Bounded + monotone sequences

Def * A sequence (a_n) is bounded from above if there is a number M such that
$$a_n \leq M \quad \text{for all } n.$$

* A sequence (b_n) is bounded from below if there is a number N such that
$$a_n \geq N \quad \text{for all } n.$$

Ex * $\frac{1}{n}$ is bounded above by 1 and below by 0.

* $a_n = (-1)^n$ is bounded above by 1 and below by -1.
* $a_n = n^2$ is bounded below by 0 but is unbounded above.

Def * A seq. (a_n) is monotonically increasing if $a_n < a_{n+1}$ for all n.

* A seq. (a_n) is monotonically decreasing if $a_n > a_{n+1}$ for all n.

Thm A sequence that is monotonically increasing + bounded above converges. A seq. that is monotonically decreasing + bounded below is convergent.

Ex $\frac{1}{n}$ is monotonically decreasing + bounded below by 0.

Ex Does the sequence $a_n = \ln(n+1) - \ln(n)$ $n \geq 1$ converge?

Testing a few numbers might lead you to believe this sequence is decreasing. How do we show this?

Consider $f(x) = \ln(x+1) - \ln(x)$, then

$$f'(x) = \frac{1}{x+1} - \frac{1}{x} = -\frac{1}{x(x+1)} < 0$$

when $x > 0$. Thus the sequence is monotonically decreasing.

How can we show that it is bounded below?

ln is an increasing $f^{\underline{n}}$ so $\ln(n+1) > \ln(n)$ thus

$a_n \geq 0 \implies a_n$ converges.

Ex $a_n = \sqrt{n^2+2n} - n$

Again, we can show that a_n is increasing by looking at

$$f(x) = \sqrt{x^2+2x} - x$$

$$f'(x) = \frac{x+1}{\sqrt{x^2+2x}} - 1$$

we see that since $(x+1)^2 \geq x^2+2x$

$$\cancel{\frac{x+1}{\sqrt{x^2+2x}} = \frac{x+1}{\sqrt{x}\sqrt{x+2}}}$$

$$\frac{(x+1)^2}{x^2+2x} \geq 1$$

i.e. $\frac{x+1}{\sqrt{x^2+2x}} \geq 1$ so $f'(x) \geq 0$.

Thus (a_n) is increasing.

~~Now $\sqrt{n^2+2n} \geq \sqrt{n^2} = n$ so $a_n \geq 0$ for all n.~~

Since $n^2+2n \leq (n+1)^2$, $\sqrt{n^2+2n} \leq n+1$ thus

$a_n = \sqrt{n^2+2n} - n \leq 1$ so (a_n) is an increasing seq. bounded above and thus converges.