This week on the problem set you will get practice with continuous random variables. Especially challenging questions, or questions that are not appropriate for an exam, are indicated with one or more asterisks.

1. From the textbook, chapter 2, problems 1, 2.

2. From the supplementary problems, chapter 2, problem 2.

3. Which of the following functions can be a probability density function for some continuous random variable. Explain your reasoning.

   (a)
$$f(t) = \begin{cases} e^t + 4t - e, & \text{for } 0 \leq t \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

   (b)
$$f(t) = \begin{cases} e^{-t}, & \text{for } t \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

   (c)
$$f(t) = e^{\sin(t)}, \text{ for all } t \in \mathbb{R}.$$

   (d)
$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ \frac{1}{2}, & \text{for } 0 \leq t \leq 1 \\ 0, & \text{for } 1 < t < 2 \\ \frac{1}{2}, & \text{for } 2 \leq t \leq 3 \\ 0, & \text{for } 3 < t. \end{cases}$$

4. Continuous random variable X has probability density function $f_X(t)$ such that
$$f_X(t) = \begin{cases} C\sin(\pi t/n), & \text{for } 0 \leq t \leq n \\ 0, & \text{otherwise.} \end{cases}$$
   Find the value of the constant C. Then find $\mathbb{E}(X)$ and $\text{var}(X)$.

5. Let $p > 0$ be a real number. Consider function f which has values $f(t) = Ct^{-p}$ for $t \geq 1$, and $f(t) = 0$ for $t < 1$. For which values of p we can select constant C so that this function f is a probability density function for some continuous random variable? For which values of p will this random variable have finite expectation? For which values of p will this random variable have finite variance?

6. Let X be a positive continuous random variable with the probability density function $f(t)$. If $g(t) = tf(t)$ is also a probability density function of some random variable Y what is $\mathbb{E}(X)$? Express $\text{var}(X)$ in terms of $\mathbb{E}(Y)$.

7. You are operating a train. Ticket for this train costs \$10. The train is late to the destination T minutes, where T is an exponential random variable with parameter 1. If the train is more than 2 minutes and less than 4 minutes late then each customer gets half of the ticket refunded. If the train is more than 4 and less than 10 minutes late each customer gets the full price of the ticket refunded. If the train is more than 10 minutes late each customer gets the full price of the ticket refunded and in addition gets \$$n$. For what values of n would your expected profit be positive? For what values of n would you expect to earn at least \$5 per ticket.

8. If X is a continuous random variable which attains only positive values (that is $X \geq 0$). Show that
$$\mathbb{E}(X) = \int_0^{\infty} \mathbb{P}(X \geq t)\ dt.$$
   Note that this also holds for discrete random variables, you can try to prove this formula for discrete random variables as well.