Final practice 3

UCLA: Math 115A, Spring 2020

Instructor: Noah White
Date:
Version: 1

- This exam has 6 questions, for a total of 60 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____________________

ID number: ____________________

Question 2 is multiple choice. Indicate your answers in the table below. *The following three pages will not be graded, your answers must be indicated here.*

Question	Points	Score
1	10	
2	10	
3	10	
4	9	
5	10	
6	11	
Total:	60	

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

Clarification on notation: Let $T : V \longrightarrow W$ be a linear map. The *kernel* of T is the same thing as the *nullspace* of T, i.e. $\ker T = \mathsf{N}(T)$. Similarly the *image* of T is the same thing as the *range* of T, i.e. $\operatorname{im} T = \mathsf{R}(T)$.

1. In each of the following questions, fill in the blanks to complete the statement of the definition or theorem.

   (a) (2 points) *Definition:* A subset $B \subset V$ of a vector space is called a *basis* if it is ________________ and ________________.

   (b) (2 points) *Definition:* A scalar $\lambda \in \mathbb{F}$ is an *eigenvalue* of a linear map $T : V \longrightarrow V$ if there exists a ________________ vector $v \in V$ such that

   $$\underline{\hspace{6cm}}.$$

   (c) (2 points) *Definition:* Suppose $T : V \longrightarrow V$ is a linear operator on a finite dimensional vector space with an eigenvalue of λ. The λ-eigenspace is defined to be

   $$E_\lambda = \ker \underline{\hspace{6cm}}$$

   and the geometric multiplicity of λ is

   $$\underline{\hspace{6cm}}.$$

   (d) (2 points) *Theorem:* Let V be a finite dimensional vector space over a field $\mathbb{F}$. A linear map $T : V \longrightarrow V$ is diagonalisable if and only if

   - ____, and
   - for every eigenvalue $\lambda \in \mathbb{F}$, ____.

   (e) (2 points) *Definition:* Let V be a finite dimensional inner product space. The adjoint of a linear map $T : V \longrightarrow V$ is the unique linear map $T^* : V \longrightarrow V$ such that for any ____________________ we have

   $$\underline{\hspace{10cm}}.$$

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) Consider the following subspace of $\mathbb{C}_3[x]$ (polynomials of degree at most 3),

   $$U = \{p \in \mathbb{C}_3[x] \mid p(-1) = 0\}$$

   The dimension of U is

   A. 0.
   B. 1.
   C. 2.
   D. 3.

   (b) (2 points) As a subset of $\mathbb{R}^3$, the set

   $$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} \right\}$$

   A. is a spanning set but not linearly independent.
   B. is linearly independent but not spanning.
   C. is neither spanning nor linearly independent.
   D. is a basis.

   (c) (2 points) A linear operator $T : V \longrightarrow V$ is called *idempotent* if $T^2 = T$. What eigenvalues can an idempotent operator possibly have?

   A. Only 0.
   B. Only 1.
   C. 0 or 1.
   D. It could have any eigenvalue.

(d) (2 points) Let $V = \mathbb{R}_1[x]$, be an inner product space with the inner product

$$\langle p, q\rangle = p(0)q(0) + p'(0)q'(0)$$

Consider the map $T : V \longrightarrow V$ given by $T(p) = 2p(\frac{1}{2}) + p(2)x$. Which of the following is *not* true.

A. T is a linear map.
B. T is self adjoint.
C. T has a basis of orthonormal eigenvectors.
D. T has an eigenspace of dimension 2.

(e) (2 points) Which of the following is *not* a linear map.

A. $P : \mathrm{Mat}_{m\times n}(\mathbb{F}) \longrightarrow \mathrm{Mat}_{n\times m}(\mathbb{F})$ such that $P(M) = M^t$.
B. $Q : \mathrm{Mat}_{n\times n}(\mathbb{F}) \longrightarrow \mathbb{F}$ such that $Q(M) = \det M$.
C. $R : \mathrm{Mat}_{n\times n}(\mathbb{F}) \longrightarrow \mathbb{F}$ such that $R(M) = \mathrm{tr}\, M$.
D. $S : \mathrm{Mat}_{m\times n}(\mathbb{F}) \longrightarrow \mathbb{F}^m$ such that $S(M) = Mv$, for a fixed $v \in \mathbb{F}^n$.

3. Consider the vector space over $\mathbb{R}$,

$$V = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathbb{R}^4 \;\middle|\; x_1 - x_2 + x_3 - x_4 = 0 \right\}$$

and the linear map $T : V \longrightarrow V$ given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \\ x_4 \\ x_3 \end{pmatrix}$$

(a) (1 point) What is the characteristic polynomial of T?

(b) (5 points) Compute the eigenvalues of T and their algebraic multiplicity.

(c) (2 points) Write down an eigenvector for each eigenspace.

(d) (2 points) Is T diagonalisable? If so, find a basis B such that $[T]_B^B$ is diagonal. If not, find B, so that the above matrix is upper triangular.

4. Consider the vector space $V = \mathbb{R}^3$ and the matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

We can define an inner product on V by

$$\langle v, w \rangle = v^t M w.$$

where v^t indicates the transpose. *Please note this is NOT the standard dot product. It is a different inner product.*

(a) (5 points) Apply the Gram-Schmidt process to the basis $E = \{e_1, e_2, e_3\}$ (the standard basis) to find an orthogonal basis B.

(b) (4 points) Let $v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Compute the coordinate vector $[v]^B$. *Note that B is not orthonormal.*

5. Let V be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $T : V \longrightarrow V$ be a normal linear operator (i.e. $T^*T = TT^*$).

   (a) (3 points) Prove for all $v \in V$ that $\|T(x)\| = \|T^*(v)\|$.

   (b) (3 points) Prove that $T - \alpha\,\mathrm{id}_V$ is normal for any $\alpha \in \mathbb{F}$

   (c) (4 points) Prove that if v is a λ-eigenvector for T, then v is also a $\overline{\lambda}$-eigenvector for T^*. *Hint: use both previous parts.*

6. Let V be a finite dimensional vector space over a field $\mathbb{F}$, and $T : V \longrightarrow V$ a linear operator. Suppose that $T^n = 0$ for some $n > 1$ (well call T *nilpotent* in this case) but that $T^{n-1} \neq 0$. Fix a vector $x \in V$ such that $T^{n-1}(x) \neq 0$.

   (a) (2 points) What are the eigenvalues of T? Justify your answer.

   (b) (1 point) Is it possible for T to be an isomorphism? Justify your answer.

   (c) (3 points) Suppose $n = 2$. Prove that $\{x, T(x)\}$ are linearly independent.

(d) (5 points) For any $n > 1$, prove that $\{x, T(x), T^2(x), \ldots, T^{n-1}(x)\}$ is linearly independent.

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