Math 3B: Lecture 9

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October 16, 2018

The fundamental theorem of calculus

Theorem

For any a,

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\,\mathrm{d}t = f(x)$$

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- $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is a function of x.
- every input x produces a number as an output.

A consequence (corrollary)

Corollary

For any antiderivative $F(x)$ of $f(x)$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

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Why?

Well $F(x) = \int_a^x f(t)\,\mathrm{d}t + C$ for some a and C. So

$$\begin{aligned} F(b) - F(a) &= \int_a^b f(t)\,\mathrm{d}t + C - \int_a^a f(t)\,\mathrm{d}t - C \\ &= \int_a^b f(t)\,\mathrm{d}t \end{aligned}$$

Example 1

Question

Evaluate the definite integral

$$\int_0^1 x^2 - 4 \, \mathrm{d}x$$

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Solution

An antiderivative of $x^2 - 4$ is $\frac{1}{3}x^3 - 4x$ so

$$\begin{aligned} \int_0^1 x^2 - 4 \, \mathrm{d}x &= \frac{1}{3} \cdot 1^3 - 4 - \frac{1}{3} \cdot 0^3 + 4 \cdot 0 \\ &= \frac{1}{3} - 4 = -\frac{11}{3} \end{aligned}$$

Example 2

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Evaluate the definite integral

$$\int_0^\pi \sin x \, \mathrm{d}x$$

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$$\int_0^{\pi} \sin x \, \mathrm{d}x$$

Solution

An antiderivative of $\sin x$ is $-\cos x$ so

$$\begin{aligned} \int_0^{\pi} \sin x \, \mathrm{d}x &= -\cos \pi + \cos 0 \\ &= -(-1) + 1 = 2 \end{aligned}$$

Why is the FTC true?

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

We want to show that $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.

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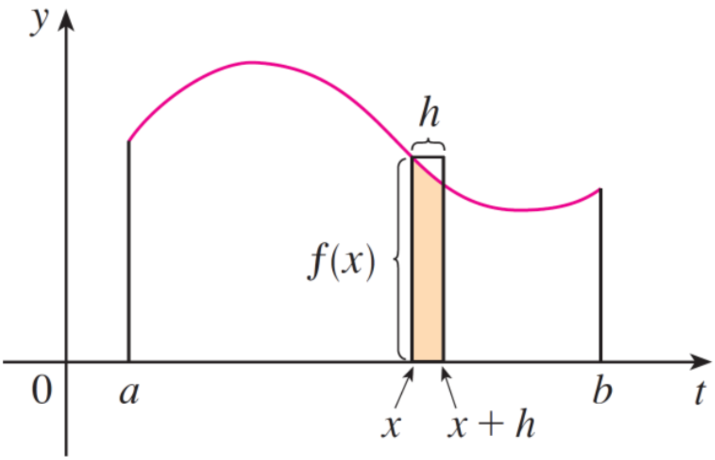
When h is very small $\displaystyle\int_x^{x+h} f(t) \approx$



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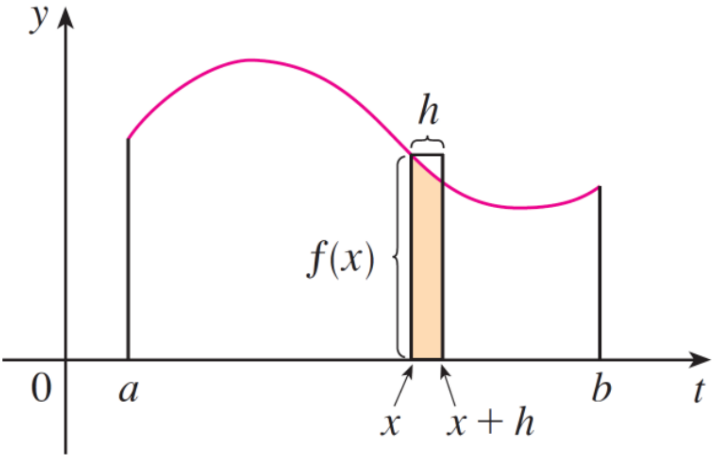

