Final practice 2

UCLA: Math 115A, Winter 2019

Instructor: Noah White
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Version: 1

- This exam has 6 questions, for a total of 60 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
Total:	60	

Question 2 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. In each of the following questions, fill in the blanks to complete the statement of the definition or theorem.
   (a) (2 points) *Definition:* The dimension of a vector space V is defined to be the ____**number**____ of elements in any ____**basis**____ of V.

   (b) (2 points) *Definition:* The characteristic polynomial of a linear map T is defined to be

$$p_T(t) = \det(T_B^B - t\,\mathrm{id})$$

for any ____**basis** B____ of V.

Part I

Theorem: Suppose $T : V \longrightarrow W$ is a linear map between finite dimensional vector spaces. Then

$$\dim\ker T + \dim\operatorname{im} T = \dim V.$$

Part II

Theorem: Let V be a finite dimensional vector space over a field $\mathbb{F}$. A linear map $T : V \longrightarrow V$ is diagonalisable if and only if

- ____**the characteristic polynomial splits**____, and
- for every eigenvalue $\lambda \in \mathbb{F}$, ____**the algebraic multiplicity equals** $\dim E_\lambda$____.

Part III

Definition: Let V be a finite dimensional inner product space. The adjoint of a linear map $T : V \longrightarrow V$ is the unique linear map $T^* : V \longrightarrow V$ such that for any ____$v, w \in V$____ we have

$$\langle T(v), w\rangle = \langle v, T^*(w)\rangle.$$

Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) Consider the following subspace of $\mathbb{R}^3$,

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; a + b + c = 0 \right\}$$

The dimension of U is

A. 0.
B. 1.
C. 2.
D. 3.

(b) (2 points) As a subset of $\mathbb{C}[x]$, the set

$$\{ 1 + x^2, x - x^2, 2 + x + x^2 \}$$

A. is a spanning set.
B. is linearly independent.
C. is neither spanning nor linearly independent.
D. is a basis.

(c) (2 points) What is the dimension of the subspace

$$\{ M \in \mathrm{Mat}_{n\times n}(\mathbb{F}) \mid Me_1 = 0 \} \subset \mathrm{Mat}_{n\times n}(\mathbb{F})$$

where e_i is the i^{th} standard basis vector.

A. $n^2 - 1$
B. $n^2 - n$
C. $n^2 + 1$
D. n

(d) (2 points) Let $V = \mathbb{R}_1[x]$, consider the map $T : V \longrightarrow V$ given by $T(p) = p(1) + p(-1)x$. Which of the following is *not* true.

A. T is a linear map.

B. The characteristic polynomial of T splits.

C. T is diagonalisable.

D. T has an eigenspace of dimension 2.

(e) (2 points) Consider again, the map T given above. Suppose V has the inner product given by

$$\langle p, q \rangle = p(1)q(1) + p'(1)q'(1)$$

What is $T^*(x)$?

A. $-1 - 4x$.

B. $3 + 4x$.

C. $1 - x$.

D. x.

Consider the vector space $V = \mathbb{R}_2[x]$ with its standard ordered basis

$$E = \left\{1, x, x^2\right\}$$

and the linear map

$$T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x],\ T(p) = p(x-1) - p(0)x^2$$

(a) (1 point) What is $[T]_E^E$?

Solution:

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (1 point) Is T invertible?

Solution: No.

(c) (6 points) Compute the eigenvalues of T and their algebraic multiplicity.

Solution:

(d) (2 points) Is T diagonalisable? If so, find a matrix Q such that $Q^{-1}[T]_E^E Q$ is diagonal. If not, find Q, so that the above matrix is upper triangular.

Solution: Yes.

Consider the vector space $V = \mathbb{R}_2[x]$ with its standard ordered basis

$$E = \left\{1, x, x^2\right\}$$

with an inner product given by

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt.$$

(a) (6 points) Use the Gram-Schmidt process to find an orthogonal basis B'.

