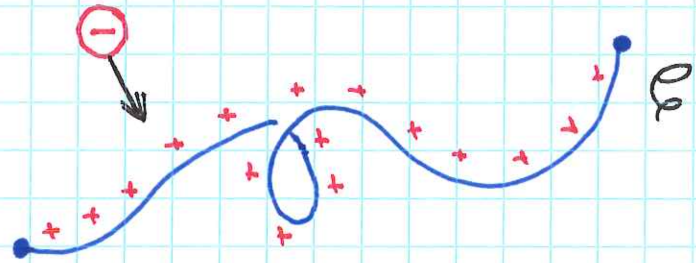
Lecture 14

1. Electric potentials and fields

- Suppose that an electric charge is distributed along a curve with charge density



- This iduces an electric field in the surrounding area, i.e. the force felt by charged particles nearby.

- Coulomb's law tells us this field is conservative and given by

$$\underline{E} = -\nabla V$$

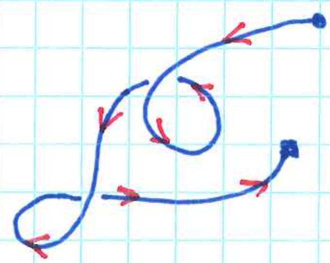
where V is the potential given by

$$V(P) = k \int_{\mathcal{C}} \frac{\delta(x\ y\ z)}{d_P(x\ y\ z)}\, ds$$

$d_P(x\ y\ z) =$ distance from P to $(x\ y\ z)$

2. Vector line integrals

- Suppose $\mathcal{C}$ is an oriented curve



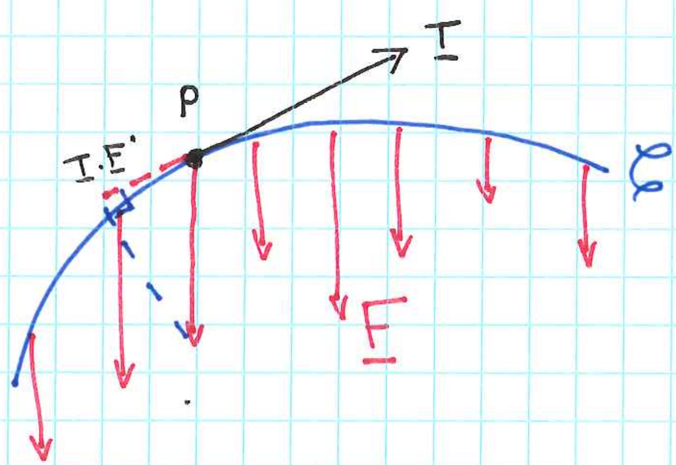
i.e. we have a given direction of travel.

- If the curve is moving through a vector field $\underline{F}$ then if $\underline{T}(P)$ is the unit tangent to $\mathcal{C}$ at the point $P \in \mathcal{C}$ then

$$(\underline{F} \cdot \underline{T})(P)$$

is a scalar function defined on P

- $\underline{F} \cdot \underline{T}$ is the component of $\underline{F}$ in the direction of $\mathcal{C}$.



Def If $\underline{F}$ is a vector field, ~~&~~ $\mathcal{C}$ a piecewise smooth oriented curve then

$$\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} := \int_{\mathcal{C}} (\underline{F} \cdot \underline{T})\, ds$$

scalar line integral.

- We again use parametrisation to ~~compl~~ compute these integrals.

- If $\underline{r} : [a, b] \longrightarrow \mathbb{R}^3$ is a parametrisation of $\mathcal{C}$ then

$$\frac{\underline{r}'(t)}{|\underline{r}'(t)|}$$

is equal to the unit tangent vector only if letting t increase for a to b traces out $\mathcal{C}$ in its given orientation

Ex Let $\mathcal{C}$ be the parabola $y = x^2$ in $\mathbb{R}^2$, ~~with orien~~ between $x = 0$ and $x = 1$, with orientation from 0 to 1.
Let $\underline{F} = \langle e^x + y, x \rangle$. Compute $\int_{\mathcal{C}} \underline{F} \cdot d\underline{r}$

