This week on the problem set we are practice finding maxima and minima of functions. The first two questions you should be able to master quickly. Questions 3-9 are more difficult and require thought. Especially challenging questions are indicated with an asterisk, these may not be appropriate for exams but are good practice nonetheless.

You will only need to hand in a small selection of the questions for homework, however I recommend that you at least attempt them all by the end of the quarter as some may appear on exams!

Homework: The first homework will be due on Friday 13 October, at 8am, the *start* of the lecture. It will consist of questions:

<p align="center">5, 8 and 9</p>

*Numbers in parentheses indicate the question has been taken from the textbook:

<p align="center">S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,</p>

and refer to the section and question number in the textbook.

1. (4.2) Find the critical points and classify them (as global/local maxima/minima or neither) using the *first derivative test*.

   (a) (4.2-6) $f(x) = 10 + 6x - x^2$

   (b) (4.2-7) $f(t) = t^2 e^{-t}$

   (c) (4.2-9) $f(x) = \frac{x}{1-x}$

   (d) (4.2-12) $y = e^{t^2 - 2t + 1}$

2. (4.2) Find the critical points and classify them (as global/local maxima/minima or neither) using the *second derivative test*.

   (a) (4.2-14) $y = 1 - \exp\left(-x^2\right)$

   (b) (4.2-15) $y = x + \frac{1}{2+x}$

   (c) (4.2-16) $y = \frac{2x^2 - x^4}{4}$

3. (4.2-35) In Problem 30 of 4.1 (see problem 9 on Problem Set 1), we saw that the weekly mortality rate during the outbreak of the Black Plague in Bombay (1905-1906) can be reasonably well described by the function

$$f(t) = 890 \operatorname{sech}^2(0.2t - 3.4) \quad \text{deaths/week}$$

   where t is measured in weeks. Find the global maximum of this function. Recall that

$$\operatorname{sech}(x) = \frac{2}{e^x + e^{-x}}.$$

4. (4.2-36,*) A particular species of plant (for example, bamboo) flowers once and then dies. A well-known formula for the average growth rate r of a semelparous species (a species that breeds only once) that breeds at age x is

$$r(x) = \frac{\ln\left[s(x)n(x)p\right]}{x}$$

   where $s(x)$ represents the proportion of plants that survive from germination to age x, $n(x)$ is the number of seeds produced at age x, and p is the proportion of seeds that germinate.

   (a) Find the age of reproduction that maximizes r in terms of the parameters a, b, c and p where

$$s(x) = e^{-ax} \quad a > 0$$

   and

$$n(x) = bx^c \quad b > 0$$

   $0 < c < 1$.

(b) Sketch the graph of $y = r(x)$ for the case where $a = 0.2, b = 3, c = 0.8$, and $p = 0.5$.

5. (4.2-40) An epidemic spreads through a community in a such a say that t weeks after its outbreak, the number of residents who have been infected is given by a function of the form

$$f(t) = \frac{A}{1 + Ce^{kt}}$$

where A is the total number of susceptible residents. Show that the epidemic is spreading most rapidly when half the susceptible residents have been infected.

(Hint: Read the question carefully, which function should you be maximising/minimising? Is it f or a function related to f? The key word is "spreading").

Solution: We would like to know when the infection is *spreading* most rapidly. Another way to say this is that we would like to know when the infection *rate* is at its maximum, i.e. we would like to find the maximum of $f'(t)$. First we find the relevant derivatives. Note that $\frac{d}{dt}Ce^{kt} = kCe^{kt}$, so using the chain rule,

$$
\begin{aligned}
f'(t) &= -\frac{A}{(1 + Ce^{kt})^2} \cdot kCe^{kt} \\
&= -\frac{kACe^{kt}}{(1 + Ce^{kt})^2},
\end{aligned}
$$

and using the product rule,

$$
\begin{aligned}
f''(t) &= -\frac{k^2 ACe^{kt}}{(1 + Ce^{kt})^2} + \frac{2k^2 AC^2 e^{2kt}}{(1 + Ce^{kt})^3} \\
&= \frac{-k^2 ACe^{kt} - k^2 AC^2 e^{2kt} + 2k^2 AC^2 e^{2kt}}{(1 + Ce^{kt})^3} \\
&= \frac{-k^2 ACe^{kt} + k^2 AC^2 e^{2kt}}{(1 + Ce^{kt})^3} \\
&= \frac{k^2 ACe^{kt}(Ce^{kt} - 1)}{(1 + Ce^{kt})^3}.
\end{aligned}
$$

