This week on the problem set you will get practice thinking about potential functions and calculating line integrals.

Homework: The homework will be due on Wednesday 27 May. It will consist of questions 3, 4 and 5 below. *Numbers in parentheses indicate the question has been taken from the textbook:

J. Rogawski, C. Adams, *Calculus, Multivariable*, 3rd Ed., W. H. Freeman & Company,

and refer to the section and question number in the textbook.

1. (Section 17.4) 2, 3, 5, 8, 9, 10, 13, 14, 17, 18, 27, 30, 34, 37, 40, 41*, 46* 48*. (questions are the same in previous versions)

2. (Section 17.5) 1, 6, 7, 12, 17, 18, 21, 22, 31*, 35. (questions are the same in previous versions)

3. Consider the line segment $(x, 0, 0)$ where $x \in [-1, 1]$ in $\mathbb{R}^3$. Imagine this line segment moving up with its centre on the z-axis, rotating parallel to the xy-plane at constant speed. It completes one full revolution when it gets to $z = 2\pi$. What surface area is swept out by the rotating line segment? You may wish to use the fact that
$$\frac{d}{dt}\left(t\sqrt{1+t^2} + \sinh^{-1}(t)\right) = 2\sqrt{1+t^2}$$
and that $\sinh^{-1} t$ is an odd function and $\sinh^{-1}(1) = \ln(1+\sqrt{2})$.

Solution: We parameterise the surface using the following strategy. Say that at time t the line segment is at height $z = t$, so $G(s,t) = (?, ?, t)$. At $z = t$ for $t \in [0, 2\pi]$ we know that out line segment as rotated t radians from it's starting point. So it's projection onto the xy-plane is $(s\cos t, s\sin t)$. Thus our parameterisation is
$$G(s,t) = (s\cos t, s\sin t, t) \text{ for } (s,t) \in \mathcal{D} = [-1,1] \times [0, 2\pi].$$
From this we calculate
$$\begin{aligned}\mathbf{T}_s &= \langle \cos t, \sin t, 0\rangle\\ \mathbf{T}_t &= \langle -s\sin t, s\cos t, 1\rangle\\ \mathbf{N} &= \langle \sin t, -\cos t, s\rangle\end{aligned}$$
and so
$$\|\mathbf{N}\| = \sqrt{1+s^2}$$
Thus the surface area is
$$\begin{aligned}\iint_{\mathcal{S}} 1\, dS &= \iint_{\mathcal{D}} \sqrt{1+s^2}\, dA_{st}\\ &= \int_0^{2\pi}\int_{-1}^{1} \sqrt{1+s^2}\, ds\, dt\\ &= \pi\left[s\sqrt{1+s^2} + \sinh^{-1} s\right]_{-1}^{1}\\ &= 2\pi\left(\sqrt{2} + \sinh^{-1}(1)\right) = 2\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right)\end{aligned}$$

4. The velocity vector field of a fluid is given by $\mathbf{F}(x,y,z) = \dfrac{\langle x, y, z\rangle}{x^2+y^2+z^2}$ measured in meters per second. What is the volume of fluid flowing each second through the open cylinder of radius 1 and height 1

centered along the z-axis, i.e. the cylinder $C = \{(x, y, z) : x^2 + y^2 = 1, 0 \le z \le 1\}$, with outward orientation?

> **Solution:** We are computing $\iint_S \mathbf{F} \cdot \, d\mathbf{n}$. We can use the parameterization of the cylinder given by $G(\theta, z) = (\cos\theta, \sin\theta, z)$. For this parameterization, $\mathbf{n} = \langle \cos\theta, \sin\theta, 0\rangle$. So, $\iint_S \mathbf{F} \cdot \, d\mathbf{n} = \int_0^{2\pi} \int_0^1 \frac{\langle \cos\theta, \sin\theta, z\rangle}{1+z^2} \cdot \langle \cos\theta, \sin\theta, 0\rangle \, dz \, d\theta$. This is $2\pi \int_0^1 \frac{1}{1+z^2} \, dz$ which is $2\pi(\arctan 1 - \arctan 0) = \pi^2/2$.

