Midterm 2 practice 2

UCLA: Math 32B, Fall 2019

Instructor: Noah White
Date:

- This exam has 5 questions, for a total of 37 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	9	
2	8	
3	7	
4	5	
5	8	
Total:	37	

Here are some formulas that you may find useful as some point in the exam.

$$\int \cos^2 x \, \mathrm{d}x = \frac{1}{2}\left(x + \cos x \sin x\right)$$

$$\int \sin^2 x \, \mathrm{d}x = \frac{1}{2}\left(x - \cos x \sin x\right)$$

$$\int \sin x \cos x \, \mathrm{d}x = \frac{1}{2}\sin^2 x$$

Spherical coordinates are given by

$$\begin{aligned} x(\rho,\theta,\phi) &= \rho\cos\theta\sin\phi \\ y(\rho,\theta,\phi) &= \rho\sin\theta\sin\phi \\ z(\rho,\theta,\phi) &= \rho\cos\phi \end{aligned}$$

The Jacobian for the change of coordinates is $J = \rho^2 \sin\phi$.

1. Let $\mathcal{E}$ be the solid region defined by

$$x^2 + y^2 + z^2 \leq a, \quad x, y, z \geq 0,$$

for a fixed constant $a > 0$.

(a) (2 points) Find the volume of $\mathcal{E}$ as an iterated integral.

Solution: First we express the region as follows

$$\mathcal{E} = \{ \ (\rho, \theta, \phi) \ \mid \ 0 \leq \rho \leq \sqrt{a}, \theta, \phi \in [0, \pi/2] \ \}$$

where we have used spherical coordinates. Note that the Jacobian is $\rho^2 \sin\phi$ which is always positive. Thus the iterated integral is

$$\mathrm{Vol}(\mathcal{E}) = \iiint_{\mathcal{E}} 1 \ dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\sqrt{a}} \rho^2 \sin\phi \ d\rho \ d\phi \ d\theta.$$

(b) (2 points) Find the volume of $\mathcal{E}$.

Solution:

$$\begin{aligned} \mathrm{Vol}(\mathcal{E}) &= \iiint_{\mathcal{E}} 1 \ dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\sqrt{a}} \rho^2 \sin\phi \ d\rho \ d\phi \ d\theta \\ &= \frac{\pi}{2} \left(\int_0^{\pi/2} \sin\phi \ d\phi \right) \left(\int_0^{\sqrt{a}} \rho^2 \ d\rho \right) \\ &= \frac{\pi}{2} \cdot 1 \cdot \frac{a^{\frac{3}{2}}}{3} = \frac{\pi a^{\frac{3}{2}}}{6} \end{aligned}$$

(c) (3 points) Let $V = \mathrm{Vol}(\mathcal{E})$. Express $C_x = \frac{1}{V}\iiint_{\mathcal{E}} x\, dV$, $C_y = \frac{1}{V}\iiint_{\mathcal{E}} y\, dV$, and $C_z = \frac{1}{V}\iiint_{\mathcal{E}} z\, dV$ as iterated integrals.

Solution: Using the above description of $\mathcal{E}$ and spherical coordinates

$$C_x = \frac{6}{\pi a^{\frac{3}{2}}} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\sqrt{a}} \rho^3 \cos\theta \sin^2\phi \, d\rho \, d\phi \, d\theta$$
$$C_y = \frac{6}{\pi a^{\frac{3}{2}}} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\sqrt{a}} \rho^3 \sin\theta \sin^2\phi \, d\rho \, d\phi \, d\theta$$
$$C_z = \frac{6}{\pi a^{\frac{3}{2}}} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\sqrt{a}} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta$$

(d) (2 points) Evaluate C_z.

