This week you will get practice with slope fields.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

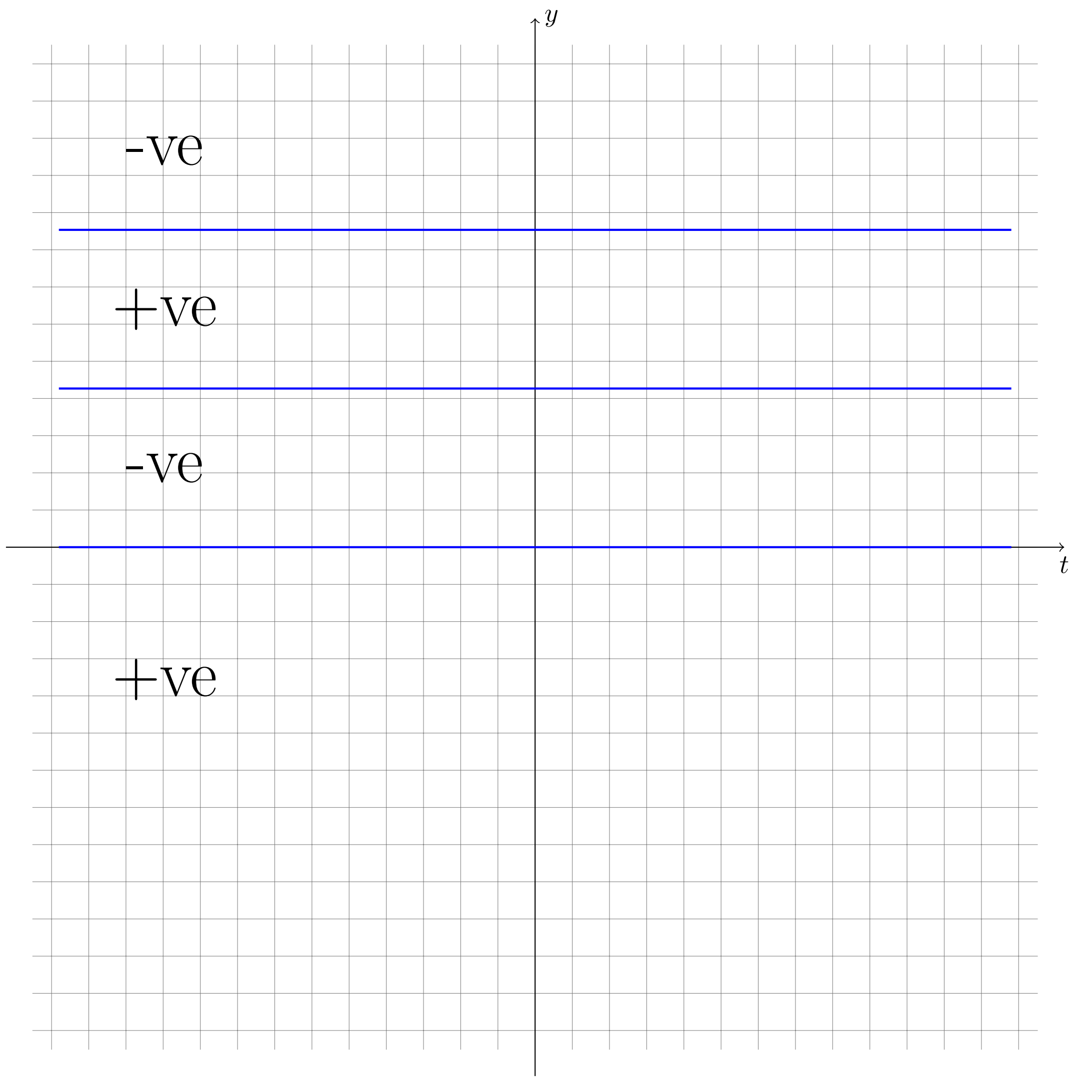
Homework: The homework will be due on Monday 4 March, at 8am, the *start* of the lecture. It will consist of questions

question 4 and question 5.

