Final exam practice

UCLA: Math 170A, Fall 2017

Instructor: Noah White
Date:
Version: 1

- This exam has 8 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Question 1 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) A box contains 4 fair coins and 3 fake coins where both sides are head. A coin is selected from the box and tossed. The result is heads. What is the probability that it is a fair coin?

   A. 0.2
   B. 0.4
   C. 0.3
   D. 0.25

   (b) (2 points) Let A and B be events. If $\mathbb{P}(A|B^c) = 1$ and $\mathbb{P}(B|A) = 2$, what is $\dfrac{\mathbb{P}(B^c)}{\mathbb{P}(A)}$?

   A. 0.6
   B. 0.3
   C. 1.2
   D. 0.4

   (c) (2 points) Let A, B be events. Then $(A \cap B) \cup (A^c \cap B)^c =$

   A. $A^c \cap B^c$
   B. $A \cup B^c$
   C. $A \cap B$
   D. A

(d) (2 points) A company has 3 departments and there are 50 people in each department. How many ways can a committee of 12 people be formed so that there are exactly 4 people from each department? (The order of the 12 people does not matter.)

A. $\binom{50}{4}^3$

B. $\binom{12}{4}\binom{150}{12}$

C. $\binom{50}{3}^4$

D. $\binom{50}{4}\binom{150}{12}$

(e) (2 points) A box contains 4 blue balls and 6 red balls. 2 balls are selected from the box simultaneously. What is the probability that both are blue given that at least one of them is blue?

A. 0.05

B. 0.2

C. 0.1

D. 0.25

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(3)$ are independant and satisfy

   $$\text{var}(X - 2Y) = 28$$

   the $\lambda =$

   A. 40
   B. 4
   C. 16
   D. 28

   (b) (2 points) Let X and Y be independant random variables. Which of the following statements is/are true?

   1. If $X \sim \text{binomial}(8, 0.3)$ and $Y \sim \text{Bernoulli}(0.3)$ then $X + Y \sim \text{binomial}(9, 0.3)$.
   2. If $X \sim \text{Geometric}(0.8)$ then $\mathbb{P}(X \geq 5) = 1 - 0.2^4$.
   3. If $X \sim \text{binomial}(100, 0.03)$ and $Y \sim \text{Poisson}(3)$ then X and Y have exactly the same variance.

   A. Only 1 and 2 are true.
   B. Only 2 and 3 are true.
   C. Only 1 is true.
   D. 1, 2 and 3 are true.

   (c) (2 points) A fair six sided die is rolled 6 times. Let X be the number of times 2 appears. Find $\mathbb{E}(X^2)$.

   A. 5/6
   B. 7/6
   C. 9/6
   D. 11/6

(d) (2 points) Let $X_11, \ldots, X_5$ be independent Geometric random variables with parameter e^{-2}. Find

$$\mathbb{P}(\min(X_1, \ldots, X_5) = 1).$$

A. $1 - e^{-2}$
B. $1 - e^{-5}$
C. $1 - e^{-1}$
D. $1 - e^{-10}$

(e) (2 points) The lifetime in hours of an electronic tube is a random variable having a probability density function given by $f_X(x) = xe^{x}$ for $x \geq 0$ and $f_X(x) = 0$ otherwise. The expected lifetime of such a tube is

A. 1
B. 2
C. 3
D. 4

3. (a) (2 points) State Bayes theorem.

Solution: Let $(\Sigma, \mathbb{P})$ be a probability space, $A_1, A_2, \ldots, A_n$ a partition of Σ and B an event. Then

$$\mathbb{P}(A_1|B) = \frac{\mathbb{P}(B|A_1)\mathbb{P}(A_1)}{\mathbb{P}(B|A_1)\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\mathbb{P}(A_2) + \ldots + \mathbb{P}(B|A_n)\mathbb{P}(A_n)}.$$

(b) (8 points) There are n boxes, the 1st one containing 1 glass, the 2nd box containing 2 glasses, the 3rd box containing 3 glasses, ... , kth box containing k glasses, ... nth box containing n glasses. In each box every glass is broken with probability $1/2$ independently of all other glasses. A contestant chooses a box uniformly at random. Given that he found no broken glasses in this box, find the probability that he choose the 1st box. Simplify your answer.

