Lecture 11-12

1. Vector fields

- A vector field is an assingment of a vector to every point on $\mathbb{R}^2$ (or $\mathbb{R}^3$).

- ie a vector field on $\mathbb{R}^2$ is a function

$$\underline{F} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

which assigns to $(x\ y)$ the vector

$$\underline{F}(x,y) = \langle F_1(x\ y), F_2(x\ y) \rangle$$

- Similarly a vector field on $\mathbb{R}^3$ is a $f^{\underline{n}}$

$$\underline{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

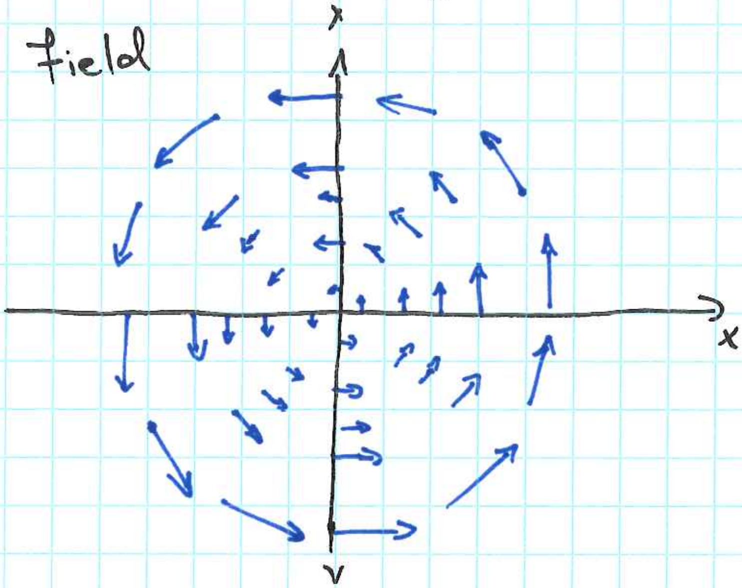
assigning to ~~ver every poi~~ the point $(x\ y\ z)$

$$\underline{F}(x\ y\ z) = \langle F_1(x y z), F_2(x\ y\ z), F_3(x\ y\ z) \rangle.$$

$$= F_1\,\underline{i} + F_2\,\underline{j} + F_3\,\underline{k}$$

- Vector fields model things like:
  - fluid velocity
  - gravitational/electric/magnetic fields

Ex The function $F(x\ y) = \langle -y, x \rangle$ defines the vector field



- We say a vector field $\underline{F}$ is a
  * unit vector field if $\|\underline{F}(P)\| = 1 \quad \forall P$
  * radial vector field if
    - $\|\underline{F}(P)\| = \|F(Q)\|$ whenever $\|P\| = \|Q\|$
    - $\dfrac{\underline{F}(P)}{\|\underline{F}(P)\|} = \pm \overrightarrow{OP}$ for all P.

Ex * $\underline{F}(x\ y) = \langle x, y \rangle$ is radial

* $\underline{F}(x\ y) = \langle -\cos(y), \sin(x) \rangle$ is unit.

2. Manipulating vector fields

- We define two important operations on vector fields
- Recall that

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle \quad \text{in } \mathbb{R}^2$$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \quad \text{in } \mathbb{R}^3$$

Def If $\underline{F}$ is a vector field then the divergence of $\underline{F}$ is

$$\operatorname{div}(\underline{F}) = \nabla \cdot \underline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

scalar product.

(similarly in $\mathbb{R}^2$).

Def If $\underline{F}$ is a vector field then the curl of $\underline{F}$ is

$$\operatorname{curl}(\underline{F}) = \nabla \times \underline{F} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

scalar product. (only 3D!).

Rmk ~~We can think of~~ Given a 2D vector field $\underline{F}(x\ y) = \langle F_1(x\ y), F_2(x\ y) \rangle$ we can construct a 3D vector field

$$\hat{\underline{F}}(x\ y\ z) = \langle F_1(x\ y), F_2(x\ y), 0 \rangle$$

i.e. $\cancel{\hat{F}_1}\ \cancel{\in F}$

$$\hat{F}_1(x\ y\ z) = F_1(x\ y)$$
$$\hat{F}_2(x\ y\ z) = F_2(x\ y)$$
$$\hat{F}_3(x\ y\ z) = 0$$

and if we restrict $\underline{\hat{F}}$ to the xy-plane, it coincides with $\underline{F}$.

Thus we can define

$$\mathrm{curl}(\underline{F}) = \mathrm{curl}(\underline{\hat{F}}) = \nabla \times \underline{\hat{F}}$$
$$= \left\langle 0,\ 0,\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle.$$

Ex $\quad \underline{F} = \langle xy,\ yz,\ xz \rangle$

$$\mathrm{div}(\underline{F}) = \nabla \cdot \underline{F} = y + z + x = \cancel{2x+y}.\ x + y + z.$$

$$\mathrm{curl}(\underline{F}) = \nabla \times \underline{F} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ xy & yz & xz \end{pmatrix}$$

$$= -y\,\underline{i} - (z-0)\,\underline{j} + (0-x)\,\underline{k}$$
$$= \langle -y,\ -z,\ -x \rangle.$$

Ex $\underline{F}(x\ y) = \langle -y, x \rangle$.

$\operatorname{div}(\underline{F}) = \nabla \cdot \underline{F} = 0 + 0 = 0$

$\operatorname{curl}(\underline{F}) = \nabla \times \langle F_1\ F_2, 0 \rangle$

$= \nabla \times \langle -y\ \cancel{y}\ x, 0 \rangle$

$= \partial_x x - \partial_y(-y) = 2.$

3. The gradient vector field

- We have an easy ~~wh~~ way to ~~t~~ produce vector fields:

given a function $f(x, y, z)$ we can look
scalar

of the gradient vector field

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle.$$

Def A vector field $\underline{F}$ is called conservative if we can find a function $f(x\ y\ z)$ s.t.

$$\underline{F} = \nabla f.$$

f is called the potential of $\underline{F}$

Rmk If $\underline{F}$ is conservative then

$$\underline{F} = \nabla f$$

for some $\cancel{E}$ f. Hence

$$\begin{aligned}
\text{curl}(\underline{F}) &= \nabla \times \nabla f \\
&= \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \cancel{\partial_x} f_x & f_y & f_z \end{pmatrix} \\
&= (f_{yz} - f_{yz})\underline{i} - (f_{zx} - f_{xz})\underline{j} + (f_{yx} - f_{xy})\underline{k} \\
&= 0
\end{aligned}$$

So a 3D vector field is conservative only if $\nabla \times \underline{F} = 0$.

- In 2D: If $\underline{F}$ is conservative then $\underline{F} = \nabla f = \langle f_x, f_y \rangle$.

Then

$$\begin{aligned}
\text{curl}(\underline{F}) &= \nabla \times \langle f_x, f_y, 0 \rangle \\
&= \det \begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & \cancel{f_z}\,0 \end{pmatrix} \\
&= (f_{xy} - f_{yx})\underline{k} = 0.
\end{aligned}$$

Thm A vector field $\underline{F}$ is conservative only if

$$\text{curl}(\underline{F}) = 0$$

(converse not true!).