This week you will get practice drawing and understanding bifurcation diagrams.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

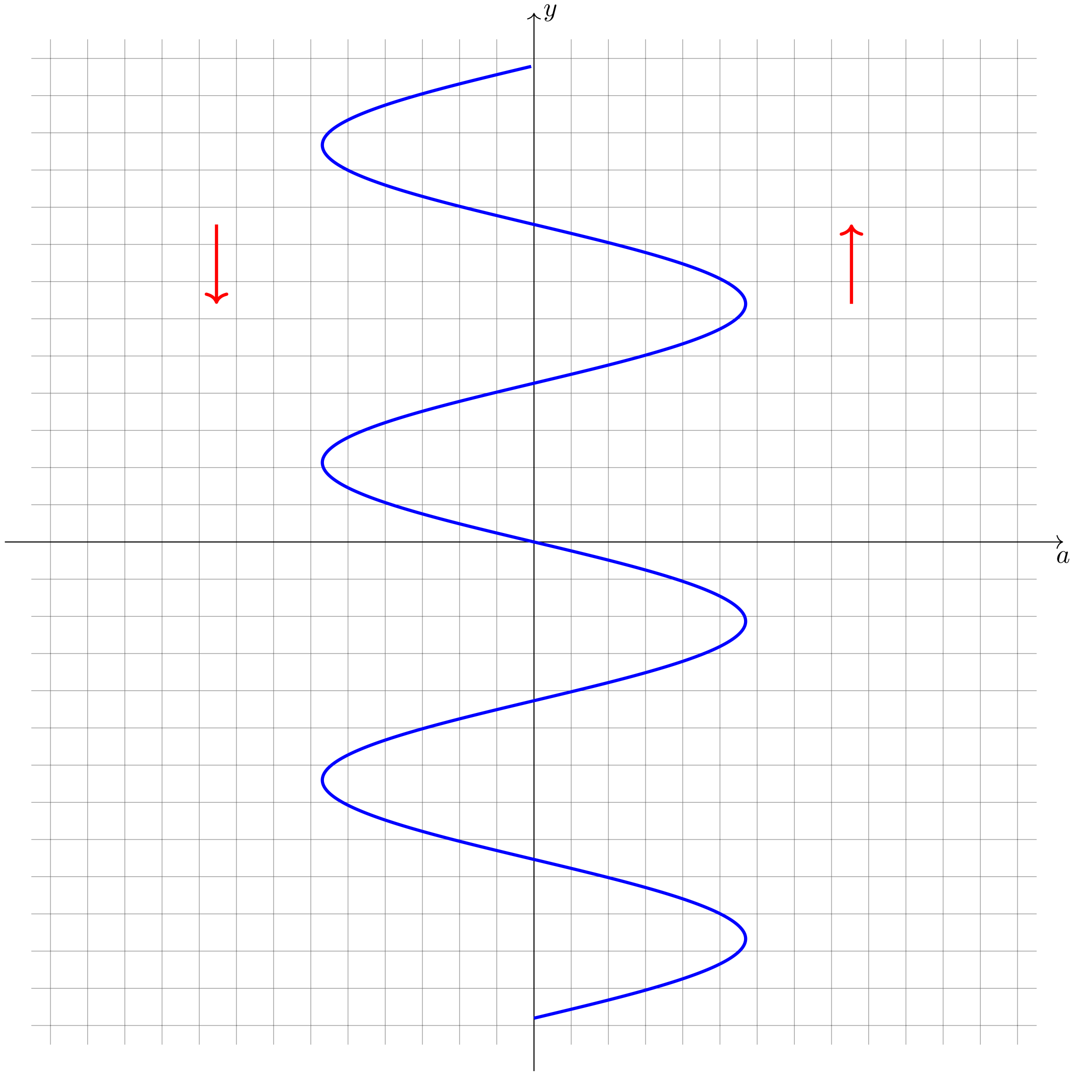
and refer to the section and question number in the textbook.

