

Critical Exponents of the Ising Model

In the vicinity of the critical temperature for $T < T_c$, the spontaneous magnetization scales as

$$M_S(T) \propto (T_c - T)^\beta. \quad (16)$$

For $T = T_c$ and $H \rightarrow 0$, we find the following scaling

$$M(T = T_c, H) \propto H^{1/\delta}. \quad (17)$$

The exponents β and δ are so-called *critical exponents* and characterize together with other exponents the underlying phase transition.

- 2D: $\beta = 1/8$ and $\delta = 15$
- 3D: $\beta = 0.326$ and $\delta = 4.790$

Similarly to the power-law scaling of the spontaneous magnetization defined in Eq. (16), we find for the magnetic susceptibility in the vicinity of T_c

$$\chi(T) \propto |T_c - T|^{-\gamma} \quad (23)$$

$$C(T) \propto |T_c - T|^{-\alpha}, \quad (24)$$

- 2D: $\gamma = 7/4$ and $\alpha = 0^1$
- 3D: $\gamma \approx 1.24$ and $\alpha \approx 0.11$

¹An exponent of $\alpha = 0$ corresponds to a logarithmic decay since $\lim_{s \rightarrow 0} \frac{|x|^{-s}-1}{s} = -\ln|x|$.

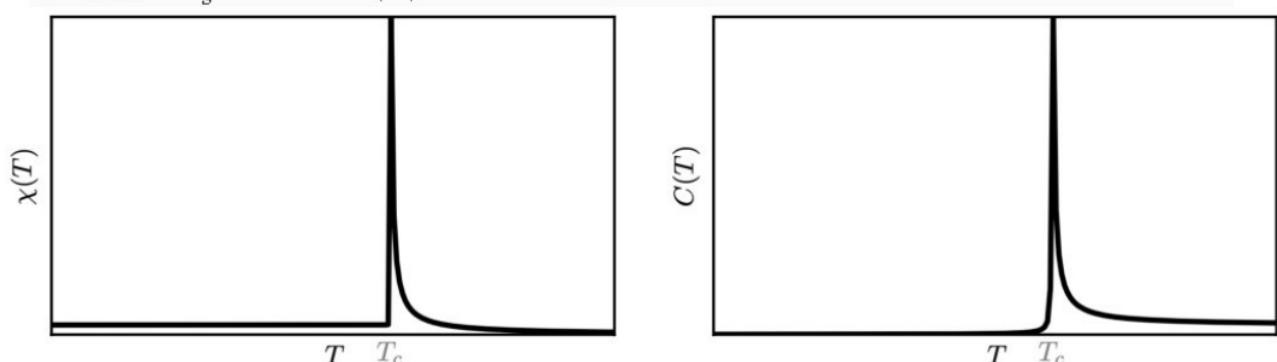


Figure 5: Susceptibility and specific heat as a function of temperature for the three dimensional Ising model. Both quantities diverge at the critical temperature T_c in the thermodynamic limit. [Böttcher,

Critical exponents and universality

The aforementioned six critical exponents are connected by four scaling laws

$$\alpha + 2\beta + \gamma = 2 \quad (\text{Rushbrooke}), \quad (29)$$

$$\gamma = \beta(\delta - 1) \quad (\text{Widom}), \quad (30)$$

$$\gamma = (2 - \eta)\nu \quad (\text{Fisher}), \quad (31)$$

$$2 - \alpha = d\nu \quad (\text{Josephson}), \quad (32)$$

which have been derived in the context of the phenomenological scaling theory for ferromagnetic systems. Due to these relations, the number of independent exponents reduces to two.

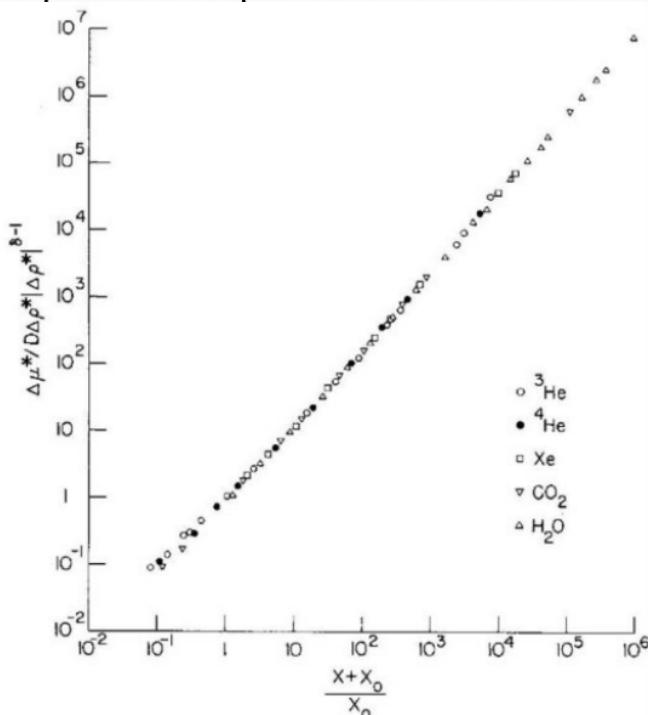


Figure 6: Universal scaling for five different gases. The scaling variable is defined as $x = \Delta T |\Delta \rho|^{-1/\beta}$ and x_0 depends on the amplitude B of the power-law for the coexistence curve $\Delta \rho = B \Delta T^\beta$

