Universität St.Gallen

Stochastic Methods in Finance

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Pricing an Asian Call Option

Students:

Luca Boschung, luca.boschung@student.unisg.ch

Ludovico Sbarra, ludovico.sbarra@student.unisg.ch

Odile Zimmerman, odile.zimmermann@student.unisg.ch

Noa Diego Frei, noafrei@student.unisg.ch

Professor:

Prof. Ph.D. Enrico Giovanni De Giorgi

enrico.degiorgi@unisg.ch

Universität St. Gallen

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1 Introduction

This report shall explain the goal and the methods used to solve the assignment, i.e. to price a European Asian Call option using the binomial model. However, before diving into the methodology and the results, an overview shall be done on the type of derivative we are considering in this report.

Ever since a viable method has been developed to price them, financial derivatives have created a trillion-dollar market. The most common types of derivatives are forward/future contracts and options. What gives these assets their name is that their value depends on the state of an underlying asset, typically its price. Here, we differentiate between call and put options: a call gives the right, but not the obligation, to buy the underlying asset at a strike price at or before a pre-agreed date (maturity), while the put gives the right, but not the obligation, to sell the underlying at the strike at or before maturity.

The value of the option depends on where the price of the underlying is with respect to the strike price at a given point in time. For both a call or a put, it is said that they are At The Money (ATM) if the stock price is equal to the strike price. A European or American call is said to be In The Money (ITM) if the price of the underlying is above the strike price, while it is Out of The Money (OTM) if the price of the underlying is below the strike price. The inverse relation holds for puts. It is only in the case where the option is ITM that it has intrinsic value, as exercising it leads to a profit.

In this report, we study the pricing of a type of exotic option: the European Asian option, whose value depends on the average price of the underlying from the moment the option was created up to maturity, when it can be exercised. We do this with a binomial model of price evolution. The report will proceed as follows. First it will focus on the mathematics and implementation of the binomial model, the calculation of the payoffs of the option in different possible states, and its pricing with backwards induction. Then it will show the results with extensive robustness checks, to ensure the price of the option varies reasonably when the relevant parameters change. In addition, a simplified pricing approach will be illustrated that uses the normal approximation of the binomial distribution.

2 Approach and Methodology

The major difficulty in pricing an Asian option is that its payoff depends on the average price of the underlying. Unlike a vanilla European or American call options an Asian option requires keeping track of the asset's price, creating a path-dependence that is more difficult to model. The pricing problem was tackled by building a binomial tree that tracks both the stock's possible price paths and the accumulated sum of those prices. Once the tree was completed, backwards induction was applied to work with the computed payoffs at maturity, to go back to $t = 0$, and calculate the option's fair value today. The following subsections break down the steps to price the options, illustrating the financial reasoning to build the algorithm.

2.1 Defining the Payoff

Let A_T be the value at maturity of the Asian call option that exists over the timespan $[0, T]$ and S be the price of the underlying. It is assumed that we observe the S at n equally spaced discrete moments t over the interval $[0, T]$ such that $t = 0, 1, \dots, n$. Let K be the strike price of the Asian option, then

$$A_T = \max\left(\frac{1}{n+1}\sum_{t=0}^{n} S_t - K,\ 0\right). \tag{1}$$

2.2 Capturing Path Dependence

In the standard binomial model for European options, the value depends only on the final possible prices of the underlying at maturity. For Asian options, however, we must track the entire price path since the payoff depends on the average price across the observation period.

At each node in our binomial tree, we need to maintain not just the current stock price, but also sufficient information to compute the relevant average. For example, at time $t = 2$, we need both the current price S_2 and the running sum of prices observed so far $(S_0 + S_1 + S_2)$.

For our implementation, at each node we track:

- The current time step t
- The number of up-moves in the path
- The current stock price S_t
- The running sum of stock prices $\sum_{i=0}^{t} S_i$

By maintaining this running sum throughout the tree, we can efficiently compute the required average at maturity with a single division operation: $\frac{1}{n+1}\sum_{i=0}^{n} S_i$.

