Math 3B : midterm 1 review 23.10.17

Ex 1 graph the function $f(x) = \frac{x^2-4}{x^2-9}$

(add info to pic as you go)

x/y-intercepts the y-int, $y = f(0) = \frac{4}{9}$

The x-int occurs when $f(x) = 0$ ie when $x^2 - 4 = 0$
thus $x = \pm 2$

horizontal asymptotes

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{x^2-4}{x^2-9} = 1$$

Thus hor. asymp. @ $y = 1$ in both $+\infty$ and $-\infty$ directions

slanted asymptotes: none

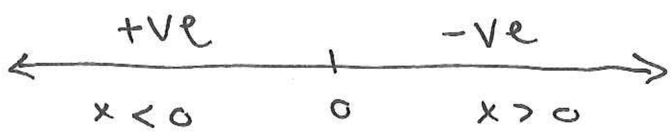
vertical asymptote: $f(x)$ undefined when $x = \pm 3$, to see what happens when we approach

$$\lim_{x \to 3^+} \frac{x^2-4}{x^2-9} = \infty \qquad \lim_{x \to -3^+} \frac{x^2-4}{x^2-9} = -\infty$$

$$\lim_{x \to 3^-} \frac{x^2-4}{x^2-9} = -\infty \qquad \lim_{x \to -3^-} \frac{x^2-4}{x^2-9} = \infty$$

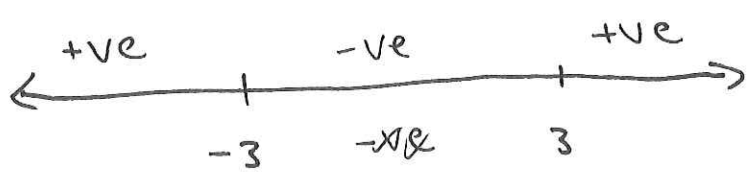
1st derivative $f'(x) = -\dfrac{10x}{(x^2-9)^2}$

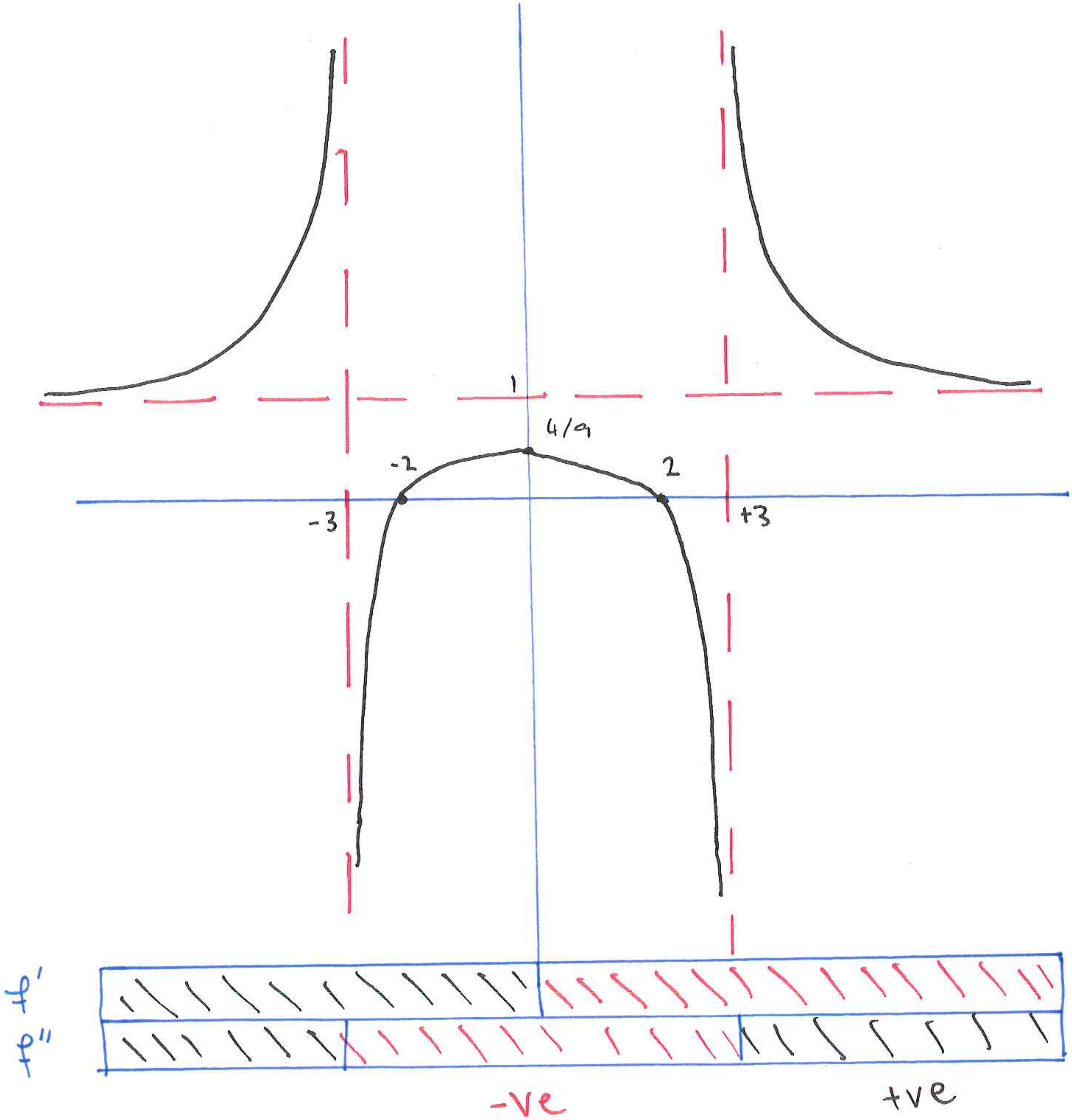
If we want to know where f' is +ve/-ve only need to look at numerator, (denominator is always positive :



