Power Series

- A power ~~ser~~ series is a series involving powers of a variable x, of the form

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

where c is a real number.

- We say F(x) has centre c.

- Note that if we fix a value for x then

$$F(x) = \sum a_n (x - \cancel{x}c)^n$$

is just a regular series. If it converges, we ~~&~~ can think of F(x) as a function (ie it takes in a number, if it converges, it spits out a number, otherwise it is undefined).

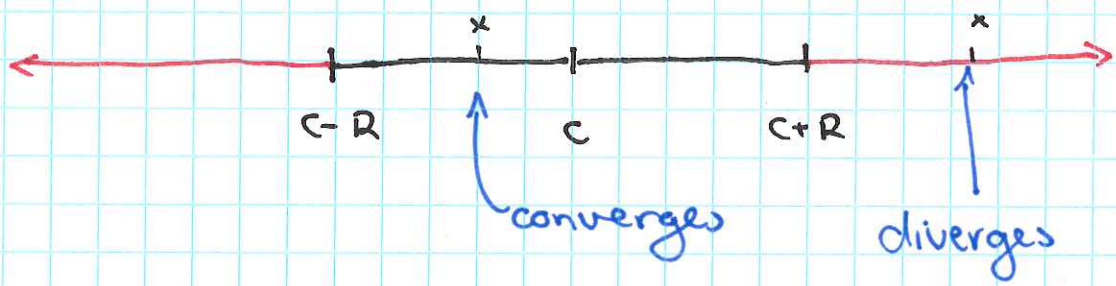
Key question: For which x does

$$F(x) = \sum a_n (x-c)^n$$

converge?

Def For a power series $F(x) = \sum a_n (x-c)^n$, its domain of ~~ab~~ ~~absolu~~ convergence is the set of all x values s.t. $F(x)$ converges.

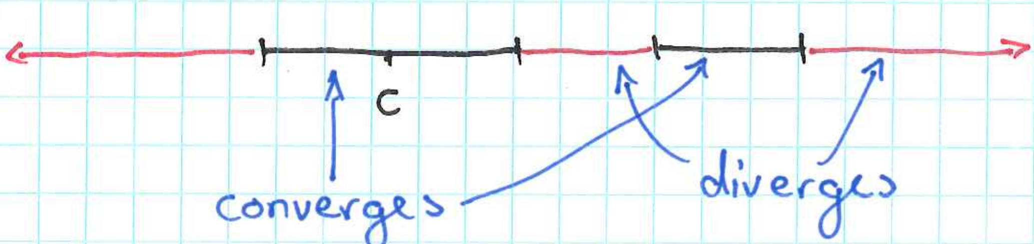
- The remarkable fact says that the domain of convergence always looks like



Ex For $F(x) = \sum x^n$, the domain of convergence is $(-1, 1)$ and ~~its~~ ~~root~~ $R = 1$.

Def R is called the radius of convergence

- The remarkable fact says that it is impossible to find a ~~poer~~ power series with a domain of convergence that looks like



Ex $F(x) = \sum_{n=0}^{\infty} x^n$ (ie $c=0$ and $a_n = 1$).

Using the ration test,

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} |x| = |x|$$

so $F(x)$ converges as long as $\rho = |x| < 1$ and diverges if $|x| > 1$.

Checking manually

$$F(1) = \sum 1^n = \sum 1$$

diverges and

$$F(-1) = \sum (-1)^n$$

diverges. Thus

$$|x| < 1 \Rightarrow F(x) \text{ converges}$$

$$|x| \geq 1 \Rightarrow F(x) \text{ diverges.}$$

Remarkable Fact We can always find a number $R \geq 0$ (or $R = \infty$) so that

$$|x-c| < R \Rightarrow F(x) \text{ converges absolutely}$$

$$|x-c| > R \Rightarrow F(x) \text{ diverges.}$$

(We don't know what happens when $|x-c| = R$).

- We find the radius of convergence by asking, "when does $F(x) = \sum a_n(x-c)^n$ converge absolutely"?

- Usually we answer this by using the ratio test (though sometimes the root test may work).

Ex $F(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n} \right| = \lim_{n\to\infty} \left| \frac{x}{n+1} \right| = 0$$

this is less than 1 for any x, so $R = \infty$.

Ex Determine the radius of convergence and the domain of convergence for

$$\sum_{n=0}^{\infty} \cancel{x^n} \frac{(x-1)^n}{n}$$

$$\rho = \lim_{n\to\infty} \left| \frac{(x-1)^{n+1}}{n+1} \cdot \frac{n}{(x-1)^n} \right|$$

$$= \lim_{n\to\infty} \frac{n}{n+1} |x-1| = |x-1|$$

we would like $\rho < 1$ so $|x-1| < 1$ ie $R = 1$

To find the domain of convergence, we test the endpoints:

when $x=0$:

$$\sum \frac{(-1)^n}{n} \quad \text{converges.}$$

when $x=2$

$$\sum \frac{1}{n} \quad \text{diverges,}$$

so the domain of convergence is

$$[0, 2).$$