This weeks problem set focuses on eigenvalues and eigenvectors of Matrices. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

1. From section 2.2, problems 4, 9.
2. From section 2.3, problems 12.
3. From section 2.4, problems 7, 16.
4. From section 2.5, problems 4, 8.
5. From section 5.1, problems 1, $2a, c, e$, $3a, c$, $4a, d, h$, 6, 7^*, 14^*, 15, 16, $22a$, 23.
6. From section 5.2, problems 1, $3a, d, e$, 8, 9, 10, 11, 18^*, 19, $20^\dagger$.
7. Let $\theta \in [0, 2\pi)$ and $R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be the linear map that takes a vector and rotates it by θ in the counterclockwise direction. Let E be the standard basis.
   (a) Find $[R_\theta]_E^E$.
   (b) Is R_θ diagonalisable?
   (c) Consider the linear map $R_\theta^{\mathbb{C}} : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$ given by $R_\theta^{\mathbb{C}}(v) = [R_\theta]_E^E v$. Is this linear map diagonalisable?
8. Let U, W be subspaces of V such that $V = U \oplus W$. Let $\mathrm{pr}_U^W : V \longrightarrow V$ be the projection of V onto U along W. Is pr_U^W diagonalisable? *Hint: find a suitable basis for V.*

9.† Let V be a vector space and $E = \{ v_1, \ldots, v_n \}$ a collection of eigenvectors for a linear map $T : V \longrightarrow V$ such that the eigenvalues are all distinct. Prove that E is a linearly independent set. *Hint: use induction on n.*

Solution: We start with $n = 1$. In this case the statement is true since E is a set of one element. For general n, let λ_i be the eigenvalue of v_i. Now consider an arbitrary linear combination

$$0 = \mu_1 v_1 + \cdots + \mu_n v_n$$

Call this equation A. Applying T to both sides we get

$$\begin{aligned} 0 = T(0) &= T(\mu_1 v_1 + \cdots + \mu_n v_n) \\ &= \mu_1 T(v_1) + \cdots + \mu_n T(v_n) \\ &= \mu_1 \lambda_1 v_1 + \cdots + \mu_n \lambda_n v_n. \end{aligned}$$

Call this equation B. Now consider $\lambda_n A - B$

$$0 = \mu_1(\lambda_n - \lambda_1) v_1 + \cdots \mu_{n-1}(\lambda_n - \lambda_{n-1}) v_{n-1}.$$

By induction $\{v_1, \ldots, n_{n-1}\}$ is a linearly independent set, so $\mu_i(\lambda_n - \lambda_i) = 0$ for $1 \leq i \leq n-1$. But by assumption $\lambda_n \neq \lambda_i$ when $1 \leq i \leq n-1$ so we must have that $\mu_i = 0$ for $1 \leq i \leq n-1$.

Thus equation A becomes

$$0 = \mu_n v_n$$

which means $\mu_n = 0$. Thus E is a linearly independent set.