Lecture 17 Applications + accumulated change continued.

What do these examples have in common?

- Add up up a quantity that is changing (eg. patients remaining / volume of disk).
- There is a "direction of change" (e.g. time / height)

Steps to ~~sto~~ solving such a problem:

- decide on the "direction of the change"
- discretize / subdivide using n subintervals
- assume all the ~~s~~ change happens suddenly at the beggining of ~~ech~~ each subinterval.
- Calculate the contribution for each subinterval
- Add together and let $n \longrightarrow \infty$
- Interpret as Riemann sum
- Solve integral.

Work

- Work is measured in Joules
- 1J = amount of energy expended moving a mass 1 metre using 1 Newton of force.
- From wikipedia : - 1J = energy required to lift a 100 g mass 1 meter above the Earth's surface.
  - Heat required to raise the temp of 1g of water by 0.24°C.

If F newtons of force are applied to move a mass d meters then the work done is

$$W = Fd \text{ J}.$$

Small example : Work done lifting 30 kg by 20 meters.

Solution Acceleration due to gravity = $-9.8 \, m/s^2$

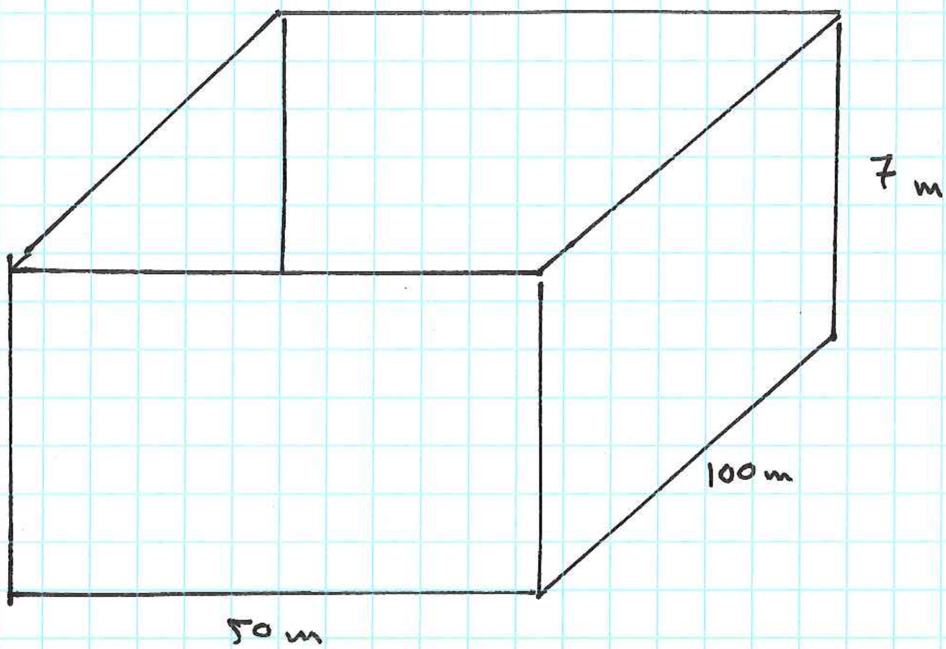
So force need to lift = $(9.8) \cdot 30$

$$W = (9.8) \cdot 30 \cdot 20$$
$$= 5886 \text{ J}.$$

Example 3

- A hole, $100\,m \times 50\,m$ big, $7\,m$ deep to be dug.
- Assume $1\,m^3$ of dirt weights 1000 kg.

Q How ~~to~~ much work is being done by ~~dirt~~ digging the hole?



Silly solution: There is $5000 \times 7\ m^3$ of dirt

$= 35\,000\,000$ kg of dirt

- Needs to be moved 7 m up
- Force needed $= (9.8) \cdot 35\,000\,000$

$= 343\,000\,000$ N

- Work $W = 343\,000\,000 \cdot 7$

$= 2\,401\,000\,000$ J.

But we shouldn't have to lift all the dirt the entire 7m up! E.g. the top layer only needs to be lifted ~ 0m

<u>Important</u>: How far below the surface the dirt is, determines how far it needs to be lifted!

- Direction of change = distance below surface = depth! = d.
- Subdivide into layers (n layers) each one $\Delta d = \frac{7}{n}$ m thick.
- The k^{th} layer (starting at $k=0$) is

$$d_k = k \cdot \Delta d = \frac{7k}{n} \text{ m deep}$$

- The k^{th} layer contains

$$100 \cdot 50 \cdot \Delta d \text{ m}^3 \text{ of dirt}$$

So weighs $5000\,000\,\Delta d$ kg, thus the work is

$$W_k = \underbrace{\underbrace{(9.8)}_{\text{acceleration}}\underbrace{5000\,000\,\Delta d}_{\text{mass}}}_{\text{Force}} \cdot \underbrace{d_k}_{\text{distance}}$$

- Adding together and letting $n \to \infty$

$$W = \lim_{n \to \infty} \sum_{k=0}^{n-1} 49000\,000\,\Delta d\, d_k$$

• Interpret as integral:

$$W = \int_0^7 49\,000\,000\, d \; dd$$

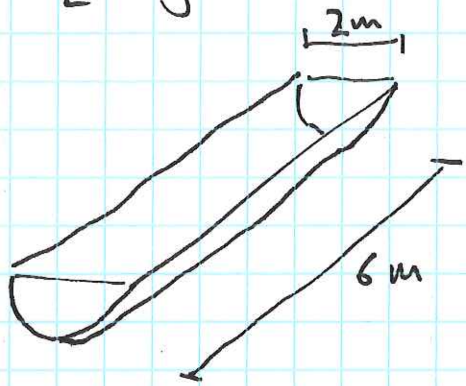
oops! terrible variable choice!

$$= \left[\frac{1}{2} 49\,000\,000\, d^2 \right]_0^7$$

$$= 1\,200\,500\,000 \text{ J}.$$

Example 4

What about a $\frac{1}{2}$ cylindrical trench?



