This weeks problem set focuses on the ideas of bases and linear transformations. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

1. From section 1.6, problems 1, $2a, e$, $3a, c$, 4, 6, 14, 15, 20^*, 26, $28^\dagger$, 33, 34^*, 35^*.

2. From section 2.1, problems 1, 2, 5, 6, 9^*, 14, $14b$.

3.† Let $V = \mathbb{F}^n$ for some field $\mathbb{F}$. If $v \in V$ (i.e. v is a column vector) a *permutation* of v is any column vector obtained from v by rearranging the entries. For example

$$\begin{pmatrix} v_1 \\ v_3 \\ v_4 \\ v_2 \end{pmatrix} \text{ is a permutation of } \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

We say that a subspace $U \subseteq V$ is *permutation invariant* if for any $v \in U$ then any permutation of v is also in U.

(a) Give an example of a one dimensional, permutation invariant subspace when $n = 2$.

Solution: We can take $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\} = \operatorname{span}\{e_1 + e_2\}$.

(b) Give an example of a one dimensional, permutation invariant subspace for any n. Call this subspace T.

Solution: Similarly, we can take $T = \operatorname{span}\{e_1 + e_2 + \cdots e_n\}$.

(c) Show that the subspace $\Sigma_n \subseteq V$ is permutation invariant.

Solution: Suppose that $v \in \Sigma_n$. Then if

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

we have by definition that $v_1 + v_2 + \cdots + v_n = 0$. If we rearrange the entries of v this sum does not change so any permutation of v is in Σ_n.

(d) Suppose that U is a permutation invariant subspace that does not contain $e_1 - e_2$. Then the first two entries of any vector in U are equal.

Solution: Proof by contradiction. Suppose that U contains a vector v with the first two entries v_1 and v_2 different. Let u be the vector with the first two entries swapped. Then $v - u = (v_1 - v_2)(e_2 - e_2)$. Since U is a subspace, it is closed under scalar multiplication and since $v_1 - v_2 \neq 0$ we can divide by it. Hence U contains $e_1 - e_2$ which is a contradiction.

(e) Suppose that U is a permutation invariant subspace such that the first two entries of any vector in U are equal. Show that $U = \{0\}$ or T.

Solution: Let $v \in U$ and suppose that

$$v = \begin{pmatrix} x \\ x \\ x_3 \\ \vdots \\ x_n \end{pmatrix}$$

Then we can swap the 2^{nd} and the i^{th} entries to get the vector

$$u = \begin{pmatrix} x \\ x_i \\ x_3 \\ \vdots \\ x_n \end{pmatrix}.$$

But U is permutation invariant so $u \in U$, thus the first two entries of u are equal. I.e. $x = x_i$ for any i. Thus all the entries of v are equal!

The only subspaces with this property are $\{0\}$ and T.

(f) List all the permutation invariant subspaces. *Hint: this is tricky, you will need to use the previous two parts.*

Solution: First we have the following permutation invariant subspaces $\{0\}, T, \Sigma_n, V$. We claim these are the only ones.

To see this, suppose we had another permutation invariant subspace U not on this list. If $e_1 - e_2 \in U$ then we also have $e_1 - e_i \in U$ for any $2 \leq i \leq n$ (since we can rearrange the entries). This is a basis of Σ_n so we must have that $\Sigma_n \subseteq U$. This is only possible if $U = \Sigma_n$ or V.

Now suppose that $e_1 - e_2 \notin U$. Then by the previous part we must have that $U = \{0\}$ of T. Hence the above is a complete list.

(g) Is it possible to always have two non-trivial, permutation invariant subspaces U, W such that $U \oplus W = V$? *Hint: you will need a condition on the characteristic of the field!*

Solution: It is clear for dimension reasons that we must have $U = T$ and $W = \Sigma_n$ (or visa versa). We have a problem when the characteristic of $\mathbb{F}$ divides n. In this case $T \subset U$ so we cannot have a direct sum!

Suppose that the characteristic of $\mathbb{F}$ does not divide n. Then T is not a subset of Σ_n and since it is one dimensional we must have that $T \cap \Sigma = \{0\}$. Furthermore, it is not hard to see that

$$\{e_1 + e_2 + \cdots + e_n, e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n\}$$

is a basis and so every vector in V can be written as a sum of a vector in T and a vector in Σ_n.