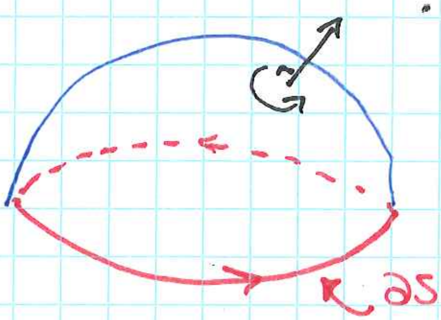
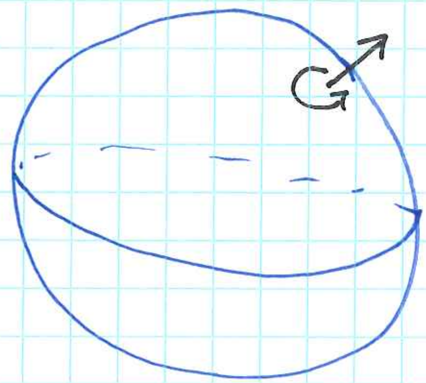
Lecture 22 + 23

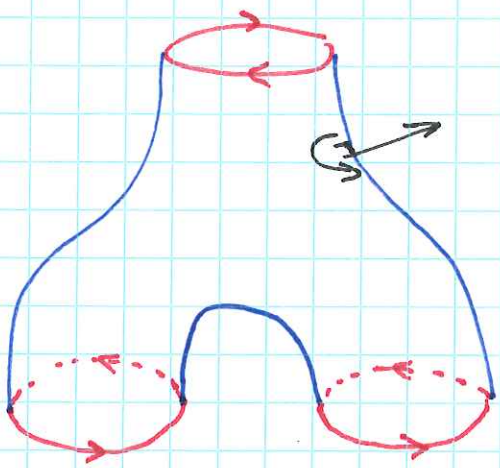
Surfaces with boundary

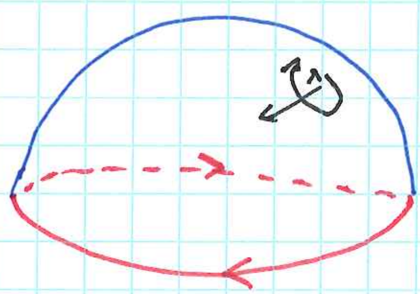
- Instead of defining rigorously, we will draw some pictures





$\partial S = \phi$





- Our surfaces are oriented and give ∂S the "boundary orientation".

Thm (Stoke's theorem)

If S is an oriented surface with piecewise smooth, simple boundary ∂S then

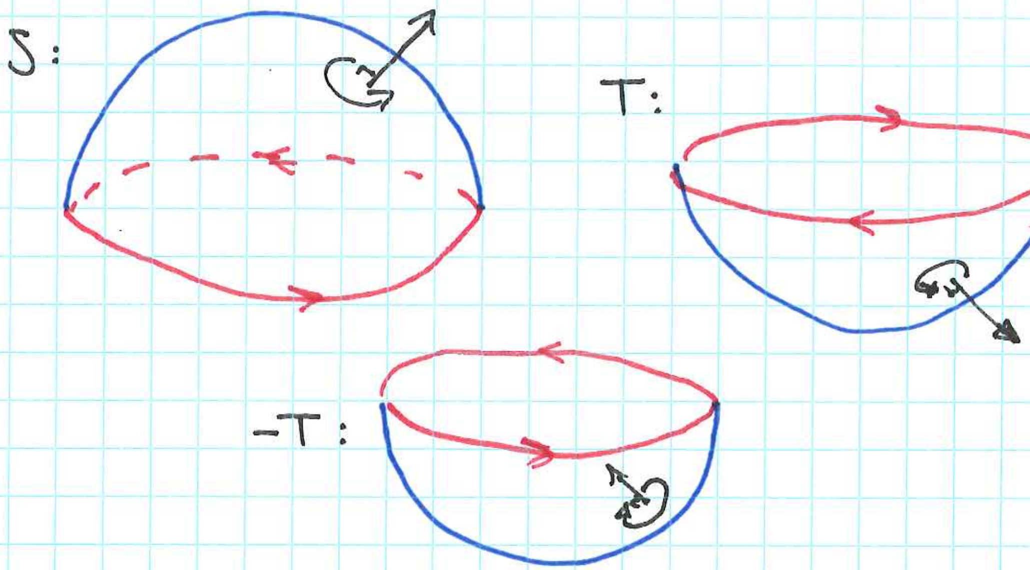
$$\oint_{\partial S} \underline{F} \cdot d\underline{r} = \iint_S \operatorname{curl}(\underline{F}) \cdot d\underline{S}$$