Solution:

$$\begin{aligned} C_z &= \frac{6}{\pi a^{\frac{3}{2}}} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\sqrt{a}} \rho^3 \cos\phi \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \frac{6}{\pi a^{\frac{3}{2}}} \cdot \frac{\pi}{2} \left(\int_0^{\pi/2} \cos\phi \sin\phi \, d\phi \right) \left(\int_0^{\sqrt{a}} \rho^3 \, d\rho \right) \\ &= \frac{3}{a^{\frac{3}{2}}} \cdot \left[\frac{1}{2} \sin^2\phi \right]_0^{\pi/2} \cdot \left[\frac{1}{4}\rho^4 \right]_0^{\sqrt{a}} \\ &= \frac{3}{a^{\frac{3}{2}}} \cdot \frac{1}{2} \cdot \frac{a^2}{4} = \frac{3\sqrt{a}}{8}. \end{aligned}$$

2. Consider the helix $\mathcal{C}$, given by the parameterisation

$$\mathbf{r}(t) = \left(\cos t, \sin t, \frac{1}{2\pi} t\right) \quad t \in [0, 4\pi],$$

so that $\mathcal{C}$ is oriented with the z coordinate increasing.

(a) (4 points) Compute the length of $\mathcal{C}$.

Solution: First we compute the tangent

$$\mathbf{r}'(t) = \left\langle -\sin t, \cos t, \frac{1}{2\pi} \right\rangle$$

thus the speed is

$$\|\mathbf{r}'(t)\| = \sqrt{1 + \frac{1}{4\pi^2}}$$

Now we can evaluate"

$$\int_{\mathcal{C}} 1 \, ds = \int_0^{4\pi} \sqrt{1 + \frac{1}{4\pi^2}} \, dt = 4\pi \sqrt{1 + \frac{1}{4\pi^2}}$$

(b) (4 points) Compute the work done by the field

$$\mathbf{F}(x,y,z) = \langle z^2, 2yz^2, 2z(x+y^2) - e^z \rangle$$

on a particle constrained to move on the curve $\mathcal{C}$.

Solution:
The work done by the field is

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

We first check to see if the field is conservative. Suppose we have $f(x,y,z)$ such that $\nabla f = \mathbf{F}$, then

$$\begin{aligned} f &= xz^2 + \alpha(y,z) \\ f &= y^2z^2 + \beta(x,z) \\ f &= z^2(x+y^2) - e^z + \gamma(x,y) \end{aligned}$$

so we can take $\alpha = y^2z^2 - e^z$, $\beta = z^2x - e^z$ and $\gamma = 0$. Thus we have a potential function $f = z^2(x+y^2) - e^z$. Thus

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= f(\mathbf{r}(4\pi)) - f(\mathbf{r}(0)) \\ &= f(1,0,2) - f(1,0,0) = 4 - e^4 - (-e^0) = 5 - e^4. \end{aligned}$$

3. For this question consider the vector field

$$\mathbf{F}(x,y) = \frac{1}{r^2}\langle y(r^2-1), x(r^2+1)\rangle,$$

where $r = \sqrt{x^2+y^2}$. This vector field is defined everywhere apart from the origin.

(a) (4 points) Is $\mathbf{F}$ conservative on the domain described above? Justify your answer.

Solution: First we check to see if the curl of the field is zero. Note that $\partial_x(r^{-2}) = -2xr^{-4}$ and $\partial_y(r^{-2}) = -2yr^{-4}$.

$$\nabla \times \mathbf{F} = 1 + \frac{1}{r^2} - \frac{2x^2}{r^4} - \left(1 - \frac{1}{r^2} + \frac{2y^2}{r^4}\right) = \frac{2}{r^2} - \frac{2(x^2+y^2)}{r^4} = 0.$$

This means we cannot rule out the field being conservative. Unfortunately the domain is not simply connected so we cannot just check the curl of the vector field. We can either attempt to find a potential function (which is extremely difficult in general!) or we can try and find a loop around which, the integral of the vector field is non zero. A good guess is to consider the counter-clockwise circle of radius one, around the origin. Let $\mathcal{C}$ be this curve. A parametrisation is $\mathbf{r}(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. The velocity is

$$\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$$

Notice that $r = 1$ for any point on $\mathcal{C}$ so $\mathbf{F}(\mathbf{r}(t)) = \langle 0, 2\cos t\rangle$. Hence

$$\begin{aligned}\oint_{\mathcal{C}} \mathbf{F}\cdot\, d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt \\ &= \int_0^{2\pi} 2\cos^2 t\, dt > 0\end{aligned}$$

This is strictly greater than zero since the function $\cos^2 t$ is positive between 0 and 2π. Thus the integral cannot be zero and hence $\mathbf{F}$ cannot be conservative.

