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section:

PROBLEM SET #12

[1]. $W = \frac{1}{\sqrt{2}}(X-Y) \quad Z = \frac{1}{\sqrt{2}}(X+Y)$

(a) $X, Y$ independent $N(0,1)$

first prove for $X_1 \sim (\mu_1, \sigma_1^2)$, $X_2 \sim (\mu_2, \sigma_2^2)$, $X_1, X_2$ independent, we have $X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$:

$f_1(x) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp(-\frac{(x-\mu_1)^2}{2\sigma_1^2})$, $f_2(x) = \frac{1}{\sqrt{2\pi}\sigma_2} \exp(-\frac{(x-\mu_2)^2}{2\sigma_2^2})$

$f(x) = f_1(x) * f_2(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x-t) dt$

$= \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} \exp(-\frac{1}{2}[\frac{(t-\mu_1)^2}{\sigma_1^2} + \frac{(x-t+\mu_2)^2}{\sigma_2^2}])dt$

$\frac{(t-\mu_1)^2}{\sigma_1^2} + \frac{(x-t-\mu_2)^2}{\sigma_2^2} = \frac{1}{\sigma_1^2+\sigma_2^2}(x-\mu_1-\mu_2)^2 + (at-b)^2$, $a = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}$, $b = \frac{\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}[\frac{\mu_1}{\sigma_1^2} + \frac{x-\mu_2}{\sigma_2^2}]$

so $f(x) = \frac{1}{2\pi\sigma_1\sigma_2} \exp(-\frac{(x-\mu_1-\mu_2)}{2(\sigma_1^2+\sigma_2^2)}) \underline{\int_{-\infty}^{+\infty} e^{\frac{1}{2}(at-b)^2} dt}$ guassian integration

$= I$, $I^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} e^{\frac{1}{2}[(au-b)^2+(av-b)^2]} du\, dv$, consider $u = \frac{r\sin\theta + b}{a}$, $v = \frac{r\cos\theta + b}{a} \Rightarrow I = \sqrt{2\pi}\, a = \frac{\sqrt{2\pi}\,\sigma_1\sigma_2}{\sqrt{\sigma_1^2+\sigma_2^2}}$

$= \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}} \exp(-\frac{(x-\mu_1-\mu_2)^2}{2(\sigma_1^2+\sigma_2^2)})$

So $X_1 + X_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$, furthermore, $\frac{1}{a}(X_1+X_2) \sim N(\frac{\mu_1+\mu_2}{a}, \frac{\sigma_1^2+\sigma_2^2}{a^2})$

for $W = \frac{1}{\sqrt{2}}(X-Y)$, consider $(-Y)$ as the variable $\sim N(0,1)$

so $W \sim N(\frac{1}{\sqrt{2}}(0+0), \frac{1}{2}(1+1)) \rightarrow W \sim N(0,1)$

for $Z = \frac{1}{\sqrt{2}}(X+Y)$, $Z \sim N(\frac{1}{\sqrt{2}}(0+0), \frac{1}{2}(1+1)) \rightarrow Z \sim N(0,1)$

At last, prove for independent of $W, Z$:

linear mapping: $A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ $\det A = 1$, $A^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$

$f_{W,Z}(\alpha,\beta) = f_{X,Y}\begin{pmatrix} \frac{1}{\sqrt{2}}(\alpha+\beta) \\ \frac{1}{\sqrt{2}}(\beta-\alpha) \end{pmatrix} = f_X(\frac{1}{\sqrt{2}}(\alpha+\beta)) \cdot f_Y(\frac{1}{\sqrt{2}}(\beta-\alpha)) = f_W(\alpha) \cdot f_Z(\beta)$

so $W, Z$ is independent

(b) $f_{W,Z}(\alpha,\beta) = f_{X,Y}(\frac{1}{\sqrt{2}}(\alpha+\beta), \frac{1}{\sqrt{2}}(\beta-\alpha)) = f_X(\frac{1}{\sqrt{2}}(\alpha+\beta)) \cdot f_Y(\frac{1}{\sqrt{2}}(\beta-\alpha)) = \frac{1}{2\pi} \cdot \frac{1}{2\sqrt{2}} \cdot \exp(-\frac{\frac{1}{2}(\alpha+\beta)^2}{4} - \frac{\frac{1}{2}(\beta-\alpha)^2}{8})$

however, $W \sim N(0, \frac{1}{2}(2+4)) = N(0,3)$, $Z \sim N(0, \frac{1}{2}(2+4)) = N(0,3)$

but $f_{W,Z}(\alpha,\beta) \neq f_W(\alpha) \cdot f_Z(\beta)$

so $\alpha, \beta$ not independent

[2].

