Midterm 2 practice

UCLA: Math 32B, Winter 2017

Instructor: Noah White
Date: February, 2017
Version: practice

- This exam has 4 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: Solutions.

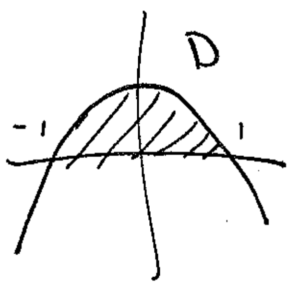
ID number: ______________________

Discussion section (please circle):

Day/TA	Ben	Gyu Eun	Robbie
Tuesday	3A	3C	3E
Thursday	3B	3D	3F

Question	Points	Score
1	10	
2	8	
3	8	
4	14	
Total:	40	

1. (a) (5 points) Compute the center of mass of the region in the xy-plane above the x-axis and below the curve $y = 1 - x^2$. Assume a constant mass density of 1.



$$(x_{cm}\ y_{cm}) = \frac{1}{\text{Area}} \iint_D (x\ y) \cdot \underset{\text{density}}{1}\, dA$$

$$D = \{(x\ y) \mid -1 \le x \le 1,\ 0 \le y \le 1 - x^2\}.$$

$$\text{Area}(D) = \iint_D 1\, dA = \int_{-1}^{1} \int_0^{1-x^2} dy\, dx$$

$$= \int_{-1}^{1} 1 - x^2\, dx = \left[x - \tfrac{1}{3}x^3\right]_{-1}^{1} = 2 - \tfrac{2}{3}$$

$$= \frac{4}{3}$$

$$\iint_D (x\ y)\, dA = \int_{-1}^{1} \int_0^{1-x^2} (x, y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[xy, \tfrac{1}{2}y^2\right]_0^{1-x^2} dx$$

$$= \int_{-1}^{1} \left(x - x^3, \tfrac{1}{2}(1-x^2)^2\right) dx$$

$$= \left[\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4, \tfrac{1}{2}\left(x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right)\right]_{-1}^{1}$$

$$= \left(0, \tfrac{1}{2}\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right) + \tfrac{1}{2}\left(1 - \tfrac{2}{3} + \tfrac{1}{5}\right)\right) = \left(0, \tfrac{8}{15}\right)$$

so

$$(x_{cm}, y_{cm}) = \left(0, \frac{2}{5}\right).$$

(b) (5 points) Determine the surface area of the paraboloid

$$x^2 + y^2 = 2z, \quad 0 \le z \le 1.$$

$z = \frac{1}{2}(x^2+y^2)$ so we parametrise by

$$G(u,v) = \left(u, v, \tfrac{1}{2}(u^2+v^2)\right).$$

If ~~we~~ $z \in [0,1]$ then $0 \le x^2+y^2 \le 2$ so

$(u,v) \in$ Disk radius $\sqrt{2} = \mathcal{D}$.

$\underline{T}_u = (1, 0, u)$ $\quad \underline{N}(u,v) = (-u, -v, 1)$

$\underline{T}_v = (0, 1, v)$ $\quad \|\underline{N}(u,v)\| = \sqrt{u^2+v^2+1}$

Thus

$$\iint_S 1 . dS = \iint_{\mathcal{D}} \sqrt{u^2+v^2+1}\, dA_{uv}$$

$$= \int_0^{\cancel{\sqrt{2}}\, 2\pi} \int_0^{\sqrt{2}} r\sqrt{r^2+1}\, dr\, d\theta \quad \leftarrow \text{change to polar.}$$

$$= \int_0^{2\pi} \left[\tfrac{1}{3}(r^2+1)^{3/2}\right]_0^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{3} 3^{3/2} - \tfrac{1}{3}\, d\theta$$

$$= 2\pi\left(\sqrt{3} - \tfrac{1}{3}\right)$$

2. (8 points) Consider the region $\mathcal{E}$ given by

$$0 \leq z \leq (y-x^2)^2, \quad x^2 \leq y \leq x.$$

Use the change of variables

$$x = u, y = v + u^2, z = wv^2,$$

to evaluate

$$\int_{\mathcal{E}} \frac{1}{y-x^2}\, \mathrm{d}V.$$

The region becomes

$$0 \cancel{\leq wv^2 \leq (y-x^2)^2}$$

$$0 \leq wv^2 \leq v^2 \quad \text{i.e.} \quad 0 \leq w \leq 1$$

$$u^2 \leq v+u^2 \leq u \quad \text{ie} \quad 0 \leq v \leq u-u^2$$

Thus $0 \leq u \leq 1$.

