Lecture 2

1. Iterated integrals

- An iterated integral is an integral of the form

$$\int_a^b \underbrace{\left(\int_c^d f(x\ y)\, dy \right)}_{\text{hold } y \text{ const.}} dx$$

- Usually we don't write the brackets
- To solve, hold ~~y~~ the relavent variables const and integrate one at a time

Eg

$$\int_0^1 \int_0^1 y e^{xy}\, dx\, dy = \int_{y=0}^1 \Big[e^{xy} \Big]_{x=0}^1 dy$$

$$= \int_0^1 e^y - 1\, dy$$

$$= \Big[e^y - y \Big]_0^1$$

$$= e - 1 - (1 - 0)$$

$$= e - 2$$

2. Fubini's theorem

Thm Let $R = [a\ b] \times [c\ d]$, then

$$\iint_R f(x\ y)\, dA = \int_{x=a}^{b} \int_{y=c}^{d} f(x\ y)\, dy\, dx = \int_{y=c}^{d} \int_{x=a}^{b} f(x\ y)\, dx\, dy$$

proof sketch

$$\iint_R f(x\ y)\, dA = \lim_{M,N \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{M} f(x_i\ y_i)\, \Delta y\, \Delta x$$

$$\cancel{= \lim_{N \to \infty}}$$

$$= \lim_{N \to \infty} \sum_{i=1}^{N} \left(\lim_{M \to \infty} \sum_{j=1}^{M} f(x_i\ y_i)\, \Delta y \right) \Delta x$$

$$= \lim_{N \to \infty} \sum_{i=1}^{N} \left(\int_{y=c}^{d} f(x_i\ y)\, dy \right) \Delta x$$

$$= \int_a^b \int_c^d f(x, y)\, dy\, dx$$

- We could also have swapped the ~~limits~~. summations.

Ex $\displaystyle\iint_R \frac{1}{(x+y)^2}\,dx\,dy$, $R = [0,1]\times[1,2]$

$$= \int_{y=1}^{2}\int_{x=0}^{1} \frac{1}{(x+y)^2}\,dx\,dy$$

$$= \int_1^2 \left[-\frac{1}{x+y}\right]_0^1 dy$$

$$= \int_1^2 \frac{1}{y} - \frac{1}{1+y}\,dy$$

$$= \Big[\ln y - \ln(1+y)\Big]_1^2$$

$$= \ln 2 - \ln 3 - \ln 1 + \ln 2$$

$$= -\ln 3 \quad //.$$

3. Seperating variabls.

Thm If $f(x,y) = g(x)\cdot h(y)$ then

$$\iint_R f(x\,y)\,dA = \left(\int_a^b g(x)\,dx\right)\left(\int_c^d h(y)\,dy\right).$$

proof By Fubini's thm

$$\iint_R f(x\,y)\,dA = \int_a^b \int_c^d g(x)\,h(y)\,dy\,dx$$

$$= \int_a^b g(x)\,\overbrace{\left(\int_c^d h(y)\,dy\right)}^{\text{scalar}}\,dx$$

Double integrals practice

* $\iint_R x^2 y \, dA \qquad R = [0, 2] \times [0\ 1]$

* $\iint_R x^2 \sin xy \, dA \qquad R = [0, 2] \times [0, \frac{\pi}{2}]$

* $\iint_R 1 - x^2 - y^2 \, dA \qquad R = [0\ 1]^2$