1. (6.4) Sketch the slope fields and a few solutions for the differential equations given

   (a) (6.4.12) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(4-y)(y-2)$

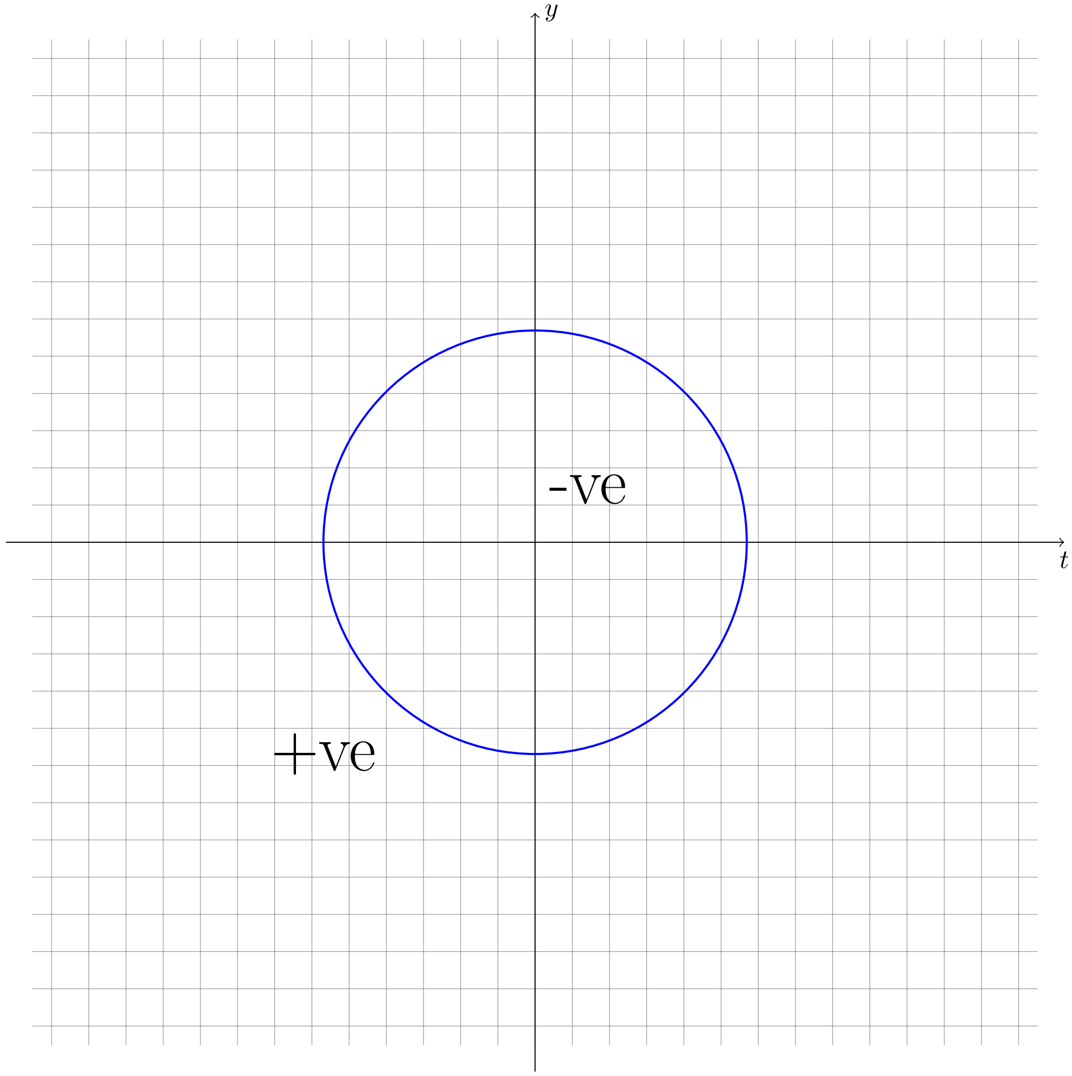
   Solution:

   (b) (6.4.14) $\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - y$

(c) (6.4.16) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + t^2 - 1$

Solution:



(d) (6.4.17) $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{y}{t}$

Hint: feel free to use technology, just make sure you know how to draw a solution if you are given a slope field.

