Midterm 2 practice 2

UCLA: Math 3B, Fall 2017

Instructor: Noah White
Date: 27 February 2017

- This exam has 3 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____________________

ID number: ____________________

Discussion section (please circle):

Day/TA	Kevin	Bohyun	Ryan
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	13	
2	14	
3	13	
Total:	40	

1. A large hole is filled with water. The hole has the shape of an upside down cone, with a depth of 5 meters at its deepest and a diameter of 20 meters at ground level. In this question we will calculate how much work is needed to pump all of the water out of the hole. You may assume that water has a density of 1000 kilograms per meter cubed and that the acceleration due to gravity is 10 m/s^2.

   (a) (1 point) Denote by x the depth below ground in meters, so that $x = 0$ is ground level and $x = 5$ is the bottom of the conical hole. We divide the hole into n horizontal slices. Let Δx be the thickness of each slice. What is Δx in terms of n?

   Solution:

   $$\Delta t = \frac{5}{n}$$

   (b) (1 point) Let x_k be the depth of the bottom of the k^{th} slice. What is x_k in terms of n and k?

   Solution:

   $$x_k = k\frac{5}{n} = k\Delta x$$

   (c) (2 points) The base of the k^{th} slice is a circle. What is its radius (leave your answer in terms of x_k)?

   Solution:

(d) (1 point) If we assume that n is large, then each slice is approximately a cylinder with height Δx and radius given by your answer to part (c). What is the volume of this cylinder? Leave your answer in terms of Δx and x_k.

Solution:

(e) (2 points) Approximately how much work is needed to raise the water in the k^{th} slice to ground level?

Solution:

(f) (3 points) Write a Riemann sum that calculates the amount of worked needed to drain the hole.

Solution:

(g) (3 points) Use an integral to evaluate this sum.

Solution: We can convert the above sum to an integral and solve:

2. (a) (3 points) Use long division to write the following fraction in the form $d(x) + \frac{r(x)}{q(x)}$ where the degree of $r(x)$ is less than the degree of $q(x)$.

$$\frac{p(x)}{q(x)} = \frac{x^4 - 2x^2 - 4x + 1}{x^3 + x^2 - x - 1}$$

Solution:

$$\begin{array}{r|rrrrr}
 & & & & x & -1 \\ \hline
x^3 + x^2 - x - 1 & x^4 & & -2x^2 & -4x & +1 \\
 & -x^4 & -x^3 & +x^2 & +x & \\ \hline
 & & -x^3 & -x^2 & -3x & +1 \\
 & & x^3 & +x^2 & -x & -1 \\ \hline
 & & & & -4x &
\end{array}$$

So

$$\frac{x^4 - 2x^2 - 4x + 1}{x^3 + x^2 - x - 1} = x - 1 - \frac{4x}{x^3 + x^2 - x - 1}$$

(b) (4 points) Calculate the integral $\int \frac{x^4-2x^2-4x+1}{x^3+x^2-x-1}\, \mathrm{d}x$.

Solution: Using part (a) we can write the integral as

$$\int x - 1 - \frac{4x}{x^3 + x^2 - x - 1}\, \mathrm{d}x.$$

We will use partial fractions to write

$$\frac{4x}{x^3 + x^2 - x - 1} = \frac{6}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Clearing denominators we get

$$6 = A(x+1) + B(x-1)$$

so we see easily that $A = 3$ and $B = -3$. Thus we calculate

$$\int 3x^2 + 4 - \frac{3}{x-1} + \frac{3}{x+1}\, \mathrm{d}x = x^3 + 4x - 3\ln(x-1) + 3\ln(x+1) + C.$$

(c) (3 points) Convert the following Riemann sum into an integral of the form $\int_0^4 f(x)\,\mathrm{d}x$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2}{n}\sqrt{1+32\frac{k^2}{n^2}}.$$

Solution:

$$y(t) = 4e^{x^2}$$

(d) (4 points) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = te^t\sqrt{1+y}$ if $y(0) = -0.75$.

Solution: We start by separating variables and integrating

$$\int 2y\,\mathrm{d}y = \int te^t\,\mathrm{d}t$$

We evaluate the right hand side using integration by parts. Let $u = t$ and $v' = e^t$. Then $u' = 1$ and $v = e^t$ so

$$\int te^t\,\mathrm{d}t = te^t - \int e^t\,\mathrm{d}t = te^t - e^t + C$$

Equating with the left hand side we get that

$$y^2 = te^t - e^t + C.$$

We can now use the fact that $y(1) = 1$ to obtain $C = 1$. Thus plugging this in an taking a square root,

$$y(t) = \sqrt{te^t - e^t + 1}.$$

3. Oxygen-15 is a radioactive isotope used in medicine. It has a half life of 120 seconds. During production, a solution containing 0.1 mg/ml of Oxygen-15 is added to a flask at a rate of 2 ml/s. Initially the flask contains 100 ml of pure water (i.e. initially there is no Oxygen-15 in the flask).

   (a) (1 point) How much Oxygen-15 (in mg) is being added, per second to the flask?

   Solution:

   (b) (2 points) Write a differential equation describing the total amount $y(t)$ of Oxygen-15 (in mg) in the flask at time t.

   Solution:

(c) (2 points) Solve the differential equation from part (b).

Solution:

(d) (3 points) What is the concentration of Oxygen-15 (in mg/ml) in the flask at time t?

Solution:

(e) (5 points) Now suppose that, at the same time it is being filled, the flask is also being drained at a rate of 1 ml/s. Write a differential equation describing the total amount $y(t)$ of Oxygen-15 (in mg) in the flask at time t (you do not need to solve it).

Solution:

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