# Math 3B: Lecture 9

Noah White

October 12, 2016

# Review lecture

- Poll on Piazza

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- Please vote for what you want covered in the review lecture

# Last time

- Finding area under curves

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- Definition of definite integral using Reimann Sum

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- Finding area under curves
- Definition of definite integral using Reimann Sum
- Using summation identities to calculate definite integrals

# The definite integral

## Defintion

The definite integral of a function $f(x)$ is defined to be

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=0}^{n-1} f(a + k \cdot \Delta x)$$

# Example 2

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Evaluate the definite integral

$$\int_0^1 x^2 - 4 \,\mathrm{d}x$$

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$$= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{k^2}{n^2} - 4$$

# Example 2

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&= \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \frac{k^2}{n^2} - 4 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{k=0}^{n-1} k^2 - \frac{1}{n} \sum_{k=0}^{n-1} 4
\end{aligned}$$

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&= \frac{1}{3} - 4 = -\frac{11}{3}
\end{aligned}$$

# The fundamental theorem of calculus

### Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t) \, \mathrm{d}t = f(x)$$

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That is, $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of $f(x)$!

**Note**

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**Note**

- $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is a function of $x$.
- every input $x$ produces a number as an output.

# Why is the FTC true?

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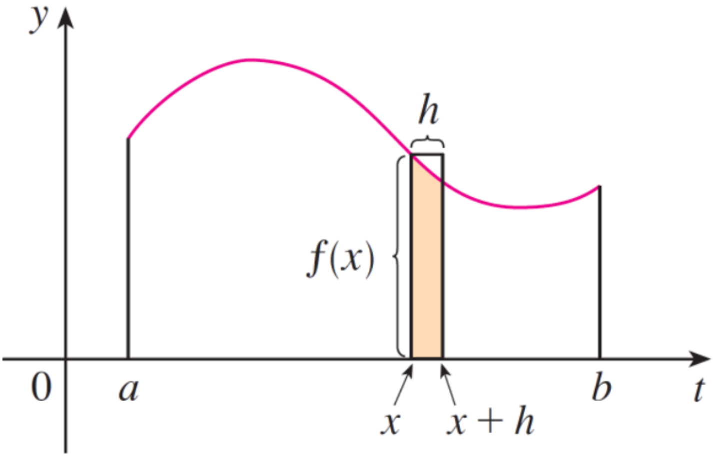
When $h$ is very small $\displaystyle\int_x^{x+h} f(t) \approx$



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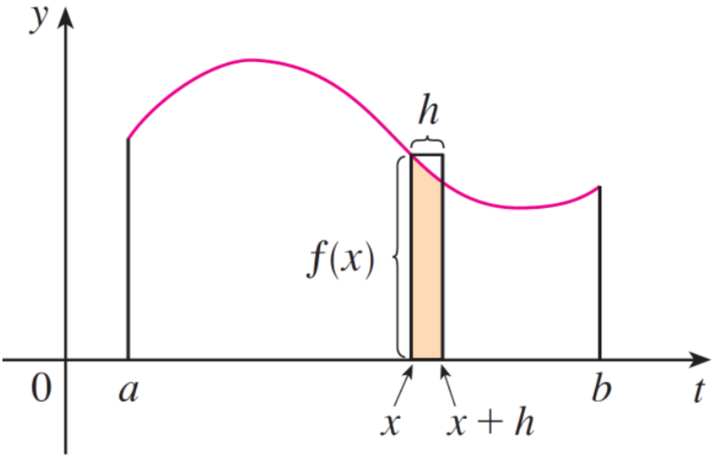

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# A consequence (corrollary)

## Corollary

For any antiderivative $F(x)$ of $f(x)$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

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## Why?

Well $F(x) = \int_a^x f(t)\,\mathrm{d}t + C$ for some $a$ and $C$. So

$$\begin{aligned} F(b) - F(a) &= \int_a^b f(t)\,\mathrm{d}t + C - \int_a^a f(t)\,\mathrm{d}t - C \\ &= \int_a^b f(t)\,\mathrm{d}t \end{aligned}$$

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Evaluate the definite integral

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## Solution

An antiderivative of $x^2 - 4$ is $\frac{1}{3}x^3 - 4x$ so

$$\begin{aligned}\int_0^1 x^2 - 4 \, \mathrm{d}x &= \frac{1}{3} \cdot 1^3 - 4 - \frac{1}{3} \cdot 0^3 + 4 \cdot 0 \\ &= \frac{1}{3} - 4 = -\frac{11}{3}\end{aligned}$$

# Example 2

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Evaluate the definite integral

$$\int_0^{\pi} \sin x \, \mathrm{d}x$$

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## Solution

An antiderivative of $\sin x$ is $-\cos x$ so

$$\begin{aligned} \int_0^\pi \sin x \, \mathrm{d}x &= -\cos \pi + \cos 0 \\ &= -(-1) + 1 = 2 \end{aligned}$$

# The indefinite integral

We also use the following notation for the general antiderivative:

**Definition**

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**Example**

$$\int \sin(x) - x\,\mathrm{d}x = -\cos(x) - \frac{1}{2}x^2 + C$$

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## Substitution

Suppose $u = g(x)$, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

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We use the substitution $u = x^2 + 1$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we can write the integral

$$\int 2 \cdot \sqrt{x^2+1} \cdot 2x\,\mathrm{d}x = 2\int \sqrt{u}\,\mathrm{d}u$$

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## Substitution for definite integrals

Suppose $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

# Integration by substitution (definite integrals)

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## Example

$$\int_0^1 4x\sqrt{x^2+1}\,\mathrm{d}x = 2\int_1^2 \sqrt{u}\,\mathrm{d}u$$

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Suppose $u = g(x)$, then

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## Example

$$\begin{aligned}\int_0^1 4x\sqrt{x^2+1}\,\mathrm{d}x &= 2\int_1^2 \sqrt{u}\,\mathrm{d}u \\ &= 2\left(\frac{2}{3}2^{\frac{3}{2}} - \frac{2}{3}1^{\frac{3}{2}}\right) = \frac{4}{3}(2\sqrt{2}-1)\end{aligned}$$