This weeks problem set focuses on the ideas of bases and linear transformations. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

1. From section 1.6, problems 1, 2a, e, 3a, c, 4, 6, 14, 15, 20*, 26, 28†, 33, 34*, 35*.

2. From section 2.1, problems 1, 2, 5, 6, 9*, 14, 14b.

3.† Let $V = \mathbb{F}^n$ for some field $\mathbb{F}$. If $v \in V$ (i.e. v is a column vector) a *permutation* of v is any column vector obtained from v by rearranging the entries. For example

$$\begin{pmatrix} v_1 \\ v_3 \\ v_4 \\ v_2 \end{pmatrix} \text{ is a permutation of } \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

We say that a subspace $U \subseteq V$ is *permutation invariant* if for any $v \in U$ then any permutation of v is also in U.

(a) Give an example of a one dimensional, permutation invariant subspace when $n = 2$.

(b) Give an example of a one dimensional, permutation invariant subspace for any n.

(c) Show that the subspace $\Sigma_n \subseteq V$ is permutation invariant.

(d) Suppose that U is a permutation invariant subspace that does not contain $e_1 - e_2$. Then the first two entries of any vector in U are equal.

(e) Suppose that U is a permutation invariant subspace such that the first two entries of any vector in U are equal. Show that $U = \{0\}$ or T.

(f) List all the permutation invariant subspaces. *Hint: this is tricky, you will need to use the previous two parts.*

(g) Is it possible to always have two non-trivial, permutation invariant subspaces U, W such that $U \oplus W = V$? *Hint: you will need a condition on the characteristic of the field!*