Math 3B: Lecture 3

Noah White

September 28, 2016

Last time

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- Graphing using calculus

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- Horizontal asymptotes

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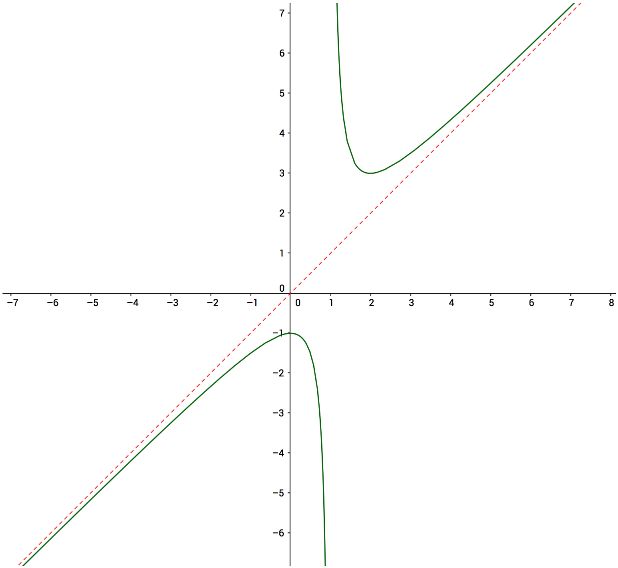
Note: The quiz will start at the beginning of the discussion section next time.

Example time

... On the board.

Slanted asymptotes



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$$\lim_{x \to \pm\infty} f'(x) = m$$
- We then know the function has a slanted asymptote $y = mx + b$.
- To find b:
$$b = \lim_{x \to \pm\infty} (f(x) - mx)$$

Example time

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Functions

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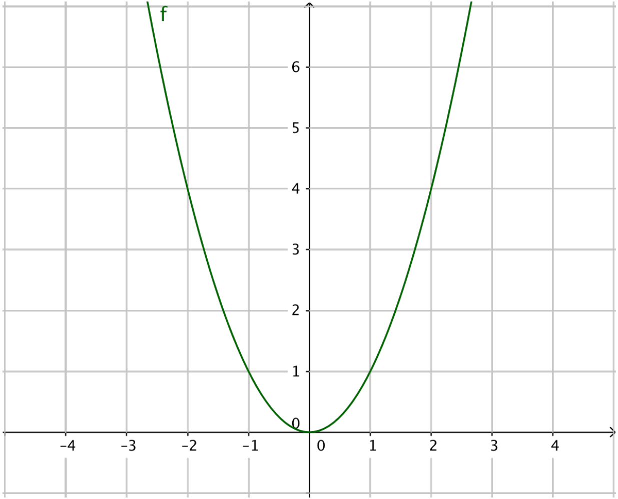
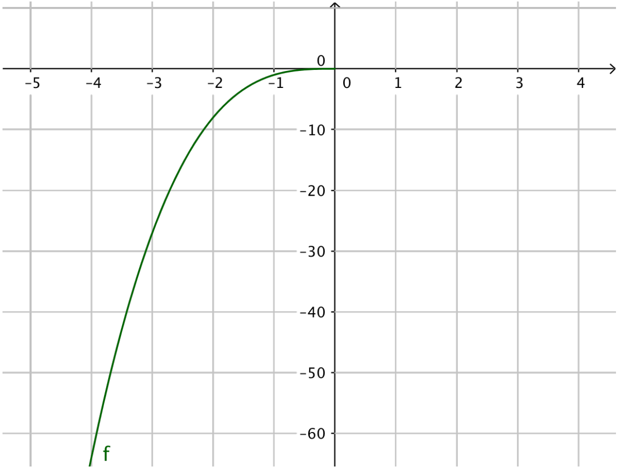


