Math 3B: Lecture 8

Noah White

October 10, 2016

Quiz this week

There is a quiz this week!

- Get to class on time - it will start shortly after 2pm

Quiz this week

There is a quiz this week!

- Get to class on time - it will start shortly after 2pm
- Look at problems 1,2 (and have a quick look at 3)

Midterm 1 next week

The first midterm will run next Monday in the lecture

- Please arrive on time, no extra time will be granted

Midterm 1 next week

The first midterm will run next Monday in the lecture

- Please arrive on time, no extra time will be granted
- Review rules re calculators and cheatsheets

Midterm 1 next week

The first midterm will run next Monday in the lecture

- Please arrive on time, no extra time will be granted
- Review rules re calculators and cheatsheets
- A list of topics and practice questions will be avaiable towards the end of the week

Last time

Last time we looked at

- differential equations

Last time

Last time we looked at

- differential equations
- antiderivatives

Last time

Last time we looked at

- differential equations
- antiderivatives
- accumulated change

Last time

Last time we looked at

- differential equations
- antiderivatives
- accumulated change
- area under curves

Example 1

Question

A truck notices a car stopped on the road ahead and starts breaking hard, decelerating at a constant rate of $-8000\mathrm{km/h}^2$. If the truck was travelling at $100\mathrm{km/h}$, how far does it travel before it stops?

Example 1

Question

A truck notices a car stopped on the road ahead and starts breaking hard, decelerating at a constant rate of $-8000\mathrm{km}/\mathrm{h}^2$. If the truck was travelling at $100\mathrm{km}/\mathrm{h}$, how far does it travel before it stops?

Solution

The velocity of the truck is given by

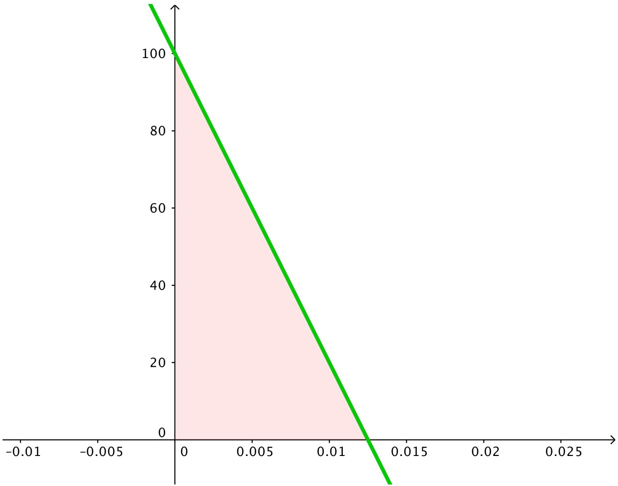
$$v(t) = 100 - 8000t$$

So the truct is stationary when $v(t) = 0$, i.e. when

$$100 = 8000t$$

so when $t = 0.0125$ hours.

Example 1



The area under the function is $0.5 \cdot 0.0125 \cdot 100 = 0.625\mathrm{km}$.

Example 2

Question

You measure the production of an oil well over the space of a 24 hour period by measuring the instantaneous flow of oil at

$$F(t) = 50 - \sqrt{144 - (x - 12)^2}$$

barrels per hour. How much oil did the well produce in the 24 hour period?

Example 2

Question

You measure the production of an oil well over the space of a 24 hour period by measuring the instantaneous flow of oil at

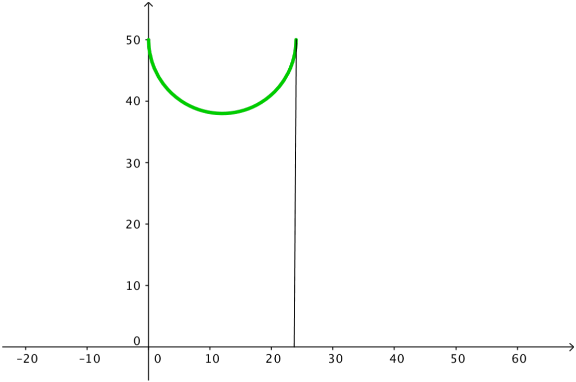
$$F(t) = 50 - \sqrt{144 - (x - 12)^2}$$

barrels per hour. How much oil did the well produce in the 24 hour period?