2. (6.4) Sketch the slope fields and the solution passing through the specified point for the differential equations given

   (a) (6.4.19) $\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - y^2, (t, y) = (0, 0)$

   (b) (6.4.20) $\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5y(1 - y), (t, y) = (0, 0.1)$

   (c) (6.4.21) $\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{\frac{t}{y}}, (t, y) = (4, 1)$

   (d) (6.4.22) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2\sqrt{t}, (t, y) = (9, -1)$

3. (6.4.37) A population subject to seasonal fluctuations can be described by the logistic equation with an

oscillating carrying capacity. Consider, for example,

$$\frac{dP}{dt} = P\left(1 - \frac{P}{100 + 50\sin 2\pi t}\right)$$

Although it is difficult to solve this differential equation, it is easy to obtain a qualitative understanding.

(a) Sketch a slope field over the region $0 \le t \le 5$ and $0 \le P \le 200$.

Solution: See attached.

(b) Sketch solutions that satisfy $P(0) = 0$, $P(0) = 10$, and $P(0) = 200$, use technology if you like.

Solution: See attached.

(c) Comment on the behaviour of the solutions.

Solution: Since $P = 0$ is an equilibrium, the solution when $P(0) = 0$ remains zero forever. When $P(0) = 10$ the solution increases and then oscillates about approximately 100. When $P(0) = 200$ the solution decreases and then oscillates about approximately 100.

4. (6.4.40) A population, in the absence of harvesting, exhibits the following growth

$$\frac{dN}{dt} = N\left(\frac{N}{100} - 1\right)\left(1 - \frac{N}{1000}\right)$$

where N is abundance and t is time in years.

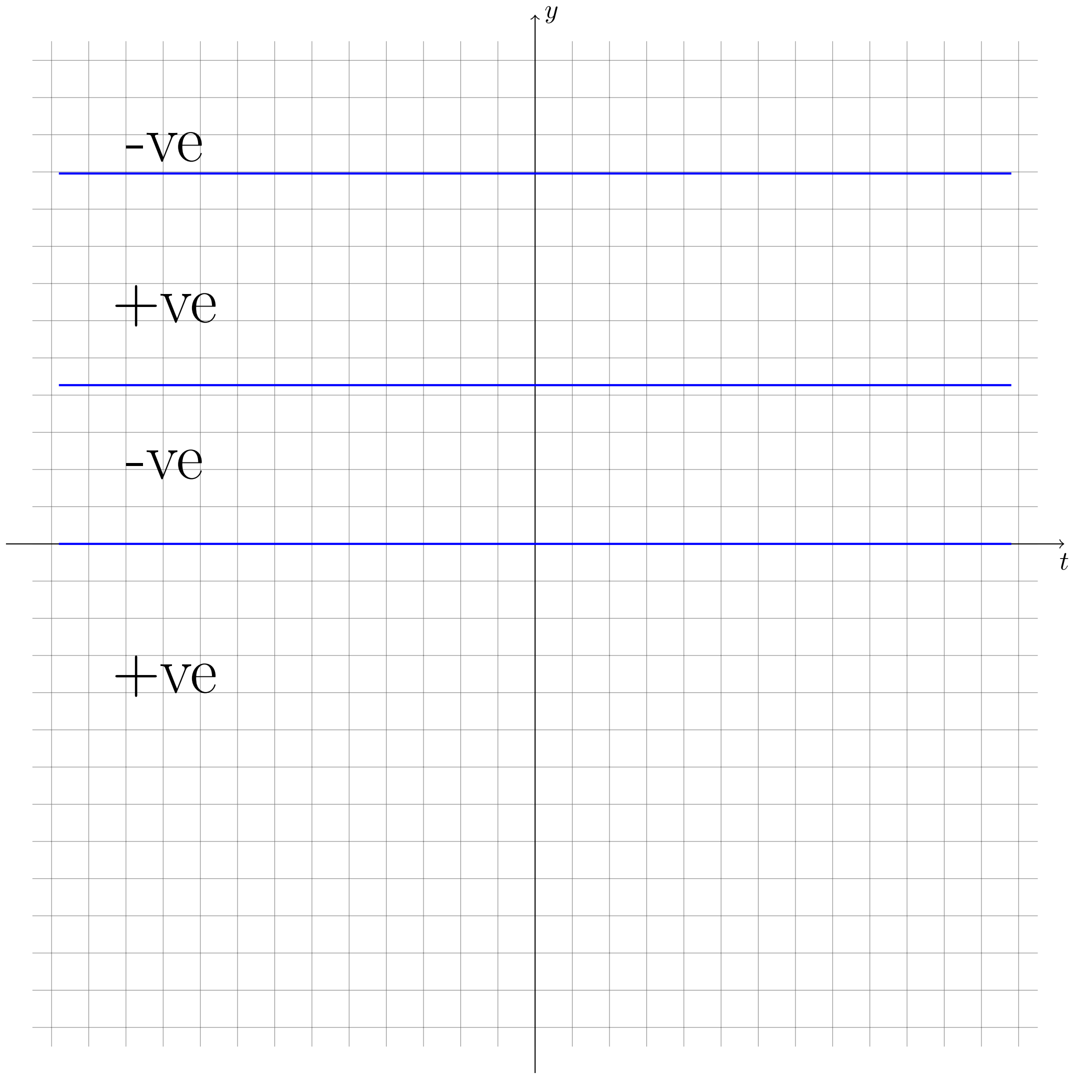
(a) Write an equation that corresponds to harvesting the population at a rate of 0.5% per day.