Surface independance

- Suppose $\underline{F} = \operatorname{curl}(\underline{E})$. We should think of curl as a derivative, so $\underline{F}$ "has an antiderivative" called its vector potential $\underline{E}$
- ~~Stoke~~ Suppose we have two oriented surfaces S, T. We use $\partial S = \partial T$ to mean that they share a common boundary and that their boundary orientations ~~to~~ match.
- E.g. If S = the top hemisphere of the sphere $x^2+y^2+z^2=1$ and T = bottom ~~the~~ hemisphere, both with orientation provided by outwards pointing normals. Then

$$\partial S \neq \partial T \quad \text{but} \quad \partial S = \partial(-T)$$

where $-T$ is the surface with opposite orientation. Indeed:



<u>Thm</u> If $\underline{F}$ has a vector potential and $\partial S = \partial T$ then

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_T \underline{F} \cdot d\underline{S}$$

i.e. The integral is independant of the surface and depends only on the boundary.

<u>proof</u>: since $\underline{F} = \text{curl}(\underline{E})$

$$\iint_S \underline{F} \cdot d\underline{S} = \iint_S \text{curl}(\underline{E}) d\underline{S}$$

$$= \oint_{\partial S} \underline{E} \cdot d\underline{r} \quad \text{(by Stokes' thm)}$$

$$= \oint_{\partial T} \underline{E} \cdot d\underline{r} \quad (\text{since } \partial S = \partial T)$$

$$= \iint_T \text{curl}(\underline{E}) d\underline{S} \quad \text{(by Stokes thm)}$$

$$= \iint_T \underline{F} \cdot d\underline{S}$$

$\boxtimes$

Rmk - This should be thought of as a higher dimensional analogue of path ind. for conservative vector fields

- A vector field with a vector potential should be thought of as an analogue of conservative vector fields.

- For conservative vector fields we had the statement that conservative $\Longrightarrow$ curl $= 0$ (or ~~not~~ curl $\neq 0 \Longrightarrow$ not conservative)

Prop If $\underline{F}$ has a vector potential, then $\operatorname{div}(\underline{F}) = 0$

proof: easy calculation $\nabla \cdot (\nabla \times \underline{F}) = 0$

Rmk - One way to use this: if $\operatorname{div}(\underline{F}) \neq 0$ then $\underline{F}$ cannot have a vector potential.

- The result "$\underline{F}$ is conservative on a simply connected domain if and only if $\operatorname{curl}(\underline{F}) = 0$" does not generalise in a straight forward way. E.g. $\mathbb{R}^3 \setminus \{(0,0,0)\}$ is simply connected ~~to~~ and

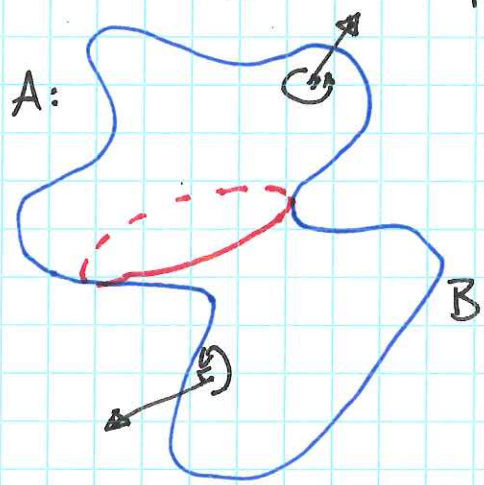
$$\operatorname{div}\left(\frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3}\right) = 0 \qquad r = \sqrt{x^2+y^2+z^2}$$

but this ~~is~~ does not have a vector potential (Exercise: show this!).

Cor Suppose S is a closed surface ($\partial S = \phi$) and $\underline{F}$ has a vector potential, then

$$\iint_S \underline{F} \cdot d\underline{S} = 0$$

proof Split S into two pieces A, B:



Notice that $\partial A = \partial(-B)$ surface w/ opp. orientation

By surface independance

$$\iint_A \underline{F} \cdot d\underline{S} = \iint_{-B} \underline{F} \cdot d\underline{S} = -\iint_B \underline{F} \cdot d\underline{S}$$

But
$$\iint_S \underline{F} \cdot d\underline{S} = \iint_A \underline{F} \cdot d\underline{S} + \iint_B \underline{F}\, d\underline{S}$$
$$= \iint_A \underline{F}\, d\underline{S} - \iint_{-B} \underline{F}\, d\underline{S} = 0$$

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