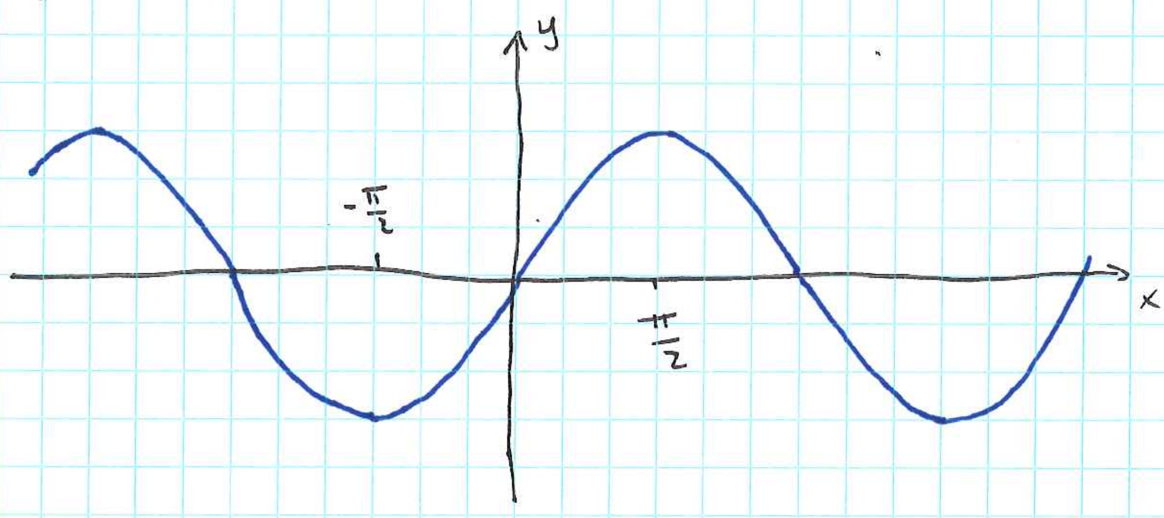
Inverse trig functions

- The function $\mathbb{R} \to \mathbb{R}$, $\sin(x)$ is not injective so cannot possibly have an inverse

- To fix this we restrict to a domain on which $\sin x$ is inj. To figure out what the domain should be we can look at the graph of $\sin x$:



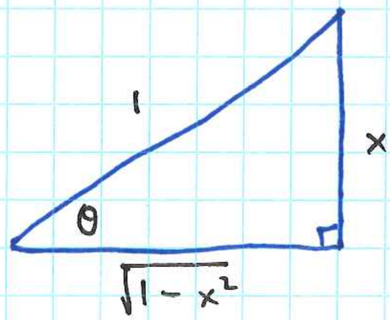
- We see that the function

$$\sin x \, ; \, \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \longrightarrow [-1, 1]$$

is bijective so an inverse exists.

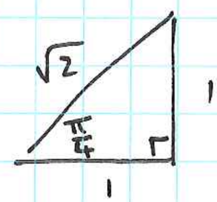
Def The function $\sin^{-1} x \, ; \, [-1, 1] \longrightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is defined to be the inverse to $\sin x$.

– If $x > 0$ how do we find $\sin^{-1} x$?



$\sin^{-1} x = \theta$, the unique angle such that $\sin\theta = x$.

Ex $\sin^{-1} \frac{1}{\sqrt{2}}$. We know from the Δ:



that $\sin \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ so $\sin^{-1} \frac{1}{\sqrt{2}} = \frac{\pi}{4}$. In particular

$$\sin^{-1}\left(\sin \frac{\pi}{4}\right) = \frac{\pi}{4}$$

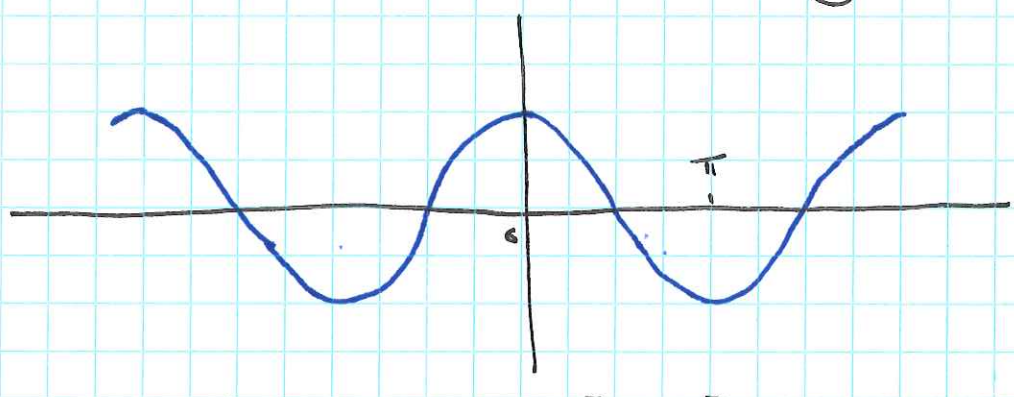
Ex $\sin^{-1}\left(\sin \frac{3\pi}{2}\right) \neq \frac{3\pi}{2}$!! $\sin^{-1} x$ is the inverse to $\sin x$ <u>only</u> when restricted to $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

