This weeks problem set provides practice with the Gram-Schmidt process and adjoint operators. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 6: due Friday 16 March: questions $22a$ from 6.2 and $4a, b$ below.

1. From section 6.2, problems 1, $2b, g, i, k$, 5^*, 6, 7, 9, 13^*, 17^*, 22.
2. From section 6.3, problems 1, $2a$, $3a, c$, 4, 6, 8^*.
3. From section 6.4, problems 1, $2a, c$, 4, 7, 11, 24.
4. Let V be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
   (a) Fix $y \in V$ and suppose $\langle x, y\rangle = 0$ for all $x \in V$. Show that $y = 0$.

   Solution: Let $B = \{v_1, \dots, v_n\}$ be an orthonormal basis for V. Then
   $$y = \sum_{i=1}^{n} \langle y, v_i\rangle v_i = 0$$
   since $\langle y, v_i\rangle = 0$ for all i.

   (b) Let $T : V \longrightarrow V$ be a linear map such that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all pairs $x, y \in V$ (we call such a map an *isometry*). Prove that T is an isomorphism.

   Solution: First we show that T is injective. Suppose that $T(x) = T(y)$. One one hand we have
   $$||T(x)|| = \langle T(x), T(x)\rangle = \langle x, x\rangle.$$
   On the other hand,
   $$||T(x)|| = \langle T(x), T(x)\rangle = \langle T(x), T(y)\rangle = \langle x, y\rangle.$$
   Thus $\langle x, x\rangle = \langle x, y\rangle$, i.e. $\langle 0, x - y\rangle = 0$. Thus by the above, $x - y = 0$ or $x = y$.
   Now, since T is an injective map from V to V, it must be surjective by the dimension theorem. Thus it is an isomorphism.

   (c) (not for homework)† Find all isometries $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that have $\det T = 1$.

Solution: *Solution to 22 from 6.2.* The question asks us to consider the vector space $\mathcal{C}([0, 1], \mathbb{R})$ of continuous functions on $[0, 1]$ into $\mathbb{R}$ with inner product, $\langle f, g\rangle = \int_0^1 f(t)g(t)\ dt$, and to use the Gram Schmidt process to find an orthonormal basis for the subspace span$\{t, \sqrt{t}\}$.
Part a). We use the Gram-Schmidt process to define first an orthogonal basis $\{f_1, f_2\}$. Set $f_1 = t$. Then
$$f_2 = \sqrt{t} - \frac{\langle \sqrt{t}, t\rangle}{||t||^2} t.$$

To get an explicit expression we do some calculations.

$$||t||^2 = \int_0^1 t^2 \, dt = \frac{1}{3}.$$
$$\langle \sqrt{t}, t \rangle = \int_0^1 t^{3/2} \, dt = \frac{2}{5}.$$
$$||\sqrt{t}||^2 = \int_0^1 t \, dt = \frac{1}{2}.$$

Putting this together we get

$$f_2 = \sqrt{t} - \frac{6}{5}t.$$

To get an orthonormal basis, we need to normalise, so we need to calculate

$$\begin{aligned} ||f_2||^2 &= \int_0^1 \left(\sqrt{t} - \frac{6}{5}t\right)^2 \, dt \\ &= \int_0^1 t - \frac{12}{5}t^{3/2} + \frac{36}{25}t^2 \, dt \\ &= \frac{1}{2} - \frac{24}{25} + \frac{36}{75} = \frac{1}{50} \end{aligned}$$

We already know that $||f_1|| = \frac{1}{\sqrt{3}}$ and now we also know $||f_2|| = \frac{1}{5\sqrt{2}}$ thus, an orthonormal basis is

$$\{g_1 = \sqrt{3}t, g_2 = \sqrt{2}\left(5\sqrt{t} - 6t\right)\}.$$

Part b). We want to project t^2 onto W. The result will be

$$\langle t^2, g_1 \rangle g_1 + \langle t^2, g_2 \rangle g_2.$$

We calculate the coefficients.

$$\begin{aligned} \langle t^2, g_1 \rangle &= \int_0^1 \sqrt{3}t^3 \, dt = \frac{\sqrt{3}}{4}. \\ \langle t^2, g_2 \rangle &= \int_0^1 \sqrt{2}\left(5t^{5/2} - 6t^{3/2}\right) \, dt \\ &= \sqrt{2}\left(\frac{10}{7} - \frac{12}{5}\right) \\ &= -\frac{34\sqrt{2}}{35}. \end{aligned}$$

Thus, the best approximation is

$$\frac{3}{4}t - \frac{68}{35}\left(5\sqrt{t} - 6t\right) = \frac{1737}{140}t - \frac{68}{7}\sqrt{t}.$$