Solution

Lets graph the function

Example 2



The area is $50 \cdot 24 = 1200$ minus the area of the semicircle: $0.5 \cdot \pi \cdot 12^2 \approx 226$. So the total oil is $1200 - 226 = 974$ barrels.

Example 3

Question

Tracked over a year, a city has a birth rate of

$$b(t) = \begin{cases} 1000 + 50t & \text{when } t \in [0, 3] \\ 1330 - 60t & \text{when } t \in [3, 9] \\ 160 + 70t & \text{when } t \in [9, 12]. \end{cases}$$

The city also has a death rate of $d(t) = 990$ individuals per month. By how much will the city grow over the year?

Example 3

Question

Tracked over a year, a city has a birth rate of

$$b(t) = \begin{cases} 1000 + 50t & \text{when } t \in [0, 3] \\ 1330 - 60t & \text{when } t \in [3, 9] \\ 160 + 70t & \text{when } t \in [9, 12]. \end{cases}$$

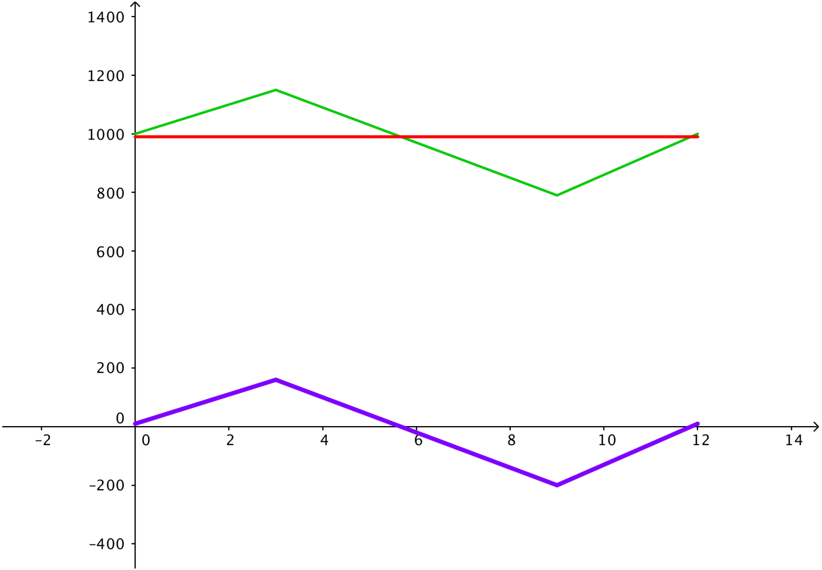
The city also has a death rate of $d(t) = 990$ individuals per month. By how much will the city grow over the year?

Solution

The net growth is given by

$$n(t) = b(t) - d(t) = \begin{cases} 10 + 50t & \text{when } t \in [0, 3] \\ 340 - 60t & \text{when } t \in [3, 9] \\ -830 + 70t & \text{when } t \in [9, 12]. \end{cases}$$

Example 3



Areas under general curves

We would like to calculate the area between a function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

Areas under general curves

We would like to calculate the area between a function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

A first approach to calculating the area under a curve is to approximate using rectangles:

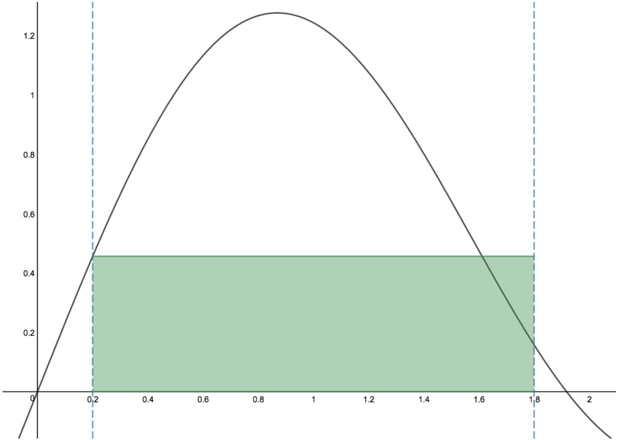
Areas under general curves

We would like to calculate the area between a function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

A first approach to calculating the area under a curve is to approximate using rectangles:

