This week you will practice writing differential equations modelling real world phenomena as well as understanding population models. You will also get practice solving separable differential equations.

Homework: The homework will be due on Friday 15 February, at 8am, the *start* of the lecture. It will consist of questions:

$7, 10.$(l) from problem set 5 and 5.(l) from this problem set below

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. Divide $p(x)$ by $q(x)$ and express the quotient as a divisor plus a remainder.
   (a) $p(x) = 2x^3 + 4x^2 - 5$, $q(x) = x + 3$
   (b) $p(x) = 15x^4 - 3x^2 - 6x$, $q(x) = 3x + 6$
   (c) $p(x) = 2x^4 - 5x^3 + 6x^2 + 3x - 2$, $q(x) = x - 2$
   (d) $p(x) = 5x^4 + 2x^3 + x^2 - 3x + 1$, $q(x) = x + 2$
   (e) $p(x) = x^6$, $q(x) = x - 1$
   (f) $p(x) = x^3 - 5x^2 + x - 15$, $q(x) = x^2 - 1$
   (g) $p(x) = x^3 - 2x^2 - 5x + 7$, $q(x) = x^2 + x - 6$
   (h) $p(x) = x^3 + 3x^2 - 6x - 7$, $q(x) = x^2 + 2x - 8$
   (i) $p(x) = 2x^3 - 8x^2 + 8x - 4$, $q(x) = 2x^2 - 4x + 2$
   (j) $p(x) = 3x^4 - x^3 - 2x^2 + 5x - 1$, $q(x) = x + 1$
   (k) $p(x) = 4x^5 + 7x^4 - 9x^3 + 2x^2 - x + 3$, $q(x) = x^2 - 4x + 3$
   (l) $p(x) = 4x^5 + 7x^4 - 9x^3 + 2x^2 - x + 3$, $q(x) = x^3 + x^2 - 5x + 3$
   (m) Make up your own! Pick random polynomials and divide!

2. Use the method of partial fractions to break up these rational functions.
   (a) $\frac{2}{(x-2)x}$
   (b) $\frac{5}{(x-2)(x+3)}$
   (c) $\frac{7}{(x+6)(x-1)}$
   (d) $\frac{5x}{(x-1)(x+4)}$
   (e) $\frac{x}{(x+1)(x+2)}$
   (f) $\frac{12x-6}{(x-3)(x+3)}$
   (g) $\frac{x-1}{(x+2)(x+1)}$
   (h) $\frac{1}{x^2-x-6}$
   (i) $\frac{11}{x^2-3x-28}$
   (j) $\frac{10}{x^2+2x-24}$
   (k) $\frac{4x}{x^2+6x+5}$
   (l) $\frac{3x}{x^2-7x+10}$
   (m) $\frac{1}{x^3-2x^2-5x+6}$

(n) $\frac{4x^2-x}{x^3-4x^2-x+4}$

3. Use the method of partial fractions to break up these rational functions.
   (a) $\frac{x}{(x+1)^2}$
   (b) $\frac{2x-1}{(x+3)^2}$
   (c) $\frac{1-3x}{(x-1)^2}$
   (d) $\frac{1+3x}{(x-2)^2}$
   (e) $\frac{2x^2}{(x-1)^3}$
   (f) $\frac{x-1}{(x-2)^3}$
   (g) $\frac{x-3}{(x+2)^2(x-2)}$
   (h) $\frac{x}{(x-1)(x+3)^2}$
4. Integrate the functions in question 2 and question 3.
5. Calculate $\int \frac{p(x)}{q(x)}\,\mathrm{d}x$ for each part of question 1.

Solution: For 5(l): First we apply polynomial long division to obtain

$$\frac{4x^5+7x^4-9x^3+2x^2-x+3}{x^3+x^2-5x+3} = 4x^2+3x+8+\frac{-3x^2+30x-21}{x^3+x^2-5x+3}$$

Now we wish to apply the method of partial fractions to the fraction on the right hand side. First we note that $x=1$ is a root of x^3+x^2-5x+3. Thus we can factorise to get

$$x^3+x^2-5x+3=(x-1)(x^2+2x-3)=(x-1)^2(x+3).$$

The method of partial fractions tells us that there must be A, B, C such that

$$\frac{-3x^2+30x-21}{x^3+x^2-5x+3} = \frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+3}.$$

