ECE 310 Fall 2023

Lecture 18

Ideal analog-to-digital conversion

Corey Snyder

Learning Objectives

After this lecture, you should be able to:

- State the resulting DTFT of a discrete-time signal obtained via ideal sampling of a continuous-time signal.
- Explain Nyquist criterion for sampling continuous-time signals.
- Identify the Nyquist rate of a continuous-time signal.

Recap from previous lecture

We reviewed magnitude and phase response in the previous lecture to conclude our discussion of the DTFT. We now will start working with sampling and reconstruction for analog-to-digital and digital-to-analog conversion. In this lecture, we turn our attention to ideal analog-to-digital conversion and carefully derive the relationship between the continuous-time and discrete-time Fourier spectra of an analog signal and its digital counterpart after ideal sampling.

1 Ideal sampling of analog signals

Recall from lecture 1 that the ideal sampling of continuous-time signals yields discrete-time signals as

$$x[n] = x(nT), \quad -\infty < n < \infty. \tag{1}$$

Ideal sampling is accomplished by *instantaneously* measuring the value of $x(t)$ at a uniform interval given by the *sampling period* T then converting these measurements to a discrete-time sequence. We may alternatively say we sample $x(t)$ with *sampling frequency*

$$f_s = \frac{1}{T}. \tag{2}$$

In order to instantaneously sample a continuous-time signal, we first consider the *sifting property* of the Dirac delta signal $\delta(t)$:

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0). \tag{3}$$

The sifting property tells us that we can measure the values of a continuous-time signal at a single point in time, t_0, by multiplying with a Dirac delta centered at t_0. Consider now if we multiply our continuous-time signal $x(t)$ with a sequence of Dirac delta functions uniformly spaced by T:

$$x_{\text{sampled}}(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \tag{4}$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT). \tag{5}$$

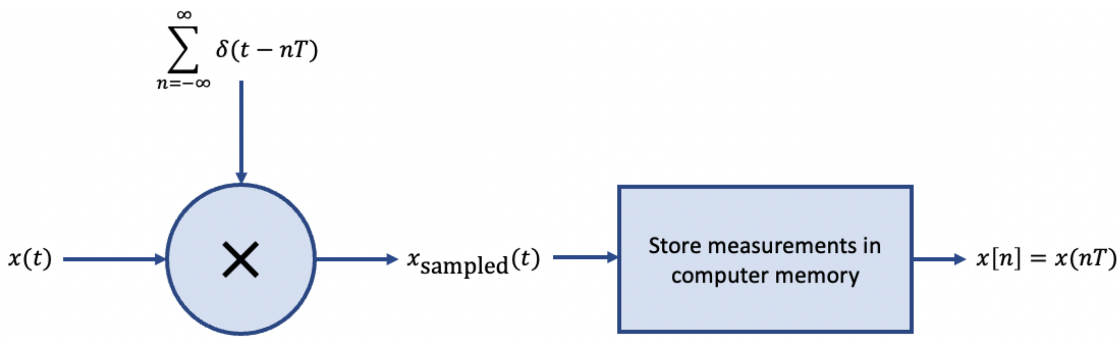


Figure 1: Ideal analog-to-digital conversion procedure.

We see that multiplying with this sequence of Dirac delta signals, known also as an *impulse train*, provides the desired regular measurement of our continuous-time signal. It is important to stress that $x_{\text{sampled}}(t)$ is still a continuous-time signal. What we have accomplished thus far is to represent the original signal $x(t)$ with only one non-zero value in each interval of width T, e.g. $3T/2$ to $5T/2$ is only non-zero at $t = 2T$ for $x_{\text{sampled}}(t)$. This is a critical step since we of course cannot store the infinitely many values in time of a continuous-time signal over any interval. Furthermore, Eqn. 4 will help us answer a key question: what is the resulting DTFT after ideally sampling a continuous-time signal with a given continuous-time Fourier transform (CTFT)?