2.3 Building the Tree Step by Step

To build the binomial tree, the following steps have been adopted:

1. Starting point: at time zero, there is only one node, which is the closing Microsoft stock price of April 28, 2025, defined as S_0.
2. Populating the tree: because the framework is based on a binomial tree starting from S_0, the price can either follow an up- or a down-move:

(a) In case of an up-move, the price experiences a total return of

$$u = e^{\sigma\sqrt{\Delta t}},$$

(b) and in case of a down-move, the total return is

$$d = e^{-\sigma\sqrt{\Delta t}}.$$

3. Depth of the tree: this branching is repeated $n = 25$ times, which are the periods – or the "splits" – that define the length of the tree. At this point, each path has its cumulative total, which we convert into the arithmetic average.

This process produces a non-recombining tree because the running total differs significantly between sequences of ups and downs, and the nodes at the same depth do not merge. In coding terms, we defined a node for each point in the tree in which the current time step, the count of up-moves taken so far, the corresponding stock price at that moment, and the cumulative sum of all prices along that path are saved. Because each node contains all the information it needs, the same child-generation logic can be applied recursively: every new node in turn creates its own two children, continuing until we reach the final time step. This means that we built the entire tree recursively: starting from the root node at time zero, a single function calls itself on each new child until it reaches the final step. This approach also makes tree generation surprisingly fast: using this setup, it takes no more than two seconds to build and populate the full tree for moderate step counts, whereas a similarly sized loop-based implementation would require far more indexing overhead and bookkeeping. By reaching the final step, each node already "knows" its average price (by dividing its sum by the number of steps) and can immediately compute its payoff, allowing us to apply backwards induction without any additional bookkeeping.

2.4 Calculating Payoffs at Maturity

Once the final split occurs, each path yields the final stock price and the total sum of prices from all time steps. By dividing the sum by $n + 1$, we get the average price for that path. The payoffs are calculated and recorded for each possible price path according to Equation 1.

2.5 Backwards Induction

After populating the tree, computing the arithmetic averages, and obtaining the payoffs, it is now possible to apply backwards induction to find the price of the Asian option at $t = 0$. To go back from the terminal nodes, the risk-neutral probability measure is computed

$$q = \frac{1 + r - d}{u - d}$$

where r is the risk-free interest rate. This risk-neutral probability serves to weight the Asian payoffs and compute and expected value. Finally, one must discount this expected value for the n periods in which the option exists. Since the model is discrete, it is assumed that the the interest rate compounds n times and not continuously.

2.6 Approximation through Normal Distribution

To approximate the price of the Asian option, particularly when dealing with a large number of time steps (n), we employ a normal approximation to the binomial model. The core idea is that in the binomial model, the total number of up-moves over n steps, denoted by J, follows a binomial distribution:

$$J = (\text{number of up-moves in } n \text{ steps}) \sim \text{Binomial}(n, q), \tag{2}$$

where q is the risk-neutral probability of an up-move. This distribution has a mean and variance given by:

$$E[J] = nq, \qquad \text{Var}[J] = nq(1-q). \tag{3}$$

For a sufficiently large n, the distribution of J can be well-approximated by a normal distribution:

$$J \approx \mathcal{N}(nq, nq(1-q)). \tag{4}$$

A common goal in option pricing is to find a closed-form solution, similar to the Black-Scholes formula. For Asian options on the arithmetic average, this would involve treating the arithmetic average of asset prices, $A = \frac{1}{n+1}\sum_{k=0}^{n} S_k$, as being normally distributed, $A \approx \mathcal{N}(m_A, \sigma_A^2)$, and then applying a formula like

$$E[\max(A-K, 0)] = (m_A - K)\Phi\left(\frac{m_A - K}{\sigma_A}\right) + \sigma_A \phi\left(\frac{m_A - K}{\sigma_A}\right). \tag{5}$$

However, this approach presents significant analytical challenges for arithmetic averages. The individual asset prices S_k are log-normally distributed, but their sum (and thus the arithmetic average A) does not follow a simple, known distribution. Deriving the exact moments m_A and σ_A^2 analytically is complex, and the assumption that A is strictly normal is an approximation in itself. Consequently, a precise and simple closed-form solution is generally not viable for arithmetic Asian options.