5. Let $\mathbf{F} \left\langle y(ye^{x+y^2} - 1) + x^2, 2y(1+y^2)e^{x+y^2} + x \right\rangle$ and let $\mathcal{C}$ be the portion of $y = 1 - x^2$ above the x-axis, oriented left to right.

   (a) Parameterise $\mathcal{C}$ and write $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ as a single integral. Do not try and evaluate.

   > **Solution:** A parameterisation is $\mathbf{r} = (t, 1 - t^2)$ for $t \in [-1, 1]$ and so $\mathbf{r}'(t) = \langle 1, -2t\rangle$. The integral becomes
   >
   > $$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_{-1}^{1} (1-t^2)((1-t^2)e^{t^4-2t^2+t+1} - 1) + t^2 - 2t\left(2(1-t^2)(2-2t^2+t^4)e^{t^4-2t^2+t+1} + t\right) \, dt \\ &= \int_{-1}^{1} (4t^7 - 12t^5 + t^4 + 16t^3 - 2t^2 - 8t + 1)e^{t^4-2t^2+t+1} - 1 \, dt\end{aligned}$$

   (b) Now let $\mathcal{L}$ be the straight line from $(-1, 0)$ to $(1, 0)$ oriented left to right and let $\mathcal{D}$ be the region bounded by the x-axis and $\mathcal{C}$. Use Green's theorem to relate the integrals of $\mathbf{F}$ over $\mathcal{C}$ and $\mathcal{L}$ to an integral over $\mathcal{D}$. Use this to evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

   > **Solution:** By looking at the picture we see that $\mathcal{L} - \mathcal{C}$ is a closed curve that is the boundary of $\mathcal{D}$ (including orientation matching). Thus by Green's theorem
   >
   > $$\int_{\mathcal{L}} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{L}-\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \nabla \times \mathbf{F} \, dA$$
   >
   > A calculation shows that $\nabla \times \mathbf{F} = 2$. Thus
   >
   > $$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{L}} \mathbf{F} \cdot d\mathbf{r} - \iint_{\mathcal{D}} 2 \, dA$$
   >
   > The double integral is easy to evaluate:
   >
   > $$\iint_{\mathcal{D}} 2 \, dA = 2\int_{-1}^{1} \int_{0}^{1-x^2} \, dy \, dx = \frac{8}{3}.$$
   >
   > The curve $\mathcal{L}$ is parameterised by $\mathbf{r}(t) = (t, 0)$ for $t \in [-1, 1]$. So $\mathbf{r}'(t) = \langle 1, 0\rangle$. So
   >
   > $$\int_{\mathcal{L}} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} t^2 \, dt = \frac{2}{3}.$$
   >
   > So we get $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \frac{2}{3} - \frac{8}{3} = -2$

(c) **Path (almost)-independence for non-conservative vector fields.** More generally, suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two oriented (nonintersecting) curves with the same endpoints, and $\mathbf{F}$ is a vector field that is defined everywhere on the region $\mathcal{D}$ between $\mathcal{C}_1$ and $\mathcal{C}_2$. If $\mathbf{F}$ is conservative we know that $\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = 0$. If $\mathbf{F}$ is not conservative, what is

$$\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r}?$$

Solution: One of $\mathcal{C}_1 - \mathcal{C}_2$ or $\mathcal{C}_2 - \mathcal{C}_1$ will be the boundary of $\mathcal{D}$. Suppose it is $\mathcal{C}_1 - \mathcal{C}_2$. Then by Green's theorem

$$\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{C}_1-\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{D}} \nabla\times\mathbf{F}\, dA.$$

*The questions marked with an asterisk are more difficult or are of a form that would not appear on an exam. Nonetheless they are worth thinking about as they often test understanding at a deeper conceptual level.