Homework 7: Convex programs

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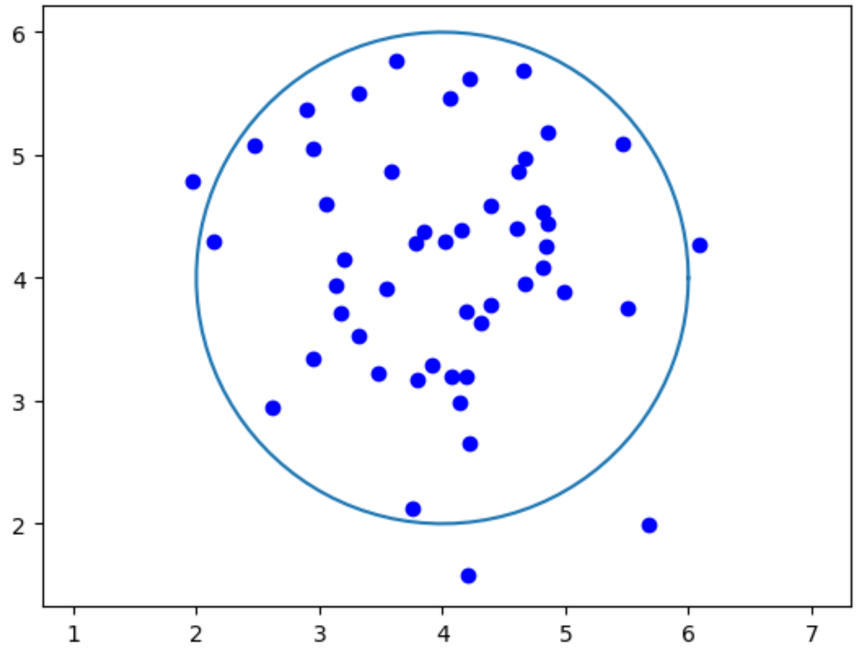
1. Enclosing circle

Generate the points and plot a circle:

In [3]:
```
using PyPlot
X = 4 + randn(2, 50)
t = linspace(0, 2*pi, 100)
r = 2;
x1 = 4; x2 = 4
plot(x1 + r*cos(t), x2 + r*sin(t))
scatter(X[1,:], X[2,:], color="blue")
axis("equal")
```



Out[3]:
```
(1.7397062263937673,6.31928593183141,1.326753812133779,6.2222718002627015)
```

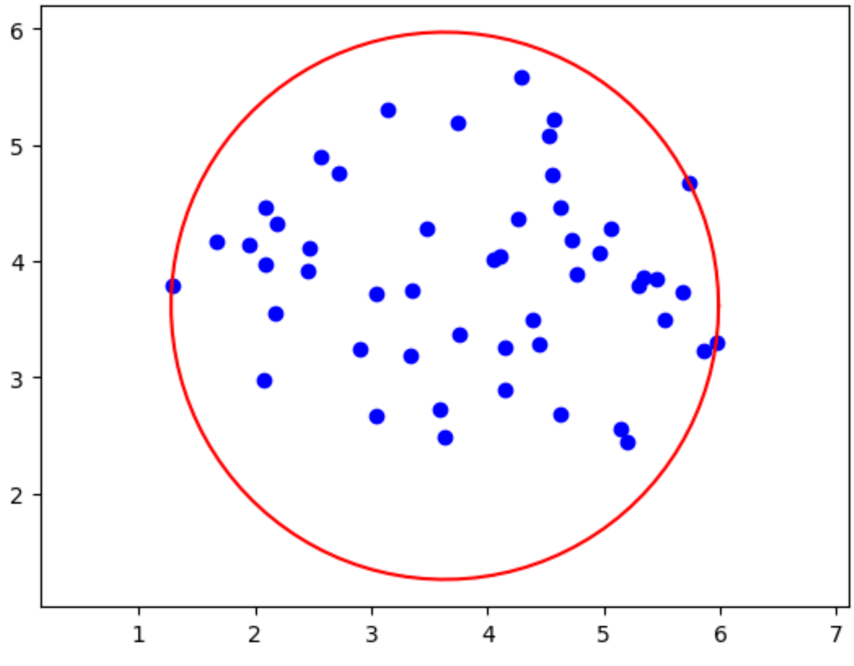
The optimization problem can be described as follows:

