Math 3B: Lecture 16

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Last time

- Checking solutions to differential equations.

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- Implicit differentiation.

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- Implicit differentiation.
- Separation of variables.

Linear models

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A first order ODE is linear if it is of the form

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Examples

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\lambda y.$$

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- concentration of a drug in bloodstream
- pollutant in water supply

General solution

Using separation of variables, we can show that the general solution to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a - by$$

is

$$y(t) = \frac{a}{b} - Ce^{-bt}$$

where C is an arbitrary constant.

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$$M\left(\frac{1}{2}\right)^{t/2} = Me^{-0.5t\ln(2)} \text{ mg left}$$

- Thus the rate at which the drug is leaving (at time t) is given by

$$0.5\ln(2)Me^{-0.5t\ln(2)} = 0.5\ln(2)(\text{current concentration}) \;\; \text{mg/h}.$$

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- Thus at time t the concentration is

$$y(t) = 28.9 - 28.9e^{-0.3t} = 28.9\left(1 - e^{-0.3t}\right)$$

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$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(A - T)$$

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- Thus

$$90 = 70 - C \quad \text{so} \quad C = -20.$$

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- Rearranging we get $20e^{-0.01t} = 5$ i.e.

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- So we get

$$t = -100\ln\left(\frac{1}{4}\right) \approx 138 = 2 \text{ hours } 18 \text{ minutes.}$$

Von Bertalanffy growth model

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- Surface area is $S = 6L^2$

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$$\frac{\mathrm{d}M}{\mathrm{d}t} = 3L^2 \frac{\mathrm{d}L}{\mathrm{d}t}$$

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- We get

$$3L^2 \frac{\mathrm{d}L}{\mathrm{d}t} = 6aL^2 - bL^3$$

Von Bertalanffy growth model

- Dividing by $3L^2$ gives

$$\frac{\mathrm{d}L}{\mathrm{d}t} = \frac{6a}{3} - \frac{b}{3}L = \frac{b}{3}\left(\frac{6a}{b} - L\right)$$

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Von Bertalanffy growth model

The growth of an organism is goverened by

$$\frac{\mathrm{d}L}{\mathrm{d}t} = k(L_\infty - L)$$

where k and L_∞ are constants.