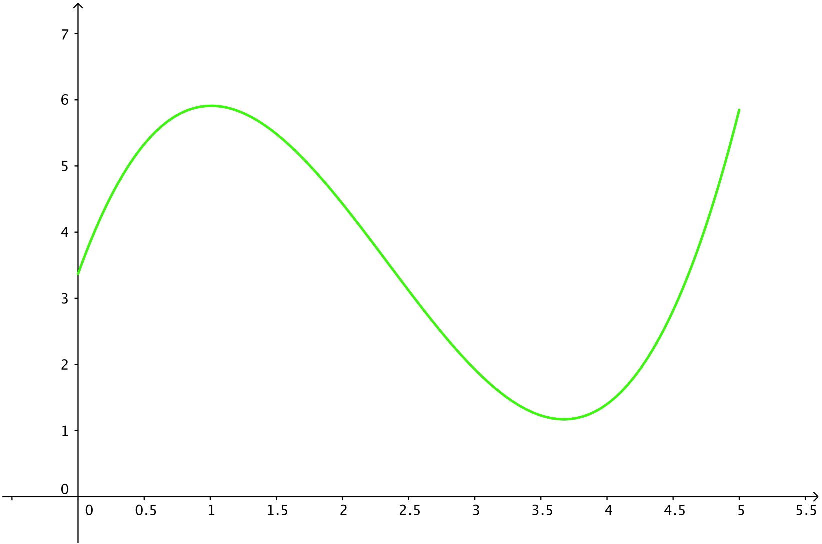
Math 3B: Lecture 10

Noah White

February 1, 2019

More complicated rates of change

Suppose we have a car whose speed is descibed by the following curve. How far has it travelled in this time?



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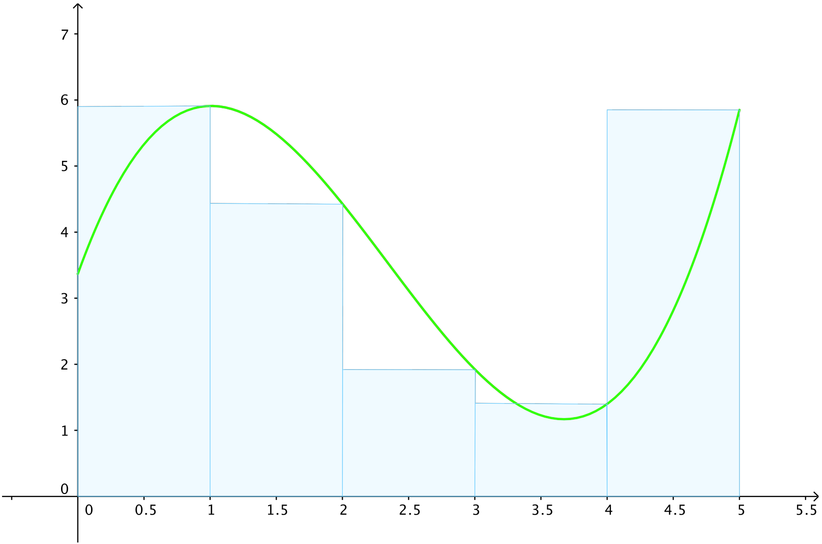

Accumulated change

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- Answer: area under $f(t)$ between a and b.

Areas under general curves

We would like to calculate the area between a function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

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(Too hard to draw, lets look at an animation)

The definite integral

Defintion

The definite integral of a function $f(x)$ is defined to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(a + k\Delta x)$$

where $\Delta x = \frac{b-a}{n}$.

