Lecture 11 Application of integration + Riemann sums to accumulated change.

- We start with a motivating example

Ex (5.8, Example 1 in text)

- A medical clinic opens
- initially 300 patients join
- Clinic gains patients continuously at a rate of 10 pats/month.
- The proportion of patients who remain members of the clinic t months after joining is

$$e^{-t/20}$$

Question: How many patients does the clinic have after 15 months?

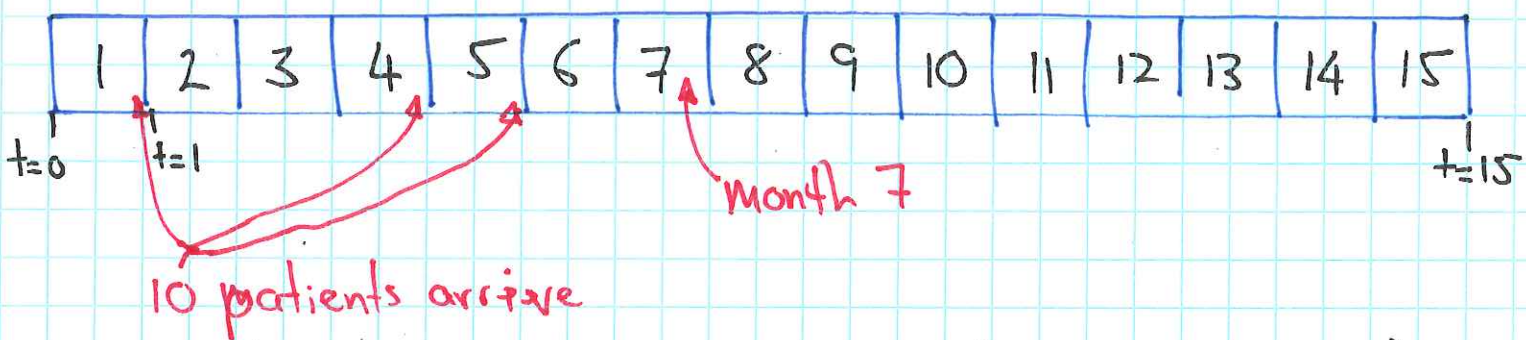
* For the initial 300 patients, this is easy.

$$300e^{-15/20}$$

remain ~~of~~ after 15 months.

* For the patients arriving continuously, it is not so straight forward. Each one has been with the clinic for a slightly different amount of time

Simplifying assumption: Lets assume patients arrive only at the end of the month, i.e. 10 patients join at the end of every month.



(at $t=a$, a months have elapsed since opening).

* In 1st month
  * 10 patients arrive at $t=1$
  * at $t=15$ these patients will have been with the clinic for 14 months
  * thus there are $10e^{-14/20}$ of these patients left at $t=15$.

* In 2^{nd} month
  * 10 patients arrive at $t=2$
  * at $t=15$ these patients have been with the clinic for 13 months
  * thus there are $10e^{-13/15}$ of these patients left at $t=15$
* In k^{th} month
  * 10 patients arrive at $t=k$
  * at $t=15$ these patients have been with the clinic for $15-k$ months
  * thus there are $10e^{-(15-k)/20}$ of these patients left at $t=15$.

– We can simply sum up the contributions:

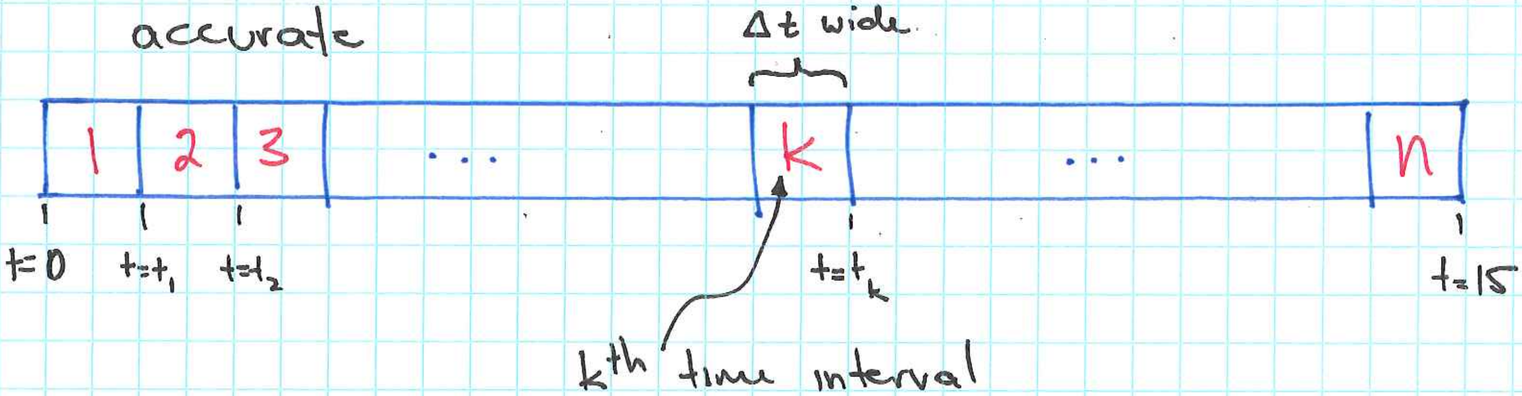
$$\text{Total \# patients at } t=15 = 300e^{-15/20} + \sum_{k=1}^{15} 10e^{-(15-k)/20}$$

$$\approx 249.9 \quad \text{(using a calculator!)}$$

– But this is just an approximation! The patients don't really arrive in blocks of 10 every month but rather are spread out over the month.

– Better approximation: Split $t=0$ to $t=15$ into n time intervals, for n really big!

- If $n \to \infty$ then our approximation becomes accurate



- Since we split the 15 months into n time intervals, ~~we~~ each one ~~is~~ is

$$\underline{\Delta t} = 15/n \text{ months}$$

long.

- We have 10 patients arriving <u>per month</u>, thus in each time interval

$$\underline{10 \cdot \Delta t} = \frac{150}{n}$$

patients arrive.

- From the end of the k^{th} time interval until $t = 15$ is $\underline{15 - t_k}$ months, where ~~Each time interval~~ $t = t_k$ is the end of the k^{th} time interval, i.e. $t_k = k\frac{15}{n}$.
- ~~tt~~Since the time intervals are small, we can assume that the $10\,\Delta t$ people arriving during the k^{th} interval ~~have $15-t_k$ months~~

will have been with the clinic for $15-t_k$ months when $t=15$

- Thus of the $10\,\Delta t$ ~~people~~ patients,

$$10\,\Delta t\, e^{-(15-t_k)/20}$$

remain when $t=15$

- Adding up all contributions and letting $n \to \infty$ we get

$$\#\text{ people remaining} = 300e^{-3/4} + \lim_{n\to\infty} \sum_{k=1}^{n} 10\Delta t\, e^{(t_k-15)/20}$$

- We recognise this sum as a Riemann sum for the function $10e^{(t-15)/20}$ over the interval

$$t_0 = 0 \quad \text{to} \quad t_n = 15$$

so

$$\#\text{ patients remaining} = 300e^{-3/4} + \int_0^{15} 10e^{(t-15)/20}\, dt$$

$$\approx 247.2$$