Some Extra Credit Questions

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My favorite part of exam-writing is coming up with fun extra credit questions for students who finish with time to spare, and I've decided to compile the (at least moderately) interesting ones here. The main reason this document exists is to be a reference so that I know which questions I've already put on an exam, but making this public for potential budding problem solvers is also an added benefit. These questions were intended mostly for lower-division math classes, but that doesn't mean they're necessarily easy. If you want to take a stab at some of these for fun, I'd be happy to receive an email of your solution!

Note #1: On exams, I gave out hints on the particularly challenging problems, but I've removed most of them here as to not spoil any potential fun.

Note #2: The questions I find particularly satisfying have been marked with *.

MAC 2312 Calculus w/ Analytic Geometry 2

Sp23 E2 #2: Consider the function $f(x) = \sin(\ln x)$ on the interval $\left(0, e^{\pi/2}\right)$. Given that the average value of $f(x)$ on this interval is $f_{\text{ave}} = 1/2$, calculate

$$\int_0^{e^{\pi/2}} \sin(\ln x)\, dx.$$

Sp23 E3 #1: Evaluate

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}.$$

***Fa23 E1 #2:** As it turns out, calculating the indefinite integral

$$\int e^{-x^2}\, dx$$

is impossible to do. However, using techniques from Calculus 3, one can deduce that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

Given this fact, evaluate

$$\int_{-\infty}^{\infty} x^2 e^{-x^2}\, dx.$$

***Fa23 E2 #3:** Show that the volume of a sphere with radius r is given by the formula

$$V = \frac{4}{3}\pi r^3.$$

Fa23 E2 #2: Prove that $0.\bar{9} = 1$.

Fa23 FE #2: Recall the Maclaurin Series representation:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Use the **first three nonzero terms** of the above series to approximate

$$\int_0^{0.1} e^{x^2}\, dx.$$

MAS 2103 Matrix Theory

Sp24 E1 #2: Calculate the following determinant.

Hint: Perform cofactor expansion. It may be easier to solve this problem by crossing out rows and columns directly on the matrix below. (Be careful doing this if you're not using a pencil.)

$$\begin{vmatrix}
1 & 7 & 3 & 4 & 5 & 1 & 2 & 1 & 2 & 2 \\
0 & 2 & 9 & 1 & 1 & 3 & 7 & 1 & 1 & 3 \\
0 & 0 & 0 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 7 & 2 & 4 & 5 & 1 & 1 & 3 & 0 \\
0 & 0 & 3 & 5 & 7 & 2 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 6 & 5 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 2 & 0 & 0 & 0 & 0 \\
0 & 2 & 2 & 3 & 1 & 2 & 0 & 4 & 0 & 0 \\
0 & 1 & 3 & 1 & 2 & 2 & 1 & 1 & 0 & 0
\end{vmatrix}$$

***Sp24 E2 #2:** The Riesz Representation Theorem is a highly abstract and useful tool in a field of mathematics called *functional analysis* (and is also your professor's favorite theorem). Prove the following, simplified version of the Riesz Representation Theorem:

If $L : \mathbb{R}^n \to \mathbb{R}$ is a linear transformation, then there exists a vector $\mathbf{v} \in \mathbb{R}^n$ such that $L(\mathbf{x}) = \mathbf{x} \cdot \mathbf{v}$, for all $\mathbf{x} \in \mathbb{R}^n$.

Sp24 FE #1: Let A be a matrix with 10,778 rows and 4,366 columns. If the nullity of A is 512, what is its rank?

Sp24 FE #2: Let A be a 1,000,000 $\times$ 1,000,000 singular matrix. List one eigenvalue of A.

Sp24 FE #3: Let W be a subspace of $\mathbb{R}^n$. Recall that the $W^\perp$ is called the **orthogonal complement** of W, where

$$W^\perp = \{x \in \mathbb{R}^n : x \cdot w = 0 \text{ for all } w \in W\}.$$

Show that, for any matrix A,

$$(\mathrm{Row}(A))^\perp = \mathrm{Nul}(A).$$

MAP 3305 Engineering Mathematics 1

Sp22 E1 #1: Solve the following initial value problem.

$$2e^{2t}\frac{1}{y} - e^{2t}\frac{y'}{y^2} = 1, \quad y(0) = 1.$$

Sp22 E2 #1: For the following differential equation, determine a suitable form for the particular solution if the **method of undetermined coefficients** is to be used (but do not solve).

$$y^{(5)} + y''' = t + \pi\sin(t)$$

Sp22 E3 #2: Calculate $\mathcal{L}^{-1}\{1\}$. *(Note: $\mathcal{L}$ denotes the Laplace Transform.)*

Sp22 FE #1: Suppose a tank originally contains 50L of water. Water containing 2 grams per liter of salt is poured into the tank at a rate of 5 L/min, and the mixture flows out at the same rate. How much salt must *initially* be in the tank so that the amount of salt in the tank remains constant throughout the entire experiment.
(Note: You can solve this without solving a differential equation, though this is not what I intended my students to do.)

***Fa22 E3 #1:** Suppose $y(t)$ is a solution to the initial value problem

$$y' = \sum_{n=1}^{\infty} \frac{1}{2^n}\delta(t-n), \qquad y(0) = 0.$$

What is $\lim_{t\to\infty} y(t)$?

Fa22 FE #1: Find a **nonzero** solution to the following differential equation.

$$y^{(2022)} - y = 0$$

Fa24 E2 #2: You may recall from class that **squigonometry** is the study of points on the unit squircle $x^4 + y^4 = 1$. More specifically, given an angle θ, one can represent the point (x, y) lying on the squircle at angle θ via the squigonometric functions $(\mathrm{cq}(\theta), \mathrm{sq}(\theta))$ (these functions are called *cosquine* and *squine*, respectively). As it turns out, the following relations hold:

$$\frac{d}{dx}\mathrm{cq}(x) = -\mathrm{sq}(x)^3 \quad \text{and} \quad \frac{d}{dx}\mathrm{sq}(x) = \mathrm{cq}(x)^3.$$

Using squigonometric substitution, calculate the following indefinite integral.

$$\int \frac{dx}{(1 - x^4)^{\frac{3}{4}}}$$

Fa24 E3 #1: Without integrating, calculate the Laplace transforms of the functions

$$f(t) = te^t \cos(t) \qquad \text{and} \qquad g(t) = te^t \sin(t).$$

Fa24 E3 #2: Find a solution to the following initial value problem

$$y' = \delta(t), \quad y(0) = 0.$$

Explain why trying to use the Laplace transform to solve the problem above yields the **wrong** answer.