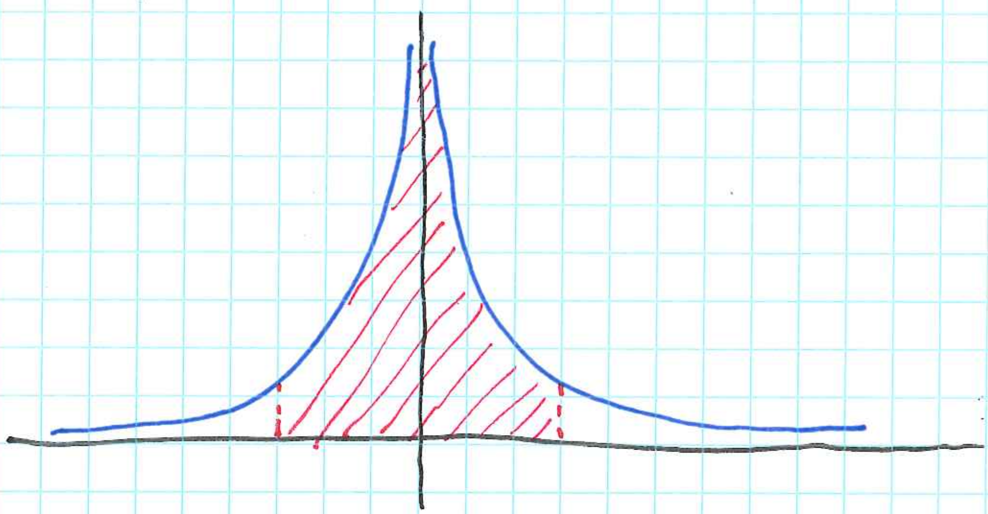
Improper integrals (discontinuities)

Lets try and evaluate the integral

$$\int_{-1}^{1} \frac{1}{x^2}\, dx$$

$$= -\frac{1}{x}\Big|_{-1}^{1} = -\frac{1}{1} - \left(-\frac{1}{-1}\right) = -2$$



But the area is positive! What went wrong? There is a discontinuity a $x = 0$, right in the middle of $[-1, 1]$.

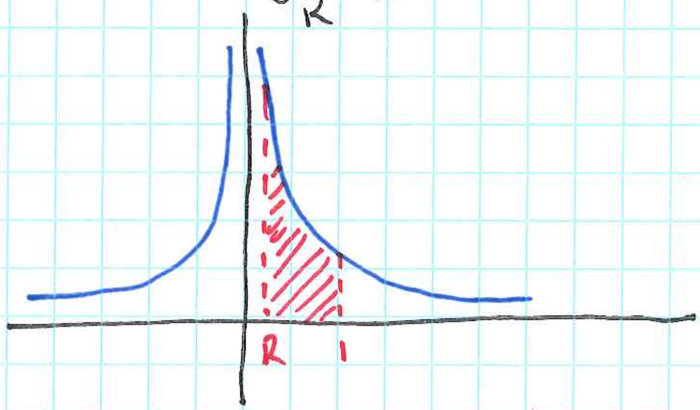
So how do we evaluate this integral. The first step is to write it as

$$\int_{-1}^{1} \frac{1}{x^2}\, dx = \int_{-1}^{0} \frac{1}{x^2}\, dx + \int_{0}^{1} \frac{1}{x^2}\, dx$$

So we no longer have discontinuities in the domain. How can we evaluate

$$\int_0^1 \frac{1}{x^2}\,dx \ ?$$

If $0<R<1$ then $\int_R^1 \frac{1}{x^2}\,dx$ is an approximation.



The smaller we make R, the better the approx

so:

$$\int_0^1 \frac{1}{x^2}\,dx := \lim_{R\to 0^+} \int_R^1 \frac{1}{x^2}\,dx$$

Thus

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{R\to 0^+} -\frac{1}{x}\Big|_R^1$$

$$= \lim_{R\to 0^+} -1 + \frac{1}{R}$$

$$= \infty$$

Similarly

$$\int_{-1}^0 \frac{1}{x^2}\,dx = \lim_{R\to 0^-} \int_{-1}^R \frac{1}{x^2}\,dx$$

$$= \lim_{R \to 0^-} -\frac{1}{x}\Big|_{-1}^{R}$$

$$= \lim_{R \to 0^-} -\frac{1}{R} - 1$$

$$= \infty$$

So $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,dx = \int_{-1}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$

$$\text{"} = \infty + \infty \text{"}$$

So the area is infinite!

We say "the integral does not converge"

Def If $f(x)$ is continuous on $[a, b)$ and has a discontinuity at $x = b$ then

$$\int_a^b f(x)\,dx := \lim_{R \to b^-} \int_a^R f(x)\,dx$$

Similarly if f cts on $(a, b]$, discts at a

$$\int_a^b f(x)\,dx := \lim_{R \to a^+} \int_R^b f(x)\,dx$$

and if f cts on $[a, b]$ and discts at $x = c \in (a, b)$ then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Rmk * These integrals are called "improper integrals"

* If the integral equals a finite number we say it "converges"
* Otherwise it "does not converge"

Ex $$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{R\to 0^+} \int_R^1 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{R\to 0^+} 2\sqrt{x}\Big|_R^1$$

$$= \lim_{R\to 0^+} 2 - 2\sqrt{R} = 2$$

Ex $$\int_0^1 \frac{1}{x}\,dx = \lim_{R\to 0^+} \int_R^1 \frac{1}{x}\,dx$$

$$= \lim_{R\to 0^+} \ln x\Big|_R^1$$

$$= \lim_{R\to 0^+} 0 - \ln R$$

$$= \infty$$

Ex

$$\int_0^e \ln x \, dx = \lim_{R \to 0^+} \int_R^e \ln x \, dx$$

$$= \lim_{R \to 0^+} x \ln x \Big|_R^e - \int_R^e dx$$

$$= \lim_{R \to 0^+} e - R \ln R - e + R$$

$$= \lim_{R \to 0^+} R(1 - \ln R)$$

$$= \lim_{R \to 0^+} \frac{1 - \ln R}{R^{-1}}$$

$$= \lim_{R \to 0^+} \frac{-1/R}{-1/R^2}$$

$$= \lim_{R \to 0^+} R = 0.$$