Final exam practice

UCLA: Math 170A, Fall 2017

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Date:
Version: 1

- This exam has 12 questions, for a total of 100 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	0	
10	10	
11	10	
12	0	
Total:	100	

Question 1 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) A box contains 4 fair coins and 3 fake coins where both sides are head. A coin is selected from the box and tossed. The result is heads. What is the probability that it is a fair coin?

   A. 0.2
   B. 0.4
   C. 0.3
   D. 0.25

   (b) (2 points) Let A and B be events. If $\mathbb{P}(A|B^c) = \frac{1}{2}$ and $\mathbb{P}(B|A) = \frac{2}{5}$, what is $\dfrac{\mathbb{P}(B^c)}{\mathbb{P}(A)}$?

   A. 0.6
   B. 0.3
   C. 1.2
   D. 0.4

   (c) (2 points) Let A, B be events. Then $(A \cap B) \cup (A^c \cap B)^c =$

   A. $A^c \cap B^c$
   B. $A \cup B^c$
   C. $A \cap B$
   D. A

(d) (2 points) A company has 3 departments and there are 50 people in each department. How many ways can a committee of 12 people be formed so that there are exactly 4 people from each department? (The order of the 12 people does not matter.)

A. $\binom{50}{4}^3$
B. $\binom{12}{4}\binom{150}{12}$
C. $\binom{50}{3}^4$
D. $\binom{50}{4}\binom{150}{12}$

(e) (2 points) A box contains 4 blue balls and 6 red balls. 2 balls are selected from the box simultaneously. What is the probability that both are blue given that at least one of them is blue?

A. 0.05
B. 0.2
C. 0.1
D. 0.25

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(3)$ are independant and satisfy

   $$\text{var}(X - 2Y) = 28$$

   the $\lambda =$

   A. 40
   B. 4
   C. 16
   D. 28

   (b) (2 points) Let X and Y be independant random variables. Which of the following statements is/are true?

   1. If $X \sim \text{binomial}(8, 0.3)$ and $Y \sim \text{Bernoulli}(0.3)$ then $X + Y \sim \text{binomial}(9, 0.3)$.
   2. If $X \sim \text{Geometric}(0.8)$ then $\mathbb{P}(X \geq 5) = 1 - 0.2^4$.
   3. If $X \sim \text{binomial}(100, 0.03)$ and $Y \sim \text{Poisson}(3)$ then X and Y have exactly the same variance.

   A. Only 1 and 2 are true.
   B. Only 2 and 3 are true.
   C. Only 1 is true.
   D. 1, 2 and 3 are true.

   (c) (2 points) A fair six sided die is rolled 6 times. Let X be the number of times 2 appears. Find $\mathbb{E}(X^2)$.

   A. 5/6
   B. 7/6
   C. 9/6
   D. 11/6

(d) (2 points) Let $X_1 1, \ldots, X_5$ be independent Geometric random variables with parameter e^{-2}. Find

$$\mathbb{P}(\min(X_1, \ldots, X_5) = 1).$$

A. $1 - e^{-2}$
B. $1 - e^{-5}$
C. $1 - e^{-1}$
D. $1 - e^{-10}$

(e) (2 points) The lifetime in hours of an electronic tube is a random variable having a probability density function given by $f_X(x) = xe^{-x}$ for $x \geq 0$ and $f_X(x) = 0$ otherwise. The expected lifetime of such a tube is

A. 1
B. 2
C. 3
D. 4

3. (a) (2 points) State Bayes theorem.

   (b) (8 points) There are n boxes, the 1st one containing 1 glass, the 2nd box containing 2 glasses, the 3rd box containing 3 glasses, ... , kth box containing k glasses, ... nth box containing n glasses. In each box every glass is broken with probability 1/2 independently of all other glasses. A contestant chooses a box uniformly at random. Given that he found no broken glasses in this box, find the probability that he choose the 1st box. Simplify your answer.

4. (a) (2 points) If X is a geometric random variable with parameter p, what is the range of X and what is $p_X(k)$?

