ECE 310 Fall 2023

Lecture 23

Spectral analysis with the DFT

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Learning Objectives

After this lecture, you should be able to:

- State the DTFT spectrum of a finite-length sinusoidal signal and how the length of the signal affects the corresponding DFT.
- Explain what zero-padding is and how it helps better visualize the DFT of a discrete-time signal.
- Discuss the challenges with separating multiple sinusoids present in a given finite-length signal.

Recap from previous lecture

We finished defining the DFT and its key properties in the previous lecture. We now turn our attention to an important application of the DFT: spectral analysis. This is a large topic, so we will take two lectures to discuss spectral analysis as well as similar applications of the DFT. In this lecture, we will define and motivate the challenges of spectral analysis.

1 Spectrum of a finite-length sinusoid

We begin by considering a finite-length cosine signal $x[n]$. The math is very similar with sine, but cleaner with cosine since we avoid the additional j factor.

$$x[n] = A\cos(\omega_0 n),\ 0 \le n \le N-1. \tag{1}$$

1.1 DTFT of a finite-length sinusoid

The resulting DTFT spectrum of $x[n]$ is

$$\begin{aligned}
X(\omega) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} && (2)\\
&= \sum_{n=0}^{N-1} A\cos(\omega_0 n)e^{-j\omega n} && (3)\\
&= \frac{A}{2}\sum_{n=0}^{N-1} e^{-j(\omega-\omega_0)n} + e^{-j(\omega+\omega_0)n} && (4)\\
&= \frac{A}{2}\left[\frac{1-e^{-j(\omega-\omega_0)N}}{1-e^{-j(\omega-\omega_0)}} + \frac{1-e^{-j(\omega+\omega_0)N}}{1-e^{-j(\omega+\omega_0)}}\right] && (5)\\
&= \frac{A}{2}\left[\left(\frac{e^{-j\frac{\omega-\omega_0}{2}N}}{e^{-j\frac{\omega-\omega_0}{2}}}\right)\left(\frac{e^{j\frac{\omega-\omega_0}{2}N} - e^{-j\frac{\omega-\omega_0}{2}N}}{e^{j\frac{\omega-\omega_0}{2}} - e^{-j\frac{\omega-\omega_0}{2}}}\right) + \left(\frac{e^{-j\frac{\omega+\omega_0}{2}N}}{e^{-j\frac{\omega+\omega_0}{2}}}\right)\left(\frac{e^{j\frac{\omega+\omega_0}{2}N} - e^{-j\frac{\omega+\omega_0}{2}N}}{e^{j\frac{\omega+\omega_0}{2}} - e^{-j\frac{\omega+\omega_0}{2}}}\right)\right] && (6)
\end{aligned}$$

