Lecture 6

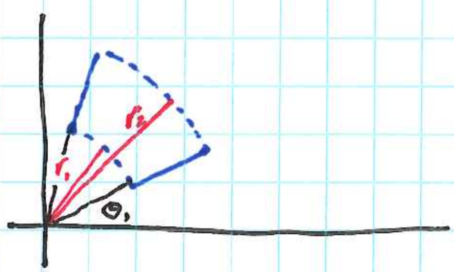
1. Integration in polar coordinates

- Sometimes it is much easier to integrate a function expressed in polar coordinates than in rectangular coordinates.

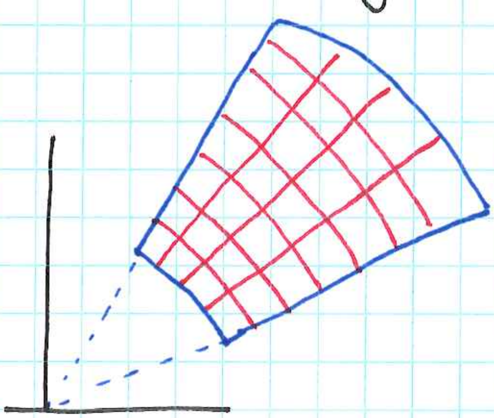
- How do we do this? Suppose we have a function $f(x\ y)$ which we express in polar coordinates $f(r\cos\theta, r\sin\theta)$.

- We integrate f on a polar rectangle

$$R = [p\ q] \times [\varphi, \psi]$$



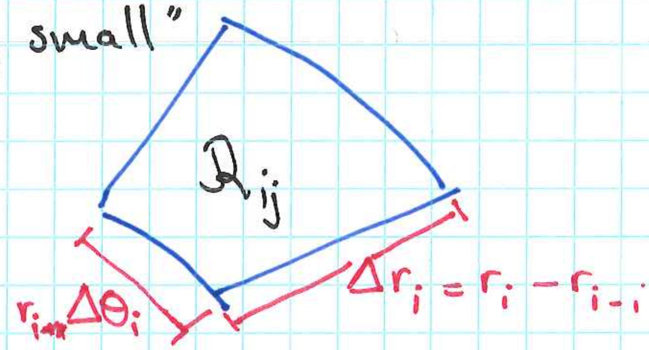
- Partition the rectangle:



by $p = r_1 < r_2 < \_\_\_ < r_m = q$

$\varphi = \theta_1 < \theta_2 < \_\_\_ < \theta_n = \psi$

- The rectangle $\mathcal{R}_{ij} = [r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j]$ is "very small"



- Is approx a rectangle, thus has area approx

$$r_i \, \Delta r_i \, \Delta\theta_i$$

~~- Thus volume~~

- Choose sample points $P_{ij} \in \mathcal{R}_{ij}$
- Volume under f, over $\mathcal{R}_{ij}$ is approx

$$f(P_{ij})\, r_i \, \Delta r_i \, \Delta\theta_i$$

Thm
$$\iint_{\mathcal{R}} f(x\, y)\, dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

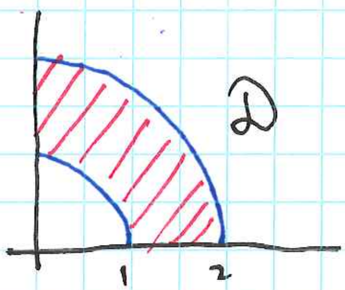
$$= \lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij})\, r_i \, \Delta r_i \, \Delta\theta_i$$

Thm For any region

$$\iint_D f(x\ y)\,dA = \iint_{D_0} f(r\cos\theta, r\sin\theta)\, r\,dA$$

region in ~~pol~~ polar coordinates

Ex Integrate $x+y$ over the annulus with inner radius 1 and outer radius 2, in the first quadrant.



This is the polar rectangle $R = [1\ 2] \times [0\ \frac{\pi}{2}]$

so

$$\iint_R x+y\,dA = \int_0^{\frac{\pi}{2}} \int_1^2 r(\cos\theta + \sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{3} r^3 (\cos\theta + \sin\theta)\right]_1^2 d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{7}{3}(\cos\theta + \sin\theta)\,d\theta$$

$$= \left[\frac{7}{3}(\sin\theta - \cos\theta)\right]_0^{\frac{\pi}{2}}$$

$$= \frac{7}{3}(1-0) - \frac{7}{3}(0-1) = \frac{14}{3}.$$

- Using the same argument as for vertically / horizontally simple regions we can see, if

$$r_1(\theta) \quad \text{and} \quad r_2(\theta)$$

are two functions, and D is the region

$$D = \{(r\ \theta) \mid \theta \in [\varphi, \psi],\ r_1(\theta) \leq r \leq r_2(\theta)\}$$

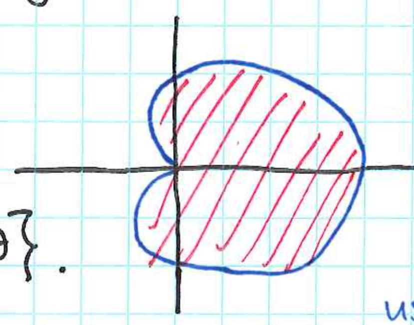
then

<u>Thm</u>

$$\iint_D f(x\ y)\, dA = \int_{\varphi}^{\psi} \int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

<u>Ex</u> Find the area D enclosed by the curve

$$r = 1 + \cos\theta$$



- Note that

$$D = \{(r\ \theta) \mid 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1 + \cos\theta\}.$$

so

$$\text{Area} = \iint_D 1\, dA = \int_0^{2\pi} \int_0^{1+\cos\theta} r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2\, d\theta = \overbrace{\int_0^{2\pi} \frac{1}{2}\, d\theta}^{\pi} + \overbrace{\int_0^{2\pi} \cos\theta\, d\theta}^{0} + \overbrace{\frac{1}{2}\int_0^{2\pi} \cos^2\theta\, d\theta}^{\frac{\pi}{2}}$$

use double angle formula

$$\cancel{\left[\theta + \sin\theta\right]_0^{2\pi} = 2\pi} = \frac{3\pi}{2}.$$