To eval. the above we not that $\frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$

so $\sin \frac{3\pi}{2} = \sin -\frac{\pi}{2}$ so

$$\sin^{-1}\left(\sin \frac{3\pi}{2}\right) = \sin^{-1}\left(\sin -\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

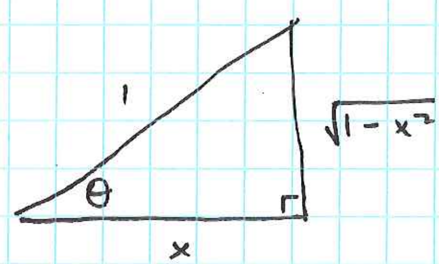
- We can do ~~the~~ the same thing ~~a~~ with $\cos x$



We choose the domain $[0, \pi]$. So $\cos x$ is bij. as a function $[0,\pi] \longrightarrow [-1,1]$

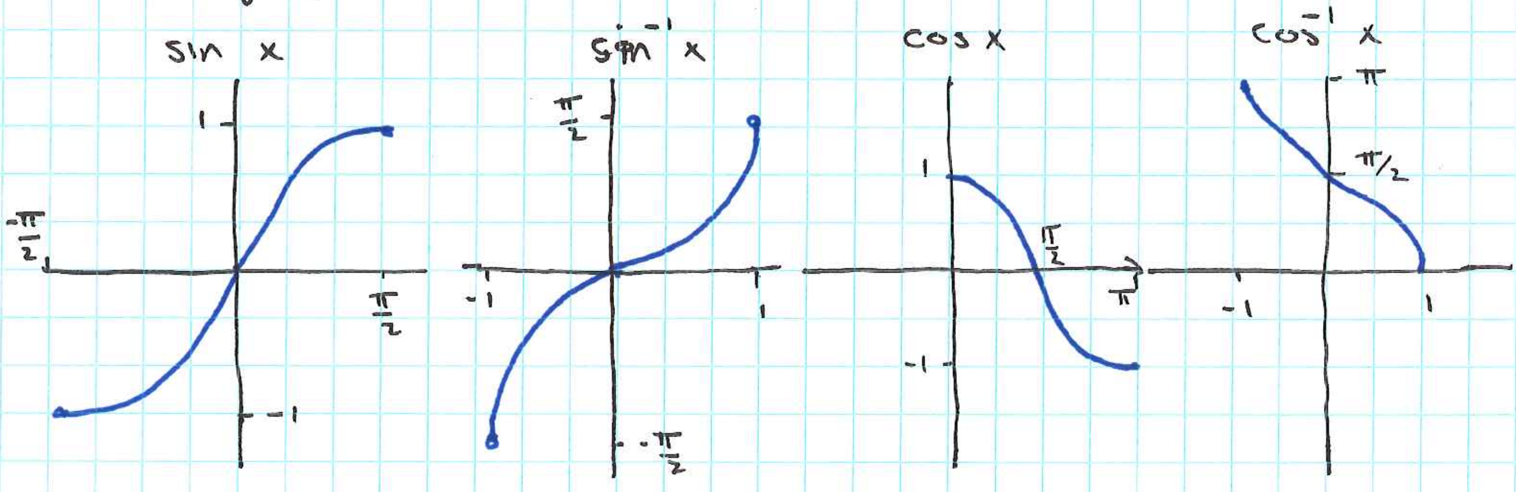
Def $\cos^{-1} x ; [-1,1] \longrightarrow [0,\pi]$ is the function inverse to $\cos x$.

- As before we can calculate $\cos^{-1} x$ using the triangle

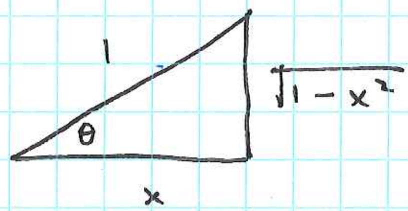


$\cos^{-1} x = \theta$.

- The graphs:



Ex What is $\tan(\cos^{-1} x)$? We use the triangle



where $\theta = \cos^{-1} x$. So $\tan\theta = \tan(\cos^{-1} x) = \frac{1}{x}\sqrt{1-x^2}$.