Solution: We will need the following integrals

$$\int_{-1}^{1} x^n \, dx = \begin{cases} \frac{2}{n+1} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd.} \end{cases}$$

We let $v_1 = 1$. Then

$$v_2 = x - \frac{\langle x, 1\rangle}{\|\|1\|\|^2} 1 = x - \frac{\int_{-1}^{1} x \, dx}{\int_{-1}^{1} 1 \, dx} 1 = x.$$

$$v_3 = x^2 - \frac{\langle x^2, 1\rangle}{\|\|1\|\|^2} 1 - \frac{\langle x^2, x\rangle}{\|\|x\|\|^2} 1 = x^2 - \frac{\int_{-1}^{1} x^2 \, dx}{\int_{-1}^{1} 1 \, dx} 1 - \frac{\int_{-1}^{1} x^3 \, dx}{\int_{-1}^{1} x^2 \, dx} x = x^2 - \frac{1}{3}.$$

So the orthogonal basis is $1, x, x^2 - \frac{1}{3}$.

(b) (2 points) Give an orthonormal basis B of $V = \mathbb{R}_2[x]$.

Solution: We simply normalise the basis we found above

$$\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)$$

where we have used the integrals given above and the integral

$$\int_{-1}^{1}\left(x^2 - \frac{1}{3}\right)^2 dx = \left[\frac{1}{5}x^5 - \frac{2}{9}x^3 + \frac{1}{9}x\right]_{-1}^{1} = \frac{2}{5} - \frac{4}{9} + \frac{2}{9} = \frac{2}{5} - \frac{2}{9} = \frac{8}{45}.$$

(c) (2 points) Let $f = 1 + x + x^2$. Compute the coordinate vector $[f]_B$.

Solution: We need to compute $\langle f, v_i\rangle$ for $i = 1, 2, 3$.

$$\langle f, \frac{1}{\sqrt{2}}\rangle = \frac{1}{\sqrt{2}}\int_{-1}^{1} 1 + x + x^2\, dx = \frac{1}{\sqrt{2}}\left(2 + \frac{2}{3}\right) = \frac{8}{3\sqrt{2}}.$$

$$\langle f, \sqrt{\frac{3}{2}}x\rangle = \sqrt{\frac{3}{2}}\int_{-1}^{1} x + x^2 + x^3\, dx = \sqrt{\frac{3}{2}}\frac{2}{3} = \sqrt{\frac{2}{3}}$$

$$\langle f, \frac{3\sqrt{5}}{2\sqrt{2}}\left(x^2 - \frac{1}{3}\right)\rangle = \frac{3\sqrt{5}}{2\sqrt{2}}\int_{-1}^{1} x^2 + x^3 + x^4 - \frac{1}{3} - \frac{1}{3}x - \frac{1}{3}x^2\, dx = \frac{3\sqrt{5}}{2\sqrt{2}}\left(\frac{2}{5} + \frac{4}{9} - \frac{2}{3}\right) = \frac{3\sqrt{5}}{2\sqrt{2}}\frac{32}{45} = \frac{16}{3\sqrt{10}}$$

So

$$[f]^B = \begin{pmatrix} \frac{8}{3\sqrt{2}} \\ \sqrt{\frac{2}{3}} \\ \frac{16}{3\sqrt{10}} \end{pmatrix}$$

Let V be a finite dimensional vector space over a field $\mathbb{F}$. Consider the set $\mathcal{L}(V, V)$ of linear maps from V to V. Fix a linear map $S \in \mathcal{L}(V, V)$. Define the subset

$$C(S) = \{\, T \in \mathcal{L}(V, V) \mid ST = TS \,\}\,.$$

(a) (3 points) Prove or disprove that $C(S)$ is a subspace.

> **Solution:** Suppose $T, T' \in C(S)$, then $S(T + T') = ST + ST' = TS + T'S = (T + T')S$. Thus $T + T' \in C(S)$, and so $C(S)$ is closed under addition. Similarly if $T \in C(S)$ and $\lambda \in \mathbb{F}$ then $S(\lambda T) = \lambda ST = \lambda TS = (\lambda T)S$ so $\lambda T \in C(S)$ which is therefore closed under scalar multiplication. Clearly the zero map commutes with S and therefore $0 \in C(S)$. Thus it is a subspace.

(b) (1 point) Determine $C(\mathrm{id})$.

> **Solution:** Since everything commutes with id we have $C(\mathrm{id}) = \mathcal{L}(V, V)$.

(c) (3 points) Show that if v is an eigenvector for S and $T \in C(S)$, then $T(v)$ is as well.

> **Solution:** $S(T(v)) = T(S(v))$ since $T \in C(S)$. But $T(S(v)) = T(\lambda v) = \lambda T(v)$ where λ is the eigenvalue of v. Thus $S(T(v)) = \lambda T(v)$ and hence $T(v)$ is an λ-eigenvector for S.