Table 1: The critical exponents of the Ising model in two and three dimensions [Pelissetto, Vicari, Phys. Rep. 368, 549–727 (2002)]

Exponent	$d = 2$	$d = 3$
α	0	0.110(1)
β	1/8	0.3265(3)
γ	7/4	1.2372(5)
δ	15	4.789(2)
η	1/4	0.0364(5)
ν	1	0.6301(4)

In terms of a Markov chain, the transition probability from one state to another is given by the probability of a new state to be proposed (T) and the probability of this state to be accepted (A). Namely, $T(X \rightarrow Y)$ is the probability that a new configuration Y is proposed, starting from configuration X . The transition probability fulfills three conditions:

1. *Ergodicity*: any configuration in the phase space must be reachable within a finite number of steps,
2. *Normalization*: $\sum_Y T(X \rightarrow Y) = 1$,
3. *Reversibility*: $T(X \rightarrow Y) = T(Y \rightarrow X)$.

Once a configuration is proposed, we can accept the new configuration with probability $A(X \rightarrow Y)$ or reject it with probability $1 - A(X \rightarrow Y)$. The *probability of the Markov chain* is then given by

$$W(X \rightarrow Y) = T(X \rightarrow Y) \cdot A(X \rightarrow Y). \quad (1)$$

We denote the probability to find the system in a certain configuration X at virtual time τ by $p(X, \tau)$. The *master equation* describes the time evolution of $p(X, \tau)$ and is given by

$$\frac{dp(X, \tau)}{d\tau} = \sum_Y p(Y)W(Y \rightarrow X) - \sum_Y p(X)W(X \rightarrow Y). \quad (2)$$

A stationary state p_{st} is reached if $\frac{dp(X, \tau)}{d\tau} = 0$. The probability of the Markov chain fulfills the following properties:

1. *Ergodicity*: any configuration must be reachable: $\forall X, Y : W(X \rightarrow Y) \geq 0$,
2. *Normalization*: $\sum_Y W(X \rightarrow Y) = 1$,
3. *Homogeneity*: $\sum_Y p_{\text{st}}(Y)W(Y \rightarrow X) = p_{\text{st}}(X)$.

Markov Chains stationary states

A stationary state p_{st} is reached if $\frac{dp(X,\tau)}{d\tau} = 0$. The probability of the Markov chain fulfills the following properties:

1. *Ergodicity*: any configuration must be reachable: $\forall X, Y : W(X \rightarrow Y) \geq 0$,
2. *Normalization*: $\sum_Y W(X \rightarrow Y) = 1$,
3. *Homogeneity*: $\sum_Y p_{\text{st}}(Y)W(Y \rightarrow X) = p_{\text{st}}(X)$.

It then follows from the stationary state condition ($\frac{dp(X,\tau)}{d\tau} = 0$) that

$$\sum_Y p_{\text{eq}}(Y)W(Y \rightarrow X) = \sum_Y p_{\text{eq}}(X)W(X \rightarrow Y).$$

A sufficient condition for this to be true is

$$p_{\text{eq}}(Y)W(Y \rightarrow X) = p_{\text{eq}}(X)W(X \rightarrow Y), \quad (4)$$

which is referred to as a *detailed balance condition*.

As an example, in a canonical ensemble at fixed Temperature T , the equilibrium distribution is given by the Boltzmann factor

$$p_{\text{eq}}(X) = \frac{1}{Z_T} \exp \left[-\frac{E(X)}{k_B T} \right] \quad (5)$$

with the partition function $Z_T = \sum_X \exp \left[-\frac{E(X)}{k_B T} \right]$.

$M(RT)^2$ algorithm

One possible choice of the acceptance probability fulfilling the detailed balance condition is given by

$$A(X \rightarrow Y) = \min \left[1, \frac{p_{\text{eq}}(Y)}{p_{\text{eq}}(X)} \right]. \quad (6)$$

which can be obtained by rewriting Eq. (4).

In the case of the canonical ensemble with

$p_{\text{eq}}(X) = \frac{1}{Z_T} \exp \left[-\frac{E(X)}{k_B T} \right]$, the acceptance probability becomes

$$A(X \rightarrow Y) = \min \left[1, \exp \left(-\frac{\Delta E}{k_B T} \right) \right], \quad (7)$$

where $\Delta E = E(Y) - E(X)$. The last equation implies that the step is always accepted if the energy decreases, and if the energy increases, it is accepted with probability $\exp \left(-\frac{\Delta E}{k_B T} \right)$.

