UCLA: Math 32B **Problem set 2 (solutions)** **Winter, 2020**

This week on the problem set we will see examples of integrals over more general regions.

You will only need to hand in a small selection of the questions for homework, however I recommend that you at least attempt them all by the end of the quarter as some may appear on exams!

Homework: due Friday 10 April, uploaded to Gradescope before 11:59pm. It will consist of questions $3, 4, 5,$ and 6 below.
Note that the references to the textbook are for the 4^{th} edition, *late transcendentals* version. Any differences between the 3^{rd} and 4^{th} editions is noted in parentheses.

1. From 16.2 in the textbook: 4, 8, 14, 20, 21, 23, 29, 31, 45, 48, 49 (Question 21 is different in the two versions, but both are fine.).

2. From 16.3 in the textbook: 3, 5, 6, 7.

3. Consider an integral over the domain $\mathcal{D}$ that is the part of the first quadrant bounded by $y = -(x-1)^2+1$ and $y = 1/x$. We can write an integral over this domain as: $\displaystyle\int_1^{\frac{1+\sqrt{5}}{2}} \int_{1/x}^{-(x-1)^2+1} f(x, y)\, dy\, dx$. Change the order of integration to write this as an integral where you integrate in the order $dx\, dy$.

Solution: We first graph the region in the question.

Now we find the intersection points buy setting

$$\begin{aligned} \frac{1}{x} &= 1 - (x-1)^2 \\ 1 &= -x^3 + 2x^3 \\ x^3 - 2x^2 + 1 &= 0 \end{aligned}$$

We can easily see that $x = 1$ is a solution, and factorising we see that $x^3-2x^2+1 = (x-1)(x^2-x-1)$, so we see that the intersection points are

$$(1, 1), \left(\frac{1+\sqrt{5}}{2}, \frac{2}{1+\sqrt{5}}\right), \left(\frac{1-\sqrt{5}}{2}, \frac{2}{1-\sqrt{5}}\right).$$

Only the first two are in the first quadrant, so these are the ones we are looking for. This allows us to give a vertically simple description,

$$\mathcal{D} = \left\{ (x, y) \;\middle|\; \frac{1}{x} \leq y \leq 1 - (x-1)^2,\ 1 \leq x \leq \frac{1+\sqrt{5}}{2} \right\},$$

which is used to show that

$$\iint_{\mathcal{D}} f(x, y)\ dA = \int_1^{\frac{1+\sqrt{5}}{2}} \int_{1/x}^{-(x-1)^2+1} f(x, y)\ dy\ dx$$

but we can also give a horizontally simple description

$$\mathcal{D} = \left\{ (x, y) \;\middle|\; \frac{1}{y} \leq x \leq 1 + \sqrt{(1-y)},\ \frac{2}{1+\sqrt{5}} \leq y \leq 1 \right\},$$

where we have used the fact that the bounding curves can be rearranged to $x = 1/y$ and $x = 1 + \sqrt{(1-y)}$. This allows us to change the order of integration and give

$$\iint_{\mathcal{D}} f(x, y)\ dA = \int_{\frac{2}{1+\sqrt{5}}}^{1} \int_{1/y}^{1+\sqrt{(1-y)}} f(x, y)\ dx\ dy$$

4. Consider the function $E(s) = \displaystyle\int_0^s e^{-x^2}\ dx$. This is an incredibly important function in applied mathematics (and therefore physics, chemistry, etc). Unfortunately it is impossible to express the antiderivative of e^{-x^2} in terms of functions you already know. So how can we calculate $E(s)$? It turns out, that its value at infinity,

$$E(\infty) := \lim_{s\to\infty} E(s) = \int_0^\infty e^{-x^2}\ dx,$$

cal be calculated using a trick which this question will guide you through. In fact, we will calculate $E(\infty)^2$.

(a) Express $E(\infty)^2 = \left(\displaystyle\int_0^\infty e^{-x^2}\ dx\right)\left(\displaystyle\int_0^\infty e^{-y^2}\ dy\right)$ as a double integral and therefore as an iterated integral, in the order $dx\ dy$. Make sure to describe the region in $\mathbb{R}^2$ we are integrating over precisely. *Hint: consider the separation of variables formula.*

Solution: We use separation of variables in reverse.

$$\begin{aligned} E(\infty)^2 &= \left(\int_0^\infty e^{-x^2}\ dx\right)\left(\int_0^\infty e^{-y^2}\ dy\right) \\ &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\ dx\ dy \\ &= \iint_{\mathcal{R}} e^{-(x^2+y^2)}\ dA, \end{aligned}$$

where $\mathcal{R} = [0, \infty) \times [0, \infty)$ is the first quadrant in the plane.