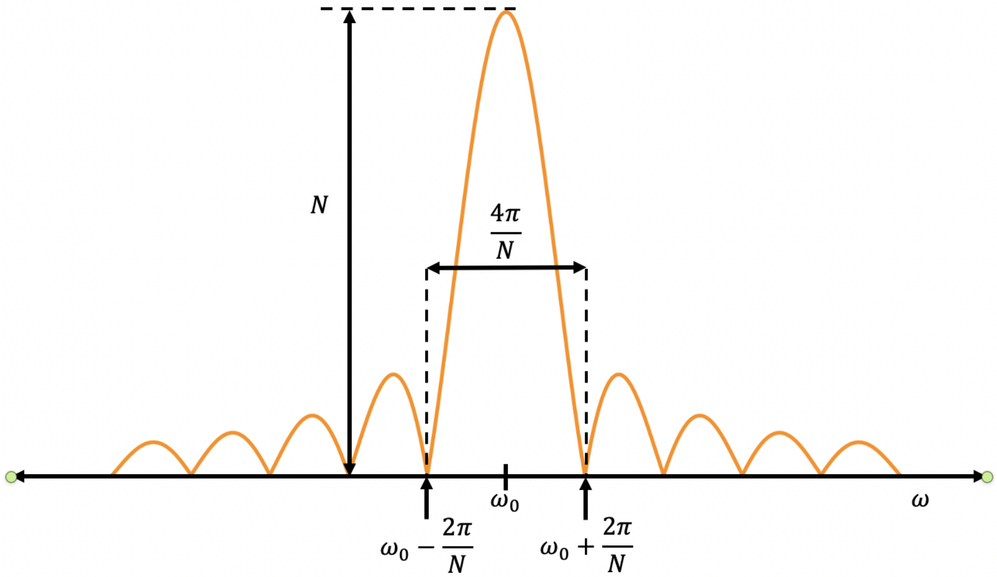


Figure 1: Magnitude spectrum $|C(\omega - \omega_0)|$

$$= \frac{A}{2}\left[e^{-j\frac{\omega-\omega_0}{2}(N-1)}\left(\frac{\sin\left(\frac{N}{2}(\omega-\omega_0)\right)}{\sin\left(\frac{1}{2}(\omega-\omega_0)\right)}\right) + e^{-j\frac{\omega+\omega_0}{2}(N-1)}\left(\frac{\sin\left(\frac{N}{2}(\omega+\omega_0)\right)}{\sin\left(\frac{1}{2}(\omega+\omega_0)\right)}\right)\right] \tag{7}$$

Above, line 5 follows from the sum of a finite-length geometric sequence. The final result we obtain is quite messy, so let's unpack everything. Let $C(\omega)$ be the following spectrum:

$$C(\omega) = e^{-j\frac{\omega}{2}(N-1)}\left(\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}\right), \tag{8}$$

where

$$\frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)} = \begin{cases} N, & \omega = 0 \\ \frac{\sin\left(\frac{N}{2}\omega\right)}{\sin\left(\frac{1}{2}\omega\right)}, & \omega \neq 0 \end{cases}. \tag{9}$$

It is also helpful to note that $C(\omega)$ is zero at all integer multiples of $\frac{2\pi}{N}$ except $\omega = 0$.

We can use $C(\omega)$ to write the spectrum $X(\omega)$ much more neatly:

$$X(\omega) = \frac{A}{2}C(\omega - \omega_0) + \frac{A}{2}C(\omega + \omega_0). \tag{10}$$

We see then that $X(\omega)$ has useful symmetry with two shifted copies of $C(\omega)$. Thus, we can understand $X(\omega)$ by examining $C(\omega)$. Figure 1 depicts the magnitude spectrum of $C(\omega)$. The shape of this spectrum may remind us of a sinc function and in fact we often refer to $C(\omega)$ as the *periodic sinc function* or *Dirichlet function*. We see that there is one large *main lobe* surrounded by smaller, symmetrical *side lobes*. The height of the main lobe is given by the length of $x[n]$, N, and the width of the main lobe is $\frac{4\pi}{N}$. Intuitively, as N becomes larger, the main lobe gets taller and narrower, approaching a Dirac delta function in the limit as $N \to \infty$ since the height will become infinite and the width goes to zero. This is consistent with the DTFT of an infinite-length sinusoid!

Lecture exercise: Consider how we may derive $X(\omega)$ using the DTFT of a rectangular signal and the modulation property of the DTFT.

1.2 DFT of a finite-length sinusoid

We can quickly derive the DFT of a finite-length sinusoid $x[n]$ by following our conversion between the DFT and DTFT of a finite-length signal:

$$X[k] = X(\omega)\big|_{\omega=\frac{2\pi k}{N}}. \tag{11}$$

Thus, we have

$$X[k] = \frac{A}{2} C\left(\frac{2\pi k}{N} - \omega_0\right) + \frac{A}{2} C\left(\frac{2\pi k}{N} + \omega_0\right). \tag{12}$$

The length of $x[n]$ will play a crucial role in what our DFT looks like in practice. Let's look at a specific example where

$$x_1[n] = \cos\left(\frac{\pi}{3} n\right), 0 \leq n < 24 \tag{13}$$

$$x_2[n] = \cos\left(\frac{\pi}{3} n\right), 0 \leq n < 22. \tag{14}$$

The period of both $x_1[n]$ and $x_2[n]$ is 6 samples. The key difference is that $x_1[n]$ contains an integer number of periods while $x_2[n]$ samples a fracional number of periods from the sinusoidal signal. We visualize the magnitude of the resulting DFT spectrum for these two signals in Fig. 2.