In summary, the steps of the $M(RT)^2$ algorithm applied to the Ising model are

M(RT)² algorithm

- Randomly choose a lattice site i ,
- Compute $\Delta E = E(Y) - E(X) = 2J\sigma_i h_i$,
- Flip the spin if $\Delta E \leq 0$, otherwise accept it with probability $\exp \left(-\frac{\Delta E}{k_B T} \right)$,

with $h_i = \sum_{\langle i,j \rangle} \sigma_j$ and $E = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j$.

Glauber dynamics general

The Metropolis algorithm is not the only possible choice to fulfill the detailed balance condition. Another acceptance probability given by

$$A_G(X \rightarrow Y) = \frac{\exp\left(-\frac{\Delta E}{k_B T}\right)}{1 + \exp\left(-\frac{\Delta E}{k_B T}\right)} \quad (8)$$

has been suggested by Glauber in 1963.

In contrast to the M(RT)² acceptance probability, updates with $\Delta E = 0$ are not always accepted but with probability 1/2.

To prove that Eq. (8) satisfies the condition of detailed balance, we have to show that

$$p_{\text{eq}}(Y)A_G(Y \rightarrow X) = p_{\text{eq}}(X)A_G(X \rightarrow Y) \quad (9)$$

since $T(Y \rightarrow X) = T(X \rightarrow Y)$.

The previous equation is equivalent to

$$\frac{p_{\text{eq}}(Y)}{p_{\text{eq}}(X)} = \frac{A_G(X \rightarrow Y)}{A_G(Y \rightarrow X)} \quad (10)$$

which is fulfilled since

$$\frac{p_{\text{eq}}(Y)}{p_{\text{eq}}(X)} = \exp\left(-\frac{\Delta E}{k_B T}\right) \quad (11)$$

and

$$\frac{A_G(X \rightarrow Y)}{A_G(Y \rightarrow X)} = \frac{\exp\left(-\frac{\Delta E}{k_B T}\right)}{1 + \exp\left(-\frac{\Delta E}{k_B T}\right)} \left[\frac{\exp\left(\frac{\Delta E}{k_B T}\right)}{1 + \exp\left(\frac{\Delta E}{k_B T}\right)} \right]^{-1} = \exp\left(-\frac{\Delta E}{k_B T}\right) \quad (12)$$

Glauber dynamics for the Ising model

As in the M(RT)² algorithm, only the local configuration around the lattice site is relevant for the update procedure.

Furthermore, with $J = 1$, the probability to flip spin σ_i is

$$A_G(X \rightarrow Y) = \frac{\exp\left(\frac{-2\sigma_i h_i}{k_B T}\right)}{1 + \exp\left(\frac{-2\sigma_i h_i}{k_B T}\right)} \quad (13)$$

with $h_i = \sum_{\langle i,j \rangle} \sigma_j$ being the local field and $X = \{\dots, \sigma_{i-1}, \sigma_i, \sigma_{i+1}, \dots\}$ and $Y = \{\dots, \sigma_{i-1}, -\sigma_i, \sigma_{i+1}, \dots\}$ the initial and final configuration, respectively.

We abbreviate the probability defined by Eq. (13) as p_i . The spin flip and no flip probabilities can then be expressed as

$$p_{\text{flip}} = \begin{cases} p_i & \text{for } \sigma_i = -1 \\ 1 - p_i & \text{for } \sigma_i = +1 \end{cases} \quad \text{and} \quad p_{\text{no-flip}} = \begin{cases} 1 - p_i & \text{for } \sigma_i = -1 \\ p_i & \text{for } \sigma_i = +1 \end{cases} \quad (14)$$

A possible implementation is

$$\sigma_i(\tau + 1) = -\sigma_i(\tau) \cdot \text{sign}(p_i - z), \quad (15)$$

with $z \in (0, 1)$ being a uniformly distributed random number, or

$$\sigma_i(\tau+1) = \begin{cases} +1 & \text{with probability } p_i \\ -1 & \text{with probability } 1 - p_i \end{cases} \quad \text{and} \quad p_i = \frac{\exp(2\beta h_i)}{1 + \exp(2\beta h_i)}. \quad (16)$$

This method does not depend on the spin value at time t and is called *heat-bath Monte Carlo*.

Creutz Algorithm

Figure 3: An algorithm to perform microcanonical Monte Carlo simulations, i.e., system at constant energy.

The movement in phase space is in fact not strictly constrained to a subspace of constant energy but there is a certain additional volume in which we can freely move. The condition often constant energy is softened by introducing a so-called *demon* which corresponds to a small reservoir of energy E_D that can store a certain maximum energy E_{\max} .

