Math 3B: Lecture 15

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February 13, 2019

Integrating $\frac{1}{x}$.

Recall that

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$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

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Using substitution this gives the formula

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + C.$$

Integrating $\frac{1}{x^k}$.

Recall that if $k > 1$

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Integrating rational functions $p(x)/q(x)$

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1. Express $\frac{p(x)}{q(x)}$ in the form

$$d(x) + \frac{r(x)}{q(x)}$$

   using polynomial long division.

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$$\frac{A}{(ax+b)^k}$$
using partial fractions

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3. Integrate all these pieces seperately.

Example 1

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So

$$I = \frac{1}{3}x^2 - 2x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

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$$\frac{x^3 - 2x^2 + 4x}{(x-1)^3} = 1 + \frac{x^2 + x + 1}{(x-1)^3} = 1 + \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3}$$

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So

$$I = x + \ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + C.$$

Differential equations (motivation)

An (ordinary) differential equation (or ODE) is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

$$x^2 y'' + xy' + x^2 y = 0$$

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The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

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$$y'' + y' = x$$

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And so on.

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Note

The right hand side of the equation does not have any y's.

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- draw solutions for many other ODEs
- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)
- understand how sensitive ODEs are to their parameters.

Initial value problems

- If we try to solve the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2 - \sin t$$

we get (by integrating)

$$y(t) = t^3 + \cos t + C.$$

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- Then we see that $y(0) = 1 + C$, so $C = 1$.

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- If we want to draw the graph of $y(t)$ then we look at $g(0, 1)$.
- If this is positive we go up, negative we go down!