Math 3B: Lecture 14

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How to factorize polynomials

The normal method for factorizing a polynomial $p(x)$ is to find a root α and then writing

$$p(x) = q(x)(x - \alpha).$$

How to factorize polynomials

The normal method for factorizing a polynomial $p(x)$ is to find a root α and then writing

$$p(x) = q(x)(x - \alpha).$$

What if we want to divide a polynomial $p(x)$ by another polynomial $q(x)$? We want to write

$$p(x) = q(x)d(x) + r(x)$$

for a polynomial $d(x)$ (the divisor) and a remainder $r(x)$.

Long division

We know how to do this with numbers! Long division.

$$
\begin{array}{r}
176 \\
34\,\overline{\smash{)}\,6000} \\
\underline{3400} \\
2600 \\
\underline{2380} \\
220 \\
\underline{204} \\
16
\end{array}
$$

Long division

We know how to do this with numbers! Long division.

$$
\begin{array}{r}
176 \\
34 \overline{\smash{)}\,6000} \\
\underline{3400} \\
2600 \\
\underline{2380} \\
220 \\
\underline{204} \\
16
\end{array}
$$

So $6000 = 34 \cdot 176 + 16$ or $\frac{6000}{34} = 176 + \frac{16}{34}$.

Why?

Lets rewrite the equation $p(x) = q(x)d(x) + r(x)$

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

Why?

Lets rewrite the equation $p(x) = q(x)d(x) + r(x)$

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

E.g.

$$\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 1} = x + \frac{1}{(x+1)^2}.$$

Why?

Lets rewrite the equation $p(x) = q(x)d(x) + r(x)$

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

E.g.

$$\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 1} = x + \frac{1}{(x+1)^2}.$$

The left hand side is difficult to integrate. The right hand side is easy!

How?

$$x+3\overline{\smash{\big)}\quad x^2+5x+4}$$

How?

$$
\begin{array}{r}
x \\
x+3 \overline{\smash{\big)}\; x^2+5x+4}
\end{array}
$$

How?

$$
\begin{array}{r|l}
 & x \\
\cline{2-2}
x+3 & x^2 + 5x + 4 \\
 & \underline{-\, x^2 - 3x} \\
\end{array}
$$

How?

$$
\begin{array}{r}
x \phantom{{}+4} \\
x+3 \overline{\smash{\big)}\, x^2 + 5x + 4} \\
\underline{-x^2 - 3x \phantom{{}+4}} \\
2x + 4
\end{array}
$$

How?

$$
\begin{array}{r}
x+2 \\
x+3 \overline{\big) \quad x^2+5x+4} \\
-x^2-3x \\
\hline
2x+4
\end{array}
$$

How?

$$
\begin{array}{r}
x + 2 \\
x + 3 \overline{\big) \quad x^2 + 5x + 4} \\
\underline{- x^2 - 3x } \\
2x + 4 \\
\underline{- 2x - 6}
\end{array}
$$

How?

$$
\begin{array}{rr}
 & x + 2 \\ \cline{2-2}
x + 3 \big) & x^2 + 5x + 4 \\
 & \underline{-x^2 - 3x\phantom{{}+4}} \\
 & 2x + 4 \\
 & \underline{-2x - 6} \\
 & -2
\end{array}
$$

How?

$$
\begin{array}{r}
x + 2 \\
x+3\overline{\smash{\big)}\; x^2 + 5x + 4} \\
\underline{-\,x^2 - 3x \phantom{{}+4}} \\
2x + 4 \\
\underline{-\,2x - 6} \\
-\,2
\end{array}
$$

So

$$\frac{x^2 + 5x + 4}{x + 3} = x + 2 - \frac{2}{x + 3}.$$

Example 1

$$x - 3\overline{\smash{)}\; x^3 - 12x^2 \qquad\qquad - 42}$$

Example 1

$$
\begin{array}{r@{}l}
 & x^2 \\
\cline{2-2}
x - 3 \big) & \; x^3 - 12x^2 \qquad\qquad - 42
\end{array}
$$

Example 1

$$
\begin{array}{r@{}l}
 & x^2 \\
\cline{2-2}
x - 3\big) & x^3 - 12x^2 \qquad\qquad - 42 \\
 & -\,x^3 + 3x^2 \\
\cline{2-2}
\end{array}
$$

Example 1

$$
\begin{array}{r@{}l}
 & \quad\; x^2 \\
\cline{2-2}
x-3\big) & \; x^3 - 12x^2 \qquad\qquad\quad - 42 \\
 & - x^3 \;\; + 3x^2 \\
\cline{2-2}
 & \qquad\;\; - 9x^2
\end{array}
$$