Solution: Let A_k be the event that the contestant chose the kth box, then $\mathbb{P}(A_k) = 1/n$, for any $k = 1, 2, \ldots, n$. Let X be the number of broken glasses in the box chosen by the contestant. Given the event A_k there are k glasses in the box opened, $\mathbb{P}(X = 0|A_k) = (1/2)^k$. Then use the Bayes' rule

$$\begin{aligned}
\mathbb{P}(A_1|X=0) &= \frac{\mathbb{P}(X=0|A_1)\mathbb{P}(A_1)}{\mathbb{P}(X=0|A_1)\mathbb{P}(A_1) + \mathbb{P}(X=0|A_2)\mathbb{P}(A_2) + \cdots + \mathbb{P}(X=0|A_n)\mathbb{P}(A_n)} \\
&= \frac{(1/2)\cdot(1/n)}{(1/2)\cdot(1/n) + (1/2)^2\cdot(1/n) + \cdots + (1/2)^n\cdot(1/n)} = \frac{1/2}{1/2 + 1/2^2 + \cdots + 1/2^n} \\
&= \frac{1/2}{1 - 1/2^n} = \frac{2^{n-1}}{2^n - 1}.
\end{aligned}$$

4. (a) (2 points) If X is a geometric random variable with parameter p, what is the range of X and what is $p_X(k)$?

Solution: Range is $\{1, 2, \dots\}$ and $p_X(k) = (1-p)^{k-1}p$.

(b) (8 points) You arrive at the opening of a new cafe which has only two chairs. Unfortunately, there are three people ahead of you in the line. You know that the number of minutes that a guest spends in a cafe is a geometric random variable with mean 2, and that the time each guest spends in the cafe is independent of all other guests. What is the probability you will have to wait more than n minutes.

Solution: You will wait more than n minutes if either:

1) each of the 1st two guests spends more than n minutes in the cafe,

2) one of the first two guests spends more than n minutes in the cafe and the other spends $k \le n$ minutes, but then the 3rd guest who occupies his table spends more than $n-k$ minutes in the cafe.

These events are disjoint and the sum of their probabilities gives us the answer. To compute their probabilities observe that the probability that a geometric parameter $1/2$ random variable X is strictly larger than k is

$$\frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \frac{1}{2^{k+3}} + \cdots = \frac{1}{2^k}.$$

Therefore, since the guest times are independent the event in 1) has probability $\frac{1}{2^{2n}}$. The event in 2) has probability

$$2\left(\sum_{k=1}^{n-1} \frac{1}{2^k}\frac{1}{2^{n-k}}\right) \times \frac{1}{2^n} = \frac{n-1}{2^{2n-1}}.$$

(the first $\frac{1}{2^k}$ in the sum is the probability that a guest spends exactly k minutes in the cafe and the $\frac{1}{2^{n-k}}$ term is the probability of the event that the next who occupies the table spends strictly more than $n-k$ minutes, also we have to multiply by 2 in front since the person who leaves before n minutes could be either one of the first two guests) Therefore, the probability we wait more than n minutes is

$$\frac{1}{2^{2n}} + \frac{n-1}{2^{2n-1}} = \frac{2n-1}{2^{2n}}.$$

5. The number of cars that arrive in a certain mechanic shop is a Poisson random variable with the parameter $\lambda = 1$. For $k \geq 1$, given that exactly k cars arrive, the total time T that it takes to repair all these cars is a continuous uniform random variable on the interval $[0, k+1]$.

   (a) (5 points) Find the probability $\mathbb{P}(T \leq 1)$. Simplify your answer.

   Solution: Use the total probability theorem. Let X be the number of cars that arrived. Then

   $$\mathbb{P}(T \leq 1) = \sum_{k=0}^{\infty} \mathbb{P}(T \leq 1 | X = k)\mathbb{P}(X = k).$$

   If $X = 0$ then $T = 0$ so $\mathbb{P}(T \leq 1 | X = 0) = 1$. For $k \geq 1$ we have $\mathbb{P}(T \leq 1 | X = k) = \frac{1}{k+1}$. Then

   $$\mathbb{P}(T \leq 1) = e^{-1} + \sum_{k=1}^{\infty} \frac{1}{k+1} \frac{e^{-1}}{k!} = e^{-1}\Big(1 + \sum_{k=1}^{\infty} \frac{1}{(k+1)!}\Big) = e^{-1}\Big(e - \frac{1}{1!}\Big) = 1 - e^{-1}.$$

   (b) (5 points) Find the expectation $\mathbb{E}(T)$. Simplify your answer.