Example of a global maximum

$f : (-\infty, 0\,] \longrightarrow \mathbb{R}; f(x) = x^3$ has a max at $x = 0$



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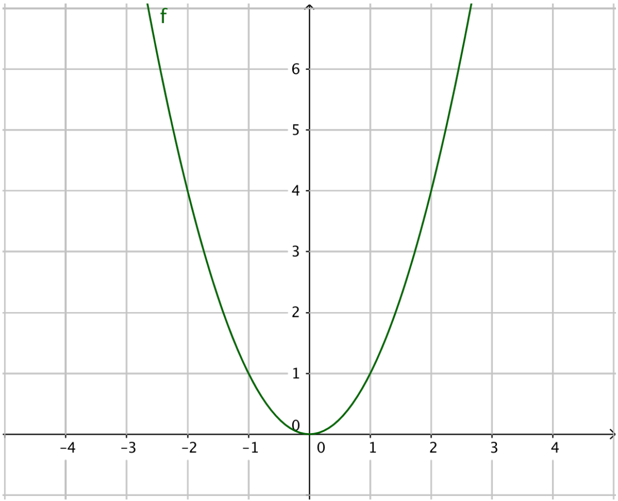
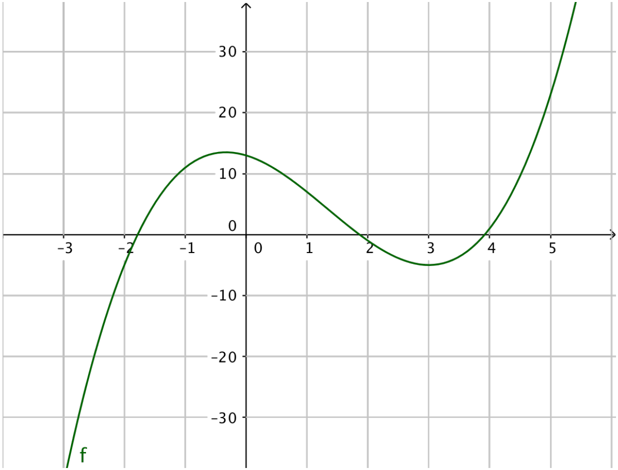


Example of a local maximum

$$f : \mathbb{R} \longrightarrow \mathbb{R}; f(x) = x^3 - 4x^2 - 3x + 13 \text{ has a local max at } x = -\frac{1}{3}$$



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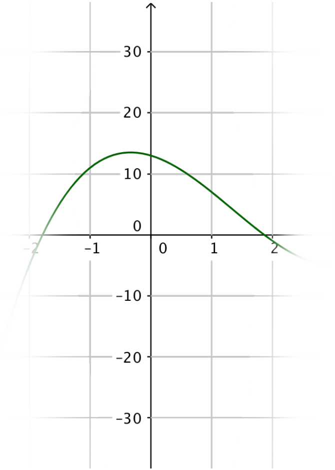

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- $f(x) = \sin x$ has a critical point at $x = \frac{\pi}{2}$ (since $f'(x) = \cos x$)
- $f(x) = e^x$ doesn't have any critical points since $f'(x) = e^x$ can never be zero

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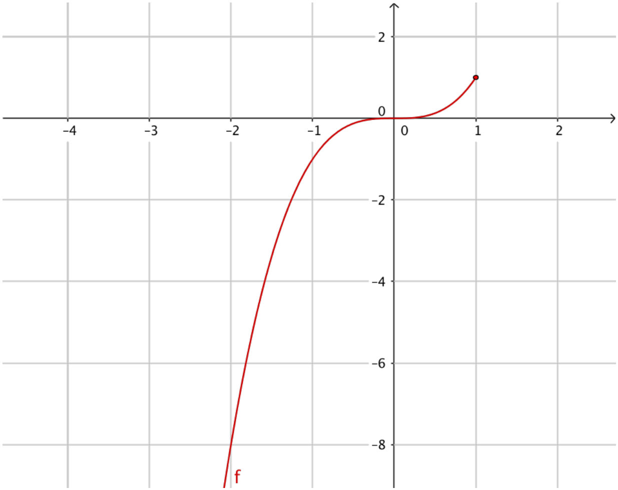
Example

$f : (-\infty, 1] \longrightarrow \mathbb{R}$; $f(x) = x^3$ has critical points at

$$x = 0 \text{ and } 1$$

Example

$f'(x) = 3x^2$ so $f'(0) = 0$ and $f'(1)$ is undefined.



