

This week on the problem set you will get practice thinking about potential functions and calculating line integrals.

**Homework:** The homework will be due on Wednesday 27 May. It will consist of questions 3, 4 and 5 below.

\*Numbers in parentheses indicate the question has been taken from the textbook:

J. Rogawski, C. Adams, *Calculus, Multivariable*, 3<sup>rd</sup> Ed., W. H. Freeman & Company,

and refer to the section and question number in the textbook.

1. (Section 17.4) 2, 3, 5, 8, 9, 10, 13, 14, 17, 18, 27, 30, 34, 37, 40, 41\*, 46\* 48\*. (questions are the same in previous versions)
2. (Section 17.5) 1, 6, 7, 12, 17, 18, 21, 22, 31\*, 35. (questions are the same in previous versions)
3. Consider the line segment  $(x, 0, 0)$  where  $x \in [-1, 1]$  in  $\mathbb{R}^3$ . Imagine this line segment moving up with its centre on the  $z$ -axis, rotating parallel to the  $xy$ -plane at constant speed. It completes one full revolution when it gets to  $z = 2\pi$ . What surface area is swept out by the rotating line segment? You may wish to use the fact that

$$\frac{d}{dt} \left( t\sqrt{1+t^2} + \sinh^{-1}(t) \right) = 2\sqrt{1+t^2}$$

and that  $\sinh^{-1} t$  is an odd function and  $\sinh^{-1}(1) = \ln(1 + \sqrt{2})$ .

4. The velocity vector field of a fluid is given by  $\mathbf{F}(x, y, z) = \frac{\langle x, y, z \rangle}{x^2 + y^2 + z^2}$  measured in meters per second. What is the volume of fluid flowing each second through the open cylinder of radius 1 and height 1 centered along the  $z$ -axis, i.e. the cylinder  $C = \{(x, y, z) : x^2 + y^2 = 1, 0 \leq z \leq 1\}$ , with outward orientation?
5. Let  $\mathbf{F} \left\langle y(ye^{x+y^2} - 1) + x^2, 2y(1 + y^2)e^{x+y^2} + x \right\rangle$  and let  $\mathcal{C}$  be the portion of  $y = 1 - x^2$  oriented left to right.

(a) Parameterise  $\mathcal{C}$  and write  $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$  as a single integral. Do not try and evaluate.

(b) Now let  $\mathcal{L}$  be the straight line from  $(-1, 0)$  to  $(1, 0)$  oriented left to right and let  $\mathcal{D}$  be the region bounded by the  $x$ -axis and  $\mathcal{C}$ . Use Green's theorem to relate the integrals of  $\mathbf{F}$  over  $\mathcal{C}$  and  $\mathcal{L}$  to an integral over  $\mathcal{D}$ . Use this to evaluate  $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ .

(c) **Path (almost)-independence for non-conservative vector fields.** More generally, suppose  $\mathcal{C}_1$  and  $\mathcal{C}_2$  are two oriented (nonintersecting) curves with the same endpoints, and  $\mathbf{F}$  is a vector field that is defined everywhere on the region  $\mathcal{D}$  between  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . If  $\mathbf{F}$  is conservative we know that  $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 0$ . If  $\mathbf{F}$  is not conservative, what is

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} - \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}?$$

\*The questions marked with an asterisk are more difficult or are of a form that would not appear on an exam. Nonetheless they are worth thinking about as they often test understanding at a deeper conceptual level.