Lecture 3

1 More general regions

- We would like to be able to define/calculate integrals over more general regions $D \subseteq \mathbb{R}^2$

- We allow the following types of regions:
  * D is closed (contains its boundary)
  * The boundary of D is smooth or piecewise smooth. and simple

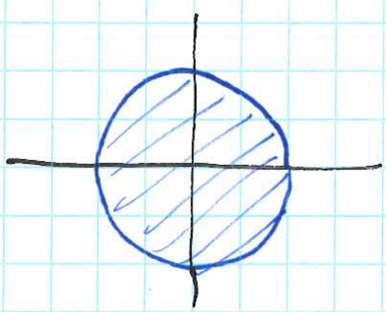
    no self intersection.

Ex - We could have a disk:

$$D = \{(x\ y) \mid x^2 + y^2 \leq 1\}.$$

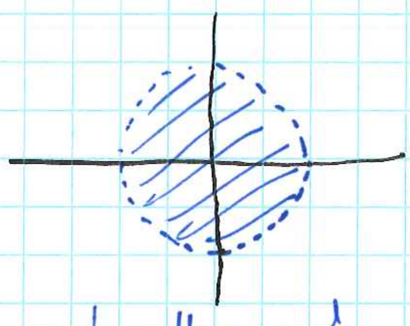
- We cannot have an open disk:

$$\{(x\ y) \mid x^2 + y^2 < 1\}.$$



Allowed

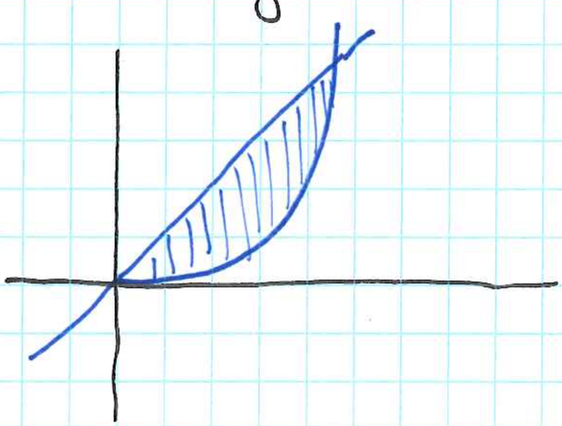
VS



not allowed

- We can have regions defined in the following form:

D is the region bounded by
- $y = x$
- $y = x^2$



$$D = \{(x\ y) \mid y \leq x\} \cap \{(x\ y) \mid y \geq x^2\}$$

2. Integrals over general regions.

- If $f : D \longrightarrow \mathbb{R}$ is a function on a simple closed domain with piecewise smooth boundary ~~tet~~ and $\mathcal{R}$ is a rectangle containing D then let

$$\tilde{f}(x\, y) = \begin{cases} f(x y) & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \notin D \end{cases}$$

Def The double integral of $f(x\, y)$ over D is defined as

$$\iint_D f(x\, y)\, dA := \iint_{\mathcal{R}} \tilde{f}(x\, y)\, dA$$

We say f is integrable on D if $\tilde{f}$ is integrable on $\mathcal{R}$.

Note: $\iint_D f(x y)\, dA$ does not depend on $\mathcal{R}$!

Thm If $f(x\, y)$ is continuous on a closed domain with piecewise smooth, simple boundary then f is integrable, i.e. $\iint_D f(x y)\, dA$ exists.

Rmk - linearity still holds
- $\iint_D 1\, dA = \text{Area}(D)$.

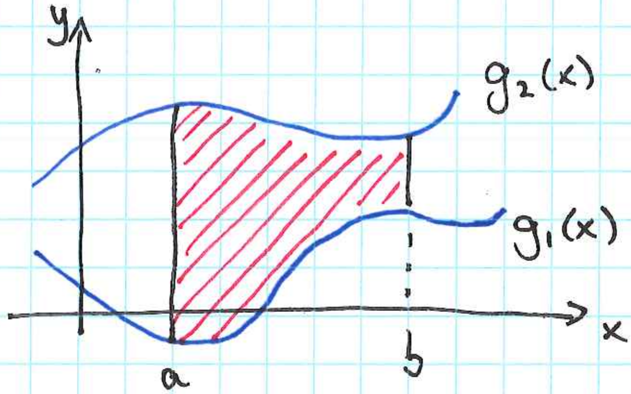
- If $D_1, D_2, \ldots, D_k$ are all sensible regions that do not overlap exept possibly on their boundary then if $D = D_1 \cup D_2 \cup \ldots \cup D_n$

$$\iint_D f(x\,y)\,dA = \iint_{D_1} f(x\,y)\,dA + \ldots + \iint_{D_n} f(x\,y)\,dA.$$