Given these complexities, our implemented approach directly utilizes the normal approximation of J to re-evaluate the probabilities associated with the outcomes generated by an n-step binomial tree. Instead of attempting to derive an analytical form for the distribution of A, we perform the following steps:

1. Generate Paths and Outcomes: We first construct an n-step binomial tree to determine all possible paths. For each unique path ending at a terminal node i, we record the total

number of up-moves (J_i) and the corresponding arithmetic average of asset prices along that path (A_i).

2. Approximate Probabilities using Normal PDF of J: For each unique count of up-moves J_i observed at the terminal nodes, we calculate a pseudo-probability density value using the probability density function (PDF) of the normal distribution $\mathcal{N}(nq, nq(1-q))$. Let $\sigma_J = \sqrt{nq(1-q)}$ be the standard deviation of J. The value for each J_i is:

$$\varphi_i = \frac{1}{\sigma_J\sqrt{2\pi}} \exp\left(-\frac{1}{2}\left(\frac{J_i - nq}{\sigma_J}\right)^2\right). \tag{6}$$

   These φ_i values represent the likelihood of observing J_i up-moves under the normal approximation for J.

3. Normalize Probabilities: The raw φ_i values do not necessarily sum to one. Therefore, we normalize them to obtain a valid set of probabilities p_i:

$$p_i = \frac{\varphi_i}{\sum_k \varphi_k}, \tag{7}$$

   such that $\sum_i p_i = 1$. Each p_i now represents the approximated probability of achieving an outcome associated with J_i up-moves.

4. Calculate Expected Payoff: The expected option payoff under this normal approximation is then computed as the weighted average of the individual path payoffs, $\max(A_i - K, 0)$, using these normalized probabilities p_i:

$$\text{Expected Payoff} = \sum_i p_i \cdot \max(A_i - K, 0). \tag{8}$$

   This sum is taken over all terminal states (or unique J_i values, with A_i being the average price corresponding to that J_i if multiple paths lead to the same J_i but different A_i, though in a standard tree A_i is path-dependent and thus unique for each path). In our code, this is implemented by associating each of the 2^n path-specific average prices with the normal-approximated probability derived from its corresponding count of up-moves.

5. Discounting: Finally, the calculated expected payoff is discounted back to the present value using the risk-free interest rate r over n periods:

$$\text{Option Price} = \text{Expected Payoff} \times (1 + r_{\text{period}})^{-n}. \tag{9}$$

This method provides an alternative way to incorporate the normal approximation without needing to derive the full distribution of the arithmetic average A.

3 Results

3.1 Base Case

When running our code, the output looks as follows:

Input Parameters		
Initial Stock Price, S_0	:	391.16
Strike Price, K	:	391.16
Annual Risk-Free Rate, r	:	0.0100 (1.00%)
Volatility, σ	:	0.2703
Time to Maturity, T	:	0.50 years
Number of Steps, n	:	25
Derived Model Parameters		
Time Step, $\Delta t = T/n$	:	0.020000
Interest Rate per Step, $r\Delta t$	:	0.000400
Up Factor, $u = e^{\sigma\sqrt{\Delta t}}$	:	1.038969
Down Factor, $d = 1/u$	:	0.962493
Risk-Neutral Probability, q	:	0.493059
Calculated Results		
Exact Option Price	:	17.5758
Approximate Option Price	:	14.0407

Table 1: Asian Call Option Pricing (Binomial Tree)

Our base-case parameters are the following: an annual interest rate of $r_{annual} = 1\%$ an annualized volatility of $\sigma_{annual} \approx 27\%$, and a strike price equal to the initial stock price of $S_0 = K = \$391.16$, as it has been assumed the call to be ATM. The maturity of the call option is $T = 6$ months. Under these conditions, the six-month Asian call is valued at \$17.58. This figure falls well within the typical range observed for similarly-matured options on similar underlying, confirming that our model produces a plausible market-consistent price.

