This week you will practice writing differential equations modelling real world phenomena as well as understanding population models. You will also get practice solving separable differential equations.

Homework: The homework will be due on Friday 15 February, at 8am, the *start* of the lecture. It will consist of questions:

$7, 10.$(l) from problem set 5 and 5.(l) from this problem set below

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. Divide $p(x)$ by $q(x)$ and express the quotient as a divisor plus a remainder.
   (a) $p(x) = 2x^3 + 4x^2 - 5$, $q(x) = x + 3$
   (b) $p(x) = 15x^4 - 3x^2 - 6x$, $q(x) = 3x + 6$
   (c) $p(x) = 2x^4 - 5x^3 + 6x^2 + 3x - 2$, $q(x) = x - 2$
   (d) $p(x) = 5x^4 + 2x^3 + x^2 - 3x + 1$, $q(x) = x + 2$
   (e) $p(x) = x^6$, $q(x) = x - 1$
   (f) $p(x) = x^3 - 5x^2 + x - 15$, $q(x) = x^2 - 1$
   (g) $p(x) = x^3 - 2x^2 - 5x + 7$, $q(x) = x^2 + x - 6$
   (h) $p(x) = x^3 + 3x^2 - 6x - 7$, $q(x) = x^2 + 2x - 8$
   (i) $p(x) = 2x^3 - 8x^2 + 8x - 4$, $q(x) = 2x^2 - 4x + 2$
   (j) $p(x) = 3x^4 - x^3 - 2x^2 + 5x - 1$, $q(x) = x + 1$
   (k) $p(x) = 4x^5 + 7x^4 - 9x^3 + 2x^2 - x + 3$, $q(x) = x^2 - 4x + 3$
   (l) $p(x) = 4x^5 + 7x^4 - 9x^3 + 2x^2 - x + 3$, $q(x) = x^3 + x^2 - 5x + 3$
   (m) Make up your own! Pick random polynomials and divide!

2. Use the method of partial fractions to break up these rational functions.
   (a) $\frac{2}{(x-2)x}$
   (b) $\frac{5}{(x-2)(x+3)}$
   (c) $\frac{7}{(x+6)(x-1)}$
   (d) $\frac{5x}{(x-1)(x+4)}$
   (e) $\frac{x}{(x+1)(x+2)}$
   (f) $\frac{12x-6}{(x-3)(x+3)}$
   (g) $\frac{x-1}{(x+2)(x+1)}$
   (h) $\frac{1}{x^2-x-6}$
   (i) $\frac{11}{x^2-3x-28}$
   (j) $\frac{10}{x^2+2x-24}$
   (k) $\frac{4x}{x^2+6x+5}$
   (l) $\frac{3x}{x^2-7x+10}$
   (m) $\frac{1}{x^3-2x^2-5x+6}$

(n) $\frac{4x^2-x}{x^3-4x^2-x+4}$

3. Use the method of partial fractions to break up these rational functions.
   (a) $\frac{x}{(x+1)^2}$
   (b) $\frac{2x-1}{(x+3)^2}$
   (c) $\frac{1-3x}{(x-1)^2}$
   (d) $\frac{1+3x}{(x-2)^2}$
   (e) $\frac{2x^2}{(x-1)^3}$
   (f) $\frac{x-1}{(x-2)^3}$
   (g) $\frac{x-3}{(x+2)^2(x-2)}$
   (h) $\frac{x}{(x-1)(x+3)^2}$
4. Integrate the functions in question 2 and question 3.
5. Calculate $\int \frac{p(x)}{q(x)}\,\mathrm{d}x$ for each part of question 1.
6. (6.1) Write a differential equation to model the situations described below. Do not try to solve.
   (a) (6.1-1) The number of bacteria in a culture grows at a rate that is proportional to the number of bacteria present.
   (b) (6.1-2) A sample of radium decays at a rate that is proportional to the amount of radium present in the sample.
   (c) (6.1-5) According to Benjamin Gompertz (1779-1865) the growth rate of a population is proportional to the number of individuals present, where the factor of proportionality is an exponentially decreasing function of time.
   (d) (6.1-7) The rate at which an epidemic spreads through a community of P susceptible people is proportional to the product of the number of people y who have caught the disease and the number $P - y$ who have not.
   (e) (6.1-8) The rate at which people are implicated in a government scandal is proportional to the product of the number N of people already implicated and the number of people involved who have not yet been implicated.
7. (6.1) A population model is given by
$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(100 - P).$$
   (a) (6.1-9) For what values is the population at equilibrium?
   (b) (6.1-10) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$?
   (c) (6.1-11) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$?
   (d) (6.1-12) Describe how the fate of the population depends on the initial density.
8. (6.1) A population model is given by
$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(P-1)(100 - P).$$
   (a) (6.1-13) For what values is the population at equilibrium?
   (b) (6.1-14) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$?
   (c) (6.1-15) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$?

(d) (6.1-16) Describe how the fate of the population depends on the initial density.

9. (6.1) Radioactive decay: Certain types of atoms (e.g. carbon-14, xenon-133, lead-210, etc.) are inherently unstable. They exhibit random transitions to a different atom while emitting radiation in the process. Based on experimental evidence, Rutherford found in the early 20th century that the number, N, of atoms in a radioactive substance can be described by the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$

where t is measured in years and $\lambda > 0$ is known as the *decay constant*. The decay constant is found experimentally by measuring the half life, τ of the radioactive substance (i.e. the time it takes for half of the substance to decay). Use this information in the following problems.

(a) (6.1-18) Find a solution to the decay equation assuming that $N(0) = N_0$.

(b) (6.1-19) For xenon-133, the half-life is 5 days. Find λ. Assume t is measured in days.

(c) (6.1-20) For carbon-14 the half life is $5,568$ years. Find the decay constant λ, assuming t is measured in years.

(d) (6.1-21) How old is a piece of human bone which contains just 60% of the amount of carbon-14 expected in a sample of bone from a living person, assuming the half life of carbon-14 is $5,568$ years?

(e) (6.1-22) The Dead Sea Scrolls were written on parchment at about 100 B.C. What percentage of carbon-14 originally contained in the parchment remained when the scrolls were discovered in 1947?

10. (6.1-30) Hyperthyroidism is caused by a new growth of tumor-like cells that secrete thyroid hormones in excess to the normal hormones. If left untreated, a hyperthyroid individual can exhibit extreme weight loss, anorexia, muscle weakness, heart disease intolerance to stress, and eventually death. The most successful and least invasive treatment option is radioactive iodine-131 therapy.

This involves the injection of a small amount of radioactivity into the body. For the type of hyperthyroidism called Graves' disease, it is usual for about $40 - 80\%$ of the administered activity to concentrate in the thyroid gland. For functioning adenomas ("hot nodules"), the uptake is closer to $20 - 30\%$. Excess iodine-131 is excreted rapidly by the kidneys. The quantity of radioiodine used to treat hyperthyroidism is not enough to injure any tissue except the thyroid tissue, which slowly shrinks over a matter of weeks to months. Radioactive iodine is either swallowed in a capsule or sipped in solution through a straw. A typical dose is $5 - 15$ millicures. The half-life of iodine-131 is 8 days.

(a) Suppose that it takes 48 hours for a shipment of iodine-131 to reach a hospital. How much of the initial amount shipped is left once it arrives at the hospital?

(b) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left one week later?

(c) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left 30 days later?