This week you will get practice solving separable differential equations, as well as some practice with linear models

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.2) Solve the following differential equations.
   (a) $\frac{\mathrm{d}y}{\mathrm{d}t} = 5y$
   (b) $\frac{\mathrm{d}y}{\mathrm{d}t} = -y$
   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y$
   (d) $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.2y$
   (e) (6.2-17) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^3$
   (f) (6.2-18) $\frac{\mathrm{d}y}{\mathrm{d}t} = y \sin t$
   (g) (6.2-20) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y}$
   (h) (6.2-24) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}\sqrt{1+x^2}$
   (i) (6.2-26) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x}{\cos y}$
   (j) (6.2-30) $\frac{\mathrm{d}y}{\mathrm{d}t} = yt$ with $y(1) = -1$
   (k) (6.2-32) $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-y}t$ with $y(-2) = 0$
   (l) (6.2-34) $\frac{\mathrm{d}y}{\mathrm{d}t} = ty^2 + 3t^2y^2$ with $y(-1) = 2$

Solution: We begin by factorising the right hand side,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (t + 3t^2)y^2.$$

We can now separate variables and integrate:

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int t + 3t^2\,\mathrm{d}t$$

We integrate both sides using the power rule,

$$-\frac{1}{y} = \frac{1}{2}t^2 + t^3 + C$$

for an arbitrary constant C. Rearranging,

$$y(t) = -\frac{2}{t^2 + 2t^3 + C}.$$

Now we use the fact that $y(-1) = 2$:

$$2 = -\frac{2}{1 - 2 + C}$$

so $C = 0$ an the solution is

$$y(t) = -\frac{2}{t^2 + 2t^3}.$$

(m) $\frac{\mathrm{d}y}{\mathrm{d}x} = y \sin x + \frac{y}{(x+1)^2}$ with $y(0) = 1$

Solution: We begin by factorising the right hand side,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(\sin x + \frac{1}{(x+1)^2}\right).$$

We can now separate variables and integrate:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \sin x + \frac{1}{(x+1)^2}\,\mathrm{d}x$$

We integrate both sides,

$$\ln(y) = -\cos x - \frac{1}{(x+1)} + C$$

for an arbitrary constant C. Exponentiating both sides,

$$y(t) = C\exp\left(-\cos x - \frac{1}{(x+1)}\right).$$

Now we use the fact that $y(0) = 1$:

$$1 = C\exp(-1-1) = Ce^{-2}$$

so $C = e^2$ and the solution is

$$y(t) = \exp\left(2 - \cos x - \frac{1}{(x+1)}\right).$$

(n) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}e^{-x^2}$ with $y(0) = 1$

(o) $\frac{\mathrm{d}y}{\mathrm{d}x} = y + ye^x$ with $y(0) = e$

2. (6.2-44) Populations may exhibit seasonal growth in response to seasonal fluctuations in resource availability. A simple model accounting for seasonal fluctuations in the abundance N of a population is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (R + \cos t)N$$

where R is the average per-capita growth rate and t is measured in years.

(a) Assume $R = 0$ and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

Solution: When $R = 0$ the equation is $N' = N\cos t$. Using separation of variables we find the solution $N(t) = Ce^{\sin t}$ and since $N(0) = N_0$ we see that $C = N_0$. As $N \to \infty$, this fluctuates between N_0e^{-1} and N_0e.

(b) Assume $R = 1$ (more generally $R > 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

Solution: When $R = 1$ the equation is $N' = N(1 + \cos t)$. Using separation of variables we find the solution $N(t) = Ce^{t+\sin t}$ and since $N(0) = N_0$ we see that $C = N_0$. As $N \to \infty$, the t dominates the $\sin t$ and the population grows exponentially.

(c) Assume $R = -1$ (more generally $R < 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

> **Solution:** When $R = -1$ the equation is $N' = N(-1 + \cos t)$. Using separation of variables we find the solution $N(t) = Ce^{-t+\sin t}$ and since $N(0) = N_0$ we see that $C = N_0$. As $N \to \infty$, the e^{-t} dominates the $e^{\sin t}$ and the population decreases to zero.

3. (6.3-25) In 1990 the gross domestic product (GDP) of the United States was $5,464 billion. Suppose the growth rate from 1989 to 1990 was 5.08%. Predict the GDP in 2003.

   (Hint: You should assume that the percentage growth rate is constant - not very realistic!)

4. (6.3-28) According to the Department of Health and Human Services, the annual growth rate in the number of marriages per year in 1990 in the United States was 9.8% and there were 2,448,000 marriages that year. How many marriages will there be in 2004 if the annual growth rate in the number of marriages per year is constant?

5. (6.3-30) Calculate the infusion rate in milligrams per hour required to maintain a long-term drug concentration of 50 mg/L (i.e., the rate of change of drug in the body equals zero when the concentration is 50 mg/L). Assume that the half-life of the drug is 3.2 hours and that the patient has 5 L of blood.

6. (6.3-31) Calculate the infusion rate in milligrams per hour required to maintain a desired drug concentration of 2 mg/L. Assume the patient has 5.6 L of blood and the half-life of the drug is 2.7 hours.

> **Solution:** The amount of drug (in mg) in the body $y(t)$ at time t will obey a differential equation of the form
>
> $$\frac{\mathrm{d}y}{\mathrm{d}t} = \text{ rate in } - \text{ rate out } .$$
>
> If the drug is being infused at a rate of a mg/h then this is the rate in. If the drug has a half-life of 2.7 hours, this means, after t hours, the fraction of the drug that is left in the body is given by
>
> $$\left(\frac{1}{2}\right)^{\frac{t}{2.7}} = e^{-\frac{\ln 2}{2.7}t}.$$
>
> Thus, in the absence of any infusion, the drug is being expelled by the body at a rate of
>
> $$\frac{\mathrm{d}}{\mathrm{d}t}e^{-\frac{\ln 2}{2.7}t} = -\frac{\ln 2}{2.7}e^{-\frac{\ln 2}{2.7}t} = -\frac{\ln 2}{2.7}(\text{current level of drug}).$$
>
> Thus if $y(t)$ is the current level of drug in the body, then at time t the drug is being expelled at a rate of $-\frac{\ln 2}{2.7}y(t)$ mg/h. This is the rate out. Our differential equation becomes
>
> $$\frac{\mathrm{d}y}{\mathrm{d}t} = a - \frac{\ln 2}{2.7}y.$$
>
> Over the long term, the solution of this equation will approach the equilibrium solution $y(t) = \frac{2.7a}{\ln 2}$. Over the long term we would like the concentration of the drug to be 2 mg/L, since the patient has 5.6 L of blood, that means we would like there to be 11.2 mg of drug in the body in the long term. I.e. we want
>
> $$\frac{2.7a}{\ln 2} = 11.2$$
>
> Rearranging, we get
>
> $$a = \frac{11.2 \ln 2}{2.7} \approx 2.88 \text{ mg/h}.$$

7. (6.3-34) A drug is given at an infusion rate of 50 mg/h. The drug concentration value determined at 3 hours after the start of the infusion is 8 mg/L. Assuming the patient has 5 L of blood, estimate the half-life of this drug.

8. (6.3-37) After one hydrodynamic experiment, a tank contains 300 L of a dye solution with a dye concentration of 2 g/L. To prepare for the next experiment, the tank is to be rinsed with water flowing in at a rate of 2 L/min, with the well-stirred solution flowing out at the same rate. Write an equation that describes the amount of dye in the container. Be sure to identify variables and their units.