ECE 310 Fall 2023

Lecture 15

Discrete-time Fourier transform properties

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Learning Objectives

After this lecture, you should be able to:

- Explain the relationship between the z-transform and the DTFT.
- Apply key properties of the DTFT to compute transform pairs and analyze signals in both the time-domain and frequency-domain.

Recap from previous lecture

We built on our understanding of continuous-time Fourier analysis to explore discrete-time signals in the previous lecture. This allowed us to develop the discrete-time Fourier series (DTFS) and discrete-time Fourier transform (DTFT) as tools for describing the frequency content of periodic and aperiodic digital signals. In this lecture, we will continue defining the DTFT by explaining its connections to the z-transform and several key properties.

1 DTFT and the z-transform

The DTFT and z-transform both describe the content of discrete-time systems with respect to particular classes of signals. We demonstrated in lecture 6 that the z-transform expresses how similar a given signal is to any complex exponential of the form $z^n,\ z \in \mathbb{C}$. In the previous lecture, we used the DTFS to show that the DTFT indicates the content of each frequency in a given periodic or aperiodic signal. It is natural then to ask: how are the DTFT and z-transform related?

To answer this question, consider which signals describe what we know about both the z-transform and the DTFT. In other words, which signals are both complex exponentials of the form z^n and periodic? The answer is periodic complex exponential signals of the form $e^{j\omega n},\ \omega \in \mathbb{R}$. This implies that the case of $z = e^{j\omega}$ must be special for the z-transform. Suppose we have a discrete-time signal $x[n]$. The corresponding z-transform, as we well know, is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}. \tag{1}$$

Now, let's evaluate at $z = e^{j\omega}$:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}. \tag{2}$$

This is exactly our summation for the DTFT! This also explains why you will see the DTFT of a signal written as $X(e^{j\omega})$ in the *Manolakis and Ingle* textbook. *The DTFT is defined as the z-transform evaluated along the unit-circle in the complex z-domain.* We denote the DTFT as $X(\omega)$ instead for notational simplicity, but both are valid.

Another reasonable question to ask then is: why did we bother with the z-transform all that time if we can just use the DTFT? First, the z-transform still expresses the content of a discrete-time signal with respect to a broader class of signals. The DTFT only tells us how $|z| = 1$ is present in each $x[n]$. The z-transform characterizes any choice of z. Consider, for example,

$$x[n] = 2^n u[n]. \tag{3}$$

The corresponding z-transform is given by

$$X(z) = \frac{1}{1 - 2z^{-1}}, \ |z| > 2. \tag{4}$$

Conversely, the DTFT of this $x[n]$ does not exist since our convergence condition of

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty \tag{5}$$

is not satisfied and the DTFT sum does not converge. This should remind us of our discussions on BIBO stability. Remember that an LTI system is BIBO stable if and only if the region of convergence (ROC) of its z-transform contains the unit circle. Together, this gives us another equivalent condition for stability. *An LTI system is BIBO stable if the DTFT of its impulse response $h[n]$ exists.* This condition is equivalent to our second definition of BIBO stability:

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty. \tag{6}$$

In fact, Eqns. 5 and 6 are the same! Altogether, this demonstrates the important connections between the DTFT and z-transform.

1.1 Linear algebra intuition

Recall in lecture 6 how we demonstrated (in an optional reading section) the linear algebra intuition of the z-transform. We showed that each value $X(z)$ for the z-transform of $x[n]$ can be written as the dot product between two vectors

$$X(z) = v^* u \tag{7}$$

where

$$u = x[n] \tag{8}$$
$$v^* = z^{-n} \tag{9}$$

and v^* denotes the conjugate transpose of a complex-valued vector. Thus, the z-transform tells us how similar $x[n]$ is to the exponential signal that has complex conjugate of z^{-n}. We can immediately extend this reasoning to the DTFT from our recent discussion of how the z-transform and DTFT are related. We may now write each value of the DTFT as

$$X(\omega) = v^* u \tag{10}$$

where

$$u = x[n] \tag{11}$$
$$v = e^{j\omega n}. \tag{12}$$

This further reinforces the logic we built up in the previous two lectures from our discussion of Fourier analysis. The DTFT tells us how similar a discrete-time signal is to complex exponential signals of any frequency. In other words, how much each frequency is present in the signal.