$$\min_{xc,yc,R} \ R$$

$$s.t. \ (Xi_x - xc)^2 + (Xi_y - yc)^2 \leq R \quad for \ i = 1:50$$

R is the square of the circle radius.
xc and yc is the center of the smallest circle.
Xi_x is the x coordinate of the ith point and Xi_y is the y coordinate of the ith point.

In [14]:
```
using JuMP, Gurobi

num_points = 50

# generate random points first
X = 4 + randn(2, num_points)

# do optimization
m = Model(solver=GurobiSolver(OutputFlag=0))

@variable(m, xc)   # x coordinate of the circle center
@variable(m, yc)   # y coordinate of the circle center
@variable(m, R)    # radius^2 the circle

for i in 1:num_points
    @constraint(m, (X[1,i]-xc)^2 + (X[2,i]-yc)^2 <= R)
end

@objective(m, Min, R)

solve(m)
xc = getvalue(xc)
yc = getvalue(yc)
r = sqrt(getvalue(R))
println("xc:", xc)
println("yc:", yc)
println("r:", r)

# plot
plot(xc + r*cos(t), yc + r*sin(t), color="red")
scatter(X[1,:], X[2,:], color="blue")
axis("equal")
```




```
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xc:3.6323320279308056
yc:3.6159516552125144
r:2.3553265533066687
```
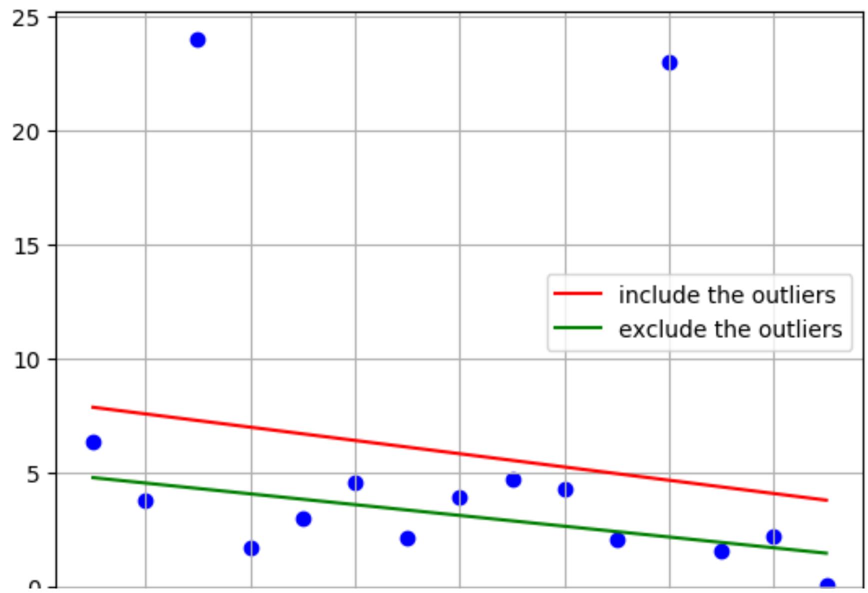

Out[14]:
```
(1.014707246922618,6.235260999828116,1.025418564050432,6.2064847463745965)
```

2. The Huber loss

(a)