2nd derivative $f''(x) = \dfrac{30(x^2+3)}{(x^2-9)^3}$

The numerator is always +ve, but the denominator changes sign:





1
4/9
-2
2
-3
+3
f'
f''
-ve
+ve

Ex 2 graph $f(x) = \dfrac{2x^2 - 3x + 2}{x - 1}$

where it is given that

$$f'(x) = \frac{2x^2 - \cancel{3}4x + 1}{(x-1)^2} \qquad f''(x) = \frac{2}{(x-1)^3}$$

x/y-int $f(0) = -2$, and $f(x) = 0$ when

$2x^2 - 3x + 2 = 0$ ← discrimant $b^2 - 4ac = 9 - 4 \cdot 2 \cdot 2 < 0$

so no solutions. Thus

y-int at $y = -2$, no x-ints.

horizontal asymp. $\lim_{x \to \pm\infty} f(x) = \infty$ so no

hor. asymps.

vertical asymp. Possible when $x = 1$.

$$\lim_{x \to 1^+} \frac{2x^2 - 3x + 2}{x - 1} = \infty, \quad \lim_{x \to 1^-} \frac{2x^2 - 3x + 2}{x - 1} = -\infty$$

Slanted asymp. Reminder, we have a slanted asymptote if

$$m := \lim_{x \to \pm\infty} f'(x) \quad \text{is const}$$

then if

$$b := \lim_{x \to \pm\infty} \left(f(x) - mx\right)$$

the slanted asymp. is $y = mx + b$.

In the $+\infty$ direction

$$m = \lim_{x\to\infty} \frac{2x^2-4x-1}{(x-1)^2} = 2 \qquad \text{(by L'Hop)}$$

$$b = \lim_{x\to\infty} \frac{2x^2-3x+2}{x-1} - 2x$$

$$= \lim_{x\to\infty} \frac{2x^2-3x+2-2x(x-1)}{x-1}$$

$$= \lim_{x\to\infty} \frac{-x+2}{x-1} = -1 \qquad \text{(by L'Hop)}$$

Thus slanted asymp of $y = 2x-1$

In the $-\infty$ direction: same calculation.

Alternatively: for this example we can see that

$$f(x) = \frac{2x^2-3x+2}{x-1} = 2x-1+\frac{1}{x-1} \approx 2x-1$$

when $x \to \pm\infty$.

<u>1^{st} derivative</u> denom. is always +ve.

