

This week on the problem set you will get practice thinking about potential functions and calculating line integrals.

**Homework:** The second homework will be due on Monday 11 May and will consist of 3, 4, 5, and 6 below.

\*Numbers in parentheses indicate the question has been taken from the textbook:

J. Rogawski, C. Adams, *Calculus, Multivariable*, 3<sup>rd</sup> Ed., W. H. Freeman & Company,

and refer to the section and question number in the textbook.

- (Section 17.1) Questions 13 – 17, 22, 26, 28, 29, 38, 42, 44, 47, 52, 56\*. (Use the following translations 4<sup>th</sup>  $\mapsto$  3<sup>rd</sup> editions: 47  $\mapsto$  45, 52  $\mapsto$  50, 56  $\mapsto$  54, otherwise the questions are the same).
- (Section 17.2) 3, 10, 12, 13, 21, 24, 28, 43, 44, 46, 47, 54, 55, 57, 63, 64, 67. (Use the following translations 4<sup>th</sup>  $\mapsto$  3<sup>rd</sup> editions: 43  $\mapsto$  41, 44  $\mapsto$  42, 46  $\mapsto$  44, 47  $\mapsto$  45, 54  $\mapsto$  52, 55  $\mapsto$  53, 57  $\mapsto$  55, otherwise the questions are the same).
- A parameterized curve  $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^2$  (the codomain could be  $\mathbb{R}^3$  as well) is a *flow line* for the vector field  $\mathbf{F}$  if for all  $t \in (a, b)$  we have that  $\mathbf{F}(\mathbf{r}(t)) = \mathbf{r}'(t)$ . A flow line for a vector field is the path that a particle would follow if the vector field was a velocity vector field for a fluid.
  - Consider the vector field  $\mathbf{F}(x, y) = \langle x, y \rangle$ . Find a flow line for  $\mathbf{F}$  (note that a vector field will have many different flow lines).
  - Find a collection of flow lines for  $\mathbf{F}$  so that every point  $(x, y)$  is contained in exactly one of the flow lines in the collection.  
But we are missing the flow line through  $(0, 0)$  which is  $\mathbf{r}(t) = (0, 0)$ .
  - Consider the vector field  $\mathbf{G}(x, y) = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle$ . Find a collection of flow lines for  $\mathbf{G}$  so that every point  $(x, y)$  is contained in exactly one of the flow lines in the collection.
  - What do the flow lines look like in  $\mathbb{R}^2$  for the vector fields  $\mathbf{F}$  and  $\mathbf{G}$ ? Relate how the flow lines are similar and different to how the vector fields  $\mathbf{F}$  and  $\mathbf{G}$  are similar and different.
  - A particle is dropped into the plane at the point  $(-1, 1)$  at time  $t = 0$ . If the particle is located at  $(x, y)$  in the plane its velocity vector is  $(1, 2x)$ . What is the position of the particle at time  $t = 3$ ?
- Consider the vector field  $\mathbf{F} = \left\langle \frac{1-y}{x^2 + (y-1)^2} + \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + (y-1)^2} + \frac{x}{x^2 + y^2} \right\rangle$ 
  - Show that the curl of  $\mathbf{F}$  is zero.
  - Show that  $\mathbf{F}$  is not conservative on the largest domain on which it is defined.
  - Show that  $\mathbf{F}$  is conservative on the right half plane and find a potential function.

**Hint:** For all of these, it will be useful to think of  $\mathbf{F}$  as the sum of two more familiar vector fields.

- A vector field  $\mathbf{F} = \langle F_1, F_2 \rangle$  is called *holomorphic* if it satisfies the *Cauchy-Riemann* equations:

$$\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y} \quad \text{and} \quad \frac{\partial F_2}{\partial x} = -\frac{\partial F_1}{\partial y}.$$

For example, the vector field  $\mathbf{G} = \langle x + p, y + q \rangle$  is holomorphic for any fixed  $p, q \in \mathbb{R}$ , but  $\langle x^2, F_2 \rangle$  is not holomorphic, no matter what  $F_2$  is! The purpose of this question is to convince you that holomorphic vector fields satisfy some spooky properties. (If you want to know more about this, take Math 132).

- Give an example of a holomorphic vector field  $\mathbf{H} = \langle H_1, H_2 \rangle$  (other than simple multiples or additions of the above examples). This is in general quite difficult, so to get to started, look for a vector field where

$$H_1 = x^2 + ay^2 + y$$

You will need to work out an appropriate value of  $a$ . Include verification that your example is in fact holomorphic.

- (b) Let  $C$  be the unit circle centred at  $(a, b) \in \mathbb{R}^2$  oriented counter clockwise. An amazing fact is that if  $\mathbf{F}$  is holomorphic then

$$F_1(a, b) = \frac{1}{2\pi} \oint_C F_1 \, ds \quad \text{and} \quad F_2(a, b) = \frac{1}{2\pi} \oint_C F_2 \, ds$$

Notice the right hand side only involves values of  $\mathbf{F}$  on  $C$ , and yet, somehow this knows about the value of  $\mathbf{F}$  inside  $C$ ! Verify that the formulas hold for the example  $\mathbf{G}$  and your example  $\mathbf{H}$  from part (a).

- (c) Maybe you aren't yet convinced that there is something strange going on with holomorphic vector fields. Well it turns out the values of  $\mathbf{F}$  on a circle know even a little more, namely

$$\frac{\partial F_1}{\partial x}(a, b) = \frac{1}{2\pi} \oint_C (x - a)F_1 + (y - b)F_2 \, ds \quad \text{and} \quad \frac{\partial F_2}{\partial x}(a, b) = \frac{1}{2\pi} \oint_C (x - a)F_2 - (y - b)F_1 \, ds$$

Verify these formulas as well. Note the above aren't as symmetrical as the previous formulas. You will most likely need to use some formulas for the antiderivative of things like  $\sin^n t$ . You can look these up.

- (d) What are the analogous formulas for the  $y$ -derivatives of  $F_1$  and  $F_2$ . In general, we will have formulas for all  $x$  and  $y$  derivatives but they become very complicated, the language of complex analysis allows us to understand this phenomenon in a clearer way.
6. Let  $C$  be the portion of the curve defined by the intersection of the surfaces  $y = x^2$  and  $x = y + z$  where  $z \geq 0$ . Take the orientation to be away from the origin. Let

$$\mathbf{F} = \left\langle yz + \frac{1}{\sqrt{2 - x^3}}, xz - \frac{1}{\sqrt{2 - y^3}}, xy \right\rangle$$

- (a) What is  $\text{curl}(\mathbf{F})$ ?
- (b) Parameterise the curve and write out the integral of  $\mathbf{F}$  as a single integral. There is no need to evaluate this integral.
- (c) Calculate  $\int_C \mathbf{F} \cdot d\mathbf{r}$ . *Hint: the integral in the previous question is very difficult so you should find another way to compute this.*

\*The questions marked with an asterisk are more difficult or are of a form that would not appear on an exam. Nonetheless they are worth thinking about as they often test understanding at a deeper conceptual level.