Another important thing to notice is the fact that the price approximated through the normal distribution is lower than the one computed with the binomial tree. This difference may be due to the excessive smoothing out of the ups and downs performed by our model. By treating the average price as if it followed a bell-curve, we miss capturing the higher average prices, which are those rare, big upside moves that push the true option value higher. In other words, the full tree keeps track of every twist and turn in the stock path, including the unlikely but valuable ones, while the normal approximation flattens those extremes and so gives a slightly lower, more conservative price.

3.2 Robustness Check

In this subsection, we try to stress our model by changing some of the initial conditions. For example, we are interested in finding out how the price of the Asian Call Option changes when changing parameters such as the annual interest rate, the volatility, or the strike price. For simplicity, we focused on just three strike-price scenarios: one ITM, our base-case for the ATM option, and an OTM option. Regarding the annual interest rate and the volatility, we had more options: we have $\sigma \in [\sigma * 0.6, \sigma * 0.7, \sigma * 0.8, \sigma, \sigma * 1.2, \sigma * 1.4, \sigma * 1.6]$, where σ is our base case volatility equal to $\sigma \approx 0.27$. For the annual interest rate: $r \in [0.005, 0.01, 0.02, 0.05, 0.1, 0.2]$.

Depicted below you can find a shorter version, with fewer parameters combinations: The

Varying σ		
σ	Exact Price	Approximated Price
0.216	14.165	11.234
0.270	17.576	14.041
0.324	20.984	16.873
Varying Interest Rate		
r_{annual}	Exact Price	Approximated Price
0.005	17.358	13.944
0.010	17.576	14.041
0.020	18.016	14.235
Varying Strike Price		
Strike Price	Exact Price	Approximated Price
312.928	79.011	78.949
391.160	17.576	14.041
469.392	1.015	0.259

Table 2: Robustness of Asian Call Option Pricing: Varying one parameter at a time. This analysis assesses the impact of varying a single input parameter (σ, r_{annual}, or Strike Price) while holding others at their default values: $\sigma = 0.27$, $r_{annual} = 0.01$, and Strike Price = 391.16. The table compares the Exact and Normal Approximation pricing methods.

“Exact Price” refers to the value obtained from our full binomial-tree calculation, while the “Approximate Price” is the result of the Normal-approximation shortcut. Because it can be hard to see overall trends in a large table, we have also generated a series of plots to illustrate how the price changes across different parameters using the ”Exact Price”:

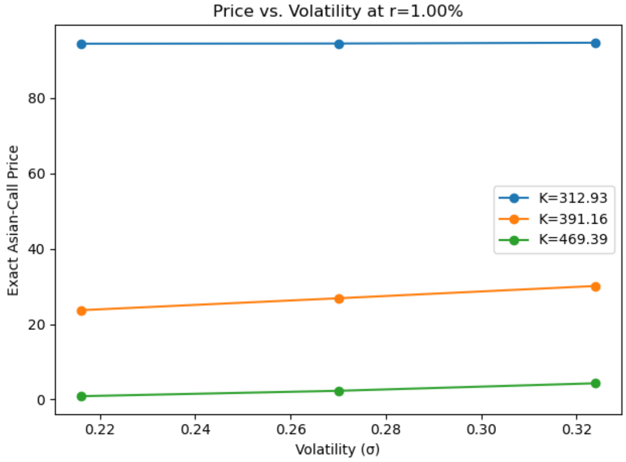


Figure 1: Price vs volatility, r_{annual}=1%

Figure 1 shows how the option prices rise with increasing volatility. Intuitively, higher volatility means the stock is more likely to make large upward moves, which directly benefits the holder of a call option. At the same time, if the stock falls sharply, the worst outcome for the option buyer is simply to walk away and let the option expire worthless. In other words, greater uncertainty on the upside carries no extra downside risk for the option holder, so more volatility always boosts the option's value. We also note that raising the strike price drives the option's value down: a higher K makes it less likely that the average price will exceed the strike, so the payoff becomes rarer. Conversely, a lower strike, especially one below the current stock price, pushes the call deep in the money and guarantees positive payoff in more scenarios, which naturally implies a higher option premium.

