Infinite Series

Suppose we are given a sequence (a_n). We want to make sense of

$$\sum_{n=0}^{\infty} a_n .$$

Ex $a_n = (-1)^n$

$$\sum_{n=0}^{\infty} (-1)^n = \underbrace{1 - 1}_{=0} + \underbrace{1 - 1}_{=0} + \underbrace{1 - 1}_{=0} + \dots$$

$$\overset{?}{=} 0$$

or

$$= 1 \underbrace{- 1 + 1}_{=0} \underbrace{- 1 + 1}_{=0} \underbrace{- 1 + \dots}_{=0}$$

$$\overset{?}{=} 1$$

Something went wrong! We need to be careful!

If we have a sequence (a_n), define the <u>N^{th} partial sum</u> to be

$$S_N := \sum_{n=0}^{N} a_n$$

Rmk * $S_{N+1} = S_N + a_{N+1}$

* As N gets very large we should imagine that S_N is getting close to $\sum_{n=0}^{\infty} a_n$.

* (S_N) is a perfectly good sequence!

Ex * $(a_n) = (-1)^n$. then

$$S_N = \underbrace{1 - 1 + 1 - 1 + \dots \pm 1}_{N \text{ times}}$$

$$= \begin{cases} 1 & \text{if } N \text{ even} \\ 0 & \text{if } N \text{ odd} \end{cases}$$

* $a_n = \frac{1}{2^n}$ then $S_N = 1 + \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^N}$

Def $\sum_{n=0}^{\infty} a_n \overset{\text{Def}}{=} \lim_{N \to \infty} S_N$

If this limit exists we say $\sum a_n$ converges, otherwise we say it diverges.

Ex * $a_n = (-1)^n$, $S_N = \begin{cases} 0 & N \text{ even} \\ 1 & N \text{ odd} \end{cases}$

$\lim_{N \to \infty} S_N$ does not exist, so $\sum a_n$ div.

* $a_n = 1$ then $S_N = 1 + 1 + \ldots + 1 = N$

so $\sum_{n=1}^{\infty} a_n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} N = \infty$

so $\sum a_n$ diverges.

Ex A geometric series is one of the form

$$\sum_{n=0}^{\infty} cr^n.$$

Let's find an expression for S_N:

$$S_N = c + cr + cr^2 + \ldots + cr^N$$

Here's the sneaky trick

$$rS_N = cr + cr^2 + cr^3 + \ldots + cr^{N+1}$$

So
$$S_N - rS_N = c - cr^{N+1}$$

$$(1-r)\,S_N = c(1-r^{N+1})$$

$$S_N = \frac{c(1-r^{N+1})}{1-r}$$

Now

$$\sum_{n=0}^{\infty} cr^n = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{c(1-r^{N+1})}{1-r}$$

From this we can see that if $-1 < r < 1$ we get convergece, if $|r| > 1$ then divergence. If $r = 1$ then $S_N = c + c + \dots + c = Nc$ so divergence. and if $r = -1$ then $S_N = c - c + c - c + \dots \pm c$ which also does not converge. So

$$\sum_{n=0}^{\infty} cr^n = \begin{cases} \dfrac{c}{1-r} & |r| < 1 \\ \text{diverges} & |r| \geq 1 \end{cases}$$

<u>Ex</u> THE MOST IMPORTANT EXAMPLE IN THE COURSE!!!

The harmonic series is $\sum_{n=\phi}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$

We will show that it diverges. Consider

$$S_2 = 1 + \frac{1}{2} \geq \frac{1}{2} + \frac{1}{2} = 1$$

$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \geq \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{3}{2}$$

$$S_8 = \underbrace{1 + \frac{1}{2}} + \underbrace{\frac{1}{3} + \frac{1}{4}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}$$

$$\geq \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{2} + \frac{1}{2}$$

so $S_{2^N} = S_{2^{N-1}} + \frac{1}{2^{N+1}+1} + \frac{1}{2^{N-1}+2} + \dots + \frac{1}{2^N}$

$\geq S_{2^{N-1}} + \frac{1}{2}$.

Thus the terms S_{2^N} are growing without bound. Hence the seq. S_N diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

$\boxed{\text{very important}}$

Thm If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges

RMK It is not true that $\sum_{n=1}^{\infty} a_n$ converges if $\lim_{n\to\infty} a_n = 0$, eg. the ~~hark~~ harmonic series.

Thm We can use the limit rules for seq's to show, if $\sum a_n, \sum b_n$ convergent an $\sum c_n$ divergent,

1. $\sum k a_n = k\left(\sum a_n\right)$

2. $\sum (a_n + b_n) = \sum a_n + \sum b_n$

3. $\sum a_n + c_n$ diverges.

Ex Find $\sum_{n=1}^{\infty} \frac{8+2^{n+5}}{5^n}$.

$$= \sum_{n=1}^{\infty} \left[\frac{8}{5^n} + \frac{2^{n+5}}{5^n} \right]$$

$$= \sum_{n=1}^{\infty} \frac{8}{5^n} + \sum_{n=1}^{\infty} \frac{2^{n+5}}{5^n}$$

$$= 8\left(\sum_{n=1}^{\infty} \frac{1}{5^n} \right) + 2^5 \left(\sum_{n=1}^{\infty} \left(\frac{2}{5}\right)^n \right)$$

$$= 8 \cdot \frac{5^{-1}}{1-5^{-1}} + 2^5 \frac{2/5}{1-2/5}$$

$$= \frac{8/5}{4/5} + 32 \cdot \frac{2/5}{3/5}$$

$$= 2 + 64/3 = 70/3.$$