Creutz algorithm

- Choose a site,
- Compute ΔE for the spin flip,
- Accept the change if $E_{\max} \geq E_D - \Delta E \geq 0$.

Pro: Besides the fact that we can randomly choose a site, this method involves no random numbers and is thus said to be completely deterministic and therefore reversible.

Con: The temperature of the system is not known.

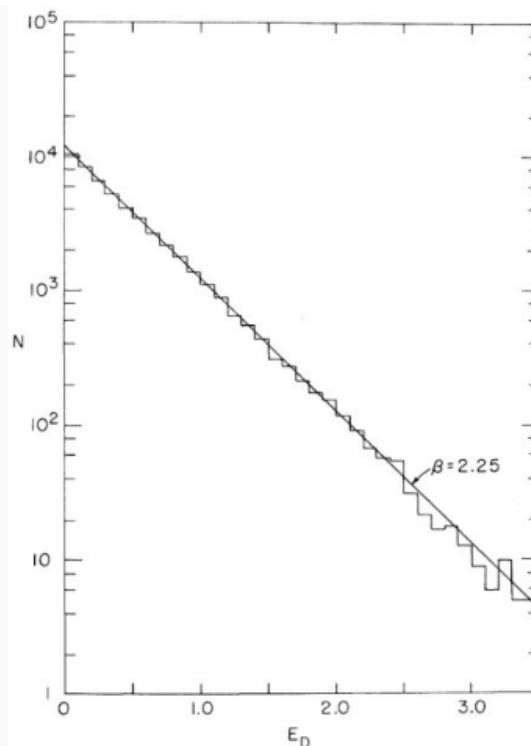


Figure 4: The distribution of the demon energy E_D is exponentially distributed. Based on the Boltzmann factor, it is possible to extract the inverse temperature $\beta = (k_B T)^{-1} = 2.25$. The figure is taken from Ref. shown in Figure 3.

Temporal Correlations

According to the definition of a Markov chain, the dependence of a quantity A on virtual time τ is given by

$$\langle A(\tau) \rangle = \sum_X p(X, \tau) A(X) = \sum_X p(X, \tau_0) A(X(\tau)). \quad (17)$$

In the second step of the latter equation, we used the fact that the average is taken over an ensemble of initial configurations $X(\tau_0)$ which evolve according to Eq. (2).

For some $\tau_0 < \tau$, the *non-linear correlation function*

$$\Phi_A^{\text{nl}}(\tau) = \frac{\langle A(\tau) \rangle - \langle A(\infty) \rangle}{\langle A(\tau_0) \rangle - \langle A(\infty) \rangle} \quad (18)$$

is a measure to quantify the deviation of $A(\tau)$ from $A(\infty)$ relative to the deviation of $A(\tau_0)$ from $A(\infty)$.

The linear correlation function of two values A, B is defined as

$$\Phi_{AB}(\tau) = \frac{\langle A(\tau_0)B(\tau) \rangle - \langle A \rangle \langle B \rangle}{\langle AB \rangle - \langle A \rangle \langle B \rangle} \quad (22)$$

with

$$\langle A(\tau_0)B(\tau) \rangle = \sum_X p(X, \tau_0) A(X(\tau_0)) B(X(\tau)).$$

As τ goes to infinity, $\Phi_{AB}(\tau)$ decreases from unity to zero.

If $A = B$, we call Eq. (22) the *autocorrelation function*. For the spin-spin correlation in the Ising model we obtain

$$\Phi_\sigma(\tau) = \frac{\langle \sigma(\tau_0)\sigma(\tau) \rangle - \langle \sigma(\tau_0) \rangle^2}{\langle \sigma^2(\tau_0) \rangle - \langle \sigma(\tau_0) \rangle^2}$$

Non-Linear Correlation

The *non-linear* correlation time τ_A^{nl} describes the relaxation towards equilibrium and is defined as¹

$$\tau_A^{\text{nl}} = \int_0^\infty \Phi_A^{\text{nl}}(\tau) d\tau. \quad (20)$$

¹If we consider an exponential decay of $\Phi_A^{\text{nl}}(\tau)$, we find that this definition is meaningful since

$$\int_0^\infty \exp\left(-\tau/\tau_A^{\text{nl}}\right) d\tau = \tau_A^{\text{nl}}. \quad (19)$$

For some $\tau_0 < \tau$, the *non-linear correlation function*

$$\Phi_A^{\text{nl}}(\tau) = \frac{\langle A(\tau) \rangle - \langle A(\infty) \rangle}{\langle A(\tau_0) \rangle - \langle A(\infty) \rangle} \quad (18)$$

is a measure to quantify the deviation of $A(\tau)$ from $A(\infty)$ relative to the deviation of $A(\tau_0)$ from $A(\infty)$.