Figure 2 plots a heat map of the distribution of up-moves across all 25-step paths. Most paths feature between 11 and 14 up-moves, indicating that the underlying asset tends to drift upward over the period. This upward bias makes a call option more likely to finish in the money, since a larger number of up-moves raises the average price and increases the chance of a positive payoff.

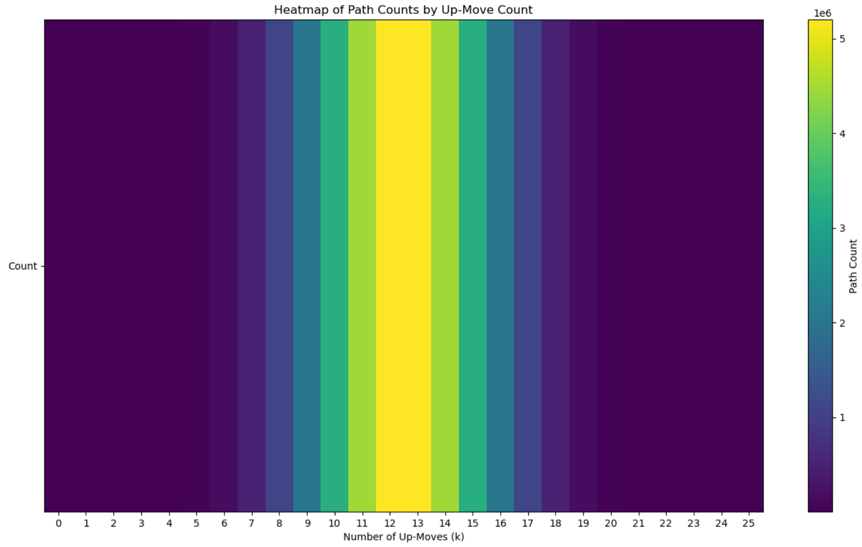


Figure 2: Heat Map for Number of Up-Moves

Figure 3 shows the distribution of the final Stock price and the average price at the leaf level.

`./immagini/histogram.pdf`

Figure 3: Distributions of the Stock Prices and the Average Path Prices

The histogram peaks near \$400, but the distribution exhibits a long right tail, there are more extreme values above the initial price $S_0 = \$391.16$ than below it, suggesting a left-skewed distribution. Those high–end outliers boost the likelihood of a "big" average price, making the call option more valuable.

Finally, Figure 4 shows a surface plotting volatility w.r.t. to the strike price and the "Exact Price", illustrating how the Asian call option's value varies jointly with volatility and strike price. The surface clearly rises as volatility increases, reflecting greater upside potential, and slopes downward as the strike price climbs, reflecting the lower likelihood of finishing in the money.