2 DTFT pairs

We can use the relationship between the z-transform and the DTFT to derive many useful transform pairs. However, we can only do this for signals that have an ROC containing the unit-circle, i.e. where $z = e^{j\omega}$ is defined in the z-domain. This means the DTFT of signals like cosine, sine, $u[n]$, and $e^{j\omega n}$ do not formally exist since they are not absolutely summable. We do make exceptions for these signals, however, since they are so fundamental to signal processing and their DTFTs should make intuitive sense.

We start by considering $x[n] = e^{j\omega_0 n}$. This signal is defined as only containing exactly one frequency. Therefore, the corresponding DTFT $X(\omega)$ should look something like

$$X(\omega) = \begin{cases} \text{Non-zero}, & \omega = \omega_0 + 2\pi k, \ k \in \mathbb{Z} \\ 0, & \text{otherwise} \end{cases}. \tag{13}$$

To define the Fourier transform (continuous-time or discrete-time) for periodic exponential signals, we define a new signal known as the *Dirac delta function* denoted by $\delta(t)$, $t \in \mathbb{R}$:

$$\delta(t-a) = \begin{cases} \infty, \ t = a \\ 0, \ \text{otherwise} \end{cases}. \tag{14}$$

Thus, the Dirac delta function has infinite height with infinitesimal width at one fixed point over the real line. Strictly speaking in mathematical terms, we say that the Dirac delta is a probability distribution of a continuous-valued random variable that can only take on exactly one value. Thus, by the axioms of probability, the Dirac delta integrates to one. We also define the following *sifting property* of the Dirac delta:

$$x(t-a) = \int_{-\infty}^{\infty} x(t)\delta(t-a)dt. \tag{15}$$

It is important to note that we use parentheses and a continuous argument for the Dirac delta $\delta(t)$ while we use square brackets and a discrete argument for the Kronecker delta $\delta[n]$. These are not the same signals!

Using the Dirac delta and the sifting property, we can derive the DTFT of $x[n] = e^{j\omega_0 n}$. We have already argued that $X(\omega)$ will be of the form $c\delta(\omega - \omega_0)$ where c is some scalar. We can then use the inverse DTFT to solve for c:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}d\omega \tag{16}$$

$$e^{j\omega_0 n} = \frac{1}{2\pi}\int_{2\pi} c\delta(\omega - \omega_0)e^{j\omega n}d\omega \tag{17}$$

$$2\pi e^{j\omega_0 n} = ce^{j\omega_0 n} \tag{18}$$

$$c = 2\pi. \tag{19}$$

Thus,

$$e^{j\omega_0 n} \xleftrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0). \tag{20}$$

Applying Euler's identities, we also obtain the DTFT pairs for cosine and sine:

$$\cos(\omega_0 n) \xleftrightarrow{\mathcal{F}} \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] \tag{21}$$

$$\sin(\omega_0 n) \xleftrightarrow{\mathcal{F}} -j\pi\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right] \tag{22}$$

Table 1 summarizes some useful DTFT pairs. Do not forget that all DTFTs are 2π periodic, thus we can also replace each ω with $\omega + 2\pi m, \ m \in \mathbb{Z}$. Note that we use the following new signals in this table:

$$\text{sinc}(Ln) = \frac{\sin(Ln)}{Ln} \tag{23}$$

$$\text{rect}\left(\frac{n-k}{L}\right) = \begin{cases} 1, & k - \frac{L}{2} < n \le k + \frac{L}{2} \\ 0, & \text{otherwise} \end{cases} \tag{24}$$

$$\text{triangle}\left(\frac{n}{L}\right) = \Delta\left(\frac{n}{L}\right) = \begin{cases} 1 - \left|\frac{n}{L}\right|, & |n| \le L \\ 0, & \text{otherwise} \end{cases}. \tag{25}$$

Table 1: Common DTFT pairs.

