ECE 310 Fall 2023

Lecture 9

Transfer functions and LTI system response: Part 1

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Learning Objectives

After this lecture, you should be able to:

- Compute the response of an LTI system to an input signal using the z-transform.
- Write the transfer function of an LTI system described by an LCCDE.
- Identify FIR and IIR systems based on characteristics of an LCCDE or transfer function.

Recap from previous lecture

We covered the inverse z-transform in the previous lecture and common techniques for computing it by hand. We now have all the tools we need to describe signals in the complex z-domain along with their time-domain counterparts. In this lecture, we will apply this knowledge to explore how the response of LTI systems can be described using the z-transform.

1 LTI system response with the z-transform

For an LTI system with impulse response $h[n]$, let $H(z)$ denote its z-transform or *transfer function*. The response $y[n]$ of this LTI system to an input signal $x[n]$ is then given by

$$y[n] = x[n] * h[n]. \tag{1}$$

Directly applying the convolution property of the z-transform, we obtain the following key result:

$$\begin{aligned} &Y(z) = H(z)X(z), \\ &\text{ROC at least } R_x \cap R_h. \end{aligned} \tag{2}$$

We now have a second general approach for computing the response of an LTI system to any input signal! We can use the convolution sum or multiply in the z-domain and take the inverse z-transform. Rearranging Eqn. 2, we arrive at a useful result for describing the system response $H(z)$:

$$H(z) = \frac{Y(z)}{X(z)}. \tag{3}$$

1.1 Transfer functions and LCCDEs

Recall from Lecture 6 that we commonly use linear constant-coefficient difference equations (LCCDEs) to describe LTI systems. We can express any causal LCCDE as

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M-1} b_k x[n-k]. \tag{4}$$

where N specifies the number of *feedback* terms and M gives the number of input terms. Note that the following discussion holds just as well for non-causal systems, and we are focusing on causal systems for notational simplicity. When $N > 0$, it can be challenging to identify the impulse response of our system and thus difficult to compute system outputs. Instead, we can try working in the z-domain!

$$y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M-1} b_k x[n-k]$$

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M-1} b_k z^{-k} X(z) \tag{5}$$

$$Y(z)\left(1 + \sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M-1} b_k z^{-k}\right) \tag{6}$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{\sum_{k=0}^{M-1} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \tag{7}$$

In line 5, we have taken the z-transform of both sides of the LCCDE by using the linearity and time shifting properties of the z-transform. We regroup the terms in line 6 and then use our definition of a transfer function from Eqn. 3 to obtain the final result. This result is incredibly useful because we can now

1. Find the impulse response of any LCCDE by computing the transfer function then taking the inverse z-transform.

2. Calculate the response of an LTI system described in the time-domain without using the impulse response or convolution sum directly.

Let's briefly revisit an example from Lecture 5 to reinforce this first point and to motivate the next section.

Exercise 1: Compute the impulse response of the following causal LCCDE:

$$y[n] = \frac{1}{2} y[n-1] + x[n]. \tag{8}$$

We stated in Lecture 5 that $h[n] = \left(\frac{1}{2}\right)^n u[n]$, but now let's prove it using the z-transform.

$$Y(z) = \frac{1}{2} z^{-1} Y(z) + X(z) \tag{9}$$

$$Y(z)\left(1 - \frac{1}{2} z^{-1}\right) = X(z) \tag{10}$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1 - \frac{1}{2} z^{-1}}, \ |z| > \frac{1}{2} \tag{11}$$

$$H(z) \overset{\mathcal{Z}}{\mapsto} h[n] = \left(\frac{1}{2}\right)^n u[n]. \tag{12}$$

The last line follows by inspection from our transform pairs and the system being causal.

1.2 Characterizing transfer functions

Our general form of transfer functions in Eqn. 7 allows us to easily define key features of any transfer function.