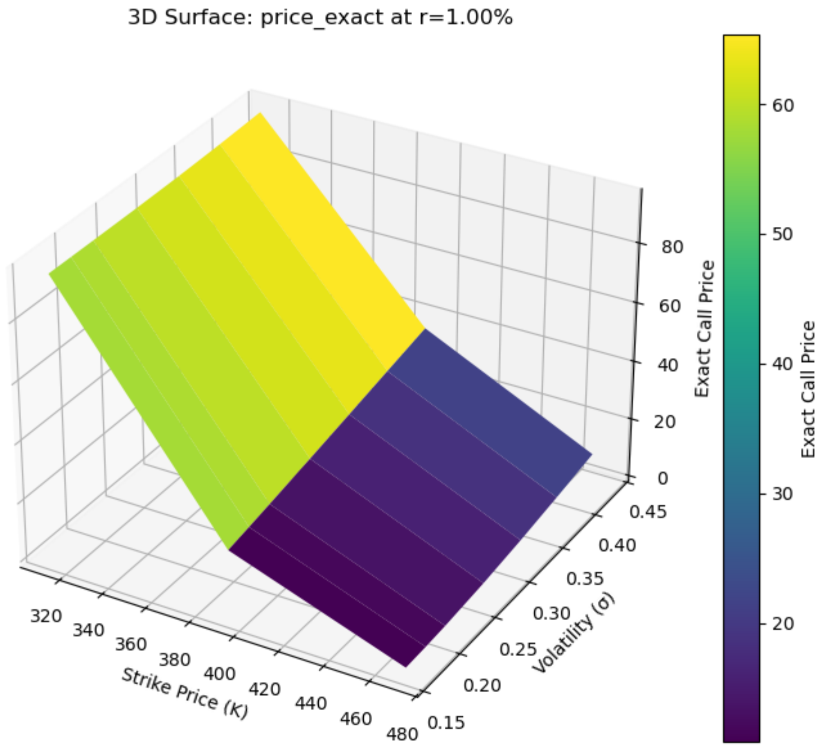


Figure 4: Price for different volatility and strike price combinations

A Complete table

Table 3: Complete results of Asian Call Option Pricing with varying parameters

σ	r_{annual}	Strike	$\text{Price}_{\text{exact}}$	$\text{Price}_{\text{approx}}$
0.162	0.005	312.928	78.525	78.495
0.162	0.005	391.160	10.520	8.343
0.162	0.005	469.392	0.019	0.001
0.162	0.010	312.928	78.817	78.545
0.162	0.010	391.160	10.751	8.452
0.162	0.010	469.392	0.020	0.001
0.162	0.020	312.928	79.396	78.644
0.162	0.020	391.160	11.222	8.670
0.162	0.020	469.392	0.023	0.001
0.162	0.050	312.928	81.109	78.929
0.162	0.050	391.160	12.701	9.341
0.162	0.050	469.392	0.032	0.001
0.162	0.100	312.928	83.873	79.372
0.162	0.100	391.160	15.369	10.511
0.162	0.100	469.392	0.057	0.002
0.162	0.200	312.928	89.081	80.189
0.162	0.200	391.160	21.353	13.055
0.162	0.200	469.392	0.162	0.005
0.189	0.005	312.928	78.529	78.572
0.189	0.005	391.160	12.230	9.734
0.189	0.005	469.392	0.091	0.009
0.189	0.010	312.928	78.821	78.622
0.189	0.010	391.160	12.458	9.840
0.189	0.010	469.392	0.095	0.009
0.189	0.020	312.928	79.400	78.720
0.189	0.020	391.160	12.920	10.052
0.189	0.020	469.392	0.104	0.009
0.189	0.050	312.928	81.111	79.004
0.189	0.050	391.160	14.361	10.700
0.189	0.050	469.392	0.137	0.012
0.189	0.100	312.928	83.875	79.442
0.189	0.100	391.160	16.930	11.818

(Table 3 continued on next page)

Table 3: Complete results of Asian Call Option Pricing with varying parameters

σ	r_{annual}	Strike	$\text{Price}_{\text{exact}}$	$\text{Price}_{\text{approx}}$
0.189	0.100	469.392	0.211	0.016
0.189	0.200	312.928	89.082	80.226
0.189	0.200	391.160	22.611	14.204
0.189	0.200	469.392	0.469	0.031
0.216	0.005	312.928	78.551	78.661
0.216	0.005	391.160	13.940	11.131
0.216	0.005	469.392	0.260	0.039
0.216	0.010	312.928	78.841	78.711
0.216	0.010	391.160	14.165	11.234
0.216	0.010	469.392	0.270	0.040
0.216	0.020	312.928	79.419	78.809
0.216	0.020	391.160	14.619	11.440
0.216	0.020	469.392	0.290	0.042
0.216	0.050	312.928	81.125	79.091
0.216	0.050	391.160	16.027	12.066
0.216	0.050	469.392	0.360	0.050
0.216	0.100	312.928	83.884	79.525
0.216	0.100	391.160	18.513	13.140
0.216	0.100	469.392	0.508	0.065
0.216	0.200	312.928	89.085	80.287
0.216	0.200	391.160	23.948	15.396
0.216	0.200	469.392	0.965	0.106
0.270	0.005	312.928	78.726	78.900
0.270	0.005	391.160	17.358	13.944
0.270	0.005	469.392	0.989	0.254
0.270	0.010	312.928	79.011	78.949
0.270	0.010	391.160	17.576	14.041
0.270	0.010	469.392	1.015	0.259
0.270	0.020	312.928	79.578	79.045
0.270	0.020	391.160	18.016	14.235
0.270	0.020	469.392	1.067	0.268
0.270	0.050	312.928	81.258	79.321
0.270	0.050	391.160	19.367	14.822
0.270	0.050	469.392	1.238	0.299
0.270	0.100	312.928	83.979	79.744

