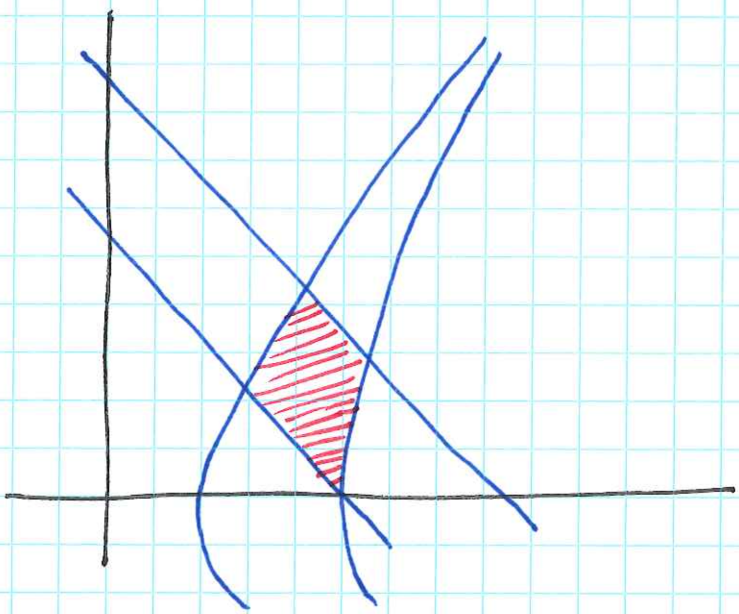
Lecture 7

1. Motivation

- Suppose we have some complicated region D

eg. $x^2 - y^2 \geq 1$ $\quad$ $y + x \geq 2$

$x^2 - y^2 \leq 4$ $\quad$ $y + x \leq 4$



- Suppose we let $u = x^2 - y^2$ $\quad v = y + x$ then

$$(u\ v) \in [1\ 4] \times [2\ 4] = R,$$

a rectangle!

- Aim: Write $\iint_D f(x\ y)\, dA$ in the form

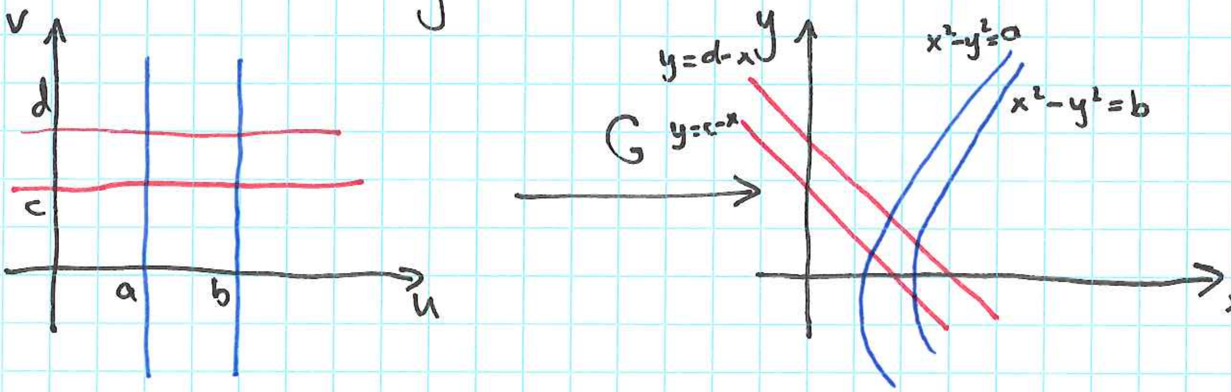
$$\int \int \ldots ? \ldots \, d \quad d$$

- Let $G(u\ v) = (x(u,v),\ y(u\ v))$ where

$$x(u\ v) = \tfrac{1}{2}v + \tfrac{1}{2}uv^{-1}$$

$$y(u\ v) = \tfrac{1}{2}v - \tfrac{1}{2}uv^{-1}$$

- This defines a map from the uv-plane to the xy plane mapping the line $u = a$ to the ~~line~~ curve $x^2 - y^2 = a$, the line $v = b$ to the ~~line~~ ~~curve~~ line $y = b - x$.