(b) Use the change of variables $t = x/y$ to transform the inner integral. Express $E(\infty)^2$ as an iterated integral in the order $dy\ dt$.

Solution: We are concentrating on the integral $\int_0^\infty e^{-(x^2+y^2)}\ dx$, where y is held constant. To make the change of variables observe $dt = \frac{1}{y}dx$ and the limits remain the same. Thus

$$\int_0^\infty e^{-(x^2+y^2)}\ dx = \int_0^\infty e^{-(y^2t^2+y^2)}y\ dt = \int_0^\infty ye^{-y^2(t^2+1)}\ dt.$$

Now since $\mathcal{R}$ is a rectangle, we can simply swap the order of integration, so

$$E(\infty)^2 = \int_0^\infty \int_0^\infty ye^{-y^2(t^2+1)}\ dt\ dy = \int_0^\infty \int_0^\infty ye^{-y^2(t^2+1)}\ dy\ dt.$$

(c) Evaluate the iterated integral.

Solution: The inner integral can now be evaluated:

$$\begin{aligned}\int_0^\infty ye^{-y^2(t^2+1)}\ dy &= \lim_{s\to\infty}\left[-\frac{e^{-y^2(t^2+1)}}{2(t^2+1)}\right]_0^s \\ &= \lim_{s\to\infty} -\frac{e^{-s^2(t^2+1)}}{2(t^2+1)} + \frac{1}{2(t^2+1)} = \frac{1}{2(t^2+1)}.\end{aligned}$$

Now we can evaluate the iterated integral:

$$\begin{aligned}E(\infty)^2 &= \int_0^\infty \int_0^\infty ye^{-y^2(t^2+1)}\ dy\ dt \\ &= \int_0^\infty \frac{1}{2(t^2+1)}\ dt \\ &= \lim_{s\to\infty}\left[\frac{1}{2}\arctan(t)\right]_0^s = \lim_{s\to\infty}\frac{1}{2}\arctan(s) = \frac{\pi}{4}.\end{aligned}$$

(d) Determine whether $E(\infty)$ is positive or negative. Find the value of $E(\infty)$.

Solution: The function e^{-x^2} is positive for all values of x and so its graph lies wholly above the x-axis. Thus any integral of this function will always be positive. In particular $E(s) > 0$ and so $E(\infty) > 0$. Thus we have that $E(\infty)$ is the positive square root of $E(\infty)^2$, so $E(\infty) = \frac{\sqrt{\pi}}{2}$.

(e) Explain why this method does not allow you to calculate $E(s)$ for more general $s < \infty$.

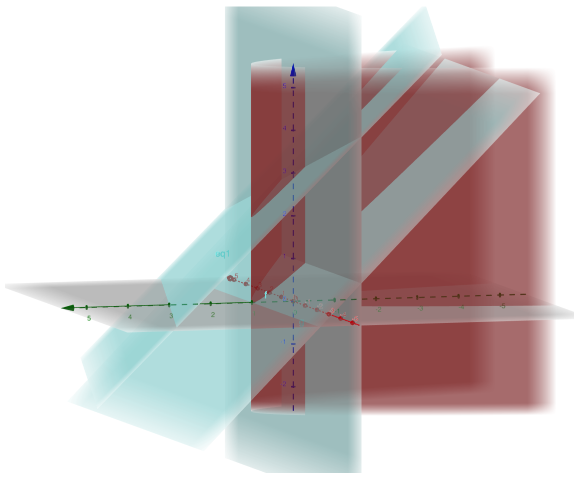
Solution: In part c), if we did not take the limit $\lim_{s\to\infty}$, we would have to integrate the function $\frac{e^{s^2(t^2+1)}}{t^2+1}$ which does not have an elementary description.