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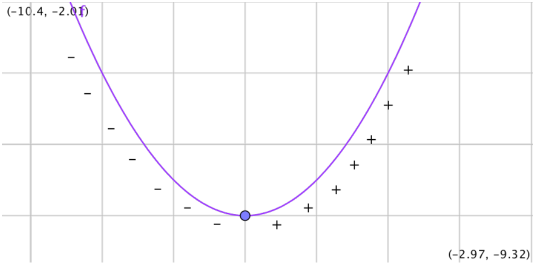

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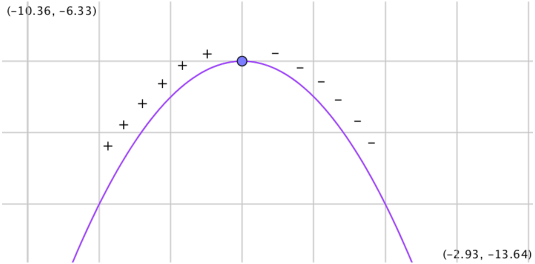

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Note: If $f''(a) = 0$ then we cannot conclude anything! E.g x^3 or x^4.

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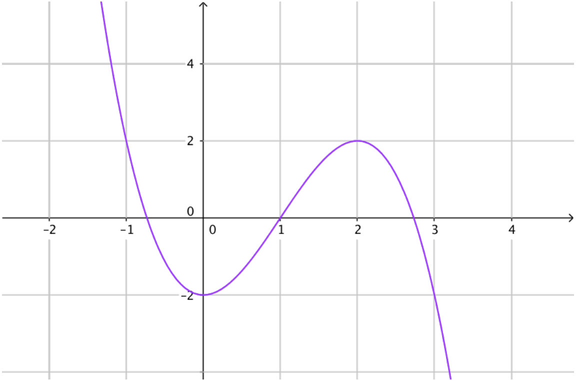

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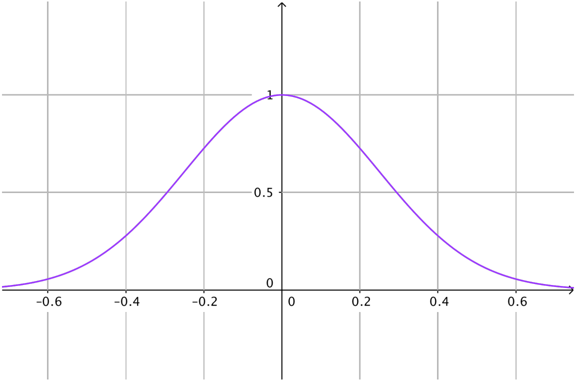

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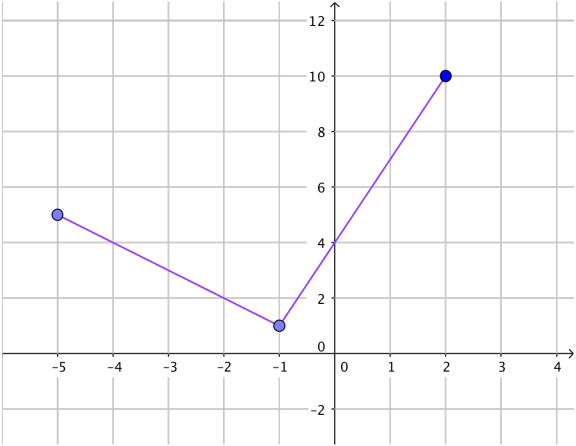

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Suppose we create a rectangle with length x feet and width y feet. They must satisfy

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We would like to find the value of x which maximises this function!