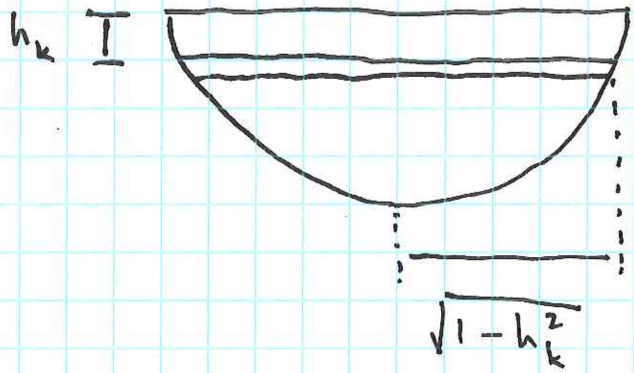
• Now the size of each layer is ~~no~~ changing w/ the depth as well.

• Let h = depth below ground.

• divide into n layers, $\Delta h = \frac{1}{n}$ m thick

• The k^{th} layer

  – $h_k = k\frac{1}{n}$ m below ground.



so k^{th} layer is

$$6\cdot 2\sqrt{1-h_k^2}\ \Delta h\ m^3 = 12000\sqrt{1-h_k^2}\,\Delta h\, kg$$

of dirt.

- Work needed to lift k^{th} slice:

$$(9.8)\cdot 12000\sqrt{1-h_k^2}\ \Delta h\cdot h_k$$

- Adding

$$W = \lim_{n\to\infty}\sum_{k=0}^{n-1} 117600\, h_k\sqrt{1-h_k^2}\ \Delta h$$

- As an integral

$$= \int_0^1 117600\, h\sqrt{1-h^2}\ dh$$

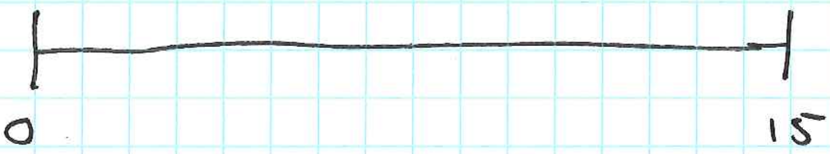
$$= \int_1^0 -58800\sqrt{u}\ du$$

$$= \left[58800\cdot 2\cdot\frac{1}{3}\, u^{3/2}\right]_0^1$$

$= 39\,200$ J.

Examples revisited

1.



t = time

- When clinic is t months old:

  $10\,dt$ patients start

- They have $15 - t$ months left so

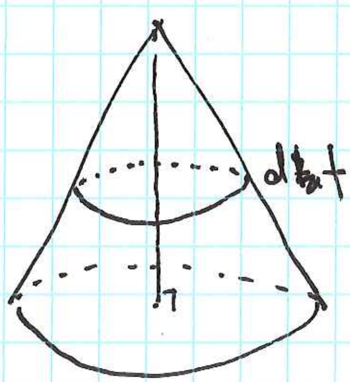
  $$10e^{(t-15)/20}\,dt$$

  of these patients remain

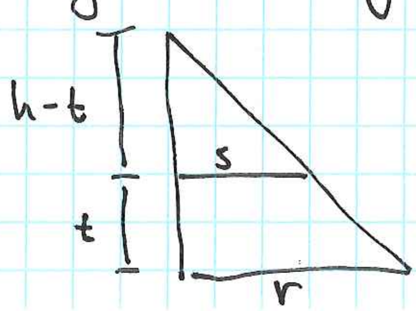
- "Summing infinitely"

$$\text{Total left} = \int_0^{15} 10e^{(t-15)/20}\,dt$$

2.



- Divide cone into infinitely many layers of thick-ness dt
- Layer at height t



$$\frac{s}{r} = \frac{h-t}{h}$$

So layer has volume

$$\pi s^2 \, dt = \pi \frac{r^2}{h^2}(h-t)^2 \, dt$$

- "Adding infinitely" from $t=0$ to $t=h$

$$\text{Vol} = \int_0^h \pi \frac{r^2}{h^2}(h-t)^2 \, dt$$

$$\vdots$$

$$= \frac{1}{3} h \pi r^2$$

<u>3</u>
- Divide pit into infinitely many layers of thickness dh.
- layer at depth h has weight

  $5000\cancel{8}000 \, dh$ kg

  so work needed to lift it h meters is

  $$49\,000\,000 h \, dh$$
- Summing infinitely from $h=0$ to $h=7$

$$W = \int_0^7 49\,000\,000 h \, dh.$$