We see a striking difference between the two magnitude spectra. The magnitude spectrum for $X_1[k]$ is only non-zero when $k = 4$ and $k = 20$. These values of k correspond to $\frac{\pi}{3}$ and $\frac{5\pi}{3} \equiv -\frac{\pi}{3}$. Thus, we capture the peaks of the two main lobes in the corresponding DTFT $X_1(\omega)$ and easily identify the frequency of this single sinusoid. The other magnitude spectrum, $|X_2[k]|$, also has its largest values at frequencies close to $\frac{\pi}{3}$ and $\frac{5\pi}{6}$; however, most other samples of $X_2[k]$ are non-zero. The phenomenon we see in $X_2[k]$ is known as *spectral leakage* where the energy of a central peak spreads out to adjacent frequencies. Spectral leakage is a consequence of periodic extension from the DFT and also truncating infinite-length sequences to a finite length. Recall that the DFT implies N-periodic extension in both the time and DFT domain. In the case of $x_1[n]$, the periodic extension of $x_1[n]$ will be a smooth and infinitely long representation of $\cos(\frac{\pi}{3}n)$. Thus, we only see the frequencies $\pm\frac{\pi}{3}$ in the periodically extended $x_1[n]$. Extending $x_2[n]$ creates a problem since $x_2[n]$ captures 3 complete periods and one fractional period of $\cos(\frac{\pi}{3}n)$. When we periodically extend $x_2[n]$, we will have periodic discontinuities between each extended copy due to the fractional period at the end of each copy. These discontinuities interrupt the smooth representation of the frequency $\pi/3$ and introduces new frequency content into the DFT. The result is that the energy of the main lobes spreads out to adjacent frequencies and we have a less precise representation of the true frequencies in the signal.

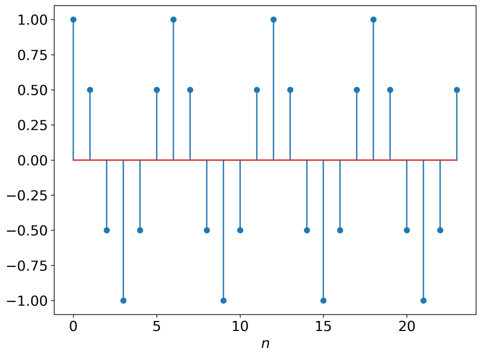
Another way to understand the differences in $X_1[k]$ and $X_2[k]$ is to consider the shape of the $C(\omega)$ function that expresses each peak in the DTFT of the finite-length $x[n] = \cos(\omega_0 n)$. If there exists an integer k such that $\omega_0 = 2\pi k/N$, then the DFT will evaluate the DTFT at the respective peaks of each main lobe. We also know that any integer multiple of $2\pi/N$ away from the center of each main lobe evaluates to zero. This is the case for $X_1[k]$ in the previous example (i.e. when $k = 4$ and $k = 20$); thus, we only have two non-zero values in the DFT. For $x_2[n]$, there does not exist an integer k such that $\omega_0 = 2\pi k/N$, i.e. the corresponding value of k is $11/3$. Thus, $X_2[k]$ evaluates the DTFT of $x_2[n]$ away from the peaks of the main lobes and away from the nulls of each main/side lobe.

It is reasonable to ask then: can we choose the length of our signals to obtain cleaner spectra? The answer, unfortunately, is no because most interesting signals have many spectral components and we don't know their frequencies ahead of time. Thus far, we have considered signals with only one spectral component, i.e. one sinusoidal component. In the next section, we will consider the more general spectral analysis problem.

