Final practice 3

UCLA: Math 115A, Winter 2020

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Date:
Version: 1

- This exam has 6 questions, for a total of 60 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question 2 is multiple choice. Indicate your answers in the table below. *The following three pages will not be graded, your answers must be indicated here.*

Question	Points	Score
1	10	
2	10	
3	10	
4	9	
5	10	
6	11	
Total:	60	

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

Clarification on notation: Let $T : V \longrightarrow W$ be a linear map. The *kernel* of T is the same thing as the *nullspace* of T, i.e. $\ker T = \mathsf{N}(T)$. Similarly the *image* of T is the same thing as the *range* of T, i.e. $\operatorname{im} T = \mathsf{R}(T)$.

1. In each of the following questions, fill in the blanks to complete the statement of the definition or theorem.

   (a) (2 points) *Definition:* A subset $B \subset V$ of a vector space is called a *basis* if it is **linearly independent** and **spanning**.

   (b) (2 points) *Definition:* A scalar $\lambda \in \mathbb{F}$ is an *eigenvalue* of a linear map $T : V \longrightarrow V$ if there exists a **nonzero** vector $v \in V$ such that

   $$T(v) = \lambda v.$$

   (c) (2 points) *Definition:* Suppose $T : V \longrightarrow V$ is a linear operator on a finite dimensional vector space with an eigenvalue of λ. The λ-eigenspace is defined to be

   $$E_\lambda = \ker T - \lambda \, \mathrm{id}$$

   and the geometric multiplicity of λ is

   $$\dim E_\lambda.$$

   (d) (2 points) *Theorem:* Let V be a finite dimensional vector space over a field $\mathbb{F}$. A linear map $T : V \longrightarrow V$ is diagonalisable if and only if

   - **the characteristic polynomial splits**, and
   - for every eigenvalue $\lambda \in \mathbb{F}$, **the algebraic multiplicity equals** $\dim E_\lambda$.

   (e) (2 points) *Definition:* Let V be a finite dimensional inner product space. The adjoint of a linear map $T : V \longrightarrow V$ is the unique linear map $T^* : V \longrightarrow V$ such that for any $v, w \in V$ we have

   $$\langle T(v), w \rangle = \langle v, T^*(w) \rangle.$$

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) Consider the following subspace of $\mathbb{C}_3[x]$ (polynomials of degree at most 3),

$$U = \{p \in \mathbb{C}_3[x] \mid p(-1) = 0\}$$

The dimension of U is

A. 0.
B. 1.
C. 2.
D. 3.

(b) (2 points) As a subset of $\mathbb{R}^3$, the set

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\}$$

A. is a spanning set but not linearly independent.
B. is linearly independent but not spanning.
C. is neither spanning nor linearly independent.
D. is a basis.

(c) (2 points) A linear operator $T : V \longrightarrow V$ is called *idempotent* if $T^2 = T$. What eigenvalues can an idempotent operator possibly have?

A. Only 0.
B. Only 1.
C. 0 or 1.
D. It could have any eigenvalue.

(d) (2 points) Let $V = \mathbb{R}_1[x]$, be an inner product space with the inner product

$$\langle p, q\rangle = p(0)q(0) + p'(0)q'(0)$$

Consider the map $T : V \longrightarrow V$ given by $T(p) = 2p(\frac{1}{2}) + p(2)x$. Which of the following is *not* true.

A. T is a linear map.
B. T is self adjoint.
C. T has a basis of orthonormal eigenvectors.
D. T has an eigenspace of dimension 2.

(e) (2 points) Which of the following is *not* a linear map.