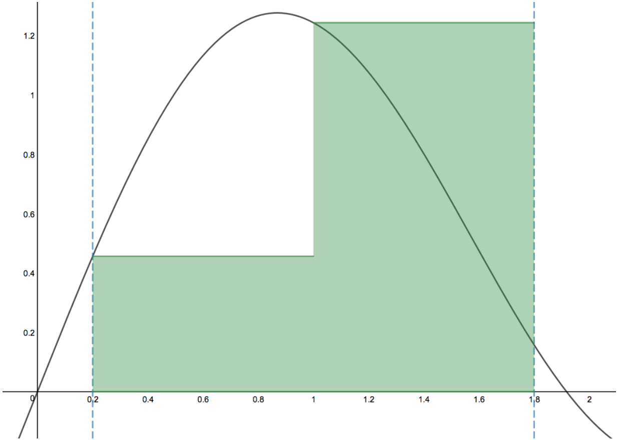
(Too hard to draw, lets look at an animation)

A general formula (n=1)



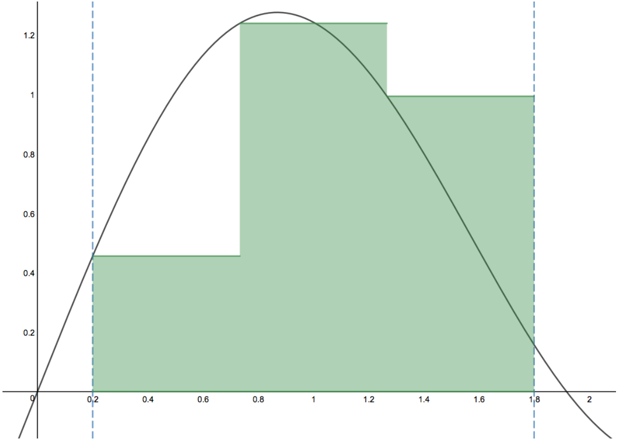
$$A = (b - a)f(a)$$

A general formula (n=2)



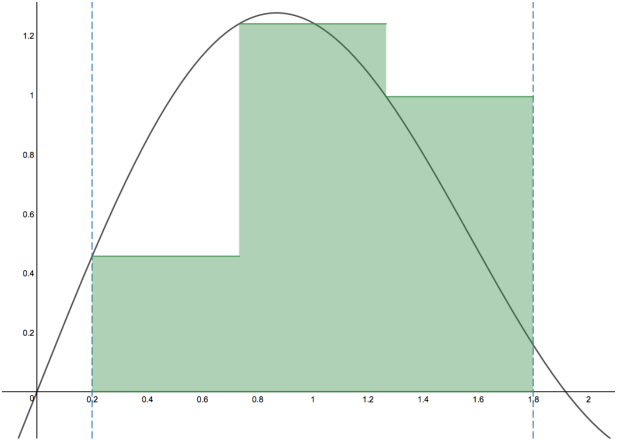
$$\Delta x = \frac{b-a}{2} \qquad A = \frac{b-a}{2} f(a) + \frac{b-a}{2} f(a + \Delta x)$$

A general formula (n=3)



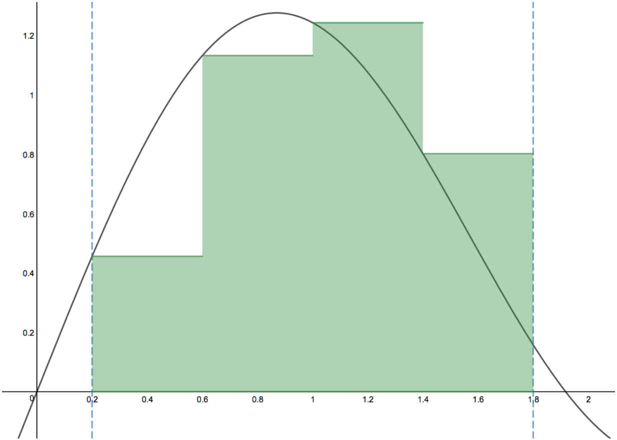
$$\Delta x = \frac{b-a}{3} \qquad A = \frac{b-a}{3} f(a) + \frac{b-a}{3} f(a+\Delta x) + \frac{(b-a)}{3} f(a+2\Delta x)$$

A general formula (n=3)