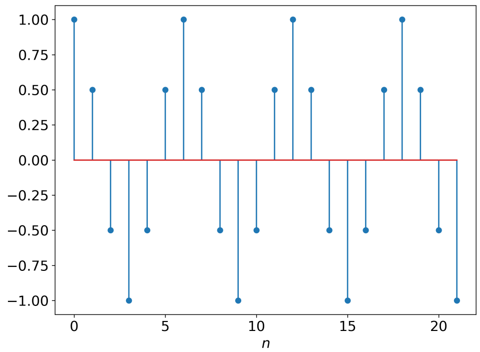
2 Spectral analysis

Spectral analysis is the problem where we seek to identify some known (or potentially unknown) number of sinusoidal components in a given signal. Thus, for finite-length signal $x[n]$, we assume

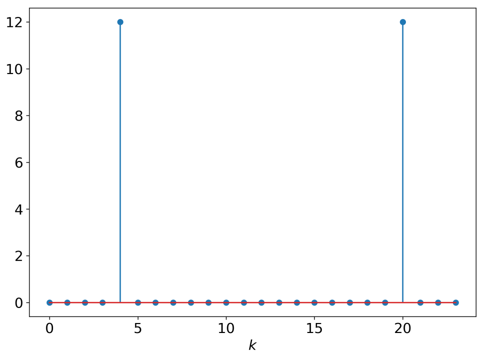
$$x[n] = \sum_{m=1}^{M} A_m \cos(\omega_m n),\ 0 \leq n \leq N-1. \tag{15}$$



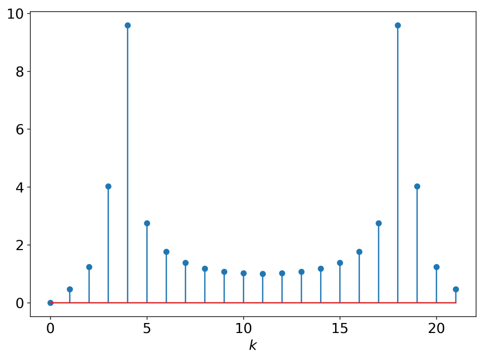
(a) $x_1[n]$



(b) $x_2[n]$



(c) $|X_1[k]|$



(d) $|X_2[k]|$

Figure 2: Depictions of $\cos\left(\frac{\pi}{3}n\right)$ with different lengths N. The first column corresponds to $x_1[n]$ and $N = 24$ while the second column corresponds to $x_2[n]$ and $N = 22$.

We see that both the amplitudes and frequencies of each sinusoidal component are unknown ahead of time. Our primary goal is to identify all the M frequencies $\{\omega_m\}_{m=1}^{M}$ present in $x[n]$.

We can simplify the problem and develop all the same tools if we consider a signal with $M = 2$ sinusoidal components. Thus, let $s[n]$ be

$$s[n] = A_1 \cos(\omega_1 n) + A_2 \cos(\omega_2 n), \ 0 \leq n \leq N - 1. \tag{16}$$

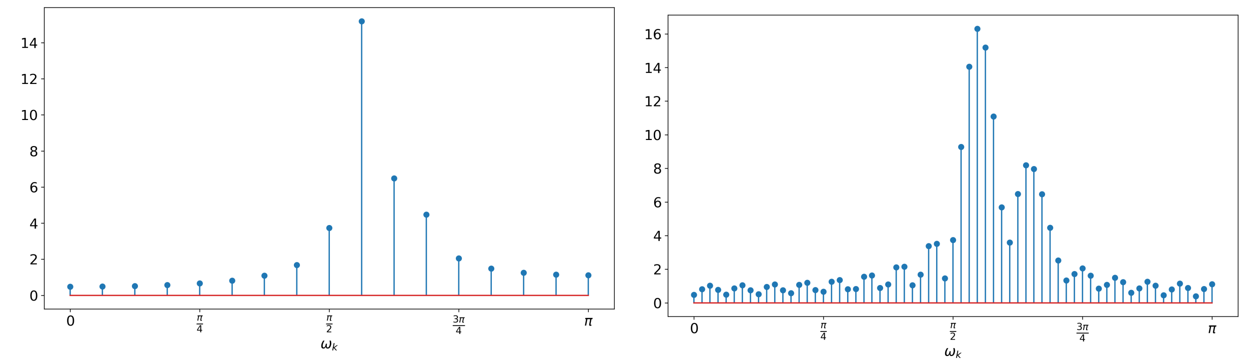
In order to define the key challenges of spectral analysis, let's consider an example signal $s[n]$. Let

