Final exam (practice)

UCLA: Math 32B, Winter 2017

Instructor: Noah White
Date:

- This exam has 7 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section: ______________________________

Question	Points	Score
1	12	
2	12	
3	13	
4	12	
5	9	
6	8	
7	14	
Total:	80	

Questions 1 and 2 are multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following four pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) The iterated integral $\int_0^1 \int_3^5 1 \, \mathrm{d}x \, \mathrm{d}y$ is equal to

   A. 2
   B. 4
   C. 1
   D. -1

   (b) (2 points) Integrate $f(x, y) = \sin x$ in the region $\mathcal{R} = [-1, 1] \times [0, 4]$

   A. π
   B. $-\pi$
   C. 1
   D. 0

   (c) (2 points) Integrate the function $f(x) = 4xy$ on the triangle $\mathcal{D}$ with vertices $(0, 0), (1, 0)$ and $(0, 2)$.

   A. 2/3
   B. 2
   C. $-4/3$
   D. 4

(d) (2 points) The Jacobian of the function $G(u, v, w) = (u^2, v - u, ve^w)$

A. ue^w

B. $(v - u)e^w$

C. we^w

D. $2uve^w$

(e) (2 points) Integrate $e^{\sqrt{x^2+y^2+z^2}}$ over the ball $x^2 + y^2 + z^2 \leq 4$.

A. $8\pi(e^2 - 1)$

B. $8\pi^2$

C. $8\pi e^2$

D. $4(e^2 - 1)$

(f) (2 points) Calculate the line integral of $f(x, y) = 1$ along the circle $x^2 + y^2 = 4$.

A. 4π**.**

B. 2π.

C. -4π.

D. -2π.

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) Suppose $f(t)$ is a function defined on all of $\mathbb{R}$. Let $\mathcal{C}$ be the oriented curve given by the parametrisation $\mathbf{r}(t) = (t, f(t))$ for $a \leq t \leq b$ (where $a < b$). The flux through $\mathcal{C}$ of the vector field $\langle 0, e^{x^2+y^2} \rangle$ is

   A. greater than and sometimes equal to zero.

   B. less than and sometimes equal to zero.

   C. always greater than zero.

   D. always less than zero.

   (b) (2 points) The vector field $r^{-2}\langle -y, x, 0\rangle$ where $r^2 = x^2+y^2$, has domain $\mathbb{R}^3 - \{(x, y, z) \mid x = y = 0\}$. The vector field

   A. has zero curl and is conservative.

   B. has non zero curl and is conservative.

   C. has zero curl and is not conservative.

   D. has non zero curl and is not conservative.

   (c) (2 points) Let $\varphi(x, y)$ be a scalar function defined on the plane $\mathbb{R}^2$ and let $\mathbf{F} = \nabla\varphi$. Suppose $\int_{\mathcal{C}'} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 3$ where $\mathcal{C}'$ is the straight line from $(-3, 0)$ to $(3, 0)$. What is the value of $\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ if $\mathcal{C}$ is the top half of the ellipse

   $$\frac{x^2}{9} + \frac{y^2}{2} = 1$$

   oriented counter clockwise.

   A. 0

   B. 1.

   C. 3.

   D. -3.

(d) (2 points) Consider the surface $\mathcal{S}$ parametrised by $G(u, v) = (u - v, u^2, v)$. What is the normal vector to the surface at the point $(0, 4, 2)$?

A. $\langle 2, -1, 2\rangle$

B. $\langle 8, -1, 8\rangle$.

C. $\langle 8, 1, 8\rangle$.

D. $\langle 4, -1, 4\rangle$**.**

(e) (2 points) Suppose $\mathbf{F}$ is an *incompressible* vector field (i.e. it has $\mathrm{div}(\mathbf{F}) = 0$). What is the (outward) flux of $\mathbf{F}$ through the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1?$$

A. 0**.**

B. πabc.

C. $-\pi abc$.

D. $\sqrt{a^2 + b^2 + c^2}$.

(f) (2 points) Let $\mathcal{C}$ be the unit circle, oriented counter clockwise, and $\mathcal{D}$ be the unit disk, both centered at the origin. If $\mathbf{F} = \langle \log(x^2 + y^2 + 1), \log(x^2 + y^2 + 1)\rangle$, the integral $\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$ is equal to

A. $\iint_{\mathcal{D}} \frac{2(x+y)}{x^2+y^2+1}\, \mathrm{d}S$

B. $-\iint_{\mathcal{D}} \frac{2(x+y)}{x^2+y^2+1}\, \mathrm{d}S$

C. $\iint_{\mathcal{D}} \frac{2(x-y)}{x^2+y^2+1}\, \mathrm{d}S$

D. $-\iint_{\mathcal{D}} \frac{2(x-y)}{x^2+y^2+1}\, \mathrm{d}S$

3. Consider the vector field

$$\mathbf{F}(x,y,z) = \frac{1}{u^2}\langle -z, z, x-y\rangle \quad \text{where } u^2 = z^2 + (x-y)^2.$$

(a) (2 points) What is the largest domain on which $\mathbf{F}$ is defined?

