Math 3B: Lecture 11

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Introduction

Midterm 1

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Homework

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- Two problems to hand in Friday 3 November

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- Two problems to hand in Friday 3 November
- Problem 7, problem set 4

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- Problem 7, problem set 4
- Problem 3, problem set 5

Integration by substitution

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Suppose $u = g(x)$, then

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$$\int f(g(x))\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x = \int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

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Question

$$\int 4x\sqrt{x^2+1}\,\mathrm{d}x$$

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Solution

We use the substitution $u = x^2 + 1$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we can write the integral

$$\int 2 \cdot \sqrt{x^2+1} \cdot 2x\,\mathrm{d}x = 2\int \sqrt{u}\,\mathrm{d}u$$

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$$\begin{aligned}\int 2 \cdot \sqrt{x^2+1} \cdot 2x\,\mathrm{d}x &= 2\int \sqrt{u}\,\mathrm{d}u \\ &= 2 \cdot \frac{2}{3}u^{\frac{3}{2}} + C\end{aligned}$$

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&= 2 \cdot \frac{2}{3}u^{\frac{3}{2}} + C \\
&= \frac{4}{3}(x^2+1)^{\frac{3}{2}} + C
\end{aligned}$$

Integration by substitution (definite integrals)

Substitution for definite integrals

Suppose $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

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Example

$$\int_0^1 4x\sqrt{x^2+1}\,\mathrm{d}x = 2\int_1^2 \sqrt{u}\,\mathrm{d}u$$

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$$\begin{aligned}\int_0^1 4x\sqrt{x^2+1}\,\mathrm{d}x &= 2\int_1^2 \sqrt{u}\,\mathrm{d}u \\ &= 2\left[\frac{2}{3}u^{\frac{3}{2}}\right]_1^2\end{aligned}$$

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The product rule

Just like integration by substitution reverses the chain rule, integration by parts "reverses" the product rule:

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

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writen another way

$$(uv)' = u'v + uv'$$

Integration by parts

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Lets integrate both sides

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By the fundamental theorem of calculus

$$uv = \int u'v \,\mathrm{d}x + \int uv' \,\mathrm{d}x$$

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Rearranging...

Integration by parts

The integration by parts formula

$$\int uv' \,\mathrm{d}x = uv - \int u'v \,\mathrm{d}x$$

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Alternative statement

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u$$

Examples

One the board. . .

How to deal with rational functions?

How can we integrate something like

$$\int \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} \, \mathrm{d}x$$

or

$$\int \frac{x+2}{x^3 - x} \, \mathrm{d}x?$$

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

using polynomial long division.

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This is still not something we can integrate so we need to go further.

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots$$

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How do we reverse this process?

Answer: partial fractions

When the denominator is $(ax+b)(cx+d)\cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$$

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we can always find constants $A_1, A_2, \dots, n$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots \frac{A_n}{a_nx + b_n}$$

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Multiplying both sides by $(x-1)(x+1)$

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x+1} + \frac{B(x-1)(x+1)}{x-1} \\ &= A(x-1) + B(x+1) \\ &= (A+B)x + (B-A) \end{aligned}$$

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Comparing coefficients

$$A + B = 0 \qquad \text{and} \qquad B - A = 1$$

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So

$$-2A = 1 \quad \text{hence} \quad A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$

Repeated factors

What if $q(x)$ contains repeated factors? E.g. if $q(x) = (x-1)^2$ or $q(x) = (x-1)(x+2)^3$?

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For every factor $(ax+b)^k$ in $q(x)$, the partial fraction expansion has terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}.$$

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$$A = 1 \quad \text{and} \quad B = 1.$$

Side note: integrating $\frac{1}{x}$.

Recall that

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Using substitution this gives the formula

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + C.$$

Side note: integrating $\frac{1}{x^k}$.

Recall that if $k > 1$

Fact

$$\int \frac{1}{x^k}\,\mathrm{d}x = -\frac{1}{(k-1)x^{k-1}} + C$$

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Integrating rational functions $p(x)/q(x)$

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1. Express $\frac{p(x)}{q(x)}$ in the form

$$d(x) + \frac{r(x)}{q(x)}$$

   using polynomial long division.

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2. Write $\frac{r(x)}{q(x)}$ as a sum of fractions of the form

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2. Write $\frac{r(x)}{q(x)}$ as a sum of fractions of the form

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3. Integrate all these pieces seperately.

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Solution

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Using long division and partial fractions

$$\frac{x^4 - 3x^2 + 3}{x^2 - 1} = x^2 - 2 + \frac{1}{x^2 - 1} = x^2 - 2 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

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So

$$I = \frac{1}{3}x^2 - 2x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

Example 2

$$I = \int \frac{x^3 - 2x^2 + 4x}{(x-1)^3}\,\mathrm{d}x$$

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Using long division

$$\frac{x^3 - 2x^2 + 4x}{(x-1)^3} = 1 + \frac{x^2 + x + 1}{(x-1)^3}$$

Example 2

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Solution

Using long division and partial fractions

$$\frac{x^3 - 2x^2 + 4x}{(x-1)^3} = 1 + \frac{x^2 + x + 1}{(x-1)^3} = 1 + \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3}$$

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So

$$I = x + \ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + C.$$