Lecture 4

1. Triple integrals

- $f(x\ y\ z)$ is a function $\mathbb{R}^3 \to \mathbb{R}$ of 3 variables
- We would like to integrate f over regions $\mathcal{H} \subseteq \mathbb{R}^3$.
- As in the 2D case, we start with "rectanges" (boxes):

$$\mathcal{B} = \underset{x}{[a, b]} \times \underset{y}{[c, d]} \times \underset{z}{[p\ q]}$$

- Partition into invervals:

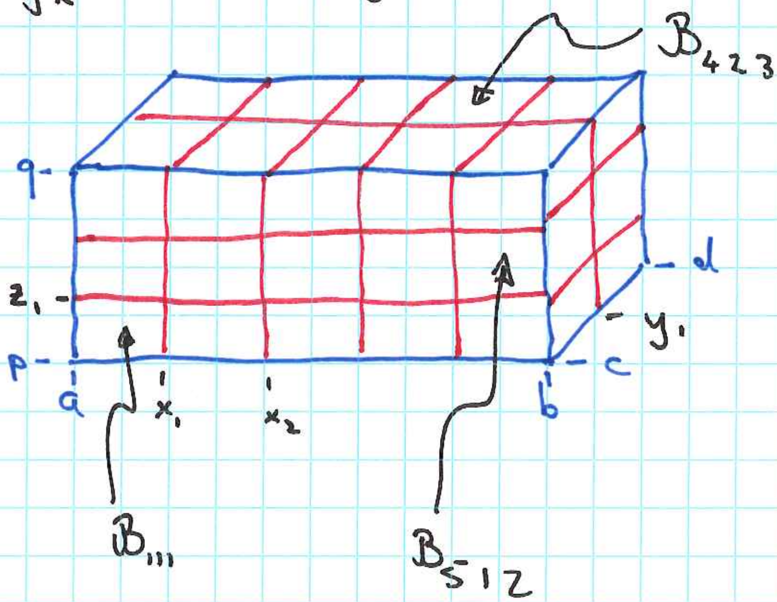
$$a = x_0 < x_1 < \ \_\_\_ \ < x_\ell = b$$

$$c = y_0 < y_1 < \ \_\_\_ \ < y_m = d$$

$$p = z_0 < z_1 < \ \_\_\_ \ < z_n = q$$

- This divides $\mathcal{B}$ into "subboxes"

$$\mathcal{B}_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}\ y_j] \times [z_{k-1}\ z_k]$$



- Choose a point P_{ijk} in each subbox B_{ijk}
- We can estimate the "4D volume" of the graph of $f(x\ y\ z)$ in $\mathbb{R}^4$ over B by

$$\sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(P_{ijk}) \underbrace{\Delta z_i \Delta y_j \Delta x_k}_{\Delta V_{ijk}}$$

where $\Delta x_k = x_k - x_{k-1}$

$\Delta y_j = y_j - y_{j-1}$

$\Delta z_i = z_i - z_{i-1}$

so that $\Delta V_{ijk} = \text{Volume}(B_{ijk})$.

- Denote the partition by $\mathcal{P}$ and let

$$\|\mathcal{P}\| = \max\{\Delta x_i, \Delta y_i, \Delta z_i\}.$$

Def The triple integral is defined to be

$$\iiint_B f(x\ y\ z)\, dV := \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(P_{ijk}) \Delta z_k \Delta y_j \Delta x_i$$

Thm (Fubini)

$$\iiint_B f(x\ y\ z)\, dV = \int_{x=a}^{b} \int_{y=c}^{d} \int_{z=p}^{q} f(x\ y\ z)\, dz\, dy\, dx$$

Exercise What is the analogus "separation of variables" result?

Ex $\iiint_B xyz\, dV$ for $B = [0\ 1]\times[-1\ 1]\times[-1,0]$

$$= \int_{x=0}^{1}\int_{y=-1}^{1}\int_{z=-1}^{0} xyz\, dz\,dy\,dx$$

$$= \int_0^1\int_{-1}^{1}\left[\tfrac{1}{2}xyz^2\right]_{-1}^{0} dy\,dx$$

$$= \int_0^1\int_{-1}^{1} \tfrac{1}{2}xy\, dy\,dx$$

$$= \int_0^1\left[\tfrac{1}{4}xy^2\right]_{-1}^{1} dx$$

$$= \int_0^1 0\, dx$$

$$= 0$$

~~We also have an analogue of simple regions~~

2 More general regions $E \subseteq \mathbb{R}^3$

- We require E to be closed
- The boundary of E should be "simple" + "smooth"

we won't say precisely what these mean in this context, rather lets see some examples

- If $z_1(x\ y)$ and $z_2(x\ y)$ are smooth functions then the region

$$\mathcal{E} = \left\{ (x\ y\ z) \mid (x\ y) \in \mathcal{D}\ ,\ z_1(x\ y) \leq z \leq z_2(x\ y) \right\} \quad (*)$$

where $\mathcal{D} \subseteq \mathbb{R}^2$ is a closed region with simple p/w smooth boundary, then $\mathcal{E}$ satisfies the conditions