Solution: $\mathbf{F}$ is defined unless $u^2 = z^2 + (x-y)^2 = 0$ this happens only when $z = 0$ and $x = y$. Thus $\mathbf{F}$ has largest domain

$$\mathbb{R}^3 - \{(a,a,0) \mid a \in \mathbb{R}\}.$$

(b) (4 points) Calculate the curl of $\mathbf{F}$.

Solution: To make our calculation a little easier let us first notice that

$$\partial_x \frac{1}{u^2} = -\frac{2(x-y)}{u^4}, \partial_y \frac{1}{u^2} = \frac{2(x-y)}{u^4}, \partial_z \frac{1}{u^2} = -\frac{2z}{u^4}.$$

This means that

$$\begin{aligned}
\mathrm{curl}(\mathbf{F})_x &= -\frac{1}{u^2} + (x-y)\frac{2(x-y)}{u^4} - \frac{1}{u^2} + \frac{2z^2}{u^4} \\
&= \frac{-2u^2 + 2(x-y)^2 + 2z^2}{u^4} = 0 \\
\mathrm{curl}(\mathbf{F})_y &= -\frac{1}{u^2} + (x-y)\frac{2(x-y)}{u^4} - \frac{1}{u^2} + \frac{2z^2}{u^4} \\
&= \frac{-2u^2 + 2(x-y)^2 + 2z^2}{u^4} = 0 \\
\mathrm{curl}(\mathbf{F})_z &= -\frac{2z(x-y)}{u^4} + \frac{2z(x-y)}{u^4} = 0.
\end{aligned}$$

So the curl is zero.

(c) (5 points) Show that $\mathbf{F}$ is not conservative.

Solution: To show $\mathbf{F}$ is not conservative consider the circle $z^2 + (x-y)^2 = 1$ on the plane $x = -y$. This is parametrised by

$$\mathbf{r}(t) = \langle \frac{1}{2}\cos t, -\frac{1}{2}\cos t, \sin t\rangle$$

This is a convenient circle since for every point on the circle $u^2 = 1$. Thus

$$\mathbf{F} = \langle -\sin t, \sin t, \cos t\rangle$$

and

$$\mathbf{r}'(t) = \langle -\frac{1}{2}\sin t, \frac{1}{2}\sin t, \cos t\rangle.$$

Hence

$$\begin{aligned}
\oint_{\mathcal{C}} \mathbf{F}\cdot\mathbf{r} &= \int_0^{2\pi} \langle -\sin t, \sin t, \cos t\rangle \cdot \langle -\frac{1}{2}\sin t, \frac{1}{2}\sin t, \cos t\rangle \,\mathrm{d}t \\
&= \int_0^{2\pi} \frac{1}{2}\sin^2 t + \frac{1}{2}\sin^2 t + \cos^2 t \,\mathrm{d}t
\end{aligned}$$

$$\int_0^{2\pi} 1 \,\mathrm{d}t = 2\pi \neq 0.$$

Hence **F** cannot be conservative.

(d) (2 points) Calculate $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ if $\mathcal{C}$ is the curve given by the intersection of the sphere $(x-2)^2 + (y+2)^2 + z^2 = 1$ and the surface $z = (x-2)^2$.

Solution: The curve is contained in the domain where $x > y$, and since this domain is simply connected **F** is conservative and thus the integral is zero.

4. (a) (4 points) Let $\mathcal{S}$ be an oriented surface and let $\mathbf{F}$ be a vector field which has continuous partial derivatives on an open region containing $\mathcal{S}$. State Stokes' theorem for $\mathbf{F}$.

Solution:
$$\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}.$$

(b) (3 points) Let $\mathbf{F} = \langle xe^{y+z} - y, ye^{y+z} + x, e^{x+y} \rangle$. Calculate the curl of $\mathbf{F}$.

