

This week on the problem set you will get practice thinking about potential functions and calculating line integrals.

Homework: The second homework will be due on Monday 11 May and will consist of 3, 4, 5, and 6 below.

*Numbers in parentheses indicate the question has been taken from the textbook:

J. Rogawski, C. Adams, *Calculus, Multivariable*, 3rd Ed., W. H. Freeman & Company,

and refer to the section and question number in the textbook.

- (Section 17.1) Questions 13 – 17, 22, 26, 28, 29, 38, 42, 44, 47, 52, 56*. (Use the following translations 4th \mapsto 3rd editions: 47 \mapsto 45, 52 \mapsto 50, 56 \mapsto 54, otherwise the questions are the same).
- (Section 17.2) 3, 10, 12, 13, 21, 24, 28, 43, 44, 46, 47, 54, 55, 57, 63, 64, 67. (Use the following translations 4th \mapsto 3rd editions: 43 \mapsto 41, 44 \mapsto 42, 46 \mapsto 44, 47 \mapsto 45, 54 \mapsto 52, 55 \mapsto 53, 57 \mapsto 55, otherwise the questions are the same).
- A parameterized curve $\mathbf{r} : [a, b] \rightarrow \mathbb{R}^2$ (the codomain could be \mathbb{R}^3 as well) is a *flow line* for the vector field \mathbf{F} if for all $t \in (a, b)$ we have that $\mathbf{F}(\mathbf{r}(t)) = \mathbf{r}'(t)$. A flow line for a vector field is the path that a particle would follow if the vector field was a velocity vector field for a fluid.
 - Consider the vector field $\mathbf{F}(x, y) = \langle x, y \rangle$.
 - Find a flow line for \mathbf{F} (note that a vector field will have many different flow lines).
 - Find a collection of flow lines for \mathbf{F} so that every point (x, y) is contained in exactly one of the flow lines in the collection.
 - Consider the vector field $\mathbf{G}(x, y) = \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right\rangle$. Find a collection of flow lines for \mathbf{F} so that every point (x, y) is contained in exactly one of the flow lines in the collection.
 - What do the flow lines look like in \mathbb{R}^2 for the vector fields \mathbf{F} and \mathbf{G} ? Relate how the flow lines are similar and different to how the vector fields \mathbf{F} and \mathbf{G} are similar and different.
 - A particle is dropped into the plane at the point $(-1, 1)$ at time $t = 0$. If the particle is located at (x, y) in the plane its velocity vector is $(1, 2x)$. What is the position of the particle at time $t = 3$?
- Consider the vector field $\mathbf{F} = \left\langle \frac{1-y}{x^2 + (y-1)^2} + \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + (y-1)^2} + \frac{x}{x^2 + y^2} \right\rangle$
 - Show that the curl of \mathbf{F} is zero.
 - Show that \mathbf{F} is not conservative on the largest domain on which it is defined.
 - Show that \mathbf{F} is conservative on the right half plane and find a potential function.

Hint: For all of these, it will be useful to think of \mathbf{F} as the sum of two more familiar vector fields.

- A vector field $\mathbf{F} = \langle F_1, F_2 \rangle$ is called *holomorphic* if it satisfies the *Cauchy-Riemann* equations:

$$\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y} \quad \text{and} \quad \frac{\partial F_2}{\partial x} = -\frac{\partial F_1}{\partial y}.$$

For example, the vector field $\mathbf{G} = \langle x + p, y + q \rangle$ is holomorphic for any fixed $p, q \in \mathbb{R}$, but $\langle x^2, F_2 \rangle$ is not holomorphic, no matter what F_2 is! The purpose of this question is to convince you that holomorphic vector fields satisfy some spooky properties. (If you want to know more about this, take Math 132).

- Give an example of a holomorphic vector field $\mathbf{H} = \langle H_1, H_2 \rangle$ (other than simple multiples or additions of the above examples). This is in general quite difficult, so to get to started, look for a vector field where

$$H_1 = e^x \cos y + ax^2 + y^2$$

You will need to work out an appropriate value of a . Include verification that your example is in fact holomorphic.

- (b) Let C be the unit circle centred at $(a, b) \in \mathbb{R}^2$ oriented counter clockwise. An amazing fact is that if \mathbf{F} is holomorphic then

$$F_1(a, b) = \frac{1}{2\pi} \oint_C F_1 \, ds \quad \text{and} \quad F_2(a, b) = \frac{1}{2\pi} \oint_C F_2 \, ds$$

(there are similar formulas for $F_1(a, b)$ and $F_2(a, b)$). Notice the right hand side only involves values of \mathbf{F} on C , and yet, somehow this knows about the value of \mathbf{F} inside C ! Verify that the formulas hold for the example \mathbf{G} and your example \mathbf{H} from part (a).

- (c) Maybe you aren't yet convinced that there is something strange going on with holomorphic vector fields. Well it turns out the values of \mathbf{F} on a circle know even a little more, namely

$$\frac{\partial F_1}{\partial x}(a, b) = \frac{1}{2\pi} \oint_C (x-a)F_1 + (y-b)F_2 \, ds \quad \text{and} \quad \frac{\partial F_1}{\partial x}(a, b) = \frac{1}{2\pi} \oint_C (x-a)F_2 - (y-b)F_1 \, ds$$

Verify these formulas as well. Note the above aren't as symmetrical as the previous formulas.

- (d) What are the analogous formulas for the y -derivatives of F_1 and F_2 . In general, we will have formulas for all x and y derivatives but they become very complicated, the language of complex analysis allows us to understand this phenomenon in a clearer way.
6. Let \mathcal{C} be the portion of the curve defined by the intersection of the surfaces $y = x^2$ and $x = y + z$ where $z \geq 0$. Take the orientation to be away from the origin. Let

$$\mathbf{F} = \left\langle yz + \frac{1}{\sqrt{1-x^3}}, xz - \frac{1}{\sqrt{1-y^3}}, xy \right\rangle$$

- (a) What is $\text{curl}(\mathbf{F})$?
- (b) Parameterise the curve and write out the integral of \mathbf{F} as a single integral. There is no need to evaluate this integral.
- (c) Calculate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. *Hint: the integral in the previous question is very difficult so you should find another way to compute this.*

*The questions marked with an asterisk are more difficult or are of a form that would not appear on an exam. Nonetheless they are worth thinking about as they often test understanding at a deeper conceptual level.