Clearing denominators,

$$-3x^2+30x-21 = A(x-1)(x+3)+B(x+3)+C(x-1)^2$$

By letting $x=1$ we see that $6=4B$ and letting $x=-3$ we get that $-138=16C$. Thus

$$B=\frac{3}{2}, C=-\frac{69}{8}.$$

We need to find the value of A so we look at the x^2 coefficient: $-3=A+C=A-69/8$ so

$$A=\frac{45}{8}.$$

Thus

$$\frac{-3x^2+30x-21}{x^3+x^2-5x+3} = \frac{45}{8(x-1)}+\frac{3}{2(x-1)^2}-\frac{69}{8(x+3)}.$$

Finally, we can integrate to get

$$\begin{aligned}\int \frac{4x^5+7x^4-9x^3+2x^2-x+3}{x^3+x^2-5x+3}dx &= \int 4x^2+3x+8+\frac{-3x^2+30x-21}{x^3+x^2-5x+3}dx \\ &= \int 4x^2+3x+8+\frac{45}{8(x-1)}+\frac{3}{2(x-1)^2}-\frac{69}{8(x+3)}dx \\ &= \frac{4}{3}x^2+\frac{3}{2}x+8x+\frac{45}{8}\ln|x-1|-\frac{3}{2(x-1)}+\frac{69}{8}\ln|x+3|+C.\end{aligned}$$

6. (6.1) Write a differential equation to model the situations described below. Do not try to solve.

   (a) (6.1-1) The number of bacteria in a culture grows at a rate that is proportional to the number of bacteria present.

   > **Solution:** Lets say that the rate is a times the number of bacteria (i.e. this is the constant of proportionality). Let $P(t)$ be the number of bacteria at time t. Then the rate at which the number grows is $\frac{\mathrm{d}P}{\mathrm{d}t}$. So the fact that the rate is a times the number of bacteria is just expressed as
   > $$\frac{\mathrm{d}P}{\mathrm{d}t} = aP$$
   > .

   (b) (6.1-2) A sample of radium decays at a rate that is proportional to the amount of radium present in the sample.

   > **Solution:** This is similar to the first example. However the sample is decaying. Let $N(t)$ be the amount of radium present, and λ the proportion. Since it is decaying, the rate should be negative so
   > $$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$

   (c) (6.1-5) According to Benjamin Gompertz (1779-1865) the growth rate of a population is proportional to the number of individuals present, where the factor of proportionality is an exponentially decreasing function of time.

   > **Solution:** This is similar to the first example. However now, the proportionality a changes over time. In particular a is exponenitally decreasing. Thus $a = Ae^{-kt}$ for some A and some k. Thus
   > $$\frac{\mathrm{d}P}{\mathrm{d}t} = aP = Ae^{-kt}P.$$

   (d) (6.1-7) The rate at which an epidemic spreads through a community of P susceptible people is proportional to the product of the number of people y who have caught the disease and the number $P - y$ who have not.

   (e) (6.1-8) The rate at which people are implicated in a government scandal is proportional to the product of the number N of people already implicated and the number of people involved who have not yet been implicated.

7. (6.1) A population model is given by
   $$\frac{\mathrm{d}P}{\mathrm{d}t} = P(100 - P).$$

   (a) (6.1-9) For what values is the population at equilibrium?

   > **Solution:** The population is at equilibrium when the right hand side is zero. I.e. either $P = 0$ or $P = 100$.

   (b) (6.1-10) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$?

   > **Solution:** When $0 < P < 100$.

   (c) (6.1-11) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$?

> **Solution:** When $P > 100$ or when $P < 0$ (however this does not make physical sense).

(d) (6.1-12) Describe how the fate of the population depends on the initial density.

> **Solution:** If $P(0) = 0$ or 100 then the population will stay at that level forever. If $0 < P(0) < 100$ then the population will increase towards and approach 100. If $P(0) > 100$ then the population decreases to and approaches 100.

8. (6.1) A population model is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(P-1)(100-P).$$

(a) (6.1-13) For what values is the population at equilibrium?

> **Solution:** The equilibrium solutions of the model occur when $P' = 0$. That is, when $P = 0, 1, 100$.

(b) (6.1-14) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$?

> **Solution:** By testing points we see that $P' > 0$ when $1 < P < 100$.

(c) (6.1-15) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$?

> **Solution:** By testing points we see that $P' < 0$ when $0 < P < 1$ and when $P > 100$.

(d) (6.1-16) Describe how the fate of the population depends on the initial density.