(a) $Var(X+2Y) = Cov(X+2Y, X+2Y)$

$= Var(X) + 4Var(Y) + 4Cov(X,Y)$

$Var(X-2Y) = Cov(X-2Y, X-2Y)$

$= Var(X) + 4Var(Y) - 4Cov(X,Y)$

so $Var(X+2Y) = Var(X-2Y) \Rightarrow Cov(X,Y) = 0$, $X, Y$ is uncorrelated.

(b) $Var(X) = Var(Y)$: $X, Y$ can be correlated.

consider $X, Y$ both describe the same event: a coin flip for once, head for 1.

thus $Var(X) = Var(Y)$, but $X, Y$ is the same event, should be correlated.

(c) $Cov(5X+3, 2X+3Y-1)$

$= 5Cov(X, 2X+3Y)$

$= 10Var(X) + 15Cov(X,Y)$

[3]. signal $M$, observation $M_1, M_2$ on earth, same $\mu, \sigma^2$

SNR (signal-to-noise-ratio): $\frac{\mu}{\sigma^2}$ , $S = \frac{M_1+M_2}{2}$

(a) $\mu_S = \frac{1}{2}(\mu_1+\mu_2)$, for $Var_S$, since $M_1, M_2$ is uncorrelated:

$Var_S = \frac{\sigma_1^2+\sigma_2^2}{4}$

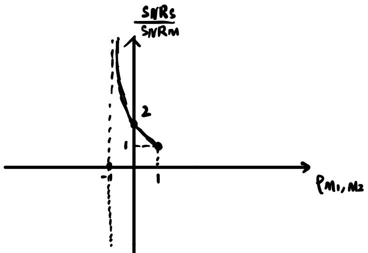
so $SNR_S = \frac{\mu_S}{Var_S} = \frac{\frac{1}{2}(\mu_1+\mu_2)}{\frac{\sigma_1^2+\sigma_2^2}{4}} = 2\frac{\mu}{\sigma^2}$

that is, SNR of $S$ is twice that of the individual observations.

(b) $Var_S = \frac{1}{4}Var(M_1+M_2) = \frac{1}{4}[Var(M_1) + Var(M_2) + 2Cov(M_1, M_2)]$

$\frac{SNR_S}{SNR_M} = \frac{\mu}{Var_S} \cdot \frac{\sigma^2}{\mu} = \frac{\sigma^2}{Var_S} = 4 \cdot \frac{\sigma^2}{2\sigma^2 + 2Cov(M_1,M_2)} = 2 \cdot \frac{1}{1+\rho_{M_1,M_2}}$

$\rho_{M_1,M_2} \in (-1, 1]$



(c) $SNR_S = 1.5 SNR_M \Rightarrow \rho_{M_1, M_2} = \frac{1}{3}$

when $\rho_{M_1, M_2}$ close to $-1$, the $SNR_S$ can be arbitrarily high.

this condition is not practical, since the observation for the same signal should have positive correlation

[4]. $X_1 \sim X_{50}$, $\mu = 10$, $\sigma^2 = 1$, $|Cov(X_i, X_j)| \leq 0.05$ if $|i-j| \leq 2$, $i \neq j$ $|Cov(X_i, X_j)| = 0$ if $|i-j| > 2$

$S_{50} = X_1 + \cdots + X_{50}$

(a) $Var(S_{50}) = \sum_{i=1}^{50} Var(X_i) + 2 \cdot \sum_{1 \leq i < j \leq 50} Cov(X_i, X_j)$

$\leq 50 \times 1 + [48 \times 2 \times 0.05 + 0.05] \cdot 2$ for $i = 1 \sim 48$, each have 2 terms that $Cov(X_i, X_j) \leq 0.05$, for $i = 49$, $Cov(49, 50) \leq 0.05$

$= 59.7$

(b) $P\{|\frac{S_{50}}{50} - 10| \geq 0.5\} \leq \frac{\sigma^2}{0.5^2} \cdot \frac{1}{50^2} = \frac{59.7}{25^2} = 0.09552$