Alternating series + absolute convergence

Today we consider series

$$\sum a_n$$

where a_n is not necessarily positive.

Def We will say $\sum_{n=0}^{\infty} a_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

Ex The series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}$ converges absolutely since

$$\sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges (by the integral test, for example).

Ex The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is not absolutely convergent since $\sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ does not converge.

So how do we decide if a series like $\sum \frac{(-1)^n}{n}$ is convergent?

Def We say a series $\sum a_n$ is conditionally convergent if it is convergent but not absolutely convergent (ie $\sum |a_n|$ does not converge)

Thm If $\sum a_n$ is absolutely convergent then $\sum a_n$ converges

proof: This follows from the fact that

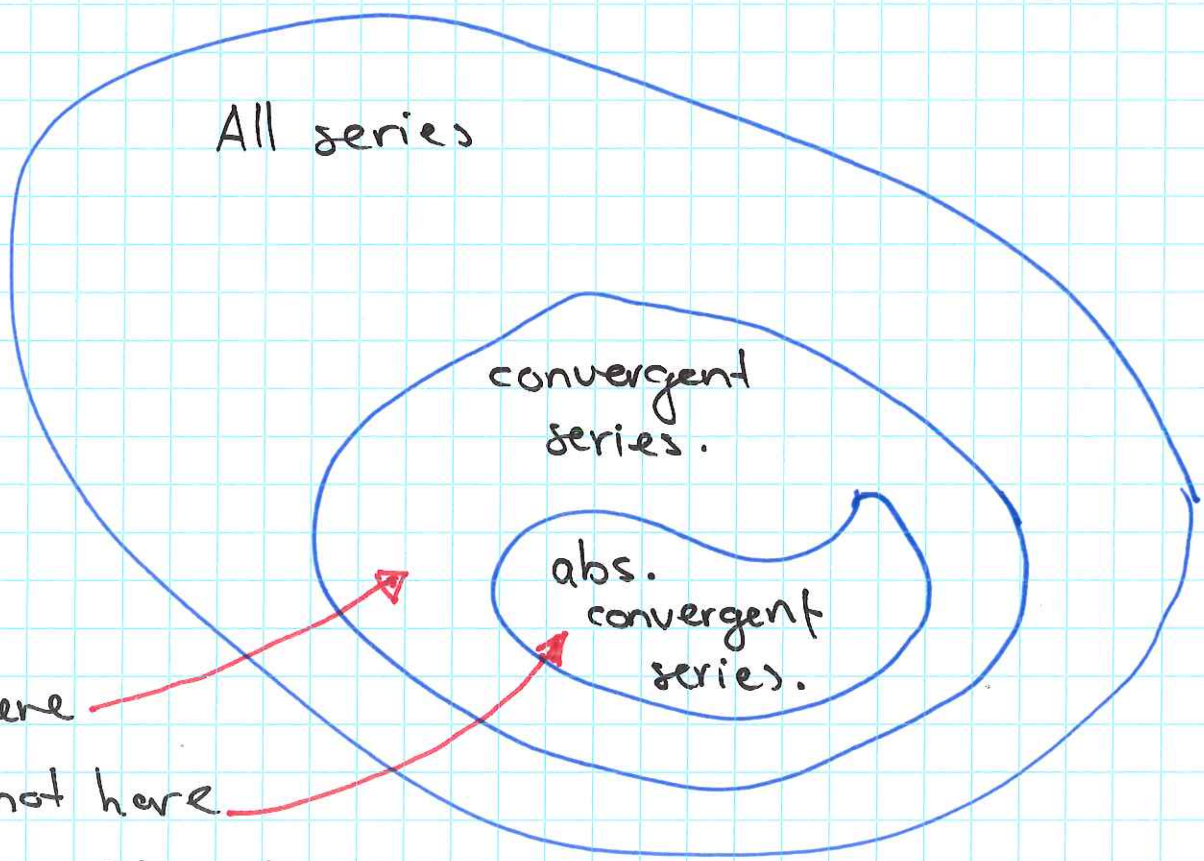
$$-|a_n| \le a_n \le |a_n|$$

and so

$$S_{4N}$$

$$-|a_1| \dots - |a_{N-1}| - |a_N| \le S_N \le |a_1| + \dots + |a_N|$$

and since $\lim\limits_{N \to \infty} |a_1| + \dots + |a_N|$ exists so does $\lim\limits_{N \to \infty} S_N$.



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but not here

are conditionally convergent series.

Thm (Alternating series test) Suppose (a_n) is a decreasing sequence such that

$$\lim_{n\to\infty} a_n = 0$$

Then $\sum \cancel{a}(-1)^n a_n$ converges.

Ex Does $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converge?

Let $a_n = \frac{1}{n}$. Then a_n is decreasing and $\displaystyle\lim_{n\to\infty} \frac{1}{n} = 0$ so $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.