

# Final practice 1

UCLA: Math 115A, Spring 2019

*Instructor:* Noah White

*Date:*

*Version:* practice

- This exam has 8 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: \_\_\_\_\_

ID number: \_\_\_\_\_

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Question 2 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

*Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!*

**Question 2.**

<i>Part</i>	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. In each of the following questions, fill in the blanks to complete the statement of the definition or theorem.

(a) (2 points) *Definition:* Let  $V$  be a vector space. A subset  $B$  is called a basis of  $V$  if  $B$  is

- \_\_\_\_\_ and
- is a \_\_\_\_\_ set.

(b) (2 points) *Definition:* Let  $V$  and  $W$  be vector spaces over a field  $\mathbb{F}$ . A function  $T : V \rightarrow W$  is a linear map if

- \_\_\_\_\_ for every  $v, w \in V$ , and
- \_\_\_\_\_ for every  $\lambda \in \mathbb{F}$  and  $v \in V$ .

(c) (2 points) *Theorem:* Suppose  $\dim V = n$ . If  $T : V \rightarrow V$  is a linear map with \_\_\_\_\_ distinct eigenvalues, then  $T$  is \_\_\_\_\_.

(d) (2 points) *Definition:* A basis  $B$  for a vector space  $V$  equipped with an inner product is called orthonormal if

- \_\_\_\_\_ for every  $v \in B$ , and
- \_\_\_\_\_ for every  $v, w \in B$ .

(e) (2 points) *Definition:* The Frobenius inner product is an inner product defined on the vector space \_\_\_\_\_ and is defined by the formula

\_\_\_\_\_.

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) The subset

$$V_\lambda = \{ p \in \mathbb{R}[x] \mid p'(\lambda) = 0 \}$$

is a subspace

- A. for any choice of  $\lambda \in \mathbb{R}$ .
- B. only for  $\lambda = 0$ .
- C. whenever  $\lambda > 0$ .
- D. for no choice of  $\lambda$ .

(b) (2 points) As a subset of  $\text{Mat}_{2 \times 2}(\mathbb{C})$ , the set

$$\left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\}$$

- A. is a spanning set.
- B. is linear independent.
- C. is neither spanning nor linearly independent.
- D. is a basis.

(c) (2 points) Suppose  $n \geq 2$ . Consider the function  $T : \text{Mat}_{n \times n}(\mathbb{C}) \longrightarrow \text{Mat}_{n \times n}(\mathbb{C})$  given by  $T(M) = M^t$ , the transpose matrix. Which of the following is *not* true.

- A.  $T$  is a linear map.
- B.  $T^2 = -\text{id}$ .
- C.  $T$  has eigenvalues 1 and  $-1$ .
- D.  $T$  has an eigenspace of dimension  $\frac{1}{2}n(n+1)$ .

(d) (2 points) Consider again, the map  $T$  given above. Which of the following is true.

- A.  $T$  is diagonalisable, only when  $n = 2$ .
- B.  $T$  is diagonalisable for any  $n$ .
- C.  $T$  is never diagonalisable.
- D. The characteristic polynomial for  $T$  does not split.

(e) (2 points) Which of the following pairs of matrices are orthogonal in  $\text{Mat}_{2 \times 2}(\mathbb{C})$  equipped with the Frobenius inner product?

- A.  $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$
- B.  $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$
- C.  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$
- D.  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

3. Consider the vector space  $V = P_2(\mathbb{R})$  with its standard ordered basis

$$\beta = \{1, x, x^2\}$$

and the linear maps

$$T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R}), \quad T(f) = f(1) + f(-1)x + f(0)x^2$$

$$S : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R}), \quad S(ax^2 + bx + c) = cx^2 + bx + a.$$

- (a) (2 points) What is  $[T]_\beta$  and  $[S]_\beta$ ? Show that

$$[TS]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- (b) (6 points) Compute  $[(TS)^{-1}]_\beta$ .

- (c) (2 points) What is  $(TS)^{-1}(x^2 + x + 1)$ ?

4. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \in M_{3 \times 3}(\mathbb{R}),$$

- (a) (2 points) Compute the characteristic polynomial of  $A$  and determine the eigenvalues and their algebraic multiplicity.
- (b) (6 points) Is  $A$  diagonalizable? If yes, compute a basis  $\beta$  of eigenvectors of  $A$ .
- (c) (2 points) Compute  $[L_A]_\beta$ , where the  $L_A$  is the linear transformation given by

$$L_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3, v \mapsto Av.$$

5. Consider the vector space  $V = \mathbb{R}^4$  with its standard inner product. Consider the linearly independent subset

$$S = \{w_1 = (1, 0, 1, 0), w_2 = (1, 1, 1, 1), w_3 = (2, 2, 0, 2)\}.$$

- (a) (6 points) Apply the Gram-Schmidt orthogonalization algorithm to  $S$  to compute an orthogonal basis  $\beta'$  of  $\text{Span}(S)$ .
- (b) (2 points) Use your result to compute an orthonormal basis  $\beta$  of  $\text{Span}(S)$ .
- (c) (2 points) Let  $x = (1, 2, 3, 2) \in \text{Span}(S)$ . Compute the coordinate vector  $[x]_\beta$ .