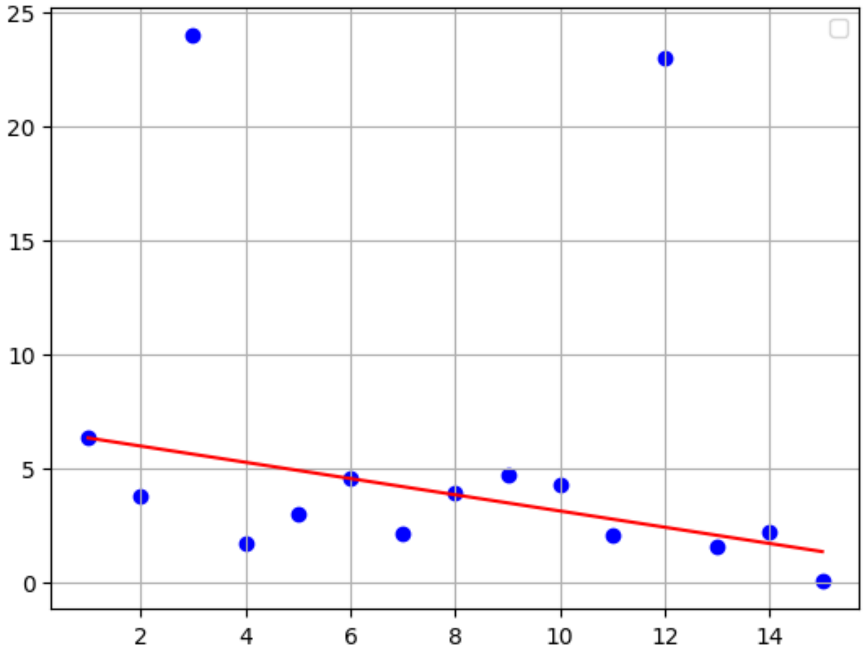
In [4]:
```
x = [1:15;]
y = [6.31 3.78 24 1.71 2.99 4.53 2.11 3.88 4.67 4.25 2.06 23 1.58 2.17 0.02]
x_drop_outliers = [1 2 4 5 6 7 8 9 10 11 13 14 15]
y_drop_outliers = [6.31 3.78 1.71 2.99 4.53 2.11 3.88 4.67 4.25 2.06 1.58 2

using JuMP, Gurobi, PyPlot

# include the outliers
m1 = Model(solver=GurobiSolver(OutputFlag=0))

@variable(m1, a1)
@variable(m1, b1)

@objective(m1, Min, sum{(y[i]-a1*x[i]-b1)^2, i=1:15})

solve(m1)

a1 = getvalue(a1)
b1 = getvalue(b1)

println("best fit of least square(include the outliers):")
println("a1:", a1)
println("b1:", b1)

# exclude the outliers
m2 = Model(solver=GurobiSolver(OutputFlag=0))

@variable(m2, a2)
@variable(m2, b2)

@objective(m2, Min, sum{(y_drop_outliers[i]-a2*x_drop_outliers[i]-b2)^2, i=

solve(m2)

a2 = getvalue(a2)
b2 = getvalue(b2)

println("best fit of least square(exclude the outliers):")
println("a2:", a2)
println("b2:", b2)

scatter(x, y, color="blue")
plot(x, a1*x+b1, color="red", label="include the outliers")
plot(x, a2*x+b2, color="green", label="exclude the outliers")
legend()
grid("on")
```



You can see that the line which includes the outliers is higher that the line which excludes the outliers. Because the red line affected by the outliers heavily.

(b)

In [7]:
```
x = [1:15;]
y = [6.31 3.78 24 1.71 2.99 4.53 2.11 3.88 4.67 4.25 2.06 23 1.58 2.17 0.02]

using JuMP, Gurobi, PyPlot

# include the outliers
m = Model(solver=GurobiSolver(OutputFlag=0))

@variable(m, a)
@variable(m, b)

# l1-norm
@variable(m, t[1:15])
for i in 1:15
    @constraint(m, y[i]-a*x[i]-b <= t[i])
    @constraint(m, -t[i] <= y[i]-a*x[i]-b)
end

@objective(m, Min, sum(t))

solve(m)

a = getvalue(a)
b = getvalue(b)

println("best fit of l1 cost function(include the outliers):")
println("a:", a)
println("b:", b)

scatter(x, y, color="blue")
plot(x, a*x+b, color="red")
legend()
grid("on")
```




```
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best fit of l1 cost function(include the outliers):
a:-0.35599999999999987
b:6.6659999999999995
```

You can see that l1 cost handle outliers better than least squares.
We can justify this point by evaluating the errors of l_2 cost function and l_1 cost function.