First, recall the windowing property of the Fourier transform (continuous-time or discrete-time):

$$x(t)w(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi} X_a(\Omega) * W_a(\Omega). \tag{6}$$

Since we are working with both continuous-time and discrete-time signal representations, we will use subscript a for spectra of analog/continuous-time signals and subscript d for spectra of digital/discrete-time signals. Also, remember that we use Ω to denote continuous-time radial frequency (radians/second) and ω for discrete-time radial frequency (radians/sample). Referring back to Eqn. 4, we can take the Fourier transform with respect to both sides:

$$x_{\text{sampled}}(t) \overset{\mathcal{F}}{\longleftrightarrow} X_s(\Omega) \tag{7}$$

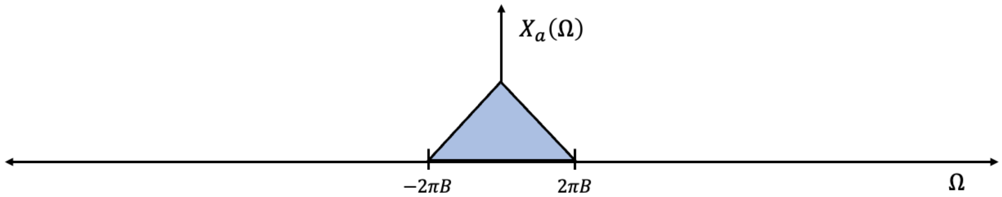
$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X_a(\Omega) \tag{8}$$

$$\sum_{n=-\infty}^{\infty} \delta(t-nT) \overset{\mathcal{F}}{\longleftrightarrow} \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\Omega - n\frac{2\pi}{T}\right) \tag{9}$$

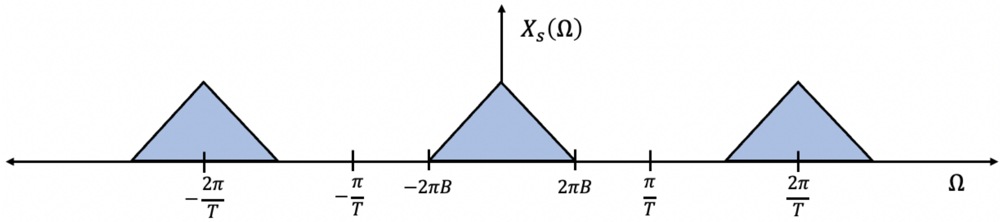
Plugging in these transform pairs, we may solve for the resulting sampled spectrum $X_s(\Omega)$:

$$\begin{aligned} X_s(\Omega) &= \frac{1}{2\pi}\left[X_a(\Omega) * \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta\left(\Omega - n\frac{2\pi}{T}\right)\right] && (10)\\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a\left(\frac{\Omega T - 2\pi n}{T}\right) && (11)\\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a\left(\Omega - n\frac{2\pi}{T}\right). && (12) \end{aligned}$$

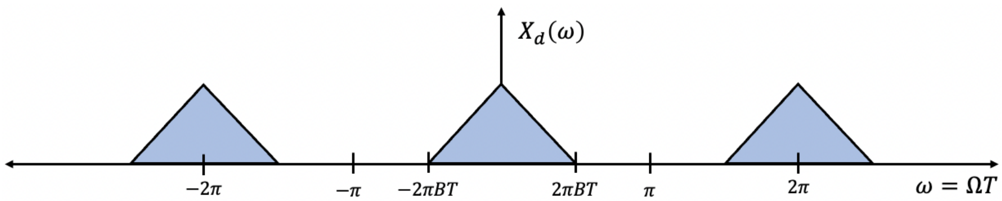
The final step in line 12 follows from the identity property of convolving with a Dirac delta function. Figure 1 depicts the ideal analog-to-digital conversion process of sampling with an impulse train and converting these measurements to a discrete-time sequence.



