

# RESEARCH STATEMENT

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## 1. INTRODUCTION

My research centres on connections between representation theory and algebraic geometry which give meaning to, or are inspired by, interesting combinatorial phenomena. Having algebraic or geometric interpretations of combinatorial phenomena can often provide a key piece of extra structure which can then be used to attack otherwise difficult problems. My research seeks to exploit this observation.

The fundamental motivating example is given by the *Littlewood-Richardson coefficients*. These are the coefficients appearing in products of *Schur polynomials* and turn out to be nonnegative integers. Representation theory and geometry both have their own explanation for this phenomenon: in representation theory these integers appear as the dimensions of certain vector spaces and in geometry they count intersection points in *Schubert varieties*. These Littlewood-Richardson coefficients satisfy important extra symmetries coming from actions of the *cactus group*. I have contributed to this picture by unifying algebraic and geometric interpretations of these symmetries and using this to move towards better understanding of some fundamental questions in combinatorial representation theory. In further work I gave an alternative algebraic description of this action and have explained how one can obtain explicit, and in a sense, canonical bijections that realise the above-mentioned coincidence of numbers.

My work falls into three main categories that I discuss in the following sections: actions of the cactus group, moduli of Bethe algebras and relations to geometry, explicit descriptions of the centre of the quantum group  $U_q(\mathfrak{gl}_r)$ . In each section I outline my track record and future directions for my research. I explain some motivation and identify a range of precise, tractable questions as well as some more ambitious ones. In Section 6 I give an indication of which questions might be suitable for students of varying levels.

## 2. SYMMETRIES OF KAZHDAN-LUSZTIG CELLS

**2.1. Motivation.** To understand the representations of finite simple groups (of Lie type) one is immediately confronted with the problem of understanding Coxeter groups (and their Hecke algebras) and more generally complex reflection groups. Kazhdan-Lusztig cells are a partitioning of a Coxeter group with deep connections to the representation theory of Hecke algebras, finite groups of Lie type,  $p$ -adic groups, primitive ideals of universal enveloping algebras, the geometry of nilpotent orbits and Schubert varieties, etc. Bonnafé and Rouquier [5] have conjectured that these cells have large groups of symmetries. Importantly this means they can be alternatively defined as the orbits under this group action. This allows the definition to be extended to complex reflection groups (where they are called Calogero-Moser cells), a long awaited development.

My research concerns investigating symmetries of these cells, in particular coming from the cactus group. The definition of Bonnafé-Rouquier is geometrically involved and difficult to calculate, where as actions coming from the cactus group can often be explicitly described, for example as Schützenberger involutions on tableaux.

**2.2. Track record.** Losev [15] and Bonnafé [4] have shown that the cactus group  $C_W$  attached to a Coxeter group  $W$ , acts on the Coxeter group with the orbits contained in the Kazhdan-Lusztig cells.

The cactus group of  $W$  is generated by elements  $s_I$  labelled by subsets  $I$  of simple reflections generating finite parabolic subgroups.

**Theorem 1** (Rouquier-White [20]). *There is a map from the cactus group to the unipotent completion of the braid group  $\hat{B}_W$ . Under this map, the elements  $s_I$  have well defined images in the Hecke algebra. Furthermore, taking their image in Lusztig’s asymptotic Hecke algebra recovers Bonnafé and Losev’s action of the cactus group as right or left multiplication on the standard basis.*

This joint work with Raphaël Rouquier [20] provides an alternative description of the cactus action, but importantly gives a linear action of the cactus group on the Hecke algebra. Using this linear action one can hope to investigate the connection between the cactus group and the representations of the Hecke algebra.

By the Tits deformation argument, the Hecke algebra is isomorphic to the group algebra of  $W$  (over  $\mathbb{C}(v)$ ). Lusztig provided an explicit isomorphism. Let  $D$  set of simple reflections in  $W$ .

**Theorem 2** (Rouquier-White [20]). *The image of  $s_D$  in the Hecke algebra coincides with the preimage of the longest element  $w_0 \in W$  under Lusztig’s isomorphism. Parabolic versions of the same statement hold as well.*

**2.3. Future work.** There is no clear definition for a cactus group for general complex reflection groups. By giving a description of cactus groups for complex reflection groups I hope to be able to provide insight into the possible definitions of cells for these groups.

