Math 3B: Lecture 14

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Repeated factors

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For every factor $(ax+b)^k$ in $q(x)$, the partial fraction expansion has terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}.$$

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So

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Side note: integrating $\frac{1}{x}$.

Recall that

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Using substitution this gives the formula

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + C.$$

Side note: integrating $\frac{1}{x^k}$.

Recall that if $k > 1$

Fact

$$\int \frac{1}{x^k}\,\mathrm{d}x = -\frac{1}{(k-1)x^{k-1}} + C$$

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using polynomial long division.
2. Write $\frac{r(x)}{q(x)}$ as a sum of fractions of the form
$$\frac{A}{(ax+b)^k}$$
using partial fractions
3. Integrate all these pieces seperately.

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So

$$I = \frac{1}{3}x^2 - 2x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

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So

$$I = x + \ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + C.$$

Differential equations (motivation)

An (ordinary) differential equation (or ODE) is an equation that involves derivatives of an unknown function.

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or

$$x^2 y'' + xy' + x^2 y = 0$$

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The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

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And so on.

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Note

The right hand side of the equation does not have any y's.

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- draw solutions for many other ODEs
- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)
- understand how sensitive ODEs are to their parameters.

Initial value problems

- If we try to solve the differential equation

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we get (by integrating)

$$y(t) = t^3 + \cos t + C.$$

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- The extra piece of data we need is called an "initial value".
- E.g. $y(0) = 2$.
- Then we see that $y(0) = 1 + C$, so $C = 1$.

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- If this is positive we go up, negative we go down!