ECE 310 Fall 2023

Lecture 30

Window method for FIR filter design

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Learning Objectives

After this lecture, you should be able to:

- Apply the window method of FIR filter design and understand the tradeoffs of different window choices for low-pass filter design.
- Create high-pass, band-pass, and band-stop filters by modifying an appropriately designed low-pass filter.

Recap from previous lecture

We began our discussion of FIR filters in the previous lecture and motivated the usefulness of linear phase FIR filters. We carefully defined the four types of linear phase FIR filters and made the distinction between linear phase and generalized linear phase. With this knowledge in hand, we will turn our attention to one popular and effective method of FIR filter design known as the window method. We will develop this technique to first design low-pass filters then use simple time-domain operations to create other canonical filter structures like high-pass, band-pass, and band-stop filters.

1 Window method

One of the most popular techniques for FIR filter design is known as the *window method*. We have actually already demonstrated the window method in action when we shifted an ideal low-pass filter to obtain a causal, symmetric impulse response with $\hat{h}_{\text{lpf}}[n]$ in the previous lecture. As a reminder, this impulse response was given by

$$\hat{h}_{\text{lpf}}[n] = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)}, \tag{1}$$

where $\alpha = \frac{N-1}{2}$. In general, the window method may be summarized in the following steps:

1. Derive the ideal (infinite-length) impulse response $d[n]$ corresponding to the desired ideal frequency response $D(\omega)$, i.e. take the inverse DTFT of $D(\omega)$.
2. Shift the ideal impulse response $d[n]$ by $\alpha = (N-1)/2$ to obtain $g[n] = d[n-\alpha]$.
3. Apply a window function $w[n]$ to the shifted impulse response $g[n]$ to obtain the final impulse response $h[n] = g[n]w[n]$.

This procedure may be used to approximate any ideal frequency response first represented by $D(\omega)$. In Eqn. 1 we implicitly followed the window method by using a rectangular window function since we use the first N causal samples of the shifted impulse response. As we discussed previously when learning about spectral analysis, the truncation of infinite-length sequences to finite-length sequences causes undesirable frequencies to appear in the DTFT and DFT of the windowed signal. We introduced the use of window functions to provide a tradeoff between main lobe width and side lobe attenuation for spectral analysis. We may apply the same intuition now for the window method of FIR filter design.

The choice of window function will affect the transition bandwidth and stopband attenuation of the resulting FIR filter. Recall that we compared the following two windows when performing spectral analysis: the rectangular window and the Hamming window.

Rectangular window.

$$w[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

Hamming window.

$$w[n] = \begin{cases} 0.54 - 0.46\cos\left(\frac{2\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}. \tag{3}$$

Other windows we may define include the Bartlett (or triangular), Hann, and Blackman windows.

Bartlett (triangular) window.

$$w[n] = \begin{cases} 2n/(N-1), & 0 \le n \le (N-1)/2 \\ 2 - 2n/(N-1), & (N-1)/2 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

Hann window.

$$w[n] = \begin{cases} 0.5 - 0.5\cos\left(\frac{2\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

Blackman window.

$$w[n] = \begin{cases} 0.42 - 0.5\cos\left(\frac{2\pi n}{N-1}\right) + 0.08\cos\left(\frac{4\pi n}{N-1}\right), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

Figure 1 shows each of these windows being used in the window method for a low-pass filter with $\omega_c = \frac{\pi}{3}$ and length $N = 25$. We see how the rectangular window has the narrowest transition band with the weakest stopband attenuation while the Hamming window gives the widest transition band and best stopband attenuation. Meanwhile, the Bartlett and Hann windows lie somewhere in between this design trade-off of narrower transition bandwidth vs. stronger stopband attenuation. In general, the choice of window functions trades off transition bandwidth and stopband attenuation. Improving transition bandwidth, i.e. achieving a narrower transition band, will degrade the stopband attenuation. Conversely, a window function with stronger stopband attenuation will widen the transition bandwidth.

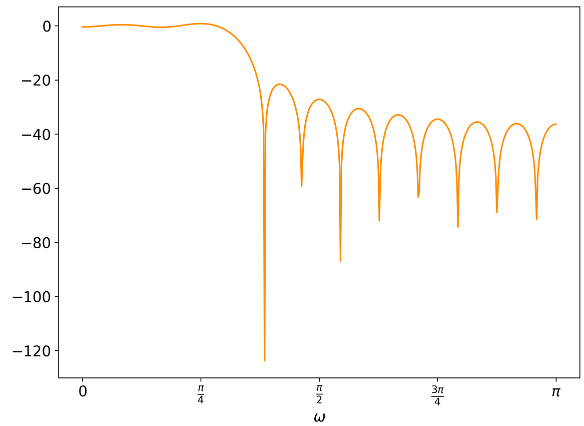
2 Window method examples

We now turn our attention to examples of applying the window method for FIR filter design. We will look at the four canonical filter types: low-pass, high-pass, band-pass, and band-stop filters, and show some shortcuts to move between the filter types.