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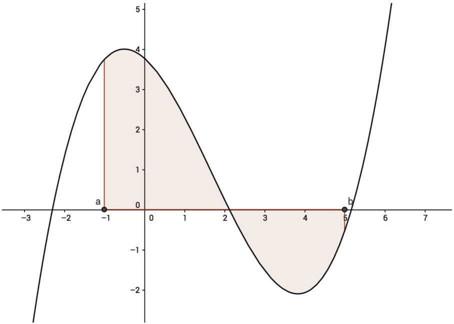
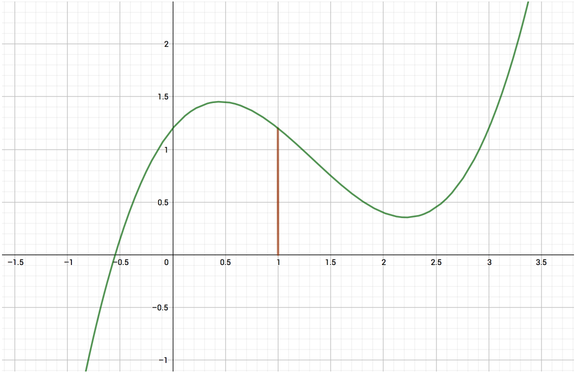


Properties of definite integrals

Zero area

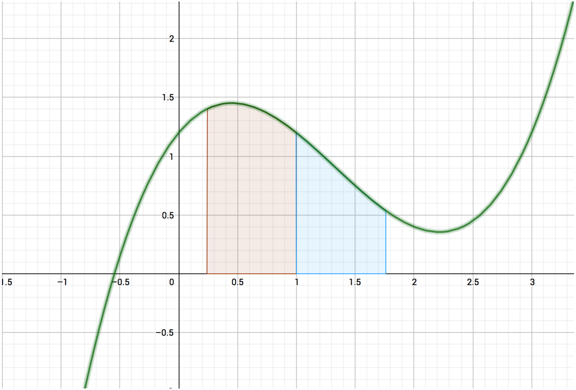
$$\int_a^a f(x)\,dx = 0$$



Properties of definite integrals

Adding areas

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$



More properties of definite integrals

Reversing the area

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

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Additivity

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

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Lineararity (scalars factor out)

$$\int_a^b \alpha f(x)\ dx = \alpha \int_a^b f(x)\ dx$$

The fundamental theorem of calculus

Theorem

For any a,

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

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- $F(x) = \int_a^x f(t)\,dt$ is a function of x.
- every input x produces a number as an output.

A consequence (corrollary)

Corollary

For any antiderivative $F(x)$ of $f(x)$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

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Why?

Well $F(x) = \int_a^x f(t)\,dt + C$ for some a and C. So

$$\begin{aligned} F(b) - F(a) &= \int_a^b f(t)\,dt + C - \int_a^a f(t)\,dt - C \\ &= \int_a^b f(t)\,dt \end{aligned}$$

Example 1

Question

Evaluate the definite integral

$$\int_0^1 x^2 - 4 \, dx$$

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Solution

An antiderivative of $x^2 - 4$ is $\frac{1}{3}x^3 - 4x$ so

$$\begin{aligned}\int_0^1 x^2 - 4 \, dx &= \frac{1}{3} \cdot 1^3 - 4 - \frac{1}{3} \cdot 0^3 + 4 \cdot 0 \\ &= \frac{1}{3} - 4 = -\frac{11}{3}\end{aligned}$$

Example 2

Question

Evaluate the definite integral

$$\int_0^{\pi} \sin x \; dx$$

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Solution

An antiderivative of $\sin x$ is $-\cos x$ so

$$\begin{aligned} \int_0^{\pi} \sin x \, dx &= -\cos \pi + \cos 0 \\ &= -(-1) + 1 = 2 \end{aligned}$$

Why is the FTC true?