(a) Original analog frequency spectrum $X_a(\Omega)$



(b) Sampled analog frequency spectrum $X_s(\Omega)$



(c) Sampled digital frequency spectrum $X_d(\omega)$

Figure 2: Illustration of the frequency spectra of an original continuous-time signal $X_a(\Omega)$, the resulting sampled continuous-time spectrum $X_s(\Omega)$ with sampling period T, and the corresponding digital frequency spectrum $X_d(\omega)$.

1.1 Converting to a discrete-time sequence

Finally, we would like to convert from the CTFT of our ideally sampled signal to its corresponding DTFT when the sampled signal is represented as a discrete-time sequence. In the same sense, we would like to replace the analog radial frequency Ω with digital radial frequency ω. To determine this conversion, we can simply refer to sampling a sinusoid. Consider sampling the continuous-time signal $x(t) = \sin(\Omega t)$. The resulting sampled signal would be

$$x[n] = \sin(\Omega n T) \tag{13}$$
$$= \sin(\omega n). \tag{14}$$

Here, we have obtained a key relationship:

$$\omega = \Omega T. \tag{15}$$

Substituting Eqn. 15 into Eqn. 12, we obtain the DTFT of our discrete-time signal after ideal analog-to-digital conversion:

$$X_d(\omega) = X_s(\Omega)\Big|_{\Omega=\omega/T} \tag{16}$$
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a\left(\frac{\omega - 2\pi n}{T}\right) \tag{17}$$

Figure 2 shows the changes in the frequency spectra from the original continuous-time signal to continuous-time sampled signal to the final discrete-time signal.

2 Nyquist criteria

The CTFT of our sampled signal, $X_s(\Omega)$, shows that the sampled spectrum is the summation of infinitely many shifted copies of the original spectrum. These spectral copies are spaced uniformly by multiples of $\frac{2\pi}{T}$ in continuous-time radial frequency Ω. If T is too large, we may not have sufficient space between spectral copies and they will overlap. To prevent overlap, we can derive a necessary condition for a particular class of signals known as *bandlimited signals*.

A bandlimited signal $x(t)$ with CTFT $X_a(\Omega)$ is any continuous-time signal such that

$$X_a(\Omega) = 0, |\Omega| > 2\pi B. \tag{18}$$

The quantity B represents the maximum *linear* frequency present in a bandlimited signal. To derive our condition for no overlap in $X_s(\Omega)$, we need only prevent overlap between the right edge of the central spectral copy (centered at $\Omega = 0$) and the left edge of the first spectral copy to its right (centered at $\Omega = 2\pi/T$).

$$\text{(Right edge of central copy) } 2\pi B < \frac{2\pi}{T} - 2\pi B \text{ (Left edge of } n = 1 \text{ copy)} \tag{19}$$

