Midterm 2 practice 1

UCLA: Math 3B, Winter 2019

Instructor: Noah White
Date:
Version: practice

- This exam has 3 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section (please circle):

Day/TA	Louis	Matthew
Tuesday	1A	1C
Thursday	1B	1D

Question	Points	Score
1	13	
2	14	
3	13	
Total:	40	

1. The Department of Homeland Security receives 500 visa applications per day. Applications take varying amounts of time to process, depending on the type and complexity of the request. Once an application is approved or denied it is termed *resolved*. Experimentally, auditors determined that if an application is currently unresolved then the probability that it will remain unresolved after t days is

$$e^{-0.01t}.$$

The department currently has $10,000$ unresolved applications. In this question we will work out how many unresolved applications the department has one year from now.

(a) (1 point) Of the $10,000$ currently unresolved applications, how many will remain unresolved in one years time (assume a year has 365 days)?

Solution:

$$10,000e^{-3.65}$$

(b) (1 point) We divide the time period of 1 year into n subintervals. Let Δt be the length in days of each subinterval. What is Δt in terms of n?

Solution:

$$\Delta t = \frac{365}{n}$$

(c) (2 points) Let $t = 0$ be the start of the time period so that $t = 365$ is one year later. Let t_k be the end of the k^{th} subinterval. What is t_k in terms of n and k?

Solution:

$$t_k = k\frac{365}{n} = k\Delta t$$

(d) (1 point) How many visa applications does the Department receive during the k^{th} subinterval (in terms of Δt)?

Solution:

$$500\Delta t$$

(e) (2 points) Assume that all of these applications arrive at the department at the end of the k^{th} subinterval. How many of these remain unresolved one year later when $t = 365$? Your answer should be in terms of Δt and t_k.

Solution: From $t = t_k$ to $t = 365$ there are $365 - t_k$ days. Thus the number of unresolved applications is

$$500e^{-0.01(365-t_k)}\Delta t.$$

(f) (3 points) Write a Riemann sum that calculates the total number of unresolved applications the Department will have in one year. Remember to include those remaining from the original $10,000$.

Solution: We sum all the contributions from all subinterval and take a limit as $n \to \infty$. We also add the result of part (a).

$$10,000e^{-3.65} + \lim_{n\to\infty} \sum_{k=1}^{n} 500e^{-0.01(365-t_k)}\Delta t.$$

(g) (3 points) Use an integral to evaluate this sum.

Solution: We can convert the above sum to an integral and solve:

$$\begin{aligned} 10,000e^{-3.65} + \lim_{n\to\infty} \sum_{k=1}^{n} 500e^{-0.01(365-t_k)}\Delta t &= 10,000e^{-3.65} + \int_0^{365} 500e^{-0.01(365-t)}\,\mathrm{d}t \\ &= 10,000e^{-3.65} + \left[\frac{500}{0.01}e^{-0.01(365-t)}\right]_0^{365} \\ &= 10,000e^{-3.65} + 50,000 - 50,000e^{-3.65} \\ &= 50,000 - 40,000e^{-3.65} \\ &\approx 48,960. \end{aligned}$$

2. (a) (3 points) Use long division to write the following fraction in the form $d(x) + \frac{r(x)}{q(x)}$ where the degree of $r(x)$ is less than the degree of $q(x)$.

$$\frac{p(x)}{q(x)} = \frac{3x^4 + x^2 - 10}{x^2 - 1}$$

Solution:

$$\begin{array}{r|l} & 3x^2 \quad + 4 \\ \hline x^2 - 1 & 3x^4 \quad + x^2 - 10 \\ & -3x^4 + 3x^2 \\ \hline & \quad 4x^2 - 10 \\ & \quad -4x^2 \quad + 4 \\ \hline & \quad\quad -6 \end{array}$$

So

$$\frac{3x^4 + x^2 - 10}{x^2 - 1} = 3x^2 + 4 - \frac{6}{x^2 - 1}$$

(b) (4 points) Calculate the integral $\int \frac{3x^4+x^2-10}{x^2-1}\,\mathrm{d}x$.

Solution: Using part (a) we can write the integral as

$$\int 3x^2 + 4 - \frac{6}{x^2 - 1}\,\mathrm{d}x.$$

We will use partial fractions to write

$$\frac{6}{x^2 - 1} = \frac{6}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Clearing denominators we get

$$6 = A(x+1) + B(x-1)$$

so we see easily that $A = 3$ and $B = -3$. Thus we calculate

$$\int 3x^2 + 4 - \frac{3}{x-1} + \frac{3}{x+1}\,\mathrm{d}x = x^3 + 4x - 3\ln(x-1) + 3\ln(x+1) + C.$$

(c) (3 points) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = 2yt$ when $y(0) = 4$.

Solution:

$$y(t) = 4e^{x^2}$$

(d) (4 points) Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{te^t}{2y}$ if $y(1) = 1$.

