This weeks problem set focuses isomorphisms and coordinate vectors and the matrices associated to linear transformations. It will be quite a large problem set, and because of the way we will be covering it in class, don't worry if you can't do some of the problems until after next Friday. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

1. From section 2.4, problems 1, $2a, c, e$, 3, 7, 14, 15^*, 17^*, $24^{*,\dagger}$.

2. From section 2.2, problems 1, $2a, c, f$, 10, $11^\dagger$, 12^*, $14^\dagger$, 16.

3. From section 2.3, problems 1, $2a$, 3, 12, 16, $17^\dagger$, 16.

   There are mathematical objects called $\mathfrak{sl}_2$*-representations* which are important in quantum mechanics and beautiful objects in their own right. We won't define what they are exactly**, but their are vector spaces that come packaged with a certain pair of linear maps. The next questions give an example.

4.† Let $V = \mathbb{C}[x, y]$ be the vector space of polynomials in two variables. So we have $x^2 - 2xy^2 + 1 \in V$ for example. Define two linear maps $E, F : V \longrightarrow V$ where

$$E(p) = x\frac{\partial p}{\partial y} \text{ and } F(p) = y\frac{\partial p}{\partial x}$$

(a) Find a formula for $H := EF - FE$.

Solution: We just calculate what H does to a polynomial, using the chain rule:

$$\begin{aligned} H(p) &= EF(p) - FE(p) \\ &= x\frac{\partial}{\partial y}y\frac{\partial p}{\partial x} - y\frac{\partial}{\partial x}x\frac{\partial p}{\partial y} \\ &= x\frac{\partial p}{\partial x} + xy\frac{\partial^2 p}{\partial y \partial x} - y\frac{\partial p}{\partial y} - xy\frac{\partial^2 p}{\partial y \partial x} \\ &= x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y}. \end{aligned}$$

(b) A subspace $U \subset V$ is called a *subrepresentation* if $E(U) \subset U$ and $F(U) \subset U$. Let $V(n) = \operatorname{span}\{\, x^{n-a}y^a \mid 0 \leq a \leq n \,\}$, this is the space of *homogeneous polynomials of degree* n, i.e. every term on the polynomial has degree n. Show that $V(n)$ is a subrepresentation, for any $n \geq 0$.

Solution: Note that $E(x^{n-a}y^a) = ax^{n-a+1}y^{a-1} \in V(n)$ and $F(x^{n-a}y^a) = (n-a)x^{n-a-1}y^{a+1} \in V(n)$. Thus, since an arbitrary element $p \in V(n)$ is simply a linear combination of these, we have that $E(p), F(p) \in V(n)$ and hence it is a subrepresentation.

(c) With the basis $x^n, x^{n-1}y, x^{n-2}y^2, \ldots, y^n$, determine the matrix corresponding to the linear maps E, H, F restricted to the subspaces $V(n)$.

Solution: We will do H first. Note that $H(x^{n-a}y^a) = (n - 2a)x^{n-a}y^a$. Hence the matrix for H is diagonal with the (i, i)-entry being $n - 2(i - 1)$.
Now $E(x^{n-a}y^a) = ax^{n-a+1}y^{a-1}$ and so the matrix for E is zero everywhere, apart from the $(i, i + 1)$-entry which is i.
Similarly, $F(x^{n-a}y^a) = (n - a)x^{n-a-1}y^{a+1}$ and so the matrix for F is zero everywhere, apart from the $(i + 1, i)$-entry which is $n - i + 1$.

Examples for $n = 3$ are

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

5.† Another example of an $\mathfrak{sl}_2$ representation is given by $W = \mathbb{C}^2$ and where E' and F' are the linear transformations given by left multiplication by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find an isomorphism $\theta : V(1) \longrightarrow W$ so that $\theta E = E'\theta$ and $\theta F = F'\theta$ as linear maps $V(1) \longrightarrow W$.

