Midterm 2 practice

UCLA: Math 3B, Fall 2016

Instructor: Noah White
Date: Monday, November 21, 2016
Version: practice.

- This exam has 3 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____________________

ID number: ____________________

Discussion section: ____________________

Question	Points	Score
1	14	
2	12	
3	14	
Total:	40	

1. A hemispherical hole needs to be dug in order to plant a tree. The hole should be 2 m in diameter. Recall that the volume of a sphere, radius r, is given by

$$V = \frac{4}{3}\pi r^3.$$

(a) (1 point) What is the total volume of the hole?

Solution:

$$\frac{1}{2} \cdot \frac{4}{3}\pi = \frac{2}{3}\pi \text{ m}^3$$

(b) (1 point) Suppose we divide the hole into n slices, horizontally, of constant thickness Δh. What is Δh in terms of n?

Solution:

$$\Delta h = \frac{1}{n}$$

(c) (1 point) Let $h = 0$ be the top of the hole and $h = 1$ be the bottom. Let h_k be the depth of the top of the k^{th} slice (where the first slice is the 0^{th} slice), this $h_0 = 0$. What is h_k in terms of Δh and k?

Solution:

$$h_k = k\Delta h$$

(d) (2 points) At a depth of $h = h_k$, what is the radius of the hole?

Solution:

$$\sqrt{1 - h_k^2}$$

(e) (2 points) If n is large enough, we can approximate each slice by a cylinder of height Δh and radius as in part (d). Using this approximation, what is the volume of of the k^{th} slice?

Solution:

$$\pi(1 - h_k^2)\Delta h \text{ m}^3$$

(f) (3 points) The work needed to lift m kg of material, d meters up is given by

$$W = 10md$$

where we have approximated the acceleration due to gravity as 10 m/s^2. How much work needs to be done in order to lift the k^{th} slice out of the hole if we assume 1 m^3 of dirt weighs 1000 kg?

Solution:

$$10000\pi(1 - h_k^2)h_k\Delta h$$

(g) (2 points) Write a Riemann sum which represents the total amount of work needed to dig the dirt out of the hole.

Solution:

$$W = \lim_{n\to\infty} \Delta h \sum_{k=0}^{n-1} 10000\pi(1 - h_k^2)h_k$$

(h) (2 points) Use an integral to evaluate the Riemann sum above.

Solution:

$$\begin{aligned}
W &= \int_0^1 10000\pi(1 - h^2)h \; \mathrm{d}h \\
&= \left[10000\pi\left(\frac{1}{2}h^2 - \frac{1}{4}h^4\right)\right]_0^1 \\
&= 10000\pi\left(\frac{1}{2} - \frac{1}{4}\right) - 10000\pi(0) \\
&= 2500\pi
\end{aligned}$$

2. Solve the following differential equations. If no initial condition is given, find the general solution.

   (a) (4 points) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{2x+4}$.

   Solution:

   $$y = C\sqrt{2x-4}$$

   (b) (4 points) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{e^{y^2}}{y}$ where $y(1) = 0$.

   Solution:

   $$y = \sqrt{\ln\left(\frac{1}{3-2t}\right)}$$

   (c) (4 points) $\frac{\mathrm{d}y}{\mathrm{d}x} = e^{x+y}$ where $y(0) = 0$.

   Solution:

   $$y(x) = -\ln(2 - e^x)$$

3. A river flows into a small lake and another river flows out of the lake such that the lake has a constant volume of 2000 m^3 (the rate of water flowing in equals the rate of water flowing out). The river flowing into the lake contains a pollutant present at 0.5 $\mathrm{g/m}^3$. In this question you will model the total amount of pollutant, $y(t)$, present at time t (Note that $y(t)$ is the total amount of pollutant in the lake and not a concentration).

   (a) (1 point) Assume that the river flowing in, flows at a constant rate of 20 m^3/h. At what rate is the pollutant flowing into the lake (in mg/h)?

   Solution: Every hour there is 0.5 grams of pollutant entering the lake *per meter cubed* of water. Since there are 20 m^3 of water entering the lake every hour, there is 10 g/h of pollutant entering the lake.

   (b) (4 points) Under the above assumption, write a differential equation describing the change in the level of pollution in the lake.

   Solution: The differential equation will take the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \text{rate in } - \text{ rate out.}$$

   Thus we need to find the rate out. There are 20 m^3 flowing out every hour. At time t the concentration of pollutant in the lake is

$$\frac{y(t)}{2000}\ \mathrm{g/m}^3.$$

   Thus at time t there is

$$\frac{20y(t)}{2000} = \frac{y(t)}{100}\ \mathrm{g/h}$$

   of pollutant leaving the lake. Thus our differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 10 - \frac{y}{100}$$

(c) (3 points) Assuming that initially there is no pollutant in the lake, solve this differential equation.

Solution: We can use the general solution of the linear model to get

$$y(t) = 1000 - Ce^{-0.01t}.$$

We assume that $y(0) = 0$ to get that

$$0 = 1000 - C$$

so the final solution is

$$y(t) = 1000\left(1 - e^{-0.01t}\right)$$

(d) (5 points) Now assume that there is some seasonal variability and that the river flowing in (and thus also the river flowing out), flow at a rate of $40\sin^2 t$ m^3/h. Write and solve a differential equation to model this situation, assuming there is initially no pollution in the lake.

Solution: Here we repeat the analysis above with the changed assumption. At time t, there is $40\sin^2 t$ m^3 of water entering the lake every hour. Thus there is $20\sin^2 t$ g/h of pollutant entering the lake at time t.

Now, at time t, there is $y(t)$ grams of pollutant in the lake and thus the concentration of pollutant is

$$\frac{y(t)}{2000}\ \mathrm{g/m^3}.$$

Thus there is

$$\frac{40y(t)\sin^2 t}{2000} = \frac{y(t)\sin^2 t}{50}\ \mathrm{g/h}$$

flowing out of the lake. The differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 20\sin^2 t - \frac{y(t)\sin^2 t}{50} = \left(20 - \frac{y}{50}\right)\sin^2 t.$$

To solve this we separate variables and integrate

$$\int \frac{50}{1000-y}\,\mathrm{d}y = \int \sin^2 t\,\mathrm{d}t.$$

We use the hint to obtain

$$-50\ln(1000-y) = \frac{1}{2}(t - \sin(t)\cos(t)) + C.$$

Rearranging we get that the solution is

$$y(t) = 1000 - C\exp\left(-0.01(t - \sin(t)\cos(t))\right)$$

We can use the fact that $y(0) = 0$ to get

$$C = 1000$$

so the final solution is

$$y(t) = 1000\left(1 - e^{-0.01(t-\sin(t)\cos(t))}\right).$$

(e) (1 point) Compare the long term behaviour of the two solutions.

Solution: In the long term, both solution approach 1000 as the $\sin(t)\cos(t)$ term becomes insignificant.

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