Midterm 2 practice 2

UCLA: Math 31B, Spring 2017

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Date:
Version: practice

- This exam has 4 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section: ______________________________

Question	Points	Score
1	8	
2	10	
3	15	
4	7	
Total:	40	

1. Calculate the following limits using any technique you like.

   (a) (4 points)
$$\lim_{x\to 0+} x(\ln x)^2$$

   (b) (4 points)
$$\lim_{x\to 0+} x^{\sin x}.$$

Solution:

(a) We turn the $0\cdot\infty$ indeterminate into an $\frac{\infty}{\infty}$ indeterminate and apply L'Hôpital's rule twice:

$$\begin{aligned}\lim_{x\to 0+} x(\ln x)^2 &= \lim_{x\to 0+}\left[\frac{(\ln x)^2}{\frac{1}{x}}\right] = \lim_{x\to 0+}\left[\frac{2(\ln x)\cdot\frac{1}{x}}{\frac{-1}{x^2}}\right]\\ &= \lim_{x\to 0+}\left[\frac{2\ln x}{\frac{-1}{x}}\right] = \lim_{x\to 0+}\left[\frac{\frac{2}{x}}{\frac{1}{x^2}}\right]\\ &= \lim_{x\to 0+} 2x = 0.\end{aligned}$$

(b) Let $y(x) = \ln\left[x^{\sin x}\right]$. Then $y(x) = (\sin x)(\ln x)$. Thus,

$$\begin{aligned}\lim_{x\to 0+} y(x) &= \lim_{x\to 0+}(\sin x)(\ln x) = \lim_{x\to 0+}\left[\frac{\ln x}{\frac{1}{\sin x}}\right] = \lim_{x\to 0+}\left[\frac{\frac{1}{x}}{\frac{-\cos x}{\sin^2 x}}\right]\\ &= -\lim_{x\to 0+}\left[\frac{\sin x}{x}\cdot\frac{\sin x}{\cos x}\right] = -\lim_{x\to 0+}\left[\frac{\sin x}{x}\right]\cdot\lim_{x\to 0+}\left[\frac{\sin x}{\cos x}\right] = -1\cdot 0 = 0.\end{aligned}$$

We conclude that $\lim_{x\to 0+} x^{\sin x} = e^0 = 1$.

2. For each of the following improper integrals say whether it converges or diverges. If the integral converges, you should say what value it converges to.
   (a) (5 points) $\int_0^\infty \frac{2x}{(1+x^2)^2}\ dx$.
   (b) (5 points) $\int_0^1 \ln x\ dx$.

Solution:

(a) It converges. We use the u-sub, $u = 1 + x^2$, so that $du = 2xdx$.

$$\begin{aligned}\int_0^\infty \frac{2x}{(1+x^2)^2}\ dx &= \lim_{S\to\infty}\int_0^S \frac{2x}{(1+x^2)^2}\ dx\\ &= \lim_{S\to\infty}\int_1^{1+S^2}\frac{1}{u^2}\ du = \lim_{S\to\infty}\left[\frac{-1}{u}\right]_1^{1+S^2}\\ &= \lim_{S\to\infty}\left[\frac{-1}{1+S^2}+1\right] = 1.\end{aligned}$$

(b) It converges. We do integration by parts. We also use the fact that $\lim_{S\to 0+} S\ln S = 0$ (use for example L'Hopital's rule),

$$\begin{aligned}\int_0^1 \ln x\ dx &= \lim_{S\to 0+}\int_S^1 \ln x\ dx\\ &= \lim_{S\to 0+}\left[x\ln x - x\right]_S^1\\ &= \lim_{S\to 0+}[-1 - S\ln S + S]\\ &= -1.\end{aligned}$$

3. For each of the following series say whether it converges or diverges. You do NOT need to justify your answer.

   Grading scheme: 0 points for wrong, 1 point for no response, 3 points for correct.

   (a) (3 points) $\sum_{n=1}^{\infty}(-1)^n \frac{2n}{1+n^2}$.

   (b) (3 points) $\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5-4}}$.

   (c) (3 points) $\sum_{n=0}^{\infty} \frac{2n}{(1+n^2)^2}$ (*Hint: once of the previous questions in the midterm might help*).

   (d) (3 points) $\sum_{n=1}^{\infty} a_n$ where the sequence of partial sums $(s_N)_{N=1}^{\infty}$ is described by

   $$s_N = N^{-\sin N^{-1}}.$$

   (e) (3 points) $\sum_{n=1}^{\infty} a_n$ where the sequence of partial sums $(s_N)_{N=1}^{\infty}$ is described by

   $$s_N = \sum_{n=1}^{N} n^{-\sin n^{-1}}.$$

Solution:

(a) Converges (use alternating series test).

(b) Converges (use limit comparison test with $b_n = 1/\sqrt{n^3}$).

(c) Converges (use integral test and question 2b).

(d) Converges (to 1, see question 1b).

(e) Diverges (since $a_n \to 1 \neq 0$).

4. (7 points) Does the series $\sum_{n=1}^{\infty} \sqrt[n]{n} - 1$ converge or diverge? Full points will only be given to solutions that justify the answer clearly.

Solution: Let $a_n = \sqrt[n]{n} - 1 = n^{1/n} - 1$. First we check that $a_n \to 0$. We can do this using L'Hopital's rule. Note that

$$\begin{aligned} \lim_{n\to\infty} \ln\left(n^{1/n}\right) &= \lim_{n\to\infty} \frac{1}{n} \ln(n) \\ &= \lim_{n\to\infty} \frac{1}{n} = 0. \end{aligned}$$

So $\lim_{n\to\infty} n^{1/n} = 1$ and $\lim_{n\to\infty} a_n = 0$. Thus the series has a chance to converge.

At this point we might be a bit stuck. It is not obvious how to integrate this function and it also may not be obvious how to use the comparison test. In situations like this it is worth trying to compare the series to a "standard" one, say $\sum \frac{1}{n}$. Let $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} n(n^{1/n} - 1) = \infty.$$

Since $\sum \frac{1}{n}$ diverges, then by the limit comparison test $\sum_{n=1}^{\infty} \sqrt[n]{n} - 1$ does as well.

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