This weeks problem set focuses isomorphisms and coordinate vectors and the matrices associated to linear transformations. It will be quite a large problem set, and because of the way we will be covering it in class, don't worry if you can't do some of the problems until after next Friday. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 2: due Friday 9 Feb: questions 4 and 5 below.

1. From section 2.4, problems 1, $2a, c, e$, 3, 7, 14, 15^*, 17^*, $24^{*,\dagger}$.
2. From section 2.2, problems 1, $2a, c, f$, 10, $11^\dagger$, 12^*, $14^\dagger$, 16.
3. From section 2.3, problems 1, $2a$, 3, 12, 16, $17^\dagger$, 16.

   There are mathematical objects called $\mathfrak{sl}_2$*-representations* which are important in quantum mechanics and beautiful objects in their own right. We won't define what they are exactly**, but their are vector spaces that come packaged with a certain pair of linear maps. The next questions give an example.

4.† Let $V = \mathbb{C}[x, y]$ be the vector space of polynomials in two variables. So we have $x^2 - 2xy^2 + 1 \in V$ for example. Define two linear maps $E, F : V \longrightarrow V$ where

$$E(p) = x\frac{\partial p}{\partial y} \text{ and } F(p) = y\frac{\partial p}{\partial x}$$

(a) Find a formula for $H := EF - FE$.

Solution: We just calculate what H does to a polynomial, using the chain rule:

$$\begin{aligned} H(p) &= EF(p) - FE(p) \\ &= x\frac{\partial}{\partial y}y\frac{\partial p}{\partial x} - y\frac{\partial}{\partial x}x\frac{\partial p}{\partial y} \\ &= x\frac{\partial p}{\partial x} + xy\frac{\partial^2 p}{\partial y \partial x} - y\frac{\partial p}{\partial y} - xy\frac{\partial^2 p}{\partial y \partial x} \\ &= x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y}. \end{aligned}$$

(b) A subspace $U \subset V$ is called a *subrepresentation* if $E(U) \subset U$ and $F(U) \subset U$. Let $V(n) = \text{span}\{\, x^{n-a}y^a \mid 0 \le a \le n \,\}$, this is the space of *homogeneous polynomials of degree* n, i.e. every term on the polynomial has degree n. Show that $V(n)$ is a subrepresentation, for any $n \ge 0$.

Solution: Note that $E(x^{n-a}y^a) = ax^{n-a+1}y^{a-1} \in V(n)$ and $F(x^{n-a}y^a) = (n-a)x^{n-a-1}y^{a+1} \in V(n)$. Thus, since an arbitrary element $p \in V(n)$ is simply a linear combination of these, we have that $E(p), F(p) \in V(n)$ and hence it is a subrepresentation.

(c) With the basis $x^n, x^{n-1}y, x^{n-2}y^2, \ldots, y^n$, determine the matrix corresponding to the linear maps E, H, F restricted to the subspaces $V(n)$.

Solution: We will do H first. Note that $H(x^{n-a}y^a) = (n-2a)x^{n-a}y^a$. Hence the matrix for H is diagonal with the (i, i)-entry being $n - 2(i - 1)$.
Now $E(x^{n-a}y^a) = ax^{n-a+1}y^{a-1}$ and so the matrix for E is zero everywhere, apart from the $(i, i+1)$-entry which is i.
Similarly, $F(x^{n-a}y^a) = (n-a)x^{n-a-1}y^{a+1}$ and so the matrix for F is zero everywhere, apart from the $(i+1, i)$-entry which is $n - i + 1$.

Examples for $n = 3$ are

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

5.† Another example of an $\mathfrak{sl}_2$ representation is given by $W = \mathbb{C}^2$ and where E' and F' are the linear transformations given by left multiplication by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find an isomorphism $\theta : V(1) \longrightarrow W$ so that $\varphi E = E'\varphi$ and $\varphi F = F'\varphi$ as linear maps $V(1) \longrightarrow W$.

6.† Show that there is no, nonzero, linear map $\theta : V(n) \longrightarrow V(m)$ so that $E\theta = \theta E$ and $F\theta = \theta F$ whenever $n \neq m$. *Hint: if such a map does exist, where does x^n get sent? Now use that $H\theta = \theta H$. This is pretty hard, let me know if you need more hints*

** Ok, if you really want to know exactly what they are here is the definition: An $\mathfrak{sl}_2$-representation is a vector space V with two linear maps $E, F : V \longrightarrow V$ such that

$$E^2F - 2EFE + FE^2 = -2E$$

and the same equation with the E's and F's swapped. There is a much more intuitive definition but one would need to know some more abstract algebra. If you are really keen, try and find more $\mathfrak{sl}_2$ representations and show me!