

This weeks problem set focuses isomorphisms and coordinate vectors and the matrices associated to linear transformations. It will be quite a large problem set, and because of the way we will be covering it in class, don't worry if you can't do some of the problems until after next Friday. A question marked with a  $\dagger$  is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a  $*$  is especially important.

**Homework 2:** due Friday 9 Feb: questions 4 and 5 below.

1. From section 2.4, problems 1,  $2a, c, e, 3, 7, 14, 15^*, 17^*, 24^{*,\dagger}$ .
2. From section 2.2, problems 1,  $2a, c, f, 10, 11^\dagger, 12^*, 14^\dagger, 16$ .
3. From section 2.3, problems 1,  $2a, 3, 12, 16, 17^\dagger, 16$ .

There are mathematical objects called  $\mathfrak{sl}_2$ -representations which are important in quantum mechanics and beautiful objects in their own right. We won't define what they are exactly\*\*, but their are vector spaces that come packaged with a certain pair of linear maps. The next questions give an example.

- 4 $^\dagger$  Let  $V = \mathbb{C}[x, y]$  be the vector space of polynomials in two variables. So we have  $x^2 - 2xy^2 + 1 \in V$  for example. Define two linear maps  $E, F : V \longrightarrow V$  where

$$E(p) = x \frac{\partial p}{\partial y} \text{ and } F(p) = y \frac{\partial p}{\partial x}$$

- (a) Find a formula for  $H := EF - FE$ .

**Solution:** We just calculate what  $H$  does to a polynomial, using the chain rule:

$$\begin{aligned} H(p) &= EF(p) - FE(p) \\ &= x \frac{\partial}{\partial y} y \frac{\partial p}{\partial x} - y \frac{\partial}{\partial x} x \frac{\partial p}{\partial y} \\ &= x \frac{\partial p}{\partial x} + xy \frac{\partial^2 p}{\partial y \partial x} - y \frac{\partial p}{\partial y} - xy \frac{\partial^2 p}{\partial y \partial x} \\ &= x \frac{\partial p}{\partial x} - y \frac{\partial p}{\partial y}. \end{aligned}$$

- (b) A subspace  $U \subset V$  is called a *subrepresentation* if  $E(U) \subset U$  and  $F(U) \subset U$ . Let  $V(n) = \text{span} \{ x^{n-a} y^a \mid 0 \leq a \leq n \}$ , this is the space of *homogeneous polynomials of degree  $n$* , i.e. every term on the polynomial has degree  $n$ . Show that  $V(n)$  is a subrepresentation, for any  $n \geq 0$ .

**Solution:** Note that  $E(x^{n-a} y^a) = ax^{n-a+1} y^{a-1} \in V(n)$  and  $F(x^{n-a} y^a) = (n-a)x^{n-a-1} y^{a+1} \in V(n)$ . Thus, since an arbitrary element  $p \in V(n)$  is simply a linear combination of these, we have that  $E(p), F(p) \in V(n)$  and hence it is a subrepresentation.

- (c) With the basis  $x^n, x^{n-1}y, x^{n-2}y^2, \dots, y^n$ , determine the matrix corresponding to the linear maps  $E, H, F$  restricted to the subspaces  $V(n)$ .

**Solution:** We will do  $H$  first. Note that  $H(x^{n-a} y^a) = (n-2a)x^{n-a} y^a$ . Hence the matrix for  $H$  is diagonal with the  $(i, i)$ -entry being  $n - 2(i - 1)$ .

Now  $E(x^{n-a} y^a) = ax^{n-a+1} y^{a-1}$  and so the matrix for  $E$  is zero everywhere, apart from the  $(i, i+1)$ -entry which is  $i$ .

Similarly,  $F(x^{n-a} y^a) = (n-a)x^{n-a-1} y^{a+1}$  and so the matrix for  $F$  is zero everywhere, apart from the  $(i+1, i)$ -entry which is  $n - i + 1$ .

Examples for  $n = 3$  are

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

5.<sup>†</sup> Another example of an  $\mathfrak{sl}_2$  representation is given by  $W = \mathbb{C}^2$  and where  $E'$  and  $F'$  are the linear transformations given by left multiplication by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find an isomorphism  $\theta : V(1) \rightarrow W$  so that  $\theta E = E'\theta$  and  $\theta F = F'\theta$  as linear maps  $V(1) \rightarrow W$ .