In the vicinity of the critical temperature T_c , we observe the so-called *critical slowing down* of our dynamics, i.e., the non-linear correlation time is described by power law

$$\tau_A^{\text{nl}} \sim |T - T_c|^{-z_A^{\text{nl}}} \quad (21)$$

with z_A^{nl} being the non-linear dynamical critical exponent. This implies that the time needed to reach equilibrium diverges at T_c !

Linear Correlation

The linear correlation function of two values A, B is defined as

$$\Phi_{AB}(\tau) = \frac{\langle A(\tau_0)B(\tau) \rangle - \langle A \rangle \langle B \rangle}{\langle AB \rangle - \langle A \rangle \langle B \rangle} \quad (22)$$

with

$$\langle A(\tau_0)B(\tau) \rangle = \sum_X p(X, \tau_0) A(X(\tau_0)) B(X(\tau)).$$

As τ goes to infinity, $\Phi_{AB}(\tau)$ decreases from unity to zero. The *linear* correlation time τ_A^{nl} describes the relaxation towards equilibrium

$$\tau_{AB} = \int_0^\infty \Phi_{AB}(\tau) d\tau. \quad (23)$$

As in the case of the non-linear correlation time, in the vicinity of T_c , we observe a *critical slowing down*, i.e.,

$$\tau_{AB} \sim |T - T_c|^{-z_A}. \quad (24)$$

with z_A being the *linear* dynamical critical exponent.

The dynamical exponents for spin correlations turn out to be

$$z_\sigma = 2.16 \text{ (2D)},$$

$$z_\sigma = 2.09 \text{ (3D)}.$$

There is a conjectured relation between the Ising critical exponents and the critical dynamical exponents for spin σ and energy correlations E . The relations

$$z_\sigma - z_\sigma^{\text{nl}} = \beta, \quad (25)$$

$$z_E - z_E^{\text{nl}} = 1 - \alpha, \quad (26)$$

$$(27)$$

are numerically well-established, however, not yet analytically proven.

Decorrelated configurations

Connecting this behavior with the one observed for the correlation time described by Eq. (23) yields

$$\tau_{AB} \sim |T - T_c|^{-z_{AB}} \sim L^{\frac{z_{AB}}{\nu}} \quad (28)$$

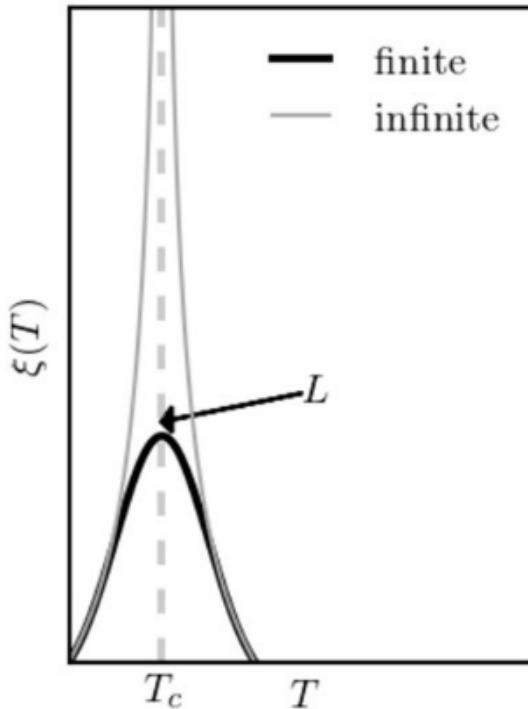


Figure 5: The correlation length diverges in an infinite system at T_c according to the definition of the correlation length from last week's lecture. In a finite system, however, we observe a round off and the correlation length approaches the system size L at T_c .

To ensure not to sample correlated configurations one should

- first reach equilibrium (discard $n_0 = c\tau^{\text{nl}}(T)$ configurations),
- only sample every $n_e^{th} = c\tau(T)$ configuration,
- and at T_c use $n_0 = cL^{\frac{z_{\text{nl}}}{\nu}}$ and $n_e = cL^{\frac{z}{\nu}}$

where $c \approx 3$ is a "safety factor" to make sure to discard enough samples.

Finite size methods

Divergent behavior at T_c as described by

$$\chi(T) \sim |T_c - T|^{-\gamma} \quad (1)$$

$$C(T) \sim |T_c - T|^{-\alpha}, \quad (2)$$

$$\xi(T) \sim |T - T_c|^{-\nu} \quad (3)$$

The larger the system size, the more pronounced is the divergence.