**Question 1.** *Can one define a (pure) cactus group for an arbitrary complex reflection group? Does this cactus group have an action on the complex reflection group which preserves the Calogero-Moser cells? My prior research puts me in a unique position to tackle the possible approaches to this question outlined below.*

The cactus group of a Coxeter group  $W$  can be thought of as a kind of asymptotic limit of the braid group and can be defined in two ways. Combinatorially it can be defined using certain symmetries of the associated Coxeter diagram. The second definition is topological. De Concini and Procesi [9] define a compactification of  $\mathbb{P}\mathfrak{h}^{\text{reg}}$ , denoted  $\overline{\mathbb{P}\mathfrak{h}^{\text{reg}}}$ . The pure cactus group  $PC_W$  is defined to be the fundamental group of the real locus  $\mathbb{P}\mathfrak{h}_{\mathbb{R}}^{\text{reg}}$ . Note that  $W$  preserves the real locus of  $\mathbb{P}\mathfrak{h}^{\text{reg}}$  since it is a Coxeter group. The cactus group  $C_W$  is defined to be the fundamental group of the stack  $[\overline{\mathbb{P}\mathfrak{h}^{\text{reg}}}/W]$ .

Another approach would be to try and define these groups combinatorially. There exist Coxeter-like diagrams which display a set of generators and relations for a complex reflection group (see for example [7]). Yet another approach may be possible via the ramification of Gaudin algebras for rational Cherednik algebras.

For a general Coxeter group  $W$ , the orbits of the cactus group  $C_W$  on  $W$  are not the entire Kazhdan-Lusztig cells (for example type  $B_2$ ). One can see this as a problem of the cactus group not being “big enough”. Theorem 1 gives a starting point on where to look for extra symmetries.

**Question 2.** *Are there naturally defined elements of the Hecke algebra or asymptotic algebra that enlarge the cactus action so that its orbits are the entire cells? Even in type  $B_2$  this would be interesting. What is the representation theoretic meaning of the cactus group orbits in  $W$ ?*

The link with Lusztig’s isomorphism given in Theorem 2 motivates the question as to whether one can provide explicit formulas (at least for dihedral groups and in type  $A$ ) for the isomorphism.

**Question 3.** *Provide explicit formulas for the images of the generators  $s_I$  in the Hecke algebra and in the asymptotic algebra.*

Currently, we have an explicit description of the action of the cactus group on  $W$  in very few cases, only in type  $A$  and for dihedral groups. It is desirable give combinatorial descriptions in other types.

**Question 4.** *Give combinatorial descriptions of the action of the cactus group on  $W$  on a case by case basis. In type  $B$  one can expect this to be related to domino tableaux and domino insertion. There is also the possibility of a connection with the combinatorics of fully commutative elements and Temperley-Lieb algebras. In affine type  $A$  one hopes there are straightforward connection to finite type  $A$  combinatorics.*

We also expect that these techniques may extend to wreath product groups, giving combinatorial descriptions of the Calogero-Moser cells, at least for asymptotic parameters.

### 3. CACTUS GROUP ACTIONS ON CRYSTALS AND BETHE VECTORS IN TYPE $A$

**3.1. Motivation.** This project focuses on the action of the cactus group on the symmetric group (type  $A$ ). Here the action is given Schützenberger involutions on tableaux via the RSK correspondence (i.e. it can be algorithmically computed). This can be viewed as an action coming from *crystals*. Crystals are combinatorial models for representations of  $\mathfrak{gl}_r$  and their tensor products carry actions of the cactus group (see [12]).

My work focused on giving a geometric description of this action via monodromy of *Bethe vectors*, eigenvectors of certain Hamiltonians for a quantum integrable system. This was motivated by the expectation that such an interpretation would help provide important understanding towards a proof of a conjecture of Bonnafé-Rouquier relating Kazhdan-Lusztig cells to the geometry of Calogero-Moser space in type  $A$ .

