This week on the problem set you will get practice thinking about potential functions and calculating line integrals.

Homework: The homework will be due on Monday 25 November. It will consist of questions:

17.4.41 and 17.5.22.

*Numbers in parentheses indicate the question has been taken from the textbook:

J. Rogawski, C. Adams, *Calculus, Multivariable*, 3rd Ed., W. H. Freeman & Company,

and refer to the section and question number in the textbook.

1. (Section 17.4) 2, 3, 5, 8, 9, 10, 13, 14, 17, 18, 27, 30, 34, 37, 40, 41*, 46* 48*. (questions are the same in previous versions)

2. (Section 17.5) 1, 6, 7, 12, 17, 18, 21, 22, 31*, 35. (questions are the same in previous versions)

3. Consider the line segment $(x, 0, 0)$ where $x \in [-1, 1]$ in $\mathbb{R}^3$. Imagine this line segment moving up with its centre on the z-axis, rotating parallel to the xy-plane at constant speed. It completes one full revolution when it gets to $z = 2\pi$. What surface area is swept out by the rotating line segment? You may wish to use the fact that
$$\frac{d}{dt}\left(t\sqrt{1+t^2} + \sinh^{-1}(t)\right) = 2\sqrt{1+t^2}$$
and that $\sinh^{-1}$ is an odd function and $\sinh(1) = \ln(1 + \sqrt{2})$.

Solution: We parameterise the surface using the following strategry. Say that at time t the line segment is at height $z = t$, so $G(s, t) = (?, ?, t)$. At $z = t$ for $t \in [0, 2\pi]$ we know that out line segment as rotated t radians from it's starting point. So it's projection onto the xy-plane is $(s\cos t, s\sin t)$. Thus our parameterisation is

$$G(s, t) = (s\cos t, s\sin t, t) \text{ for } (s, t) \in \mathcal{D} = [-1, 1] \times [0, 2\pi].$$

From this we calculate

$$\begin{aligned}
\mathbf{T}_s &= \langle \cos t, \sin t, 0\rangle \\
\mathbf{T}_t &= \langle -s\sin t, s\cos t, 1\rangle \\
\mathbf{N} &= \langle \sin t, -\cos t, s\rangle
\end{aligned}$$

and so

$$\|\mathbf{N}\| = \sqrt{1+s^2}$$

Thus the surface area is

$$\begin{aligned}
\iint_{\mathcal{S}} 1\, dS &= \iint_{\mathcal{D}} \sqrt{1+s^2}\, dA_{st} \\
&= \int_0^{2\pi} \int_{-1}^{1} \sqrt{1+s^2}\, ds\, dt \\
&= \pi \left[s\sqrt{1+s^2} + \sinh^{-1} s\right]_{-1}^{1} \\
&= 2\pi\left(\sqrt{2} + \sinh^{-1}(1)\right) = 2\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right)
\end{aligned}$$

4. Let $\mathbf{F}\left\langle y(ye^{x+y^2}-1)+x^2, 2y(1+y^2)e^{x+y^2}+x\right\rangle$ and let $\mathcal{C}$ be the portion of $y=1-x^2$ oriented left to right.

   (a) Parameterise $\mathcal{C}$ and write $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ as a single integral. Do not try and evaluate.

   Solution: A parameterisation is $\mathbf{r}=(t,1-t^2)$ for $t\in[-1,1]$ and so $\mathbf{r}'(t)=\langle 1,-2t\rangle$. The integral becomes

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} &= \int_{-1}^{1} (1-t^2)((1-t^2)e^{t^4-2t^2+t+1}-1)+t^2-2t\left(2(1-t^2)(2-2t^2+t^4)e^{t^4-2t^2+t+1}+t\right)\,dt \\ &= \int_{-1}^{1} (4t^7-12t^5+t^4+16t^3-2t^2-8t+1)e^{t^4-2t^2+t+1}-1\,dt\end{aligned}$$

   (b) Now let $\mathcal{L}$ be the straight line from $(-1,0)$ to $(1,0)$ oriented left to right and let $\mathcal{D}$ be the region bounded by the x-axis and $\mathcal{C}$. Use Green's theorem to relate the integrals of $\mathbf{F}$ over $\mathcal{C}$ and $\mathcal{L}$ to an integral over $\mathcal{D}$. Use this to evaluate $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$.

   Solution: By looking at the picture we see that $\mathcal{L}-\mathcal{C}$ is a closed curve that is the boundary of $\mathcal{D}$ (including orientation matching). Thus by Green's theorem

$$\int_{\mathcal{L}} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{L}-\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \iint_{\mathcal{D}} \nabla\times\mathbf{F}\,dA$$

   A calculation shows that $\nabla\times\mathbf{F}=2$. Thus

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{\mathcal{L}} \mathbf{F}\cdot d\mathbf{r} - \iint_{\mathcal{D}} 2\,dA$$

   The double integral is easy to evaluate:

$$\iint_{\mathcal{D}} 2\,dA = 2\int_{-1}^{1}\int_{0}^{1-x^2} dy\,dx = \frac{8}{3}.$$

   The curve $\mathcal{L}$ is parameterised by $\mathbf{r}(t)=(t,0)$ for $t\in[-1,1]$. So $\mathbf{r}'(t)=\langle 1,0\rangle$. So

$$\int_{\mathcal{L}} \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1} t^2\,dt = \frac{2}{3}.$$

   So we get $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \frac{2}{3}-\frac{8}{3}=-2$

   (c) **Path (almost)-independence for non-conservative vector fields.** More generally, suppose $\mathcal{C}_1$ and $\mathcal{C}_2$ are two oriented curves with the same endpoints, and $\mathbf{F}$ is a vector field that is defined everywhere on the region $\mathcal{D}$ between $\mathcal{C}_1$ and $\mathcal{C}_2$. If $\mathbf{F}$ is conservative we know that $\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = 0$. If $\mathbf{F}$ is not conservative, what is

$$\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r}?$$

Solution: One of $\mathcal{C}_1 - \mathcal{C}_2$ or $\mathcal{C}_2 - \mathcal{C}_1$ will be the boundary of $\mathcal{D}$. Suppose it is $\mathcal{C}_1 - \mathcal{C}_2$. Then by Green's theorem

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_1 - \mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \nabla \times \mathbf{F} \, dA.$$

*The questions marked with an asterisk are more difficult or are of a form that would not appear on an exam. Nonetheless they are worth thinking about as they often test understanding at a deeper conceptual level.