The critical points of $f'(t)$ occur when $f''(t) = 0$ which is precisely when $Ce^{kt} - 1 = 0$, i.e. when $e^{kt} = C^{-1}$. This means the only critical points is at $t = -k^{-1}\ln C$. Using the first derivative test we easily see this is a maximum.

Now consider

$$
\begin{aligned}
f(-k^{-1}\ln C) &= \frac{A}{1 + Ce^{k(-k^{-1})\ln C}} \\
&= \frac{A}{1 + Ce^{-\ln C}} \\
&= \frac{A}{1 + C \cdot C^{-1}} \\
&= \frac{A}{2}.
\end{aligned}
$$

Thus when half the susceptible residents are infected, the epidemic is spreading most rapidly.

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6. (4.3-24) In a species of fish, the growth rate function is given by $G(x) = 1.5x(1 - x/K)$ where $K = 6000$ metric tons (i.e. the population of fish, x, is measured in metric tons rather than number of individuals).

The price a fisherman can get is $p = \$600$ per metric ton. If the amount the fisherman can harvest is determined by the function $H = hx$, where teach unit of h costs the fisherman $c = \$100$, what is the maximum amount of money the fisherman can expect to make on a sustainable basis. (Hint: The fishermans sustainable income is given by $pH - ch$ where H is a sustainable harvesting rate).

(Further hint: read Example 2 in 4.2 and the preceding two paragraphs.)

7. (4.3-32) During the winter, a species of bird migrates from the coast of a mainland to an island 500 miles southeast. If the energy the bird requires to fly one mile over the water is twice more than the amount of energy it requires to fly over the land, determine what path the species should fly to minimize the amount of energy used.

   (Hint: there is an ambiguity in this question which the text book does not clarify. You should assume that the shoreline is a straight line and runs either north-south or east-west - either choice results in the same answer.)

8. (*) You are a Biologist and you are studying the population of a species of fish that has just been reintroduced to a local river system. You believe the number of fish is increasing linearly over time. That is, you think the number of fish $P(t)$ at time t is of the form

$$P(t) = mt + I$$

for some number m and where I is the initial number of fish introduced. To test this theory you make n measurements.

t	$P(t)$
t_1	p_1
t_2	p_2
$\vdots$	$\vdots$
t_n	p_n

Your statistician friend tells you that the error between your hypothesised function $P(t) = mt + I$ and the data you actually measured is given by

$$E(m) = \sum_{i=1}^{n} (p_i - mt_i - I)^2 .$$

(a) What m should you pick to best model the population of fish? Your answer should be given in terms of the t_i's, p_i's and I.

> **Solution:** The best m to choose is the one which minimises the error between the line and the data points, that is, we want to minimise $E(m)$. To do this first find the critical points. We have, using the chain rule,
>
> $$\begin{aligned} E'(m) &= \sum_{i=1}^{n} \frac{\mathrm{d}}{\mathrm{d}m}(p_i - mt_i - I)^2 \\ &= \sum_{i=1}^{n} -2t_i(p_i - mt_i - I) \\ &= -2\sum_{i=1}^{n} t_i(p_i - I) + 2m\sum_{i=1}^{n} t_i^2 . \end{aligned}$$

Setting this equal to zero we get

$$-2\sum_{i=1}^{n} t_i(p_i - I) + 2m\sum_{i=1}^{n} t_i^2 = 0$$

$$2m\sum_{i=1}^{n} t_i^2 = 2\sum_{i=1}^{n} t_i(p_i - I)$$

$$m = \frac{\sum_{i=1}^{n} t_i(p_i - I)}{\sum_{i=1}^{n} t_i^2}.$$

We can check this is a minimum by finding the second derivative:

$$E''(m) = 2\sum_{i=1}^{n} t_i^2.$$

Since this is always positive (and since $E(m)$ is a parabola), we should pick

$$m = \frac{\sum_{i=1}^{n} t_i(p_i - I)}{\sum_{i=1}^{n} t_i^2}$$

which is a global minimum of $E(m)$.

