What do these examples have in common?

- Add up quantity that is changing (eg. patients remaining / volume of disk)
- There is a "direction of change" (eg time / height)

Steps to solving these problems

- Identify "direction of change"
- divide into n subintervals
- assume all the change happens suddenly at the <u>end</u> of each subinterval
- Add together contributions from each subinterval and let $n \longrightarrow \infty$
- interpret as ~~a~~ Riemann sum
- Convert to integral
- solve!

Work

- Work is measured in Joules
- 1J = amount of energy expended moving a mass 1 metre using 1 Newton of force.
- From wikipedia: - 1J = energy required to lift a 100 g mass 1 meter above the Earth's surface.
  - Heat required to raise the temp of 1g of water by 0.24°C.

If F newtons of force are applied to move a mass d meters then the work done is

$$W = Fd \quad J.$$

Small example: Work done lifting 30 kg by 20 meters.

Solution Acceleration due to gravity $= -9.8 \, m/s^2$

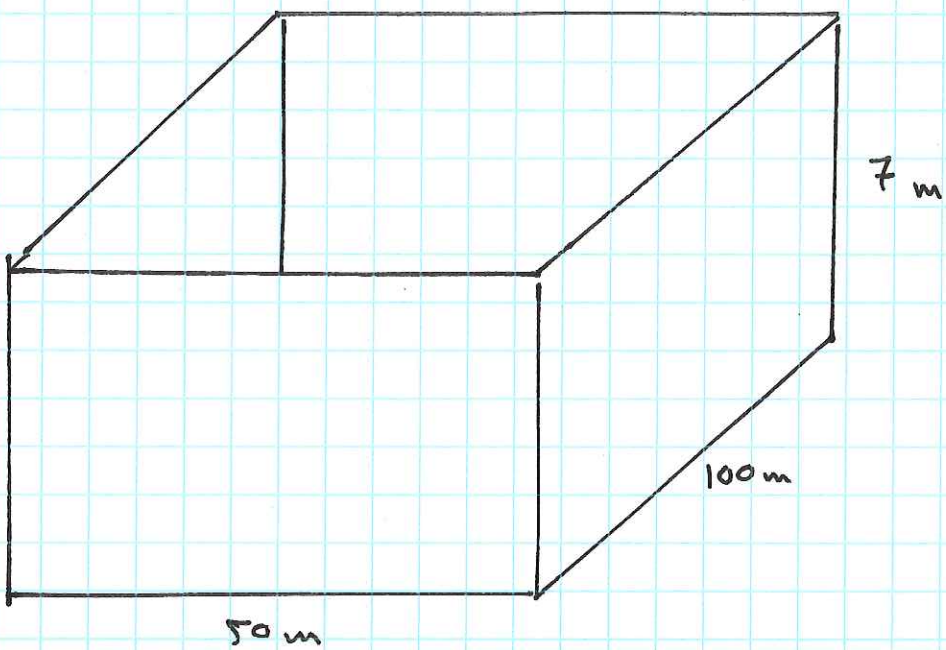
So force need to lift $= (9.8) \cdot 30$

$$W = (9.8) \cdot 30 \cdot 20$$
$$= 5886 \text{ J}.$$

Example 3

- A hole, 100 m x 50 m big, 7 m deep to be dug.
- Assume $1 m^3$ of dirt weights 1000 kg.

Q How ~~to~~ much work is being done by ~~dirt~~ digging the hole?



Silly solution: There is $5000 \times 7 \ m^3$ of dirt

$= 35\,000\,000$ kg of dirt

- Needs to be moved 7 m up
- Force needed $= (9.8) \cdot 35\,000\,000$

$= 343\,000\,000$ N

- Work $W = 343\,000\,000 \cdot 7$

$= 2\,401\,000\,000$ J.

But we shouldn't have to lift all the dirt the entire 7m up! E.g. the top layer only needs to be lifted ~ 0m

<u>Important</u>: How far below the surface the dirt is, determines how far it needs to be lifted!

