Midterm 1 practice 3

UCLA: Math 115A, Spring 2020

Instructor: Noah White
Date:

- This exam has 4 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section (please circle):

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
Total:	20	

Question 1 is multiple choice. Indicate your answers in the table below. *The following three pages will not be graded, your answers must be indicated here.*

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

☐ I wish to opt out of having my exam graded using Gradescope.

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (1 point) If V is a vector space over the field $\mathbb{C}$ and $v \in V$ then

   $$(2-3)\cdot(v+w)+(5-3)\cdot v+w$$

   equals

   A. 1
   B. v
   C. 0
   D. $2v$

   The following two questions concern the subsets

   $$A = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \;\middle|\; \lambda a^2 + 2(c - a^2) = 0 \right\} \subseteq \mathbb{R}^3$$

   $$B = \{\, p \in \mathbb{R}[x] \mid p' = \lambda \,\} \subseteq \mathbb{R}[x]$$

   for some $\lambda \in \mathbb{R}$.

   (b) (1 point) Which of the following is a true statement.

   A. Both A and B are subspaces regardless of the value of $\lambda \in \mathbb{R}$.
   B. A is not a subspace for any λ and B is a subspace when $\lambda = 0$.
   C. Both are subspaces when $\lambda = 0$.
   D. A is a subspace when $\lambda = 2$.

   (c) (1 point) When $\lambda = 0$, the subspace B has dimension

   A. 1
   B. 2
   C. 3
   D. 4

(d) (1 point) Let $\mathbb{F}$ be one of the fields $\mathbb{Z}_p$ for $p = 2, 3, 5, 7$. Consider the vectors

$$\begin{pmatrix} [1] \\ [1] \end{pmatrix} \text{ and } \begin{pmatrix} [-1] \\ [5] \end{pmatrix}.$$

For which fields are the two vectors linearly dependent?

A. Only for $\mathbb{Z}_2$.
B. Only for $\mathbb{Z}_5$
C. For $\mathbb{Z}_2$ and $\mathbb{Z}_3$.
D. For $\mathbb{Z}_2$ and $\mathbb{Z}_5$.

(e) (1 point) Fix $\lambda \in \mathbb{R}$. Consider the subspaces

$$U = \left\{ \begin{pmatrix} -a \\ \lambda a \end{pmatrix} \middle| \; a \in \mathbb{R} \right\} \subset \mathbb{R}^2$$

$$W = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \middle| \; a + b = 0 \right\} \subset \mathbb{R}^2$$

When is $U \oplus W = \mathbb{R}^2$? (i.e. when is $\mathbb{R}^2$ a direct sum of these two subspaces).

A. For any $\lambda \in \mathbb{R}$.
B. Never.
C. When $\lambda \neq 1$.
D. When $\lambda \neq 0$.

2. Give (simple) examples of all of the following situations.

(a) (2 points) An infinite dimensional vector space V over $\mathbb{C}$ and an infinite dimensional subspace W such that $W \neq V$.

> **Solution:** $V = \mathbb{C}[x]$ and $W = \mathrm{span}\{x^{2n} \mid n \geq 0\}$

(b) (2 points) A linearly dependant subset $\{v_1, v_2, v_3\} \subseteq V$ consisting of 3 elements such that no element is a scalar multiple of another (i.e. $v_i \neq \lambda v_j$ for any $\lambda \in \mathbb{C}$ and $i \neq j$).

> **Solution:** $v_1 = 1, v_2 = x, v_3 = 1 + x$

(c) (1 point) A basis for W.

> **Solution:** $\{x^{2n} \mid n \geq 0\}$.

3. (5 points) Let $\mathbb{C}_3[x]$ be the vector space consisting of polynomials of degree less than 3 (i.e constant, linear and quadratic polynomials only). Let $S = \{1 + x, x - x^2, 1 + x + x^2\} \subset \mathbb{C}_3[x]$. Prove or disprove that S is a basis of $\mathbb{C}_3[x]$.