   (b) (8 points) You arrive at the opening of a new cafe which has only two chairs. Unfortunately, there are three people ahead of you in the line. You know that the number of minutes that a guest spends in a cafe is a geometric random variable with mean 2, and that the time each guest spends in the cafe is independent of all other guests. What is the probability you will have to wait more than n minutes.

5. The number of cars that arrive in a certain mechanic shop is a Poisson random variable with the parameter $\lambda = 1$. For $k \geq 1$, given that exactly k cars arrive, the total time T that it takes to repair all these cars is a continuous uniform random variable on the interval $[0, k+1]$.
   (a) (5 points) Find the probability $\mathbb{P}(T \leq 1)$. Simplify your answer.
   (b) (5 points) Find the expectation $\mathbb{E}(T)$. Simplify your answer.

6. (a) (1 point) If S and T are independant frandom variables and f, g are functions of a real variable, then $f(S)$ and $g(T)$ are independant random variables. True or False?

   (b) (9 points) Let X be a standard Normal random variable, and Y a geometric random variable with parameter 1/4. If X and Y are independent compute $\mathbb{E}(e^X)$, $\mathbb{E}(e^Y)$ and $\mathbb{E}(e^{X+Y})$.

7. Let X be an exponential random variable with parameter 1. Given $X = t$, let Y be uniformly distributed on the interval $[0, e^t]$.
   (a) (3 points) Compute $\mathbb{E}(Y|X = 1)$.
   (b) (3 points) Compute the joint PDF of X and Y.
   (c) (4 points) Compute the PDF of Y.

8. Let T be a triangle with vertices $(-1,0)$, $(1,0)$ and $(0,1)$. Let the random variables X and Y be jointly continuous with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} C|xy|, & \text{for } (x,y) \in T, \\ 0, & \text{for } (x,y) \notin T. \end{cases}$$

   (a) (3 points) Find the value of the constant C.

   (b) (4 points) Find the PMFs of X and Y.

   (c) (3 points) Are X and Y independant?

   (d) Find the CDF of the random variable $Z = X + Y$ (compute the integral, but there is no need to simplify the resulting expression).

9. You are playing a game where bottles are arranged in a grid, a distance d apart. You must throw a ring over the bottles and you win if the ring lands on a bottle. Assume that the bottles have a circular top of radius r and the ring has an inner radius R. For simplicity, assume that the grid of bottles is infinite and that the ring is equally likey to land anywhere. We also assume that $R - r < d/2$ (so that the ring can't fit over two bottles at once). What is the probability that we win?

10. (10 points) Let X be an exponential random variable with parameter 1. Find the CDF and the PDF of the random variables

    (1) $Y = \sqrt{X}$,

    (2) $Z = X^2$.

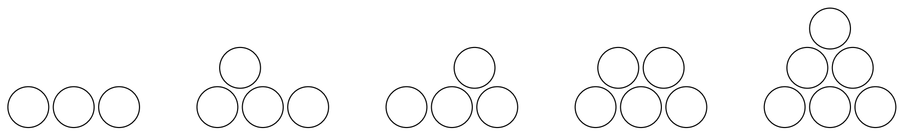
11. (10 points) Let A be the subset of the plane $\mathbb{R}^2$ defined as the intersection of the first quadrant and the annulus between the circles of radii 1 and 2 centered at the origin. In other words

$$A = \{(x, y) \mid x \geq 0,\ y \geq 0,\ 1 \leq x^2 + y^2 \leq 4\}.$$

Let (X, Y) be a uniformly chosen point in the region A. Find the marginal PDF and the expected value of the first coordinate X.

12. (*Bonus question*) This question is quite difficult, and quite a way beyond the scope of this course, however there is a very beautiful solution which I will show you on the last day of class. If you come up with a solution, please share it with me!

    Let C_n be the number of ways to stack circles in partial pyramid shapes, as shown below, so that the base has n circles. Below you can see that $C_3 = 5$ (and an easy calculation shows that $C_0 = 1, C_1 = 1, C_2 = 2$) Find a formula for C_n.

    Here is a hint: try and come up with a recursive formula for C_n (i.e. a formula that depends on $C_0, C_1, \ldots, C_{n-1}$).

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