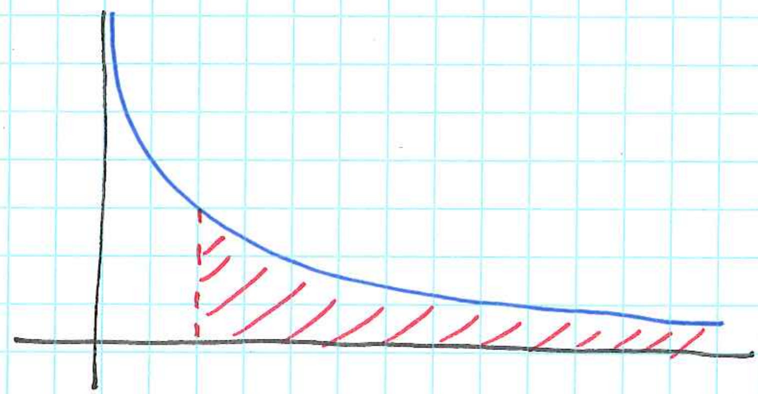
Improper integrals (infinities)

How do we integrate over infinite intervals?

E.g. the area



$$\int_1^\infty \frac{1}{x^2}\,dx \quad ?$$

Def

* $\displaystyle\int_a^\infty f(x)\,dx := \lim_{R\to\infty} \int_a^R f(x)\,dx$

* $\displaystyle\int_{-\infty}^a f(x)\,dx := \lim_{R\to-\infty} \int_R^a f(x)\,dx$

Ex $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx$

$$= \lim_{R\to\infty} -\frac{1}{R} + 1$$

$$= 1$$

Ex $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx = \lim_{R\to\infty} \int_1^R \frac{1}{x}\,dx$

$$= \lim_{R\to\infty} \ln x \Big|_1^R$$

$$= \lim_{R\to\infty} \ln R = \infty$$

Ex $\displaystyle\int_{-\infty}^{1} e^x\,dx = \lim_{R\to-\infty} \int_R^1 e^x\,dx$

$$= \lim_{R\to-\infty} e^x \Big|_R^1$$

$$= \lim_{R\to-\infty} e - e^R$$

$$= e$$

What about $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$

Def $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} f(x)\,dx$

Ex $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_0^{\infty} \frac{1}{1+x^2}\,dx + \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx$

$$\int_0^{\infty} \frac{dx}{1+x^2} = \lim_{R\to\infty} \int_0^R \frac{1}{1+x^2}\,dx = \lim_{R\to\infty} \tan^{-1} x \Big|_0^R$$

$$= \lim_{R\to\infty} \tan^{-1} R = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2} = \lim_{R\to-\infty} \int_R^0 \frac{1}{1+x^2}\,dx = \lim_{R\to-\infty} \tan^{-1}x \Big|_R^0$$

$$= \lim_{R\to-\infty} -\tan^{-1}R = \frac{\pi}{2}$$

so $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$

Thm (p-test)

* $$\int_a^\infty \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{p-1} & p > 1 \\ \text{diverges} & p \leq 1 \end{cases}$$

* $$\int_0^a \frac{dx}{x^p} = \begin{cases} \dfrac{a^{1-p}}{1-p} & p < 1 \\ \text{diverges} & p \geq 1 \end{cases}$$

Thm (comparison test) Assume $f(x) \geq g(x) \geq 0$ for all $x \geq a$ then

* If $\int_a^\infty f(x)\,dx$ converges, so does $\int_a^\infty g(x)\,dx$

* If $\int_a^\infty g(x)\,dx$ diverges, so does $\int_a^\infty f(x)\,dx$

Ex Does $\displaystyle\int_0^\infty \frac{1}{\cosh x + e^x}\,dx$ converge?

Since $\cosh x \geq 0$ for $x \geq 0$, $\cosh x + e^x \geq e^x$

so
$$\frac{1}{\cosh x + e^x} \leq \frac{1}{e^x}.$$

By ~~the comparison test~~

Consider $\displaystyle\int_0^\infty e^{-x}\,dx = \lim \int_0^R e^{-x}\,dx = \lim_{R\to\infty} -e^{-R} + 1 = 1$

so by the comparison test $\displaystyle\int_0^\infty \frac{1}{\cosh x + e^x}\,dx$ converges.