Solution: We start by separating variables and integrating

$$\int 2y \,\mathrm{d}y = \int te^t \,\mathrm{d}t$$

We evaluate the right hand side using integration by parts. Let $u = t$ and $v' = e^t$. Then $u' = 1$ and $v = e^t$ so

$$\int te^t \,\mathrm{d}t = te^t - \int e^t \,\mathrm{d}t = te^t - e^t + C$$

Equating with the left hand side we get that

$$y^2 = te^t - e^t + C.$$

We can now use the fact that $y(1) = 1$ to obtain $C = 1$. Thus plugging this in an taking a square root,

$$y(t) = \sqrt{te^t - e^t + 1}.$$

3. Chlorine is added to swimming pools to keep the water free from hazardous bacteria. It is recommended that the level of chlorine in a pool be kept at about 2 mg/L (grams per liter). However, chlorine, will *degas* from the water over time. This means the chlorine will leave the water. In fact the half-life of chlorine in water is about 4 hours.

   You have just purchased a fancy new chlorinator for your pool. This device pumps water out of the pool and pumps it back in, with an extra a mg/L of chlorine. It pumps water at a rate of 100 L/h. You can control the rate a by adjusting the settings on the chlorinator. Your pool has a volume of $75,000$ L.

   (a) (2 points) How much chlorine (in mg) is being added, per hour by the chlorinator? Leave your answer in terms of a.

   Solution: There are 100 L being pumped through every hour. To each liter, a mg is being added, so in total we are adding $100a$ mg/L. Your equation should depend on a.

   (b) (4 points) Write a differential equation describing the total level $y(t)$ of chlorine (in mg) in the pool at time t.

   Solution: The differential equation will take the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \text{rate in } - \text{ rate out.}$$

   The rate in is $100a$ from above. Thus we need to find the rate out. The chlorine has a half-life of 4 hours. So have t hours the fraction of the chlorine that is left is

$$\left(\frac{1}{2}\right)^{\frac{t}{4}} = e^{-\frac{\ln 2}{4}t}.$$

   Differentiating this we see that the chlorine is leaving the pool at a rate of $\frac{\ln(2)}{4}y(t)$ mg/h. Thus the differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 100a - \frac{\ln 2}{4}y$$

(c) (3 points) What should a be to ensure that over the long term, the pool has roughly 2 mg/L of chlorine?

Solution: The equilibrium solution of the differential equation above is

$$y(t) = \frac{400a}{\ln 2}.$$

The solution will eventually approach the equilibrium so we would like this equilibrium to be the desired amount of chlorine. We would like 2 mg/L and the pool contains 75,000 L so we want 150,000 mg of chlorine in the pool. I.e.

$$\begin{aligned}\frac{400a}{\ln 2} &= 150,000\\ a &= 375\ln 2 \approx 260 \text{ mg/L}.\end{aligned}$$

(d) (4 points) Assume that the pool initially contains 0.5 mg/L of chlorine. Solve the differential equation.

Solution: We separate variables and integrate

$$\int \frac{1}{37,500\ln 2 - \frac{\ln 2}{4}y}\,\mathrm{d}y = \int 1\,\mathrm{d}t.$$

We can use the substitution $u = -\frac{\ln 2}{4}y$ to evaluate the left hand side. Thus

$$\frac{\mathrm{d}u}{\mathrm{d}u} = -\frac{\ln 2}{4}$$

so our integral becomes

$$\begin{aligned}\int \frac{1}{37,500\ln 2 - \frac{\ln 2}{4}y}\,\mathrm{d}y &= \int \frac{-\frac{4}{\ln 2}}{37,500\ln 2 - \frac{\ln 2}{4}y}\left(-\frac{\ln 2}{4}\right)\,\mathrm{d}y\\ &= -\frac{4}{\ln 2}\int \frac{1}{37,500\ln 2 + u}\,\mathrm{d}u\\ &= -\frac{4}{\ln 2}\ln(37,500\ln 2 + u)\\ &= -\frac{4}{\ln 2}\ln\left(37,500\ln 2 - \frac{\ln 2}{4}y\right).\end{aligned}$$

Thus we get

$$-\frac{4}{\ln 2}\ln\left(37,500\ln 2 - \frac{\ln 2}{4}y\right) = t + C.$$

Rearranging and taking an exponential we get

$$37,500\ln 2 - \frac{\ln 2}{4}y = Ce^{-\frac{\ln 2}{4}t}.$$

So

$$y(t) = 150,000 - Ce^{-\frac{\ln 2}{4}t}.$$

Now to find C we use the fact that initially the pool has a concentration of 0.5 mg/L of chlorine. Since the pool contains 75,000 L this means $y(0) = 37,500$ mg. So

$$37,500 = 150,000 - C$$

so $C = 112,500$ and our final solution is

$$y(t) = 150,000 - 112,500e^{-\frac{\ln 2}{4}t}$$

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