Homework: The homework will be due on Friday 11 March, at 8am, the *start* of the lecture. It will consist of

question 1, 2(e) from PS 9, and question 2 from PS 10.

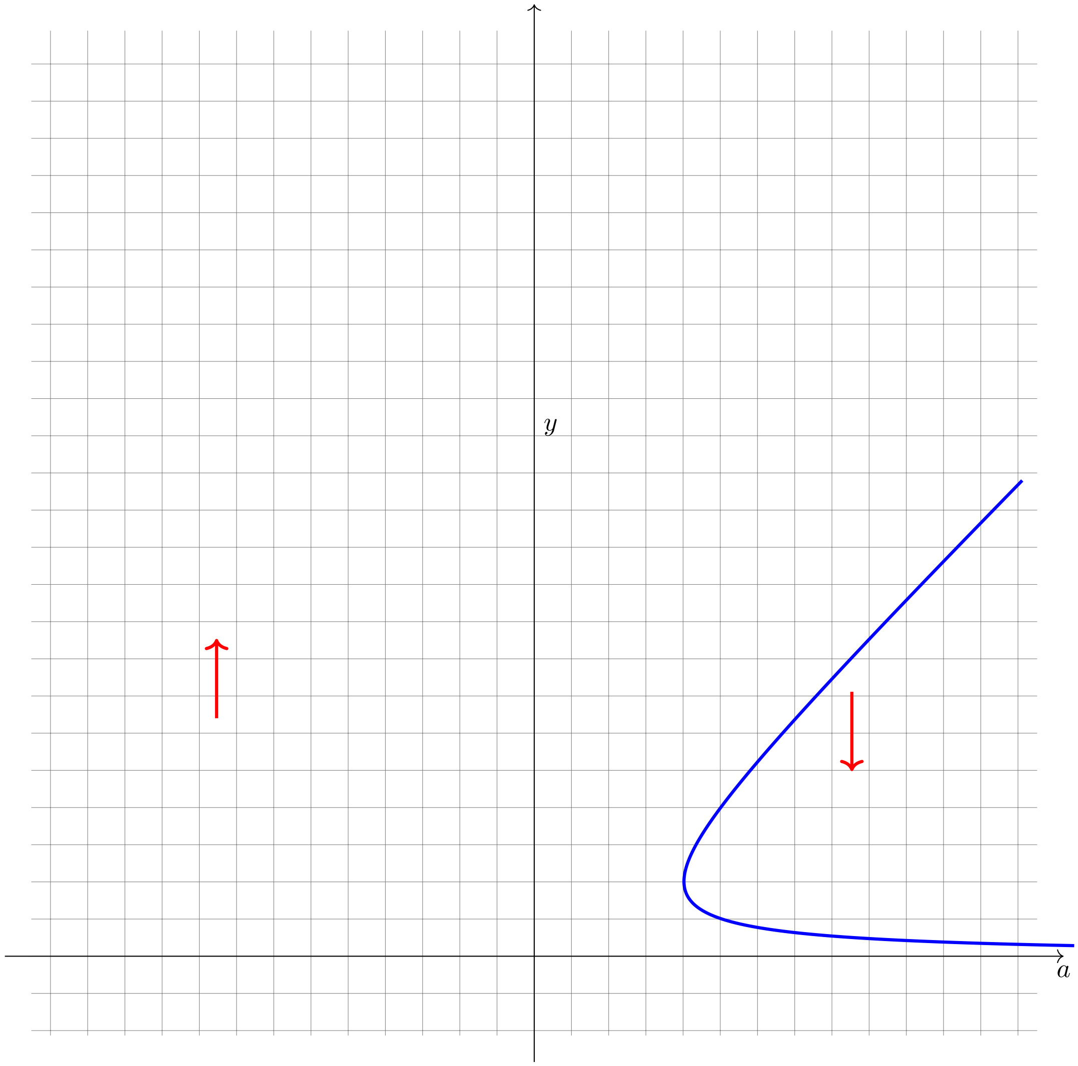
1. (6.6) Sketch the bifurcation diagrams for the equations in the following.
   (a) (6.6-7) $\frac{\mathrm{d}y}{\mathrm{d}t} = ay - y^2$
   (b) (6.6-10) $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - ay^2$
   (c) (6.6-11) $\frac{\mathrm{d}y}{\mathrm{d}t} = \sin y + a$

Solution:



(d) (6.6-12) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 - ay + 2$ for $a > 0$.