(Table 3 continued on next page)

Table 3: Complete results of Asian Call Option Pricing with varying parameters

σ	r_{annual}	Strike	$\text{Price}_{\text{exact}}$	$\text{Price}_{\text{approx}}$
0.270	0.100	391.160	21.722	15.816
0.270	0.100	469.392	1.573	0.355
0.270	0.200	312.928	89.133	80.469
0.270	0.200	391.160	26.770	17.864
0.270	0.200	469.392	2.463	0.495
0.324	0.005	312.928	79.186	79.277
0.324	0.005	391.160	20.772	16.782
0.324	0.005	469.392	2.261	0.783
0.324	0.010	312.928	79.461	79.323
0.324	0.010	391.160	20.984	16.873
0.324	0.010	469.392	2.304	0.793
0.324	0.020	312.928	80.010	79.415
0.324	0.020	391.160	21.410	17.055
0.324	0.020	469.392	2.392	0.814
0.324	0.050	312.928	81.636	79.676
0.324	0.050	391.160	22.713	17.605
0.324	0.050	469.392	2.673	0.879
0.324	0.100	312.928	84.280	80.076
0.324	0.100	391.160	24.961	18.528
0.324	0.100	469.392	3.197	0.995
0.324	0.200	312.928	89.318	80.754
0.324	0.200	391.160	29.711	20.404
0.324	0.200	469.392	4.481	1.264
0.378	0.005	312.928	79.978	79.859
0.378	0.005	391.160	24.182	19.645
0.378	0.005	469.392	4.004	1.671
0.378	0.010	312.928	80.241	79.901
0.378	0.010	391.160	24.387	19.730
0.378	0.010	469.392	4.064	1.688
0.378	0.020	312.928	80.765	79.984
0.378	0.020	391.160	24.801	19.901
0.378	0.020	469.392	4.185	1.721
0.378	0.050	312.928	82.321	80.222
0.378	0.050	391.160	26.059	20.414
0.378	0.050	469.392	4.565	1.823

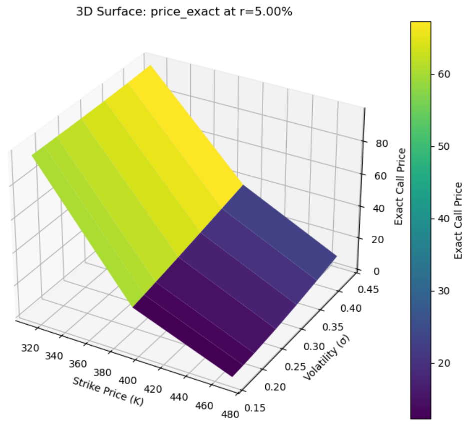
(Table 3 continued on next page)

