

This weeks problem set focuses on the ideas of bases and linear transformations. A question marked with a \dagger is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a $*$ is especially important.

Homework 2: due Monday 4 Feb: question 18 and 19 from section 2.1. Note that the notation textbook uses $N(T)$ instead of $\ker T$ and $R(T)$ instead of $\operatorname{im} T$. The questions are typed out below.

1. From section 2.1, problems 15, 17, 18, 19, 24, 26*, 28, 31 \dagger , 40*.
2. From section 2.1, problems 1, 2, 5, 6, 9*, 14, 14b.
- 3* Let V be a finite dimensional vector space over \mathbb{F} and $B\{v_1, \dots, v_n\}$ a basis. Let W be another vector space and w_1, \dots, w_n a collection of elements. Show that there is a unique linear map such that $T(v_i) = w_i$.
- 4* Let V and W be vector spaces over \mathbb{F} . Define $\operatorname{Hom}(V, W)$ to be the set of linear maps from V to W .
 - (a) Show that $\operatorname{Hom}(V, W)$ is itself a vector space.
 - (b) If V is finite dimensional and B is a basis for V , construct a basis for $V^* = \operatorname{Hom}(V, \mathbb{F})$. The vector space V^* is called the *dual space* to V .
- 5* Let $T : V \longrightarrow W$ be an injective linear map. Show that, if we consider T , instead, as a linear map $V \longrightarrow \operatorname{im} T$ (just restrict what we consider to be the codomain), then it defines an isomorphism and shows that $V \cong \operatorname{im} T$.
- 6* Let V and W be vector spaces over \mathbb{F} . Define the set

$$V \times W = \{ (v, w) \mid v \in V \text{ and } w \in W \}.$$

This is called the *product* of the vector spaces.

- (a) Show that $V \times W$ is a vector space.
 - (b) Define a map $\iota_V : V \rightarrow V \times W$ by $\iota_V(v) = (v, 0)$. Show that ι_V is an injective linear map. Note that we can define a similar map ι_W .
 - (c) If $U \subset V$ is a subspace, show that $U \times W$ is a subspace of $V \times W$.
 - (d) Show that $V \times W = (V \times \{0\}) \oplus (\{0\} \times W)$.

Note that we can consider $V \times \{0\}$ as a copy of V in $V \times W$. For this reason, often mathematicians write $V \oplus W$ instead of $V \times W$ and call it the external direct product. Though this is a little confusing so we won't talk about it in this way in this class.
- 7* (2.1.18) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\ker T = \operatorname{im} T$.
 - 8* (2.1.19) Give an example of distinct linear transformations T and U such that $\ker T = \ker U$ and $\operatorname{im} T = \operatorname{im} U$.