~~above~~

- If $f(x\ y\ z)$ is a function on $\mathcal{E}$ and $\mathcal{B}$ is a box $[a\ b]\times[c\ d]\times[p\ q]$ containing $\mathcal{E}$ define

$$\tilde{f}(x\ y\ z) = \begin{cases} f(x\ y\ z) & \text{if } (x\ y\ z) \in \mathcal{E} \\ 0 & \text{o/w} \end{cases}$$

<u>Def</u> $$\iiint_{\mathcal{E}} f(x\ y\ z)\, dV := \iiint_{\mathcal{B}} \tilde{f}(x\ y\ z)\, dV$$

With $\mathcal{E}$ as in $*$ we have

<u>Thm</u>

$$\iiint_{\mathcal{E}} f(x\ y\ z)\, dV = \iint_{\mathcal{D}} \left(\int_{z_1(x\ y)}^{z_2(x\ y)} f(x\ y\ z)\, dz \right) dA$$

Rmk $\iiint_{\mathcal{E}} 1 \, dV = \text{Volume}(\mathcal{E})$

Ex Integrate y over the region $\mathcal{E}$ bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane.

- We can express $\mathcal{E}$ as

$$\mathcal{E} = \{ (x\ y\ z) \mid (x\ y) \in D,\ 0 \leq z \leq 1 - x^2 - y^2 \}$$

where

$$D = \{ (x\ y) \mid x^2 + y^2 \leq 1 \}. \text{ (the unit disk)}$$

- Thus

$$\iiint_{\mathcal{E}} \cancel{f(x\ y\ z)}\, y \, dV = \iint_D \int_0^{1-x^2-y^2} y \, dz \, dA$$

But $D = \{ (x\ y) \mid -1 \leq x \leq 1,\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \}$

so for a function $f(x\ y)$

$$\iint_D f(x\ y) \, dA = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x\ y) \, dy \, dx$$

- Thus

$$\iiint_{\mathcal{E}} \cancel{f}\, y \, dV = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} y \, dz \, dy \, dx$$

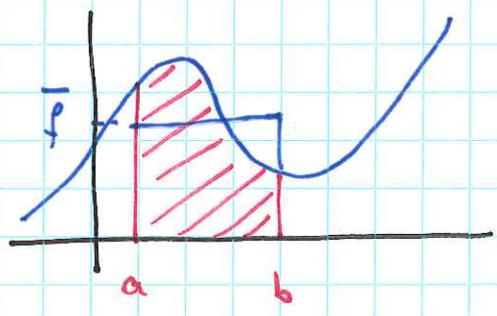
- Now we just have an iterated integral to solve (exercise).

3. Average value + MVT

- In 1D: define the average value of $f(x)$ on $[a\ b] \subseteq \mathbb{R}$ to be

$$\bar{f} := \frac{1}{b-a} \int_a^b f(x)\, dx$$

(Note that $b - a = \int_a^b 1\, dx$)



we have $\bar{f} \cdot (b-a) = \int_a^b f(x)\,dx$

Thm (Mean value thm) If $f(x)$ is cts on $[a\ b]$ then $\exists$ a point $P \in D$ s.t.

$$\int_a^b f(x)\,dx = f(P)\,(b-a)$$

- The function must take on its average value somewhere in the domain

- In 2D: define the average value of $f(x\,y)$ on $D \subseteq \mathbb{R}^2$ to be

$$\bar{f} := \frac{1}{\text{Area}(D)} \iint_D f(x\,y)\,dA$$

(Note $\text{Area}(D) = \iint_D 1\,dA$)

- We have $\text{Area}(D) \cdot \bar{f} = \iint_D f(x,y)\,dA$

Thm (MVT) If $f(x\,y)$ is cts on D, an D is connected, $\exists$ a point $P \in D$ s.t.

$$\iint_D f(x\,y)\,dA = f(P) \cdot \text{Area}(D)$$

- In 3D: define the average value of $f(x\,y\,z)$ on $\mathcal{E} \subseteq \mathbb{R}^3$ to be

$$\bar{f} := \frac{1}{\text{Vol}(\mathcal{E})} \iiint_{\mathcal{E}} f(x\,y\,z)\,dV$$

(Note $\text{Vol}(\mathcal{E}) = \iiint_{\mathcal{E}} 1\,dV$)

- We have $\text{Vol}(\mathcal{E}) \cdot \bar{f} = \iiint_{\mathcal{E}} f(x\,y\,z)\,dV$

Thm (MVT) If $f(x\,y\,z)$ is cts on $\mathcal{E}$ and $\mathcal{E}$ is connected, $\exists$ a point $P \in \mathcal{E}$ s.t.

$$\iiint_{\mathcal{E}} f(x\,y\,z)\,dV = f(P) \cdot \text{Vol}(\mathcal{E}).$$