Math 3B: Lecture 18

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November 13, 2017

Checking solutions

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$$\int \frac{1}{f(y)}\,\mathrm{d}y = \int g(x)\,\mathrm{d}x.$$
4. solve for y!

Examples

On the board. . .

Linear models

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Examples

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\lambda y.$$

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- concentration of a drug in bloodstream
- pollutant in water supply

General solution

Using separation of variables, we can show that the general solution to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a - by$$

is

$$y(t) = \frac{a}{b} - Ce^{-bt}$$

where C is an arbitrary constant.

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$$M\left(\frac{1}{2}\right)^{t/2} = Me^{-0.5t\ln(2)} \text{ mg left}$$

- Thus the rate at which the drug is leaving (at time t) is given by

$$0.5\ln(2)Me^{-0.5t\ln(2)} = 0.5\ln(2)(\text{current concentration}) \;\text{ mg/h}.$$

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- Thus at time t the concentration is

$$y(t) = 28.9 - 28.9e^{-0.3t} = 28.9\left(1 - e^{-0.3t}\right)$$

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$$\frac{\mathrm{d}T}{\mathrm{d}t} = k(A - T)$$

General solution

$$T(t) = A - Ce^{-kt}.$$

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An object takes 20 minutes to cool from 90° to 86° in a room which is 70°. At what time will it be 75°?

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- We know $T(0) = 90$ and $T(20) = 86$.
- Thus

$$90 = 70 - C \quad \text{so} \quad C = -20.$$

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- Thus

$$e^{-20k} = \frac{86-70}{20} = \frac{4}{5} \quad \text{so} \quad k = -\frac{1}{20}\ln\left(\frac{4}{5}\right) \approx -0.01.$$

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- Rearranging we get $20e^{-0.01t} = 5$ i.e.

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- Applying a logarithm

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- So we get

$$t = -100\ln\left(\frac{1}{4}\right) \approx 138 = 2 \text{ hours } 18 \text{ minutes.}$$