Signal $x[n]$	DTFT $X(\omega)$
$\delta[n]$	1
$u[n]$	$\frac{1}{1-e^{-j\omega}} + \pi\delta(\omega)$
$a^n u[n]$	$\frac{1}{1-ae^{-j\omega}},\ 0 < \lvert a \rvert < 1$
$e^{j\omega_0 n}$	$2\pi\delta(\omega - \omega_0)$
$\cos(\omega_0 n)$	$\pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$
$\sin(\omega_0 n)$	$-j\pi\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$
$\text{rect}\left(\frac{n-k}{L}\right)$	$\frac{\sin\left(\frac{L\omega}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} e^{-j\omega k}$
$\text{sinc}(Ln)$	$\frac{\pi}{L}\text{rect}\left(\frac{\omega}{2L}\right)$
$\text{sinc}^2(Ln)$	$\frac{\pi}{L}\Delta\left(\frac{\omega}{2L}\right)$

3 Properties of the DTFT

There are many important properties of the DTFT. Some of these properties may be directly inherited from our z-transform properties by substituting $z = e^{j\omega}$. In this section, we will introduce and prove some of the unique properties to the DTFT, while Table 2 provides a summary of DTFT properties.

Hermitian symmetry. We may decompose any discrete-time signal $x[n]$ as follows into its real part and imaginary part:

$$x[n] = \text{Re}\{x[n]\} + j\text{Im}\{x[n]\}. \tag{26}$$

The resulting DTFT $X(\omega)$ would then be

$$X(\omega) = \sum_{n=\infty}^{\infty} \left(\text{Re}\{x[n]\} + j\text{Im}\{x[n]\}\right) e^{-j\omega n}. \tag{27}$$

Using Euler's identity of $e^{-j\omega n} = \cos(\omega n) - j\sin(\omega n)$, we can separate the DTFT into its real and imaginary parts:

$$\text{Re}\{X(\omega)\} = \sum_{n=-\infty}^{\infty} \text{Re}\{x[n]\}\cos(\omega n) + \text{Im}\{x[n]\}\sin(\omega n) \tag{28}$$

$$\text{Im}\{X(\omega)\} = \sum_{n=-\infty}^{\infty} \text{Im}\{x[n]\}\cos(\omega n) - \text{Re}\{x[n]\}\sin(\omega n). \tag{29}$$

Note that we are utilizing the even and odd symmetry of cosine and sine, respectively, when invoking Euler's identity above. If our signal is real-valued, i.e. $\text{Im}\{x[n]\} = 0$, we will have the real part of the DTFT composed of even-symmetric cosines and the imaginary part composed of odd-symmetric sines. Thus,

$$X(\omega) = \sum_{n=-\infty}^{\infty} \text{Re}\{x[n]\}\cos(\omega n) - j\text{Re}\{x[n]\}\sin(\omega n). \tag{30}$$

Equation 30 allows us to then state the overall symmetries of the DTFT for real-valued signals:

$$X^*(\omega) = X(-\omega) \tag{31}$$

$$|X(\omega)| = |X(-\omega)| \tag{32}$$

$$\angle X(\omega) = -\angle X(-\omega). \tag{33}$$

The above property is referred to as *Hermitian symmetry*. We see that if $x[n]$ is real-valued, the magnitude spectrum will be even-symmetric and the phase spectrum will be odd-symmetric. Because we almost always work with real-valued signals, Hermitian symmetry is one of the most important properties of the DTFT.

We most commonly represent the DTFT spectrum over the range of $\omega \in [-\pi, \pi]$ since the symmetry of the spectrum will make plotting and visualizing easier. Thus, we often will perform our DTFT math over this interval as well.