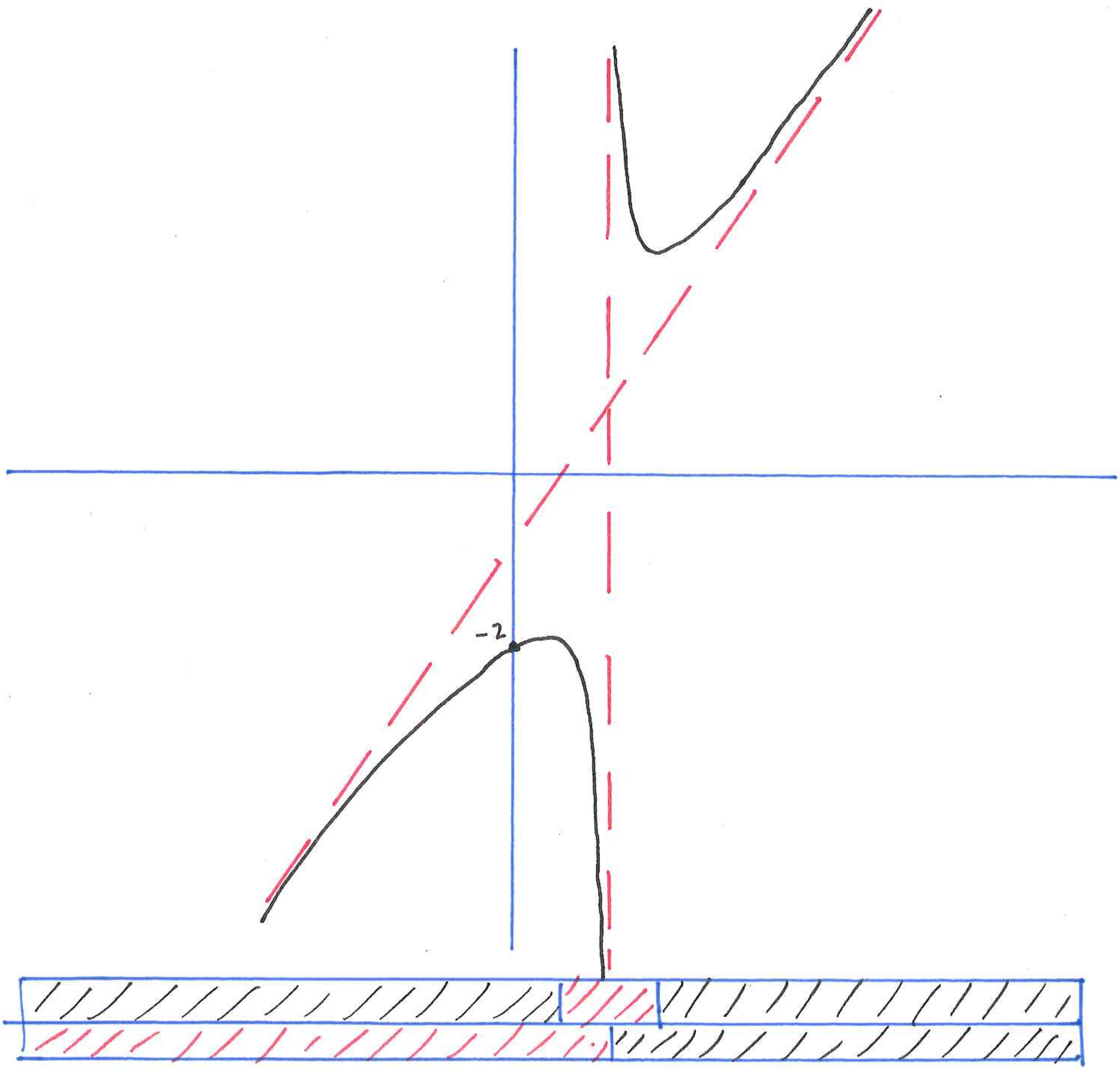
$2x^2-4x+1=0$ when $x = 1+\frac{1}{\sqrt{2}}$ and $1-\frac{1}{\sqrt{2}}$

~~Tho~~ Testing points

+ve : −ve : +ve

$0 \quad 1-\frac{1}{\sqrt{2}} \quad 1+\frac{1}{\sqrt{2}}$

2nd derivative: numerator always positive



Ex (10. on problem set 2)

What point ~~on~~ $(x\ y)$, on the parabola $y = x^2$ is closest to $(16, \frac{1}{2})$?

Solution: the distance from $(x\ y)$ to $(16, \frac{1}{2})$ is

$$d(x,y) = \sqrt{(x-16)^2 + (y-\tfrac{1}{2})^2}$$

We know that $y = x^2$ so

$$d(x) = \sqrt{(x-16)^2 + (x^2-\tfrac{1}{2})^2} \quad \left(\text{Domain } (0,\infty)\right)$$

This is the function we wish to minimize.