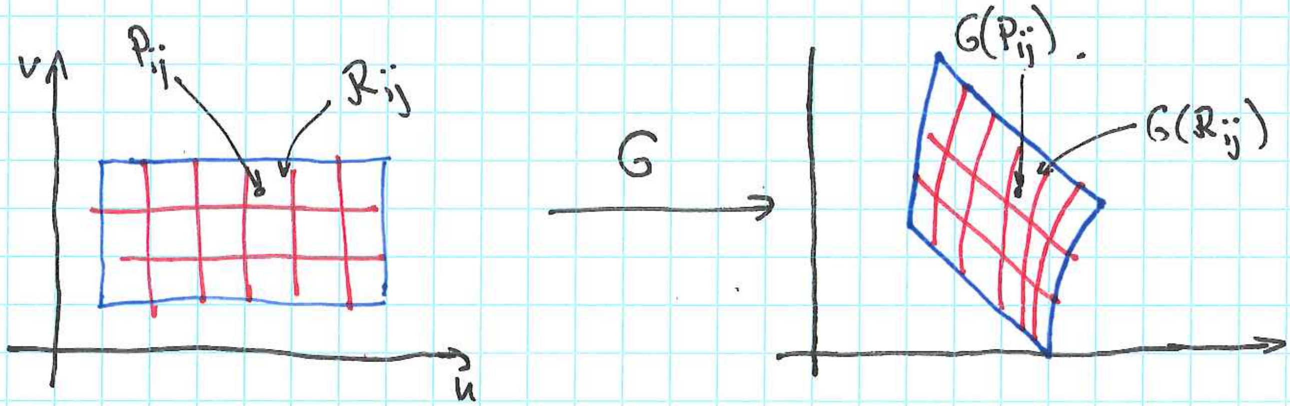
- In particular, it maps $\mathcal{R}$ to $\mathcal{D}$!
  that is, $G(\mathcal{R}) = \mathcal{D}$.
- ~~To~~ To find an iterated integral as above, we try to construct a Riemann sum:
  * partition $[1\ 4]$ and $[2\ 4]$
    (or more generally $[a,b] \times [c,d]$ in uv-plane)

    $$a = u_0 < u_1 < \dots < u_m = b$$

    $$c = v_0 < v_1 < \dots < v_n = d$$

  * Let $\mathcal{R}_{ij} = [u_{i-1}\ u_i] \times [v_{j-1}\ v_j]$ and $\Delta u_i = u_i - u_{i-1}$
    $\Delta v_j = v_j - v_{j-1}$
  * Choose $P_{ij} \in \mathcal{R}_{ij}$, call the data the partition $\mathcal{P}$

- This subdivides $\mathcal{R}$ into subrectangles $\mathcal{R}_{ij}$
But also subdivides $\mathcal{D}$ into subregions $G(\mathcal{R}_{ij})$:



- Thus we have a Riemann sum:

$$\iint_{\mathcal{D}} f(x\,y)\,dA = \lim_{\|P\|\to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f\big(G(P_{ij})\big)\cdot \text{Area}\big(G(\mathcal{R}_{ij})\big)$$

- To get any further, we need an expression for

$$\text{Area}\,(G(\mathcal{R}_{ij})).$$

2. Area + the Jacobian

- First we consider linear changes of variable

$$G(u,v) = (Au+Cv,\ Bu+Dv)$$

$A, B, C, D \in \mathbb{R}$

Note: We call such a mapping G, a change of variable mapping

- We want to answer, given a rectangle

$$\mathcal{R} = [a\ b] \times [c\ d]$$

What ~~is~~ is the area of $G(\mathcal{R})$?

