Functions + inverses

- In one way or another, in every area of mathematics, functions are the central object of study.
- Why? In almost any application, we use some kind of function to represent the quantity we want to study
- eg. $P(t)$ = potulation at time t

  $T(t)$ = temperature over time

  $x(t)$ = position of an object over time

  etc
- Single variable calc is about the study of functions that take in a real number $x \in \mathbb{R}$ and spit out a real number $f(x) \in \mathbb{R}$.

- Mostly we are interested in continuous functions (i.e. a small change in the input produces only a small change in the output).
- This course will be about: a careful study of functions. We will ~~reas~~ realise, it is very difficult to write down examples of continuous functions

so we develop better methods ~~of~~ for representing functions eg.

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Def A function $f: D \longrightarrow R$ consists of three pieces of data

* A set D called the domain
* A set R called the codomain (or range)
* A rule, f, that assigns to every element x of D a single element $f(x)$ of R.

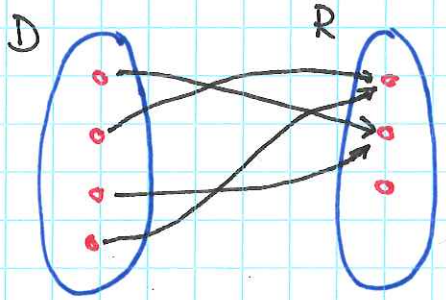
Rmk - Every one of these pieces of data is very important. The following three functions are all different

* $f: \mathbb{R} \longrightarrow \mathbb{R}$; $f(x) = x^2$
* $f: \mathbb{R} \longrightarrow \mathbb{R}_{\geq 0}$; $f(x) = x^2$
* $f: \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$; $f(x) = x^2$

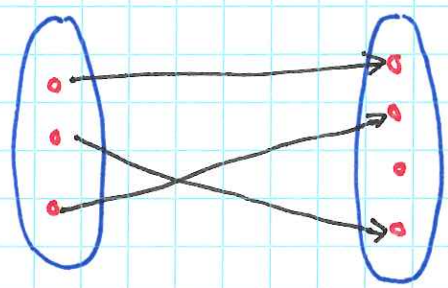
Def A function $f: D \to R$ is called

* **injective** if no two inputs, map to the same output; i.e. if $f(a) = f(b) \implies a = b$
* **surjective** if every output gets mapped to, i.e. if for ever $y \in R$ there exists an $x \in D$ such that $f(x) = y$
* **bijective** if it is both injective + surjective

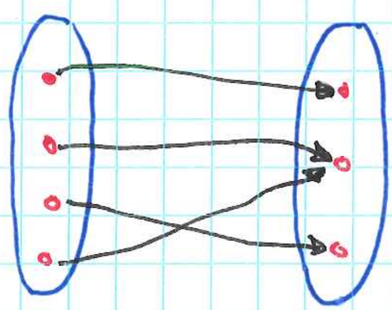
Some pictures



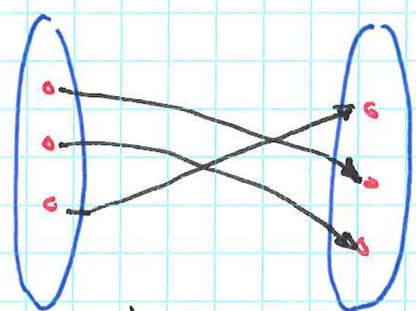
not injective
not surjective



injective
not surjective



not injective
surjective



injective
surjective

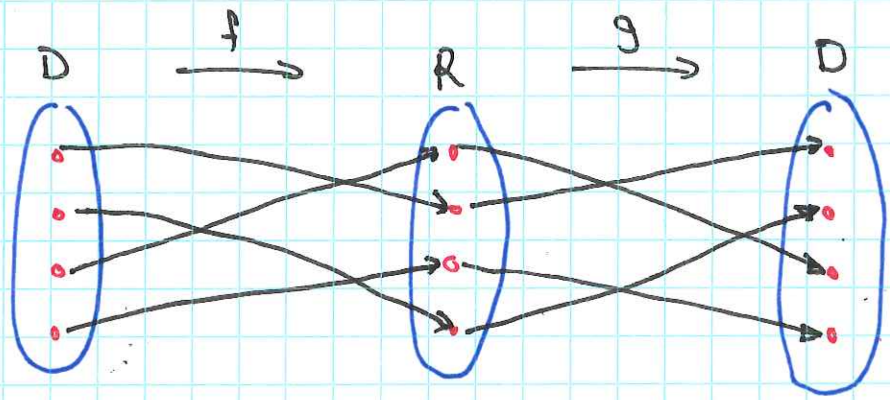
Def A function $g : R \longrightarrow D$ is an inverse to the function $f : D \longrightarrow R$ if

$$g(f(x)) = x$$

for every $x \in D$ and

$$f(g(y)) = y$$

for every $y \in R$.



Examples

* $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x + 1$

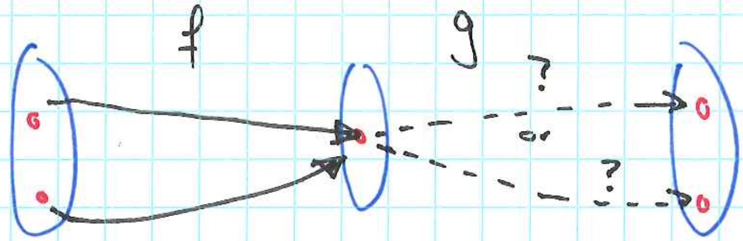
  $g : \mathbb{R} \to \mathbb{R}$; $g(x) = x - 1$

* $f : \mathbb{R} \to \mathbb{R}$; $f(x) = 3x - 1$

  $g : \mathbb{R} \to \mathbb{R}$; $g(x) = \frac{1}{3}(x + 1)$

- What about $f : \mathbb{R} \to \mathbb{R}$; $f(x) = x^2$. What is it's inverse? Where to send 4? $4 \xrightarrow{g}$ i.e. $g(4) = ?$ Obviously $g(4) = 2$. What about $f(-2)$? Well $f(-2) = 4$ so $g(f(-2)) = 2 \neq -2$ so f cannot have an inverse. What was the problem?

- f was not injective!



Fact f ~~only~~ has an inverse if and only if • f is bijective.

- Lets consider $f : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$; $f(x) = x^2$, then this has an inverse $g : \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ $g(x) = \sqrt{x}$