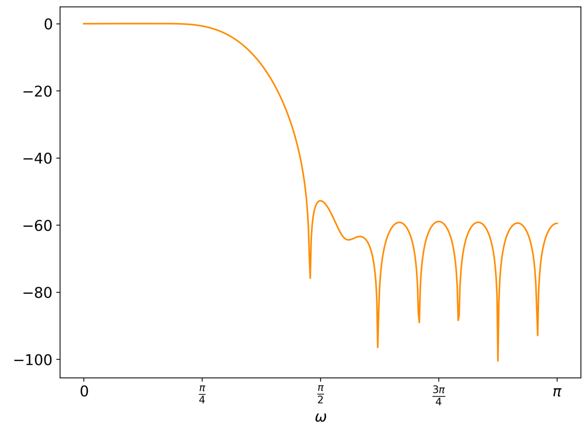
2.1 Low-pass filters

We have already covered low-pass filter design with the window method, but we can step through the math again as a reminder when looking at the other filter types. We start by noting the ideal frequency response $D_{\text{lpf}}(\omega)$ with cutoff frequency ω_c.

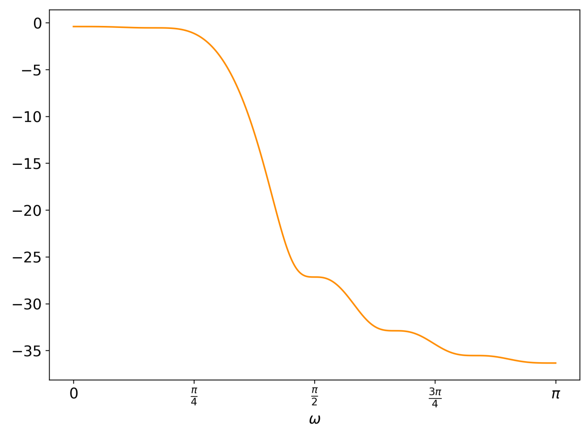
$$D_{\text{lpf}}(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}. \tag{7}$$



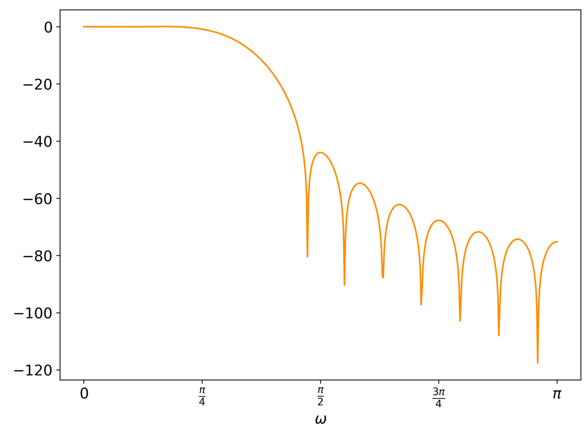
(a) Rectangular window



(b) Hamming window



(c) Bartlett window



(d) Hann window

Figure 1: Choice of different window functions for designing FIR filter via the window method. The given filter is a low-pass filter with length $N = 25$ and $\omega_c = \frac{\pi}{3}$.

We perform the inverse DTFT to obtain the ideal impulse response $d[n]$:

$$
\begin{aligned}
d_{\text{lpf}}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_{\text{lpf}}(\omega)e^{j\omega n}d\omega && (8)\\
&= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega && (9)\\
&= \frac{1}{2\pi}\left(\frac{e^{j\omega n}}{jn}\right)\Bigg|_{-\omega_c}^{\omega_c} && (10)\\
&= \frac{1}{\pi n}\left(\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right) && (11)\\
&= \frac{\sin(\omega_c n)}{\pi n}. && (12)
\end{aligned}
$$

This confirms our earlier result stated in the previous lecture. Next, we shift by $\alpha = \frac{N-1}{2}$ then apply the window function. Thus, the final low-pass filter is given by

$$
h_{\text{lpf}}[n] = \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)}w[n] \tag{13}
$$

for any choice of window function $w[n]$.

2.2 High-pass filters

The ideal frequency response $D_{\text{hpf}}(\omega)$ with cutoff frequency ω_c for a high-pass filter is given by:

$$D_{\text{hpf}}(\omega) = \begin{cases} 0, & |\omega| \leq \omega_c \\ 1, & |\omega| > \omega_c \end{cases}. \tag{14}$$

Taking the inverse DTFT,

$$\begin{aligned} d_{\text{hpf}}[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} D_{\text{hpf}}(\omega)e^{j\omega n}d\omega &(15)\\ &= \frac{1}{2\pi}\left[\int_{-\pi}^{-\omega_c} e^{j\omega n}d\omega + \int_{\omega_c}^{\pi} e^{j\omega n}d\omega\right] &(16)\\ &= \frac{1}{2\pi}\left[\left(\frac{e^{j\omega n}}{jn}\right)\Big|_{-\pi}^{-\omega_c} + \left(\frac{e^{j\omega n}}{jn}\right)\Big|_{\omega_c}^{\pi}\right] &(17)\\ &= \frac{1}{2\pi}\left[\frac{e^{j\pi n} - e^{-j\pi n} - (e^{j\omega_c n} - e^{-j\omega_c n})}{jn}\right] &(18)\\ &= \frac{1}{2\pi}\left[\frac{2j\sin(\pi n) - 2j\sin(\omega_c n)}{jn}\right] &(19)\\ &= \frac{\sin(\pi n)}{\pi n} - \frac{\sin(\omega_c n)}{\pi n} &(20)\\ &= \delta[n] - \frac{\sin(\omega_c n)}{\pi n}. &(21) \end{aligned}$$

