Math 3B: Lecture 9

Noah White

October 17, 2017

The fundamental theorem of calculus

Theorem

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- $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is a function of x.
- every input x produces a number as an output.

A consequence (corrollary)

Corollary

For any antiderivative $F(x)$ of $f(x)$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

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Why?

Well $F(x) = \int_a^x f(t)\,\mathrm{d}t + C$ for some a and C. So

$$\begin{aligned} F(b) - F(a) &= \int_a^b f(t)\,\mathrm{d}t + C - \int_a^a f(t)\,\mathrm{d}t - C \\ &= \int_a^b f(t)\,\mathrm{d}t \end{aligned}$$

Example 1

Question

Evaluate the definite integral

$$\int_0^1 x^2 - 4 \, \mathrm{d}x$$

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Solution

An antiderivative of $x^2 - 4$ is $\frac{1}{3}x^3 - 4x$ so

$$\begin{aligned} \int_0^1 x^2 - 4 \, \mathrm{d}x &= \frac{1}{3} \cdot 1^3 - 4 - \frac{1}{3} \cdot 0^3 + 4 \cdot 0 \\ &= \frac{1}{3} - 4 = -\frac{11}{3} \end{aligned}$$

Example 2

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Evaluate the definite integral

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Solution

An antiderivative of $\sin x$ is $-\cos x$ so

$$\begin{aligned}\int_0^{\pi} \sin x \, \mathrm{d}x &= -\cos \pi + \cos 0 \\ &= -(-1) + 1 = 2\end{aligned}$$

Why is the FTC true?

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$$

We want to show that $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.

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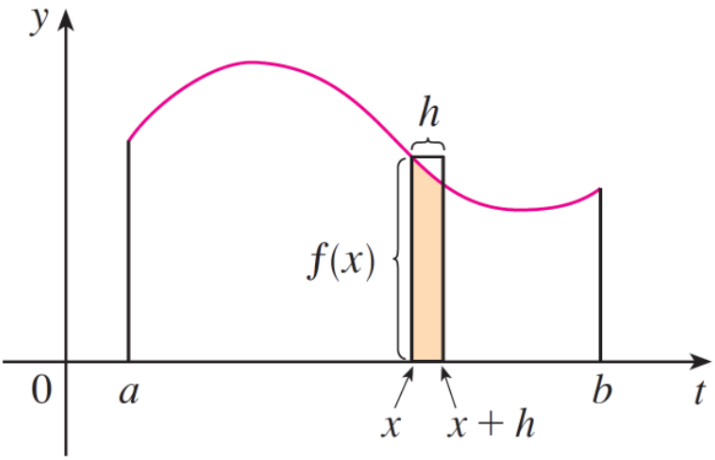
When h is very small $\int_x^{x+h} f(t) \approx$



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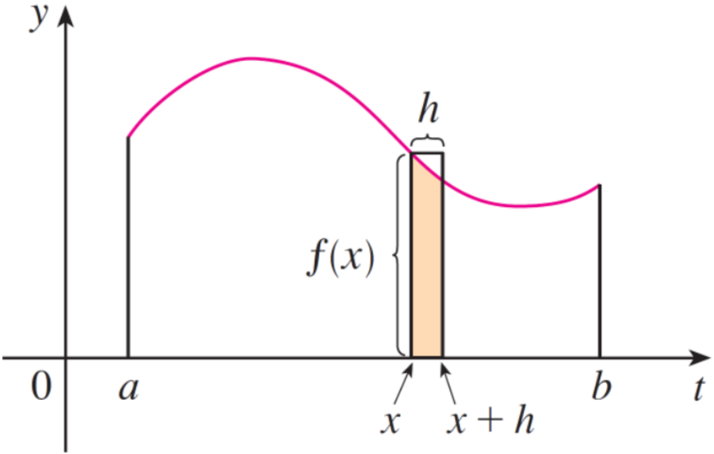

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The indefinite integral

We also use the following notation for the general antiderivative:

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Example

$$\int \sin(x) - x\,\mathrm{d}x = -\cos(x) - \frac{1}{2}x^2 + C$$

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Suppose $u = g(x)$, then

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Solution

We use the substitution $u = x^2 + 1$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we can write the integral

$$\int 2 \cdot \sqrt{x^2+1} \cdot 2x\,\mathrm{d}x = 2\int \sqrt{u}\,\mathrm{d}u$$

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&= 2 \cdot \frac{2}{3} u^{\frac{3}{2}} + C \\
&= \frac{4}{3}(x^2+1)^{\frac{3}{2}} + C
\end{aligned}$$

Integration by substitution (definite integrals)

Substitution for definite integrals

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