A. $P : \mathrm{Mat}_{m\times n}(\mathbb{F}) \longrightarrow \mathrm{Mat}_{n\times m}(\mathbb{F})$ such that $P(M) = M^t$.
B. $Q : \mathrm{Mat}_{n\times n}(\mathbb{F}) \longrightarrow \mathbb{F}$ such that $Q(M) = \det M$.
C. $R : \mathrm{Mat}_{n\times n}(\mathbb{F}) \longrightarrow \mathbb{F}$ such that $R(M) = \mathrm{tr}\, M$.
D. $S : \mathrm{Mat}_{m\times n}(\mathbb{F}) \longrightarrow \mathbb{F}^m$ such that $S(M) = Mv$, for a fixed $v \in \mathbb{F}^n$.

3. Consider the vector space over $\mathbb{R}$,

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x_1 - x_2 + x_3 - x_4 = 0 \right\}$$

and the linear map $T : V \longrightarrow V$ given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{pmatrix}$$

(a) (1 point) What is the characteristic polynomial of T?

Solution: We first need a basis of V. A convenient one, B, contains $v_1 = (1, 1, 0, 0), v_2 = (0, 0, 1, 1)$ and $v_3 = (1, 0, 0, 1)$. Then, $T(v_1) = v_1$, $T(v_2) = v_2$, $T(v_3) = v_1 + v_2 - v_3$. Thus the matrix is

$$[T]_B^B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

Hence we calculate that $p_T(t) = -(1 - t)^2(1 + t)$.

(b) (5 points) Compute the eigenvalues of T and their algebraic multiplicity.

Solution: 1 has algebraic multiplicity 2, and -1 has algebraic multiplicity 1.

(c) (2 points) Write down an eigenvector for each eigenspace.

> **Solution:** Both v_1, and v_2 are 1-eigenvectors. If we want a -1 eigenvector, we need to solve
>
> $$-av_1 - bv_2 - cv_3 = T(av_1 + bv_2 + cv_3) = (a+c)v_1 + (b+c)v_2 - cv_3$$
>
> from which we get that $-a = a + c, -b = b + c$ and $-c = -c$. Thus $2b = 2a = -c$. So $v_1 + v_2 - 2v_3$ is a -1-eigenvector.

(d) (2 points) Is T diagonalisable? If so, find a basis B such that $[T]_B^B$ is diagonal. If not, find B, so that the above matrix is upper triangular.

> **Solution:** Yes. From above we can see the geometric multiplicities match the geometric ones. The set $\{v_1, v_2, v_1 + v_2 - 2v_3\}$ is a basis of eigenvectors.

4. Consider the vector space $V = \mathbb{R}^3$ and the matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

We can define an inner product on V by

$$\langle v, w \rangle = v^t M w.$$

where v^t indicates the transpose. *Please note this is NOT the standard dot product. It is a different inner product.*

(a) (5 points) Apply the Gram-Schmidt process to the basis $E = \{e_1, e_2, e_3\}$ (the standard basis) to find an orthogonal basis B.

Solution: We begin by setting $w_1 = e_1$. Then $w_2 = e_2 - \frac{\langle e_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$. To calculate this note that $\langle e_2, w_1 \rangle = \langle e_2, e_1 \rangle = -1$ and $\langle ew_1, w_1 \rangle = \langle e_1, e_1 \rangle = 2$. So

$$w_2 = e_2 - \frac{-1}{2} e_1 = \frac{1}{2} e_1 + e_2$$

Now set

$$w_2 = e_3 - \frac{\langle e_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle e_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

Note that $\langle e_3, w_1 \rangle = \langle e_3, e_1 \rangle = 0$, and

$$\langle e_3, w_2 \rangle = \langle e_3, frac12 e_1 + e_2 \rangle = \frac{1}{2} \langle e_3, e_1 \rangle + \langle e_3, e_2 \rangle = -1$$

We also need

$$\langle w_2, w_2 \rangle = \langle \frac{1}{2} e_1 + e_2, \frac{1}{2} e_1 + e_2 \rangle = \frac{1}{4} \langle e_1, e_1 \rangle + \langle e_1, e_2 \rangle + \langle e_2, e_2 \rangle = \frac{1}{2} - 1 + 2 = \frac{3}{2}.$$

