

This week you will practice writing differential equations modelling real world phenomena as well as understanding population models. You will also get practice solving separable differential equations.

\*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.1) Write a differential equation to model the situations described below. Do not try to solve.
  - (a) (6.1-1) The number of bacteria in a culture grows at a rate that is proportional to the number of bacteria present.
  - (b) (6.1-2) A sample of radium decays at a rate that is proportional to the amount of radium present in the sample.
  - (c) (6.1-5) According to Benjamin Gompertz (1779-1865) the growth rate of a population is proportional to the number of individuals present, where the factor of proportionality is an exponentially decreasing function of time.
  - (d) (6.1-7) The rate at which an epidemic spreads through a community of  $P$  susceptible people is proportional to the product of the number of people  $y$  who have caught the disease and the number  $P - y$  who have not.
  - (e) (6.1-8) The rate at which people are implicated in a government scandal is proportional to the product of the number  $N$  of people already implicated and the number of people involved who have not yet been implicated.

2. (6.1) A population model is given by

$$\frac{dP}{dt} = P(100 - P).$$

- (a) (6.1-9) For what values is the population at equilibrium?
  - (b) (6.1-10) For what values is  $\frac{dP}{dt} > 0$ ?
  - (c) (6.1-11) For what values is  $\frac{dP}{dt} < 0$ ?
  - (d) (6.1-12) Describe how the fate of the population depends on the initial density.
3. (6.1) A population model is given by

$$\frac{dP}{dt} = P(P - 1)(100 - P).$$

- (a) (6.1-13) For what values is the population at equilibrium?
  - (b) (6.1-14) For what values is  $\frac{dP}{dt} > 0$ ?
  - (c) (6.1-15) For what values is  $\frac{dP}{dt} < 0$ ?
  - (d) (6.1-16) Describe how the fate of the population depends on the initial density.
4. (6.1) Radioactive decay: Certain types of atoms (e.g. carbon-14, xenon-133, lead-210, etc.) are inherently unstable. They exhibit random transitions to a different atom while emitting radiation in the process. Based on experimental evidence, Rutherford found in the early 20th century that the number,  $N$ , of atoms in a radioactive substance can be described by the equation

$$\frac{dN}{dt} = -\lambda N$$

where  $t$  is measured in years and  $\lambda > 0$  is known as the *decay constant*. The decay constant is found experimentally by measuring the half life,  $\tau$  of the radioactive substance (i.e. the time it takes for half of the substance to decay). Use this information in the following problems.

- (a) (6.1-18) Find a solution to the decay equation assuming that  $N(0) = N_0$ .
- (b) (6.1-19) For xenon-133, the half-life is 5 days. Find  $\lambda$ . Assume  $t$  is measured in days.
- (c) (6.1-20) For carbon-14 the half life is 5,568 years. Find the decay constant  $\lambda$ , assuming  $t$  is measured in years.
- (d) (6.1-21) How old is a piece of human bone which contains just 60% of the amount of carbon-14 expected in a sample of bone from a living person, assuming the half life of carbon-14 is 5,568 years?
- (e) (6.1-22) The Dead Sea Scrolls were written on parchment at about 100 B.C. What percentage of carbon-14 originally contained in the parchment remained when the scrolls were discovered in 1947?
5. (6.1-30) Hyperthyroidism is caused by a new growth of tumor-like cells that secrete thyroid hormones in excess to the normal hormones. If left untreated, a hyperthyroid individual can exhibit extreme weight loss, anorexia, muscle weakness, heart disease intolerance to stress, and eventually death. The most successful and least invasive treatment option is radioactive iodine-131 therapy.
- This involves the injection of a small amount of radioactivity into the body. For the type of hyperthyroidism called Graves disease, it is usual for about 40 – 80% of the administered activity to concentrate in the thyroid gland. For functioning adenomas (hot nodules), the uptake is closer to 20 – 30%. Excess iodine-131 is excreted rapidly by the kidneys. The quantity of radioiodine used to treat hyperthyroidism is not enough to injure any tissue except the thyroid tissue, which slowly shrinks over a matter of weeks to months. Radioactive iodine is either swallowed in a capsule or sipped in solution through a straw. A typical dose is 5 – 15 millicuries. The half-life of iodine-131 is 8 days.
- (a) Suppose that it takes 48 hours for a shipment of iodine-131 to reach a hospital. How much of the initial amount shipped is left once it arrives at the hospital?
- (b) Suppose a patient is given a dosage of 10 millicuries of which 30% concentrates in the thyroid gland. How much is left one week later?
- (c) Suppose a patient is given a dosage of 10 millicuries of which 30% concentrates in the thyroid gland. How much is left 30 days later?
6. (6.2) Solve the following differential equations.
- (a)  $\frac{dy}{dt} = 5y$
- (b)  $\frac{dy}{dt} = -y$
- (c)  $\frac{dy}{dx} = -3y$
- (d)  $\frac{dy}{dx} = 0.2y$
- (e) (6.2-17)  $\frac{dy}{dt} = y^3$
- (f) (6.2-18)  $\frac{dy}{dt} = y \sin t$
- (g) (6.2-20)  $\frac{dy}{dt} = \frac{t}{y}$
- (h) (6.2-24)  $\frac{dy}{dx} = \frac{x}{y} \sqrt{1 + x^2}$
- (i) (6.2-26)  $\frac{dy}{dx} = \frac{\sin x}{\cos y}$
- (j) (6.2-30)  $\frac{dy}{dt} = yt$  with  $y(1) = -1$
- (k) (6.2-32)  $\frac{dy}{dt} = e^{-y}t$  with  $y(-2) = 0$
- (l) (6.2-34)  $\frac{dy}{dt} = ty^2 + 3t^2y^2$  with  $y(-1) = 2$
- (m)  $\frac{dy}{dx} = y \sin x + \frac{y}{(x+1)^2}$  with  $y(0) = 1$
- (n)  $\frac{dy}{dx} = \frac{x}{y}e^{-x^2}$  with  $y(0) = 1$
- (o)  $\frac{dy}{dx} = y + ye^x$  with  $y(0) = e$

7. (6.2-44) Populations may exhibit seasonal growth in response to seasonal fluctuations in resource availability. A simple model accounting for seasonal fluctuations in the abundance  $N$  of a population is

$$\frac{dN}{dt} = (R + \cos t)N$$

where  $R$  is the average per-capita growth rate and  $t$  is measured in years.

- (a) Assume  $R = 0$  and find a solution to this differential that satisfies  $N(0) = N_0$ . What can you say about  $N(t)$  at  $t \rightarrow \infty$ ?
- (b) Assume  $R = 1$  (more generally  $R > 0$ ) and find a solution to this differential that satisfies  $N(0) = N_0$ . What can you say about  $N(t)$  at  $t \rightarrow \infty$ ?
- (c) Assume  $R = -1$  (more generally  $R < 0$ ) and find a solution to this differential that satisfies  $N(0) = N_0$ . What can you say about  $N(t)$  at  $t \rightarrow \infty$ ?