This weeks problem set provides practice with diagonalisable operators and the basic properties of inner products. A question marked with a $^\dagger$ is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a $^*$ is especially important.

**Homework 5:** due Monday 3 June: questions 22 from Section 6.2 (see 3 below) and question 2 below.

1. From section 6.2, problems 1, $2b, g, i, k$, $5^*$, 6, 7, 9, $13^*$, $17^*$, 22.

2. Let $V$ be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

   (a) Fix $y \in V$ and suppose $\langle x, y\rangle = 0$ for all $x \in V$. Show that $y = 0$.

   **Solution:** Let $B = \{v_1, \ldots, v_n\}$ be an orthonormal basis for $V$. Then
   $$y = \sum_{i=1}^{n} \langle y, v_i\rangle v_i = 0$$
   since $\langle y, v_i\rangle = 0$ for all $i$.

   (b) Let $T : V \longrightarrow V$ be a linear map such that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all pairs $x, y \in V$ (we call such a map an *isometry*). Prove that $T$ is an isomorphism.

   **Solution:** First we show that $T$ is injective. Suppose that $T(x) = T(y)$. One one hand we have
   $$||T(x)|| = \langle T(x), T(x)\rangle = \langle x, x\rangle.$$
   On the other hand,
   $$||T(x)|| = \langle T(x), T(x)\rangle = \langle T(x), T(y)\rangle = \langle x, y\rangle.$$
   Thus $\langle x, x\rangle = \langle x, y\rangle$, i.e. $\langle 0, x - y\rangle = 0$. Thus by the above, $x - y = 0$ or $x = y$.
   Now, since $T$ is an injective map from $V$ to $V$, it must be surjective by the dimension theorem. Thus it is an isomorphism.

   (c) $^\dagger$ Find all isometries $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that have $\det T = 1$.

   **Solution:** The map $T$ will be given by a matrix
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
   From the fact that it is an isometry we see that $|||T(e_i)||| = |||e_i||| = 1$ for $i = 1, 2$. Thus
   $$\left\|\left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|\right| = \left\|\left| \begin{pmatrix} a \\ c \end{pmatrix} \right\|\right| = a^2 + c^2 = 1$$
   and similarly $b^2 + d^2 = 1$. This means, both columns of the matrix are points on the unit circle. I.e. for some choice of $\theta, \psi \in [0, 2\pi)$ then we have that
   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\psi \\ \sin\theta & \sin\psi \end{pmatrix}$$
   Additionally we have that $\langle T(e_1), T(e_2)\rangle = \langle e_1, e_2\rangle = 0$. I.e the two columns of the matrix are at right angles to each other, so $\psi = \theta \pm \pi/2$ (modulo $2\pi$). Alternatively this can be seen since
   $$0 = \langle T(e_1), T(e_2)\rangle = \langle \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \rangle = \cos\theta\cos\psi + \sin\theta\sin\phi = \cos(\theta - \psi)$$

Which means that $\theta - \psi = \pm\frac{\pi}{2} + 2n\pi$ for $n \in \mathbb{Z}$.
The condition that the determinant is 1 is that $ad - bc = 1$ which translates to

$$1 = \cos\theta \sin\psi - \cos\psi \sin\theta = \sin(\theta - \psi)$$

hence $\theta - \psi = \pi/2$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\theta + \pi/2 \\ \sin\theta & \sin\theta + \pi/2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some choice of $\theta \in [0, 2\pi)$.

3. (22 from 6.2) Let $V = \mathcal{C}([0,1], \mathbb{R})$ be the space of real valued, continuous functions on the interval $[0,1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\ dt$. Let $W$ be the subspace spanned by the linearly independent set $\{t, \sqrt{t}\}$.
   (a) Find an orthonormal basis for $W$.
   (b) Let $h(t) = t^2$. Use the orthonormal basis obtained in (a) to obtain the "best" (closest) approximation of $h$ in $W$.

**Solution:** *Solution to 22 from 6.2.* The question asks us to consider the vector space $\mathcal{C}([0,1], \mathbb{R})$ of continuous functions on $[0,1]$ into $\mathbb{R}$ with inner product, $\langle f, g \rangle = \int_0^1 f(t)g(t)\ dt$, and to use the Gram Schmidt process to find an orthonormal basis for the subspace $\operatorname{span}\{t, \sqrt{t}\}$.

*Part a).* We use the Gram-Schmidt process to define first an orthogonal basis $\{f_1, f_2\}$. Set $f_1 = t$. Then

$$f_2 = \sqrt{t} - \frac{\langle \sqrt{t}, t \rangle}{||t||^2} t.$$

To get an explicit expression we do some calculations.

$$||t||^2 = \int_0^1 t^2\ dt = \frac{1}{3}.$$
$$\langle \sqrt{t}, t \rangle = \int_0^1 t^{3/2}\ dt = \frac{2}{5}.$$
$$||\sqrt{t}||^2 = \int_0^1 t\ dt = \frac{1}{2}.$$

Putting this together we get

$$f_2 = \sqrt{t} - \frac{6}{5}t.$$

To get an orthonormal basis, we need to normalise, so we need to calculate

$$\begin{aligned} ||f_2||^2 &= \int_0^1 \left(\sqrt{t} - \frac{6}{5}t\right)^2\ dt \\ &= \int_0^1 t - \frac{12}{5}t^{3/2} + \frac{36}{25}t^2\ dt \\ &= \frac{1}{2} - \frac{24}{25} + \frac{36}{75} = \frac{1}{50} \end{aligned}$$

We already know that $||f_1|| = \frac{1}{\sqrt{3}}$ and now we also know $||f_2|| = \frac{1}{5\sqrt{2}}$ thus, an orthonormal basis is

$$\{g_1 = \sqrt{3}t, g_2 = \sqrt{2}\left(5\sqrt{t} - 6t\right)\}.$$

*Part b).* We want to project $t^2$ onto $W$. The result will be

$$\langle t^2, g_1\rangle g_1 + \langle t^2, g_2\rangle g_2.$$

We calculate the coefficients.

$$\begin{aligned}
\langle t^2, g_1\rangle &= \int_0^1 \sqrt{3}t^3\, dt = \frac{\sqrt{3}}{4}. \\
\langle t^2, g_2\rangle &= \int_0^1 \sqrt{2}\left(5t^{5/2} - 6t^3\right)\, dt \\
&= \sqrt{2}\left(\frac{10}{7} - \frac{3}{2}\right) \\
&= -\frac{\sqrt{2}}{14}.
\end{aligned}$$

Thus, the best approximation is

$$\frac{3}{4}t - \frac{1}{7}\left(5\sqrt{t} - 6t\right) = \frac{45}{28}t - \frac{5}{7}\sqrt{t}.$$