

This weeks problem set focuses on the concept of a change of basis matrix. A question marked with a  $\dagger$  is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a  $*$  is especially important.

**Homework 2:** due Friday May 8: questions 2, 5 and 6 below.

1. From section 2.5, problems 1,  $2a, c$ ,  $3a, c$ , 5, 7,  $10^*$ ,  $13^*$ .
- 2\* Let  $V$  be a finite dimensional vector space and  $W$  a subspace. Show that  $V$  and  $W \times V/W$  are isomorphic by finding an explicit isomorphism (rather than simply computing the dimensions).
- 3\* Let  $V$  be a finite dimensional vector space and  $W$  a subspace. Show that  $\dim(V/W) = \dim V - \dim W$ .  
*Hint: consider a basis of  $W$  and extend it to  $V$ . Now find a basis for  $V/W$ . You can also prove it using the dimension theorem.*
- 4\* Let  $T : V \rightarrow W$  be a linear map.
  - (a) Show that  $\text{im } T$  and  $V/\ker T$  are isomorphic.
  - (b) Use this (and the previous exercise) to give an alternative proof of the dimension theorem.
5. A differential operator on  $\mathbb{R}_n[x]$  is a linear combination of expressions of the form  $x^a \frac{d^b}{dx^b}$  where  $a - b \leq 0$  (otherwise the degree would potentially increase!) and  $b \leq n$ . We can consider a differential operator as a linear map  $\mathbb{R}_n[x] \rightarrow \mathbb{R}_n[x]$ .
  - (a) Let  $D : \mathbb{R}_2[x] \rightarrow \mathbb{R}_2[x]$  be the differential operator given by  $2 - 4\frac{d}{dx} + 2x\frac{d^2}{dx^2}$ . Find the matrix of  $D$  relative to the basis  $\{x^2, (x-1)^2, (x+1)^2\}$ . *Note: the 2 in  $D$  means multiply by 2, so  $D(1) = 2$  and  $D(x) = 2x - 4$ .*
  - (b) Does the differential equation  $2f - 4\frac{df}{dx} + 2x\frac{d^2f}{dx^2} = 0$  have any solutions  $f \in \mathbb{R}_2[x]$ ? *Hint: what is a solution in terms of the linear map  $D$ ?*
  - (c) Suppose  $E : \mathbb{R}_2[x] \rightarrow \mathbb{R}_2[x]$  is a differential operator and that the matrix of  $E$ , relative to the basis  $\{1, x, x^2\}$  is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find  $E$ .

6. Consider the linear map  $X : \mathbb{R}_n[x] \rightarrow \mathbb{R}_n[x]$  given by  $X(p) = \frac{dp}{dx} + \frac{x^n}{n!}p(0)$ . Calculate the dimension of

$$C(X) = \{ T \in \text{Hom}(\mathbb{R}_n[x], \mathbb{R}_n[x]) \mid T \circ X = X \circ T \}.$$

*Hint: this will be quite tricky without involving matrices. It is also a very good idea to try  $n = 1, 2, 3$  before moving on to the general statement.*