This week you will get practice drawing and understanding slope fields, making qualitative statements about solutions using them and some practice applying Euler's method.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.4) Sketch the slope fields and a few solutions for the differential equations given
   (a) (6.4.12) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(4-y)(y-2)$
   (b) (6.4.14) $\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - y$
   (c) (6.4.16) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + t^2 - 1$
   (d) (6.4.17) $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{y}{x}$

   Hint: feel free to use technology, just make sure you know how to draw a solution if you are given a slope field.

2. (6.4) Sketch the slope fields and the solution passing through the specified point for the differential equations given
   (a) (6.4.19) $\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - y^2, (t, y) = (0, 0)$
   (b) (6.4.20) $\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5y(1-y), (t, y) = (0, 0.1)$
   (c) (6.4.21) $\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{\frac{t}{y}}, (t, y) = (4, 1)$
   (d) (6.4.22) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2\sqrt{t}, (t, y) = (9, -1)$

3. (6.4.33) Consider the differential equation
$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{t}$$
   (a) verify that $y(t) = \ln t$ is a solution to this differential equation satisfying $y(1) = 0$.
   (b) Use Euler's method to approximate $y(2) = \ln 2$ with $h = 0.5$.

4. (6.4.37) A population subject to seasonal fluctuations can be described by the logistic equation with an oscillating carrying capacity. Consider, for example,
$$\frac{\mathrm{d}P}{\mathrm{d}t} = P\left(1 - \frac{1}{100 + 50\sin 2\pi t}\right)$$
   Although it is difficult to solve this differential equation, it is easy to obtain a qualitative understanding.
   (a) Sketch a slope field over the region $0 \le t \le 5$ and $0 \le P \le 200$.
   (b) Sketch solutions that satisfy $P(0) = 0$, $P(0) = 10$, and $P(0) = 200$.
   (c) Use technology to obtain a better rendition of the slope field and solutions.
   (d) Comment on your solutions and compare to your work using different methods.

5. (6.4.40) A population, in the absence of harvesting, exhibits the following growth
$$\frac{\mathrm{d}N}{\mathrm{d}t} = N\left(\frac{N}{100} - 1\right)\left(1 - \frac{N}{1000}\right)$$
   where N is abundance and t is time in years.
   (a) Write an equation that corresponds to harvesting the population at a rate of 0.5% per day.

(b) Sketch the slope field for the differential equation you found in part a; by sketching solutions, describe how the fate of the population depends on its initial abundance.

6. (6.5) Draw phase lines, classify the equilibria, and sketch a solution satisfying the specified initial value for the equations in the following.
   (a) (6.5-2) $\frac{\mathrm{d}y}{\mathrm{d}t} = 2 - 3y$, $y(0) = 2$
   (b) (6.5-5) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-10)(20-y)$, $y(0) = 9$
   (c) (6.5-6) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-5)(25-y)$, $y(0) = 7$
   (d) (6.5-7) $\frac{\mathrm{d}y}{\mathrm{d}t} = \sin y$, $y(0) = 0.1$
   (e) (6.5-10) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^3 - 4y$, $y(0) = 0.1$

7. (6.5-33) To account for the effect of a generalist predator (with a type II functional response) on a population, ecologists often write differential equations of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0.1N\left(1 - \frac{N}{1,000}\right) - \frac{10N}{1+N}$$

   (a) Sketch the phase line for this system.
   (b) Discuss how the fate of the population depends on its initial abundance.

   Hint: dont worry about what the first sentence means, you dont need to know where the differential equation comes from.

8. (6.5-39) Consider a population of clonally reproducing individuals consisting of two genotypes, a and A, with per capita growth rates, r_a and r_A, respectively. If N_a and N_A denote the densities of genotypes a and A, then

$$\frac{\mathrm{d}N_a}{\mathrm{d}t} = r_a N_a \qquad \frac{\mathrm{d}N_A}{\mathrm{d}t} = r_A N_A$$

   Also, let $y = \frac{N_a}{N_a + N_A}$ be the fraction of individuals in the population that are genotype a. Show that y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (r_a - r_A)y(1-y)$$

9. (6.5-40) In the Hawk-Dove replicator equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{2}(1-y)(C(1-y) - V)$$

   if the value $V > 0$ is specified, then find the range of values of C (in terms of V) that will ensure a polymorphism exists (i.e., find conditions that ensure the existence of an equilibrium $0 < y^* < 1$ that is stable).

   (Hint: you do not need to know anything about the Hawk-Dove Replicator - though it is very interesting! - all you need to know is that V is a constant and C is a parameter. A polymorphism is a stable equilibrum between zero and one.)

10. (6.5-41) Production of pigments or other protein products of a cell may depend on the activation of a gene. Suppose a gene is *autocatalytic* and produces a protein whose presence activates greater production of that protein. Let y denote the amount of the protein (say, micrograms) in the cell. A basic model for the rate of this self-activation as a function of y is

$$A(y) = \frac{ay^b}{k^b + y^b} \text{ micrograms/minute}$$

   where a represents the maximal rate of protein production, $k > 0$ is a half saturation constant, and $b \geq 1$ corresponds to the number of protein molecules required to active the gene. On the other hand, proteins in the cell are likely to degrade at a rate proportional to y, say cy. Putting these two components together, we get the following differential equation model of the protein concentration dynamics:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay^b}{k^b + y^b} - cy$$

(a) Verify that $\lim_{y\to\infty} A(y) = a$ and $A(k) = a/2$.

(b) Verify that $y = 0$ is an equilibrium for this model and determine under what conditions it is stable.

(Hint: the definition of autocatalytic is given in the question, it is a gene that produces a protein whose presence activate greater production of that protein.)

11. (6.5-42) Consider the model of an autocatalytic gene in Problem 41 with $b = 1$, $k > 0$, $a > 0$, and $c > 0$.

(a) Sketch the phase line for this model when $ck > a$.

(b) Sketch the phase line for this model when $ck < a$.