- Direction of change = distance below surface = depth! = d.
- Subdivide into layers (n layers) each one $\Delta d = \frac{7}{n}$ m thick.
- The k^{th} layer (starting at $k = \cancel{0}1$) is
$$d_k = k \cdot \Delta d = \frac{7k}{n} \text{ m deep}$$
- The k^{th} layer contains
$$100 \cdot 50 \cdot \Delta d \text{ m}^3 \text{ of dirt}$$
So weighs $5\,000\,000\,\Delta d$ kg, thus the work is
$$W_k = \underbrace{\underbrace{(9.8)}_{\text{acceleration}}\underbrace{5\,000\,000\,\Delta d}_{\text{mass}}}_{\text{Force}} \cdot \underbrace{d_k}_{\text{distance}}$$
- Adding together and letting $n \to \infty$
$$W = \lim_{n \to \infty} \sum_{k=\cancel{0}1}^{n} 49\,000\,000\,\Delta d\, d_k$$

- Interpret as integral:

$$W = \int_0^7 49\,000\,000\, d \; dd$$

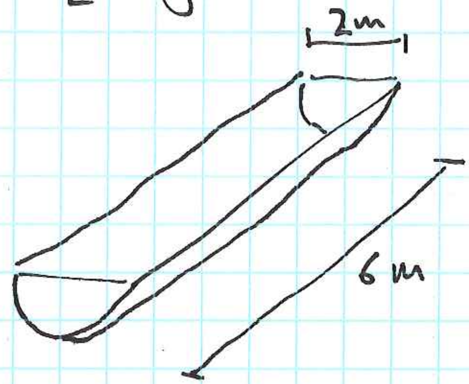
oops! terrible variable choice!

$$= \left[\tfrac{1}{2} 49\,000\,000\, d^2 \right]_0^7$$

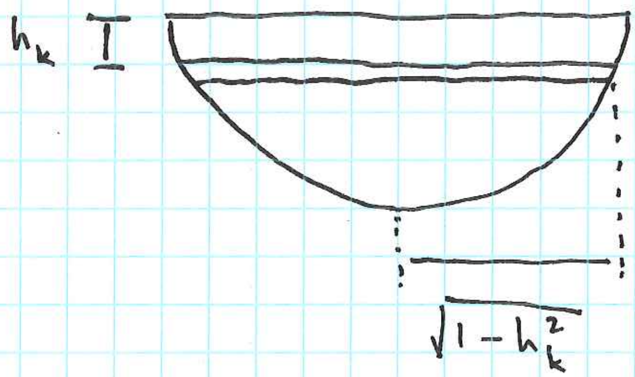
$$= 1\,200\,500\,000 \text{ J}.$$

Example 4

What about a $\frac{1}{2}$ cylindrical trench?



- Now the size of each layer is ~~co~~ changing w/ the depth as well.
- Let h = depth below ground.
- divide into n layers, $\Delta h = \frac{1}{n}$ m thick
- The k^{th} layer
  - $h_k = k\frac{1}{n}$ m below ground.



so k^{th} layer is

$$6.2\sqrt{1-h_k^2}\ \Delta h \ m^3 = 12000\sqrt{1-h_k^2}\,\Delta h \, kg$$

of dirt.

- Work needed to lift k^{th} slice:

$$(9.8)\cdot 12000\sqrt{1-h_k^2}\ \Delta h \cdot h_k$$

- Adding

$$W = \lim_{n \to \infty} \sum_{k=1}^{n} 117600\, h_k \sqrt{1-h_k^2}\ \Delta h$$

- As an integral

$$= \int_0^1 117600\, h\sqrt{1-h^2}\ dh$$

$$= \int_1^0 -58800\sqrt{u}\ du$$

$$= \left[58800 \cdot 2 \cdot \frac{1}{3} u^{3/2}\right]_0^1 = 19\,200$$