Midterm 2

UCLA: Math 3B, Fall 2018

Instructor: Noah White
Date: 19 November 2018

- This exam has 3 questions, for a total of 30 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: Solutions

ID number: ______________________

Discussion section (please circle):

Day/TA	Ben	Ryan
Tuesday	1A	1C
Thursday	1B	1D

Question	Points	Score
1	9	
2	10	
3	11	
Total:	30	

Question 1 is multiple choice. Indicate your answers in the table below. *The following three pages will not be graded, your answers must be indicated here.*

Part	A	B	C	D
(a)		X		
~~(b)~~				
(c)		X		
(d)			X	
(e)				X
(f)			X	

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (1 point) If the function $y(t)$ satisfies the differential equation $\frac{dy}{dt} = -1$ and $y(0) = 1$ then

   A. $y(t) = -1$
   B. $y(t) = 1 - t$.
   C. $y(t) = e^{-t}$.
   D. $y(t) = 0$.

   (b) (1 point) The definite integral $\int_0^{\pi} x \cos x \ dx$ is equal to

   ~~A. e.~~
   ~~B. 1.~~
   ~~C. 0.~~
   ~~D. 2.~~

   -2

   (c) (1 point) In the partial fraction expansion $\dfrac{9x - 23}{(x-3)^2(x+1)} = \dfrac{A}{x-3} + \ldots,$

   A. $A = 1$.
   B. $A = 2$.
   C. $A = -1$.
   D. $A = -2$.

(d) (2 points) If the function $y(t)$ satisfies the differential equation $\frac{dy}{dt} = e^{-y}(2t+1)$ and $y(0) = e$ then

A. $y(1) = e$
B. $y(e) = 1$
C. $y(-1) = e$
D. $y(e) = -1$

(e) (2 points) A population $y(t)$ of bacteria is found to double every hour (so $y(1) = 2y(0)$ and $y(2) = 4y(0)$). What differential equation does y satisfy?

A. $\frac{dy}{dt} = y$
B. $\frac{dy}{dt} = 2y$
C. $\frac{dy}{dt} = -y$
D. $\frac{dy}{dt} = (\ln 2)y$

(f) (2 points) A population $P(t)$ obeys the differetial equation $\frac{dP}{dt} = (P^2 - 10P + 25)(10 - P)$. What is the eventual fate of the population if $5 < P(0) < 10$?

A. The population grows forever.
B. The population stabilises at or close to 5.
C. The population stabilises at or close to 10.
D. The population dies out, i.e. $P(t)$ is eventually close to zero.

2. Park rangers in the Channel islands are studying the migration of seabirds. They notice that over the season, birds arrive on the island at a rate of 50 birds per day. When they start their study there are already 500 birds. They notice that if a bird arrives, the probability that it will be on the island in t days is

$$e^{-t/100}.$$

In this question, we will calculate the number of birds on the island in 100 days.

(a) (1 point) Of the original 500 birds, how many will be on the island in 100 days?

$500e^{-1}$

(b) (1 point) Divide the 100 day period into n subintervals. Let Δt be the length of each subinterval. What is Δt in terms of n?

$\Delta t = \frac{100}{n}$

(c) (1 point) Let t measure time, so $t = 0$ is the current moment and $t = 100$ is the end of the period we are studying. Let t_k be the end of the k^{th} subinterval. What is t_k in terms of n and k?

$t_k = \frac{100k}{n}$

(d) (1 point) How many birds arrive during the k^{th} subinterval (in terms of Δt)?

$50\,\Delta t$

(e) (2 points) Of the birds that arrive during the k^{th} subinterval, how many will remain on the island at the 100 day mark? Your answer should be in terms of Δt and t_k.

$50e^{-(100-t_k)/100}\,\Delta t$

(f) (2 points) Write a Riemann sum that calculates the total number of birds on the island on day 100. Dont forget to add the contribution due to the original birds!

$$500e^{-1} + \lim_{n\to\infty} \sum_{k=1}^{n} 50e^{-(100-t_k)/100} \Delta t$$

(g) (2 points) Use an integral to evaluate this sum.

$$= 500e^{-1} + \int_0^{100} 50e^{-(100-t)/100}\, dt$$

$$= 500e^{-1} + \left[5000\, e^{(t-100)/100} \right]_0^{100}$$

$$= 500e^{-1} + 5000 - 5000e^{-1}$$

$$= 5000 - 4500e^{-1}$$

$$= 500(10 - 9e^{-1}).$$

3. As part of an industrial process a factory has a tank or water, into which polluted water is being pumped at a rate of 2 L/min. The water being pumped in contains 3 g/L of pollutant. You may assume that the tank initially has 100 L of clean water (containing no pollutant). Water is also being drained from the tank at the same rate, 2 L/min.

   (a) (1 point) How much pollutant (in g) is being added, per minute to the tank?

   6 g/min.

   (b) (4 points) Write a differential equation describing the total amount $y(t)$ of pollutant (in grams) in the tank at time t.

   rate in $= 6$

   rate out $= 2 \cdot \frac{y}{100}$

   $$\frac{dy}{dt} = 6 - \frac{y}{50}$$

(c) (3 points) How much pollutant does the tank contain after 100 minutes? What about in the long term (i.e. as $t \to \infty$)?

- Equillibrium at $y = 300$ st

as $t \to \infty$, $y(t) \to 300$.

- General solution: $y(t) = 300 - Ce^{-t/50}$

since $y(0)=0$ $\quad 0 = 300 - C \quad C = 300$

$y(t) = 300(1 - e^{-t/50})$

$y(100) = 300(1 - e^{-2})$.

(d) (3 points) Suppose in addition to the above process, a filter is added to the tank. This filter removes 1 g of pollutant from the tank per minute. What is the amount of pollutant in the tank in the long term?

New rate out $= \frac{y}{50} + 1$

$\frac{dy}{dt} = 6 - \frac{y}{50} - 1 = 5 - y/50$

Equilibrium @ $y = 250$

As $t \to \infty$, $y(t) \to 250$.