Example 1

$$
\begin{array}{r@{}l}
 & \; x^2 \quad - 9x \\
\cline{2-2}
x - 3\big) & \; x^3 - 12x^2 \qquad\qquad - 42 \\
 & -x^3 + 3x^2 \\
\cline{2-2}
 & \qquad\; - 9x^2
\end{array}
$$

Example 1

$$
\begin{array}{r}
x^2 \quad - 9x \qquad\qquad \\
x-3\,\big)\overline{\;x^3 - 12x^2 \qquad\qquad\quad - 42} \\
\underline{-\,x^3 \;\; + 3x^2} \qquad\qquad\qquad\quad \\
-\,9x^2 \qquad\qquad\qquad\quad \\
\underline{9x^2 - 27x} \qquad\quad
\end{array}
$$

Example 1

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & x^2 & -\,9x & & \\ \cline{2-5}
x-3 & x^3 - 12x^2 & & & -\,42 \\
 & -\,x^3 \;\; +3x^2 & & & \\ \cline{2-2}
 & -\,9x^2 & & & \\
 & 9x^2 & -\,27x & & \\ \cline{2-3}
 & & -\,27x & & -\,42
\end{array}
$$

Example 1

$$
\begin{array}{rrrrr}
 & & x^2 & -\,9x & -\,27 \\
\cline{2-5}
x-3\big) & x^3 & -\,12x^2 & & -\,42 \\
 & -\,x^3 & +\,3x^2 & & \\
\cline{2-3}
 & & -\,9x^2 & & \\
 & & 9x^2 & -\,27x & \\
\cline{3-4}
 & & & -\,27x & -\,42
\end{array}
$$

Example 1

$$
\begin{array}{r|rrrr}
\multicolumn{2}{r}{x^2} & -\,9x & -\,27 \\
\cline{2-5}
x-3 & x^3 - 12x^2 & & -\,42 \\
& -\,x^3 \;\; +\,3x^2 & & \\
\cline{2-2}
& -\,9x^2 & & \\
& 9x^2 & -\,27x & \\
\cline{2-3}
& & -\,27x & -\,42 \\
& & 27x & -\,81 \\
\cline{3-4}
\end{array}
$$

Example 1

$$
\begin{array}{rrrrr}
 & x^2 & -\,9x & -\,27 \\ \cline{2-4}
x-3 \big) & x^3 - 12x^2 & & -\,42 \\
 & -\,x^3 + 3x^2 & & \\ \cline{2-2}
 & -\,9x^2 & & \\
 & 9x^2 & -\,27x & \\ \cline{2-3}
 & & -\,27x & -\,42 \\
 & & 27x & -\,81 \\ \cline{3-4}
 & & & -\,123
\end{array}
$$

Example 1

$$
\begin{array}{rrrrr}
 & & x^2 & -\,9x & -\,27 \\
\cline{2-5}
x-3 \big) & x^3 & -\,12x^2 & & -\,42 \\
 & -\,x^3 & +\,3x^2 & & \\
\cline{2-3}
 & & -\,9x^2 & & \\
 & & 9x^2 & -\,27x & \\
\cline{3-4}
 & & & -\,27x & -\,42 \\
 & & & 27x & -\,81 \\
\cline{4-5}
 & & & & -\,123
\end{array}
$$

So

$$\frac{x^3 - 12x^2 - 42}{x-3} = x^2 - 9x - 27 - \frac{123}{x-3}.$$

Example 2

$$x^2 + 1\overline{)\quad x^3 - x^2 + x - 1}$$

Example 2

$$\begin{array}{r}
x \\
x^2+1 \overline{\big) \quad x^3 - x^2 + x - 1}
\end{array}$$

Example 2

$$
\begin{array}{r}
x \phantom{{}-1} \\
x^2+1 \overline{\smash{\big)}\ x^3 - x^2 + x - 1} \\
\underline{-x^3 \phantom{{}-x^2} - x} \phantom{{}-1}
\end{array}
$$

Example 2

$$
\begin{array}{r}
x \phantom{{}-1} \\
x^2+1 \overline{\big)\; x^3 - x^2 + x - 1} \\
\underline{-x^3 \phantom{{}-x^2} - x} \phantom{{}-1} \\
-x^2 \phantom{{}+x} - 1
\end{array}
$$

Example 2

$$
\begin{array}{r}
x - 1 \\
x^2 + 1 \overline{\smash{\big)}\; x^3 - x^2 + x - 1} \\
\underline{-x^3 \qquad\quad - x \phantom{{}-1}} \\
-x^2 \qquad\quad - 1
\end{array}
$$