Solution:

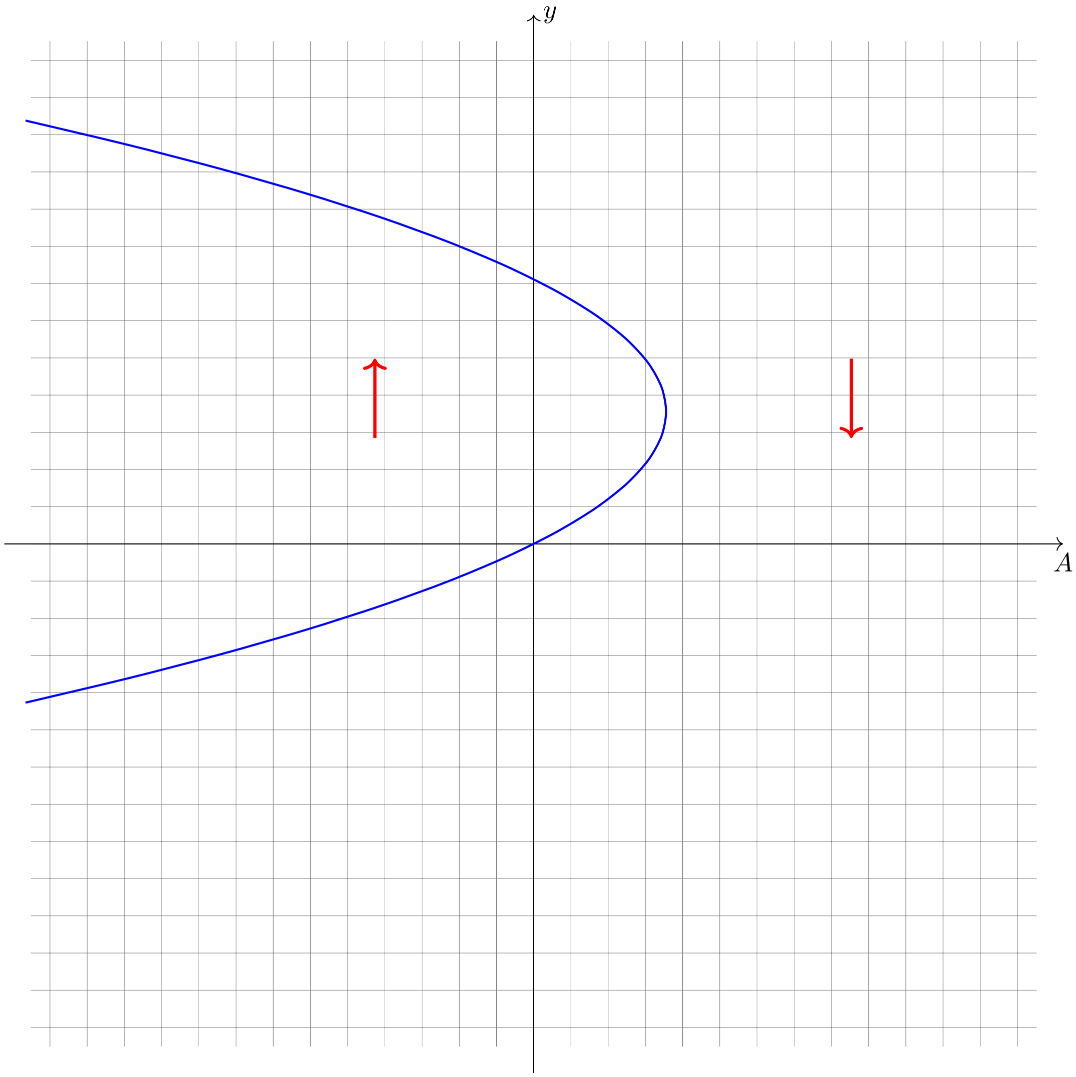


2. The fisheries management department is considering opening up an area for recreational fishing. The area has a population of abalone. The department would like to set an overall cap on the number of abalone that can be taken per year. It is known that the population of abalone grow logistically with a birth rate of 0.5 and a carrying capacity of 5000. If the department sets a cap of A individuals to be taken per year, it expects the actual number of abalone taken will be $\pm 10\%$ of this. Prior to allowing harvesting, the population of abalone is in equilibrium at 5000 individuals. What is the maximum cap that the department could set, without endangering the population of abalone?

Solution: The differential equation that models the population is

$$\frac{dy}{dt} = 0.5y\left(1 - \frac{y}{5000}\right) - A.$$

Thus the bifurcation diagram is



The points at which the parabola passes through the y-axis are $(0,0)$ and $(0,5000)$. Since the initial value of the population is $y(0) = 5000$ we see that as long as A is small enough there is a stable equilibrium and the population will decrease but stabilise. We need to work out where the vertex of the parabola is. By symmetry, this is at $y = 2500$ and thus $A = 0.5 \cdot 2500 \cdot (1 - 2500/5000) = 2500/4 = 625$.

If we set $A = 625$ however, it is possible that the quota will be overshot and thus the population will actually die out. Thus we want 625 to be, at most, 110% of the quota, thus we should let A be, a most $6250/11 \approx 568.2$.

3. (6.6) Consider the model

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay^2}{k^2 + y^2} - cy$$

of an autocatalytic gene from question 5. In each of the following cases, two of the parameters are specified. Sketch a bifurcation diagram with respect to the third parameter.

(a) (6.6-13) $k = 1$, $c = 2$ with a as the bifurcation parameter.

(b) (6.6-14) $k = 2$, $c = 1$ with a as the bifurcation parameter.

(c) (6.6-15) $a = 10$, $k = 1$ with c as the bifurcation parameter.

(d) (6.6-16) $a = 10$, $c = 1$ with k^2 as the bifurcation parameter.

4. (6.6-40) Suppose the growth rate of a whale population at density N (individuals per million square kilometers of ocean), harvested at a rate h, is given by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0.07N\left(\frac{N}{10} - 1\right)\left(1 - \frac{N}{80}\right) - h$$

where the units of t are years.

(a) Sketch a bifurcation diagram with respect to the parameter h as it varies over the interval $[0, 8]$.

