Lecture 15

- Today we find out why conservative vector fields are so important.

- We should think of vector fields as derivatives of functions of multiple variables. The derivative of $f(x\ y\ z)$ is $\nabla f = \langle \partial_x f\ \partial_y f\ \partial_z f \rangle$

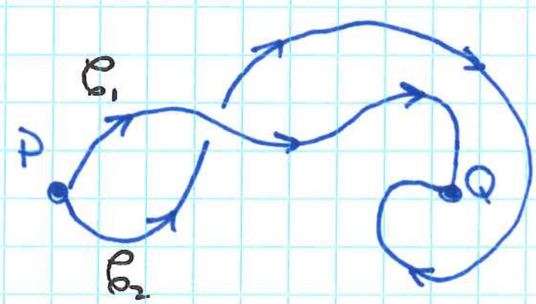
- Then, conservative vector fields are the vector fields with "antiderivatives"!

Thm (FTC for conservative vector fields)
if $\underline{F} = \nabla f$

$$\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = f(Q) - f(P)$$

where P is the starting point and Q is the endpoint of $\mathcal{C}$

Rmk (by picture)



$$\int_{\mathcal{C}_1} \underline{F} \cdot d\underline{r} = \int_{\mathcal{C}_2} \underline{F} \cdot d\underline{r}$$

- If $\mathcal{C}$ is a loop (a closed path) we use the notation

$$\oint_{\mathcal{C}} \underline{F} \cdot d\underline{r}$$

Cor If $\underline{F} = \nabla f$ $\quad \oint_{\mathcal{C}} \underline{F} \cdot d\underline{r} = 0.$

2. Potential functions

Thm If $\nabla f = \nabla g$ then $f = g + C$ ← constant.

proof The equality implies $\nabla f - \nabla g = 0$

But

$$\nabla f - \nabla g = \langle f_x - g_x, f_y - g_y, f_z - g_z \rangle$$

$$= \langle (f-g)_x, (f-g)_y, (f-g)_z \rangle$$

$$= \nabla(f-g)$$

This means $(f-g)_x = 0 \quad (*)$

$$(f-g)_y = 0 \quad (**)$$

$$(f-g)_z = 0 \quad (***)$$

The only way $(*)$ could be true is if $f-g = \alpha(y, z)$ a $f^{\underline{n}}$ depending only on $(y\ z)$

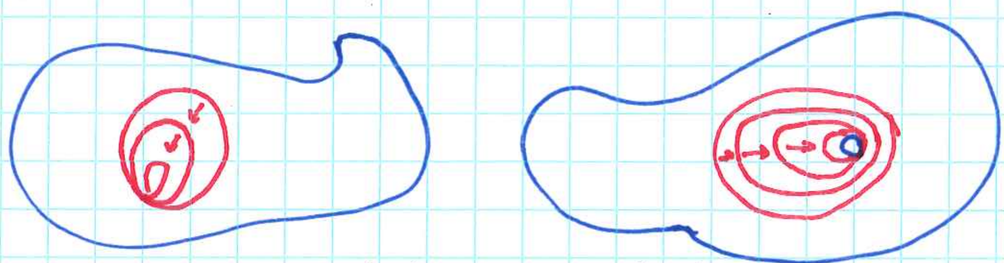
But $(**)$ says $f-g$ should only depend on x, z

But $(***)$ says $f-g$ should only depend on x, y

Thus $f-g$ must be a const. $\square$.

- How do we know if a v.f. is conservative?
- It turns out knowing $\operatorname{curl}(\underline{F})=0$ is enough if we restrict to

- A domain is simply connected if it is connected and has no holes. In $\mathbb{R}^2$ this means every loop can be shrunk to a point



simply connected   not simply connected

In $\mathbb{R}^3$, every ~~st~~ sphere can be shrunk to a point.

Thm A vector field on a simply connected domain is conservative if and only if
$$\operatorname{curl}(\underline{F})=0.$$

Ex The vortex vector field is

$$\underline{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

$$\operatorname{curl}(\underline{F}) = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

$$= \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}$$

$$= 0$$

So is $\underline{F}$ conservative? No! At lease on the domain $\mathbb{R}^2 \setminus 0$ it isn't, this is not simply connected.

3. How to find potential functions.

- Suppose we have a vector field $\underline{F}$ ~~def~~ which we know is conservative. How can we find it's potential function

- Suppose $\underline{F} = \nabla f$. Then

$$F_1 = f_x \quad F_2 = f_y \quad \cancel{F_3 = f_z}$$

Integrating

$$f = \int F_1 \, dx + g(y)$$

$$f = \int F_2 \, dy + h(x)$$

We equate these to find g and h.

<u>Ex</u> $\underline{F} = \langle 2x-2y, -2x \rangle$

$\text{curl}(\underline{F}) = \partial_y(2x-2y) - \partial_x(-2x)$

$= 0$

So $\underline{F}$ is conservative. Suppose $\underline{F} = \nabla f$ then

$f_x = 2x - 2y$

so $f_x = \int 2x - 2y \, dx$

$= x^2 - 2xy + g(y)$

$f_y = -2x$

$f = -2xy + h(x)$

So $x^2 - 2xy + g(y) = -2xy + h(x)$

so $g(y) = 0$, $h(x) = x^2$

$f = x^2 - 2xy + C.$