(d) (3 points) Suppose that S has $n = \dim V$ distinct eigenvalues. Show that any $T \in C(S)$ is diagonalisable. *Hint: what dimension do the eigenspaces of S have? Now use part c.*

Solution: Since S has n distinct eigenvalues, each of its eigenspaces must be one dimensional. From the above, if λ is an eigenvalue, and v a λ-eigenvector, then $T(v) \in E_\lambda$. Thus $T(v)$ and v are linearly dependant. Thus there is some scalar such that $T(v) = \mu v$ and so v is an eigenvector for T.

Now since S has distinct eigenvalues, it has a basis of eigenvectors, but the above shows that this is also a basis of eigenvectors for T and this it is diagonalisable.

Let V be a finite dimensional inner product space over $\mathbb{R}$, with $\dim V = n$. A linear map $S : V \longrightarrow V$ which preserves the inner product (i.e. $\langle v, w\rangle = \langle S(v), S(w)\rangle$ for any $v, w \in V$) is called a *reflection* if $S^2 = \mathrm{id}$ and the nullity of $S - \mathrm{id}$ is $n - 1$. Let S be a reflection.

(a) (1 point) Give an example of a reflection for the vector space $V = \mathbb{R}^2$ with the usual inner product. *Hint: the name should mean something!*

Solution: for example the linear map given by multiplication with the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(b) (2 points) Show that if $v \in \ker(S - \mathrm{id})^{\perp}$ then $S(v) \in \ker(S - \mathrm{id})^{\perp}$.

Solution: We need to check that for any $x \in \ker(S - \mathrm{id})$ that $\langle S(v), x\rangle = 0$. Note that $S(x) = x$.

$$\langle S(v), x\rangle = \langle S(S(v)), S(x)\rangle = \langle S^2(v), S(x)\rangle = \langle v, x\rangle = 0$$

(c) (2 points) Determine the eigenvalues of S. *Hint: if λ is an eigenvalue, then λ^2 is an eigenvalue of S^2.*

Solution: Since $S^2 = \mathrm{id}$ we see that if v is a λ-eigenvector then $v = S^2(v) = \lambda^2 v$, so $\lambda^2 = 1$ and thus $\lambda = \pm 1$. So these are the *possible* eigenvalues, but we must show they both are actually eigenvalues.

By assumption, $\dim\ker(S - \mathrm{id}) = n - 1$, thus 1 is an eigenvalue if $n > 1$. Let $v \in \ker(S - \mathrm{id})^{\perp}$. Note that $\dim\ker(S - \mathrm{id})^{\perp} = 1$ and by the above $S(v) \in \ker(S - \mathrm{id})^{\perp}$ so $S(v) = \mu v$ for some scalar μ. Thus μ is an eigenvalue and thus $\mu = \pm 1$. But if $\mu = 1$ then $v \in \ker(S - \mathrm{id})$ which is impossible, so $\mu = -1$.

Thus S always has an eigenvalue of -1 and if $n > 1$ it also has an eigenvalue of 1.

(d) (1 point) Is S diagonalisable?

Solution: Yes. The geometric multiplicities are $n-1$ and 1.

(e) (4 points) Show that there exists a vector of unit length, $v \in V$ such that for any $w \in V$,

$$S(w) = w - 2\langle w, v\rangle v$$

Solution: Let v be a unit length -1-eigenvector (which exists by the previous part). Then we can extend this to a basis of eigenvectors $B = \{v_1, \ldots, v_{n-1}, v\}$. Note that the v_i are 1-eigenvectors. We can always write $w = \lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1} + \lambda v$, then

$$\begin{aligned} S(w) &= S(\lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1} + \lambda v) \\ &= \lambda_1 S(v_1) + \cdots + \lambda_{n-1} S(v_{n-1}) + \lambda S(v) \\ &= \lambda_1 v_1 + \cdots + \lambda_{n-1} v_{n-1} - \lambda v \\ &= w - 2\lambda v. \end{aligned}$$

Now notice that $\langle w, v\rangle = \lambda$ (since the $\ker(S - \mathrm{id})^{\perp}$ is the -1-eigenspace). Thus $S(w) = w - 2\langle w, v\rangle v$.

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