This week on the problem set you will get practice applying and understanding Green's theorem and Stokes' theorem.

Homework: The homework will be due on Friday 5 June. It will consist of questions $3, 4, 5$ below.
*Numbers in parentheses indicate the question has been taken from the textbook:

J. Rogawski, C. Adams, *Calculus, Multivariable*, 3rd Ed., W. H. Freeman & Company,

and refer to the section and question number in the textbook.

1. (Section 18.1) 3, 7, 8, 9, 12, 19, 20, 21, 23, 24 25, 29, 36*, 41, 45. (Use the following translations 4th $\mapsto$ 3rd editions: $7 \mapsto 5$, $8 \mapsto 6$, $9 \mapsto 7$, $12 \mapsto 10$, $19 \mapsto 15$, $20 \mapsto 16$, $21 \mapsto 17$, $23 \mapsto 19$, $24 \mapsto 20$, $25 \mapsto 21$, $29 \mapsto 25$, $36 \mapsto 32$, $41 \mapsto 37$, $45 \mapsto 41$ otherwise the questions are the same).

2. (Section 18.2) 5, 8, 9, 18, 19. (Use the following translations 4th $\mapsto$ 3rd editions: $18 \mapsto 16$, $19 \mapsto 17$, otherwise the questions are the same).

3. Let $\mathbf{F}(x, y, z) = \langle x, x + y^3, x^2 + y^2 - z\rangle$ and let S be the surface $z = x^2 - y^2$ where $x^2 + y^2 \le 1$ with upward orienation and boundary $\mathcal{C}$ (with the usual boundary orientation). Find $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

> **Solution:** We can use Stoke's theorem, which says that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F}\, dS$. Note that the orientation of $\mathcal{C}$ matches the orientation of S. The curl of $\mathbf{F}$ is $\langle 2y, -2x, 1\rangle$. Using the parameterization $G(u, v) = (u, v, u^2 - v^2)$ where $u^2 + v^2 \le 1$ which has $\mathbf{N}(u, v) = \langle -2u, 2v, 1\rangle$ we see that $\iint_S \nabla \times \mathbf{F}\, dS = \iint_{u^2+v^2\le 1} \langle 2v, -2u, 1\rangle \cdot \langle -2u, 2v, 1\rangle\, dA$.
>
> Converting to polar coordinates, this is $\int_0^{2\pi} \int_0^1 (-8r^2 \cos\theta \sin\theta + 1) r\, dr\, d\theta$.
>
> This can now be done with single variable calculus techniques, after integrating with respect to r we have $\int_0^{2\pi} -2\cos\theta\sin\theta + 1/2\, d\theta$ which is $\cos^2\theta|_0^{2\pi} + \pi = \pi$.

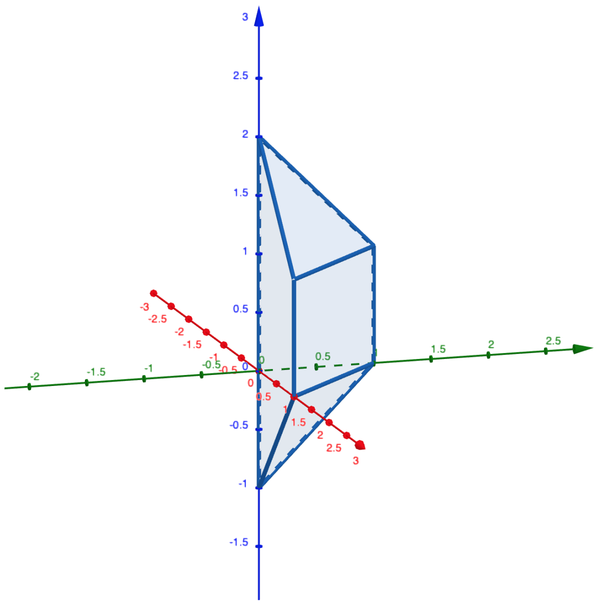
4. Let $\mathbf{F} = \langle x, y, -2z + e^{x^4+y^2}\rangle$ and let S be the part of the hyperboloid $x^2 + y^2 = 1 + z^2$ where $z^2 \le 3$ oriented so that at points with positive z values the z coordinate of the normal vector is negative (i.e. with outward pointing normal). What is $\iint_S \mathbf{F} \cdot dS$?

   Hint: Find a simpler surface with the same boundary.

> **Solution:** Note that the divergence of $\mathbf{F}$ is zero, so $\mathbf{F} = \nabla \times G$ for some G. So, by Stoke's theorem if S' is any other oriented surface with $\partial S' = \partial S$ with the same orientation then $\iint_S \mathbf{F} \cdot dS = \iint_{S'} \mathbf{F} \cdot dS$.
>
> We can use the surface S' that is the cylinder $x^2 + y^2 = 4$ where $-\sqrt{3} \le z \le \sqrt{3}$ with outward pointing normal. This has paramaterization $G(\theta, z) = (2\cos\theta, 2\sin\theta, z)$ where $0 \le \theta \le 2\pi$ and $-\sqrt{3} \le z \le \sqrt{3}$. The normal vector is $N(\theta, z) = (2\cos\theta, 2\sin\theta, 0)$.
>
> Therefore:
>
> $$\iint_{S'} \mathbf{F} \cdot dS = \int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{2\pi} \langle 2\cos\theta, 2\sin\theta, z - e^{16\cos^4\theta+\sin^2\theta}\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle\, d\theta\, dz.$$

This simplies to $\iint_{S'} \mathbf{F} \cdot dS = \int_{-\sqrt{3}}^{\sqrt{3}} \int_0^{2\pi} 4 \, d\theta \, dz = 16\pi\sqrt{3}$.

5. Consider the 3 dimensional polyhedron pictured below with vertices



$(0, 0, 2)$
$(0, 0, -1)$
$(0, 1, 0)$
$(1, 0, 0)$
$(0, 1, 1)$
$(1, 0, 1)$

with outward pointing orientation. Find the flux of $\mathbf{F} = \langle 2x^2 - 3xy^2, xz^2e^z + y^3, \sin(x^2 + y^2)\rangle$ through $\mathcal{S}$.

Solution: We note that $\mathbf{F}$ has continuous partial derivatives everywhere on the interior of the polyhedron (we will call this region $\mathcal{E}$). We note that $\partial\mathcal{E} = \mathcal{S}$ and $\nabla \cdot \mathbf{F} = 2x$. Thus by Stokes' theorem

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{E}} 4x \, dV$$

The region $\mathcal{E}$ is z-simple and can be described by

$$\mathcal{E} = \{\ (x, y, z) \mid (x, y) \in \mathcal{D}, x + y - 1 \leq x \leq 2 - x - y\}$$

Where $\mathcal{D}$ is the region where $x, y \geq 0$ and $x + y \leq 1$. Thus the flux equals

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \iiint_{\mathcal{E}} 4x \, dV \\
&= \iint_{\mathcal{D}} \int_{x+y-1}^{2-x-y} 4x \, dz \, dA_{xy} \\
&= \iint_{\mathcal{D}} 4x(3 - 2x - 2y) \, dA \\
&= \int_0^1 \int_0^{1-x} 4x(3 - 2x) - 8xy \, dy \, dx \\
&= \int_0^1 4x(1 - x)(3 - 2x) - 4x(1 - x)^2 \, dx = 1
\end{aligned}$$

*The questions marked with an asterisk are more difficult or are of a form that would not appear on an exam. Nonetheless they are worth thinking about as they often test understanding at a deeper conceptual level.