Math 3B: Lecture 19

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Linear models

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Examples

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\lambda y.$$

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Newton's Law of Cooling

The temperature T of a body changes at a rate propotional to the difference between the ambient temperature A and T.

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$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(A - T)$$

General solution

$$T(t) = A - Ce^{-kt}.$$

Example 2

An object takes 20 minutes to cool from 90° to 86° in a room which is 70°. At what time will it be 75°?

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- Thus

$$90 = 70 - C \quad \text{so} \quad C = -20.$$

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$$75 = 70 + 20e^{-0.01t}.$$

- Rearranging we get $20e^{-0.01t} = 5$ i.e.

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- Applying a logarithm

$$-0.01t = \ln\left(\frac{1}{4}\right)$$

- So we get

$$t = -100\ln\left(\frac{1}{4}\right) \approx 138 = 2 \text{ hours } 18 \text{ minutes.}$$

Slope fields

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Key tool

Slope fields. At every point on the *yt*-plane we draw a small line segment (a vector) with slope $f(y, t)$.

Examples

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If we want to draw a slope field, we cannot actually draw a line segment for every point. Instead we pick a grid of points in the plane.

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Examples

Lets use Geogebra! Here is the command we will use:

`SlopeField[f(x,y)]` will produce a slope field for the equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x, y)$$

Sketching solutions

Using the slope field we can sketch rough pictures of the solution, given a starting point (an initial condition).

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Examples

Lets use Geogebra again.

Nullclines

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Examples

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Drawing slope fields by hand

Drawing slope fields by hand can be difficult! But we can use the nullclines to get an approximate picture

Examples

Lets draw some on the board.