Step 1: find critical points

$$d'(x) = \frac{(x-16) + 2x(x^2-\frac{1}{2})}{\sqrt{(x-16)^2 + (x^2-\frac{1}{2})^2}}$$

$$= \frac{2x^3 - 16}{\sqrt{(x-16)^2 + (x^2-\frac{1}{2})^2}}$$

Note: denominator is always > 0

So crit pts occur when $2x^3 - 16 = 0$

i.e. when $x^3 = 8$, $\boxed{x = 2}$ ← only crit point.

Step 2 There are several ways to check this is a global ~~max~~ min:

1. open interval method:

~~f(~~ $d(2) = \sqrt{14^2 + 3.75^2} \sim \sqrt{210}$

$L = \lim_{x \to 0} d(x) = \sqrt{16^2 + \frac{1}{4}} \sim \sqrt{256}$

$M = \lim_{x \to \infty} d(x) = \infty$

so 2 is the global min.

2. Notice, using 1st derivative test that $x = 2$ is a local ~~max~~ min. Since it is the only crit pt. it must be the global max.

~~Poit~~ Point on $y = x^2$ closest to $(16, \frac{1}{2})$ is

$(2, 4)$.

Example

A campaign manager for the Democratic candidate running for the NH senate seat needs to decide how to spend \$10 mil. in TV and radio ~~ads~~ advertising money. The data analytics firm employed by the campaign has experimentally determined that in NH's 1st and 2nd congressional districts, \$x mil spent would result in

$$R_1(x) = 1860\ln(1+x)$$
$$R_2(x) = 790\ln(1+x)$$

undecided voters switching to your candidate.

Q How much money should you spend in each district?

A We will spend all of the money so if we spend \$x mil in district 1, the total increase will be

$$R(x) = R_1(x) + R_2(10-x).$$

$$= 1860\ln(1+x) + 790\ln(1+10-x)$$

We want to maximise this function.
Note: domain $= [0,10]$.

Find the critical points

$$R'(x) = \frac{1860}{1+x} - \frac{790}{1+10-x}$$

setting equal to zero:

$$(11-x)1860 = (1+x)790$$

$$19670 = 2650x$$

$$x = \frac{1967}{265} \sim 7.46$$

We have critical points at

$$x = 0, 10, \frac{1967}{265}$$

Using 1st der test: $x = 0, 10$ are minimums.

$$R''(x) = -\frac{1860}{(1+x)^2} - \frac{790}{(11-x)^2} < 0$$

So $x = \frac{1976}{265}$ is a max!

Spend \$7.46 mil in district 1.
\$2.54 mil in district 2.

~~Mat~~ Max $= R(7.46) \sim 4970$

~ 0.5% of Registered voters.

Ex Express the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k\pi^2}{n^2} \sin\left(\frac{k\pi}{n}\right)$$

as a definite integral $\int_0^{\pi} f(x)\,dx$

Solution We know that

$$\int_0^{\pi} f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(0+k\Delta x)$$

where $\Delta x = \frac{\pi - 0}{n} = \frac{\pi}{n}$, thus $k\Delta x = \frac{k\pi}{n}$, and

$$\sum_{k=1}^{n} \frac{k\pi^2}{n^2} \sin\left(\frac{k\pi}{n}\right) = \underbrace{\frac{\pi}{n}}_{\Delta x} \sum_{k=1}^{n} \underbrace{\underbrace{\frac{k\pi}{n}}_{k\Delta x} \sin\left(\underbrace{\frac{k\pi}{n}}_{k\Delta x}\right)}_{= f(k\Delta x)}$$

so we guess $f(x) = x\sin x$.

ans

$$\int_0^{\pi} x\sin x\,dx.$$

Ex Express $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k}{n^2+nk}$ as a definite integral $\int_1^2 f(x)\,dx$.

Solution $\int_1^2 f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta x\, f(x_k)$

where $\Delta x = \frac{2-1}{n} = \frac{1}{n}$ and $x_k = a + k\Delta x$

$$= 1 + \frac{k}{n}.$$

Now

$$\sum_{k=1}^{n} \frac{k}{n^2+nk} = \underbrace{\frac{1}{n}}_{\Delta x} \sum_{k=1}^{n} \underbrace{\frac{k}{n+k}}_{f(x_k)}$$

so $f\left(1+\frac{k}{n}\right) = \frac{k}{n+k} = \frac{k/n}{1+k/n}$

$$= \frac{1+k/n-1}{1+\frac{k}{n}}$$

$$= \frac{x_k - 1}{x_k}$$

So $f(x) = \frac{x-1}{x}$ ans: $\int_1^2 \frac{x-1}{x}\,dx$.