Thus

$$w_3 = e_3 - \frac{-1}{\frac{3}{2}} (\frac{1}{2} e_1 + e_2) = \frac{1}{3} e_1 + \frac{2}{3} e_2 + e_3.$$

(b) (4 points) Let $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Compute the coordinate vector $[v]^B$. *Note that B is not orthonormal.*

Solution:

$$\frac{\langle v, w_1 \rangle}{\langle w_1, w_1 \rangle} = \frac{\langle e_1 + e_2 + e_3, e_1 \rangle}{\langle e_1, e_1 \rangle} = \frac{1}{2}$$

$$\frac{\langle v, w_2 \rangle}{\langle w_2, w_2 \rangle} = \frac{\langle e_1 + e_2 + e_3, \frac{1}{2}e_1 + e_2 \rangle}{3/2} = \frac{1/2 + 0}{3/2} = \frac{1}{3}$$

$$\frac{\langle v, w_3 \rangle}{\langle w_3, w_3 \rangle} = \frac{\langle e_1 + e_2 + e_3, \frac{1}{3}e_1 + \frac{2}{3}e_2 + e_3 \rangle}{\langle \frac{1}{3}e_1 + \frac{2}{3}e_2 + e_3, \frac{1}{3}e_1 + \frac{2}{3}e_2 + e_3 \rangle} = \frac{1/3 + 0 + 1}{2/9 - 2/9 + 0 - 2/9 + 4/9 - 2/3 + 0 - 2/3 + 2}$$

$$= \frac{1/3}{8/9} = \frac{3}{8}$$

5. Let V be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $T : V \longrightarrow V$ be a normal linear operator (i.e. $T^*T = TT^*$).

   (a) (3 points) Prove for all $v \in V$ that $\|T(x)\| = \|T^*(v)\|$.

   Solution: $\|T(x)\|^2 = \langle T(x), T(x)\rangle = \langle T^*T(x), x\rangle = \langle TT^*(x), x\rangle = \langle T^*(x), T^*(x)\rangle = \|T^*(x)\|^2$

   (b) (3 points) Prove that $T - \alpha\,\mathrm{id}_V$ is normal for any $\alpha \in \mathbb{F}$

   Solution: First we note that $(T - \alpha\,\mathrm{id})^* = T^* - \overline{\alpha}\,\mathrm{id}$, thus

$$\begin{aligned}(T - \alpha\,\mathrm{id})(T - \alpha\,\mathrm{id})^* &= (T - \alpha\,\mathrm{id})(T^* - \overline{\alpha}\,\mathrm{id}) \\ &= TT^* - \overline{\alpha}T - \alpha T^* + \alpha\overline{\alpha}\,\mathrm{id} \\ &= T^*T - \overline{\alpha}T - \alpha T^* + \alpha\overline{\alpha}\,\mathrm{id} \\ &= (T^* - \overline{\alpha}\,\mathrm{id})(T - \alpha\,\mathrm{id}) \\ &= (T - \alpha\,\mathrm{id})^*(T - \alpha\,\mathrm{id}).\end{aligned}$$

   (c) (4 points) Prove that if v is a λ-eigenvector for T, then v is also a $\overline{\lambda}$-eigenvector for T^*. *Hint: use both previous parts.*

   Solution: Suppose v is a λ-eigenvector for T. Then by part a) and b) we have that

$$0 = \|(T - \lambda\,\mathrm{id})(v)\| = \|(T - \lambda\,\mathrm{id})^*(v)\| = \|(T^* - \overline{\lambda}\,\mathrm{id})(v)\|$$

   Thus $(T^* - \overline{\lambda}\,\mathrm{id})(v) = 0$ and so v is a $\overline{\lambda}$-eigenvector for T^*.