> **Solution:** Using the above information we see that if $P(0) < 1$ the population eventually dies out. When $1 < P(0)$ the population eventually stabilises at $P = 100$.

9. (6.1) Radioactive decay: Certain types of atoms (e.g. carbon-14, xenon-133, lead-210, etc.) are inherently unstable. They exhibit random transitions to a different atom while emitting radiation in the process. Based on experimental evidence, Rutherford found in the early 20th century that the number, N, of atoms in a radioactive substance can be described by the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$

where t is measured in years and $\lambda > 0$ is known as the *decay constant*. The decay constant is found experimentally by measuring the half life, τ of the radioactive substance (i.e. the time it takes for half of the substance to decay). Use this information in the following problems.

(a) (6.1-18) Find a solution to the decay equation assuming that $N(0) = N_0$.

> **Solution:** Using separation of variables we get
>
> $$N(t) = Ce^{-\lambda t}$$
>
> Using the intitial value gives $C = N_0$ so
>
> $$N(t) = N_0 e^{-\lambda t}$$

(b) (6.1-19) For xenon-133, the half-life is 5 days. Find λ. Assume t is measured in days.

> **Solution:** We know that after 5 days we will have half the amount left, i.e. $N(5) = N_0/2$ this we have the equation
> $$\frac{N_0}{2} = N_0 e^{-\lambda 5}$$
> We can cancel the N_0 and rearrange (by taking a log) to get
> $$\lambda = \frac{\ln 2}{5}.$$

(c) (6.1-20) For carbon-14 the half life is 5,568 years. Find the decay constant λ, assuming t is measured in years.

> **Solution:** Very similar to the above, we get $\lambda = \frac{\ln 2}{5568}$

(d) (6.1-21) How old is a piece of human bone which contains just 60% of the amount of carbon-14 expected in a sample of bone from a living person, assuming the half life of carbon-14 is 5,568 years?

> **Solution:** We know that it obeys the equation $N(t) = N_0 e^{-\lambda t}$ where λ is as given above. The question is telling us to find t such that $N(t) = 0.6N_0$, we get an equation
> $$0.6N_0 = N_0 e^{-\frac{\ln 2}{5568} t}.$$
> We can rearrange this and solve for t (the N_0 cancels)
> $$t = -\frac{5568 \ln 0.6}{\ln 2}$$

(e) (6.1-22) The Dead Sea Scrolls were written on parchment at about 100 B.C. What percentage of carbon-14 originally contained in the parchment remained when the scrolls were discovered in 1947?

> **Solution:** If originally there was N_0 cabon-14 then after 2047 years there would be
> $$N(2047) = N_0 e^{-\frac{\ln 2}{5568} 2047}$$
> left. As a percentage of N_0 this would be $100e^{-\frac{\ln 2}{5568} 2047} \approx 77.5\%$.

10. (6.1-30) Hyperthyroidism is caused by a new growth of tumor-like cells that secrete thyroid hormones in excess to the normal hormones. If left untreated, a hyperthyroid individual can exhibit extreme weight loss, anorexia, muscle weakness, heart disease intolerance to stress, and eventually death. The most successful and least invasive treatment option is radioactive iodine-131 therapy.

    This involves the injection of a small amount of radioactivity into the body. For the type of hyperthyroidism called Graves' disease, it is usual for about $40 - 80\%$ of the administered activity to concentrate in the thyroid gland. For functioning adenomas ("hot nodules"), the uptake is closer to $20 - 30\%$. Excess iodine-131 is excreted rapidly by the kidneys. The quantity of radioiodine used to treat hyperthyroidism is not enough to injure any tissue except the thyroid tissue, which slowly shrinks over a matter of weeks to months. Radioactive iodine is either swallowed in a capsule or sipped in solution through a straw. A typical dose is $5 - 15$ millicures. The half-life of iodine-131 is 8 days.

(a) Suppose that it takes 48 hours for a shipment of iodine-131 to reach a hospital. How much of the initial amount shipped is left once it arrives at the hospital?

Solution: After t days the fraction of the substance that will remain is

$$\left(\frac{1}{2}\right)^{t/8} = e^{-t(\ln 2)/8}$$

Thus after 2 days there will be $e^{-0.25\ln 2}$ of it remaining.

(b) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left one week later?

Solution:

$$3e^{-\frac{7}{8}\ln 2} \text{ millicures}$$

(c) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left 30 days later?

Solution:

$$3e^{-\frac{15}{4}\ln 2} \text{ millicures}$$