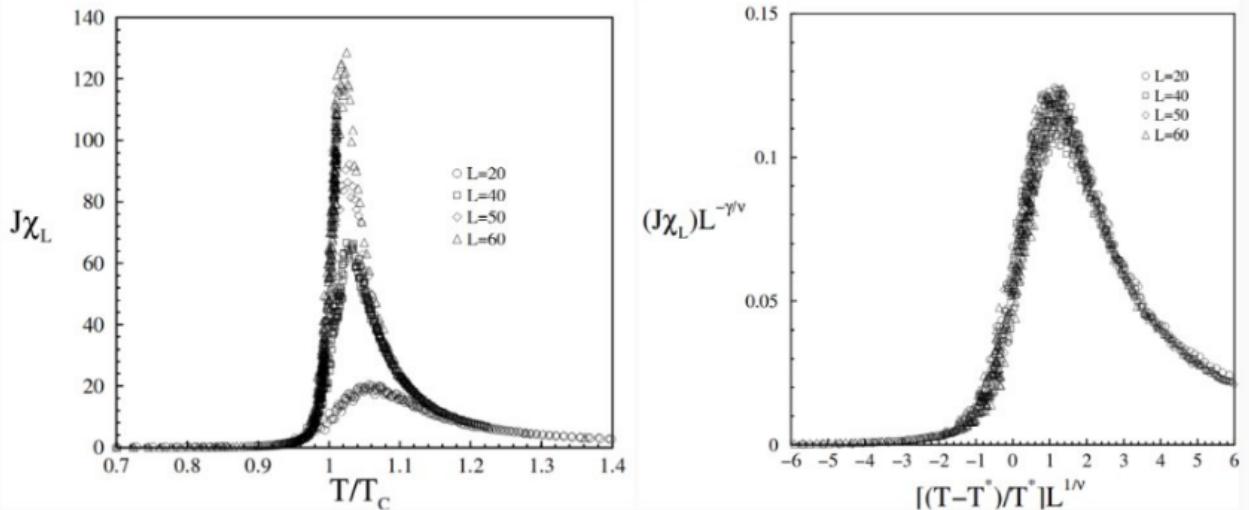


Figure 3: The system size dependence of the susceptibility and the corresponding finite size scaling. The figure is taken from [Da Silva et al., Braz. J. Phys. 32, 2002].

The finite size scaling relation of the susceptibility is given by

$$\chi(T, L) = L^{\frac{\gamma}{\nu}} F_\chi \left[(T - T_c) L^{\frac{1}{\nu}} \right], \quad (4)$$

where F_χ is called susceptibility *scaling function*¹.

¹Based on Eq. (1), we can infer that $F_\chi \left[(T - T_c) L^{\frac{1}{\nu}} \right] \sim (|T - T_c| L^{\frac{1}{\nu}})^{-\gamma}$ as $L \rightarrow \infty$.

In the case of the magnetization, the corresponding finite size scaling relation is

$$M_S(T, L) = L^{-\frac{\beta}{\nu}} F_{M_S} \left[(T - T_c) L^{\frac{1}{\nu}} \right]. \quad (5)$$

Binder Cumulant

We still need a way to determine T_c more precisely. To do that, we make use of the so-called *Binder cumulant*

$$U_L = 1 - \frac{\langle M^4 \rangle_L}{3 \langle M^2 \rangle_L^2}, \quad (6)$$

which is independent of the system size L at T_c since

$$\frac{\langle M^4 \rangle_L}{3 \langle M^2 \rangle_L^2} = \frac{L^{-\frac{4\beta}{\nu}} F_{M^4} \left[(T - T_c) L^{\frac{1}{\nu}} \right]}{\left\{ L^{-\frac{2\beta}{\nu}} F_{M^2} \left[(T - T_c) L^{\frac{1}{\nu}} \right] \right\}^2} = F_C \left[(T - T_c) L^{\frac{1}{\nu}} \right]. \quad (7)$$

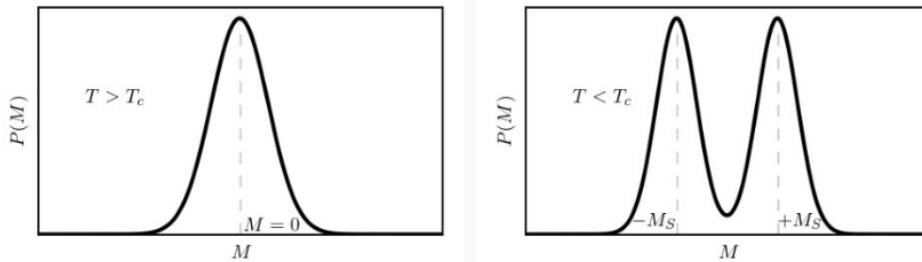


Figure 4: The distribution $P(M)$ of the magnetization M above and below the critical temperature T_c .