Example 2

$$
\begin{array}{r}
x - 1 \\
x^2 + 1 \overline{\big) \; x^3 - x^2 + x - 1} \\
\underline{-x^3 \phantom{{}-x^2} - x \phantom{{}-1}} \\
-x^2 \phantom{{}+x} - 1 \\
\underline{x^2 \phantom{{}+x} + 1}
\end{array}
$$

Example 2

$$
\begin{array}{r}
x - 1 \\
x^2 + 1 \overline{\smash{\big)}\, x^3 - x^2 + x - 1} \\
\underline{- x^3 \qquad\;\; - x \quad} \\
- x^2 \qquad\;\; - 1 \\
\underline{x^2 \qquad\;\; + 1} \\
0
\end{array}
$$

Example 2

$$
\begin{array}{r}
x - 1 \\
x^2 + 1 \overline{\big) \; x^3 - x^2 + x - 1} \\
\underline{-x^3 \qquad\quad - x \quad} \\
-x^2 \qquad - 1 \\
\underline{x^2 \qquad + 1} \\
0
\end{array}
$$

So

$$\frac{x^3 - x^2 + x - 1}{x^2 + 1} = x - 1.$$

Example 3

$$x^2 + x + 1 \overline{\big) \quad x^3 \qquad\qquad\qquad - 1}$$

Example 3

$$
\begin{array}{r}
x \\
x^2+x+1 \overline{\smash{\big)}\; x^3 \qquad\qquad -1}
\end{array}
$$

Example 3

$$
\begin{array}{r}
x \\
x^2+x+1 \overline{\smash{\big)}\, x^3 -1} \\
\underline{-x^3-x^2-x}
\end{array}
$$

Example 3

$$\begin{array}{r}
x \phantom{{}-1} \\
x^2+x+1\overline{\big)\, x^3 \phantom{{}-x^2-x} -1} \\
\underline{-x^3-x^2-x\phantom{{}-1}} \\
-x^2-x-1
\end{array}$$

Example 3

$$
\begin{array}{r}
x - 1 \\
x^2 + x + 1 \overline{\big) \; x^3 \qquad\qquad\qquad - 1} \\
\underline{-x^3 - x^2 - x \quad\;\;} \\
-x^2 - x - 1
\end{array}
$$

Example 3

$$
\begin{array}{r}
x - 1 \\
x^2 + x + 1 \overline{\big) \quad x^3 \qquad\qquad\quad - 1} \\
\underline{- x^3 - x^2 - x \quad\;} \\
- x^2 - x - 1 \\
\underline{x^2 + x + 1}
\end{array}
$$

Example 3

$$
\begin{array}{r@{}l}
 & \; x - 1 \\ \cline{2-2}
x^2 + x + 1 \big) & \; x^3 \quad - 1 \\
 & -x^3 - x^2 - x \\ \cline{2-2}
 & - x^2 - x - 1 \\
 & \; x^2 + x + 1 \\ \cline{2-2}
 & \phantom{-x^3 - x^2 - x - {}} 0
\end{array}
$$

Example 3

$$
\begin{array}{r|l}
 & x - 1 \\
\cline{2-2}
x^2 + x + 1 & x^3 - 1 \\
 & \underline{-x^3 - x^2 - x} \\
 & -x^2 - x - 1 \\
 & \underline{x^2 + x + 1} \\
 & 0
\end{array}
$$

So

$$\frac{x^3 - 1}{x^2 + x + 1} = x - 1.$$

Example 4

$$3x - 1 \overline{\smash{)}\; 2x^3 \quad - 4x^2 \qquad\qquad + 1}$$

Example 4

$$\begin{array}{r|l} & \frac{2}{3}x^2 \\ \cline{2-2} 3x-1 & 2x^3 - 4x^2 \qquad\qquad + 1 \end{array}$$

Example 4

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & \frac{2}{3}x^2 & & \\
\cline{2-5}
3x-1) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
\end{array}$$

Example 4

$$
\begin{array}{r@{}llll}
 & & \frac{2}{3}x^2 & & \\ \cline{2-5}
3x-1 \big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{10}{3}x^2 & &
\end{array}
$$

Example 4

$$
\begin{array}{r@{}l}
 & \tfrac{2}{3}x^2 - \tfrac{10}{9}x \\
\cline{2-2}
3x - 1 \big) & \;2x^3 \;-4x^2 \qquad\qquad +1 \\
 & -2x^3 + \tfrac{2}{3}x^2 \\
\cline{2-2}
 & \qquad -\tfrac{10}{3}x^2
\end{array}
$$