In [77]:
```
error_l2 = 0
error_l1 = 0

for i in 1:13
    error_l2 = error_l2 + (y_drop_outliers[i]-a1*x_drop_outliers[i]-b1)^2
    error_l1 = error_l1 + (y_drop_outliers[i]-a*x_drop_outliers[i]-b)^2
end
println("error_l2:", error_l2)
println("error_l1:", error_l1)
```

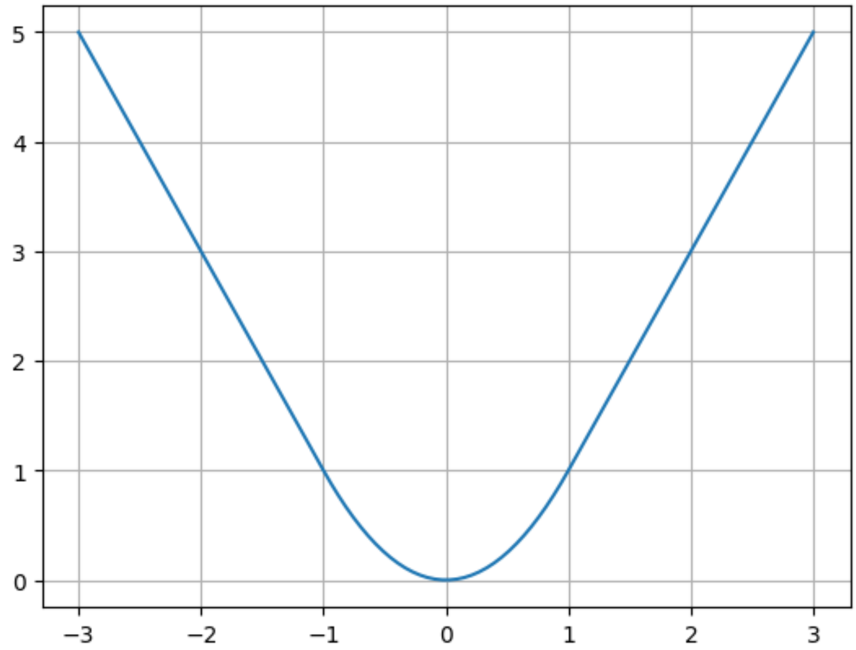
```
error_l2:115.98970497417054
error_l1:30.457700000000003
```

(c)

First plot the Huber loss function of the convex QP form:

In [8]:

```
using JuMP, Gurobi, PyPlot

M = 1

X = linspace(-3, 3, 100)
y = []

m = Model(solver=GurobiSolver(OutputFlag=0))

function HuberLoss(x)
    @variable(m, v >= 0)
    @variable(m, w <= M)

    @constraint(m, x <= w+v)
    @constraint(m, -w-v <= x)

    @objective(m, Min, w^2 + 2*M*v)

    solve(m)

    return getobjectivevalue(m)
end

for x in X
    push!(y, HuberLoss(x))
end

plot(X, y)
grid("on")
```




```
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```

Then find the best linear fit to the data using Huber loss with M=1.
The optimization problem can be written as follows:

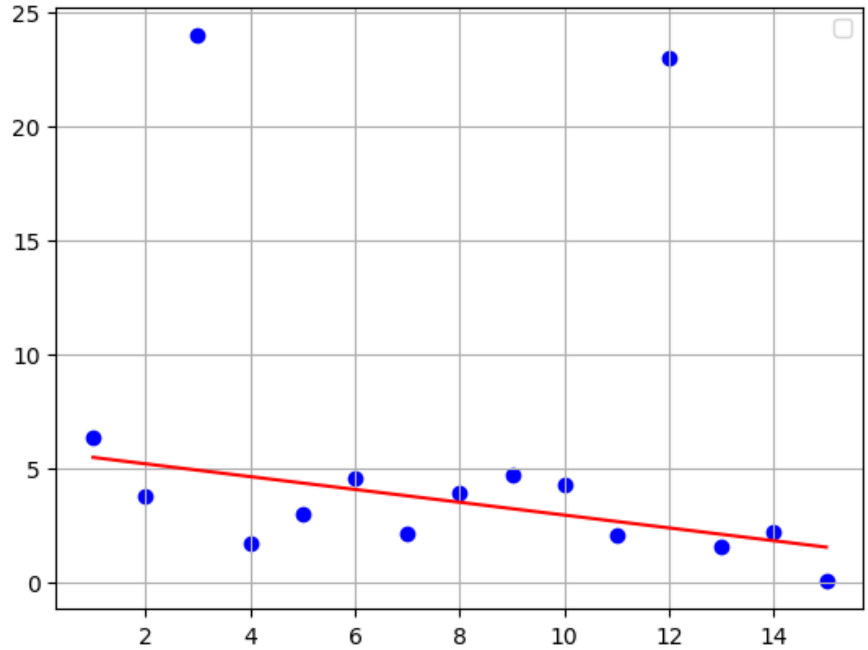
$$\min_{a,b,v,w} \sum_{i=1}^{15} {w_i}^2 + 2Mv_i$$

$$s.t.\ |y_i - ax_i - b| \leq w_i + v_i \quad for\ i = 1:15$$

$$v_i \geq 0$$

$$w_i \leq M$$

In [10]:
```
x = [1:15;]
y = [6.31 3.78 24 1.71 2.99 4.53 2.11 3.88 4.67 4.25 2.06 23 1.58 2.17 0.02]

using JuMP, Gurobi, PyPlot

M = 1
mh = Model(solver=GurobiSolver(OutputFlag=0))

@variable(mh, a)
@variable(mh, b)
@variable(mh, v[1:15] >= 0)
@variable(mh, w[1:15] <= M)

for i in 1:15
    @constraint(mh, y[i]-a*x[i]-b <= w[i]+v[i])
    @constraint(mh, -w[i]-v[i] <= y[i]-a*x[i]-b)
end

@objective(mh, Min, sum{w[i]^2+2*M*v[i], i=1:15})

solve(mh)

a = getvalue(a)
b = getvalue(b)

println("best fit of Huber loss(include the outliers):")
println("a:", a)
println("b:", b)