5. Find the volume of the region bounded by $y = 1 - x^2$, $z + y = 1, y = 0$ and $4z + 4y + x = 12$.

Solution: We can describe this region $\mathcal{E}$ by $1 - y \le z \le 3 - y - \frac{1}{4}x$ and $(x, y) \in \mathcal{D}$ where

$$\mathcal{D} = \left\{(x, y) \,\middle|\, \le 0 \le y \le 1 - x^2, -1 \le x \le 1\right\}$$

It helps to visualise this:



Now we can use a triple integral to calculate the volume.

$$\begin{aligned}
\mathrm{Vol}(\mathcal{E}) &= \iiint_{\mathcal{E}} 1 \, dV \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} \int_{1-y}^{3-y-\frac{1}{4}x} 1 \, dz \, dy \, dx \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} 2 - \frac{1}{4}x \, dy \, dx \\
&= \int_{-1}^{1} \left[\left(2 - \frac{1}{4}x\right) y\right]_0^{1-x^2} dx \\
&= \int_{-1}^{1} \frac{1}{4}(8 - x)(1 - x^2) \, dx \\
&= \frac{1}{4}\left[8x - \frac{1}{2}x^2 - \frac{8}{3}x^3 + \frac{1}{4}x^4\right]_{-1}^{1} \\
&= \frac{1}{4}\left(16 - \frac{16}{3}\right) = \frac{8}{3}.
\end{aligned}$$

6. Compute the integral $\iiint_{\mathcal{W}} xy \, dV$ where $\mathcal{W}$ is the part of the first octant inside the elliptical cyclinder $(x/2)^2 + (z/3)^2 = 1$ and inside the ellipsoid $(x/4)^2 + (y/4)^2 + (z/5)^2 = 1$.

Solution: Fist we see that the region $\mathcal{W}$ is y-simple, we can express it as

$$\mathcal{W} = \left\{ (x, y, z) \,\middle|\, 0 \le y \le \sqrt{1 - (x/4)^2 - (z/5)^2},\ (x, z) \in \mathcal{D}\right\}$$

where $\mathcal{D}$ is the region bounded by the ellipse $(x/2)^2+(x/3)^2=1$ in the first quadrant of the xz-plane. Thus we get

$$\begin{aligned}\iiint_{\mathcal{W}} xy\ dV &= \iint_{\mathcal{D}} \int_0^{4\sqrt{1-(x/4)^2-(z/5)^2}} xy\ dy\ dA_{xz} \\ &= \iint_{\mathcal{D}} 8x\left(1-\left(\frac{x}{4}\right)^2-\left(\frac{z}{5}\right)^2\right)\ dA_{xz}.\end{aligned}$$

Now we can describe the region $\mathcal{D}$ as vertically simple:

$$\mathcal{D}=\left\{\ (x,z)\ \middle|\ 0\le z\le 3\sqrt{1-(x/2)^2},\ 0\le x\le 2\ \right\}$$

and thus out integral becomes

$$\begin{aligned}\iiint_{\mathcal{W}} xy\ dV &= \int_0^2\int_0^{3\sqrt{1-(x/2)^2}} 8x\left(1-\left(\frac{x}{4}\right)^2-\left(\frac{z}{5}\right)^2\right)\ dz\ dx \\ &= \int_0^2 \left[8xz\left(1-\left(\frac{x}{4}\right)^2\right)-\frac{8xz^3}{3\cdot 25}\right]_0^{3\sqrt{1-(x/2)^2}}\ dx \\ &= \int_0^2 24x\sqrt{1-\frac{x^2}{4}}\left(1-\frac{x^2}{16}-\frac{3}{25}\left(1-\frac{x^2}{4}\right)\right)\ dx \\ &= \int_0^2 24x\sqrt{1-\frac{x^2}{4}}\left(\frac{22}{25}-\frac{13}{400}x^2\right)\ dx\end{aligned}$$

This is an integral we can solve using the substitution $u=1-\dfrac{x^2}{4}$. We have $du=-\dfrac{1}{2}x\ dx$ or $x\ dx=-2\ du$. Note that $x^2=1-4u$. Our integral becomes

$$\begin{aligned}\iiint_{\mathcal{W}} xy\ dV &= -48\int_1^0 \sqrt{u}\left(\frac{3}{4}+\frac{13}{100}u\right)\ du \\ &= \frac{12}{25}\int_0^1 \sqrt{u}\,(75+13u)\ du \\ &= \frac{12}{25}\int_0^1 75u^{\frac{1}{2}}+13u^{\frac{3}{2}}\ du \\ &= \frac{12}{25}\left[50u^{\frac{3}{2}}+\frac{26}{5}u^{\frac{5}{2}}\right]_0^1 \\ &= \frac{12}{25}\left(50+\frac{26}{5}\right)=\frac{3312}{125}=26.496\end{aligned}$$