This weeks problem set focuses on the ideas of bases and linear transformations. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 2: due end of February 5 January: questions 3, 4b and 6$b-d$ below.

1. From section 2.1, problems 15, 17, 18, 19, 24, 26*, 28, 31†, 40*.

2. From section 2.1, problems 1, 2, 5, 6, 9*, 14, 14b.

3.* Let V be a finite dimensional vector space over $\mathbb{F}$ and $B\{v_1, \ldots, v_n\}$ a basis. Let W be another vector space and $w_1, \ldots, w_n$ a collection of elements. Show that there is a unique linear map such that $T(v_i) = w_i$.

Solution: Since B is a basis, if we have any vector $v \in V$ we can write

$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n.$$

Define $T(v) = \lambda_1 T(v_1) + \ldots + \lambda_n T(v_n) = \lambda_1 w_1 + \cdots + \lambda_n w_n$. We can check that it is linear by supposing that $u, v \in V$ and that

$$\begin{aligned} u &= \mu_1 v_1 + \ldots + \mu_n v_n \\ v &= \lambda_1 v_1 + \ldots + \lambda_n v_n. \end{aligned}$$

Then we have $u + v = v = (\mu_1 + \lambda_1)v_1 + \ldots + (\mu_n + \lambda_n)v_n$, so

$$\begin{aligned} T(u+v) &= (\mu_1 + \lambda_1)w_1 + \ldots + (\mu_n + \lambda_n)w_n, \text{ but} \\ T(u) + T(v) &= \mu_1 w_1 + \ldots + \mu_n w_n + \lambda_1 w_1 + \cdots + \lambda_n w_n. \end{aligned}$$

Thus $T(u+v) = T(u) + T(v)$. Now lets suppose $\mu \in \mathbb{F}$, then $\mu v = \mu\lambda_1 v_1 + \cdots + \mu\lambda_n v_n$, so

$$\begin{aligned} T(\mu v) &= \mu\lambda_1 w_1 + \cdots + \mu\lambda_n w_n, \text{ but} \\ \mu T(v) &= \mu(\lambda_1 w_1 + \cdots + \lambda_n w_n). \end{aligned}$$

Thus $T(\mu v) = \mu T(v)$ and so T is linear.

To see that T is unique, suppose that $SLV \longrightarrow W$ is another linear map such that $S(v_i) = w_i$. Now let $v \in V$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then by linearity

$$\begin{aligned} S(v) &= S(\lambda_1 v_1 + \cdots + \lambda_n v_n) \\ &= \lambda_1 S(v_1) + \cdots + \lambda_n S(v_n) \\ &= \lambda_1 w_1 + \cdots + \lambda_n w_n = T(v). \end{aligned}$$

Thus $S = T$.

4.* Let V and W be vector spaces over $\mathbb{F}$. Define $\mathrm{Hom}(V, W)$ to be the set of linear maps from V to W.

(a) Show that $\mathrm{Hom}(V, W)$ is itself a vector space.

Solution: We define the linear map $T + S$ by $(T + S)(v) = T(v) + S(v)$ and λT by $(\lambda T)(v) = \lambda T(V)$. The zero element is the map $O : V \longrightarrow W$ defined by $O(v) = 0$. It is easy to check all of the axioms.

(b) If V is finite dimensional and B is a basis for V, construct a basis for $V^* = \mathrm{Hom}(V, \mathbb{F})$. The vector space V^* is called the *dual space* to V.

> **Solution:** Suppose $B = \{ v_1, \dots, v_n \}$. For any $1 \le i \le n$ define the linear map $\varepsilon_i : V \longrightarrow \mathbb{F}$ by $\varepsilon_i(v_j) = \delta_{ij}$. Here $\delta_{ij} = 0$ is $i \ne j$ and $\delta_{ii} = 1$.
> We claim that $B^* = \{ \varepsilon_1, \dots, \varepsilon_n \}$ is a basis for V^*. Here is a sketch of the proof. To see that B^* spans V^*, take an arbitrary linear map $\chi : V \longrightarrow \mathbb{F}$. One can see that this map is completely determined by the values $\chi(v_i)$. Then
>
> $$\chi = \chi(v_1)\varepsilon_1 + \ldots + \chi(v_n)\varepsilon_n.$$
>
> To see that it is a linearly independent subset, take an arbitrary linear combination
>
> $$\lambda_1 \varepsilon_1 + \lambda_n \varepsilon_n = 0.$$
>
> This means that by evaluating this map at v_i, we get
>
> $$\lambda_i = \lambda_1 \varepsilon_1(v_i) + \cdots + \lambda_n \varepsilon_n(v_i) = 0$$

5.* Let $T : V \longrightarrow W$ be an injective linear map. Show that, if we consider T, instead, as a linear map $V \longrightarrow \operatorname{im} T$ (just restrict what we consider to be the codomain), then it defines an isomorphism and shows that $V \cong \operatorname{im} T$.

6.* Let V and W be vector spaces over $\mathbb{F}$. Define the set

$$V \times W = \{ (v, w) \mid v \in V \text{ and } w \in W \}.$$

This is called the *product* of the vector spaces.

(a) Show that $V \times W$ is a vector space.

> **Solution:** We define addition and scalar multiplication componentwise. So $(v, w) + (v', w') = (v + v', w + w')$ and $\lambda(v, w) = (\lambda v, \lambda w)$. The axioms are now not hard to check.

(b) Define a map $\iota_V : V \to V \times W$ by $\iota_V(v) = (v, 0)$. Show that ι_V is an injective linear map. Note that we can define a similar map ι_W.

> **Solution:** We have
>
> $$\iota_V(v + w) = (v + w, 0) = (v, 0) + (w, 0) = \iota_V(v) + \iota_V(w)$$
>
> and
>
> $$\iota_V(\lambda v) = (\lambda v, 0) = \lambda(v, 0) = \lambda \iota_V(v).$$

(c) If $U \subset V$ is a subspace, show that $U \times W$ is a subspace of $V \times W$.

> **Solution:** Let $x, y \in U$, thus $x = (u, w)$ and $y = (u', w')$ for some $u, u' \in U$ and $w, w' \in W$. Then $x + y = (u + u', w + w') \in U \times W$. If $\lambda \in \mathbb{F}$, then $\lambda x = (\lambda u, \lambda w) \in U \times W$. Hence $U \times W$ is closed under scalar multiplication and addition and is thus a subspace.

(d) Show that $V \times W = (V \times \{ 0 \}) \oplus (\{ 0 \} \times W)$.

Solution: First of all, it is clear that $(V \times \{0\}) \cap (\{0\} \times W) = \{0\}$. Now observe that $(v,w) = (v,0) + (0,w)$ so $V \times W = (V \times \{0\}) + (\{0\} \times W)$.

Note that we can consider $V \times \{0\}$ as a copy of V in $V \times W$. For this reason, often mathematicians write $V \oplus W$ instead of $V \times W$ and call it the external direct product. Though this is a little confusing so we won't talk about it in this way in this class.

7.* (2.1.18) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\ker T = \operatorname{im} T$.

8.* (2.1.19) Give an example of distinct linear transformations T and U such that $\ker T = \ker U$ and $\operatorname{im} T = \operatorname{im} U$.