Solution: To show S is a basis we need to show it is linearly independent and that it spans $\mathbb{C}_3[x]$. To see that it is linearly independent, consider a linear combination

$$\lambda_1(1 + x) + \lambda_2(x - x^2) + \lambda_3(1 + x + x^2) = 0,$$

for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$. Expanding the LHS we get

$$(\lambda_1 + \lambda_3) + (\lambda_1 + \lambda_2 + \lambda_3)x + (\lambda_3 - \lambda_2)x^2 = 0.$$

From the constant term we see that $\lambda_3 = -\lambda_1$. From the coefficient of x^2 we see that $\lambda_2 = \lambda_3 = -\lambda_1$. Thus the coefficient of x gives the equation $-\lambda_1 = 0$. Thus we see that $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and so S must be linearly independent.

To see that S spans $\mathbb{C}_3[x]$ we could simply observe that $\mathbb{C}_3[x]$ is a 3 dimensional space, so any linearly independent set (such as S) with three elements must be a basis.

We can also show S spans $\mathbb{C}_3[x]$ directly. Suppose $p \in \mathbb{C}_3[x]$. Then $p = a + bx + cx^2$ for some $a, b, c \in \mathbb{C}$. In order for S to span we need to be able to find $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$ such that

$$\begin{aligned} a + bx + cx^2 &= \lambda_1(1 + x) + \lambda_2(x - x^2) + \lambda_3(1 + x + x^2) \\ &= (\lambda_1 + \lambda_3) + (\lambda_1 + \lambda_2 + \lambda_3)x + (\lambda_3 - \lambda_2)x^2. \end{aligned}$$

We get the equations

$$\begin{aligned} a &= \lambda_1 + \lambda_3 \\ b &= \lambda_1 + \lambda_2 + \lambda_3 \\ c &= \lambda_3 - \lambda_2. \end{aligned}$$

We could write these as a linear system and invert a matrix to solve, but its small enough to do manually. $\lambda_1 = a - \lambda_3$, and $\lambda_2 = \lambda_3 - c$. We also have

$$\begin{aligned} \lambda_3 = b - \lambda_1 - \lambda_2 & \\ &= b - (a - \lambda_3) - (\lambda_3 - c) \\ &= b - a + c. \end{aligned}$$

Thus $\lambda_1 = 2a - b - c$ and $\lambda_2 = b - a$. Since there is a solution S spans $\mathbb{C}_3[x]$.

4. Let V be a vector space over a field $\mathbb{F}$ and W be a subspace.

(a) (2 points) Consider the map $\pi : V \longrightarrow V/W$ given by $\pi(v) = v + W$. Show that π is a linear map.

Solution: First we check that $\pi(u+v) = \pi(u) + \pi(v)$ if $u, v \in V$:

$$\pi(u+v) = (u+v) + V = (u+V) + (v+V) = \pi(u) + \pi(v)$$

where the second equality follows by the definition of V/W. Now we show that $\pi(\lambda v) = \lambda\pi(v)$ for $\lambda \in \mathbb{F}$ and $v \in V$:

$$\pi(\lambda v) = (\lambda v) + V = \lambda(v + V) = \lambda\pi(v)$$

where, again, the second equality follows by the definition of V/W.

(b) (3 points) Let $V = \mathbb{R}^3$ and

$$W = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in V \;\middle|\; a-b=0, b-c=0, c-a=0 \right\}$$

Find the dimension of W and V/W.

Solution: Note that if $a-b=0, b-c=0, c-a=0$ this is the same as $b=a$, $c=b=a$. Thus

$$W = \left\{ \begin{pmatrix} a \\ a \\ a \end{pmatrix} \in V \;\middle|\; a \in \mathbb{R} \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

and so $\dim W = 1$.
Consider the set

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + W, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + W \right\} \subset V/W.$$

We claim this is a basis (and thus $\dim V/W = 2$). To see they are linearly independent note that if

$$\lambda_1 \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + W \right) + \lambda_2 \left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + W \right) = 0 = W$$

This would mean that

$$\begin{pmatrix} \lambda_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ 0 \end{pmatrix} \in W$$

but this is clearly not true unless $\lambda_1 = \lambda_2 = 0$.
To see that this set spans V/W, observe that if

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + W \in V/W$$

then

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} + W = \begin{pmatrix} a-c \\ b-c \\ 0 \end{pmatrix} + W$$

which is clearly in the span of S. Thus S is a basis.

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