Math 3B: Lecture 7

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January 23, 2019

Antiderivatives

We will be concentrating on solving differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

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A solution $y = F(x)$ is called an antiderivative of $f(x)$.

Example 1

Question

What is the antiderivative of $f(x) = 2x$?

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What is the antiderivative of $f(x) = 2x$?

Solution

$$F(x) = x^2$$

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What is the antiderivative of $f(x) = 2x$?

Solution

$$F(x) = x^2 + 4$$

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$$F(x) = x^2 + 8$$

Example 1

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What is the antiderivative of $f(x) = 2x$?

Solution

$$F(x) = x^2 + C$$

Example 2

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What is the antiderivative of $f(x) = x^3 + 4x - 1$?

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$$F(x) = \frac{1}{4}x^4 + 2x^2 - x + C$$

Example 3

Question

What is the antiderivative of $f(x) = e^{2x}$?

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Solution

$$F(x) = \quad e^{2x}$$

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What is the antiderivative of $f(x) = e^{2x}$?

Solution

$$F(x) = \frac{1}{2}e^{2x}$$

Example 4

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What is the antiderivative of $f(x) = \frac{1}{x}$ (for $x > 0$)?

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Solution

$$F(x) = \ln x$$

Example 5

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Solution

Note that $f(x) = (1+x)^{-2}$. So

$$F(x) = \quad \frac{1}{1+x}$$

Example 5

Question

What is the antiderivative of $f(x) = \frac{1}{(1+x)^2}$?

Solution

Note that $f(x) = (1 + x)^{-2}$. So

$$F(x) = -\frac{1}{1 + x}$$

Example 6

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What is the antiderivative of $f(x) = 2x\cos x^2$?

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What is the antiderivative of $f(x) = 2x \cos x^2$?

Solution

$$F(x) = \sin x^2$$

Example 7

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Example 7

Question

What is the antiderivative of $f(x) = \frac{1}{\sqrt{x}}$?

Solution

$$f(x) = x^{-\frac{1}{2}}$$

$$F(x) = 2x^{\frac{1}{2}}$$

The general antiderivative/indefinite integral

In general, if $F(x)$ is any antiderivative of $f(x)$, then $F(x) + C$ is an antiderivative for any constant C

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We call $F(x) + C$ the general antiderivative.

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Example

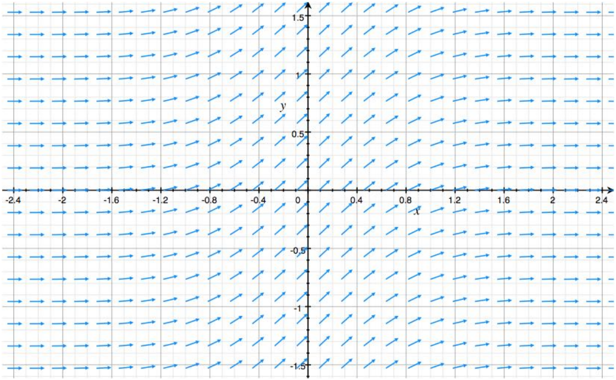
$$\int \frac{1}{x^2}\, dx = -\frac{1}{x} + C$$

Slope fields

In some cases it is impossible to find the antiderivative (without special functions). E.g.

$$f(x) = e^{-x^2}$$

But we can still (approximately) graph the antiderivative! First we draw the slope field



Slope fields (how to draw)

How to draw a slope field for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

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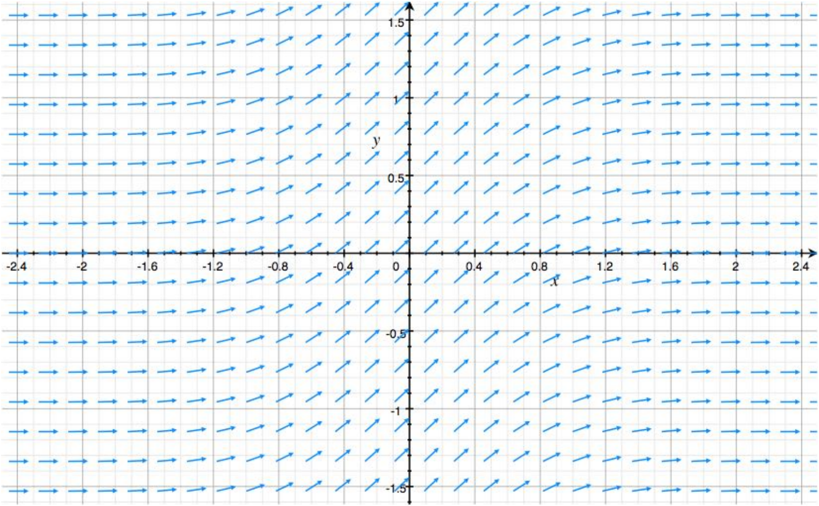
3. Answer given by differential equation above, slope is $f(x)$
4. Draw a small arrow with slope $f(x)$ and the point (x, y)
5. Do this for a grid of points on the xy-plane.