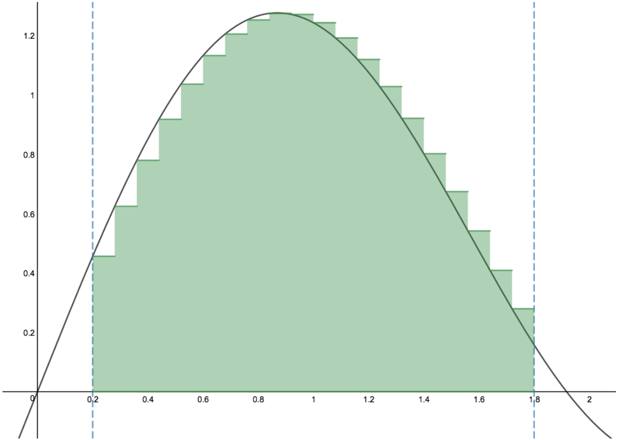
$$\Delta x = \frac{b-a}{3} \qquad A = \frac{b-a}{3}\left(f(a) + f\left(a + \Delta x\right) + f\left(a + 2\Delta x\right)\right)$$

A general formula (n=4)



$$A = \frac{b-a}{4}\left(f(a) + f\left(a + \Delta x\right) + f\left(a + 2\Delta x\right)\right) + f(a + 3\Delta x)$$

A general formula for the Riemann sum



$$\Delta x = \frac{b-a}{n} \qquad A = \frac{b-a}{n} \sum_{k=0}^{n-1} f(a + \Delta x)$$

The definite integral

Defintion

The definite integral of a function $f(x)$ is defined to be

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f(a + k \cdot \Delta x)$$

Two properties of the definite integral

These are just consequences of how area works!

Sums of areas

$$\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$$

Two properties of the definite integral

These are just consequences of how area works!

Sums of areas

$$\int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x$$

No area at all!

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

Some useful facts

Theorem

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{1}{2} n(n+1)$$

$$\sum_{k=1}^{n} k^2 = 1 + 2^2 + \cdots + n^2 = \frac{1}{6} n(n+1)(2n+1)$$

$$\sum_{k=1}^{n} k^3 = 1 + 2^3 + \cdots + n^3 = \frac{1}{4} n^2(n+1)^2$$

Example 1

Question

Evaluate the definite integral

$$\int_0^8 x \, \mathrm{d}x$$

Example 1

Question

Evaluate the definite integral

$$\int_0^8 x \, \mathrm{d}x$$

Solution

$$\int_0^8 x \, \mathrm{d}x = \lim_{n\to\infty} \frac{8-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{8}{n})$$

Example 1

Question

Evaluate the definite integral

$$\int_0^8 x \, \mathrm{d}x$$

Solution

$$\begin{aligned}\int_0^8 x \, \mathrm{d}x &= \lim_{n\to\infty} \frac{8-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{8}{n}) \\ &= \lim_{n\to\infty} \frac{8}{n} \sum_{k=0}^{n-1} \frac{8k}{n}\end{aligned}$$

Example 1

$$\int_0^8 x \, \mathrm{d}x = \lim_{n \to \infty} \frac{8-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{8}{n})$$

Example 1

$$\begin{aligned}\int_0^8 x \, \mathrm{d}x &= \lim_{n\to\infty} \frac{8-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{8}{n}) \\ &= \lim_{n\to\infty} \frac{8}{n} \sum_{k=0}^{n-1} \frac{8k}{n}\end{aligned}$$

Example 1

$$
\begin{aligned}
\int_0^8 x \, \mathrm{d}x &= \lim_{n\to\infty} \frac{8-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{8}{n}) \\
&= \lim_{n\to\infty} \frac{8}{n} \sum_{k=0}^{n-1} \frac{8k}{n} \\
&= \lim_{n\to\infty} \frac{8^2}{n^2} \sum_{k=0}^{n-1} k
\end{aligned}
$$

Example 1

$$\begin{aligned}
\int_0^8 x \, \mathrm{d}x &= \lim_{n\to\infty} \frac{8-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{8}{n}) \\
&= \lim_{n\to\infty} \frac{8}{n} \sum_{k=0}^{n-1} \frac{8k}{n} \\
&= \lim_{n\to\infty} \frac{8^2}{n^2} \sum_{k=0}^{n-1} k \\
&= \lim_{n\to\infty} \frac{8^2}{n^2} \cdot \frac{1}{2}(n-1)n
\end{aligned}$$

