

Math 3B: Lecture 14

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Repeated factors

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For every factor $(ax + b)^k$ in $q(x)$, the partial fraction expansion has terms of the form

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \cdots + \frac{A_k}{(ax + b)^k}.$$

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So

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Side note: integrating $\frac{1}{x}$.

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Using substitution this gives the formula

$$\int \frac{1}{ax + b} dx = \frac{1}{a} \ln |ax + b| + C.$$

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Recall that if $k > 1$

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2. Write $\frac{r(x)}{q(x)}$ as a sum of fractions of the form

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3. Integrate all these pieces separately.

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$$I = \frac{1}{3}x^3 - 2x + \frac{1}{2} \ln |x - 1| - \frac{1}{2} \ln |x + 1| + C.$$

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So

$$I = x + \ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + C.$$

Differential equations (motivation)

An (ordinary) **differential equation** (or **ODE**) is an equation that involves derivatives of an unknown function.

$$\frac{d^2y}{dx^2} = y - 3y^2$$

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The challenge is to find all the functions $y = f(x)$ (or even just one) that satisfy a given equation.

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And so on.

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Note

The right hand side of the equation does not have any y 's.

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- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)
- understand how sensitive ODEs are to their parameters.

Initial value problems

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- Then we see that $y(0) = 1 + C$, so $C = 1$.

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- If this is positive we go up, negative we go down!