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&= \lim_{h\to 0} \frac{1}{h} h f(x) \\
&= f(x)
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The indefinite integral

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Definition

$$\int f(x)\,\mathrm{d}x := F(x) + C$$

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Example

$$\int \sin(x) - x\,\mathrm{d}x = -\cos(x) - \frac{1}{2}x^2 + C$$

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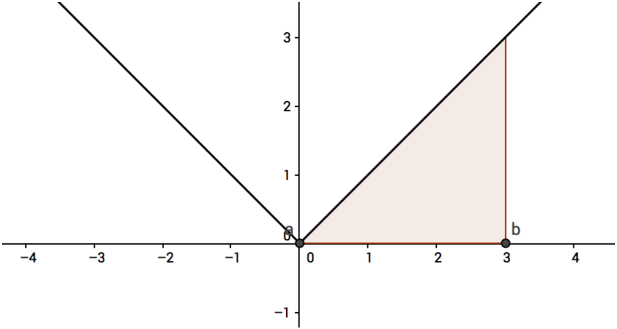
$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$
is an antiderivative for any choice of a.
- Lets use $a = 0$.
- How should we calculate $F(x)$?

Example

Use the defintition!

$$F(x) = \int_0^x |t| \, \mathrm{d}t$$

is the area under $|t|$!



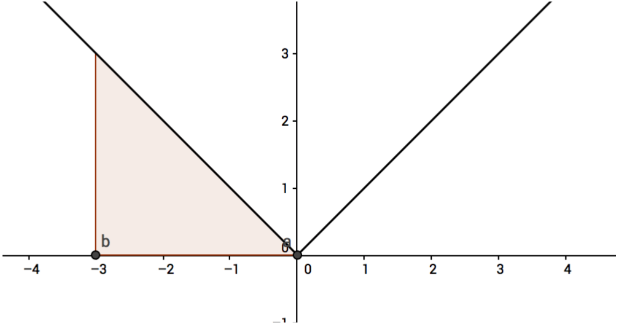
$$F(x) = \frac{1}{2}x^2 \quad \text{if } x \geq 0$$

Example

If $x \leq 0$ then

$$F(x) = \int_0^x |t| \, \mathrm{d}t = -\int_x^0 |t| \, \mathrm{d}t$$

is the negative of the area under $|t|$!



$$F(x) = -\frac{1}{2}x^2 \quad \text{if } x \leq 0$$

Example

In summary

$$F(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 & \text{if } x \leq 0 \end{cases}$$

or

$$F(x) = \frac{1}{2}x\,|x|$$

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Substitution

Suppose $u = g(x)$, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

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We use the substitution $u = x^2 + 1$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we can write the integral

$$\int 2 \cdot \sqrt{x^2+1} \cdot 2x\,\mathrm{d}x = 2\int \sqrt{u}\,\mathrm{d}u$$

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$$\begin{aligned}\int 2 \cdot \sqrt{x^2+1} \cdot 2x\,\mathrm{d}x &= 2\int \sqrt{u}\,\mathrm{d}u \\ &= 2 \cdot \frac{2}{3}u^{\frac{3}{2}} + C\end{aligned}$$

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Integration by substitution (definite integrals)

Substitution for definite integrals

Suppose $u = g(x)$, then

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$$\int_0^1 4x\sqrt{x^2+1}\,\mathrm{d}x = 2\int_1^2 \sqrt{u}\,\mathrm{d}u$$

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The product rule

Just like integration by substitution reverses the chain rule, integration by parts "reverses" the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

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written another way

$$(uv)' = u'v + uv'$$

Integration by parts

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By the fundamental theorem of calculus

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Rearranging...

Integration by parts

The integration by parts formula

$$\int uv' \, dx = uv - \int u'v \, dx$$

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The integration by parts formula

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Alternative statement

$$\int u \, dv = uv - \int v \, du$$

Examples

One the board...