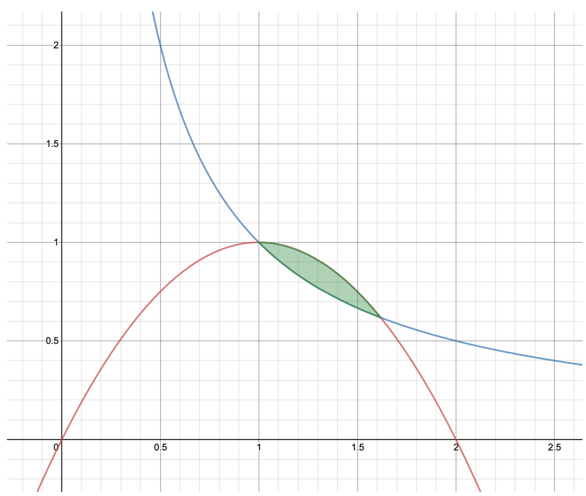
This week on the problem set we will see examples of integrals over more general regions.

You will only need to hand in a small selection of the questions for homework, however I recommend that you at least attempt them all by the end of the quarter as some may appear on exams!

Homework: due Friday 10 April, uploaded to Gradescope before 11:59pm. It will consist of questions $3, 4, 5$, and 6 below.
Note that the references to the textbook are for the 4^{th} edition, *late transcendentals* version. Any differences between the 3^{rd} and 4^{th} editions is noted in parentheses.

1. From 16.2 in the textbook: 4, 8, 14, 20, 21, 23, 29, 31, 45, 48, 49 (Question 21 is different in the two versions, but both are fine.).
2. From 16.3 in the textbook: 3, 5, 6, 7.
3. Consider an integral over the domain $\mathcal{D}$ that is the part of the first quadrant bounded by $y = -(x-1)^2+1$ and $y = 1/x$. We can write an integral over this domain as: $\displaystyle\int_1^{\frac{1+\sqrt{5}}{2}} \int_{1/x}^{-(x-1)^2+1} f(x, y)\, dy\, dx$. Change the order of integration to write this as an integral where you integrate in the order $dx\, dy$.

Solution: We first graph the region in the question.



Now we find the intersection points buy setting

$$\frac{1}{x} = 1 - (x-1)^2$$
$$1 = -x^3 + 2x^3$$
$$x^3 - 2x^2 + 1 = 0$$

We can easily see that $x = 1$ is a solution, and factorising we see that $x^3 - 2x^2 + 1 = (x-1)(x^2 - x - 1)$, so we see that the intersection points are

$$(1, 1), \left(\frac{1+\sqrt{5}}{2}, \frac{2}{1+\sqrt{5}}\right), \left(\frac{1-\sqrt{5}}{2}, \frac{2}{1-\sqrt{5}}\right).$$

Only the first two are in the first quadrant, so these are the ones we are looking for. This allows us to give a vertically simple description,

$$\mathcal{D} = \left\{ (x,y) \;\middle|\; \frac{1}{x} \le y \le 1-(x-1)^2,\ 1 \le x \le \frac{1+\sqrt{5}}{2} \right\},$$

which is used to show that

$$\iint_{\mathcal{D}} f(x,y)\, dA = \int_1^{\frac{1+\sqrt{5}}{2}} \int_{1/x}^{-(x-1)^2+1} f(x,y)\, dy\, dx$$

but we can also give a horizontally simple description

$$\mathcal{D} = \left\{ (x,y) \;\middle|\; \frac{1}{y} \le x \le 1+\sqrt{(1-y)},\ \frac{2}{1+\sqrt{5}} \le y \le 1 \right\},$$

where we have used the fact that the bounding curves can be rearranged to $x = 1/y$ and $x = 1+\sqrt{(1-y)}$. This allows us to change the order of integration and give

$$\iint_{\mathcal{D}} f(x,y)\, dA = \int_{\frac{2}{1+\sqrt{5}}}^{1} \int_{1/y}^{1+\sqrt{(1-y)}} f(x,y)\, dx\, dy$$

4. Consider the function $E(s) = \displaystyle\int_0^s e^{-x^2}\, dx$. This is an incredibly important function in applied mathematics (and therefore physics, chemistry, etc). Unfortunately it is impossible to express the antiderivative of e^{-x^2} in terms of functions you already know. So how can we calculate $E(s)$? It turns out, that its value at infinity,

$$E(\infty) := \lim_{s\to\infty} E(s) = \int_0^\infty e^{-x^2}\, dx,$$

cal be calculated using a trick which this question will guide you through. In fact, we will calculate $E(\infty)^2$.

(a) Express $E(\infty)^2 = \left(\displaystyle\int_0^\infty e^{-x^2}\, dx\right)\left(\displaystyle\int_0^\infty e^{-y^2}\, dy\right)$ as a double integral and therefore as an iterated integral, in the order $dx\, dy$. Make sure to describe the region in $\mathbb{R}^2$ we are integrating over precisely. *Hint: consider the separation of variables formula.*

Solution: We use separation of variables in reverse.

$$\begin{aligned} E(\infty)^2 &= \left(\int_0^\infty e^{-x^2}\, dx\right)\left(\int_0^\infty e^{-y^2}\, dy\right) \\ &= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy \\ &= \iint_{\mathcal{R}} e^{-(x^2+y^2)}\, dA, \end{aligned}$$

where $\mathcal{R} = [0,\infty) \times [0,\infty)$ is the first quadrant in the plane.

(b) Use the change of variables $t = x/y$ to transform the inner integral. Express $E(\infty)^2$ as an iterated integral in the order $dy\, dt$.

