

# Math 3B: Lecture 16

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# Differential equations (motivation)

An (ordinary) **differential equation** (or **ODE**) is an equation that involves derivatives of an unknown function.

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The challenge is to find all the functions  $y = f(x)$  (or even just one) that satisfy a given equation.

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And so on.

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## Note

The right hand side of the equation does not have any  $y$ 's.

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- draw solutions for many other ODEs
- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)
- understand how sensitive ODEs are to their parameters.

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- E.g.  $y(0) = 2$ .
- Then we see that  $y(0) = 1 + C$ , so  $C = 1$ .

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- If we want to draw the graph of  $y(t)$  then we look at  $g(0, 1)$ .
- If this is positive we go up, negative we go down!

# Modelling using differential equations

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- The goal is to write down a function  $y(t)$  that describes something we are interested in (e.g. population/mass/etc)
- as some other variable changes (usually time)
- We can't do this directly, but we can write down an ODE that  $y$  satisfies instead.

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The total change in population at time  $t$  is

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In real life we would determine  $b$  and  $d$  experimentally. Let  $r = b - d$ . the **instinsic growth rate**. So our model is

$$\frac{dN}{dt} = rN.$$

and we know  $N(0) = 100$ .

# Behaviour of solutions

$$\frac{dN}{dt} = rN.$$

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The population never grows or shrinks, it always stays the same (so  $N(t) = 100$  for all  $t$ ).

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The population is increasing indefinitely.

## Case 3: $r < 0$

The population is decreasing indefinitely.

# Solution to a simple ODE

## Theorem

For any constant  $a$ , if  $y$  is a solution to the ODE

$$\frac{dy}{dx} = ay$$

then  $y$  is given by

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## Next time

We will see why, but for now we can verify it is actually a solution:

$$\frac{dy}{dx} = \frac{d}{dx} Ce^{ax} = C \frac{d}{dx} e^{ax} = Cae^{ax} = ay.$$

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$$100 = Ce^{(b-d) \cdot 0} \quad \text{so} \quad C = 100e^{(d-b)}.$$

## Logistic growth

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$$(d \propto N(t)).$$

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$$\begin{aligned}\frac{dN}{dt} &= bN - (d + kN)N \\ &= (b - d - kN)N = (r - kN)N \\ &= r \left(1 - \frac{kN}{r}\right) N = r \left(1 - \frac{N}{K}\right) N\end{aligned}$$

Where  $K = r/k$ .

# Logistic growth

The equation

$$\frac{dN}{dt} = r \left( 1 - \frac{N}{K} \right) N$$

is called the **Logistic equation** and  $K$  is the **carrying capacity**.

# Behaviour of logistic growth

Assume that  $r > 0$  and  $K > 0$ .

$$\frac{dN}{dt} = r \left( 1 - \frac{N}{K} \right) N$$

Case 1.  $N(0) = 0$

In this case the growth rate is 0 initially, so  $N(t)$  does not increase or decrease, so remains 0.

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## Key takeaway

Both  $N(t) = 0$  and  $N(t) = K$  are solutions to the ODE. They are called **equilibrium solutions**.

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Case 4.  $N(0) \geq K$

In this case  $N$  is initially decreasing but decreases slower and slower as it gets close to  $K$ .