Math 3B: Lecture 17

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Last time

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- Partial fractions and long division
- Accumulated change and Riemann sums

Differential equations (motivation)

An (ordinary) differential equation (or ODE) is an equation that involves derivatives of an unknown function.

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or

$$x^2 y'' + xy' + x^2 y = 0$$

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The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

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And so on.

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Note

The right hand side of the equation does not have any y's.

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- draw solutions for many other ODEs
- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)
- understand how sensitive ODEs are to their parameters.

Modelling using differential equations

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- The goal is to write down a function $y(t)$ that describes something we are interested in (e.g. population/mass/etc)
- as some other variable changes (usually time)
- We can't do this directly, but we can write down an ODE that y satisfies instead.

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- Number of deaths is proportional to the total number of people. So

  $$dN(t) \text{ deaths per year, for some } d$$

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The total change in population at time t is

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= \text{ births at } t - \text{ deaths at } t \\ &= bN(t) - dN(t) \\ &= (b-d)N(t).\end{aligned}$$

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In real life we would determine b and d experimentally. Let $r = b - d$. the instinsic growth rate. So our model is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

and we know $N(0) = 100$.

Behaviour of solutions

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

Case 1: $r = 0$

The population never grows or shrinks, it always stays the same (so $N(t) = 100$ for all t).

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Case 2: $r > 0$
The population is increasing indefinitely.

Case 3: $r < 0$
The population is decreasing indefinitely.

Solution to a simple ODE

Theorem

For any constant a, if y is a solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay$$

then y is given by

$$y = Ce^{ax}$$

for some constant C.

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Next time

We will see why, but for now we can verify it is actually a solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}Ce^{ax} = C\frac{\mathrm{d}}{\mathrm{d}x}e^{a}x = Cae^{ax} = ay.$$

Back to example 1

We know our population model was governed by

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$$100 = Ce^{(b-d)} \quad \text{so} \quad C = 100e^{(d-b)}.$$

Logistic growth

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Where $K = r/k$.

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$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= bN - (d + kN)N \\ &= (b - d - kN)N = (r - kN)N \\ &= r\left(1 - \frac{kN}{r}\right)N = r\left(1 - \frac{N}{K}\right)N\end{aligned}$$

Where $K = r/k$.

Logistic growth

The equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

is called the Logistic equation and K is the carrying capacity.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 1. $N(0) = 0$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains 0.

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Key takeaway

Both $N(t) = 0$ and $N(t) = K$ are solutions to the ODE. They are called equalibrium solutions.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 3. $0 \leq N(0) \leq K$

In this case, N is initially increasing and so becomes more positive, slowing down as it gets close to K.

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Case 3. $0 \leq N(0) \leq K$

In this case, N is initially increasing and so becomes more positive, slowing down as it gets close to K.

Case 4. $N(0) \geq K$

In this case N is initially decreasing but decreases slower and slower as it gets close to K.

Logistic growth with outside effects

We can also modify the logistic equation to get something which models an outside effect. For example harvesting.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N - h(N)$$

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This would model a population growing logistically but where we are harvesting at a rate of $h(N)$. E.g. we decide to continually harvest 3% of the population then

$$h(N) = 0.03N.$$