Solution: We are concentrating on the integral $\int_0^\infty e^{-(x^2+y^2)}\,dx$, where y is held constant. To make the change of variables observe $dt = \frac{1}{y}dx$ and the limits remain the same. Thus

$$\int_0^\infty e^{-(x^2+y^2)}\,dx = \int_0^\infty e^{-(y^2t^2+y^2)}y\,dt = \int_0^\infty ye^{-y^2(t^2+1)}\,dt.$$

Now since $\mathcal{R}$ is a rectangle, we can simply swap the order of integration, so

$$E(\infty)^2 = \int_0^\infty \int_0^\infty ye^{-y^2(t^2+1)}\,dt\,dy = \int_0^\infty \int_0^\infty ye^{-y^2(t^2+1)}\,dy\,dt.$$

(c) Evaluate the iterated integral.

Solution: The inner integral can now be evaluated:

$$\begin{aligned}\int_0^\infty ye^{-y^2(t^2+1)}\,dy &= \lim_{s\to\infty}\left[-\frac{e^{-y^2(t^2+1)}}{2(t^2+1)}\right]_0^s \\ &= \lim_{s\to\infty} -\frac{e^{-s^2(t^2+1)}}{2(t^2+1)} + \frac{1}{2(t^2+1)} = \frac{1}{2(t^2+1)}.\end{aligned}$$

Now we can evaluate the iterated integral:

$$\begin{aligned}E(\infty)^2 &= \int_0^\infty \int_0^\infty ye^{-y^2(t^2+1)}\,dy\,dt \\ &= \int_0^\infty \frac{1}{2(t^2+1)}\,dt \\ &= \lim_{s\to\infty}\left[\frac{1}{2}\arctan(t)\right]_0^s = \lim_{s\to\infty}\frac{1}{2}\arctan(s) = \frac{\pi}{4}.\end{aligned}$$

(d) Determine whether $E(\infty)$ is positive or negative. Find the value of $E(\infty)$.

Solution: The function e^{-x^2} is positive for all values of x and so its graph lies wholly above the x-axis. Thus any integral of this function will always be positive. In particular $E(s) > 0$ and so $E(\infty) > 0$. Thus we have that $E(\infty)$ is the positive square root of $E(\infty)^2$, so $E(\infty) = \frac{\sqrt{\pi}}{2}$.

(e) Explain why this method does not allow you to calculate $E(s)$ for more general $s < \infty$.

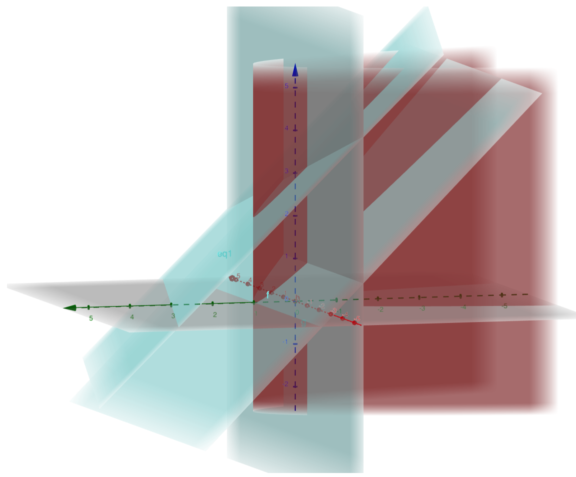
Solution: In part c), if we did not take the limit $\lim_{s\to\infty}$, we would have to integrate the function $\frac{e^{s^2(t^2+1)}}{t^2+1}$ which does not have an elementary description.

5. Find the volume of the region bounded by $y = 1 - x^2$, $z + y = 1, y = 0$ and $4z + 4y + x = 12$.

Solution: We can describe this region $\mathcal{E}$ by $1-y \leq z \leq 3-y-\frac{1}{4}x$ and $(x,y) \in \mathcal{D}$ where

$$\mathcal{D} = \left\{(x,y) \;\middle|\; \leq 0 \leq y \leq 1-x^2, -1 \leq x \leq 1\right\}$$

It helps to visualise this:



Now we can use a triple integral to calculate the volume.

$$\begin{aligned}
\mathrm{Vol}(\mathcal{E}) &= \iiint_{\mathcal{E}} 1 \; dV \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} \int_{1-y}^{3-y-\frac{1}{4}x} 1 \; dz \; dy \; dx \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} 2 - \frac{1}{4}x \; dy \; dx \\
&= \int_{-1}^{1} \left[\left(2-\frac{1}{4}x\right) y\right]_0^{1-x^2} dx \\
&= \int_{-1}^{1} \frac{1}{4}(8-x)(1-x^2) \; dx \\
&= \frac{1}{4}\left[8x - \frac{1}{2}x^2 - \frac{8}{3}x^3 + \frac{1}{4}x^4\right]_{-1}^{1} \\
&= \frac{1}{4}\left(16 - \frac{16}{3}\right) = \frac{8}{3}.
\end{aligned}$$

6. Compute the integral $\iiint_{\mathcal{W}} xy \; dV$ where $\mathcal{W}$ is the part of the first octant inside the elliptical cyclinder $(x/2)^2 + (z/3)^2 = 1$ and inside the ellipsoid $(x/4)^2 + (y/4)^2 + (z/5)^2 = 1$.

Solution: This is the integral $\int_0^3 \int_0^{2\sqrt{1-z^2/9}} \int_0^{4\sqrt{1-(x/4)^2-(z/5)^2}} xy \; dy \; dx \; dz$. Integrating once gives $\displaystyle\int_0^3 \int_0^{2\sqrt{1-z^2/9}} 8x(1-(x/4)^2-(z/5)^2)\,dx\,dz.$