6. Let V be a finite dimensional vector space over a field $\mathbb{F}$, and $T : V \longrightarrow V$ a linear operator. Suppose that $T^n = 0$ for some $n > 1$ (well call T *nilpotent* in this case) but that $T^{n-1} \neq 0$. Fix a vector $x \in V$ such that $T^{n-1}(x) \neq 0$.

   (a) (2 points) What are the eigenvalues of T? Justify your answer.

   Solution: If λ is an eigenvalue, then there exists a nonzero $v \in V$ such that $T(v) = \lambda v$. Thus $0 = T^n(v) = \lambda^n v$. Hence $\lambda^n = 0$ which means that $\lambda = 0$. Thus, T has a single eigenvalue of 0.

   (b) (1 point) Is it possible for T to be an isomorphism? Justify your answer.

   Solution: No. It has a zero eigenvalue so there is a non zero vector in the kernel, so it is not injective.

   (c) (3 points) Suppose $n = 2$. Prove that $\{x, T(x)\}$ are linearly independent.

   Solution: Consider the equation $ax + bT(x) = 0$. Apply T. We get $aT(x) + bT^2(x) = aT(x) = 0$. Thus since $T(x) \neq 0$ we must have that $a = 0$. Thus $bT(x) = 0$ and so $b = 0$ too. Hence the set is linearly independent.

(d) (5 points) For any $n > 1$, prove that $\{x, T(x), T^2(x), \ldots, T^{n-1}(x)\}$ is linearly independent.

Solution: We will prove that the set $\{T^{n-k}(x), \ldots, T^{n-1}(x)\}$ is linearly independent for $k = 1, \ldots, n$. For $k = n$ we get our result. We do this by induction on k. The result is clearly true for $k = 1$. Thus let us assume we know the result is true for $k - 1$, i.e. the set $\{T^{n-k+1}(x), \ldots, T^{n-1}(x)\}$ is linearly independent.
Now consider the set $\{T^{n-k}(x), T^{n-k+1}(x), \ldots, T^{n-1}(x)\}$ and the linear combination

$$\lambda_{n-k}T^{n-k}(x) + \lambda_{n-k+1}T^{n-k+1}(x) + \cdots + \lambda_{n-1}T^{n-1}(x) = 0$$

Applying T to both sides we get

$$\lambda_{n-k}T^{n-k+1}(x) + \lambda_{n-k+1}T^{n-k+2}(x) + \cdots + \lambda_{n-2}T^{n-1}(x) = 0$$

But we know the set $\{T^{n-k+1}(x), \ldots, T^{n-1}(x)\}$ is linearly independent so we know $\lambda_{n-k} = \lambda_{n-k+1} = \cdots = \lambda_{n-1} = 0$. Thus the linear combination reduces to $\lambda_{n-1}T^{n-1}(x) = 0$, and since $T^{n-1}(x) \neq =$ we must have that $\lambda_{n-1} = 0$. Hence the result is true by induction.
Alternate solution: Consider a linear combination

$$\lambda_0 x + \lambda_1 T(x) + \cdots + \lambda_{n-1}T^{n-1}(x) = 0$$

We will prove, by induction on $k = 0, \ldots, n-1$, that $\lambda_k = 0$. First we start with the base case. Apply T^{n-1} to the above, so we get the linear combination

$$\lambda_0 T^{n-1}(x) + \lambda_1 T^n(x) + \cdots + \lambda_{n-1}T^{2n-2}(x) = 0\lambda_0 T^{n-1}(x) \qquad =$$

Thus, $\lambda_0 = 0$. Now assume that $\lambda_0 = \lambda_1 = \cdots = \lambda_{k-1} = 0$. So our linear combination is now

$$\lambda_k T^{k-1}(x) + \lambda_{k+1}T^k(x) + \cdots + \lambda_{n-1}T^{n-1}(x) = 0$$

Applying T^{n-k} to both sides we get $\lambda_k T^{n-1}(x) = 0$ and so $\lambda_k = 0$. Thus by induction we are done.

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