Solution:
$$\mathrm{curl}(\mathbf{F}) = \langle e^{x+y} - ye^{y+z}, xe^{y+z} - e^{x+y}, 2 - xe^{y+x} \rangle$$

(c) (5 points) Let $\mathcal{S}$ be the surface defined by
$$\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 \mid z = e^{-(x^2+y^2)} \text{ for } z \geq \frac{1}{e}\}$$
with upward normal, and let $\mathbf{F}$ be as above. Calculate $\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S}$

Solution: We will use Stokes' theorem. To do this lets find the boundary. When $z = \frac{1}{e}$ we have that $e^{-1} = e^{-(x^2+y^2)}$ so $x^2+y^2 = 1$. This means the boundary is the unit circle $x^2+y^2 = 1$ at $z = e^{-1}$. Using the right hand rule we can see that the boundary has orientation counter clockwise when looking down from above.
We can parametrise this curve by
$$\mathbf{r}(t) = \langle \cos t, \sin t, e^{-1} \rangle$$
and so
$$\mathbf{r}(t) = \langle -\sin t, \cos t, 0 \rangle.$$
On this curve we have that
$$\mathbf{F} = \langle \cos te^{\sin t+e^{-1}} - \sin t, \sin te^{\sin t+e^{-1}} + \cos t, e^{\cos t+\sin t} \rangle.$$
Thus we have the dot product
$$\begin{aligned}\mathbf{F} \cdot \mathbf{r}'(t) &= -\sin t\cos te^{\sin t+e^{-1}} + \sin^2 t + \sin t\cos te^{\sin t+e^{-1}} + \cos^2 t \\ &= 1.\end{aligned}$$
Thus out line integral becomes
$$\oint_{\partial\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} 1 \ \mathrm{d}t = 2\pi.$$
By Stokes' theorem $\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial\mathcal{S}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = 2\pi$.

5. (a) (3 points) Give *three* examples of vector fields $\mathbf{F}$ in $\mathbb{R}^2$ such that $\operatorname{curl}(\mathbf{F}) = 1$. *Hint: Recall that for 2D vector fields* $\operatorname{curl}(\mathbf{F}) = \partial_x F_2 - \partial_y F_1$.

Solution:

$$\mathbf{F} = \langle y, 0\rangle, \quad \langle 0, -x\rangle \quad \text{or} \quad \frac{1}{2}\langle y, -x\rangle.$$

(b) (6 points) Use Green's theorem to find the area enclosed by the parametrised curve

$$\mathbf{r}(t) = (1 - \sin t, 1 - \cos t) \quad \text{for } t \in [0, 2\pi].$$

Solution: Let $\mathcal{C}$ be the parametrised curve. Since $\mathbf{r}(t) = \langle 1, 1\rangle - \langle \sin t, \cos t\rangle$, one can see that this is simply a circle, moreover it is oriented *clockwise*. We use that fact that if we choose one of the above vector fields, say $\mathbf{F} = \langle 0, x\rangle$ then by Green's theorem

$$\text{Area}(\mathcal{D}) = \iint_{\mathcal{D}} 1 \, \mathrm{d}A = \iint_{\mathcal{D}} \operatorname{curl}(\mathbf{F}) \, \mathrm{d}A = \oint_{-\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = -\oint_{-\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r},$$

where $\mathcal{D}$ is the region it encloses. Note that $\partial\mathcal{C} = -\mathcal{C}$ (hence the negative).
Now the velocity along the curve is $\mathbf{r}'(t) = \langle -\cos t, \sin t\rangle$ and the vector field is $\mathbf{F} = \langle 0, 1 - \sin t\rangle$. So

$$\oint_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_0^{2\pi} \langle 0, 1 - \sin t\rangle \cdot \langle -\cos t, \sin t\rangle \, \mathrm{d}t$$
$$= \int_0^{2\pi} \sin t - \sin^2 t \, \mathrm{d}t = -\pi.$$

where the integral is calulated using integration by parts. Thus

$$\text{Area}(\mathcal{D}) = \oint_{-\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \pi.$$

6. (a) (4 points) Either find a function f on $\mathbb{R}^3$ such that

$$\nabla f(x, y, z) = \langle e^z, 1, xe^z \rangle$$

or show that one cannot exist.

Solution: First we check that $\operatorname{curl}(\langle e^z, 1, xe^z \rangle) = 0$, which is indeed true, so it is possible that we can find a potential function. It must be the case

$$\begin{aligned} \partial_x f = e^z &\quad \text{so} \quad f = xe^z + \alpha(y, z) \\ \partial_y f = 1 &\quad \text{so} \quad f = y + \beta(x, z) \\ \partial_z f = xe^z &\quad \text{so} \quad f = xe^z + \gamma(x, y) \end{aligned}$$

We can choose $\alpha = \gamma = y$ and $\beta = xe^z$, so that $f = xe^z + y$.