**3.2. Track record.** For a tensor product  $L(\lambda_\bullet) = L(\lambda_1) \otimes L(\lambda_2) \otimes \cdots \otimes L(\lambda_n)$  of irreducible  $\mathfrak{gl}_r$ -representations, there is a large commutative algebra  $\mathcal{B}(z)$  (the *Bethe algebra*), depending on a tuple of distinct complex numbers  $z = (z_1, z_2, \dots, z_n)$ , that acts on the multiplicity space  $\text{Hom}(L(\mu), L(\lambda_\bullet))$ . For generic values of  $z$ , the algebra has one dimensional simultaneous eigenspaces (simple spectrum). Restricting to this generic locus, gives a finite covering space (elements of the fibre at the eigenspaces) and one can ask about the monodromy in this finite covering.

**Theorem 3** (White [21]). *The monodromy of the Bethe vectors for real  $z$  is described by the action of the cactus group  $C_{S_n}$  on  $\text{Hom}_{\text{Crys}}(\mathcal{B}(\mu), \mathcal{B}(\lambda_\bullet))$ , where  $\mathcal{B}(\mu)$  is the crystal for  $L(\mu)$ .*

The above theorem depends on a connection between Bethe algebras and intersections of Schubert varieties in Grassmannians. When these intersections have dimension zero, the number of points coincides with the number of standard tableaux (and generalisations thereof) of a set shape. The above understanding of the moduli of Bethe algebras allowed us to prove the following.

**Theorem 4** (White [22]). *When taken with respect to certain flags, the points in the intersection of Schubert varieties can be canonically labelled by standard tableaux and this labelling can be described in an elementary and purely geometrical fashion. Furthermore, this labelling coincides with labellings appearing implicitly in work of Mukhin-Tarasov-Varchenko and of Speyer.*

**3.3. Future work.** In joint work with Adrien Brochier and Iain Gordon [6], we exploit this labelling to recover the RSK correspondence. More precisely, there is a larger Bethe algebra  $\mathcal{B}(z, q)$  depending on an extra  $n$ -tuple of complex parameters. This algebra acts with simple spectrum on tensor products. Using work of Mukhin, Tarasov and Varchenko, this result can be used to calculate monodromy in Calogero-Moser space and thus provide a proof of the Bonnafé-Rouquier conjecture in type  $A$ .

**Theorem 5** (Brochier-Gordon-White [6]). *The eigenvectors of  $\mathcal{B}(z, q)$  in  $V^{\otimes n}$  are naturally labelled by pairs of a standard tableau and a semistandard tableaux by taking the limit  $z, q \rightarrow 0$  (where they become eigenvectors of the Jucys-Murphy elements and the Gelfand-Tsetlin operators), and are also labelled by words of length  $n$  by taking the limit  $z, q \rightarrow \infty$ . This recovers the RSK correspondence and can be used to prove a conjecture of Bonnafé-Rouquier.*

There is also the possibility that some of this may be extended to the case of wreath product groups. In this situation one still has Schur-Weyl duality and a version of the RSK-correspondence (and thus an obvious candidate for a cactus group, or subgroup thereof).

**Question 5.** *Extend the description of monodromy of Bethe vectors to the case of the complex reflection group  $S_n \wr \mu_l$  and give a description of the Calogero-Moser cells for this complex reflection group.*

Currently the Calogero-Moser cells have only been calculated for very small ranks so answering the above question would be a major step forward in our understanding.

#### 4. THE CENTRE OF THE QUANTUM GROUP AND THE REFLECTION EQUATION ALGEBRA

**4.1. Motivation.** Together with David Jordan, I have investigated the *reflection equation algebra*  $\mathcal{A}_q$ . This algebra satisfies a pivotal dual property. On one hand it is the quantisation of the *Semenov-Tian-Shansky Poisson bracket* on the group  $\mathrm{GL}_r$ . On the other hand, it appears as a subalgebra of the quantum group  $U_q(\mathfrak{gl}_r)$ , containing the centre. The algebra  $\mathcal{A}_q$  has received a surge of attention recently, in particular because it computes the Hochschild homology of the braided monoidal category  $\mathrm{Rep}_q(\mathrm{GL}_r)$  and has thus appeared in work [8] and [3]. It is desirable to have an explicit description of the centre of  $U_q(\mathfrak{gl}_r)$  in terms of this algebra and in a paper [13] we give such a description and a Cayley-Hamilton type theorem that generators for the centre satisfy.