$$\Rightarrow \mathcal{E}_0 : \quad 0 \leq u, w \leq 1, \quad 0 \leq v \leq u-u^2$$

$$\cancel{\iiint_{\mathcal{E}} \frac{1}{y-x^2} dV} \quad \text{Jac} \overset{\det}{=} \begin{pmatrix} 1 & 0 & 0 \\ 2u & 1 & 0 \\ 0 & 2wv & v^2 \end{pmatrix} = v^2 > 0$$

so

$$\iiint_{\mathcal{E}} \frac{1}{y-x^2} dV = \iiint_{\mathcal{E}_0} \frac{1}{v} \cancel{dV_{uvw}} \cdot v^2 dV_{uvw} = \iiint_{\mathcal{E}_0} v \, dV_{uvw}$$

$$= \int_0^1 \int_0^1 \int_0^{u-u^2} v \, dv \, du \, dw$$

$$= \int_0^1 \int_0^1 \left[\frac{1}{2}v^2\right]_0^{u-u^2} du\, dw = \int_0^1 \int_0^1 \frac{1}{2}(u^2 - 2u^3 + u^4)\, du\, dw$$

$$= \int_0^1 \left[\frac{1}{2}\left(\frac{1}{3}u^3 - \frac{1}{2}u^4 + \frac{1}{5}u^5\right)\right]_0^1 dw = \int_0^1 \frac{1}{2}\left(\frac{1}{3} - \frac{1}{2} + \frac{1}{5}\right) dw$$

$$= \int_0^1 \frac{1}{60}\, dw = \frac{1}{60}$$

3. Let $\mathbf{F}$ be a vector field given by

$$\mathbf{F}(x,y,z) = (y\cos z - yze^x, x\cos z - ze^x, -xy\sin z - ye^x).$$

(a) (4 points) Show that $\mathbf{F}$ is conservative.

(b) (4 points) Find a potential function for $\mathbf{F}$.

a) $\underline{F}$ is def. on all of $\mathbb{R}^2$ which is simply connected.

$$\operatorname{curl}(\underline{F}) = \nabla \times \underline{F} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ F_1 & F_2 & F_3 \end{pmatrix}$$

$$= \Big\langle (-x\sin z - e^x) - (-x\sin z - e^x), -(-y\sin z - ye^x) + (-y\sin z - ye^x), (\cos z - ze^x) - (\cos z - ze^x) \Big\rangle$$

$$= 0$$

Thus $\underline{F}$ is conservative!

b) Suppose $\nabla\varphi = F$. Then

$$\varphi_x = y\cos z - yze^x \Longrightarrow \varphi = xy\cos z - yze^x + \alpha(y,z)$$

$$\varphi_y = x\cos z - ze^x \Longrightarrow \varphi = xy\cos z - yze^x + \beta(x,z)$$

$$\varphi_z = -xy\sin z - ye^x \Longrightarrow \varphi = xy\cos z - yze^x + \gamma(x\,y)$$

so $\alpha = \beta = \gamma$ is a const. so

$$\varphi = xy\cos z - yze^x + C.$$

4. In this question we will calculate the surface area of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + z^2 = 1$.
   (a) (4 points) Find a parameterisation of the ellipsoid given above.
   (b) (5 points) Express the surface area as a double iterated integral.
   (c) (5 points) Evaluate the surface area when $a = 2$. You may use the fact that

