This week you will get practice drawing and understanding slope fields, making qualitative statements about solutions using them and some practice applying Euler's method.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.4.33) Consider the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{t}$$
   (a) verify that $y(t) = \ln t$ is a solution to this differential equation satisfying $y(1) = 0$.

   Solution: Clearly, if $y(t) = \ln t$ then $y(1) = \ln 1 = 0$ so the initial condition is satisfied. Now we only need to check that is satisfies the given differential equation. The left hand side is $y'(t) = 1/t$. And the right hand side is the same thing, so it is indeed a solution.

   (b) Use Euler's method to approximate $y(2) = \ln 2$ with $h = 0.5$.

   Solution: We use Euler's method to approximate $\ln 2$ by incrememting twice.

n	t_n	y_n	$y_{n+1} = y_n + \frac{h}{t_n}$
0	1	0	$y_1 = 0 + 0.5/1$
1	1.5	0.5	$y_2 = 0.5 + 0.5/1.5$
2	2	7/6	

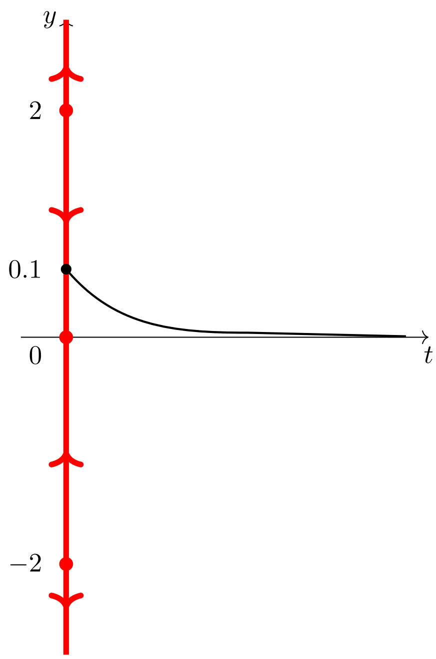
   So $\ln 2 \approx \frac{7}{6}$.

2. (6.5) Draw phase lines, classify the equilibria, and sketch a solution satisfying the specified initial value for the equations in the following.
   (a) (6.5-2) $\frac{\mathrm{d}y}{\mathrm{d}t} = 2 - 3y$, $y(0) = 2$
   (b) (6.5-5) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y - 10)(20 - y)$, $y(0) = 9$
   (c) (6.5-6) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y - 5)(25 - y)$, $y(0) = 7$
   (d) (6.5-7) $\frac{\mathrm{d}y}{\mathrm{d}t} = \sin y$, $y(0) = 0.1$
   (e) (6.5-10) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^3 - 4y$, $y(0) = 0.1$

   Solution: The equilibrium solutions will be found when $y^3 - 4y = 0$. I.e. when $y = 0$ or when $y = \pm 2$. We see that

   - when $y > 2$, then $y' > 0$,
   - when $0 < y < 2$, then $y' < 0$,
   - when $-2 < y < 0$, then $y' > 0$, and
   - when $y < -2$, then $y' < 0$.

   Thus the phase diagram is,



3. (6.5-33) To account for the effect of a generalist predator (with a type II functional response) on a population, ecologists often write differential equations of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0.1N\left(1 - \frac{N}{1,000}\right) - \frac{10N}{1+N}$$

   (a) Sketch the phase line for this system.

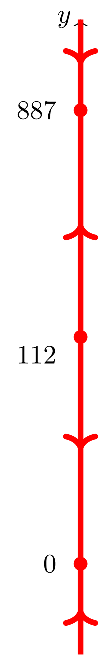
Solution: We can factorise the right hand side as

$$N\left(\frac{1}{10} - \frac{N}{10,000} - \frac{10N}{1+N}\right)$$

So we have an equilibrium at $N = 0$. The other factor can be put over a common denominator and expressed as

$$\frac{-99,000 + 999N - N^2}{10,000(1+N)}$$

To see where this is zero we use the quadratic equation and get $N \approx 112$ and 887. Thus we get the picture



(b) Discuss how the fate of the population depends on its initial abundance.

Solution: We can see from the phase diagram that if the population is initially between 0 and 112 then the population will eventually die out.

If the poulation is initially greater than 112 it will eventually stabilise at 887.

Hint: dont worry about what the first sentence means, you dont need to know where the differential equation comes from.

4. (6.5-39) Consider a population of clonally reproducing individuals consisting of two genotypes, a and A, with per capita growth rates, r_a and r_A, respectively. If N_a and N_A denote the densities of genotypes a and A, then

$$\frac{\mathrm{d}N_a}{\mathrm{d}t} = r_a N_a \qquad \frac{\mathrm{d}N_A}{\mathrm{d}t} = r_A N_A$$

Also, let $y = \frac{N_a}{N_a+N_A}$ be the fraction of individuals in the population that are genotype a. Show that y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (r_a - r_A)y(1-y)$$

5. (6.5-40) In the Hawk-Dove replicator equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{2}(1-y)(C(1-y)-V)$$

if the value $V > 0$ is specified, then find the range of values of C (in terms of V) that will ensure a polymorphism exists (i.e., find conditions that ensure the existence of an equilibrium $0 < y^* < 1$ that is stable).

(Hint: you do not need to know anything about the Hawk-Dove Replicator - though it is very interesting! - all you need to know is that V is a constant and C is a parameter. A polymorphism is a stable equilibrum between zero and one.)

6. (6.5-41) Production of pigments or other protein products of a cell may depend on the activation of a gene. Suppose a gene is *autocatalytic* and produces a protein whose presence activates greater production of that protein. Let y denote the amount of the protein (say, micrograms) in the cell. A basic model for the rate of this self-activation as a function of y is

$$A(y) = \frac{ay^b}{k^b + y^b} \text{ micrograms/minute}$$

where a represents the maximal rate of protein production, $k > 0$ is a "half saturation" constant, and $b \geq 1$ corresponds to the number of protein molecules required to active the gene. On the other hand, proteins in the cell are likely to degrade at a rate proportional to y, say cy. Putting these two components together, we get the following differential equation model of the protein concentration dynamics:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay^b}{k^b + y^b} - cy$$

   (a) Verify that $\lim_{y\to\infty} A(y) = a$ and $A(k) = a/2$.

   (b) Verify that $y = 0$ is an equilibrium for this model and determine under what conditions it is stable.

   (Hint: the definition of autocatalytic is given in the question, it is a gene that produces a protein whose presence activate greater production of that protein.)

7. (6.5-42) Consider the model of an autocatalytic gene in Problem 41 with $b = 1$, $k > 0$, $a > 0$, and $c > 0$.

   (a) Sketch the phase line for this model when $ck > a$.

   (b) Sketch the phase line for this model when $ck < a$.