Taylor series

Recall that if we have a function $f(x)$ we can approximate it using its Taylor polynomials $T_N(x)$ centered at c. We said these approximations get better and better.

Recall

$$T_N(x) = \sum_{n=0}^{N} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

So if we want a power series for $f(x)$, a good guess ~~whou~~ would by

$$\lim_{N\to\infty} T_N(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

This ~~#~~ is the Taylor series for $f(x)$.

Thm If $f(x)$ is represented by a power series centered at c, for $|x-c| < R$ with $R > 0$ then that power series is the Taylor series.

Note: This does not mean the Taylor series is always a representation of $f(x)$! It only means, if it is a rep, then it is unique.

Ex Find the Taylor series of e^x.

$$\frac{d^n}{dx^n} e^x = e^x \quad \text{so} \quad \frac{d^n}{dx^n} e^x \bigg|_{x=0} = 1$$

Thus the Taylor series is

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

as expected.

Thm Suppose there exists a $K > 0$ such that

$$|f^{(k)}(x)| \leq K \quad \text{for all } k \geq 0 \text{ and } x \in (c-R, c+R)$$

then $f(x)$ is represented by its Taylor series.

Ex Find a power series for $\cos x$.

$f(x) = \cos x$ $\qquad f(0) = 1$

$f'(x) = -\sin x$ $\qquad f'(0) = 0$

$f''(x) = -\cos x$ $\qquad f''(0) = -1$

$f'''(x) = \sin x$ $\qquad f'''(0) = 0$

so $\quad f^{(2k)}(0) = (-1)^k \quad$ and $\quad f^{(2k+1)}(0) = 0.$

so the Taylor series is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

to prove this is the power series, we need to find a K such that

$$|f^{(k)}(x)| \leq K .$$

$K = 1$ will do.

<u>Def</u> for any number a, define the binomial coefficient

$$\binom{a}{n} = \frac{a(a-1)\dots(a-n+1)}{n!} \qquad \binom{a}{0} = 1$$

<u>Ex</u>

$$\binom{4}{2} = \frac{4 \cdot 3}{2 \cdot 1} = 6$$

$$\binom{1/2}{3} = \frac{\frac{1}{2} \cdot -\frac{1}{2} \cdot -\frac{3}{2}}{3 \cdot 2 \cdot 1} = \frac{3/2^3}{3 \cdot 2} = \frac{1}{2^4} = \frac{1}{16} .$$

Ex The Taylor series for $(1+x)^a$ for any number a:

$$f(x) = (1+x)^a \qquad f(0) = 1$$

$$f'(x) = a(1+x)^{a-1} \qquad f'(0) = a$$

$$\vdots \qquad \vdots$$

$$f^{(n)}(x) = \cancel{a(a-1)\dots(a-n+1)} \; a(a-1)\cdots(a-n+1)(1+x)^{a-n} \qquad f^{(n)}(x) = a(a-1)\cdots(a-n+1)$$

so

$$\frac{f^{(n)}(0)}{n!} = \frac{a(a-1)\dots(a-n+1)}{n!} = \binom{a}{n}$$

So the Taylor series is

$$\sum_{n=0}^{\infty} \binom{a}{n} x^n$$

Thm $(1+x)^a = \sum_{n=0}^{\infty} \binom{a}{n} x^n$ for $|x| < 1$.