Example 1

$$\begin{aligned}
\int_0^8 x \, \mathrm{d}x &= \lim_{n\to\infty} \frac{8-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{8}{n}) \\
&= \lim_{n\to\infty} \frac{8}{n} \sum_{k=0}^{n-1} \frac{8k}{n} \\
&= \lim_{n\to\infty} \frac{8^2}{n^2} \sum_{k=0}^{n-1} k \\
&= \lim_{n\to\infty} \frac{8^2}{n^2} \cdot \frac{1}{2}(n-1)n \\
&= \lim_{n\to\infty} \frac{32(n-1)}{n}
\end{aligned}$$

Example 1

$$
\begin{aligned}
\int_0^8 x \,\mathrm{d}x &= \lim_{n\to\infty} \frac{8-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{8}{n}) \\
&= \lim_{n\to\infty} \frac{8}{n} \sum_{k=0}^{n-1} \frac{8k}{n} \\
&= \lim_{n\to\infty} \frac{8^2}{n^2} \sum_{k=0}^{n-1} k \\
&= \lim_{n\to\infty} \frac{8^2}{n^2} \cdot \frac{1}{2}(n-1)n \\
&= \lim_{n\to\infty} \frac{32(n-1)}{n} \\
&= 32
\end{aligned}
$$

Example 2

Question

Evaluate the definite integral

$$\int_0^1 x^2 - 4 \, \mathrm{d}x$$

Example 2

Question

Evaluate the definite integral

$$\int_0^1 x^2 - 4 \, \mathrm{d}x$$

Solution

$$\int_0^1 x^2 - 4 \, \mathrm{d}x = \lim_{n\to\infty} \frac{1-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{1}{n})$$

Example 2

Question

Evaluate the definite integral

$$\int_0^1 x^2 - 4 \,\mathrm{d}x$$

Solution

$$\int_0^1 x^2 - 4 \,\mathrm{d}x = \lim_{n\to\infty} \frac{1-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{1}{n})$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{k^2}{n^2} - 4$$

Example 2

$$\int_0^1 x^2 - 4 \, \mathrm{d}x = \lim_{n\to\infty} \frac{1-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{1}{n})$$

Example 2

$$\int_0^1 x^2 - 4 \,\mathrm{d}x = \lim_{n\to\infty} \frac{1-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{1}{n})$$

Example 2

$$\begin{aligned}\int_0^1 x^2 - 4 \,\mathrm{d}x &= \lim_{n\to\infty} \frac{1-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{1}{n}) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{k^2}{n^2} - 4 \\ &= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 - \frac{1}{n} \sum_{k=0}^{n-1} 4\end{aligned}$$

Example 2

$$
\begin{aligned}
\int_0^1 x^2 - 4 \,\mathrm{d}x &= \lim_{n\to\infty} \frac{1-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{1}{n}) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{k^2}{n^2} - 4 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 - \frac{1}{n} \sum_{k=0}^{n-1} 4 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{1}{6}(n-1)n(2n-1) - \frac{4n}{n}
\end{aligned}
$$

Example 2

$$
\begin{aligned}
\int_0^1 x^2 - 4 \,\mathrm{d}x &= \lim_{n\to\infty} \frac{1-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{1}{n}) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{k^2}{n^2} - 4 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 - \frac{1}{n} \sum_{k=0}^{n-1} 4 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{1}{6}(n-1)n(2n-1) - \frac{4n}{n} \\
&= \lim_{n\to\infty} \frac{2n^2 - 3n + 1}{6n^2} - 4
\end{aligned}
$$

Example 2

$$\begin{aligned}
\int_0^1 x^2 - 4 \,\mathrm{d}x &= \lim_{n\to\infty} \frac{1-0}{n} \sum_{k=0}^{n-1} f(0 + k \cdot \frac{1}{n}) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{k^2}{n^2} - 4 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 - \frac{1}{n} \sum_{k=0}^{n-1} 4 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \cdot \frac{1}{6}(n-1)n(2n-1) - \frac{4n}{n} \\
&= \lim_{n\to\infty} \frac{2n^2 - 3n + 1}{6n^2} - 4 \\
&= \frac{1}{3} - 4 = -\frac{11}{3}
\end{aligned}$$