Solution: The isomorphism θ will be determined by the values of $\theta(x)$ and $\theta(y)$. So lets set

$$\theta(x) = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \theta(x) = \begin{pmatrix} c \\ d \end{pmatrix}.$$

In order for $\theta E = E'\theta$ and $\theta F = F'\theta$, we must have that four things hold:

$$\theta E(x) = E'\theta(x) \tag{1}$$
$$\theta E(y) = E'\theta(y) \tag{2}$$
$$\theta F(x) = F'\theta(x) \tag{3}$$
$$\theta F(y) = F'\theta(y). \tag{4}$$

Notice that $E(x) = 0$, thus $\theta E(x) = 0$ and by 1 we must have that

$$0 = E'\theta(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

Thus $b = 0$. We do something similar and notice that $F(y) = 0$ so by we have

$$0 = F'\theta(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix}$$

so $c = 0$. Now we only need to check 2 and 3. Notice that $E(y) = x$ so $\theta E(y) = \theta(x)$. By 2,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} d \\ c \end{pmatrix}$$

Which gives us that $a = d$ and $b = c$. Checking 3 gives us the same result. So now we have that

$$\theta(x) = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \theta(x) = \begin{pmatrix} c \\ d \end{pmatrix}.$$

We just need to pick an a which ensures this will be an isomorphism. It is clear that picking any $a \neq 0$ is fine.

6.† Show that there is no, nonzero, linear map $\theta : V(n) \longrightarrow V(m)$ so that $E\theta = \theta E$ and $F\theta = \theta F$ whenever

$n \neq m$. Hint: if such a map does exist, where does x^n get sent? Now use that $H\theta = \theta H$. This is pretty hard, let me know if you need more hints

Solution: We will give a brief sketch. Consider $\theta(x^n)$. We know that $E(x^n) = 0$ so $0 = \theta(E(x^n)) = E\theta(x^n)$. I.e. we must have that $E\theta(x^n) = 0$. If

$$\theta(x^n) = \lambda_0 x^m + \lambda_1 x^{m-1} y + \ldots + \lambda_m y^m$$

then

$$E\theta(x^n) = 0 + \lambda_1 x^m + \ldots + m\lambda_m x y^{m-1}$$

The only way for this to be zero is if $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 0$. Thus $\theta(x^n) = \lambda x^m$ for some $\lambda \in \mathbb{C}$. Now lets use the fact that $H\theta = \theta H$ (this is true since $(EF - FE)H = H(EF - FE)$).

Observe that $H(x^n) = nx^n$, thus $\theta H(x^n) = \lambda n x^m$. On the other hand $H\theta(x^n) = H(\lambda x^m) = \lambda m x^m$. Hence $\lambda n x^m = \lambda m x^n$. The only way this is possible, is if either $m = n$, or if $\lambda = 0$.

If $m \neq n$ then $\lambda = 0$, so $\theta(x^n) = 0$. Now comes the somewhat challending part. How can we figure out that this means that $\theta(x^{n-a}y^a) = 0$? We consider $F^a(x^n)$. This is the result of applying F to the element x^n, a times. The result is $F^a(x^n) = \frac{n!}{(n-a)!} x^{n-a} y^a$. But we know that $\theta F^a = F^a \theta$ so we must have that $\theta(F^a(x^n)) = F^a(\theta(x^n)) = F^a(0) = 0$. But since θ is linear

$$\theta(x^{n-a}y^a) = \frac{(n-a)!}{n!} F^a(\theta(x^n)) = 0.$$

** Ok, if you really want to know exactly what they are here is the definition: An $\mathfrak{sl}_2$-representation is a vector space V with two linear maps $E, F : V \longrightarrow V$ such that

$$E^2 F - 2EFE + FE^2 = -2E$$

and the same equation with the E's and F's swapped. There is a much more intuitive definition but one would need to know some more abstract algebra. If you are really keen, try and find more $\mathfrak{sl}_2$ representations and show me!