3. Vertically + horizontally simple regions.

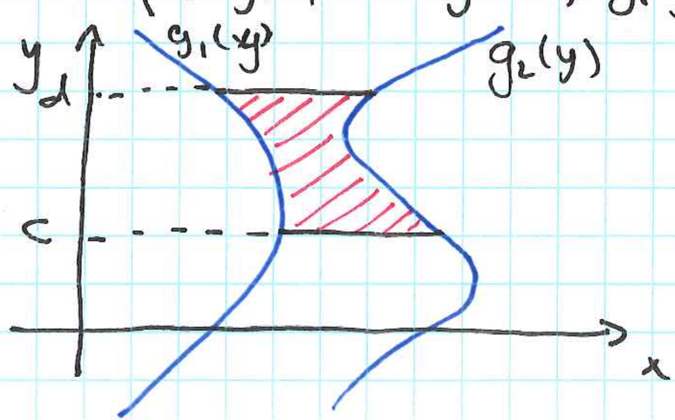
Def * A region D is vertically simple if there are functions $g_1(x)$ and $g_2(x)$ such that

$$D = \{(x,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}.$$



* A region D is horizontally simple if there are functions $g_1(y)$ and $g_2(y)$ such that

$$D = \{(x\,y) \mid c \leq y \leq d,\ g_1(y) \leq x \leq g_2(y)\}$$



Thm * If D is vertically simple such that

$$D = \{(x\,y) \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\},$$

then

$$\iint_D f(x\,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x\,y)\,dy\,dx$$

* If D is horizontally simple such that

$$D = \{(x\,y) \mid g_1(y) \leq x \leq g_2(y),\ c \leq y \leq d\},$$

then

$$\iint_D f(x\,y)\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x\,y)\,dx\,dy$$

proof If D is vert. simple and $R = [a,b] \times [c,d]$ contains D then $c \leq g_1(x) \leq g_2(x) \leq d \quad \forall x \in [a\ b]$

so

$$\iint_D f(x\,y)\,dA = \iint_R \tilde{f}(x\,y)\,dA$$

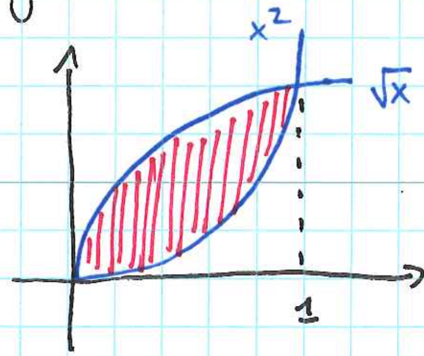
$$= \int_a^b \int_c^d \tilde{f}(x\,y)\,d\cancel{A}y\,dx$$

but $\tilde{f}(x\,y) = 0$ if $y \in [g_1(x), g_2(x)]$ thus

$$\int_c^d \tilde{f}(x\,y)\,dy = \int_{g_1(x)}^{g_2(x)} f(x\,y)\,dy$$

similarly if D is hor. simple. $\square$.

Ex $I = \iint_D x^2 y \, dA$ where D is



We can write D as

$$D = \{ (x, y) \mid 0 \le x \le 1, \ x^2 \le y \le \sqrt{x} \}.$$

$$I = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y \, dy \, dx$$

$$= \int_0^1 \left[\frac{1}{2} x^2 y^2 \right]_{x^2}^{\sqrt{x}} dx$$

$$= \int_0^1 \frac{1}{2} x^3 - \frac{1}{2} x^6 \, dx$$

$$= \left[\frac{1}{8} x^4 - \frac{1}{14} x^7 \right]_0^1$$

$$= \frac{1}{8} - \frac{1}{14} = \frac{5}{112}$$

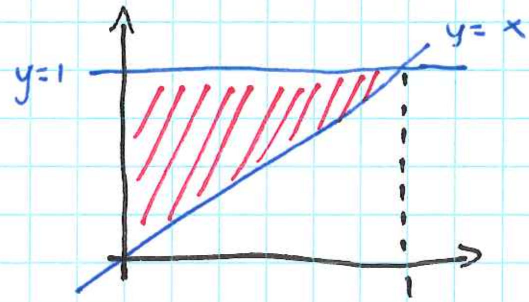
Rmk Note that we could have written

$$D = \{(x\ y) \mid 0 \le y \le 1,\ y^2 \le x \le \sqrt{y}\}$$

thus

$$\int_0^1 \int_{x^2}^{\sqrt{x}} x^2 y \, dy\, dx = \int_0^1 \int_{y^2}^{\sqrt{y}} x^2 y \, dx\, dy$$

Ex $I = \iint_D e^{y^2}\, dA$ where D is



- So D is

$$D = \{(x\ y) \mid 0 \le x \le 1,\ x \le y \le 1\}.$$

- so

$$I = \int_0^1 \int_x^1 e^{y^2} dy\, dx$$

- Now we are stuck! $\int e^{y^2} dy$ is difficult.

- lets reverse the order of integration. D is also

$$D = \{(x\ y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$$

so

$$I = \int_0^1 \int_0^y e^{y^2} dx\, dy$$

$$= \int_0^1 \left[x e^{y^2} \right]_0^y dxy$$

$$= \int_0^1 y e^{y^2} dy$$

$$= \left[\frac{1}{2} e^{y^2} \right]_0^1$$

$$= \frac{1}{2}(e-1) \quad //$$

Rmk To reverse the order of integration

* draw the region
* flip the axes.
* describe the boundaries (if possible).