**4.2. Track record.** Abstractly,  $\mathcal{A}_q$  is the algebra generated by  $a_j^i$  for  $1 \leq i, j \leq r$ , (collected into a matrix  $A$ ) subject to certain relations. Importantly  $\mathcal{A}_q$  can be embedded in  $U_q(\mathfrak{gl}_r)$  as the *locally finite part*. By restricting the adjoint action of  $U_q(\mathfrak{gl}_r)$  on itself, we obtain an adjoint action on  $\mathcal{A}_q$ . By construction  $\mathcal{A}_q$  contains the centre of  $U_q(\mathfrak{gl}_r)$ . We aim to calculate a full set of invariants for the adjoint action. By the above discussion this is the same as generators for the centre. We define the following elements of  $\mathcal{A}_q \subseteq U_q(\mathfrak{gl}_r)$ ,

$$c_k = \sum_{\substack{I \subseteq [r] \\ \#I=k}} q^{-2|I|} \sum_{\sigma \in S_I} (-1)^\sigma q^{l(\sigma)+e(\sigma)} a_{\sigma(i_1)}^{i_1} a_{\sigma(i_2)}^{i_2} \cdots a_{\sigma(i_k)}^{i_k},$$

where  $l$  and  $e$  denote the length and *excedence statistics* and  $S_I \subseteq S_r$  is the subgroup of permutations fixing  $[r] \setminus I$ .

**Theorem 6** (Jordan-White [13]). *The  $c_k$  are invariant under the adjoint action and are a full set of algebraically independent generators for the centre of  $U_q(\mathfrak{gl}_r)$ . The matrix  $A$  satisfies the Cayley-Hamilton identity:*

$$A^N - q^2 c_1 A^{N-1} + q^4 c_2 A^{N-2} - \dots + (-1)^N q^{2N} c_N I = 0.$$

A large amount of literature has been devoted to understanding the centre of  $U_q(\mathfrak{gl}_r)$ , see [19, 14, 11, 17, 18, 2]. However, these descriptions have the disadvantage of either not being explicit or not being adapted to certain applications. In addition we unify the various descriptions of the centre.

**4.3. Future work.** In the work described above, we used extensive computational experimentation using the MAGMA computer algebra package, which required us to develop a large set of code to work with  $\mathcal{A}_q$ . This will be of great benefit when exploring the many additional questions this project leads to, such as understanding the quantisation of classical invariant theory objects, describing the higher Hochschild cohomology of  $\mathcal{A}_q$  and questions about categorification of this algebra.

In the classical case, the determinant is the sum of minors. It would be interesting to quantise these minors, however, no obvious analogue of the formulas in Section 4.2 works. In the case of  $2 \times 2$  minors we can prove explicit descriptions but they do not generalise easily to  $k > 2$ .

**Theorem 7** (White). *The elements  $c(i, j|k, l) = q^{\theta(k-l)} a_k^i \tilde{a}_l^j - q^{\theta(l-k)} a_l^i \tilde{a}_k^j$  quantise  $2 \times 2$  minor formulas. The  $\tilde{a}_j^i$  are certain perturbations of the generators and  $\theta(x) = 0$  if  $x \leq 0$  and  $\theta(x) = 1$  otherwise.*

**Question 6.** *What are formulas for the  $k \times k$  minors in  $\mathcal{A}_q$ ? What is the action of  $U_q(\mathfrak{gl}_r)$  on these minors?*

This question is also interesting as it would provide a quantisation of the determinantal variety. This has been well studied in the case of the RTT-algebra (see [10]). Another tractable question is the following.

**Question 7.** *What are the semi-classical limits of the generators of the centre  $c_k$ , and can one relate this to the Capelli determinant?*

Future research includes the following questions.

**Question 8.** *What is the combinatorial meaning of the appearance of the exceedance? Can we categorify  $\mathcal{A}_q$ ? What can one say about Hochschild cohomology of  $\mathcal{A}_q$  in higher degree?*

## 5. COMBINATORICS OF GAUDIN HAMILTONIANS AND AFFINE HECKE ALGEBRAS

**5.1. Motivation.** The Gaudin Hamiltonians are commuting elements of the group algebra of the symmetric group, depending on a set of complex parameters. This project aims to understand the limits of these Hamiltonians (and higher counterparts) and the combinatorics of their eigenvalues to gain a better understanding of the Bethe algebras.

