Engaging Complexity
Quiz 4

THE LONDON INTERDISCIPLINARY SCHOOL

In this quiz we will show that the logistic map (for certain growth rates) is a complex system. We will do this by showing evidence of evolution, emergent behaviour and closeness to chaos.

In class, the logistic map was introduced as a way to model the growth of a population of bacteria. We will use x_n to denote the ratio between the population p_n at time n (hours) and its maximum carrying capacity M. In other words, $x_n = \frac{p_n}{M}$. We will also consider r to be the growth rate of the bacteria, this will be a real number between 0 and 4. Then, the logistic map is given by the following iterated function:

$$x_{n+1} = rx_n(1 - x_n) \tag{1}$$

for $n = 1, 2, 3, \ldots$ and an initial condition x_0.

1. (2 points)

   (a) Use the code `logistic_map.ipynb` (this is below the document for this Quiz on Cortex) to obtain a figure that shows the first 20 iterations of the solutions to equation (1) with $r = 0.5$ and different initial conditions $x_0 = 0.1$, $x_0 = 0.3$, $x_0 = 0.5$, $x_0 = 0.7$, $x_0 = 0.9$. This should lead to a single figure with five solutions (one for each of the initial conditions).

   (b) Repeat the previous step for $r = 1$. This should lead to a single figure with five solutions.

   (c) Comment on the two plots you obtained in the previous two questions. In particular, how do the solutions behave as n increases? Are there any equilibrium points? Does the behaviour depend on any of the parameters?

2. (2 points)

   (a) Use the code `logistic_map.ipynb` to obtain three figures that show the first 20 iterations of the solutions to (1) with $r = 1.5$, $r = 2$ and $r = 2.5$ and initial conditions $x_0 = 0.1$, $x_0 = 0.3$, $x_0 = 0.5$, $x_0 = 0.7$, $x_0 = 0.9$. This should lead to three figures, each showing five solutions.

   (b) Comment on the three plots you obtained in the previous question. In particular, how do the solutions behave as n increases? Are there any equilibrium points? Does the behaviour depend on any of the parameters?

   (c) Derive the equilibrium point(s) mathematically using equation (1). Remember that to do this you need to impose the constraint $x_n = x_{n+1}$.

   (d) How do the mathematical equilibrium points compare with the numerical solutions obtained for $0 < r < 1$ and $1 \leq r \leq 2$?

3. (2 points)

   (a) Use the code `logistic_map.ipynb` to obtain two figures that show the first 20 iterations of the solutions to (1) with $r = 3$ and $r = 3.5$, and initial conditions $x_0 = 0.1$, $x_0 = 0.3$, $x_0 = 0.5$, $x_0 = 0.7$, $x_0 = 0.9$. This should lead to two figures, each showing five solutions.

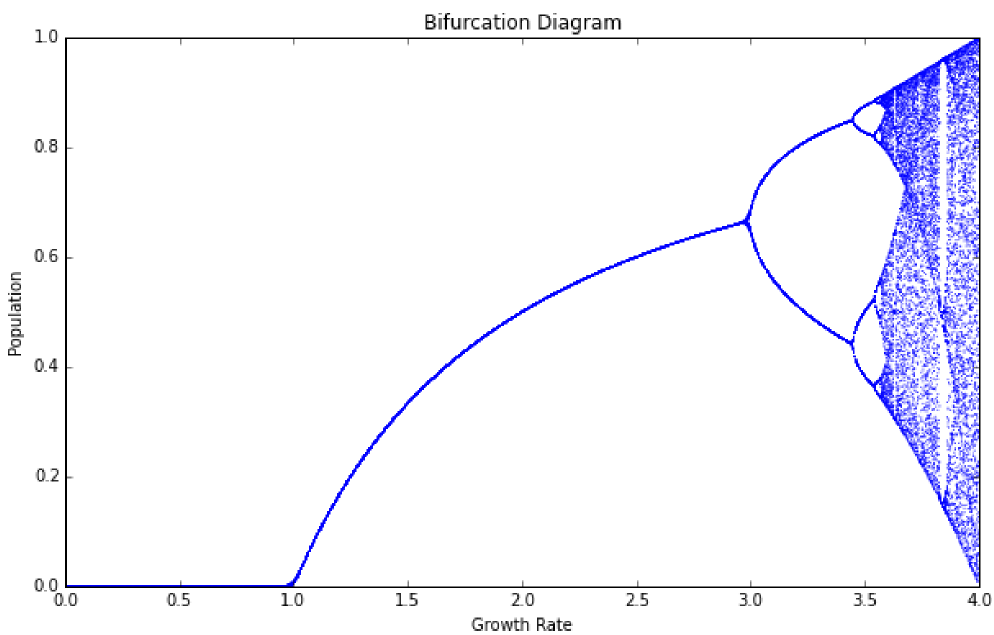
   (b) Comment on the two plots you obtained in the previous question. In particular, how do the solutions behave as n increases? Are there any equilibrium points? Does the behaviour depend on any of the parameters?

4. (2 points)

   (a) Use the code `logistic_map.ipynb` to obtain a figure that shows the first 20 iterations of the solutions to (1) with $r = 4$ and different initial conditions $x_0 = 0.1$, $x_0 = 0.101$, $x_0 = 0.1001$. This should lead to a single figure with three solutions.

   (b) Comment on the plot you obtained in the previous question. In particular, how do the solutions behave as n increases? Does the behaviour depend on any of the parameters?

5. (1 point) The figure below shows the bifurcation diagram for the logistic map. Comment on the behaviour of the attracting points of the population (x_n) as the growth rate (r) increases.



6. (1 point) Use your results in the previous questions to:

   (a) Show that the logistic map evolves as r changes.

   (b) Show that the logistic map has emergent behaviour as r changes.

   (c) Show that the logistic map is at the edge of chaos.

   (d) Conclude that the logistic map is a complex system.