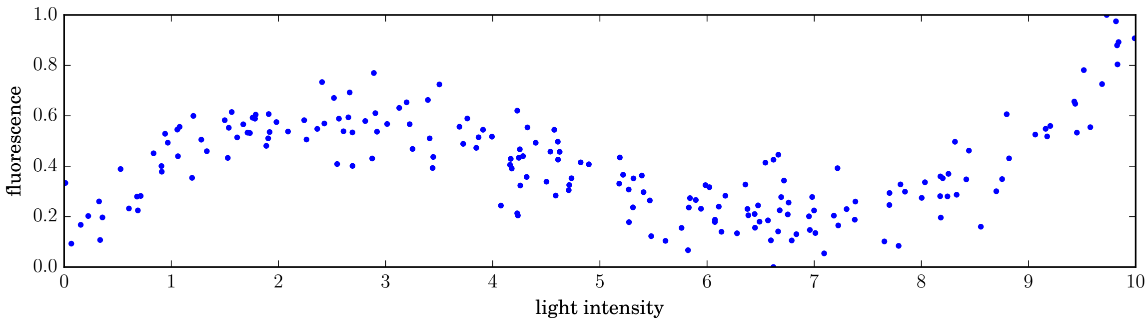
Homework 4: Least squares

Due date: 11:00pm on Sunday March 4, 2018
See the course website for instructions and submission details.

1. Spline fitting. We are running a series of experiments to evaluate the properties of a new fluorescent material. As we vary the intensity of the incident light, the material should fluoresce different amounts. Unfortunately, the material isn't perfectly uniform and our method for measuring fluorescence is not very accurate. After testing 200 different intensities, we obtained the result below (also available in `xy_data.csv`). The intensities x_i and fluorescences y_i are recorded in the first and second columns of the data matrix, respectively.



The material has interesting nonlinear properties, and we would like to characterize the relationship between intensity and fluorescence by using an approximate model that agrees well with the trend of our experimental data. Although there is noise in the data, we know from physics that *the fluorescence must be zero when the intensity is zero.* This fact must be reflected in all of our models!

a) Polynomial fit. Find the best cubic polynomial fit to the data. In other words, look for a function of the form $y = a_1x^3 + a_2x^2 + a_3x + a_4$ that has the best possible agreement with the data. Remember that the model should have (exactly) zero fluorescence when the intensity is zero! Include a plot of the data along with your best-fit cubic on the same axes.

b) Spline fit. Instead of using a single cubic polynomial, we will look for a fit to the data using two quadratic polynomials. Specifically, we want to find coefficients p_i and q_i so that our data is well modeled by the piecewise quadratic function:

$$y = \begin{cases} p_1x^2 + p_2x + p_3 & \text{if } 0 \le x < 4 \\ q_1x^2 + q_2x + q_3 & \text{if } 4 \le x < 10 \end{cases}$$

These quadratic functions must be designed so that:

- as in the cubic model, there is zero fluorescence when the intensity is zero.
- both quadratic pieces have the same value at $x = 4$.
- both quadratic pieces have the same slope at $x = 4$.

In other words, we are looking for a *smooth* piecewise quadratic. This is also known as a *spline* (this is just one type of spline, there are many other types). Include a plot of the data along with your best-fit model.

2. Moving averages. There are many ways to model the relationship between an input sequence $\{u_1, u_2, \dots\}$ and an output sequence $\{y_1, y_2, \dots\}$. In class, we saw the *moving average* (MA) model, where each output is approximated by a linear combination of the k most recent inputs:

$$\text{MA:} \qquad y_t \approx b_1 u_t + b_2 u_{t-1} + \cdots + b_k u_{t-k+1}$$

We then used least-squares to find the coefficients $b_1, \dots, b_k$. What if we didn't have access to the inputs at all, and we were asked to predict future y values based *only* on the previous y values? One way to do this is by using an *autoregressive* (AR) model, where each output is approximated by a linear combination of the ℓ most recent outputs (excluding the present one):

$$\text{AR:} \qquad y_t \approx a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_\ell y_{t-\ell}$$

Of course, if the inputs contain pertinent information, we shouldn't expect the AR method to outperform the MA method!

a) Using the same dataset from class `uy_data.csv`, plot the true y, and on the same axes, also plot the estimated $\hat{y}$ using the MA model and the estimated $\hat{y}$ using the AR model. Use $k = 5$ for both models. To quantify the difference between estimates, also compute $\|y - \hat{y}\|$ for both cases.

b) Yet another possible modeling choice is to combine both AR and MA. Unsurprisingly, this is called the *autoregressive moving average* (ARMA) model:

$$\text{ARMA:} \qquad y_t \approx a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_\ell y_{t-\ell} + b_1 u_t + b_2 u_{t-1} + \cdots + b_k u_{t-k+1}$$

Solve the problem once more, this time using an ARMA model with $k = \ell = 1$. Plot y and $\hat{y}$ as before, and also compute the error $\|y - \hat{y}\|$.

3. Hovercraft rendezvous. Alice and Bob are cruising on Lake Mendota in their hovercrafts. Each hovercraft has the following dynamics:

$$\text{Dynamics of each hovercraft:} \qquad \begin{aligned} x_{t+1} &= x_t + \tfrac{1}{3600} v_t \\ v_{t+1} &= v_t + u_t \end{aligned}$$

At time t (in seconds), $x_t \in \mathbb{R}^2$ is the position (in miles), $v_t \in \mathbb{R}^2$ is the velocity (in miles per hour), and $u_t \in \mathbb{R}^2$ is the thrust in normalized units. At $t = 1$, Alice has a speed of 20 mph going North, and Bob is located half a mile East of Alice, moving due East at 30 mph. Alice and Bob would like to rendezvous at exactly $t = 60$ seconds. The location where they meet is up to you.

a) Find the sequence of thruster inputs for Alice (u^{A}) and Bob (u^{B}) that achieves a rendezvous at $t = 60$ while minimizing the total energy used by both hovercraft:

$$\text{total energy} = \sum_{t=1}^{60} \left\|u_t^{\mathrm{A}}\right\|^2 + \sum_{t=1}^{60} \left\|u_t^{\mathrm{B}}\right\|^2$$

Plot the trajectories of each hovercraft to verify that they do indeed rendezvous.

b) In addition to arriving at the same place at the same time, Alice and Bob should also make sure their velocity vectors match when they rendezvous (otherwise, they might crash!) Solve the rendezvous problem again with the additional velocity matching constraint and plot the resulting trajectories. Is the optimal rendezvous location different from the one found in the first part?