A relatively complete description of the random effects (latent variables) for each level of a relatively simple MLTA with two timepoints and three classes is shown in Figure 1.

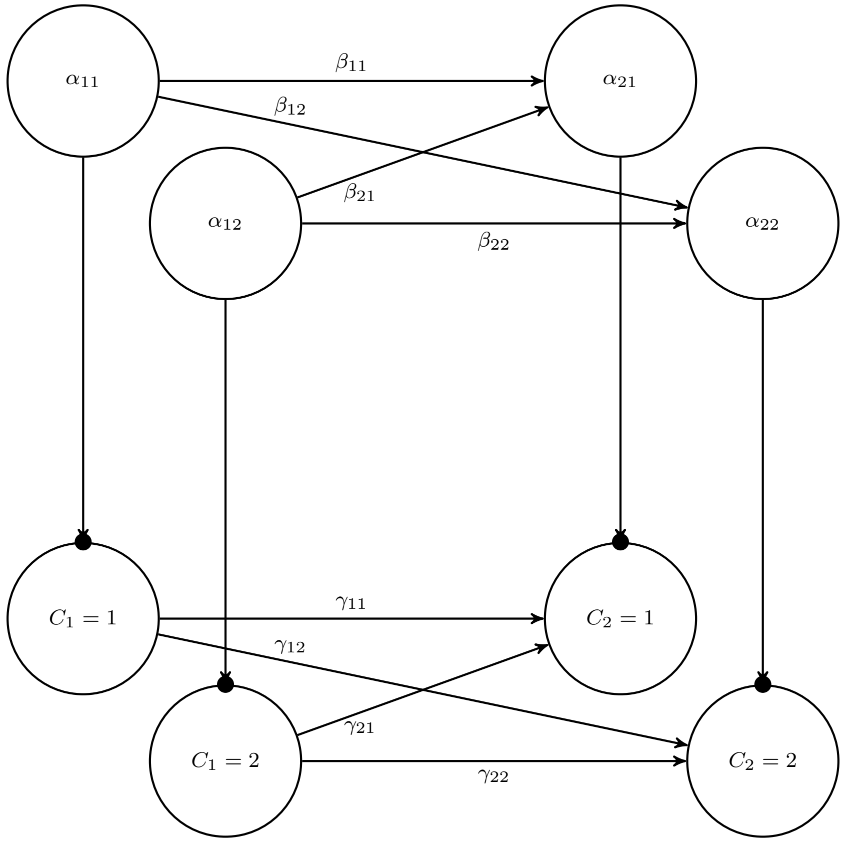

Figure 1: Conceptual diagram of latent variables in a 3-class MLTA.

Next, we have demonstrated how the variance is decomposed for the calculation of a variety of R^2 like measures associated with the latent class at time 2.

1 Variance decomposition

Some general notation.

- K = number of latent classes, $1, 2, ..., k, ..., K$, where latent class K is the reference class.
- $K - 1$ random intercepts needed to separate the classes
- α_{tk} random intercept for timepoint t of class k, so α_{11} is the random intercept of class 1 at timepoint 1.
- C_t is latent class at time t, which takes on values $C_t = 1, 2, ..., k, ..., K$

- $\beta_{k_t j_{t+1}}$ represents the regression weight associated with latent class k at timepoint t as a dummy-coded predictor of latent class j at timepoint $t+1$; ex. β_{11} is the regression weight of latent class 1 at timepoint 1 predicting latent class 1 at timepoint 2. The same scheme applies to γ parameters.
- $\frac{\pi^2}{3} = 3.29$ is the variance of the logistic distribution

Next, the decomposition is broken down. To help with showing how this works, we used the simulation condition detailed in Table 1.

Table 1: Model details for example calculations

Class Size		Transition Matrix $C_{(t-1)\backslash t}$	$C_2 = 1$	$C_2 = 2$	$C_2 = 3$
Probability Scale					
$C_1 = 1$	0.30	$C_1 = 1$	0.60	0.37	0.03
$C_1 = 2$	0.50	$C_1 = 2$	0^{b}	0.78	0.22
$C_1 = 3$	0.20	$C_1 = 3$	0^{b}	0^{b}	1
Logit Parameterization					
$C_1 = 1$	0.40	$C_1 = 1$	3.12	2.62	0^{a}
$C_1 = 2$	0.90	$C_1 = 2$	-15^{b}	1.27	0^{a}
$C_1 = 3$	0^{a}	$C_1 = 3$	-15^{b}	-15^{b}	0^{a}

Random Effects[c]	Regression Weights
$\text{var}(\alpha_1) = 0.8$	$\boldsymbol{\gamma} = (18, 17.5, 0^{\mathrm{b}}, 16)$
$\text{var}(\alpha_2) = 0.25$	$\boldsymbol{\beta} = (0.3, 0.3, 0^{\mathrm{b}}, 0.3)$

Note. [a] parameter fixed for identification; [b] parameter fixed for stage sequential development hypothesis; and [c] random effects were assumed invariant across latent class intercepts.

