Math 3B: Lecture 19

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Last time

- Modelling using differential equations

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- Exponential population growth

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- Modelling using differential equations
- Exponential population growth
- Logistic population growth

Logistic growth

The equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

is called the Logistic equation and K is the carrying capacity.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 1. $N(0) = 0$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains 0.

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Key takeaway

Both $N(t) = 0$ and $N(t) = K$ are solutions to the ODE. They are called equalibrium solutions.

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Case 3. $0 \leq N(0) \leq K$

In this case, N is initially increasing and so becomes more positive, slowing down as it gets close to K.

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In this case, N is initially increasing and so becomes more positive, slowing down as it gets close to K.

Case 4. $N(0) \geq K$

In this case N is initially decreasing but decreases slower and slower as it gets close to K.

Logistic growth with outside effects

We can also modify the logistic equation to get something which models an outside effect. For example harvesting.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N - h(t)$$

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This would model a population growing logistically but where we are harvesting at a rate of $h(N)$. E.g. we decide to continually harvest 3% of the population then

$$h(N) = 0.03N.$$

Checking solutions

The most straighforward way of checking a function $y = f(x)$ is a solution to a differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g(x, y)$$

is to simply plug it in to both sides.

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$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= e^{\sin x} \cos x \\ y \cos x &= e^{\sin x} \cos x \end{aligned}$$

Reminder: Implicit differentiation

If we have an equation relating variables y and x, e.g.

$$x^2 + y^2 = 1$$

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$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(x^2 + y^2) &= \frac{\mathrm{d}}{\mathrm{d}x}1 \\ \frac{\mathrm{d}}{\mathrm{d}x}(x^2) + \frac{\mathrm{d}}{\mathrm{d}x}(y^2) &= 0 \\ 2x + 2y'y &= 0 \end{aligned}$$

Another example

Lets differentiate

$$3x + \cos y = xy$$

To do this we apply $\frac{\mathrm{d}}{\mathrm{d}x}$ to both sides:

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3 - y' \sin y &= y + xy'.
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Note

We can rearrange this to get

$$y' = \frac{3 - y}{x + \sin y}$$

a differential equation. Whatever y is, as long as it obeys the above relation, it is a solution to this ODE!

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4. solve for y!

Examples

On the board. . .