This weeks problem set focuses on the ideas of bases and linear transformations. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

1. From section 2.1, problems 15, 17, 18, 19, 24, 26*, 28, 31†, 40*.

2. From section 2.1, problems 1, 2, 5, 6, 9*, 14, 14b.

3.* Let V be a finite dimensional vector space over $\mathbb{F}$ and $B\{\, v_1, \ldots, v_n \,\}$ a basis. Let W be another vector space and $w_1, \ldots, w_n$ a collection of elements. Show that there is a unique linear map such that $T(v_i) = w_i$.

4.* Let V and W be vector spaces over $\mathbb{F}$. Define $\operatorname{Hom}(V, W)$ to be the set of linear maps from V to W.
   (a) Show that $\operatorname{Hom}(V, W)$ is itself a vector space.
   (b) If V is finite dimensional and B is a basis for V, construct a basis for $V^* = \operatorname{Hom}(V, \mathbb{F})$. The vector space V^* is called the *dual space* to V.

5.* Let $T : V \longrightarrow W$ be an injective linear map. Show that, if we consider T, instead, as a linear map $V \longrightarrow \operatorname{im} T$ (just restrict what we consider to be the codomain), then it defines an isomorphism and shows that $V \cong \operatorname{im} T$.

6.* Let V and W be vector spaces over $\mathbb{F}$. Define the set

$$V \times W = \{\, (v, w) \mid v \in V \text{ and } w \in W \,\}.$$

   This is called the *product* of the vector spaces.
   (a) Show that $V \times W$ is a vector space.
   (b) Define a map $\iota_V : V \to V \times W$ by $\iota_V(v) = (v, 0)$. Show that ι_V is an injective linear map. Note that we can define a similar map ι_W.
   (c) If $U \subset V$ is a subspace, show that $U \times W$ is a subspace of $V \times W$.
   (d) Show that $V \times W = (V \times \{\, 0 \,\}) \oplus (\{\, 0 \,\} \times W)$.
   Note that we can consider $V \times \{\, 0 \,\}$ as a copy of V in $V \times W$. For this reason, often mathematicians write $V \oplus W$ instead of $V \times W$ and call it the external direct product. Though this is a little confusing so we won't talk about it in this way in this class.

7.* (2.1.18) Give an example of a linear transformation $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ such that $\ker T = \operatorname{im} T$.

8.* (2.1.19) Give an example of distinct linear transformations T and U such that $\ker T = \ker U$ and $\operatorname{im} T = \operatorname{im} U$.