(b) Suppose the actual data you observe is (with an initial population of 100 fish)
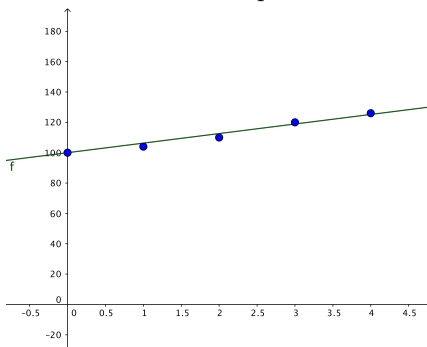
t	$P(t)$
0	100
1	104
2	110
3	120
4	126

Sketch the graph of $P(t) = mt + I$ with your chosen m from above and also plot the above data points.

Solution: From the data we see that $I = 100$ and m is given by putting the data into the formula given above:
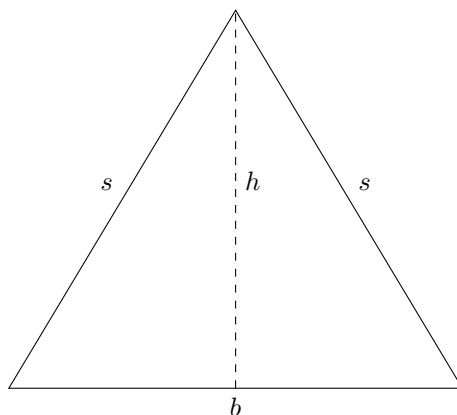
$$m = \frac{0 \cdot (100 - 100) + 1 \cdot (104 - 100) + 2 \cdot (110 - 100) + 3 \cdot (120 - 100) + 4 \cdot (126 - 100)}{0^2 + 1^2 + 3^2 + 4^2}$$

$$= \frac{188}{30} = \frac{94}{15} \approx 6.3$$

We graph the line $P(t) = 6.3t + 100$ and the data points:

9. What is the largest area you can enclose with an isosceles triangle with perimeter p?

Solution: The diagram for the problem should look like

We want to maximise the function $A = \frac{1}{2}bh$. We have two constraints on the problem. The first is that

$$p = 2s + b$$

and the second is that we can use Pythagoras' theorem to express the height in terms of the base and side length,

$$h = \sqrt{s^2 - \frac{b^2}{4}}.$$

Thus we can use the perimeter to express the height purely as a function of the base,

$$h = \sqrt{\frac{(p-b)^2}{4} - \frac{b^2}{4}} = \frac{1}{2}\sqrt{p^2 - 2pb}.$$

Substituting this into the formula for the area we obtain a function,

$$A(b) = \frac{1}{4}b\sqrt{p^2 - 2pb}$$

that we would like to maximise. First we find the derivative using the product rule.

$$
\begin{aligned}
A'(b) &= \frac{1}{4}\sqrt{p^2 - 2pb} - \frac{1}{4}bp(p^2 - 2pb)^{-\frac{1}{2}} \\
&= \frac{p^2 - 2pb - bp}{4\sqrt{p^2 - 2pb}} \\
&= \frac{p^2 - 3pb}{4\sqrt{p^2 - 2pb}}.
\end{aligned}
$$

The critical points occur when $A'(b) = 0$, i.e. when $p^2 = 3pb$, that is when $b = p/3$. The first derivative test tells us this must be a maximum.

When $b = p/3$ the perimeter formula tells us that $s = b = p/3$ and the height is

$$h = \sqrt{\frac{p^2}{9} - \frac{p^2}{36}} = \frac{p\sqrt{3}}{6}$$

So the largest area we can enclose is given by

$$A = \frac{1}{2} \cdot \frac{p}{3} \cdot \frac{p\sqrt{3}}{6} = \frac{p^2\sqrt{3}}{36}.$$

10. What point (x, y) on the parabola $y = x^2$ is closest to the point $(16, 0.5)$?