   Solution: Given $X = k$ the conditional expectation $\mathbb{E}(T|X = k) = (k+1)/2$, for $k \geq 1$ and $\mathbb{E}(T|X = 0) = 0$. We can use the formula

   $$\mathbb{E}(X) = \sum_{k=0}^{\infty} \mathbb{E}(T|X = k)\mathbb{P}(X = k) = \sum_{k=1}^{\infty} \frac{k+1}{2} \frac{e^{-1}}{k!}$$
   $$= \sum_{k=0}^{\infty} \frac{k+1}{2} \frac{e^{-1}}{k!} - \frac{e^{-1}}{2} = \mathbb{E}((X+1)/2) - \frac{e^{-1}}{2} = \frac{1+1}{2} - \frac{e^{-1}}{2} = 1 - \frac{e^{-1}}{2}.$$

   (Or you could again use the exponential series for this part too.)

6. (a) (1 point) If S and T are independant frandom variables and f, g are functions of a real variable, then $f(S)$ and $g(T)$ are independant random variables. True or False?

Solution: True.

(b) (9 points) Let X be a standard Normal random variable, and Y a geometric random variable with parameter $1/4$. If X and Y are independent compute $\mathbb{E}(e^X)$, $\mathbb{E}(e^Y)$ and $\mathbb{E}(e^{X+Y})$.

Solution:

$$\mathbb{E}(e^X) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^t e^{-t^2/2}\ dt = e^{1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(t-1)^2/2}\ dt.$$

What we did here is completed the square in the exponent. Now we will use the substitution $u = t - 1$ and then use the known fact that the normal density integrated to 1. This is quite a standard trick.

$$\mathbb{E}(e^X) = e^{1/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-u^2/2}\ du = e^{1/2}.$$

On the other hand

$$\mathbb{E}(e^Y) = \sum_{k=1}^{\infty} e^k \frac{3}{4^k} = \frac{3e}{4-e}.$$

Since X and Y are independent, so are e^X and e^Y and

$$\mathbb{E}(e^{X+Y}) = \mathbb{E}(e^X e^Y) = \mathbb{E}(e^X)\mathbb{E}(e^Y) = \frac{12e^{1/2}}{4-e}.$$

7. Let X be an exponential random variable with parameter 1. Given $X = t$, let Y be uniformly distributed on the interval $[0, e^t]$.

(a) (3 points) Compute $\mathbb{E}(Y|X = 1)$.

Solution: Given $X = 1$ random variable Y is uniform on the interval $[0, e]$, so $\mathbb{E}(Y|X = 1) = e/2$. Since the PDF of X is

$$f_X(s) = \begin{cases} 0, & s < 0 \\ e^{-s}, & s \geq 0. \end{cases}$$

and the conditional PDF of Y given X is

$$f_{Y|X}(t|s) = \begin{cases} e^{-s}, & \text{if } 0 \leq t \leq e^s, \\ 0, & \text{otherwise.} \end{cases}$$

(b) (3 points) Compute the joint PDF of X and Y.

Solution: The joint PDF of X and Y is

$$f_{X,Y}(s,t) = f_X(s) f_{Y|X}(t|s) = \begin{cases} e^{-2s}, & \text{if } s \geq 0, \text{ and } 0 \leq t \leq e^s \\ 0, & \text{otherwise..} \end{cases}$$

(c) (4 points) Compute the PDF of Y.

Solution: The marginal PDF of Y is now

$$f_Y(t) = \int_{-\infty}^{\infty} f_{X,Y}(s,t)\ ds = \begin{cases} \int_0^\infty e^{-2s}\ ds, & \text{if } 0 \leq t \leq 1 \\ \int_{\log t}^\infty e^{-2s}\ ds, & \text{if } t > 1. \end{cases}$$

So for $0 \leq t \leq 1$ we have

$$f_Y(t) = \frac{1}{2}.$$

For $t > 1$ we have

$$f_Y(t) = \frac{1}{2t^2}.$$

8. Let T be a triangle with vertices $(-1, 0)$, $(1, 0)$ and $(0, 1)$. Let the random variables X and Y be jointly continuous with joint PDF

$$f_{X,Y}(x, y) = \begin{cases} C|xy|, & \text{for } (x, y) \in T, \\ 0, & \text{for } (x, y) \notin T. \end{cases}$$

(a) (3 points) Find the value of the constant C.

Solution: To find the value of C we need to integrate the function $C|xy|$ over the triangle T. Since both the function $C|xy|$ and the triangle T are symmetric about the y axis the integral of this function over the left half of T is the same as the integral of the right half of T so

$$\iint_T C|xy|\ dydx = 2\int_0^1 \int_0^{1-x} Cxy\ dydx.$$

Here we got rid of the absolute values, because on the right half of the triangle both x and y are positive. Then the above integral is

$$\int_0^1 Cx(1-x)^2\ dx = C\int_0^1 x - 2x^2 + x^3\ dx = C(1/2 - 2/3 + 1/4) = C/12.$$

So $C = 12$.