6. Let  $S : U \rightarrow V$  and  $T : V \rightarrow W$  be linear transformations between finite dimensional vector spaces  $U, V$  and  $W$  over a field  $F$ .
- (a) (2 points) Let  $v_1, v_2, \dots, v_n \in V$  be linearly independent and  $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ , such that  $\lambda_i \neq 0$  for all  $1 \leq i \leq n$ . Show that also  $\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n$  are linearly independent.
- (b) (4 points) Let  $v, w \in V$ . Show that  $\text{span}(v, w) = \text{span}(v, v + w)$ .  
*Hint:* Proceed in two steps: Show that for all  $x \in V$ ,
1.  $x \in \text{span}(v, w)$  implies  $x \in \text{span}(v, v + w)$  and
  2.  $x \in \text{span}(v, v + w)$  implies  $x \in \text{span}(v, w)$ .
- (c) (4 points) Assume that  $\text{R}(S) = \text{N}(T)$ , i.e. the range of  $S$  is equal to the nullspace of  $T$ . Assume furthermore that  $S$  is one-to-one and  $T$  is onto. Show that

$$\dim V = \dim U + \dim W.$$

*Hint:* Use the rank-nullity formula.



7. Let  $V$  be a finite dimensional vector space over a field  $F$ . Recall that

$$\mathcal{L}(V, V) = \{T : V \rightarrow V \mid T \text{ is a linear transformation}\}$$

denotes the vector space of linear transformations from  $V$  to  $V$  (also called linear operators on  $V$ ). Fix a vector  $v \in V$  and define

$$Z = \{T \in \mathcal{L}(V, V) \mid T(v) = 0\}.$$

One calls  $Z$  the *annihilator* of  $v$  in  $\mathcal{L}(V, V)$ .

- (a) (4 points) Show that  $Z$  is a subspace of  $\mathcal{L}(V, V)$ .
- (b) (2 points) Let  $\lambda \in F$  such that  $\lambda \neq 0$ . Prove or disprove (by finding a counterexample) that

$$Z' = \{T \in \mathcal{L}(V, V) \mid T(v) = \lambda v\}$$

is a subspace of  $\mathcal{L}(V, V)$ .

- (c) (2 points) Assume that  $\beta = \{v_1, \dots, v_n\}$  is an ordered basis of  $V$ , such that  $v_1 = v$ . Let  $T \in \mathcal{L}(V, V)$ . Show that  $T \in Z$  if and only if the first column of  $A = [T]_\beta$  equals 0.
- (d) (2 points) Assuming  $v \neq 0$ , what is  $\dim(Z)$ ?

8. Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$  with an inner product  $\langle x, y \rangle \in \mathbb{R}$  for  $x, y \in V$ .

(a) (3 points) Let  $\lambda \in \mathbb{R}$  with  $\lambda > 0$ . Show that

$$\langle x, y \rangle' = \lambda \langle x, y \rangle, \text{ for } x, y \in V,$$

defines an inner product on  $V$ .

(b) (2 points) Let  $T : V \rightarrow V$  be a linear operator, such that

$$\langle T(x), T(y) \rangle = \langle x, y \rangle, \text{ for all } x, y \in V.$$

Show that  $T$  is one-to-one.

(c) (2 points) Recall that the *norm* of a vector  $x \in V$  is defined by  $\|x\| = \sqrt{\langle x, x \rangle}$ . Show that

$$\langle x, y \rangle = \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2), \text{ for all } x, y \in V.$$

Hence, the inner product can be recovered from the norm.

*Hint:* Rewrite  $\langle x + y, x + y \rangle$  using the properties of inner products. Use that  $\langle x, y \rangle \in \mathbb{R}$  is a real number by assumption.

(d) (3 points) Let  $\beta = \{v_1, \dots, v_n\}$  be a basis of  $V$ . The *Gram matrix*  $G \in M_{n,n}(\mathbb{R})$  of the inner product  $\langle -, - \rangle$  with respect to  $\beta$  is defined by

$$G_{i,j} = \langle v_i, v_j \rangle.$$

Show that  $G$  is invertible.

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