Note that the $\frac{\sin(\pi n)}{\pi n}$ term is zero for all n except n where it will evaluate to one, i.e. by following L'Hopital's rule. Shifting by α and applying the window function, we obtain

$$h_{\text{hpf}}[n] = \left(\delta[n-\alpha] - \frac{\sin(\omega_c(n-\alpha))}{\pi(n-\alpha)}\right)w[n]. \tag{22}$$

The result in Eqn. 22 is a properly design high-pass filter via the window method, but there is an even quicker way we could have designed it by utilizing DTFT properties. Another way to write $D_{\text{hpf}}(\omega)$ would be

$$D_{\text{hpf}}(\omega) = 1 - D_{\text{lpf}}(\omega). \tag{23}$$

In other words, the high-pass filter is just the complement of the low-pass filter. Using the linearity of the DTFT and the DTFT pair of 1, we can quickly find that

$$d_{\text{hpf}}[n] = \delta[n] - d_{\text{lpf}}[n]. \tag{24}$$

Note that Eqns. 21 and 24 give the same impulse response! Thus, we do not need to necessarily design a high-pass filter separately. We can instead just modify a low-pass filter accordingly.

2.3 Band-pass filters

Next, we would like to design band-pass filters. Instead of performing an inverse DTFT integral, we ask whether we can find a similar trick to convert a low-pass filter into a low-pass filter into band-pass filter similar to our trick for high-pass filters? The answer is yes!

An arbitrary band-pass filter has the following ideal frequency response:

$$D_{\text{bpf}}(\omega) = \begin{cases} 1, & -\omega_u \leq \omega \leq -\omega_l \text{ or } \omega_l \leq \omega \leq \omega_u \\ 0, & \text{otherwise} \end{cases}, \tag{25}$$

where ω_l and ω_u denote the lower and upper frequencies of the passband between $\omega \in [0, \pi]$. We can express $D_{\text{bpf}}(\omega)$ in terms of $D_{\text{lpf}}(\omega)$ as

$$D_{\text{bpf}}(\omega) = D_{\text{lpf}}(\omega - \bar{\omega}) + D_{\text{lpf}}(\omega - \bar{\omega}) \tag{26}$$

where $\bar{\omega} = \frac{\omega_l + \omega_u}{2}$ is the average between the lower and upper passband frequencies. Note that the cutoff frequency of the low-pass filter must be $\omega_c = \frac{\omega_u - \omega_l}{2}$ to set the appropriate width of the passband. To obtain Eqn. 26, first recall the modulation property of the DTFT:

$$x[n]\cos(\omega_0 n) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}X(\omega - \omega_0) + \frac{1}{2}X(\omega - \omega_0). \tag{27}$$

We can then write the ideal band-pass filter response as

$$\begin{aligned} d_{\text{bpf}}[n] &= 2\cos(\bar{\omega}n) d_{\text{lpf}}[n] &(28)\\ &= 2\cos\left(\frac{\omega_l + \omega_u}{2} n\right) \frac{\sin\left(\frac{\omega_u - \omega_l}{2} n\right)}{\pi n}. &(29) \end{aligned}$$

In summary, we may design band-pass filters by simply modulating an appropriately designed low-pass filter by a mixing frequency $\bar{w}$ according to the center of the passband. The above ideal impulse response can then be shifted by α and have a window function applied.

2.4 Band-stop filters

Finally, we may combine the previous two shortcuts to efficiently design band-stop filters from a low-pass filter. First, observe that the frequency response of a band-stop filter is given by

$$D_{\text{bsf}}(\omega) = \begin{cases} 0, & -\omega_u \leq \omega \leq -\omega_l \text{ or } \omega_l \leq \omega \leq \omega_u \\ 1, & \text{otherwise} \end{cases}. \tag{30}$$

Thus, for a band-pass filter with the same choices of ω_l and ω_l,

$$D_{\text{bsf}}(\omega) = 1 - D_{\text{bpf}}(\omega). \tag{31}$$

Following the shortcuts of the high-pass filter design and band-pass filter design, we find the ideal impulse response as

$$\begin{aligned} d_{\text{bsf}}[n] &= \delta[n] - d_{\text{bpf}}[n] &(32)\\ &= \delta[n] - 2\cos\left(\frac{\omega_l + \omega_u}{2} n\right) \frac{\sin\left(\frac{\omega_u - \omega_l}{2} n\right)}{\pi n}. &(33) \end{aligned}$$

From here, we may shift and window as per the latter two steps of the window method.