(b) (4 points) Find the PMFs of X and Y.

Solution: Compute the marginal for Y for $0 \le t \le 1$ ($f_Y(t) = 0$ for $t < 0$ and $t > 1$):

$$f_Y(t) = \int_{-\infty}^{\infty} f_{X,Y}(s, t)\ ds = \int_{-(1-t)}^{1-t} 12|st|\ ds.$$

Again since the function under the integral is even we have

$$f_Y(t) = 2\int_0^{1-t} 12|st|\ ds = 24\int_0^{1-t} st\ ds = 12t(1-t)^2.$$

For X:

$$f_X(s) = \int_{-\infty}^{\infty} f_{X,Y}(s, t)\ dt = \begin{cases} 0, & \text{for } s < -1, \text{ or } s > 1, \\ \int_0^{s+1} -12st\ dt = -6s(1+s)^2, & \text{for } -1 \le s \le 0 \\ \int_0^{1-s} 12st\ dt = 6s(1-s)^2, & \text{for } 0 < s \le 1. \end{cases}$$

In any case we can write $f_X(s) = 6|s|(1-|s|)^2$, for $-1 < s < 1$ and $f_X(s) = 0$ otherwise.

(c) (3 points) Are X and Y independant?

Solution: For X and Y to be independent we would have to have $f_{X,Y}(s, t) = f_X(s)f_Y(t)$ which is clearly not the case. So they are not independent.

(d) Find the CDF of the random variable $Z = X + Y$ (compute the integral, but there is no need to simplify the resulting expression).

Solution: To find the PDF of Z compute

$$F_Z(t) = \mathbb{P}(Z \leq t) = \iint_{T'} 12|xy| \; dxdy,$$

where T' is the triangle obtained as the intersection of the triangle T and the set $\{(x,y) : x + y \leq t\}$ (we change variables s, t to x, y because t appears as the argument of the CDF). Clearly $F_Z(t) = 0$, for $t \leq -1$ and $F(t) = 1$, for $t \geq 1$. For $-1 < t < 1$ we have that T' is the triangle with vertices $(-1, 0)$, $(t, 0)$ and $((t-1)/2, (t+1)/2)$ so we have

$$\iint_{T'} 12|xy| \; dxdy = \int_{-1}^{(t-1)/2} \int_0^{1+x} 12|xy| \; dydx + \int_{(t-1)/2}^{t} \int_0^{t-x} 12|xy| \; dydx.$$

Any x between -1 and $(t-1)/2$ is negative so the first integral is just

$$\begin{aligned}\int_{-1}^{(t-1)/2} \int_0^{1+x} 12|xy| \; dydx = \int_{-1}^{(t-1)/2} \int_0^{1+x} -12xy \; dydx = \int_{-1}^{(t-1)/2} -6x(x+1)^2 \; dydx \\ = \frac{1}{2} - \frac{3}{32}(t-1)^4 - \frac{1}{2}(t-1)^3 - \frac{3}{4}(t-1)^2.\end{aligned}$$

If $t < 0$ the other integral is over an interval $[(t-1)/2, t]$ which is left of the origin so here too $|x| = -x$ and for $t < 0$

$$\begin{aligned}\int_{(t-1)/2}^{t} \int_0^{t-x} 12|xy| \; dydx = \int_{(t-1)/2}^{t} \int_0^{t-x} -12xy \; dydx = -\int_{(t-1)/2}^{t} 6x(t-x)^2 \; dx \\ = -\left[3x^2t^2 - 4x^3t + \frac{3}{2}x^4\right]_{x=(t-1)/2}^{x=t} = \frac{3}{4}t^2(t-1)^2 - \frac{1}{2}t(t-1)^3 + \frac{3}{16}(t-1)^4 - \frac{1}{2}t^4.\end{aligned}$$

and if $t > 0$ then

$$\int_{(t-1)/2}^{t} \int_0^{t-x} 12|xy| \; dydx = -\int_{(t-1)/2}^{0} \int_0^{t-x} 12xy \; dydx + \int_0^t \int_0^{t-x} 12xy \; dydx$$

$$= -\left[3x^2t^2 - 4x^3t + \frac{3}{2}x^4\right]_{x=(t-1)/2}^{x=0} + \left[3x^2t^2 - 4x^3t + \frac{3}{2}x^3\right]_0^{x=t} = \frac{3}{4}t^2(t-1)^2 - \frac{1}{2}t(t-1)^3 + \frac{3}{16}(t-1)^4 + t^4.$$

Then $F_Z(t)$ is obtained by summing the two integrals.

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