2π **periodicity.** We have stated this property many times already, but it bears repeating since it is so essential to understanding the DTFT!

$$X(\omega) = X(\omega + 2\pi k), \ k \in \mathbb{Z}. \tag{34}$$

A short proof of 2π periodicity is as follows:

$$\begin{aligned} X(\omega + 2\pi k) &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega + 2\pi k)n} && (35) \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \underbrace{e^{-j2\pi kn}}_{=1 \ \forall k,n} && (36) \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} && (37) \\ &= X(\omega). && (38) \end{aligned}$$

Frequency shifting and modulation. We define the *frequency shifting* property as

$$e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_0). \tag{39}$$

This can be shown as follows:

$$\begin{aligned} \mathcal{F}\left\{e^{j\omega_0 n} x[n]\right\} &= \sum_{n=-\infty}^{\infty} x[n] e^{j\omega_0 n} e^{-j\omega n} && (40) \\ &= \sum_{n=-\infty}^{\infty} x[n] e^{-j(\omega - \omega_0)n} && (41) \\ &= X(\omega - \omega_0). && (42) \end{aligned}$$

We can then extend the frequency shifting property using Euler's identity to define the *modulation property*:

$$x[n]\cos(\omega_0 n) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2} X(\omega - \omega_0) + \frac{1}{2} X(\omega + \omega_0). \tag{43}$$

The modulation property is quite important, especially in the fields of digital and wireless communications.

Parseval's relation.

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{2\pi} |X(\omega)|^2 d\omega \tag{44}$$

Convolution. We may directly obtain our convolution property of the DTFT by substituting $z = e^{j\omega}$ into the corresponding property for the z-transform:

$$x[n] * h[n] \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) H(\omega). \tag{45}$$

The result tells us the same relation as before with the z-transform. Convolution in the time-domain is multiplication in our transform domain. This will be helpful soon when we discuss the frequency response and system response of LTI systems using the DTFT.

Table 2: Summary of properties of the DTFT.

Property	Signal	DTFT
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(\omega) + bX_2(\omega)$
Time shifting	$x[n-k]$	$X(\omega)e^{-jk\omega}$
Frequency shifting	$e^{j\omega_0 n}x[n]$	$X(\omega - \omega_0)$
Modulation	$x[n]\cos(\omega_0 n)$	$\frac{1}{2}X(\omega-\omega_0) + \frac{1}{2}X(\omega+\omega_0)$
Time reversal	$x[-n]$	$X(-\omega)$
Conjugation	$x^*[n]$	$X^*(-\omega)$
Differentiation	$nx[n]$	$-j\frac{dX(\omega)}{d\omega}$
Convolution	$x[n] * h[n]$	$X(\omega)H(\omega)$
Windowing	$x[n]w[n]$	$\frac{1}{2\pi}X(\omega) * W(\omega)$
Hermitian symmetry	$x[n]$ real	$X^*(\omega) = X(-\omega)$
Parseval's relation	$\sum_{n=-\infty}^{\infty} \|x[n]\|^2$	$\frac{1}{2\pi}\int_{2\pi} \|X(\omega)\|^2 d\omega$

Multiplication in time. This property states that

$$x[n]w[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}X(\omega) * W(\omega). \tag{46}$$

We see a nice symmetry here with the convolution property. Multiplication in the time-domain corresponds to convolution in the frequency domain. Let's prove this:

$$\begin{aligned}
\mathcal{F}\{x[n]w[n]\} &= \sum_{n=-\infty}^{\infty} x[n]w[n]e^{-j\omega n} &(47)\\
&= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2\pi}\int_{2\pi} X(\phi)e^{j\phi n}d\phi\right) w[n]e^{-j\omega n} &(48)\\
&= \frac{1}{2\pi}\int_{2\pi} X(\phi)\left(\sum_{n=-\infty}^{\infty} w[n]e^{j(\phi-\omega)n}\right) d\phi &(49)\\
&= \frac{1}{2\pi}\int_{2\pi} X(\phi)W(\omega-\phi)d\phi &(50)\\
&= \frac{1}{2\pi}X(\omega) * W(\omega). &(51)
\end{aligned}$$

Note that this convolution in the frequency domain is continuous convolution rather than the discrete convolution we have used thus far in the course. We use the notation of $w[n]$ for the second signal since this property is commonly called the *windowing property* as well. We will work with the windowing property and windowing functions more when we discuss the Discrete Fourier Transform and filter design in future lectures. *Please remember that the operation of windowing or multiplication in time does not describe an LTI system! We know this because the response of such a system is not given by a convolution in time, but rather multiplication.*