1. ICC for latent class at time 1. Which is actually 2 ICC values

$$
\begin{aligned}
ICC_1 &= \frac{\mathbb{V}(\alpha_{11})}{\mathbb{V}(\alpha_{11}) + \frac{\pi^2}{3}} = \frac{0.8}{0.8 + 3.29} = 0.20 \\
ICC_2 &= \frac{\mathbb{V}(\alpha_{12})}{\mathbb{V}(\alpha_{12}) + \frac{\pi^2}{3}} = \frac{0.8}{0.8 + 3.29} = 0.20
\end{aligned}
$$

2. Compute the variance of latent class at time 1 or $\mathbb{V}(C_1)$. We did this using the following: First, using the expected value of the intercepts $\alpha_{11} = 0.40$, $\alpha_{12} = 0.90$, and $\alpha_{13} = 0$

we transformed these into probabilities.

$$
\begin{aligned}
\pi_1 = Pr(C_1 = 1) &= \frac{\exp(\alpha_{11})}{\sum_{k=1}^{K} \exp(\alpha_{1k})} \\
&= \frac{\exp(\alpha_{11})}{\exp(\alpha_{11}) + \exp(\alpha_{12}) + 1} = 0.301 \\
\pi_2 = Pr(C_1 = 2) &= \frac{\exp(\alpha_{12})}{\exp(\alpha_{11}) + \exp(\alpha_{12}) + 1} = 0.497 \\
\pi_3 = Pr(C_1 = 3) &= 1 - Pr(C_1 = 1) - Pr(C_1 = 2) = 0.202
\end{aligned}
$$

Next, we use these to compute the variance using the definition of variance:

$$
\begin{aligned}
\mathbb{V}(C_1) &= \mathbb{E}(C_1^2) - [\mathbb{E}(C_1)]^2 \\
&= \left[1^2 \times Pr(C_1 = 1) + 2^2 \times Pr(C_1 = 2) + 3^2 \times Pr(C_1 = 3)\right] - \\
&\qquad [1 \times Pr(C_1 = 1) + 2 \times Pr(C_1 = 2) + 3 \times Pr(C_1 = 3)]^2 \\
&= [0.301 + 1.987 + 1.818] - [0.301 + 0.993 + 0.606]^2 \\
&= 0.493
\end{aligned}
$$

3. Compute the contribution of C_1 variance on C_2 variance. Meaning, how much of the variance of latent class at time 1 gets forwarded to latent class at time 2. This is here Figure 1 becomes really handy. The most difficult part of this is adequately accounting for the correlation among transition paths at level-1.

   Here, the variance of C_1 is transferred forward in two components: towards $C_2 = 1$ and $C_2 = 2$ and this can occur by a path from $C_1 = 1$ and $C_1 = 2$. All four regression weights (γ's) get utilized here. I denoted this $\mathbb{V}(C_{1\rightarrow 2})$, the variance of C_1 forwarded to C_2.

$$
\mathbb{V}(C_{1\rightarrow 2}) = \mathbb{V}(C_{1\rightarrow 21}) + \mathbb{V}(C_{1\rightarrow 22})
$$

   meaning that forwarded variance is the sum of the forwarded variances to each latent class.