Example 1

$$f(x) = e^{-x^2} \text{ with } F(0) = -0.25$$



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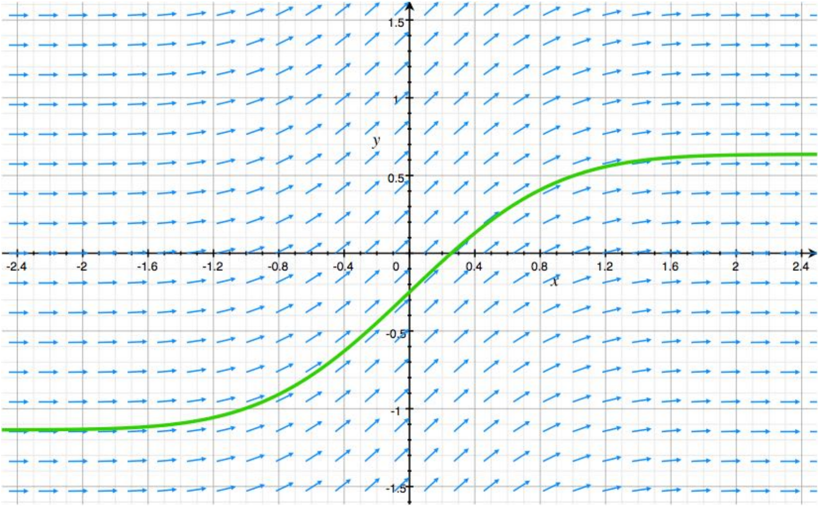
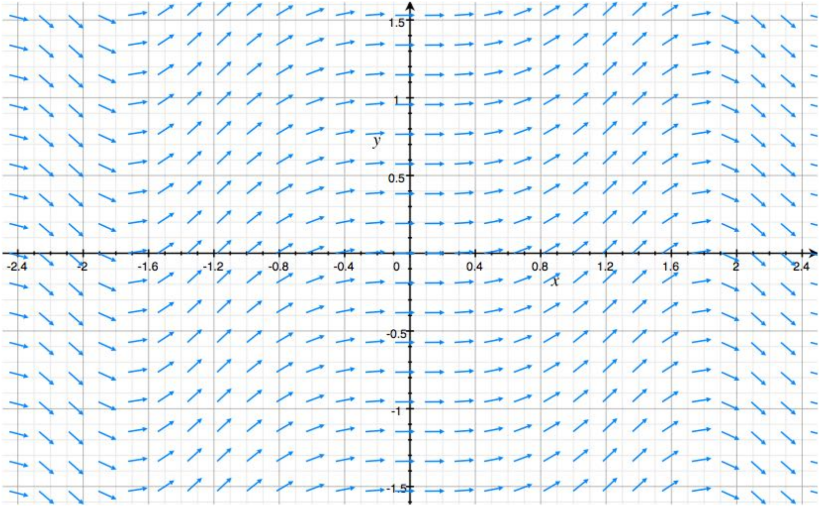


Example 2

$$f(x) = \sin(x^2) \text{ with } F(0) = 0.1$$



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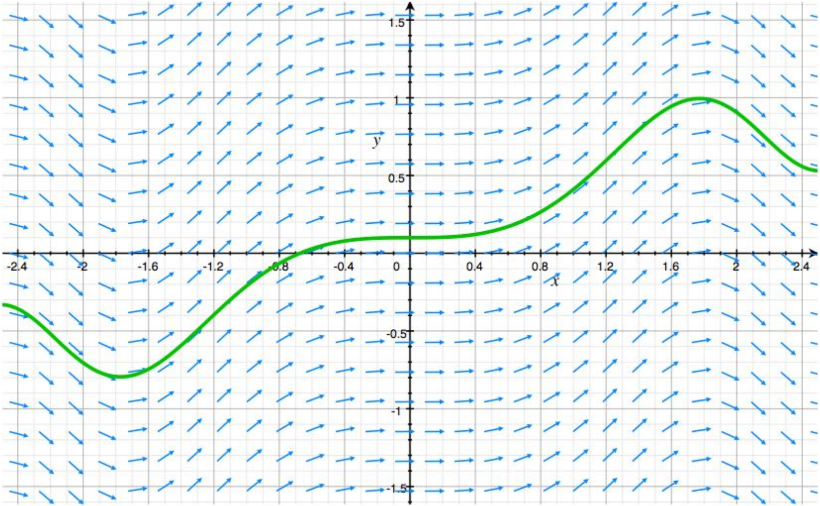

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Suppose $u = g(x)$, then

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$$\int f(g(x))\frac{\mathrm{d}u}{\mathrm{d}x} = \int f(g(x))g'(x) = \int f(u)$$

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Solution

We use the substitution $u = x^2 + 1$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we can write the integral

$$\int 2 \cdot \sqrt{x^2+1} \cdot 2x = 2\int \sqrt{u}$$

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$$\begin{aligned}\int 2 \cdot \sqrt{x^2+1} \cdot 2x &= 2\int \sqrt{u} \\ &= 2 \cdot \frac{2}{3}u^{\frac{3}{2}} + C\end{aligned}$$

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$$\begin{aligned}\int 2 \cdot \sqrt{x^2+1} \cdot 2x &= 2\int \sqrt{u} \\ &= 2 \cdot \frac{2}{3}u^{\frac{3}{2}} + C \\ &= \frac{4}{3}(x^2+1)^{\frac{3}{2}} + C\end{aligned}$$

The product rule

Just like integration by substitution reverses the chain rule, integration by parts "reverses" the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

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written another way

$$(uv)' = u'v + uv'$$

Integration by parts

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Lets integrate both sides

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Rearranging...

Integration by parts

The integration by parts formula

$$\int uv' \, dx = uv - \int u'v \, dx$$

Integration by parts

The integration by parts formula

$$\int uv' \, dx = uv - \int u'v \, dx$$

Alternative statement

$$\int u \, dv = uv - \int v \, du$$

Examples

One the board. . .