UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN
Department of Electrical and Computer Engineering

ECE 310 Digital Signal Processing – Fall 2023

Homework 9

Profs. Do, Snyder, Moustakides **Due: Friday, Oct 27, 2023 on Gradescope**

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1. Let $\{x[0], x[1], \ldots, x[N-1]\}$ be a ***real-valued*** sequence and $\{X[0], X[1], \ldots, X[N-1]\}$ denotes its length-N DFT. Suppose that $N = 2L$ where L is a positive integer.

   (a) Show that $X[0]$ and $X[L]$ are real-valued.

   (b) Using the properties of DFT, express the DFT of the following signals in terms of $X[k]$ where $\langle k \rangle_N$ denotes the modulo operation with divider N:

   i. $x_1[n] = x[n] + x[\langle n - L \rangle_N]$

   ii. $x_2[n] = x[n] - x[\langle n - L \rangle_N]$

   iii. $x_3[n] = (-1)^n x[n]$

   (c) Show that if the sequence in time is symmetric about its center, i.e. $x[n] = x[N-1-n]$, then $X[L] = 0$.

   (d) Show that if the sequence in time is anti-symmetric about its center, i.e. $x[n] = -x[N-1-n]$, then $X[0] = 0$.

2. A continuous-time signal $x_c(t) = \cos(25\pi t)$ is sampled at a rate of 60 Hz for five seconds to produce a discrete-time signal $\{x[n]\}_{n=0}^{N-1}$ with length $N = 300$.

   (a) Let $X[k]$ be the length-N DFT of $x[n]$. At what values of k will $X[k]$ have the greatest magnitude? Note that there are more than one of such k.

   (b) Suppose that $x[n]$ is zero-padded to a total length of $M = 512$. At what values of k does the length-M DFT have the greatest magnitude?

3. Let the complex sinusoid $x[n] = e^{j\omega_0 n}$ have radial frequency ω_0. We know that the DTFT $X(\omega)$ of $x[n]$ is equal to $X(\omega) = 2\pi\delta(\omega - \omega_0)$ which is concentrated on the single frequency ω_0 and in all other frequencies it is equal to 0. Consider now that we have a length-N signal $s[n] = \{x[0], \ldots, x[N-1]\}$ from the complex sinusoid. The DTFT of $s[n]$, $S(\omega)$ is given below:

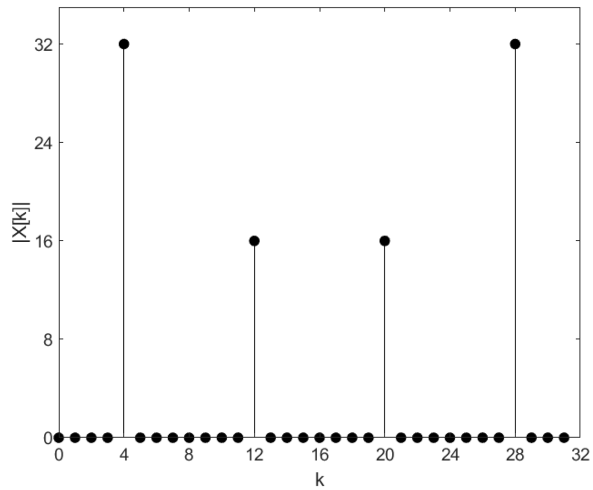
$$S(\omega) = e^{-j\frac{N-1}{2}(\omega-\omega_0)} \frac{\sin\left(\frac{N}{2}(\omega - \omega_0)\right)}{\sin\left(\frac{1}{2}(\omega - \omega_0)\right)}.$$

   Suppose $\omega_0 = \frac{2\pi\ell}{m}$ where ℓ, m positive integers with $\ell < m$ that are also mutually prime (the only common divisor is 1).

   (a) Show that in order for the frequency ω_0 to appear as some frequency $\omega_k = \frac{2\pi}{N}k$ of the corresponding DFT $S[k]$ we need to have $N = rm$ and $k = r\ell$ for some integer r. What does this tell us about the relationship between the length of the signal N and the frequency ω_0? *Hint: consider the period that corresponds to ω_0.*

(b) For this specific selection of N and k show that $\omega_k = 2\pi\frac{r\ell}{rm} = \omega_0$ is the only *nonzero* DFT sample in $S[k]$ while all other DFT samples are equal to 0.

4. A scientist is performing spectroscopy to identify the chemical composition of a material. The calculated DFT magnitude plot is shown below using data collected at a rate of 32 Terahertz for one picosecond. Assume that the measured electromagnetic signal has the form $x_c(t) = A_0 \cos(\Omega_0 t) + A_1 \cos(\Omega_1 t)$.



(a) If both frequencies Ω_0, Ω_1 are smaller than the Nyquist rate, find $A_0, A_1, \Omega_0, \Omega_1$.

(b) Suppose $x_c(t)$ were instead sampled at 64 Terahertz for one picosecond to generate 64 samples. Sketch the new DFT magnitude plot and clearly label all nonzero values.