$$A_1 = 1, \ A_2 = \frac{1}{2}, \ \omega_1 = 0.55\pi, \ \omega_2 = 0.65\pi, \ N = 32. \tag{17}$$

The resulting DFT spectrum for values of k that map between $[0, \pi]$ is shown in Fig 3a. We can clearly see a peak around $\omega_k = \frac{9\pi}{16} \approx 0.563\pi$ that represents ω_1; however, ω_2 is impossible to identify. It would be hard to know where ω_2 is, let alone know there are in fact two sinusoidal components in $s[n]$. This example is actually solved fairly easily by considering zero-padding.

2.1 Zero-padding

We may evaluate more frequencies in the DTFT with the zero-padding technique we introduced in lecture 21. Recall that if we have a length-N signal $s[n]$, we can add L zeros to the end of $s[n]$ to obtain the zero-padded signal $\tilde{s}[n]$. The DFT of $S[k]$ has a spacing of $\frac{2\pi}{N}$ between each frequency location of the DTFT. The DFT of $\tilde{S}[k]$ instead has a spacing of $\frac{2\pi}{N+L} < \frac{2\pi}{N}$ between frequency locations. Since we are only adding zeros to the signal, the underlying DTFTs $S(\omega)$ and $\tilde{S}(\omega)$ are equivalent. Thus, zero-padding will give us a denser visualization of our frequency spectrum and perhaps make the sinusoidal components clearer.



(a) $|S[k]|$

(b) $|\tilde{S}[k]|$

Figure 3: DFT magnitude spectra of $s[n]$ and zero-padded $\tilde{s}[n]$ of length $N = 128$.

Figure 3 compares the magnitude spectra $|S[k]|$ and $|\tilde{S}[k]|$ when we pad 96 zeros to $s[n]$ to obtain $\tilde{s}[n]$. With zero-padding, we are now able to identify two main lobes at $\omega_k \approx 0.547\pi$ and $\omega_k \approx 0.641\pi$. These two peaks correspond to the two sinusoidal components! Moreover, we see that the second component is roughly half the height of the first component as we would expect. *It is important to note that zero-padding does not change the underlying information in a signal since we are adding zeros to the end and thus no energy. Zero-padding only helps us obtain more values from the same DTFT using the DFT.*

2.2 Challenges of separating frequency components

We saw in the above example how zero-padding $s[n]$ allowed us to accurately distinguish between the two sinusoidal components present in the signal. However, we cannot always guarantee that sinusoidal components are separable. There are two key challenges in spectral analysis:

1. **The amplitudes of components may vary greatly.** Consider if the smaller amplitude A_2 was much smaller than A_1, e.g. $\frac{1}{10}$. In this case, the main lobe of the second component may be covered up by a side lobe of the first component. Thus, we need an effective way to minimize side lobes so that the relevant main lobes are easier to identify.

2. **The frequencies of components may be too close to separate.** We may not be able to distinguish the main lobes from two components even if they have similar amplitudes should they be too close and form one main lobe. In other words, if $|\omega_1 - \omega_2|$ is too small, the main lobes of each component will look like one combined main lobe.

In the next lecture, we will further demonstrate these two issues and discuss how we may identify and overcome each challenge.