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The second derivative $A''(x) = -2$ is always negative so this must be a maximum! Thus the dimensions for the rectangle with the largest area are

$$x = y = \frac{M}{4}$$

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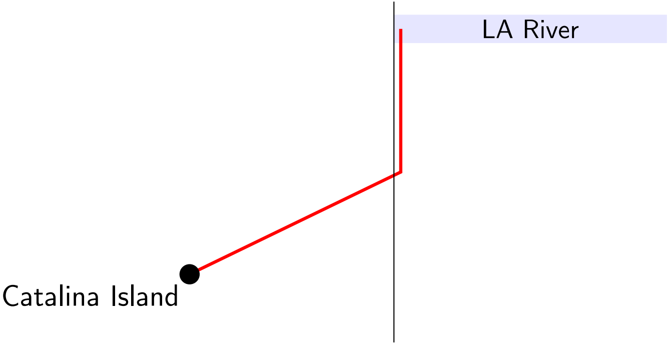

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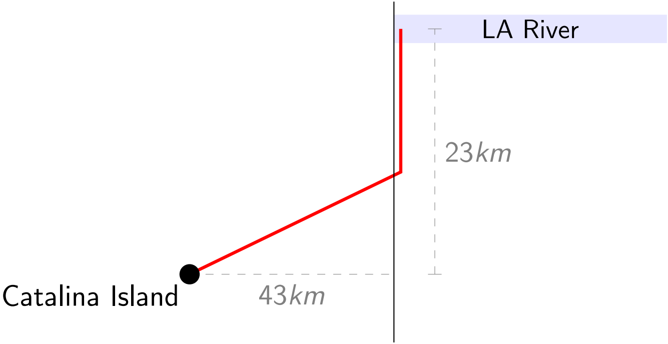

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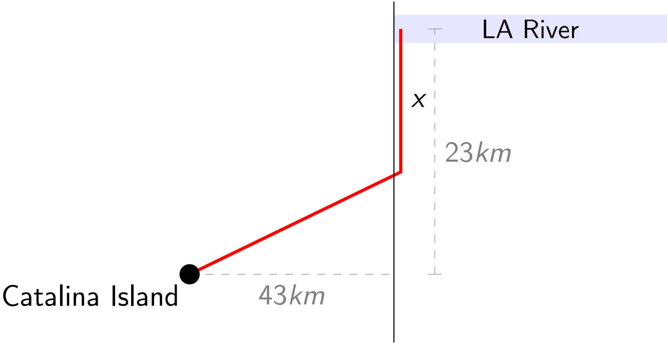
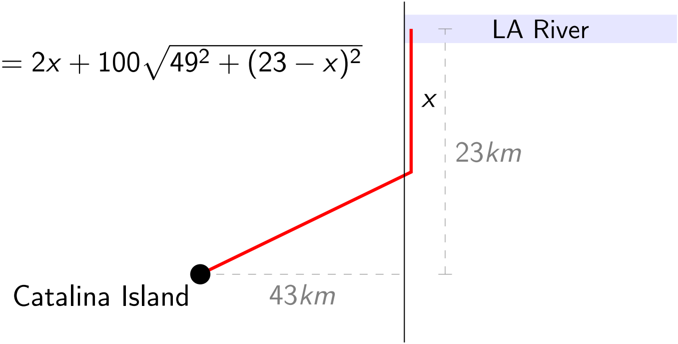


Catalina island

Question

You want to get from Catalina Island to Long Beach/LA River. An UberX costs $\$2/km$ on land, and you can charter a boat for $\$100/km$. You are very thrifty and want to save money, how far from the LA River should you come ashore?

$$C(x) = 2x + 100\sqrt{49^2 + (23 - x)^2}$$



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So the solutions are

$$x = 23 - \frac{43}{3\sqrt{11}} \quad \text{and} \quad 23 + \frac{43}{3\sqrt{11}}$$