\}_{n=0}^{8} * \{h_n\}_{n=0}^{16}$$

is to be evaluated using the DFT method. Namely,

$$\{x_n\}_{n=0}^{8} * \{h_n\}_{n=0}^{16} = \mathrm{DFT}^{-1}\{\mathrm{DFT}\{x_n\} \cdot \mathrm{DFT}\{h_n\}\} \tag{1}$$

(a) Determine the minimum number of zeros that should be padded to $\{x_n\}$ and $\{h_n\}$, respectively, before the DFTs are applied.

(b) If the DFTs are to be calculated with a radix-2 FFT algorithm, how many zeros should now be padded to $\{x_n\}$ and $\{h_n\}$, respectively.

**Solution**

**(a)** We must pad zeros onto $x$ and $h$ until their circular convolution equals their linear convolution. Denote the length of $x$ as $L = 9$ ($x_n$ defined as $\{x_n\}_{n=0}^{L-1}$), and similarly the length of $h$ as $M = 17$. We must pad zeros onto both until they are of length $N = L + M - 1 = 25$ in order for the circular convolution of these zero-padded signals to equal the linear convolution of the originals. As such, we pad $N - L = M - 1 = 16$ zeros to $x$, and $N - M = L - 1 = 8$ zeros to $h$.
**(b)** The same process as in (a) can be followed, where we now must pad the sequences to a length both greater than $L + M - 1 = 25$ *and* the length must be a *power of two*, i.e. the smallest integer $\nu$ such that $N = 2^\nu > 25$. The smallest $\nu$ such that this is the case is $\nu = 5$ yielding $N = 2^5 = 32$. As such, we pad $N - L = 32 - 9 = 23$ zeros to $x$, and $N - M = 32 - 17 = 15$ zeros to $h$.

Points were awarded as follows: 2 points were deducted for misidentifying the condition ($N_a = L + M - 1 = 25$, $N_b = 32$) for each sub-problem. One point was deducted for the mis-identification of $L$ and $M$ as 8 and 16, and each small error.

**(12 Pts.)**

6. Let $\{y_0, y_1, y_2, y_3, y_4\} = \text{DFT}^{-1}\{\text{DFT}\{x_0, x_1, 0, 0, 0\} \cdot \text{DFT}\{h_0, h_1, h_2, 0, 0\}\}$.

   (a) Determine linear convolution $\{x_0, x_1\} * \{h_0, h_1, h_2\}$ (express your results in terms of $y_0, y_1, \cdots, y_4$.)

   **Solution**:

   Since DFT length $5 \geq 2 + 3 - 1$, the linear convolution length, linear convolution will result in the same sequence as inverse DFT: $\{y_0, y_1, y_2, y_3, y_4\}$

   (6 Pt)

   (b) Determine $\text{DFT}^{-1}\{\text{DFT}\{0, 0, 0, x_0, x_1\} \cdot \text{DFT}\{h_0, h_1, h_2, 0, 0\}\}$ (express your results in terms of $y_0, y_1, \cdots, y_4$.)

   **Solution**:

$$\text{DFT}\{0, 0, 0, x_0, x_1\} = \text{DFT}\{x_0, x_1, 0, 0, 0\} \cdot e^{-j\frac{6\pi k}{5}}$$

   (3 Pt)

   So our result

$$\tilde{Y} = \text{DFT}^{-1}\{\text{DFT}\{x_0, x_1, 0, 0, 0\} \cdot \text{DFT}\{h_0, h_1, h_2, 0, 0\} \cdot e^{-j\frac{6\pi k}{5}}\} = Y[< n - 3 >_5]$$

$$\tilde{Y} = \{y_2, y_3, y_4, y_0, y_1\}$$

   (3 Pt)

**(16 Pts.)**

7. The input to a D/A converter is $\{x[n]\} = \{2, \underset{\uparrow}{1}, 0, 0, -2, 0, 4\}$ with sampling interval $T$. Determine the output of the D/A converter if the D/A converter is (a) a ZOH, and (b) an ideal D/A.

   **Solution**:

   D/A process of a signal is given by:

$$x_c(t) = \sum_{n=-\infty}^{\infty} x[n] g_a(t - nT)$$

(4 Pt)

For a ZOH, $g_a(t - nT) = rect(\frac{t-nT-\frac{T}{2}}{T})$ so that

(4 Pt)

$$x_c(t) = x[-1]rect(\frac{t - \frac{T}{2} + T}{T}) + x[0]rect(\frac{t - \frac{T}{2}}{T}) + x[3]rect(\frac{t - \frac{T}{2} - 3T}{T}) + x[5]rect(\frac{t - \frac{T}{2} - 5T}{T})$$

$$= 2rect(\frac{t - \frac{T}{2} + T}{T}) + rect(\frac{t - \frac{T}{2}}{T}) - 2rect(\frac{t - \frac{T}{2} - 3T}{T}) + 4rect(\frac{t - \frac{T}{2} - 5T}{T})$$

(2 Pt)

