Math 3B: Lecture 16

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February 15, 2019

Differential equations (motivation)

An (ordinary) differential equation (or ODE) is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

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The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

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And so on.

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Note

The right hand side of the equation does not have any y's.

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And you'll be able to

- draw solutions for many other ODEs
- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)
- understand how sensitive ODEs are to their parameters.

Initial value problems

- If we try to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - \sin t$$

we get (by integrating)

$$y(t) = t^3 + \cos t + C.$$

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- The extra piece of data we need is called an "initial value".
- E.g. $y(0) = 2$.
- Then we see that $y(0) = 1 + C$, so $C = 1$.

How to think about differential equations

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- Imagine a point starting at $(t = 0, y = 1)$.
- If we want to draw the graph of $y(t)$ then we look at $g(0, 1)$.
- If this is positive we go up, negative we go down!

Modelling using differential equations

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- The goal is to write down a function $y(t)$ that describes something we are interested in (e.g. population/mass/etc)
- as some other variable changes (usually time)
- We can't do this directly, but we can write down an ODE that y satisfies instead.

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- Number of deaths is proportional to the total number of people. So

$$dN(t) \text{ deaths per year, for some } d$$

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The total change in population at time t is

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= \text{ births at } t - \text{ deaths at } t \\ &= bN(t) - dN(t) \\ &= (b-d)N(t).\end{aligned}$$

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In real life we would determine b and d experimentally. Let $r = b - d$. the instinsic growth rate. So our model is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

and we know $N(0) = 100$.

Behaviour of solutions

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

Case 1: $r = 0$

The population never grows or shrinks, it always stays the same (so $N(t) = 100$ for all t).

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The population is increasing indefinitely.

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Case 2: $r > 0$
The population is increasing indefinitely.

Case 3: $r < 0$
The population is decreasing indefinitely.

Solution to a simple ODE

Theorem

For any constant a, if y is a solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay$$

then y is given by

$$y = Ce^{ax}$$

for some constant C.

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Next time

We will see why, but for now we can verify it is actually a solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} Ce^{ax} = C\frac{\mathrm{d}}{\mathrm{d}x} e^{a}x = Cae^{ax} = ay.$$

Back to example 1

We know our population model was governed by

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$$100 = Ce^0 \quad \text{so} \quad C = 100.$$

Logistic growth

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Where $K = r/k$.

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$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= bN - (d + kN)N \\ &= (b - d - kN)N = (r - kN)N \\ &= r\left(1 - \frac{kN}{r}\right)N = r\left(1 - \frac{N}{K}\right)N\end{aligned}$$

Where $K = r/k$.

Logistic growth

The equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

is called the Logistic equation and K is the carrying capacity.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 1. $N(0) = 0$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains 0.

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Key takeaway

Both $N(t) = 0$ and $N(t) = K$ are solutions to the ODE. They are called equalibrium solutions.

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Case 3. $0 \leq N(0) \leq K$

In this case, N is initially increasing and so becomes more positive, slowing down as it gets close to K.

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Case 4. $N(0) \geq K$

In this case N is initially decreasing but decreases slower and slower as it gets close to K.

Checking solutions

The most straighforward way of checking a function $y = f(x)$ is a solution to a differential equation

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$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= e^{\sin x} \cos x \\ y \cos x &= e^{\sin x} \cos x \end{aligned}$$

Reminder: Implicit differentiation

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$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) &= \frac{\mathrm{d}}{\mathrm{d}x}1 \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) &= 0 \\ 2x + 2y'y &= 0 \end{aligned}$$

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Lets differentiate

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3 - y' \sin y = y + xy'.
\end{gathered}$$

Note

We can rearrange this to get

$$y' = \frac{3 - y}{x + \sin y}$$

a differential equation. Whatever y is, as long as it obeys the above relation, it is a solution to this ODE!

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1. moving all the y's over to the left:
$$\frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$
2. integrating both sides (with respect to x):
$$\int \frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int g(x)\,\mathrm{d}x.$$

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1. moving all the y's over to the left:
$$\frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$
2. integrating both sides (with respect to x):
$$\int \frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int g(x)\,\mathrm{d}x.$$
3. we can use the integration by substitution formula to rewrite the left hand side:
$$\int \frac{1}{f(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x.$$

Separation of variables

Separation of variables is a very powerful technique for solving ODEs. It works, whenever we can write out an ODE in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)g(x)$$

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1. moving all the y's over to the left:
$$\frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = g(x)$$
2. integrating both sides (with respect to x):
$$\int \frac{1}{f(y)}\frac{\mathrm{d}y}{\mathrm{d}x}\,\mathrm{d}x = \int g(x)\,\mathrm{d}x.$$
3. we can use the integration by substitution formula to rewrite the left hand side:
$$\int \frac{1}{f(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x.$$
4. solve for y!

Examples

On the board. . .