Below the critical temperature ($T < T_c$), there exist one ground state with positive and one with negative magnetization and the corresponding distribution is given by

$$P_L(M) = \frac{1}{2} \sqrt{\frac{L^d}{\pi \sigma_L}} \left\{ \exp \left[-\frac{(M - M_S)^2 L^d}{\sigma_L} \right] + \exp \left[-\frac{(M + M_S)^2 L^d}{\sigma_L} \right] \right\} \quad (10)$$

For $T > T_c$, the magnetization is described by a Gaussian distribution

$$P_L(M) = \sqrt{\frac{L^d}{\pi \sigma_L}} \exp \left[-\frac{M^2 L^d}{\sigma_L} \right], \quad (8)$$

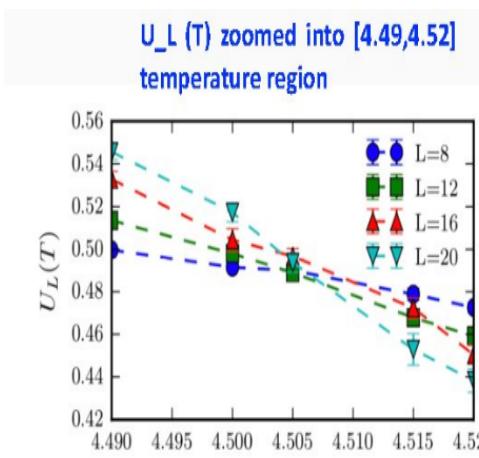
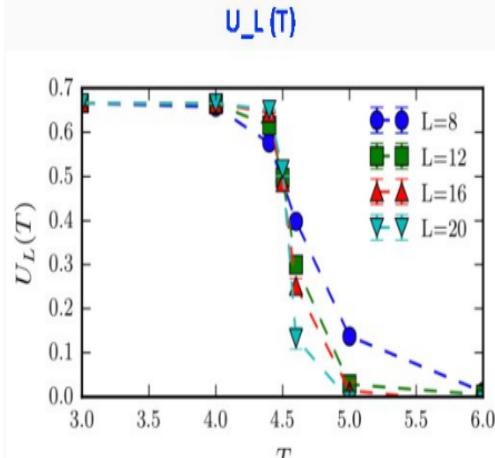
with $\sigma_L = 2k_B T \chi_L$. Since the fourth moment equals three times the second moment squared, i.e.,

$$\langle M^4 \rangle = 3 \langle M^2 \rangle_L^2, \quad (9)$$

it follows that U_L must be zero for $T > T_c$.

For this distribution, it holds that $\langle M^4 \rangle = \langle M^2 \rangle_L^2$ and therefore $U_L = \frac{2}{3}$. In summary, we demonstrated that

$$U_L = \begin{cases} \frac{2}{3} & \text{for } T < T_c \\ \text{const} & \text{for } T = T_c \\ 0 & \text{for } T > T_c \end{cases} \quad (11)$$



First Order Transition

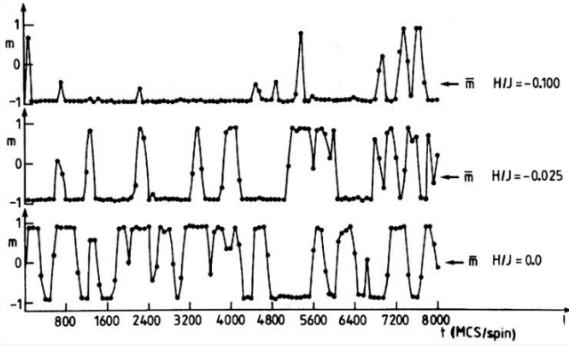


Figure 5: The magnetization exhibits a switching behavior if the field vanishes. For non-zero magnetic fields, the magnetization is driven in the direction of the field. The figure is taken from [Binder and Landau, Phys. Rev. B30 3 (1984)]

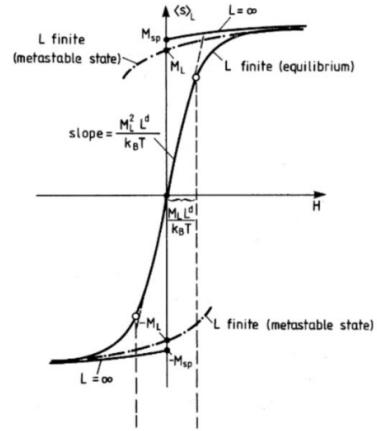


Figure 6: Hysteresis can be found by varying the field from negative to positive values and back. The figure is taken from [Binder and Landau, Phys. Rev. B30 3 (1984)]

Binder showed that the magnetization as a function of the field H is described by $\tanh(\alpha L^d)$ if the distribution of the magnetization is given by Eq. (10). Specifically, we find for the magnetization and susceptibility

$$M(H) = \chi_L^D H + M_L \tanh\left(\beta H M_L L^d\right), \quad (15)$$

$$\chi_L(H) = \frac{\partial M}{\partial H} = \chi_L^D + \frac{\beta M_L L^d}{\cosh^2(\beta H M_L L^d)}. \quad (16)$$

Similarly, to the scaling of a second order transition, we can scale the maximum of the susceptibility ($\chi_L(H=0) \sim L^d$) and the width of the peak ($\Delta\chi_L \sim L^{-d}$). To summarize, a first order phase transition is characterized by

1. A bimodal distribution of the order parameter,
2. stochastic switching between the two states in small systems,
3. hysteresis of the order parameter when changing the field,
4. a scaling of the order parameter, or response function according to Eq. (16).