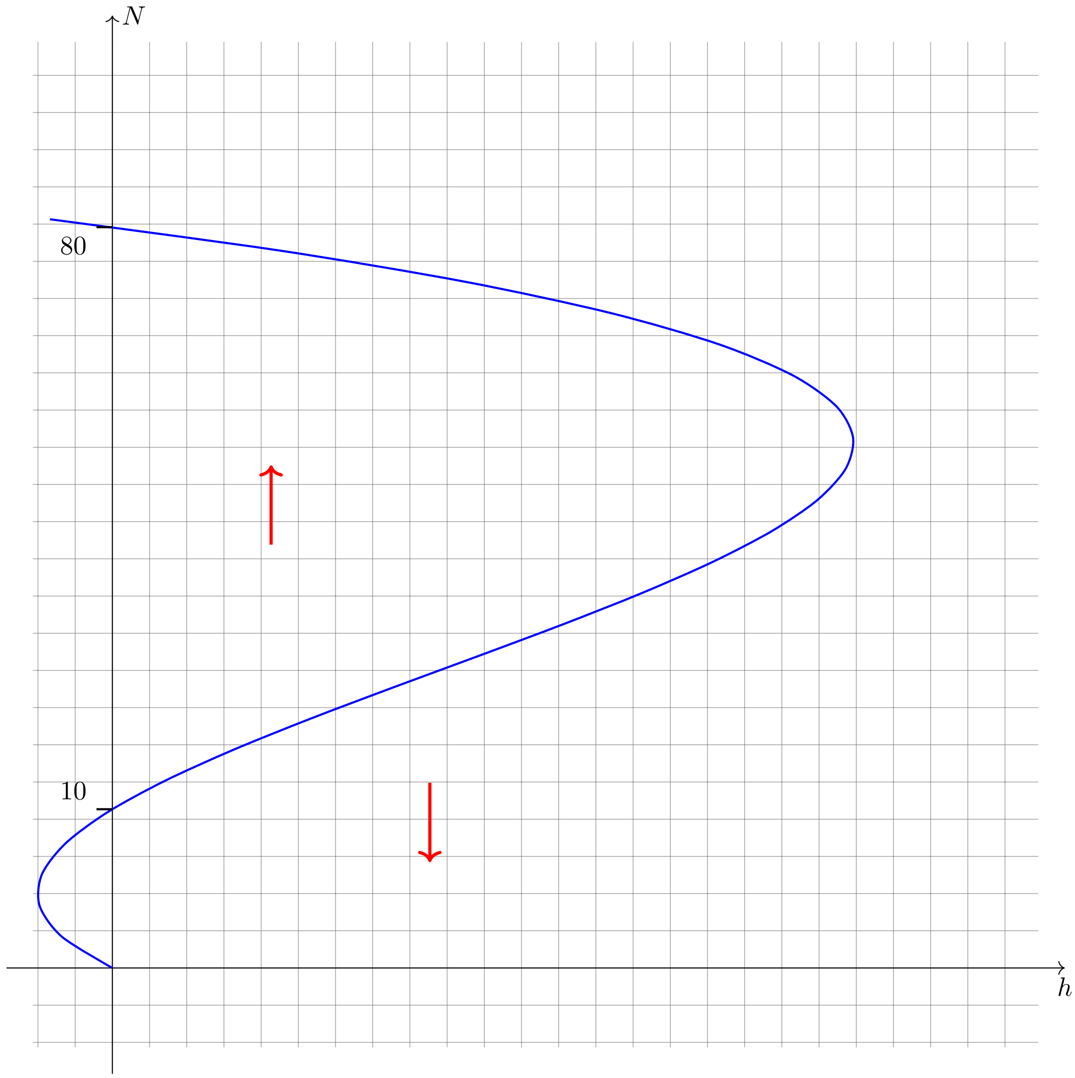
Solution: We want to sketch the points that satisfy

$$0.07N\left(\frac{N}{10} - 1\right)\left(1 - \frac{N}{80}\right) - h = 0$$

i.e. where

$$h = 0.07N\left(\frac{N}{10} - 1\right)\left(1 - \frac{N}{80}\right)$$

To do this we can sketch it in the hN-axis and then flip to the Nh-axis:



The direction of the arrows is found by subbing in points from each region (e.g. $(0, 40)$ and $(1, 0)$) into

$$0.07N\left(\frac{N}{10}-1\right)\left(1-\frac{N}{80}\right)-h$$

and determining whether it is positive or negative.

(b) If $h = vN$, then sketch a bifurcation diagram with respect to the parameter v as it varies over the interval $[0, 0.12]$.