(b) (1 point) Give a domain on which $\mathbf{F}$ is conservative.

Solution: Since the curl is zero $\mathbf{F}$ will be conservative on any simply connected domain. For example take the domain to be all points (x, y) where $x > 0$.

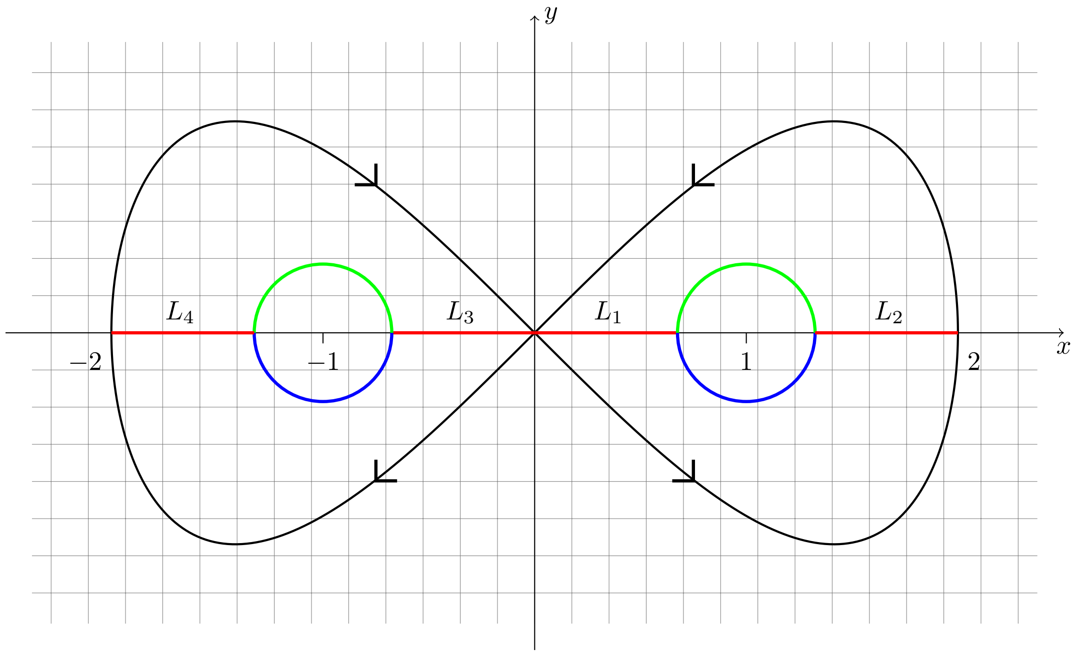
(c) (2 points) Calculate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where $\mathcal{C}$ is the ellipse $\frac{(x-4)^2}{2} + y^2 = 1$, oriented in the counter clockwise direction.

Solution: Note that $\mathbb{C}$ lies in a simply connected domain (it does not travel around the origin!), thus since it is a closed loop, the integral must be zero!

4. In this question assume that $\mathbf{E}$ is a vector field defined on the whole plane, apart from the points $(\pm 1, 0)$. Suppose that $\nabla \times \mathbf{E} = 0$. The function $\mathbf{r}(t) = (2\cos t, \sin 2t)$ for $t \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ defines the curve $\mathcal{C}$ on the graph below



(a) (1 point) Indicate on the above graph, the orientation of the curve.

(b) (4 points) Let $\mathcal{A}$ and $\mathcal{B}$ be the circles, radius $\frac{1}{2}$, and center $(1, 0)$ and $(-1, 0)$ respectively, both oriented counter clockwise. Suppose that

$$\int_{\mathcal{A}} \mathbf{E} \cdot d\mathbf{r} = 2 \quad \text{and} \quad \int_{\mathcal{B}} \mathbf{E} \cdot d\mathbf{r} = 1.$$

What is $\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r}$? Justify your answer.