Example 4

$$
\begin{array}{r}
\frac{2}{3}x^2 - \frac{10}{9}x \\
3x - 1 \overline{\big)\; 2x^3 - 4x^2 \qquad\qquad + 1} \\
\underline{-2x^3 + \frac{2}{3}x^2} \\
-\frac{10}{3}x^2 \\
\underline{\frac{10}{3}x^2 - \frac{10}{9}x}
\end{array}
$$

Example 4

$$
\begin{array}{r|l}
 & \frac{2}{3}x^2 - \frac{10}{9}x \\
\cline{2-2}
3x-1 & 2x^3 - 4x^2 \qquad\qquad + 1 \\
 & -2x^3 + \frac{2}{3}x^2 \\
\cline{2-2}
 & \qquad -\frac{10}{3}x^2 \\
 & \qquad \frac{10}{3}x^2 - \frac{10}{9}x \\
\cline{2-2}
 & \qquad\qquad -\frac{10}{9}x + 1
\end{array}
$$

Example 4

$$
\begin{array}{r|lllll}
\multicolumn{1}{r}{} & & & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} \\
\cline{2-6}
3x-1 & 2x^3 & -4x^2 & & & +1 \\
& -2x^3 & +\frac{2}{3}x^2 & & & \\
\cline{2-3}
& & -\frac{10}{3}x^2 & & & \\
& & \frac{10}{3}x^2 & -\frac{10}{9}x & & \\
\cline{3-4}
& & & -\frac{10}{9}x & & +1
\end{array}
$$

Example 4

$$
\begin{array}{r|rrrr}
 & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} & \\
\hline
3x-1 & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\hline
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\
\hline
 & & & -\frac{10}{9}x & +1 \\
 & & & \frac{10}{9}x & -\frac{10}{27} \\
\hline
\end{array}
$$

Example 4

$$
\begin{array}{rrrrr}
 & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} & \\
\cline{2-5}
3x-1 \big) \; 2x^3 & -4x^2 & & +1 & \\
-2x^3 & +\frac{2}{3}x^2 & & & \\
\cline{1-2}
 & -\frac{10}{3}x^2 & & & \\
 & \frac{10}{3}x^2 & -\frac{10}{9}x & & \\
\cline{2-3}
 & & -\frac{10}{9}x & +1 & \\
 & & \frac{10}{9}x & -\frac{10}{27} & \\
\cline{3-4}
 & & & \frac{17}{27} &
\end{array}
$$

Example 4

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} \\
\cline{2-5}
3x-1 \big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\
\cline{3-4}
 & & & -\frac{10}{9}x & +1 \\
 & & & \frac{10}{9}x & -\frac{10}{27} \\
\cline{4-5}
 & & & & \frac{17}{27}
\end{array}
$$

So

$$\frac{2x^3 - 4x^2 + 1}{3x - 1} = \frac{2}{3}\left(x^2 - \frac{5}{3}x - \frac{5}{9}\right) - \frac{17}{27(3x-1)}.$$

Example 5

$$x^2 - 2x + 5 \overline{\big) \; x^4 \qquad - x^2 \quad + x \quad - 4}$$

Example 5

$$\begin{array}{r} x^2 \\ x^2 - 2x + 5 \overline{) \; x^4 \qquad\qquad - x^2 + x - 4} \end{array}$$

Example 5

$$
\begin{array}{r}
x^2 \phantom{{}+x-4} \\
x^2-2x+5 \overline{\big)\; x^4 \phantom{{}+2x^3} - x^2 + x - 4} \\
\underline{-x^4+2x^3-5x^2} \phantom{{}+x-4}
\end{array}
$$

Example 5

$$
\begin{array}{r@{}l}
 & \phantom{-x^4+2x^3-{}} x^2 \\
\cline{2-2}
x^2-2x+5 \big) & \phantom{-{}} x^4 \phantom{{}+2x^3} - x^2 + x - 4 \\
 & -x^4+2x^3-5x^2 \\
\cline{2-2}
 & \phantom{-x^4+{}} 2x^3-6x^2+x
\end{array}
$$

Example 5

$$
\begin{array}{r|l}
 & x^2 + 2x \\
\hline
x^2 - 2x + 5 & x^4 - x^2 + x - 4 \\
 & \underline{-x^4 + 2x^3 - 5x^2} \\
 & 2x^3 - 6x^2 + x
\end{array}
$$

Example 5

$$
\begin{array}{rrrrrr}
 & & & x^2 & +2x & \\
\cline{2-6}
x^2-2x+5 \big) & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\
\cline{2-4}
 & & 2x^3 & -6x^2 & +x & \\
 & & -2x^3 & +4x^2 & -10x & \\
\cline{3-5}
\end{array}
$$