- First we determine the shape of $G(\mathcal{R})$
- The line $u = \alpha$ is mapped to:

$$(\alpha, v) \xmapsto{G} (A\alpha + Cv, B\alpha + Dv).$$

i.e. the line

$$y = \frac{D}{C}(x - A\alpha) + B\alpha$$

- The line $v = \beta$ is mapped to

$$(u, \beta) \xmapsto{G} (Au + C\beta, Bu + D\beta)$$

i.e. the line

$$y = \frac{B}{A}(x - C\beta) + D\beta$$

- Thus, the image of R under G is a parallelogram
- $\mathcal{R} = \{ (a,c) + s(b-a, 0) + t(0, d-c) \mid s,t \in [0,1] \}$.

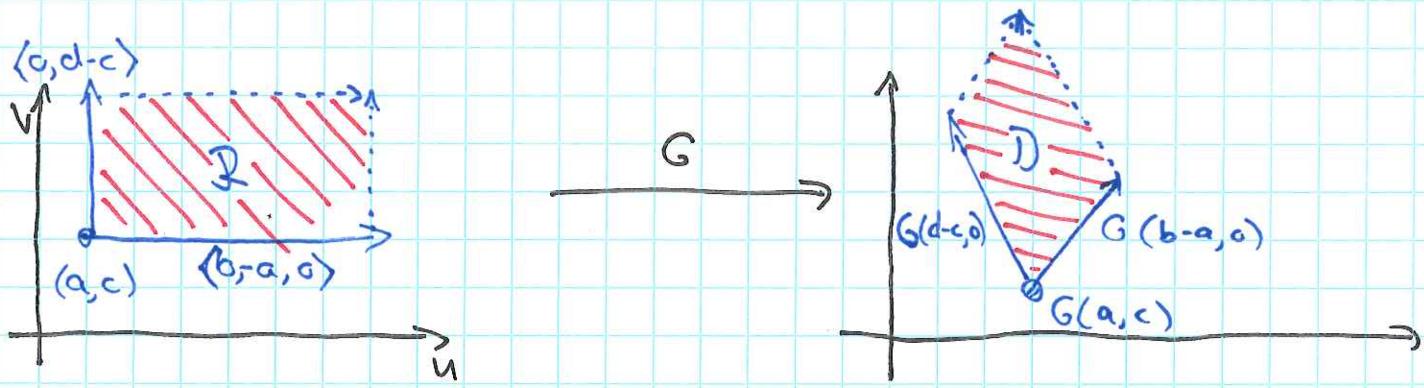
- Thus

$$G(\mathcal{R}) = \{ G(a,c) + sG(b-a,0) + tG(0,d-c) \mid s,t \in [0,1] \}$$

Which is a parallelogram spanned by the vectors

$$G(b-a,0) = (A(b-a), B(b-a))$$

$$G(0,d-c) = (C(d-c), D(d-c))$$



Exercise The parallelogram spanned by vectors

$$\langle \alpha, \beta \rangle, \langle \gamma, \delta \rangle$$

has area $\alpha\delta - \beta\gamma$.

- Thus the area of $G(R)$ is

$$\text{Area}(G(R)) = AD(b-a)(d-c) - BC(b-a)(d-c)$$
$$= (AD - BC)\,\text{Area}(R).$$

3. The Jacobian

Def For a function $G(u,v) = (x(u,v), y(u,v))$ the Jacobian is the function

$$J(G) = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

$$= \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}.$$

Ex For a linear function

$$G(u\,v) = (Au + Cv\,,\, Bu + Dv)$$

$J(G) = AD - BC$.

- So for linear function

$$\text{Area}(G(R)) = J(G)\cdot \text{Area}(R).$$

- In fact for linear functions, and ~~at~~ arbitrary regions

$$\text{Area}(G(D)) = J(G)\,\text{Area}(D).$$

(see this by approximating D using rectangles.)

- In general:

Thm Given any point $P \in D$

$$\lim_{\text{Area}(D)\to 0} \frac{\text{Area}(G(D))}{\text{Area}(D)} = J(G)(P).$$

Next time We work out an expression for

$$\iint_D f(x,y)\,dA = \iint_R f(G(u,v))\,?\,du\,dv$$