Solution: We will calculate the integral in 4 parts. Since the curl of $\mathbf{E}$ is zero, it is conservative on any simply connected domain and thus its integral will be zero. We will use the 4 quadrants as out simply connected domains. Let us split $\mathcal{C}$ into 4 pieces $\mathcal{C}_{++}, \mathcal{C}_{-+}, \mathcal{C}_{--}, \mathcal{C}_{+-}$, so that $\mathcal{C}_{++}$ is the part of $\mathcal{C}$ in the quadrant where $x, y \geq 0$ and so on. We will use the same orientations as indicated on the picture. Thus

$$\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r} = \int_{\mathcal{C}_{++}} \mathbf{E} \cdot d\mathbf{r} + \int_{\mathcal{C}_{-+}} \mathbf{E} \cdot d\mathbf{r} + \int_{\mathcal{C}_{--}} \mathbf{E} \cdot d\mathbf{r} + \int_{\mathcal{C}_{+-}} \mathbf{E} \cdot d\mathbf{r}$$

Now lets concentrate on the first quadrant. Since $\mathbf{E}$ is conservative, its integral is path independent and we can calculate it along any path that stays inside this domain. So lets define some paths that will help.

Let $\mathcal{A}_+$ and $\mathcal{A}_-$ be the top and bottom halves of $\mathcal{A}$ respectively, both oriented counter clockwise. Similarly for $\mathcal{B}_+$ and $\mathcal{B}_-$. We also define the lines L_1, L_2, L_3, L_4 as drawn on the picture and oriented to the right.

This means $\mathcal{C}_{++} = -L_2 + \mathcal{A}_+ - L_1$ (notice we have been careful with the orientations!). Thus

$$\int_{\mathcal{C}_{++}} \mathbf{E} \cdot d\mathbf{r} = -\int_{L_2} \mathbf{E} \cdot d\mathbf{r} + \int_{\mathcal{A}_+} \mathbf{E} \cdot d\mathbf{r} - \int_{L_1} \mathbf{E} \cdot d\mathbf{r}.$$

We can make similar calculations for the other parts of $\mathcal{C}$. In the end, we get

$$\begin{aligned}\int_{\mathcal{C}} &= -\int_{L_2} + \int_{\mathcal{A}_+} - \int_{L_1} + \int_{L_4} - \int_{\mathcal{B}_+} + \int_{L_3} - \int_{L_3} - \int_{\mathcal{B}_-} - \int_{L_4} + \int_{L_1} + \int_{\mathcal{A}_-} + \int_{L_2} \\ &= \int_{\mathcal{A}_+} + \int_{\mathcal{A}_-} - \int_{\mathcal{B}_+} - \int_{\mathcal{B}_-} \\ &= \int_{\mathcal{A}} - \int_{\mathcal{B}} = 2 - 1 = 1.\end{aligned}$$

5. The *hyperboloid* is Noah's favorite surface. It is given by the equation $x^2 + y^2 - z^2 = 1$.

   (a) (3 points) Find a parameterisation

   $$G(s, \theta) = (x(s, \theta), y(s, \theta), z(s, \theta)) \quad (s, \theta) \in \mathbb{R} \times [0, 2\pi]$$

   for the hyperboloid. *Hint: Let $z = s$.*

   Solution: If we follow the hint and let $z = s$, and imagine fixing this value, then we are looking to parameterise the points on $x^2 + y^2 - s^2 = 1$, i.e. $x^2 + y^2 = 1 + s^2$. This is not so hard to do in polar coordinates, and we get

   $$G(s, \theta) = (\sqrt{1 + s^2}\cos\theta, \sqrt{1 + s^2}\sin\theta, s)$$

   (b) (5 points) Find an expression for the normal vector to the surface. Indicate whether your normal vector is pointing towards the inside or outside of the surface.

   Solution: First we find the tangent vectors

   $$\begin{aligned} T_s &= \left\langle \frac{s\cos\theta}{\sqrt{1+s^2}}, \frac{s\sin\theta}{\sqrt{1+s^2}}, 1 \right\rangle \\ T_\theta &= \left\langle -\sqrt{1+s^2}\sin\theta, \sqrt{1+s^2}\cos\theta, 0 \right\rangle \end{aligned}$$

   Then

   $$N = T_s \times T_\theta = \left\langle -\sqrt{1+s^2}\cos\theta, -\sqrt{1+s^2}\sin\theta, s \right\rangle$$

   Since the z-direction is positive when z is positive and negative when z is negative, it is pointing inside.

This page has been left intentionally blank. You may use it as scratch paper. It will not be graded unless indicated very clearly here and next to the relevant question.