Solution: We want to sketch the points that satisfy

$$0.07N\left(\frac{N}{10}-1\right)\left(1-\frac{N}{80}\right)-vN=0$$

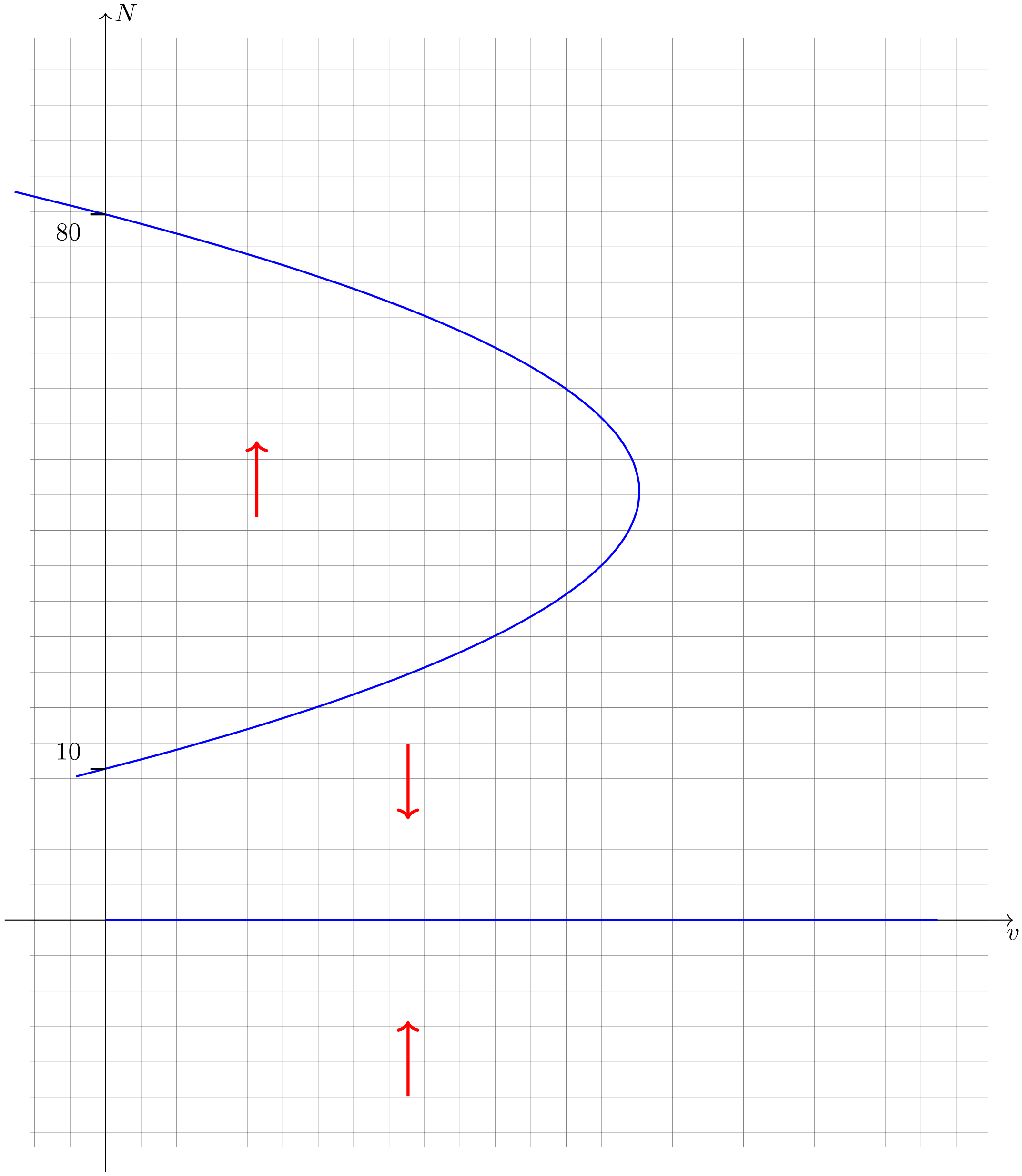
i.e. where

$$\left[0.07\left(\frac{N}{10}-1\right)\left(1-\frac{N}{80}\right)-v\right]N=0$$

Thus we want to sketch the points where either $N = 0$ or

$$v = 0.07\left(\frac{N}{10} - 1\right)\left(1 - \frac{N}{80}\right).$$

To do this we can first sketch the line $N = 0$ and then the sidewards parabola:



The direction of the arrows is found by subbing in points from each region (e.g. $(0, 40)$, $(0, 5)$ and $(0, -1)$) into

$$0.07N\left(\frac{N}{10} - 1\right)\left(1 - \frac{N}{80}\right) - vN$$

and determining whether it is positive or negative.