The Kasteleyn and Fortuin Theorem

We consider the Potts model not on a square lattice but on an arbitrary graph of nodes connected with bonds ν . Each node has q possible states and each connection leads to an energy cost of unity if two connected nodes are in a different state and of zero if they are in the same state, i.e.,

$$E = J \sum_{\nu} \epsilon_{\nu} \quad \text{with} \quad \epsilon_{\nu} = \begin{cases} 0 & \text{if endpoints are in the same state} \\ 1 & \text{otherwise} \end{cases} \quad (18)$$

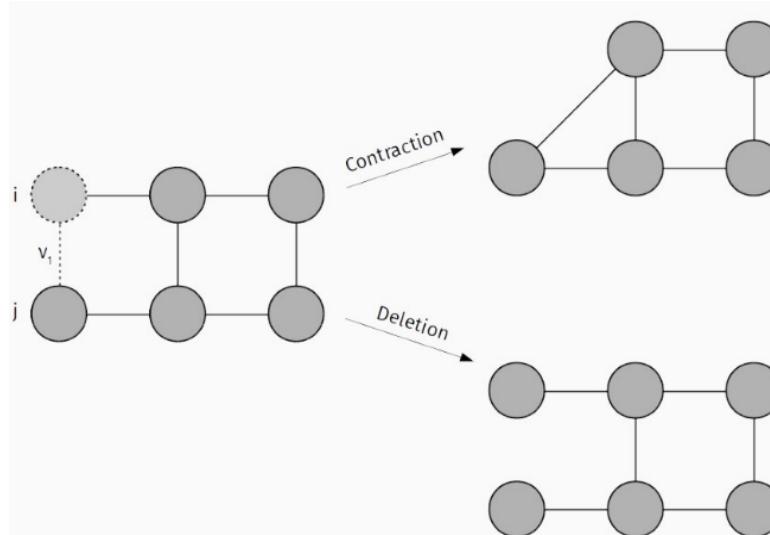


Figure 8: Contraction and deletion on a graph.

The partition function is the sum over all the possible configurations weighted by the Boltzmann factor and thus given by

$$Z = \sum_X e^{-\beta E(X)} \stackrel{(18)}{=} \sum_X e^{-\beta J \sum_{\nu} \epsilon_{\nu}} = \sum_X \prod_{\nu} e^{-\beta J \epsilon_{\nu}}. \quad (19)$$

After applying these operations to every bond, the graph is reduced to a set of separated points corresponding to clusters of nodes which are connected and in the same state out of q states. The partition function reduces to

$$Z = \sum_{\substack{\text{configurations of} \\ \text{bond percolation}}} q^{\# \text{ of clusters}} p^c (1-p)^d = \left\langle q^{\# \text{ of clusters}} \right\rangle_b, \quad (24)$$

where c and d are the numbers of contracted and deleted bonds respectively. In the limit of $q \rightarrow 1$, one obtains the partition function of bond percolation².

²In bond percolation [Broadbent, Hammersley (1957)], an edge of a graph is occupied with probability p and vacant with probability $1 - p$.

Coniglio-Klein clusters

The probability of a given cluster C to be in a certain state σ_0 is independent of the state itself, i.e.,

$$p(C, \sigma_0) = p^{c_C} (1 - p)^{d_C} \sum_{\substack{\text{bond percolation} \\ \text{without cluster } C}} q^{\# \text{ of clusters}} p^c (1 - p)^d. \quad (25)$$

This implies that flipping this particular cluster has no effect on the partition function (and therefore the energy) so that it is possible to accept the flip with probability one. This can be seen by looking at the detailed balance condition of the system

$$p(C, \sigma_1) W [(C, \sigma_1) \rightarrow (C, \sigma_2)] = p(C, \sigma_2) W [(C, \sigma_2) \rightarrow (C, \sigma_1)] \quad (26)$$

and using $p(C, \sigma_1) = p(C, \sigma_2)$.

We then obtain for acceptance probabilities

$$W[(C, \sigma_2) \rightarrow (C, \sigma_1)] = \frac{p(C, \sigma_2)}{p(C, \sigma_1) + p(C, \sigma_2)} = \frac{1}{2} \quad \text{Glauber dyn} \quad (27)$$

$$W[(C, \sigma_2) \rightarrow (C, \sigma_1)] = \min \left[1, \frac{p(C, \sigma_2)}{p(C, \sigma_1)} \right] = 1 \quad \text{Metropolis} \quad (28)$$

Based on these insights, we introduce cluster algorithms which are much faster than single-spin flip algorithms and less prone to the problem of critical slowing down.