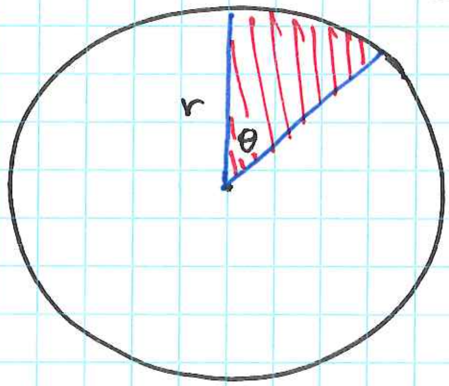
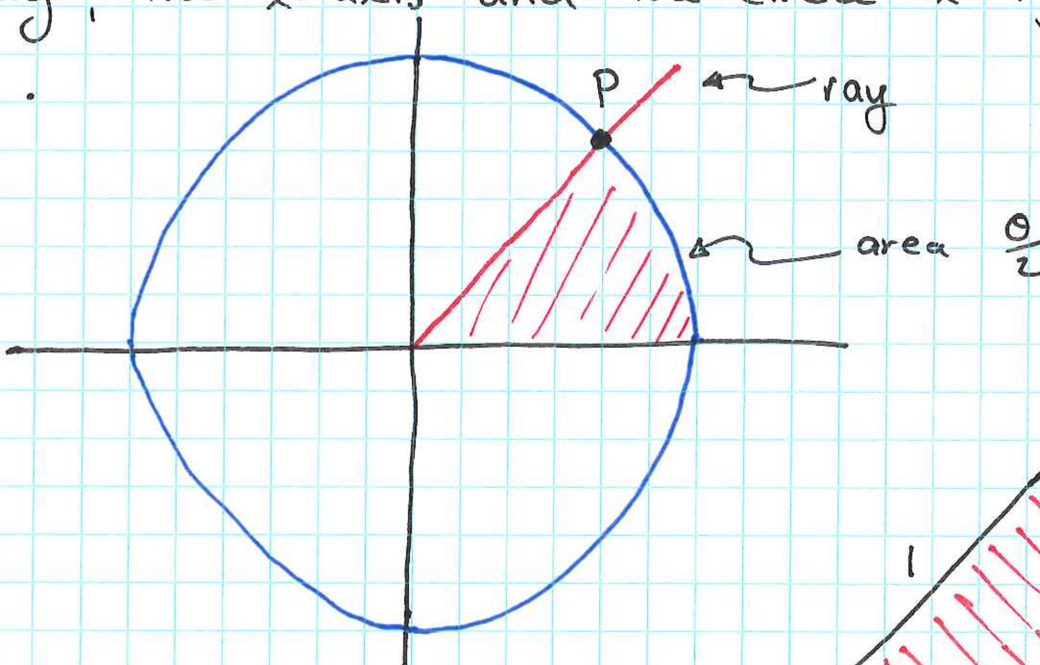
Hyperbolic trig functions

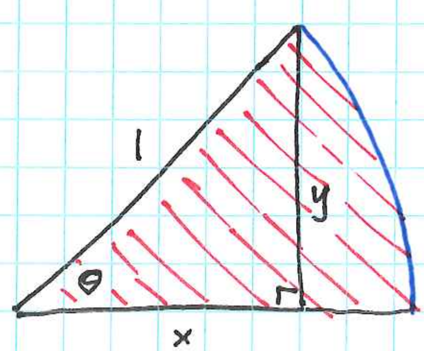
(Perhaps silly) definition of trig functions:

Recall that the sector of a circle, radius r, ~~and~~ with angle θ has area $\frac{r^2\theta}{2}$



Take a ray, from the origin, so the area between the ray, the x-axis and the circle $x^2+y^2=1$ is $\theta/2$.