Derivatives How do we find the derivatives?

$$y = \sin^{-1} x$$

then $\sin y = x$

taking derivatives wrt x of both sides:

$$\frac{dy}{dx}\cdot\cos y = 1$$

so

$$\frac{dy}{dx} = \sec y = \sec(\sin^{-1} x)$$

But using the Δ above $\cos(\sin^{-1} x) = \sqrt{1-x^2}$ so

Thm $$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}$$

<u>Ex</u> $\displaystyle\int_0^{1/4} \frac{1}{\sqrt{1-4x^2}}\, dx$

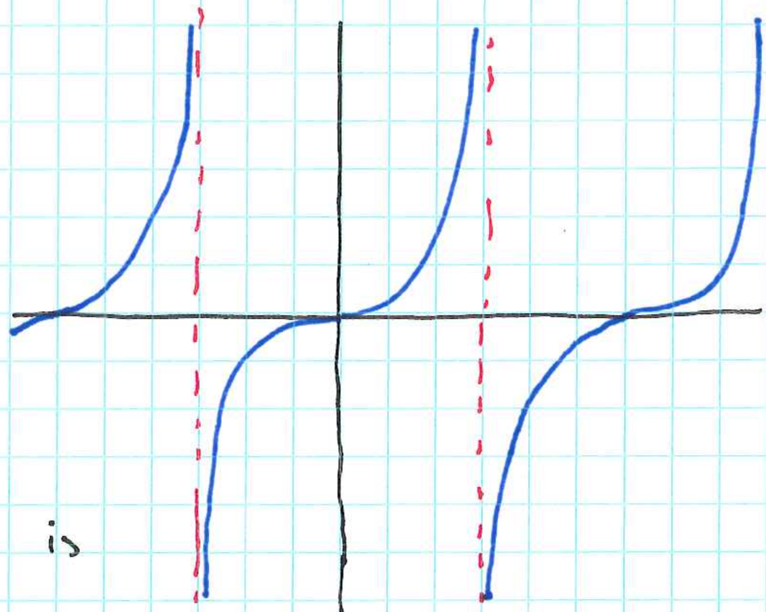
We use the substitution $u = 2x$, $du = 2\,dx$

$$= \frac{1}{2}\int_0^{1/2} \frac{1}{\sqrt{1-u^2}}\, du \qquad \left(\begin{array}{l} \text{since } x = 0 \Rightarrow u = 0 \\ x = \frac{1}{4} \Rightarrow u = \frac{1}{2} \end{array}\right)$$

$$= \frac{1}{2}\Big[\arcsin u\Big]_0^{1/2} = \frac{1}{2}\arcsin\frac{1}{2} = \frac{\pi}{12}$$

~~but~~ since $\sin\frac{\pi}{6} = \frac{1}{2}$ so $\arcsin\frac{1}{2} = \frac{\pi}{6}$

- What about $\tan x$?

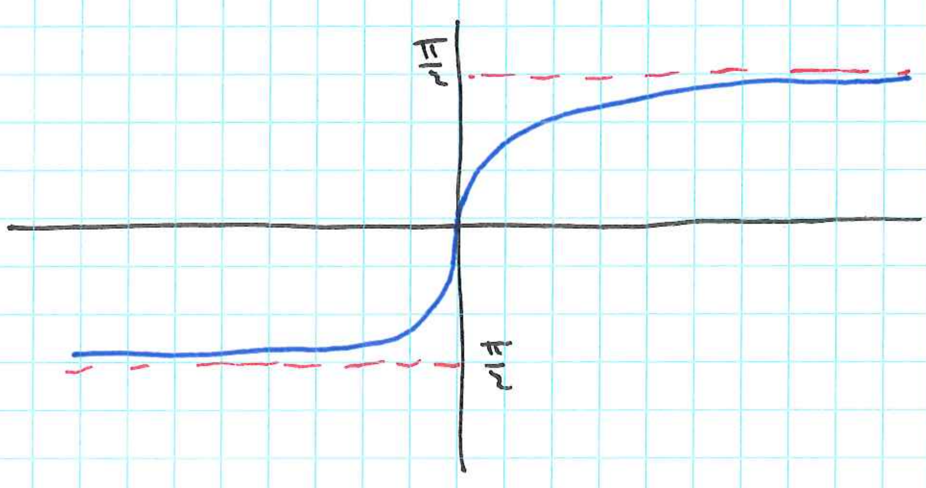


- Restrict to

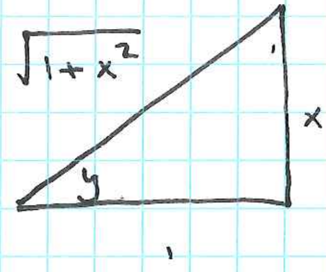
$\tan x : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$

to get a bij. function.