**Solution:** The isomorphism  $\theta$  will be determined by the values of  $\theta(x)$  and  $\theta(y)$ . So let's set

$$\theta(x) = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \theta(y) = \begin{pmatrix} c \\ d \end{pmatrix}.$$

In order for  $\theta E = E'\theta$  and  $\theta F = F'\theta$ , we must have that four things hold:

$$\theta E(x) = E'\theta(x) \tag{1}$$

$$\theta E(y) = E'\theta(y) \tag{2}$$

$$\theta F(x) = F'\theta(x) \tag{3}$$

$$\theta F(y) = F'\theta(y). \tag{4}$$

Notice that  $E(x) = 0$ , thus  $\theta E(x) = 0$  and by 1 we must have that

$$0 = E'\theta(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

Thus  $b = 0$ . We do something similar and notice that  $F(y) = 0$  so by we have

$$0 = F'\theta(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix}$$

so  $c = 0$ . Now we only need to check 2 and 3. Notice that  $E(y) = x$  so  $\theta E(y) = \theta(x)$ . By 2,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} d \\ c \end{pmatrix}$$

Which gives us that  $a = d$  and  $b = c$ . Checking 3 gives us the same result. So now we have that

$$\theta(x) = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \theta(y) = \begin{pmatrix} c \\ d \end{pmatrix}.$$

We just need to pick an  $a$  which ensures this will be an isomorphism. It is clear that picking any  $a \neq 0$  is fine.

6.<sup>†</sup> Show that there is no, nonzero, linear map  $\theta : V(n) \rightarrow V(m)$  so that  $E\theta = \theta E$  and  $F\theta = \theta F$  whenever

$n \neq m$ . *Hint: if such a map does exist, where does  $x^n$  get sent? Now use that  $H\theta = \theta H$ . This is pretty hard, let me know if you need more hints*

**Solution:** We will give a brief sketch. Consider  $\theta(x^n)$ . We know that  $E(x^n) = 0$  so  $0 = \theta(E(x^n)) = E\theta(x^n)$ . I.e. we must have that  $E\theta(x^n) = 0$ . If

$$\theta(x^n) = \lambda_0 x^m + \lambda_1 x^{m-1}y + \dots + \lambda_m y^m$$

then

$$E\theta(x^n) = 0 + \lambda_1 x^m + \dots + m\lambda_m xy^{m-1}$$

The only way for this to be zero is if  $\lambda_1 = \lambda_2 = \dots = \lambda_m = 0$ . Thus  $\theta(x^n) = \lambda x^m$  for some  $\lambda \in \mathbb{C}$ . Now let's use the fact that  $H\theta = \theta H$  (this is true since  $(EF - FE)H = H(EF - FE)$ ).

Observe that  $H(x^n) = nx^n$ , thus  $\theta H(x^n) = \lambda nx^m$ . On the other hand  $H\theta(x^n) = H(\lambda x^m) = \lambda mx^m$ . Hence  $\lambda nx^m = \lambda mx^m$ . The only way this is possible, is if either  $m = n$ , or if  $\lambda = 0$ .

If  $m \neq n$  then  $\lambda = 0$ , so  $\theta(x^n) = 0$ . Now comes the somewhat challenging part. How can we figure out that this means that  $\theta(x^{n-a}y^a) = 0$ ? We consider  $F^a(x^n)$ . This is the result of applying  $F$  to the element  $x^n$ ,  $a$  times. The result is  $F^a(x^n) = \frac{n!}{(n-a)!}x^{n-a}y^a$ . But we know that  $\theta F^a = F^a\theta$  so we must have that  $\theta(F^a(x^n)) = F^a(\theta(x^n)) = F^a(0) = 0$ . But since  $\theta$  is linear

$$\theta(x^{n-a}y^a) = \frac{(n-a)!}{n!}F^a(\theta(x^n)) = 0.$$

\*\* Ok, if you really want to know exactly what they are here is the definition: An  $\mathfrak{sl}_2$ -representation is a vector space  $V$  with two linear maps  $E, F : V \rightarrow V$  such that

$$E^2F - 2EFE + FE^2 = -2E$$

and the same equation with the  $E$ 's and  $F$ 's swapped. There is a much more intuitive definition but one would need to know some more abstract algebra. If you are really keen, try and find more  $\mathfrak{sl}_2$  representations and show me!