scatter(x, y, color="blue")
plot(x, a*x+b, color="red")
legend()
grid("on")
```




```
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best fit of Huber loss(include the outliers):
a:-0.2811079944855355
b:5.738120618284127
```

3. Heat pipe design

(a)

The flow of the fluid depends on the flow velocity and the area of the cross section, so it can be expressed as follows:

$$\alpha_4 Tr^2 * \pi r^2 = \alpha_4 \pi Tr^4$$

So the geometric program is:

$$\begin{aligned} & \max_{T,r,w} \alpha_4 \pi Tr^4 \\ s.t.\ & T_{min} \leq T \leq T_{max} \\ & r_{min} \leq r \leq r_{max} \\ & w_{min} \leq w \leq w_{max} \\ & w \leq 0.1r \\ & \alpha_1 \frac{Tr}{w} + \alpha_2 r + \alpha_3 rw \leq C_{max} \end{aligned}$$

Then it can be written as:

$$\begin{aligned} & -\alpha_4 \pi \ \min_{T,r,w} -Tr^4 \\ s.t.\ & \frac{T_{min}}{T} \leq 1 \quad \frac{T}{T_{max}} \leq 1 \\ & \frac{r_{min}}{r} \leq 1 \quad \frac{r}{r_{max}} \leq 1 \\ & \frac{w_{min}}{w} \leq 1 \quad \frac{w}{w_{max}} \leq 1 \\ & \frac{10w}{r} \leq 1 \\ & \frac{\alpha_1 Tr}{C_{max} w} + \frac{\alpha_2 r}{C_{max}} + \frac{\alpha_3 rw}{C_{max}} \leq 1 \end{aligned}$$

Apply log to both left hand side and right hand side, we will have:

$$\begin{aligned} & -\alpha_4 \pi \ \min_{T,r,w} -(\log T + 4 \log r) \\ s.t.\ & \log T_{min} - \log T \leq 0 \quad \log T - \log T_{max} \leq 0 \\ & \log r_{min} - \log r \leq 0 \quad \log r - \log r_{max} \leq 0 \\ & \log w_{min} - \log w \leq 0 \quad \log w - \log w_{max} \leq 0 \\ & \log 10 + \log w - \log r \leq 0 \\ & \log(e^{\log \frac{\alpha_1}{C_{max}} + \log T + \log r - \log w} + e^{\log \frac{\alpha_2}{C_{max}} + \log r} + e^{\log \frac{\alpha_3}{C_{max}} + \log r + \log w}) \leq 0 \end{aligned}$$

Let's move on, let $x = \log T$, $y = \log r$ and $z = \log w$, we will get:

$$
\begin{aligned}
&- \alpha_4 \pi \min_{x,y,z} -(x+4y) \\
s.t.\ & \log T_{min} - x \leq 0 \quad x - \log T_{max} \leq 0 \\
& \log r_{min} - y \leq 0 \quad y - \log r_{max} \leq 0 \\
& \log w_{min} - z \leq 0 \quad z - \log w_{max} \leq 0 \\
& \log 10 + z - y \leq 0 \\
& \log \left(e^{\log \frac{\alpha_1}{C_{max}} + x + y - z} + e^{\log \frac{\alpha_2}{C_{max}} + y} + e^{\log \frac{\alpha_3}{C_{max}} + y + z}\right) \leq 0
\end{aligned}
$$

It's a convex optimization problem.

(b)

Assume that each variable has a lower bound 0 and no upper bound and let $C_{max} = 500$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$, we will have:

$$
\begin{aligned}
&- \pi \min_{x,y,z} -(x+4y) \\
s.t.\ & \log 10 + z - y \leq 0 \\
& \log \left(e^{\log \frac{1}{500} + x + y - z} + e^{\log \frac{1}{500} + y} + e^{\log \frac{1}{500} + y + z}\right) \leq 0
\end{aligned}
$$

In [18]:
```
using JuMP, Mosek

m = Model(solver=MosekSolver(LOG=0))

@variable(m, x)
@variable(m, y)
@variable(m, z)

@constraint(m, log(10) + z - y <= 0)
@NLconstraint(m, log(exp(-log(500) + x + y -z) + exp(-log(500) + y) + exp(-

@objective(m, Min, -(x + 4y))

solve(m)

x = getvalue(x)
y = getvalue(y)
z = getvalue(z)

println("x:", x)
println("y:", y)
println("z:", z)

println()

T = exp(x)
r = exp(y)
w = exp(z)

println("optimal:")
println("T:", T)
println("r:", r)
println("w:", w)
```

```
x:-1.3862942448724969
y:5.52146090797915
z:-0.6931472456873297

optimal:
T:0.25000002906185015
r:249.99999752922588
w:0.49999996743630887
```

Heat($\alpha_4 \pi T r^4$):

In [19]:
```
pi*T*(r^4)
```

Out[19]: 3.0679618111299872e9

In [20]:
```
T*r/w + r + r*w
```

Out[20]: 500.0000095893781