This weeks problem set focuses isomorphisms and coordinate vectors and the matrices associated to linear transformations. It will be quite a large problem set, and because of the way we will be covering it in class, don't worry if you can't do some of the problems until after next Friday. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

1. From section 2.4, problems 1, $2a, c, e$, 3, 7, 14, 15^*, 17^*, $24^{*,\dagger}$.
2. From section 2.2, problems 1, $2a, c, f$, 10, $11^\dagger$, 12^*, $14^\dagger$, 16.
3. From section 2.3, problems 1, $2a$, 3, 12, 16, $17^\dagger$, 16.

4.* Let V be a finite dimensional vector space over $\mathbb{F}$ and $B\{v_1, \dots, v_n\}$ a basis. Let W be another vector space and $w_1, \dots, w_n$ a collection of elements. Show that there is a unique linear map such that $T(v_i) = w_i$.

There are mathematical objects called $\mathfrak{sl}_2$-*representations* which are important in quantum mechanics and beautiful objects in their own right. We won't define what they are exactly**, but their are vector spaces that come packaged with a certain pair of linear maps. The next questions give an example.

5.† Let $V = \mathbb{C}[x, y]$ be the vector space of polynomials in two variables. So we have $x^2 - 2xy^2 + 1 \in V$ for example. Define two linear maps $E, F : V \longrightarrow V$ where

$$E(p) = x\frac{\partial p}{\partial y} \text{ and } F(p) = y\frac{\partial p}{\partial x}$$

(a) Find a formula for $H := EF - FE$.

(b) A subspace $U \subset V$ is called a *subrepresentation* if $E(U) \subset U$ and $F(U) \subset U$. Let $V(n) = \operatorname{span}\{x^{n-a}y^a \mid 0 \le a \le n\}$, this is the space of *homogeneous polynomials of degree* n, i.e. every term on the polynomial has degree n. Show that $V(n)$ is a subrepresentation, for any $n \ge 0$.

(c) With the basis $x^n, x^{n-1}y, x^{n-2}y^2, \dots, y^n$, determine the matrix corresponding to the linear maps E, H, F restricted to the subspaces $V(n)$.

6.† Another example of an $\mathfrak{sl}_2$ representation is given by $W = \mathbb{C}^2$ and where E' and F' are the linear transformations given by left multiplication by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find an isomorphism $\theta : V(1) \longrightarrow W$ so that $\theta E = E'\theta$ and $\theta F = F'\theta$ as linear maps $V(1) \longrightarrow W$.

7.† Show that there is no, nonzero, linear map $\theta : V(n) \longrightarrow V(m)$ so that $E\theta = \theta E$ and $F\theta = \theta F$ whenever $n \neq m$. *Hint: if such a map does exist, where does x^n get sent? Now use that $H\theta = \theta H$. This is pretty hard, let me know if you need more hints*

** Ok, if you really want to know exactly what they are here is the definition: An $\mathfrak{sl}_2$-representation is a vector space V with two linear maps $E, F : V \longrightarrow V$ such that

$$E^2F - 2EFE + FE^2 = -2E$$

and the same equation with the E's and F's swapped. There is a much more intuitive definition but one would need to know some more abstract algebra. If you are really keen, try and find more $\mathfrak{sl}_2$ representations and show me!

The results in the following three exercises are called the first, second, and third isomorphism theorems respectively. The exercises are difficult (but only really because they involve unwrapping lots of definitions) but the interested student should attempt them. They turn up over and over again in algebra and there are analogues for different algebraic objects like *groups*, *rings* and *modules*. The area of mathematics that explains why the same types of theorems appear over and over again is called *category theory*.

8.[†] Suppose $T : V \longrightarrow W$ is a linear map between two vector spaces over $\mathbb{F}$. Prove that

$$V/\ker T \cong \operatorname{im} T.$$

The space on the left is the *quotient space* which you met in the first problem set.

9.[†] Let V be a vector space over $\mathbb{F}$ and let U, W be subspaces of V. Prove that

$$(U + W)/U \cong W/(U \cap W).$$

10.[†] Let V be a vector space over $\mathbb{F}$ and let $U \subseteq W$ be subspaces of V. Prove that W/U is a subspace of V/U. Furthermore prove that

$$(V/U)/(W/U) \cong V/W.$$