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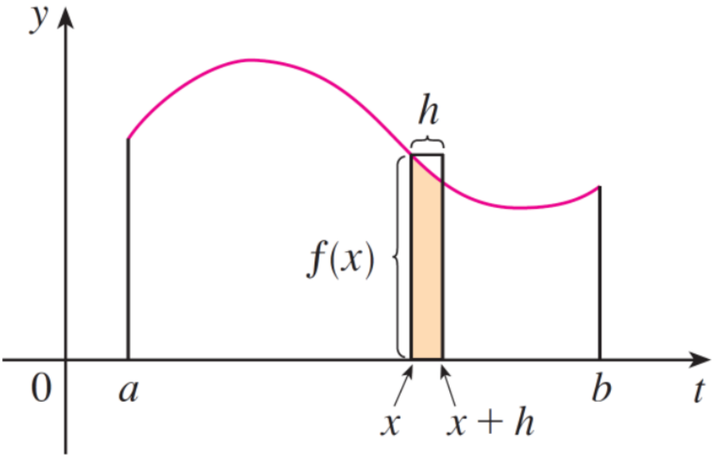
When h is very small $\int_x^{x+h} f(t) \approx$



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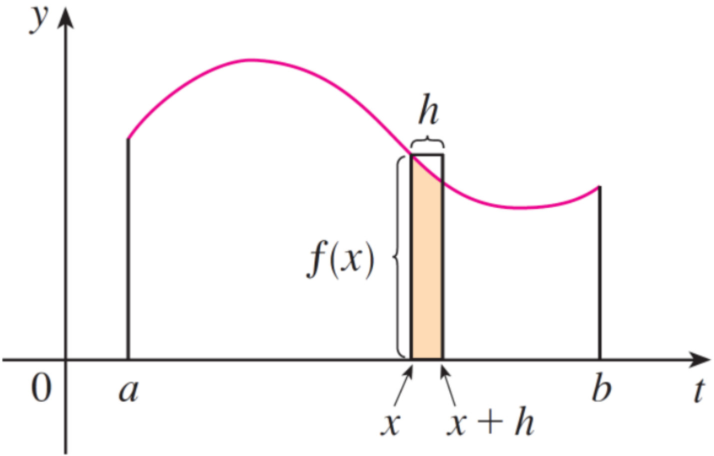

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Example

$$\int \sin(x) - x\, dx = -\cos(x) - \frac{1}{2}x^2 + C$$

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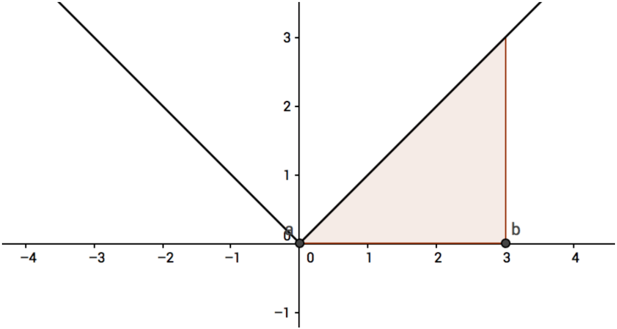
- Lets use $a = 0$.
- How should we calculate $F(x)$?

Example

Use the defintition!

$$F(x) = \int_0^x |t| \ dt$$

is the area under $|t|$!



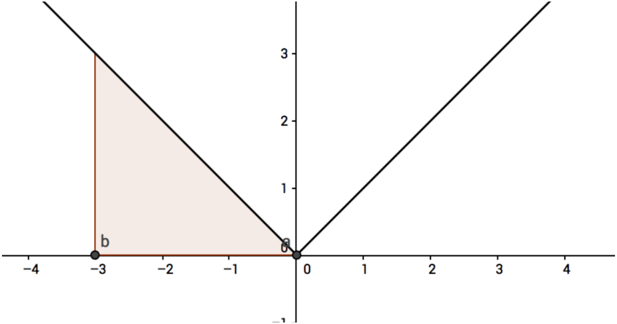
$$F(x) = \frac{1}{2}x^2 \quad \text{if } x \geq 0$$

Example

If $x \leq 0$ then

$$F(x) = \int_0^x |t| \ dt = -\int_x^0 |t| \ dt$$

is the negative of the area under $|t|$!



$$F(x) = -\frac{1}{2}x^2 \quad \text{if } x \leq 0$$

Example

In summary

$$F(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 & \text{if } x \leq 0 \end{cases}$$

or

$$F(x) = \frac{1}{2}x\,|x|$$