Midterm 2 practice 2

UCLA: Math 115A, Winter 2020

Instructor: Noah White
Date:
Version: 1

- This exam has 4 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____

ID number: ____

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
Total:	20	

Question 1 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

Clarification on notation: Let $T : V \longrightarrow W$ be a linear map. The *kernel* of T is the same thing as the *nullspace* of T, i.e. $\ker T = \mathsf{N}(T)$. Similarly the *image* of T is the same thing as the *range* of T, i.e. $\operatorname{im} T = \mathsf{R}(T)$.

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (1 point) If V is a finite dimensional vector space, with two bases, B and C then the matrix that changes B-coordinate vectors, into C-coordinate vectors is

   A. $[T]_C^B$
   B. $[T]_B^C$
   C. $[\mathrm{id}]_C^B$
   D. $[\mathrm{id}]_B^C$

   (b) (1 point) Let $E = \{1, x\}$, $C = \{x+2, x+1\}$ and $T : \mathbb{C}_1[x] \longrightarrow \mathbb{C}_1[x]$ be a linear map such that

   $$[T]_E^E = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

   Then, $[T]_C^C$ is equal to

   A. $\begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$

   B. $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

   C. $\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}$

   D. $\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$

(c) (1 point) Suppose $S : U \longrightarrow V$ and $T : V \longrightarrow W$ are linear maps between vector spaces, such that $T \circ S$ is the zero map (i.e. $T(S(u)) = 0$ for all $u \in U$). Which of the following is true?

A. $\ker S \subseteq \operatorname{im} T$
B. $\operatorname{im} S \subseteq \ker T$
C. $\ker T \subseteq \operatorname{im} S$
D. $\operatorname{im} T \subseteq \ker S$

(d) (1 point) Suppose $T : V \longrightarrow V$ is a linear map and λ is an eigenvalue of T. Which of the following is true?

A. λ^{-1} is also an eigenvalue of T.
B. λ is an eigenvalue of T^n for some $n \geq 1$.
C. λ^n is an eigenvalue of T^n for every $n \geq 1$.
D. λ is not an eigenvalue of T^n for any $n \geq 2$.

(e) (1 point) Let V be a vector space. What is the dimension of $\mathcal{L}(\{0\}, V)$?

A. 0
B. 1
C. $\dim V - 1$
D. $\dim V$

2. Let $T : V \longrightarrow W$ be a linear map between vector spaces.

(a) (2 points) Define the *rank* and *nullity* of T.

Solution: The rank is $\dim \operatorname{im} T$ and the nullity is $\dim \ker T$.

(b) (3 points) Prove the *rank-nullity theorem:* If V is finite dimensional, then $\dim V = \dim \ker T + \dim \operatorname{im} T$.

Solution: Let $n = \dim V$ and $m = \dim \ker T$. Now consider a basis $\{v_1 \ldots, v_m\}$ of $\ker T$ and extend this to a basis $\{v_1, \ldots, v_n\}$ of V. We claim that $\{T(v_{m+1}), \ldots, T(v_n)\}$ is a basis of $\operatorname{im} T$. First, we prove that it is linearly independent. Consider

$$\lambda_1 T(v_{m+1}) + \cdots + \lambda_{n-m} T(v_n) = 0.$$

By linearity

$$T(\lambda_1 v_{m+1} + \cdots + \lambda_{n-m} v_n) = 0$$

and thus $\lambda_1 v_{m+1} + \cdots + \lambda_{n-m} v_n \in \ker T$. But this means that

$$\lambda_1 v_{m+1} + \cdots + \lambda_{n-m} v_n \in \operatorname{span}\{v_1 \ldots, v_m\}$$

So we can find scalars μ_i such that

$$\lambda_1 v_{m+1} + \cdots + \lambda_{n-m} v_n = \mu_1 v_1 + \cdots + \mu_m v_m$$

Rearranging we get that

$$\lambda_1 v_{m+1} + \cdots + \lambda_{n-m} v_n - \mu_1 v_1 - \cdots - \mu_m v_m = 0$$

which means, by linear independence of $\{v_1, \ldots, v_n\}$ that $\lambda_i = 0$ for all i.
Now to see that it is spanning, assume that $w \in \operatorname{im} T$. Thus there exists $v \in V$ such that $T(v) = w$. We have a basis of V so there are scalars such that

$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n.$$

Applying T we get

$$w = T(v) = \lambda_1 T(v_1) + \cdots + \lambda_n T(v_n)$$

but $T(v_i) = 0$ for $1 \leq i \leq m$ so

$$w = T(v) = \lambda_{m+1} T(v_{m+1}) + \cdots + \lambda_n T(v_n)$$

and so $w \in \operatorname{span}\{T(v_{m+1}), \ldots, T(v_n)\}$ and hence we have proven that this is a basis for $\operatorname{im} T$. Thus $\dim \operatorname{im} T = n - m = \dim V - \dim \ker T$.

3. Consider the linear map $T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ given by

$$T(a + bx + cx) = (a + 2b + 2c) + (b + 3c)x + 2cx^2.$$

(a) (2 points) Find the characteristic polynomial and eigenvalues of T.

Solution: Choose a basis for $\mathbb{R}_2[x]$, say $E = \{1, x, x^2\}$ we have

$$[T]_E^E = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix}$$

Thus we see that $p_T(t) = (1 - t)^2(2 - t)$ and so the eigenvalues are 1 and 2.

(b) (2 points) For each eigenvalue, determine an eigenvector of T.

Solution: We start by finding the eigenvectors of $[T]_E^E$. We solve

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 1 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This gives the equations $2c = c$ (so $c = 0$), $b = b$ and $a + 2b = a$, i.e. $b = 0$ so the 1-eigenspace is

$$\operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

(and is thus 1 dimensional!)
We do the same for the 2-eigenvectors

$$\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = 2 \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

This gives the equations $2c = 2c$, $b + 3c = 2b$ (so $b = 3c$) and $a + 2b + 2c = 2a$, i.e. $a = 8c$, so the 2-eigenspace is

$$\operatorname{span}\left\{ \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix} \right\}$$

(and is thus 1 dimensional!)

(c) (1 point) Is T diagonalisable?

Solution: No, the algebraic and geometric multiplicities dont match ($2 = a_1 \neq b_1 = 1$).

4. Let V and W be vector spaces, and let $S : V \longrightarrow W$ and $T : W \longrightarrow V$ be linear maps.

   (a) (1 point) Prove that if $T \circ S$ is injective (one-to-one) then S is injective.

   > **Solution:** Suppose $T \circ S$ is injective and that $S(v) = S(w)$. Then $T \circ S(v) = T \circ S(w)$ and since this is an injective function, we must have that $v = w$, so S is injective.

   (b) (2 points) Prove that if $T \circ S$ is surjective (onto) then T is surjective.

   > **Solution:** Suppose that $T \circ S$ is surjective and that $v \in V$. Since $T \circ S$ is surjective, there exists a $v' \in V$ such that $T \circ S(v') = v$. Let $w = S(v')$, then $T(w) = v$ so T is surjective.

   (c) (2 points) Give an example of spaces V, W and linear maps S, T such that $T \circ S = \mathrm{id}_V$ but $S \circ T \neq \mathrm{id}_W$. *Hint: you shouldn't need to do anything complicated, there is a very simple example.*

   > **Solution:** Let $V = \{0\}$, and W be any non zero vector space. Now let S and T both be the zero map. Then $T \circ S : \{0\} \longrightarrow \{0\}$ is the zero map which coincides with the identity in this case, but $S \circ T : W \longrightarrow W$ is the zero map, and since W is non-zero, this is not the identity.

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