Example 5

$$
\begin{array}{r|rrrrr}
\multicolumn{1}{r}{} & & & x^2 & +2x & \\ \cline{2-6}
x^2-2x+5 & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\ \cline{2-4}
 & & 2x^3 & -6x^2 & +x & \\
 & & -2x^3 & +4x^2 & -10x & \\ \cline{3-5}
 & & & -2x^2 & -9x & -4
\end{array}
$$

Example 5

$$
\begin{array}{r|lllll}
 & & & x^2 & +2x & -2 \\
\hline
x^2-2x+5 & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\
\cline{2-4}
 & & 2x^3 & -6x^2 & +x & \\
 & & -2x^3 & +4x^2 & -10x & \\
\cline{3-5}
 & & & -2x^2 & -9x & -4
\end{array}
$$

Example 5

$$
\begin{array}{r|rrrrr}
\multicolumn{2}{r}{} & & x^2 & +2x & -2 \\
\cline{2-6}
x^2-2x+5 & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\
\cline{2-4}
 & & 2x^3 & -6x^2 & +x & \\
 & & -2x^3 & +4x^2 & -10x & \\
\cline{3-5}
 & & & -2x^2 & -9x & -4 \\
 & & & 2x^2 & -4x & +10 \\
\cline{4-6}
\end{array}
$$

Example 5

$$
\begin{array}{r}
x^2 \quad + 2x \quad - 2 \\
x^2 - 2x + 5 \overline{\big) \; x^4 \qquad\qquad - x^2 \quad + x \quad - 4} \\
\underline{- x^4 + 2x^3 - 5x^2} \qquad\qquad\quad \\
2x^3 - 6x^2 \quad + x \qquad \\
\underline{- 2x^3 + 4x^2 - 10x} \qquad \\
- 2x^2 \quad - 9x \quad - 4 \\
\underline{2x^2 \quad - 4x + 10} \\
- 13x \quad + 6
\end{array}
$$

Example 5

$$
\begin{array}{r}
x^2 \quad + 2x \quad - 2 \\
x^2 - 2x + 5 \overline{\big) \; x^4 \qquad\qquad - x^2 \quad + x \quad - 4} \\
\underline{-x^4 + 2x^3 - 5x^2} \qquad\qquad\quad \\
2x^3 - 6x^2 \quad + x \qquad\; \\
\underline{-2x^3 + 4x^2 - 10x} \qquad\; \\
-2x^2 \quad - 9x \quad - 4 \\
\underline{2x^2 \quad - 4x + 10} \\
-13x \quad + 6
\end{array}
$$

So

$$\frac{x^4 - x^2 + x - 4}{x^2 - 2x + 5} = x^2 + 2x - 2 + \frac{-13x + 6}{x^2 - 2x + 5}.$$

How to deal with rational functions?

How can we integrate something like

$$\int \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8}\, \mathrm{d}x$$

or

$$\int \frac{x+2}{x^3 - x}\, \mathrm{d}x?$$

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

using polynomial long division.

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

using polynomial long division.

This is still not something we can integrate so we need to go further.

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots$$

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots = \frac{P(x)}{Q(x)}$$

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots = \frac{P(x)}{Q(x)}$$

How do we reverse this process?

Answer: partial fractions

When the denominator is $(ax+b)(cx+d)\cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

When the denominator is $(ax + b)(cx + d) \cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_n x + b_n)$$

- If the degree of $Q(x)$ is larger than the degree of $P(x)$

When the denominator is $(ax+b)(cx+d)\cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

- If the degree of $Q(x)$ is larger than the degree of $P(x)$
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we can always find constants $A_1, A_2, \ldots, n$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots \frac{A_n}{a_nx + b_n}$$

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Multiplying both sides by $(x-1)(x+1)$

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x+1} + \frac{B(x-1)(x+1)}{x-1} \\ &= A(x-1) + B(x+1) \\ &= (A+B)x + (B-A) \end{aligned}$$

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So

$$-2A = 1 \quad \text{hence} \quad A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$

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For every factor $(ax+b)^k$ in $q(x)$, the partial fraction expansion has terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}.$$

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Side note: integrating $\frac{1}{x}$.

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Using substitution this gives the formula

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + C.$$

Side note: integrating $\frac{1}{x^k}$.

Recall that if $k > 1$

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1. Express $\frac{p(x)}{q(x)}$ in the form

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3. Integrate all these pieces seperately.

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$$I = \frac{1}{3}x^2 - 2x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

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So

$$I = x + \ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + C.$$