Midterm 1 practice 2

UCLA: Math 115A, Winter 2020

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Date:
Version: 1

- This exam has 4 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
Total:	20	

Question 1 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (1 point) In the vector space $\mathbb{C}[x]$, the set $\{ x^2 - x, x^2 + 1 \}$ is

   A. linearly dependent
   B. linearly independent
   C. a spanning set
   D. none of the above

   The following two questions concern the subsets

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; a^2 + b^2 = c \right\} \subseteq \mathbb{R}^3$$

$$V = \{ p(x) \in \mathbb{R}_2[x] \mid p(1) = \lambda \} \subseteq \mathbb{R}_2[x]$$

   for some choice of $\lambda \in \mathbb{R}$. Recall that $\mathbb{R}_2[x]$ is the space of polynomials of degree at most 2.

   (b) (1 point) Which of the following is a true statement?

   A. Both U and V are subspaces regardless of the value of $\lambda \in \mathbb{R}$.
   B. Only U is a subspace.
   C. V is a subspace for any λ.
   D. Only V is a subspace when $\lambda = 0$.

   (c) (1 point) When $\lambda = 0$, the subspace V has dimension

   A. 1
   B. 2
   C. 3
   D. 4

(d) (1 point) Let U and W be two, finite dimensional subspaces of a vector space V. Which of the following statements is true?

A. We must have $U \cap W = \{ 0 \}$.

B. If $U \cap W = \{ 0 \}$ then $U + W = V$.

C. $\dim U + \dim W \geq \dim(U + W)$**.**

D. The dimension of $U + W$ is unrelated to $\dim U$ and $\dim W$.

(e) (1 point) Which of the following definitions, makes $T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}^2$ into a surjective linear map?

A. $T(p) = \begin{pmatrix} 0 \\ p(1) \end{pmatrix}$

B. $T(p) = \begin{pmatrix} p(1) \\ p(1) \end{pmatrix}$

C. $T(p) = \begin{pmatrix} p'(1) \\ p(1) \end{pmatrix}$

D. $T(p) = \begin{pmatrix} p - 1 \\ p + 1 \end{pmatrix}$

2. Give (simple) examples of all of the following situations.

   (a) (2 points) Two subspaces U, V of $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ such that $U+V = \mathrm{Mat}_{2\times 2}(\mathbb{R})$ but $\mathrm{Mat}_{2\times 2}(\mathbb{R}) \neq U \oplus V$.

   Solution: For example take $U = \{M \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid M_{11} = 0\}$ and $V = \{M \in \mathrm{Mat}_{2\times 2}(\mathbb{R}) \mid M_{12} = M_{22} = 0\}$

   (b) (2 points) A basis for each of your subspaces U and V above.

   Solution: For U: $\{E_{12}, E_{21}, E_{22}\}$.
   For V: $\{E_{11}, E_{21}\}$

   (c) (1 point) A basis for $\mathrm{Mat}_{2\times 2}(\mathbb{R})$ that does not contain either of the bases from the previous part.

   Solution: $\{E_{11} + E_{22}, E_{12} + E_{22}, E_{21} + E_{22}, E_{22} - E_{11}\}$

3. Consider the following maps. Prove or disprove that they are linear and if linear, find the dimension of the kernel (nullspace).

   (a) (1 point) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $T(a,b) = a^2 - b^2$.

   Solution: This is not linear since $T(4,2) = 12 \neq 2 \times T(2,1) = 6$.

   (b) (4 points) $R : \mathrm{Mat}_{2\times 2}(\mathbb{C}) \longrightarrow \mathbb{C}^2$ given by

$$R(M) = M \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

   Solution: This is a linear transformation since

$$R(M+N) = (M+N)\cdot v = M\cdot v + N\cdot v = R(M) + R(N)$$

   and

$$R(\lambda M) = (\lambda M)\cdot v = \lambda(M\cdot v) = \lambda R(M).$$

4. (a) (2 points) Let V and W be vector spaces and let $T : V \longrightarrow W$ be a linear map. Prove that the kernel (nullspace), $\ker T \subset V$ is a subspace.

Solution: First note that $T(0) = 0$, so $0 \in \ker T$. Now if $u, v \in \ker T$ then $T(u + v) = T(u) + T(v) = 0$ so $u + v \in \ker T$ so the kernel is closed under addition. Similarly if $\lambda \in \mathbb{F}$ and $v \in \ker T$ then $T(\lambda v) = \lambda T(v) = 0$ so $\lambda v \in \ker T$ and the kernel is closed under scalar multiplication and thus is a subspace.

(b) (3 points) Suppose W_1 and W_2 are two subspaces of a vector space V such that $V = W_1 \oplus W_2$. If B_1 and B_2 are bases of W_1 and W_2 respectively, show that $B_1 \cup B_2$ is a basis for V.

Solution: First we show that $B = B_1 \cup B_2$ is spanning. Suppose $v \in V$. Then since $V = W_1 \oplus W_2$ there must be $w_1 \in W_1$ and $w_2 \in W_2$ such that $v = w_1 + w_2$. Now since B_1 is a basis for W_1, w_1 can be written as a linear combination of elements in B_1. Similarly w_2 can be written as a linear combination of elements in B_2 thus $v = w_1 + w_2$ can be written as a linear combination of elements in B.

For linear independence, let

$$\lambda_1 u_1 + \cdots + \lambda_m u_m + \mu_1 v_1 + \cdots + \mu_n v_n = 0$$

where $\lambda_i, \mu_i \in \mathbb{F}$ and $u_i \in B_1$ and $v_i \in B_2$. Rearranging we get that

$$\lambda_1 u_1 + \cdots + \lambda_m u_m = -\mu_1 v_1 - \cdots - \mu_n v_n$$

but this is an element of $W_1 \cap W_2$. But since $V = W_1 \oplus W_2$ we must have that $W_1 \cap W_2 = \{0\}$. Thus, on one hand

$$\lambda_1 u_1 + \cdots + \lambda_m u_m = 0$$

and so $\lambda_i = 0$ for all i since B_1 is a basis (and thus linearly independent). On the other hand we also have

$$-\mu_1 v_1 - \cdots - \mu_n v_n = 0$$

ans so $\mu_i = 0$ for all i since B_2 is a basis. Hence $B_1 \cap B_2$ is linearly independant and thus a basis.

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