(b) (4 points) Either find a vector field $\mathbf{F}$ on $\mathbb{R}^3$ such that

$$\operatorname{curl}(\mathbf{F}) = \langle y, x, z \rangle$$

or show that one cannot exist.

Solution: We calculate the divergence: $\operatorname{div}(\langle y, x, z \rangle) = 1 \neq 0$ thus it cannot possibly be the curl of a vector field.

7. Let $\mathcal{E}$ be the solid cone given by

$$z \geq \alpha\sqrt{x^2+y^2} \quad \text{and} \quad z \leq b,$$

such that the top of the cone has a radius of $a > 0$.

(a) (2 points) Express α in terms of a and b.

Solution: When $z = b$ we should have a circle of radius a. When we set $z = b$ we get

$$\frac{b}{\alpha} = \sqrt{x^2+y^2}$$

so we should set $a = b/\alpha$, i.e. $\alpha = b/a$.

(b) (5 points) Let $\mathbf{F} = \langle y^2, xy, z\rangle$. Express the triple integral of $\operatorname{div}(\mathbf{F})$ over $\mathcal{E}$ as a triple iterated integral.

Solution: First note that $\operatorname{div}(\mathbf{F}) = x + 1$. The projection of $\mathcal{E}$ onto the xy-plane is the disk D of radius a, thus we can express $\mathcal{E}$ as the region

$$\mathcal{E} = \{(x, y, z) \mid (x, y) \in D\ \frac{b}{a}\sqrt{x^2+y^2} \leq z \leq b\}.$$

So we get

$$\begin{aligned}\iiint_{\mathcal{E}} x + 1 \,\mathrm{d}V &= \iint_D \int_{\frac{b}{a}\sqrt{x^2+y^2}}^{b} x + 1 \,\mathrm{d}z\,\mathrm{d}A \\ &= \int_0^{2\pi}\int_0^a r\int_{\frac{b}{a}r}^{b} r\cos\theta + 1 \,\mathrm{d}z\,\mathrm{d}r\,\mathrm{d}\theta\end{aligned}$$

where we have changed to polar coords to evaluate the integral over the disk.

(c) (3 points) Parametrise the boundary $\partial\mathcal{E}$ of $\mathcal{E}$ as an *oriented* surface (you may parametise this as two separate surfaces.)

Solution: First we need to parametrise the disk B which makes up the top of the cone. This can be parametrised by $G_B(r,\theta) = (r\cos\theta, r\sin\theta, b)$, where $(r,\theta) \in [0, a] \times [0, 2\pi]$. It is easy to see that it has a normal of $\mathbf{N}_B = \langle 0, 0, r\rangle$ (we will need this in the next part).
The second piece, S, the side of the cone can be parametrised by $G_S(\theta, z) = (\frac{az}{b}\cos\theta, \frac{az}{b}\sin\theta, z)$ where $(\theta, z) \in [0, 2\pi] \times [0, b]$, which has normal $\mathbf{N}_S = \langle \frac{az}{b}\cos\theta, \frac{az}{b}\sin\theta, -\frac{a^2}{b^2}z\rangle$.

(d) (4 points) use the divergence theorem to express the triple integral above as the sum of two double iterated integrals.

Solution: According to the parametrisation above, on B, we have $\mathbf{F} = \langle r^2\sin^2\theta, r^2\sin\theta\cos\theta, b\rangle$. Thus the integral over B is

$$\iint_B \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi}\int_0^a rb\,\mathrm{d}r\mathrm{d}\theta.$$

According to the parametrisation above, on S, we have $\mathbf{F} = \langle \frac{a^2z^2}{b^2}\sin^2\theta, \frac{a^2z^2}{b^2}\sin\theta\cos\theta, z\rangle$. Thus the integral over S is

$$\iint_S \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi}\int_0^b 2\frac{a^3z^3}{b^3}\cos\theta\sin^2\theta - \frac{a^2z^2}{b^2}\,\mathrm{d}z\,\mathrm{d}\theta.$$

Thus

$$\iint_{\partial\mathcal{E}} \mathbf{F}\cdot \mathrm{d}\mathbf{S} = \int_0^{2\pi}\int_0^a rb\ \mathrm{d}r\mathrm{d}\theta + \int_0^{2\pi}\int_0^b 2\frac{a^3z^3}{b^3}\cos\theta\sin^2\theta - \frac{a^2z^2}{b^2}\ \mathrm{d}z\ \mathrm{d}\theta.$$

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