<u>Def</u> $\arctan x : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ is the inverse to $\tan x$



- We can find the derivative



$$y = \arctan x$$

$$\tan y = x$$

$$\sec^2 y \cdot \frac{dy}{dx} = 1 \qquad \text{but } \cos y = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{dy}{dx} = \cos^2 y = \frac{1}{1+x^2}$$

Thm $\quad \frac{d}{dx} \arctan x = \frac{1}{1+x^2}$

Ex $\quad \int \frac{1}{a^2+x^2}\,dx$ $\qquad$ ~~use~~ multiply top and bottom by $1/a^2$

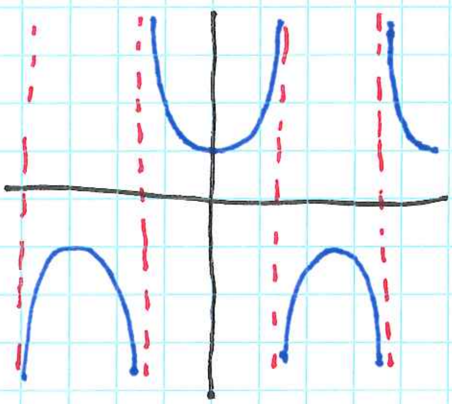
$$= \int \frac{1/a^2}{1 + x^2/a^2}\,dx \qquad u = \frac{x}{a} \qquad du = \frac{1}{a}\,dx$$

$$a\,du = dx$$

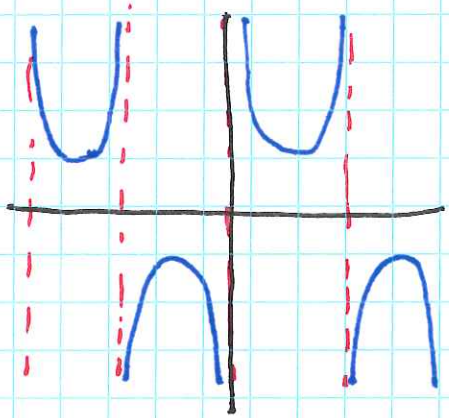
$$= \frac{1}{a}\int \frac{1}{1+u^2}\,du = \frac{1}{a}\arctan u + C$$

$$= \frac{1}{a}\arctan \frac{x}{a} + C.$$

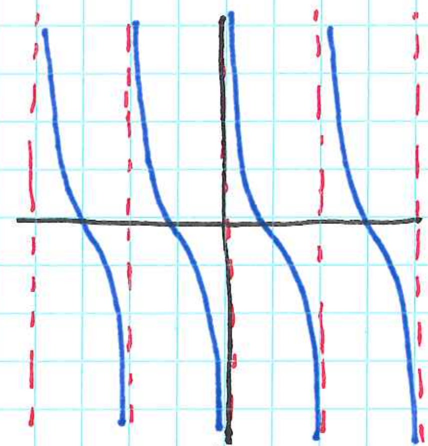
Other trig functions



sec



csc



cot

We restrict these to

$$\sec x : \left[0, \tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right] \longrightarrow (-\infty, -1] \cup [1, \infty)$$

$$\csc x : \left[-\tfrac{\pi}{2}, 0\right) \cup \left(0, \tfrac{\pi}{2}\right] \longrightarrow (-\infty, -1] \cup [1, \infty)$$

$$\cot x : (0, \pi) \longrightarrow \mathbb{R}$$

<u>Def</u> The functions

$$\operatorname{arcsec} x : (-\infty, -1] \cup [1, \infty) \longrightarrow \left[0 \tfrac{\pi}{2}\right) \cup \left(\tfrac{\pi}{2}, \pi\right]$$

$$\operatorname{arccsc} x : (-\infty, -1] \cup [1 \infty) \longrightarrow \left[-\tfrac{\pi}{2}, 0\right) \cup \left(0, \tfrac{\pi}{2}\right]$$

$$\operatorname{arccot} x : \mathbb{R} \longrightarrow (0, \pi)$$

are defined to be the inverse of the functions above.

<u>Thm</u>

$$\frac{d}{dx} \operatorname{arcsec} x = \frac{1}{|x| \sqrt{x^2 - 1}} \qquad \frac{d}{dx} \operatorname{arccsc} x = -\frac{1}{|x| \sqrt{x^2 - 1}}$$

$$\frac{d}{dx} \operatorname{arccot} x = -\frac{1}{x^2 + 1}$$