Solution: This means that 0.005 of the population is beign removed per day, so over a year, the total fraction of the population being removed is $365 \cdot 0.005 = 1.825 = 365/200$ so the DE becomes

$$\begin{aligned}
\frac{dN}{dt} &= N\left(\frac{N}{100} - 1\right)\left(1 - \frac{N}{1000}\right) - 1.825N && (1)\\
&= N\left[\left(\frac{N}{100} - 1\right)\left(1 - \frac{N}{1000}\right) - \frac{365}{200}\right] && (2)\\
&= N\left(-\frac{N^2}{100000} + \frac{N}{100} + \frac{N}{1000} - 1 - \frac{365}{200}\right) && (3)\\
&= -\frac{N}{100000}\left(N^2 - 1100N + 282500\right) && (4)
\end{aligned}$$

(b) Sketch the slope field for the differential equation you found in part a; by sketching solutions, describe how the fate of the population depends on its initial abundance.

Solution: To sketch the solution, we need to know where the right hand side is zero. One obvious place is when $N = 0$. Two others are provided by solving the quadratic equation. Approximately, the roots are $N \approx 409, 691$. Thus, the slope field looks like:



From this we can see that if the initial abundance is between 0 and 409, the population will eventually go extinct. If it is larger than 409 then the population will stabalize at 691.

5. A population of fish (measured in thousands of tons) is known to grow logistically with a carrying capacity of 100 and net birth rate 1. The population is discovered by some fishermen and they begin to harvest the population at a rate of 5 thousand tons per year. This quickly increases however, and the harvesting rate increases by 0.5 thousand tons every year. Let $y(t)$ be the size of the population (in thousands of tons) t years after harvesting begins.

   (a) Write a differential equation describing the population of fish.