For an ideal D/A, $g_a(t - nT) = sinc(\frac{\pi(t-nT)}{T})$ (Here it's the **unnormalized** sinc function)

(4 Pt)

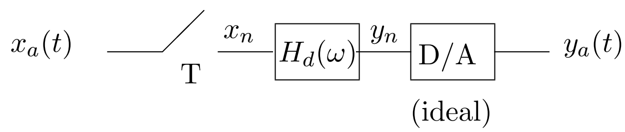
$$x_c(t) = x[-1]sinc(\frac{\pi(t + T)}{T}) + x[0]sinc(\frac{\pi t}{T}) + x[3]sinc(\frac{\pi(t - 3T)}{T}) + x[5]sinc(\frac{\pi(t - 5T)}{T})$$

$$= 2sinc(\frac{\pi(t + T)}{T}) + sinc(\frac{\pi t}{T}) - 2sinc(\frac{\pi(t - 3T)}{T}) + 4sinc(\frac{\pi(t - 5T)}{T})$$

(2 Pt)

**(10 Pts.)**

8. Consider the following system with uniform sampling




The discrete-time system $H_d(\omega)$ is an ideal low-pass filter with cutoff frequency $\frac{\pi}{4}$.

(a) If $x_a(t)$ is bandlimited to 4 kHz, what is the maximum value of $T$ that will avoid aliasing in the A/D converter?

(b) If $\frac{1}{T} = 8$ kHz, determine the maximum bandwidth of $x_a(t)$ allowed such that the overall system from $x_a(t)$ to $y_a(t)$ behaves as an LTI system.

**Solution.**
(a)
By Nyquist Criteria:
If the signal is bandlimited by $B$, or $F = \frac{B}{2\pi}$ Hz,
then we refer to twice the highest frequency in $x(t)$ as the Nyquist frequency, i.e. $2F$.

(2 Pts)

When the A/D converter satises $\frac{1}{T} \geq 2F$, i.e. $T \leq \frac{1}{2F}$,
then it is possible to exactly recover the signal $x(t)$ from its samples.

(2 Pts)

therefore, $T_{\max} = \frac{1}{8000}$ sec.

(1 Pt)

(b)
$\Omega_0 T = 2\pi B_0 T = \omega_0$

(1 Pt)

$-\Omega_0 T + 2\pi \geq$ cutoff frequency $= \frac{\pi}{4}$

(2 Pts)

$\Omega_0 \leq 14000\pi$ rad/s
or, $B_0 \leq 7000$ Hz

(2 Pts)

**(12 Pts.)**

9. A continuous-time signal $x_c(t) = \cos(8\pi t)$ is sampled at a rate of 80 Hz for five seconds to produce a discrete-time signal $x[n]$ with length $L = 400$.

   (a) Let $X[k]$ be the length-$L$ DFT of $x[n]$. At what value(s) of $k$ will $X[k]$ have the greatest magnitude?

   (b) Suppose that $x[n]$ is zero-padded to a total length of $N = 1024$. At what value(s) of $k$ does the length-$N$ DFT have the greatest magnitude?

**Solution.**
**(a)** $x[n] = x_c(nT) = cos(\frac{\pi}{10}n)$

(2 Pts)

$X_d(\omega) = \pi \sum_{-\infty}^{\infty} \delta(\omega - \frac{\pi}{10} + 2\pi l) + \delta(\omega + \frac{\pi}{10} + 2\pi l)$

(2 Pts)

Then, there will be two peaks in the DTFT $X_d(\omega)$ between $[0, 2\pi)$ which locate at $\frac{\pi}{10}$ and $\frac{19\pi}{10}$ respectively. Regarding the DFT, since it only gives samples of DTFT at frequencies $\frac{2\pi k}{L}$ for $k = 0, 1, \cdots, L-1$,

(2 Pts)

we see that $k = 20$ and $k = 380$ correspond to the two peaks respectively. Therefore, both $k = 20$ and $k = 380$ will have the greatest magnitude of $X[k]$.

(2 Pts)

**(b)** After zero-padding, now the length-$L$ DFT have samples at frequencies $\frac{2\pi k}{L}$ for $k = 0, 1, \cdots, L-1$. As a result, we don't have the samples right at the peaks $\frac{\pi}{10}$ and $\frac{19\pi}{10}$ anymore. Instead, the largest magnitude of $X[k]$ is located at the frequency which is closest to the peaks.
This leads to $2\pi \cdot \frac{51}{1024}$ (that is closest to the peak at $\frac{\pi}{10}$) and $2\pi \cdot \frac{973}{1024}$ (that is closest to the peak at $\frac{19\pi}{10}$),

(2 Pts)

and the corresponding $k$ choices are $k = 51$ and $k = 973$.

(2 Pts)

**Remark:** One might be confused that how to show both $k = 51$ and $k = 973$ share the same amount of magnitude. By symmetry, since $x[n]$ is real, we have $|X[k]| = |X[N-k]|$ and therefore $|X[51]| = |X[973]|$.