$$\int \sqrt{1+x^2}\,\mathrm{d}x = \frac{1}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left(\sqrt{1+x^2}+x\right) + C.$$

a) The sphere is parametrised by

$$G' = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

so the ellipsoid is param. by

$$G(\theta,\phi) = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, \cos\phi)$$

$$(\theta,\phi) \in [0,2\pi]\times[0,\pi].$$

b) $$\underline{T}_\theta = (-a\sin\theta\sin\phi,\ a\cos\theta\sin\phi,\ 0)$$

$$\underline{T}_\phi = (a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -\sin\phi)$$

so $$\underline{N}(\theta,\phi) = \left(-a\cos\theta\sin^2\phi,\ -a\sin\theta\sin^2\phi,\ -a^2\sin^2\theta\sin\phi\cos\phi - a^2\cos^2\theta\sin\phi\cos\phi\right)$$

$$= \left(-a\cos\theta\sin^2\phi,\ -a\sin\theta\sin^2\phi,\ -a^2\sin\phi\cos\phi\right)$$

so

$$\|\underline{N}(\theta,\phi)\|^2 = a^2\cos^2\theta\sin^4\phi + a^2\sin^2\theta\sin^4\phi + a^4\sin^2\phi\cos^2\phi$$

$$= a^2\sin^4\phi + a^4\sin^2\phi\cos^2\phi$$

$$= a^2\sin^2\phi\,(\sin^2\phi + a^2\cos^2\phi)$$

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$$= \cancel{a^2\sin^2\phi\, a^2 (\sin^2\phi}$$

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$$= a^2 \sin^2\phi \left(1 - \cos^2\phi + a^2\cos^2\phi\right)$$

$$= a^2 \sin^2\phi \left(1 + (a^2-1)\cos^2\phi\right)$$

so $\|\underline{N}(\theta,\phi)\| = a\sin\phi\sqrt{1+(a^2-1)\cos^2\phi}$

$$\iint_S 1\,dS = \iint_D a\sin\phi\sqrt{1+(a^2-1)\cos^2\phi}\;dA_{\theta\phi}$$

where $D = \{(\theta,\phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\}$.

so

$$S.A. = \int_0^{\pi}\int_0^{2\pi} a\sin\phi\sqrt{1+(a^2-1)\cos^2\phi}\;d\theta\,d\phi$$

c) Let $x = \cos\phi$. Then $\frac{dx}{d\phi} = -\sin\phi$ so

$$S.A. = \int_0^{2\pi} -\int_1^{-1} a\sqrt{1+(a^2-1)x^2}\,dx = 2\pi a\int_{-1}^{1}\sqrt{1+(a^2-1)x^2}\,dx$$

let $u = \sqrt{a^2-1}\,x$ then $\frac{du}{dx} = \sqrt{a^2-1}$ so

$$S.A. = \frac{2\pi a}{\sqrt{a^2-1}}\int_{x=-1}^{x=1}\sqrt{1+u^2}\,du$$

$$= \frac{2\pi a}{\sqrt{a^2-1}}\left[\frac{1}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(\sqrt{1+u^2}+u\right)\right]_{x=-1}^{x=1}$$

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$$= \frac{2\pi a}{\sqrt{a^2-1}}\left[\frac{1}{2}\sqrt{1+(a^2-1)x^2} + \frac{1}{2}\ln\left(\sqrt{1+(a^2-1)x^2} + \sqrt{a^2-1}\,x\right)\right]_{x=-1}^{x=1}$$

$$= \frac{\cancel{2}\pi a}{\sqrt{a^2-1}}\left[\cancel{a} + \cancel{\tfrac{1}{2}}\ln\left(a+\sqrt{a^2-1}\right) - a - \cancel{\tfrac{1}{2}}\ln\left(a-\sqrt{a^2-1}\right)\right]$$

$$= \frac{\cancel{2}\pi a}{\sqrt{a^2-1}}\ln\left(\frac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}\right)$$

When $a = 2$:

$$S.A. = \frac{2\pi}{\sqrt{3}}\ln\left(\cancel{3}\,\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$$

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