Table 3: Complete results of Asian Call Option Pricing with varying parameters

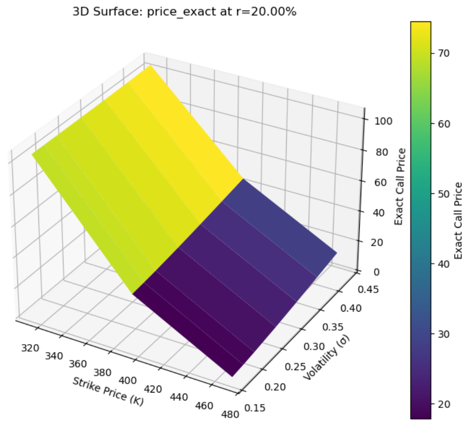
σ	r_{annual}	Strike	$\text{Price}_{\text{exact}}$	$\text{Price}_{\text{approx}}$
0.378	0.100	312.928	84.860	80.584
0.378	0.100	391.160	28.214	21.272
0.378	0.100	469.392	5.256	2.003
0.378	0.200	312.928	89.723	81.192
0.378	0.200	391.160	32.720	22.994
0.378	0.200	469.392	6.862	2.399
0.433	0.005	312.928	81.085	80.677
0.433	0.005	391.160	27.586	22.533
0.433	0.005	469.392	6.124	2.903
0.433	0.010	312.928	81.335	80.713
0.433	0.010	391.160	27.786	22.613
0.433	0.010	469.392	6.198	2.925
0.433	0.020	312.928	81.832	80.785
0.433	0.020	391.160	28.187	22.772
0.433	0.020	469.392	6.347	2.969
0.433	0.050	312.928	83.310	80.992
0.433	0.050	391.160	29.403	23.250
0.433	0.050	469.392	6.811	3.106
0.433	0.100	312.928	85.728	81.304
0.433	0.100	391.160	31.476	24.044
0.433	0.100	469.392	7.638	3.342
0.433	0.200	312.928	90.381	81.819
0.433	0.200	391.160	35.769	25.626
0.433	0.200	469.392	9.497	3.850

B Additional Plots

As shown below, increasing the annual rate to $r_{\text{annual}} = 5\%$ and 20% reproduces the same behavior observed in Figures 1 and 4.

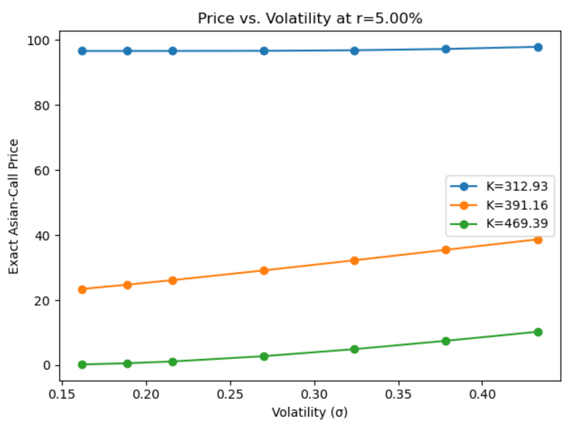


(a) "Exact Price" vs volatility vs strike price, $r_{annual} = 5\%$

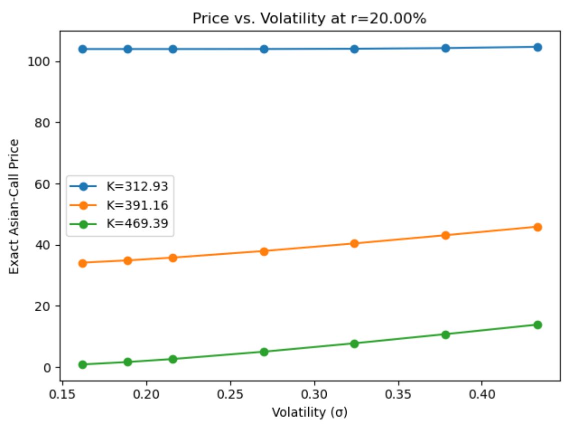


(b) "Exact Price" vs volatility vs strike price, $r_{annual} = 20\%$

Figure 5: "Exact Price" vs volatility vs strike price



(a) "Exact Price" vs volatility, $r_{annual} = 5\%$



(b) "Exact Price" vs volatility, $r_{annual} = 20\%$

Figure 6: "Exact Price" vs volatility