The representation theory of the symmetric group is understood using the *Jucys-Murphy elements*. The limit of the Gaudin Hamiltonians as  $z_1 \ll z_2 \ll \dots \ll z_n \rightarrow \infty$ , recovers the Jucys-Murphy elements. The spectrum of the Jucys-Murphy elements on irreducible representations is described using the combinatorics of tableaux and their contents and corresponds to a sequence of inductions over the chain of subgroups

$$S_1 \subseteq S_2 \subseteq \dots \subseteq S_n$$

It would also be interesting to imitate these constructions for other Coxeter groups, where analogues of the Jucys-Murphy elements are not available.

**5.2. Track record.** Other limits produce various commutative algebras analogous to the Jucys-Murphy elements. These algebras are not always maximally commutative but one can always (explicitly) produce new elements to make them so. The Gaudin Hamiltonians for the symmetric group are

$$H_a(z) = \sum_{b \neq a} \frac{(a, b)}{z_a - z_b} \in \mathbb{C}S_n$$

depending on  $n$  distinct complex parameters  $z = (z_1, z_2, \dots, z_n)$ . They generate a large commutative subalgebra  $G(z) \subseteq \mathbb{C}S_n$ . Aguirre, Felder and Veselov [1] showed that these algebras can be defined for any point in  $\overline{M}_{0, n+1}(\mathbb{C})$ , the moduli space of stable rational curves with  $n+1$  marked points. I am interested in understanding the limits of these algebras, that is, the algebras  $G(z)$  for points  $z \in \overline{M}_{0, n+1}$  corresponding to curves with the maximal number of irreducible components.

**Theorem 8** (White). *The eigenvalues of the limits of Gaudin Hamiltonians are described by the combinatorics of tableaux related to chains of parabolic subgroups in  $S_n$  and the corresponding branching graph. This extends the description of Okounkov-Vershik for Jucys-Murphy elements. Furthermore, whenever the spectrum is not simple we can add an extra set of higher Hamiltonians to achieve a*

simple spectrum. Using work in [16] we can describe these higher Gaudin Hamiltonians explicitly as limits of operators

$$X_a^{k+1}(z) = \sum_{\substack{b_1 < \dots < b_k \\ b_i \neq a}} \frac{B_{a,b_1,\dots,b_k}}{\prod_{s=1}^k (z_a - z_{b_s})}.$$

These limiting operators can be described explicitly using the combinatorics of trees. Here  $B_{a,b_1,\dots,b_k}$  is a very explicit alternating sum of elements in  $\mathbb{C}S_n$ .

### 5.3. Future work.

**Question 9.** What are the limiting eigenvalues of the operators  $X_a^{k+1}(z)$ ? Can we describe them combinatorially? Do they form a maximally commutative subalgebra? Can we extend the formulas to other Coxeter groups to provide a family of commuting operators acting with simple spectrum on irreducible representations?

The (degenerate) affine Hecke algebra, has a polynomial subalgebra, and a quotient isomorphic to  $\mathbb{C}S_n$ , in which the polynomial ring is identified with the Jucys-Murphy elements.

**Question 10.** Can one define an analogue of the affine Hecke algebra for other subalgebras  $G(z)$  and their limits?

## 6. OPPORTUNITIES FOR STUDENT RESEARCH

My research provides a rich source of opportunities for involving students. Here I outline how some of the above questions are suitable for students of varying mathematical maturity. Most of the opportunities I outline below are aimed at students at an advanced undergraduate or masters level, however some questions have room to be extended to suitable PhD projects.

The project outlined in Section 4 provides a particularly fruitful supply of questions suitable for students. Question 6 in particular would be suited to students with minimal background (linear algebra would suffice). The computer algebra package developed in the previous work would allow students to dive right in and produce data with which conjectures could be made and tested. Question 7 would be a good follow-on point for students with some background in algebra and Lie theory.

For students willing to learn some basic representation theory, the questions outlined in Section 5 would quickly provide interesting results. For slightly more advanced students, Question 1 is suitable and has the benefit of being quite computational. Students with a strong background in representation theory will find fruitful directions of research in Question 5.

I am excited about involving students in my research and view it as an important part of the next stage of my career and in continuing to develop a cohesive research program.

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