Midterm 2 practice

UCLA: Math 31B, Spring 2017

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Date:
Version: practice

- This exam has 4 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____________________

ID number: ____________________

Discussion section: ____________________

Question	Points	Score
1	8	
2	10	
3	15	
4	7	
Total:	40	

1. Calculate the following limits using any technique you like.

   (a) (5 points)

$$\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x^2}$$

   (b) (3 points)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}+e^{\frac{1}{x}}\right)^{\arctan x}$$

Solution:

(a) Let $y(x)=\ln(\frac{x+1}{x})^{x^2}=x^2\ln(1+\frac{1}{x})=\frac{\ln(1+\frac{1}{x})}{\frac{1}{x^2}}$. Then

$$\lim_{x\to\infty}y(x)=\lim_{x\to\infty}\frac{\frac{1}{1+\frac{1}{x}}\cdot(-\frac{1}{x^2})}{-\frac{2}{x^3}}=\lim_{x\to\infty}\frac{x}{2(1+\frac{1}{x})}=\infty.$$

So $\lim_{x\to\infty}(\frac{x+1}{x})^{x^2}=\lim_{x\to\infty}e^{y(x)}=\infty$.

(b) $(1+0+e^0)^{\frac{\pi}{2}}=2^{\frac{\pi}{2}}$.

2. For each of the following improper integrals say whether it converges or diverges. If the integral converges, you should say what value it converges to.

   (a) (5 points)
$$\int_{-\frac{\pi}{2}}^{0} \frac{\cos x}{\sin x}\ dx$$

   (b) (5 points)
$$\int_{1}^{\infty} \frac{\ln x}{x^2}\ dx$$

Solution:

(a)

$$\int_{-\frac{\pi}{2}}^{0} \frac{\cos x}{\sin x}\ dx = \lim_{R\to 0-} \int_{-\frac{\pi}{2}}^{R} \frac{\cos x}{\sin x}\ dx = \lim_{R\to 0-} \Big[\ln|\sin x|\Big]_{-\frac{\pi}{2}}^{R}$$
$$= \lim_{R\to 0-} \ln|\sin R|\ dx = -\infty.$$

This integral diverges.

(b)

$$\int_{1}^{\infty} \frac{\ln x}{x^2}\ dx = \lim_{R\to\infty} \int_{1}^{R} \frac{\ln x}{x^2}\ dx = \lim_{R\to\infty} \left[-\frac{\ln x + 1}{x}\right]_{1}^{R}$$
$$= \lim_{R\to\infty} \left[-\frac{\ln R + 1}{R} + 1\right] = 1.$$

This integral converges to 1.

3. For each of the following series say whether it converges or diverges. You do NOT need to justify your answer.

   Grading scheme: 0 points for wrong, 1 point for no response, 3 points for correct.

   (a) (3 points) $\sum_{n=1}^{\infty} \frac{1}{n}$.

   (b) (3 points) $\sum_{n=1}^{\infty} \frac{(n+3)^2}{n^4}$.

   (c) (3 points) $\sum_{n=45}^{\infty} \frac{(-2)^n+8^n}{3^{2n}}$.

   (d) (3 points) $\sum_{n=1}^{\infty} a_n$ where the sequence of partial sums $(s_N)_{N=1}^{\infty}$ is described by

   $$s_N = \sum_{n=1}^{N} \frac{1}{\sqrt{n}}.$$

   (e) (3 points) $\sum_{n=1}^{\infty} a_n$ where the sequence of partial sums $(s_N)_{N=1}^{\infty}$ is described by

   $$s_N = \frac{1}{\sqrt{N}}.$$

Solution:

(a) Diverges (its the harmonic series!!! see e.g. integral test)

(b) Converges (expand and split apart using limit laws)

(c) Converges (expand using series laws + geometric series)

(d) Diverges (comparison with harmonic series or integral test)

(e) Converges (by definition)

4. (7 points) Let (F_n) be the *Fibonacci* sequence, that is

$$F_0 = 0, F_1 = 1, \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

Consider the sequence (R_n) for $n \geq 1$ defined by $R_n = \frac{F_{n+1}}{F_n}$, i.e.

$$\frac{1}{1}, \frac{2}{1}, \frac{3}{2}, \frac{5}{3}, \frac{8}{5}, \frac{13}{5}, \ldots$$

You may assume that (R_n) converges to a limit L such that $1 < L < 2$. Find L.

Solution: First we notice that

$$R_n = \frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} = 1 + \frac{F_{n-1}}{F_n} = 1 + R_{n-1}^{-1}.$$

(Note: without some kind of hint guiding you to the above, I think this might be bordering on too difficult for an exam.)

Let $L = \lim_{n\to\infty} R_n$ then taking limits of both sides of the above equation gives us

$$\lim_{n\to\infty} R_n = 1 + \frac{1}{\lim_{n\to\infty} R_{n-1}}$$

that is,

$$L = 1 + \frac{1}{L}.$$

Rearranging we get the formula $L^2 - L - 1 = 0$. Using the quadratic formula we get that

$$L = \frac{1 \pm \sqrt{5}}{2}.$$

But we know that $1 < L < 2$ so $L = \frac{1}{2}\left(1 + \sqrt{5}\right)$.

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