Final practice 2

UCLA: Math 115A, Winter 2019

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Version: 1

- This exam has 1 questions, for a total of 54 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	54	
Total:	54	

Question 2 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. In each of the following questions, fill in the blanks to complete the statement of the definition or theorem.
   (a) (2 points) *Definition:* The dimension of a vector space V is defined to be the **number** of elements in any **basis** of V.

   (b) (2 points) *Definition:* The characteristic polynomial of a linear map T is defined to be

$$p_T(t) = \det(T_B^B - t\,\mathrm{id})$$

for any **basis** B of V.

Part I

Theorem: Suppose $T : V \longrightarrow W$ is a linear map between finite dimensional vector spaces. Then

$$\dim\ker T + \dim\operatorname{im} T = \dim V.$$

Part II

Theorem: Let V be a finite dimensional vector space over a field $\mathbb{F}$. A linear map $T : V \longrightarrow V$ is diagonalisable if and only if

- **the characteristic polynomial splits**, and
- for every eigenvalue $\lambda \in \mathbb{F}$, **the algebraic multiplicity equals** $\dim E_\lambda$.

Part III

Definition: Let V be a finite dimensional inner product space. The adjoint of a linear map $T : V \longrightarrow V$ is the unique linear map $T^* : V \longrightarrow V$ such that for any $v, w \in V$ we have

$$\langle T(v), w\rangle = \langle v, T^*(w)\rangle.$$

Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) Consider the following subspace of $\mathbb{R}^3$,

$$U = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; a + b + c = 0 \right\}$$

The dimension of U is

A. 0.
B. 1.
C. 2.
D. 3.

(b) (2 points) As a subset of $\mathbb{C}[x]$, the set

$$\{ 1 + x^2, x - x^2, 2 + x + x^2 \}$$

A. is a spanning set.
B. is linearly independent.
C. is neither spanning nor linearly independent.
D. is a basis.

(c) (2 points) What is the dimension of the subspace

$$\{ M \in \mathrm{Mat}_{n\times n}(\mathbb{F}) \mid Me_1 = 0 \} \subset \mathrm{Mat}_{n\times n}(\mathbb{F})$$

where e_i is the i^{th} standard basis vector.

A. $n^2 - 1$
B. $n^2 - n$
C. $n^2 + 1$
D. n

(d) (2 points) Let $V = \mathbb{R}_1[x]$, consider the map $T : V \longrightarrow V$ given by $T(p) = p(1) + p(-1)x$. Which of the following is *not* true.

A. T is a linear map.

B. The characteristic polynomial of T splits.

C. T is diagonalisable.

D. T has an eigenspace of dimension 2.

(e) (2 points) Consider again, the map T given above. Suppose V has the inner product given by

$$\langle p, q\rangle = p(1)q(1) + p'(1)q'(1)$$

What is $T^*(x)$?

A. $-1 - 4x$.

B. $3 + 4x$.

C. $1 - x$.

D. x.

Consider the vector space $V = \mathbb{R}_2[x]$ with its standard ordered basis

$$E = \left\{1, x, x^2\right\}$$

and the linear map

$$T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x],\ T(p) = p(x-1) - p(0)x^2$$

(a) (1 point) What is $[T]_E^E$?

Solution:

$$\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

(b) (1 point) Is T invertible?

Solution: No.

(c) (6 points) Compute the eigenvalues of T and their algebraic multiplicity.

Solution:

(d) (2 points) Is T diagonalisable? If so, find a matrix Q such that $Q^{-1}[T]_E^E Q$ is diagonal. If not, find Q, so that the above matrix is upper triangular.

Solution: Yes.

Consider the vector space $V = \mathbb{R}_2[x]$ with its standard ordered basis

$$E = \left\{1, x, x^2\right\}$$

with an inner product given by

$$\langle f(x), g(x)\rangle = \int_{-1}^{1} f(t)g(t)dt.$$

(a) (6 points) Use the Gram-Schmidt process to find an orthogonal basis B'.

Solution:

(b) (2 points) Give an orthonormal basis B of $V = \mathbb{R}_2[x]$.

Solution:

(c) (2 points) Let $f = 1 + x + x^2$. Compute the coordinate vector $[f]_B$.

Solution:

Let V be a finite dimensional vector space over a field $\mathbb{F}$. Consider the set $\mathcal{L}(V, V)$ of linear maps from V to V. Fix a linear map $S \in \mathcal{L}(V, V)$. Define the subset

$$C(S) = \{ T \in \mathcal{L}(V, V) \mid ST = TS \} .$$

(a) (3 points) Prove or disprove that $C(S)$ is a subspace.

Solution:

(b) (1 point) Determine $C(\mathrm{id})$.

Solution:

(c) (3 points) Show that if v is an eigenvector for S and $T \in C(S)$, then $T(v)$ is as well.

Solution:

(d) (3 points) Suppose that S has $n = \dim V$ distinct eigenvalues. Show that any $T \in C(S)$ is diagonalisable. *Hint: what dimension do the eigenspaces of S have? Now use part c.*

Solution:

Let V be a finite dimensional inner product space over $\mathbb{R}$, with $\dim V = n$. A linear map $S : V \longrightarrow V$ which preserves the inner product (i.e. $\langle v, w\rangle = \langle S(v), S(w)\rangle$ for any $v, w \in V$) is called a *reflection* if $S^2 = \mathrm{id}$ and the nullity of $S - \mathrm{id}$ is $n - 1$. Let S be a reflection.

(a) (1 point) Give an example of a reflection for the vector space $V = \mathbb{R}^2$ with the usual inner product. *Hint: the name should mean something!*

Solution:

(b) (2 points) Show that if $v \in \ker(S - \mathrm{id})^{\perp}$ then $S(v) \in \ker(S - \mathrm{id})^{\perp}$.

Solution:

(c) (2 points) Determine the eigenvalues of S. *Hint: if λ is an eigenvalue, then λ^2 is an eigenvalue of S^2.*

Solution:

(d) (1 point) Is S diagonalisable?

Solution:

(e) (4 points) Show that there exists a vector of unit length, $v \in V$ such that for any $w \in V$,

$$S(w) = w - 2\langle w, v\rangle v$$

Solution:

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