$$
\begin{aligned}
\mathbb{V}(C_{1\to 21}) &= (\alpha_{21}+\gamma_{11})^2\mathbb{V}(C_{11}) + \gamma_{21}^2\mathbb{V}(C_{12}) + \gamma_{11}\gamma_{21}\mathbb{COV}(C_{11}, C_{12}) \\
&= \gamma_{11}^2\pi_1(1-\pi_1) + \gamma_{21}^2\pi_2(1-\pi_2) - \gamma_{11}\gamma_{21}\pi_1\pi_2 \\
&= 68.2 \\
\mathbb{V}(C_{1\to 22}) &= \gamma_{12}^2\mathbb{V}(C_{11}) + \gamma_{22}^2\mathbb{V}(C_{12}) + \gamma_{12}\gamma_{22}\mathbb{COV}(C_{11}, C_{12}) \\
&= \gamma_{12}^2\pi_1(1-\pi_1) + \gamma_{22}^2\pi_2(1-\pi_2) - \gamma_{12}\gamma_{22}\pi_1\pi_2 \\
&= 86.6 \\
\mathbb{V}(C_{1\to 2}) &= \mathbb{V}(C_{1\to 21}) + \mathbb{V}(C_{1\to 22}) \\
&= \gamma_{11}^2\pi_1(1-\pi_1) + \gamma_{21}^2\pi_2(1-\pi_2) - \gamma_{11}\gamma_{21}\pi_1\pi_2 + \\
&\quad\ \gamma_{12}^2\pi_1(1-\pi_1) + \gamma_{22}^2\pi_2(1-\pi_2) - \gamma_{12}\gamma_{22}\pi_1\pi_2 \\
&= \left(\gamma_{11}^2+\gamma_{12}^2\right)\pi_1(1-\pi_1) + \left(\gamma_{21}^2+\gamma_{22}^2\right)\pi_2(1-\pi_2) - \\
&\quad\ \left(\gamma_{11}\gamma_{21}+\gamma_{12}\gamma_{22}\right)\pi_1\pi_2 \\
&= 132.7 + 64.0 - 41.9 \\
&= 154.8
\end{aligned}
$$

4. Split forwarded variance of C_1 by the ICC_k associated with each random intercept at time 1. This breaks the variance passed forward into two pieces. The first, $\mathbb{V}(C_{1\to 2}^{+})$, represents the amount of variance explained by the level-2 intercept variances. The second, $\mathbb{V}(C_{1\to 2}^{-})$, represents the amount of residual (unexplained) forwarded variance by the inclusion of the random effects.

$$
\begin{aligned}
\mathbb{V}(C_{1\to 2}^{+}) &= ICC_1 \times \left[\left(\gamma_{11}^2+\gamma_{12}^2\right)\pi_1(1-\pi_1) - \left(\gamma_{11}\gamma_{21}\right)\pi_1\pi_2\right] + \\
&\qquad ICC_2 \times \left[\left(\gamma_{21}^2+\gamma_{22}^2\right)\pi_2(1-\pi_2) - \left(\gamma_{12}\gamma_{22}\right)\pi_1\pi_2\right] \\
&= 0.20(132.7-0) + 0.20(64.0-41.9) = 30.3 \\
\mathbb{V}(C_{1\to 2}^{-}) &= (1-ICC_1) \times \left[\left(\gamma_{11}^2+\gamma_{12}^2\right)\pi_1(1-\pi_1) - \left(\gamma_{11}\gamma_{21}\right)\pi_1\pi_2\right] \\
&\qquad (1-ICC_2) \times \left[\left(\gamma_{21}^2+\gamma_{22}^2\right)\pi_2(1-\pi_2) - \left(\gamma_{12}\gamma_{22}\right)\pi_1\pi_2\right] \\
&= 0.80(132.7-0) + 0.80(64.0-41.9) = 124.5
\end{aligned}
$$

5. Next, we need to break down the contribution of α_1 on C_2. α_1 influence C_2 indirectly through C_1 and α_2.

   i. Contribution of α_1 on C_2 through C_1
      This is the piece computed in part 4: $\mathbb{V}(C_{1\to 2}^{+})$

   ii. Contribution of α_1 on C_2 through α_2
      Here, we utilize a system similar to part 3 in order to account for the variance in both intercepts. However, because these are all on the logit scale and are random effects, we have no covariance in this case. In theory, one could be specified, though.

$$\mathbb{V}(\alpha_{1\to2}) = \mathbb{V}(\alpha_{1\to21}) + \mathbb{V}(\alpha_{1\to22})$$

where we similarly pass the variance in α_1 to α_2 in two pieces.

$$\begin{aligned}
\mathbb{V}(\alpha_{1\to21}) &= \beta_{11}^2\mathbb{V}(\alpha_{11}) + \beta_{21}^2\mathbb{V}(\alpha_{12}) \\
&= 0.3^2(0.8) + 0^2(0.8) \\
&= 0.07 \\
\mathbb{V}(\alpha_{1\to22}) &= \beta_{12}^2\mathbb{V}(\alpha_{11}) + \beta_{22}^2\mathbb{V}(\alpha_{12}) \\
&= 0.3^2(0.8) + 0.3^2(0.8) \\
&= 0.14 \\
\mathbb{V}(\alpha_{1\to2}) &= \mathbb{V}(\alpha_{1\to21}) + \mathbb{V}(\alpha_{1\to22}) \\
&= 0.21
\end{aligned}$$