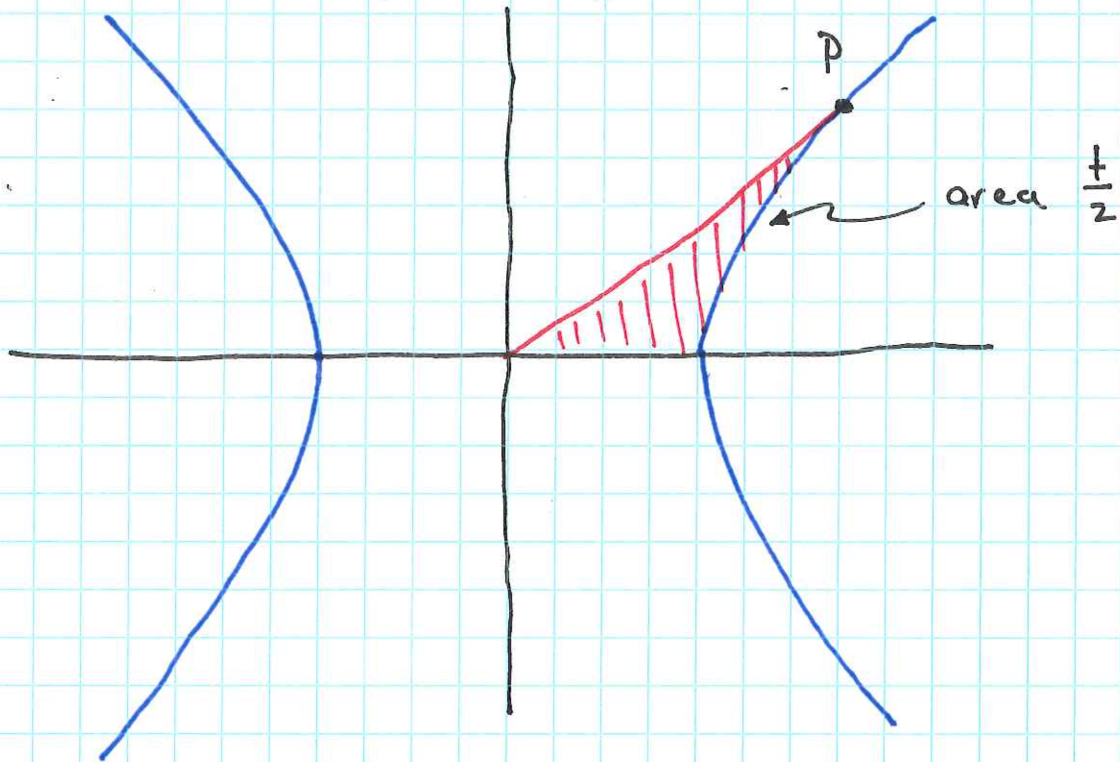


$$x^2 + y^2 = 1$$

The coordinates ~~of the~~ of P are

$$(\cos\theta, \sin\theta).$$

What if we change $x^2+y^2=1$ to $x^2 - y^2 = 1$?



Lets define P to have coordinates

<u>Def</u> $$(\cosh t, \sinh t)$$

These are the hyperbolic trig functions. In fact

<u>Thm</u> $\cosh t = \dfrac{e^x + e^{-x}}{2}$ $\qquad \sinh t = \dfrac{e^x - e^{-x}}{2}$

$\cosh t : \mathbb{R} \longrightarrow \mathbb{R}$, $\qquad \sinh t : \mathbb{R} \longrightarrow \mathbb{R}$.

<u>Def</u> $\tanh t = \dfrac{\sinh t}{\cosh t}$ $\qquad \operatorname{sech} t = \dfrac{1}{\cosh t}$

$\coth t = \dfrac{\cosh t}{\sinh t}$ $\qquad \operatorname{csch} t = \dfrac{1}{\sinh t}$

<u>Exercise</u> Use geogebra or similar to graph.

& Since we have formulas, the derivatives are easy.

<u>Thm</u>

$$\frac{d}{dx}\cosh x = \sinh x \qquad \frac{d}{dx}\sinh x = \cosh x$$

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x \qquad \frac{d}{dx}\coth x = -\operatorname{csch}^2 x$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \; \text{thanh}\, x \qquad \frac{d}{dx}\operatorname{csch} x = -\operatorname{csch} x \coth x$$

We now find the inverse functions. Looking at the graphs we see the following functions are bij.

$\cosh x : [0, \infty) \longrightarrow [1, \infty)$

$\sinh x : \mathbb{R} \longrightarrow \mathbb{R}$

$\tanh x : \mathbb{R} \longrightarrow (-1, 1)$

$\operatorname{sech} x : [0, \infty) \longrightarrow [0, 1]$

$\operatorname{csch} x : (-\infty, 0) \cup (0\ \infty) \to (-\infty, 0) \cup (0, \infty)$

$\coth x : (-\infty, 0) \cup (0\ \infty) \to (-\infty, -1) \cup (1, \infty)$

Def the inverses are

$\cosh^{-1} x : [1\ \infty) \longrightarrow [0\ \infty)$

$\sinh^{-1} x : \mathbb{R} \longrightarrow \mathbb{R}$

$\tanh^{-1} x : (-1, 1) \longrightarrow \mathbb{R}$

$\operatorname{sech}^{-1} x : (0, 1] \longrightarrow [0, \infty)$

$\operatorname{csch}^{-1} x : (-\infty, 0) \cup (0, \infty) \longrightarrow (-\infty, 0) \cup (0, \infty)$

$\coth^{-1} x : (-\infty, -1) \cup (1, \infty) \longrightarrow (-\infty, 0) \cup (0, \infty)$

Then

$$\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2 - 1}} \qquad \frac{d}{dx} \sinh x = \frac{1}{\sqrt{x^2 + 1}}$$

$$\frac{d}{dx} \tanh^{-1} x = \frac{1}{1 - x^2} \qquad \frac{d}{dx} \coth^{-1} x = \frac{1}{1 - x^2}$$

$$\frac{d}{dx} \operatorname{sech}^{-1} x = - \frac{1}{x\sqrt{1 - x^2}} \qquad \frac{d}{dx} \operatorname{csch}^{-1} x = - \frac{1}{|x| \sqrt{x^2 + 1}}$$