This weeks problem set focuses on the concept of a change of basis matrix. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 3: due Friday 10 May: questions 2 and 5 below.

1. From section 2.5, problems 1, $2a, c$, $3a, c$, 5, 7, 10^*, 13^*.

2.* Let V be a finite dimensional vector space and W a subspace. Show that V and $W \times V/W$ are isomorphic by finding an explicit isomorphism (rather than simply computing the dimensions).

3.* Let V be a finite dimensional vector space and W a subspace. Show that $\dim(V/W) = \dim V - \dim W$. *Hint: consider a basis of W and extend it to V. Now find a basis for V/W. You can also prove it using the dimension theorem.*

4.* Let $T : V \longrightarrow W$ be a linear map.
   (a) Show that $\operatorname{im} T$ and $V/\ker T$ are isomorphic.
   (b) Use this (and the previous exercise) to give an alternative proof of the dimension theorem.

5. A differential operator on $\mathbb{R}_n[x]$ is a linear combination of expressions of the form $x^a \frac{d^b}{dx^b}$ where $a - b \leq 0$ (otherwise the degree would potentially increase!). We can consider a differential operator as a linear map $\mathbb{R}_n[x] \longrightarrow \mathbb{R}_n[x]$.
   (a) Let $D : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ be the differential operator given by $2 - 4\frac{d}{dx} + 2x^2\frac{d^2}{dx^2}$. Find the matrix of D relative to the basis $\{x^2, (x-1)^2, (x+1)^2\}$.
   (b) Suppose $E : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ is a differential operator and that the matrix of E, relative to the basis $\{1, x, x^2\}$ is
   $$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$
   Find E.