Solution: The rate in is given by the model for logistic growth with $r = 1$ and $K = 100$, i.e.

$$y\left(1 - \frac{y}{100}\right)$$

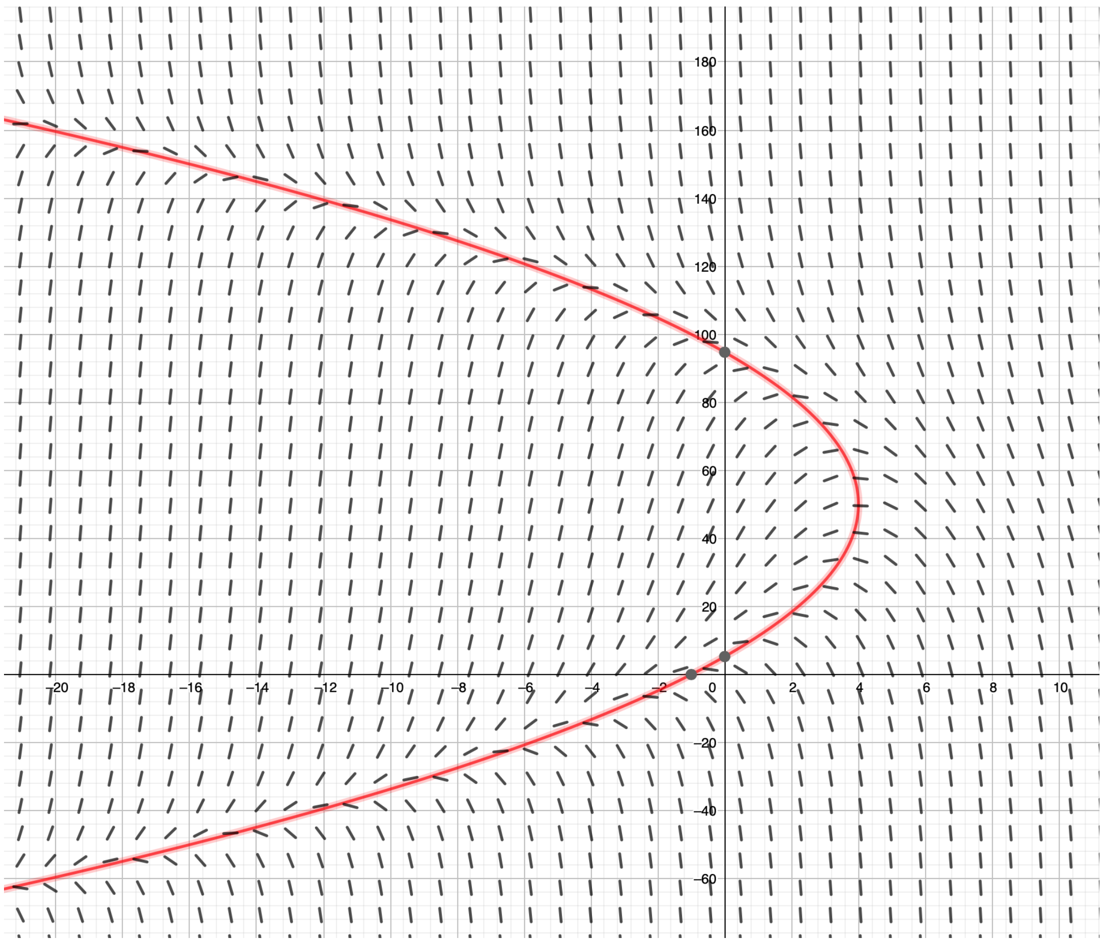
The rate out is due to harvesting and is initially 5 and increases 0.5 every year, so is $5 + 0.5t$.

Thus we have the differential equation

$$\frac{dy}{dt} = y\left(1 - \frac{y}{100}\right) - (5 + 0.5t).$$

(b) What is the eventual fate of the population, and how does it depend on its initial abundance?

Solution: We cannot solve this equation so we need some other way to analyse its solution. Perhaps the easiest way is to look at the slope field.



This clearly shows that wherever we start a solution on the y-axis (i.e. whatever the initial abundance), as we increase y, the solution will increase until it reaches the parabola, then it will start to decrease. It will never cross the parabola again since if it were to hit the parabola it would have slope zero and "bounce off" to the right. Once it passes the parabola it is clear the solution would decrease forever. Thus, whatever the initial abundance, the population eventually dies off.