First we find a parametrisation

$$\underline{r}(t) = \langle t, t^2 \rangle \qquad t \in [0,1]$$

Thus $\underline{r}'(t) = \langle 1, 2t \rangle$ so $\underline{T} = \dfrac{\underline{r}'(t)}{|\underline{r}'(t)|}$

$$\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = \int_{\mathcal{C}} \left(\underline{F} \cdot \frac{\underline{r}'(t)}{|\underline{r}'(t)|} \right) ds$$

$$= \int_0^1 \left(\underline{F} \cdot \frac{\underline{r}'(t)}{\cancel{|\underline{r}'(t)|}} \right) \cancel{|\underline{r}'(t)|}\, dt$$

$$= \int_0^1 \left(\underline{F} \cdot \underline{r}'(t) \right) dt$$

$$= \int_0^1 \langle e^x + y, x \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int_0^1 \langle e^t + t^2, t \rangle \cdot \langle 1, 2t \rangle \, dt$$

$$= \int_0^1 e^t + 3t^2 \, dt$$

$$= \left[e^t + t^3 \right]_0^1 = e$$

3. Work

- If a particle is moving along a path, in the presence of a force (eg gravity/electromagnetic/etc) Work is the amount of energy needed to overcome the force.
- Work can be negative! E.g. a falling object does not need any ~~a~~ energy at all, it is being pushed along by the ~~curve~~ force. The force is doing work.
- We are only interested in the force that is acting tangentially to the movement!

Def If $\underline{F}$ is a vector field describing a force, and a particle moves along an oriented curve $\mathcal{C}$, the work performed by $\underline{F}$

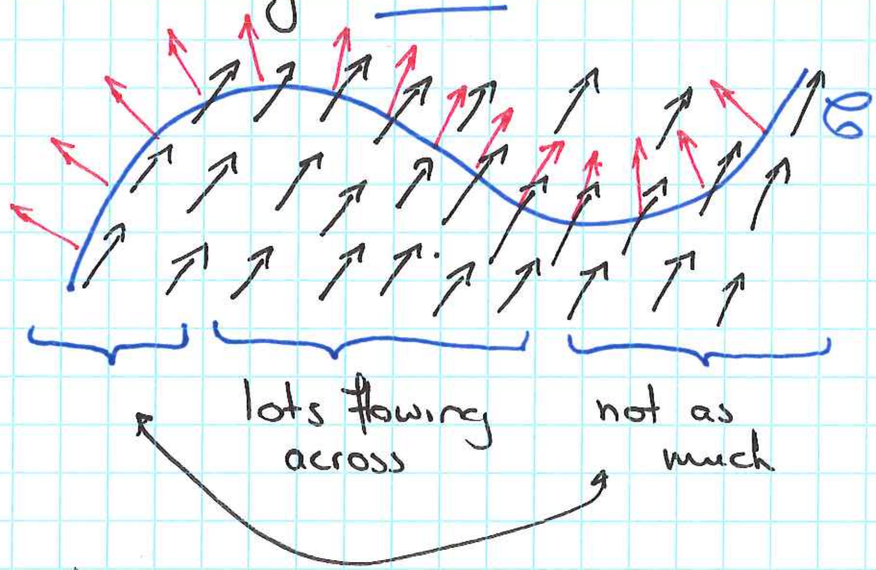
$$W = \int_{\mathcal{C}} \underline{F} \cdot d\underline{r}$$

Or the work performed by the particle against the force

$$= - \int_{\mathcal{C}} \underline{F} \cdot d\underline{r}$$

4. Flux across a curve

- How much of the vector field is flowing across the curve? This ~~st~~ only makes sense in $\mathbb{R}^2$!
- How much water is flowing over a boundary per second/hour/etc.
- Measured by **Flux**



- We integrate the normal component of the vector field.

- Let ~~N~~ $\underline{n}$ be the unit normal to $\mathcal{C}$.

Def

$$\text{Flux} = \int_{\mathcal{C}} (\underline{F} \cdot \underline{n})\, ds$$

- If $\underline{r}(t)$ is a parametrisation then

$$\underline{r}(t) = \langle x(t), y(t) \rangle$$

then

$$N(t) = \langle -y'(t), x'(t) \rangle$$

is a normal vector, so

$$\underline{n}(t) = \underline{N}(t) / |\underline{N}(t)| = \underline{N}(t) / |\underline{r}'(t)|$$

so

$$\text{Flux} = \int_a^b F(\underline{r}(t)) \cdot \underline{N}(t) \, ds$$