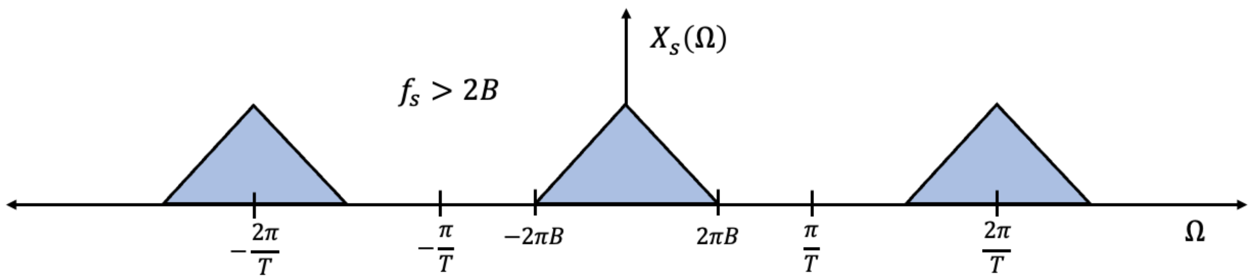
$$2B < \frac{1}{T} \tag{20}$$

$$f_s > 2B. \tag{21}$$

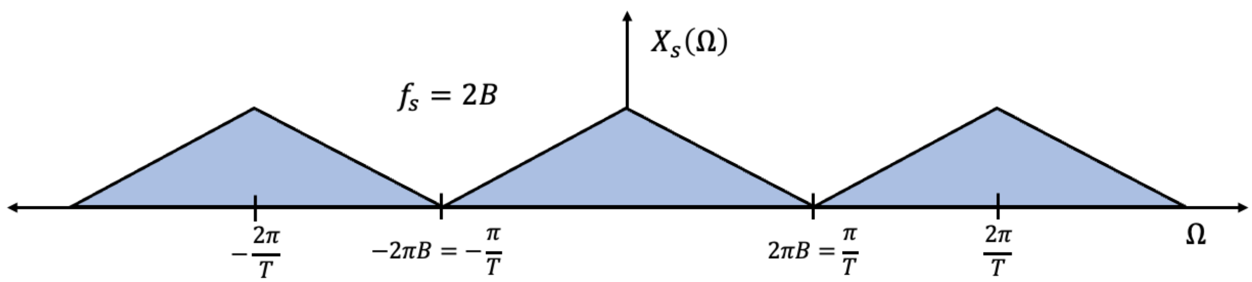
Equation 21 is known as the *Nyquist criterion* for sampling bandlimited continuous-time signals. The minimum frequency at which we must sample is known as the *Nyquist rate* of a particular bandlimited signal:

$$f_{\text{Nyquist}} = 2B. \tag{22}$$

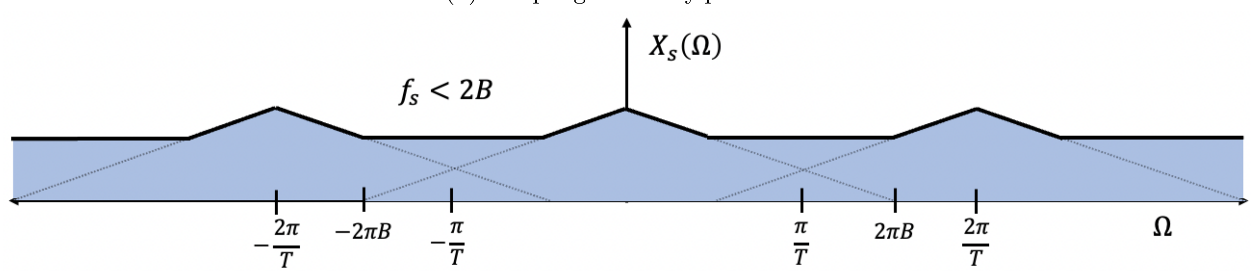
If we sampled below the Nyquist rate, we will have overlap in spectral copies. This overlap is also known as *aliasing*. We will discuss the implications of aliasing further in the next couple lectures as we cover ideal digital-to-analog conversion and the aliasing effect in greater detail. Figure 3 summarizes the effects of sampling above, below, or exactly at the Nyquist rate of a bandlimited coninuous-time signal. We see that sampling above the Nyquist rate gives us additional space or a *guard band* between spectral copies. Conversely, sampling below the Nyquist rate causes our spectral copies to collide and combine to form an entirely new shape for each copy. This change in shape for the frequency spectrum results in aliasing. Finally, sampling exactly at the Nyquist rate leaves us no guard band; however, the spectral copies do not overlap and simply touch one another at their edges.



(a) Sampling above the Nyquist rate.



(b) Sampling at the Nyquist rate.



(c) Sampling below the Nyquist rate.

Figure 3: Frequency spectra of the sampled continuous-time signal $X_s(\Omega)$ with different choices of sampling rate above, at, and below the Nyquist rate. Note the aliasing when sampling below the Nyquist rate results in overlap between spectral copies and a change in the shape of each spectral copy.