By the fundamental theorem of algebra, we will have N poles and M zeros in our system's transfer function. It is important to note that it is possible for a system to have a pole and zero at the same location. In this case, the pole and zero cancel one another and the system neither goes to zero nor infinity for this value of z. The number of feedback terms N gives us the *order* of our LTI system similar to how we describe the degree of a polynomial equation.

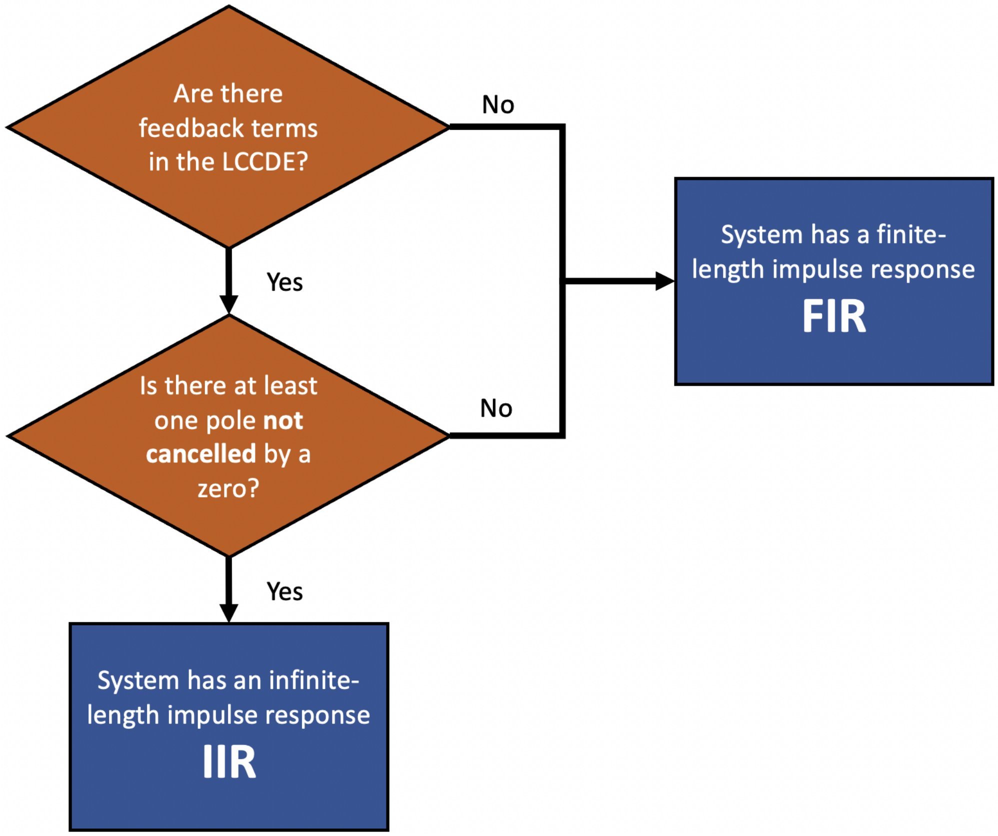

Figure 1: Process for determining if a transfer function $H(z)$ corresponds to an FIR or IIR system.

We saw in the previous exercise that a system with only two terms – one feedback term and one input term – can yield an infinitely long impulse response. This is because of the feedback term $y[n-1]$. In general, if we have any finite poles not canceled by a zero, then our system will have an *infinite impulse response* (IIR). The term finite poles refers to poles not at $z=0$ or $z=\infty$. If we have no finite poles in our system or the denominator is a monomial (has one term), then our system will have a *finite impulse response* (FIR). This most notably occurs when $N=0$ and the denominator is simply one for $H(z)$. An immediate consequence of this is that FIR systems can only have poles at $z=0$ for causal systems and $z=\infty$ for non-causal systems. This will be important when we re-visit BIBO stability using the z-transform in future lectures. Figure 1 summarizes the process for identifying FIR and IIR systems.