

This weeks problem set provides some review questions in the lead up to the second midterm. A question marked with a \dagger is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a $*$ is especially important.

Homework 4: due Friday 24 May: questions 3 and 5 below.

1. From section 5.2, problems 1, 3a, d, e, 8, 9, 10, 11, 18*, 19, 20 \dagger .
2. From section 6.1, problems 1, 2, 3, 4, 8*, 9, 12, 16, 17*, 23, 29.
3. Let $T : V \longrightarrow V$ be a diagonalisable linear operator. Let $C(T) \subseteq \text{Hom}(V, V)$ be the set of all linear maps that commute with T . I.e

$$C(T) = \{S \in \text{Hom}(V, V) \mid S \circ T = T \circ S\}.$$

- (a) If T has $n = \dim V$ distinct eigenvalues, show that any $S \in C(T)$ is diagonalisable.
 - (b) Describe explicitly $C(T)$ in the case $T = x \frac{d}{dx} : \mathbb{C}_1[x] \longrightarrow \mathbb{C}_1[x]$.
 - (c) Show that part (a) does not necessarily hold if T does not have n distinct eigenvalues.
4. Suppose U, W are subspaces of a finite dimensional vector space V and that $U + W = V$. Show that $U \oplus W = V$ if and only if $\dim U + \dim W = \dim V$.

The previous question, motivates the following definition.

Definition: If U_i , for $1 \leq i \leq k$, are subspaces of a vector space V , then we say $V = U_1 \oplus U_2 \dots \oplus U_k$ if $V = U_1 + U_2 + \dots + U_k$, i.e. every vector $v \in V$ can be written as a sum $v = \sum_{i=1}^k u_i$ with $u \in U_i$, and $\dim V = \sum_{i=1}^k \dim U_i$.

- 5* Suppose that V is a finite dimensional vector space over \mathbb{F} and $T : V \longrightarrow V$ is a linear operator, with distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Prove that

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \dots \oplus E_{\lambda_k}$$

if and only if T is diagonalisable.