Lastly, we just need to get the total contribution of α_1 on C_2 by adding these pieces together.

$$\begin{aligned}
\mathbb{V}(\alpha_1 \to C_2) &= \mathbb{V}(C^+_{1\to2}) + \mathbb{V}(\alpha_{1\to2}) + \mathbb{COV}(C^+_{1\to2}, \alpha_{1\to2}) \\
&= \mathbb{V}(C^+_{1\to2}) + \mathbb{V}(\alpha_{1\to2}) + 2\sqrt{\mathbb{V}(C^+_{1\to2}) \times \mathbb{V}(\alpha_{1\to2})} \\
&= 30.3 + 0.2 + 5.0 \\
&= 35.5
\end{aligned}$$

6. Variance of C_2 with respect to class size.

   Next, we need to get the variability in C_2 that is based on the class sizes at timepoint 2. This is tedious but not hard. First, we used the intercepts at both timepoints to compute the transition logits. That is we first get the transition probabilities then use those to get the class proportions. The transition matrix is found by $\text{logit}[\tau_{k_t j_{t+1}}]$ as the transition logit from class k at time t to class j at time $t+1$.

$$\begin{aligned}
&\text{logit}[\tau_{11}] = \alpha_{21} + \beta_{11}\alpha_{11} + \gamma_{11} = 3.12 \\
&\text{logit}[\tau_{12}] = \alpha_{21} + \beta_{12}\alpha_{11} + \gamma_{12} = 2.62 \\
&\text{logit}[\tau_{13}] = 0 \\
&\text{logit}[\tau_{21}] = \alpha_{22} + \beta_{21}\alpha_{12} + \gamma_{21} = -15 \\
&\text{logit}[\tau_{22}] = \alpha_{22} + \beta_{22}\alpha_{12} + \gamma_{22} = 1.27 \\
&\text{logit}[\tau_{23}] = 0 \\
&\text{logit}[\tau_{31}] = \alpha_{21} = -15 \\
&\text{logit}[\tau_{32}] = \alpha_{22} = -15 \\
&\text{logit}[\tau_{33}] = 0
\end{aligned}$$

   which in turn results in the transition probabilities given in Table 1, repeated here as

$C_{(t-1)\backslash t}$	$C_2 = 1$	$C_2 = 2$	$C_2 = 3$
$C_1 = 1$	$\tau_{11} = 0.60$	$\tau_{12} = 0.37$	$\tau_{13} = 0.03$
$C_1 = 2$	$\tau_{21} = 0$	$\tau_{22} = 0.78$	$\tau_{23} = 0.22$
$C_1 = 3$	$\tau_{31} = 0$	$\tau_{32} = 0$	$\tau_{33} = 1$

We are interested in getting the total class probability at time 2, this can be done by multiplying the transition matrix above ($\boldsymbol{\tau}$) by the class sizes at time 1 ($\boldsymbol{\pi}_t$). As shown next

$$\boldsymbol{\pi}_t \times \boldsymbol{\tau} = \begin{bmatrix} 0.3 & 0.5 & 0.2 \end{bmatrix} \times \begin{bmatrix} 0.6 & 0.37 & 0.03 \\ 0 & 0.78 & 0.22 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.18 & 0.50 & 0.32 \end{bmatrix}$$

which are the expected class sizes at time 2. These are used to compute the variance of C_2. Next, we use these to compute the variance using the definition of variance:

$$\begin{aligned} \mathbb{V}_{class}(C_2) &= \mathbb{E}(C_2^2) - [\mathbb{E}(C_2)]^2 \\ &= \left[1^2 \times Pr(C_2 = 1) + 2^2 \times Pr(C_2 = 2) + 3^2 \times Pr(C_2 = 3)\right] - \\ &\quad\ [1 \times Pr(C_2 = 1) + 2 \times Pr(C_2 = 2) + 3 \times Pr(C_2 = 3)]^2 \\ &= [0.18 + 1.99 + 2.87] - [0.18 + 0.99 + 0.96]^2 \\ &= 0.48 \end{aligned}$$

7. Compute total variance of C_2

$$\mathbb{V}_{total}(C_2) = \mathbb{V}_{class}(C_2) + \mathbb{V}(\alpha_1 \to C_2) + \mathbb{V}(C_{1 \to 2}) + \mathbb{V}(\alpha_2) + \frac{\pi^2}{3}$$

8. next gothrough the R^2 indices.