This week you will get practice applying the exponential and logistic models and describing their qualitative behaviour. Some of these questions take a bit of thought, they are good practice if you generally struggle with word problems. You will also get a lot of practice solving separable differential equations.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.1) Write a differential equation to model the situations described below. Do not try to solve.
   (a) (6.1-1) The number of bacteria in a culture grows at a rate that is proportional to the number of bacteria present.
   (b) (6.1-2) A sample of radium decays at a rate that is proportional to the amount of radium present in the sample.
   (c) (6.1-5) According to Benjamin Gompertz (1779-1865) the growth rate of a population is proportional to the number of individuals present, where the factor of proportionality is an exponentially decreasing function of time.
   (d) (6.1-7) The rate at which an epidemic spreads through a community of P susceptible people is proportional to the product of the number of people y who have caught the disease and the number $P - y$ who have not.
   (e) (6.1-8) The rate at which people are implicated in a government scandal is proportional to the product of the number N of people already implicated and the number of people involved who have not yet been implicated.

2. (6.1) A population model is given by
$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(100 - P).$$
   (a) (6.1-9) For what values is the population at equilibrium?
   (b) (6.1-10) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$?
   (c) (6.1-11) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$?
   (d) (6.1-12) Describe how the fate of the population depends on the initial density.

3. (6.1) A population model is given by
$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(P - 1)(100 - P).$$
   (a) (6.1-13) For what values is the population at equilibrium?
   (b) (6.1-14) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$?
   (c) (6.1-15) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$?
   (d) (6.1-16) Describe how the fate of the population depends on the initial density.

4. (6.1) Radioactive decay: Certain types of atoms (e.g. carbon-14, xenon-133, lead-210, etc.) are inherently unstable. They exhibit random transitions to a different atom while emitting radiation in the process. Based on experimental evidence, Rutherford found in the early 20th century that the number, N, of atoms in a radioactive substance can be described by the equation
$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$
   where t is measured in years and $\lambda > 0$ is known as the *decay constant*. The decay constant is found experimentally by measuring the half life, τ of the radioactive substance (i.e. the time it takes for half of the substance to decay). Use this information in the following problems.

(a) (6.1-18) Find a solution to the decay equation assuming that $N(0) = N_0$.

(b) (6.1-19) For xenon-133, the half-life is 5 days. Find λ. Assume t is measured in days.

(c) (6.1-20) For carbon-14 the half life is $5,568$ years. Find the decay constant λ, assuming t is measured in years.

(d) (6.1-21) How old is a piece of human bone which contains just 60% of the amount of carbon-14 expected in a sample of bone from a living person, assuming the half life of carbon-14 is $5,568$ years?

(e) (6.1-22) The Dead Sea Scrolls were written on parchment at about 100 B.C. What percentage of carbon-14 originally contained in the parchment remained when the scrolls were discovered in 1947?

5. (6.1-30) Hyperthyroidism is caused by a new growth of tumor-like cells that secrete thyroid hormones in excess to the normal hormones. If left untreated, a hyperthyroid individual can exhibit extreme weight loss, anorexia, muscle weakness, heart disease intolerance to stress, and eventually death. The most successful and least invasive treatment option is radioactive iodine-131 therapy.

   This involves the injection of a small amount of radioactivity into the body. For the type of hyperthyroidism called Graves disease, it is usual for about $40 - 80\%$ of the administered activity to concentrate in the thyroid gland. For functioning adenomas (hot nodules), the uptake is closer to $20 - 30\%$. Excess iodine-131 is excreted rapidly by the kidneys. The quantity of radioiodine used to treat hyperthyroidism is not enough to injure any tissue except the thyroid tissue, which slowly shrinks over a matter of weeks to months. Radioactive iodine is either swallowed in a capsule or sipped in solution through a straw. A typical dose is $5 - 15$ millicures. The half-life of iodine-131 is 8 days.

   (a) Suppose that it takes 48 hours for a shipment of iodine-131 to reach a hospital. How much of the initial amount shipped is left once it arrives at the hospital?

   (b) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left one week later?

   (c) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left 30 days later?

6. (6.2) Solve the following differential equations.

   (a) $\frac{dy}{dt} = 5y$

   (b) $\frac{dy}{dt} = -y$

   (c) $\frac{dy}{dx} = -3y$

   (d) $\frac{dy}{dx} = 0.2y$

   (e) (6.2-17) $\frac{dy}{dt} = y^3$

Solution: We begin by separating variables and integrating:

$$\int \frac{1}{y^3}\,dy = \int dt.$$

On both sides we can use the power rule to obtain

$$-\frac{1}{2y^2} = t - C$$

for an arbitrary constant C. We can rearrange this equation to obtain

$$y^2 = -\frac{1}{2(C-t)}$$

taking the square root,

$$y(t) = \frac{1}{\sqrt{2(C-t)}}.$$

(f) (6.2-18) $\frac{\mathrm{d}y}{\mathrm{d}t} = y \sin t$

(g) (6.2-20) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y}$

(h) (6.2-24) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}\sqrt{1+x^2}$

(i) (6.2-26) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x}{\cos y}$

(j) (6.2-30) $\frac{\mathrm{d}y}{\mathrm{d}t} = yt$ with $y(1) = -1$

(k) (6.2-32) $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-y}t$ with $y(-2) = 0$

(l) (6.2-34) $\frac{\mathrm{d}y}{\mathrm{d}t} = ty^2 + 3t^2y^2$ with $y(-1) = 2$

Solution: We begin by factorising the right hand side,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (t + 3t^2)y^2.$$

We can now separate variables and integrate:

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int t + 3t^2\,\mathrm{d}t$$

We integrate both sides using the power rule,

$$-\frac{1}{y} = \frac{1}{2}t^2 + t^3 + C$$

for an arbitrary constant C. Rearranging,

$$y(t) = -\frac{2}{t^2 + 2t^3 + C}.$$

Now we use the fact that $y(-1) = 2$:

$$2 = -\frac{2}{1 - 2 + C}$$

so $C = 0$ an the solution is

$$y(t) = -\frac{2}{t^2 + 2t^3}.$$

(m) $\frac{\mathrm{d}y}{\mathrm{d}x} = y \sin x + \frac{y}{(x+1)^2}$ with $y(0) = 1$

Solution: We begin by factorising the right hand side,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(\sin x + \frac{1}{(x+1)^2}\right).$$

We can now separate variables and integrate:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \sin x + \frac{1}{(x+1)^2}\,\mathrm{d}x$$

We integrate both sides,

$$\ln(y) = -\cos x - \frac{1}{(x+1)} + C$$

for an arbitrary constant C. Exponentiating both sides,

$$y(t) = C\exp\left(-\cos x - \frac{1}{(x+1)}\right).$$

Now we use the fact that $y(0) = 1$:

$$1 = C\exp(-1-1) = Ce^{-2}$$

so $C = e^2$ and the solution is

$$y(t) = \exp\left(2 - \cos x - \frac{1}{(x+1)}\right).$$

(n) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}e^{-x^2}$ with $y(0) = 1$

(o) $\frac{\mathrm{d}y}{\mathrm{d}x} = y + ye^x$ with $y(0) = e$

7. (6.2-44) Populations may exhibit seasonal growth in response to seasonal fluctuations in resource availability. A simple model accounting for seasonal fluctuations in the abundance N of a population is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (R + \cos t)N$$

where R is the average per-capita growth rate and t is measured in years.

(a) Assume $R = 0$ and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

(b) Assume $R = 1$ (more generally $R > 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

(c) Assume $R = -1$ (more generally $R < 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?