Math 3B: Lecture 6

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October 10, 2018

Differential equations (motivation)

A differential equation is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

$$x^2 y'' + xy' + x^2 y = 0$$

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The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

Newton's second law (motivation)

The original differential equation!

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If $h(t)$ measures the height of an object (maybe an apple?) above the earth then

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The force due to gravity is roughly $-10m$ Newtons, so

$$-10m = mh''(t)$$

Population growth (motivation)

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If $P(t)$ is the population at time t:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(t)$$

Some more examples of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(1-y)$$

$$y'' = \sqrt{a^2 - (y')^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(A-y)^2$$

Antiderivatives

We will be concentrating on solving differential equations of the form

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A solution $y = F(x)$ is called an antiderivative of $f(x)$.

Example 1

Question

What is the antiderivative of $f(x) = 2x$?

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Solution

$$F(x) = x^2$$

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$$F(x) = x^2 + 4$$

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$$F(x) = x^2 + 8$$

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What is the antiderivative of $f(x) = 2x$?

Solution

$$F(x) = x^2 + C$$

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Solution

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Solution

$$F(x) = \frac{1}{4}x^4 + 2x^2 - x + C$$

Example 3

Question

What is the antiderivative of $f(x) = e^{2x}$?

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Solution

$$F(x) = \quad e^{2x}$$

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Solution

$$F(x) = \frac{1}{2}e^{2x}$$

Example 4

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What is the antiderivative of $f(x) = \frac{1}{x}$ (for $x > 0$)?

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Solution

$$F(x) = \ln x$$

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Solution

Note that $f(x) = (1 + x)^{-2}$. So

$$F(x) = \quad \frac{1}{1 + x}$$

Example 5

Question

What is the antiderivative of $f(x) = \frac{1}{(1+x)^2}$?

Solution

Note that $f(x) = (1+x)^{-2}$. So

$$F(x) = -\frac{1}{1+x}$$

Example 6

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What is the antiderivative of $f(x) = 2x\cos x^2$?

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What is the antiderivative of $f(x) = 2x \cos x^2$?

Solution

$$F(x) = \sin x^2$$

Example 7

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Solution

$$f(x) = x^{-\frac{1}{2}}$$

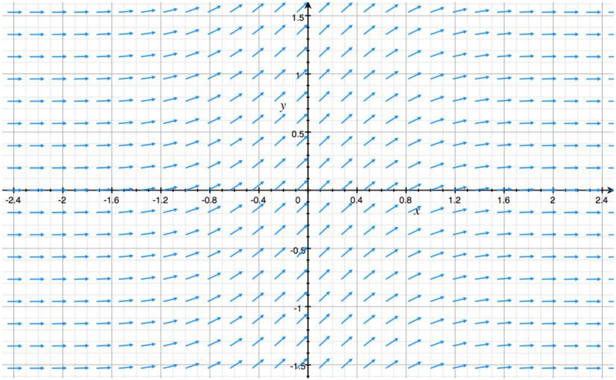
$$F(x) = 2x^{\frac{1}{2}}$$

Slope fields

In some cases it is impossible to find the antiderivative (without special functions). E.g.

$$f(x) = e^{-x^2}$$

But we can still (approximately) graph the antiderivative! First we draw the slope field



Slope fields (how to draw)

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1. Draw the xy-plane.

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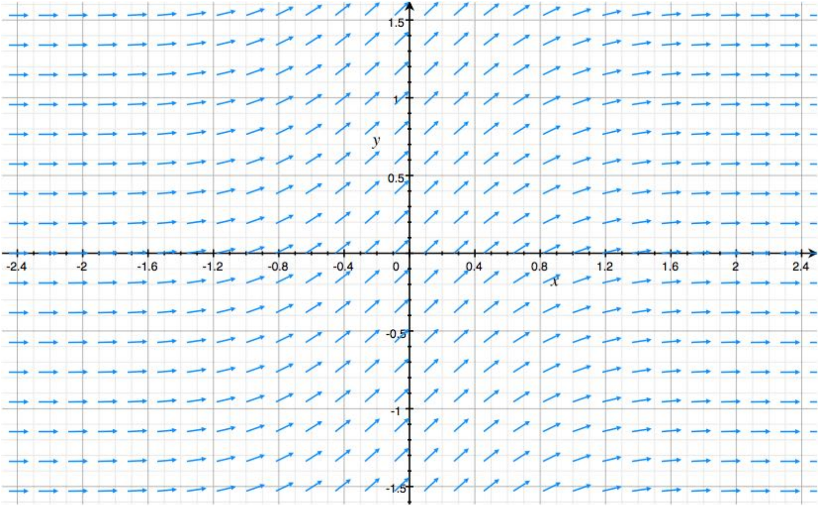
4. Draw a small arrow with slope $f(x)$ and the point (x, y)
5. Do this for a grid of points on the xy-plane.

Example 1

$$f(x) = e^{-x^2} \text{ with } F(0) = -0.25$$



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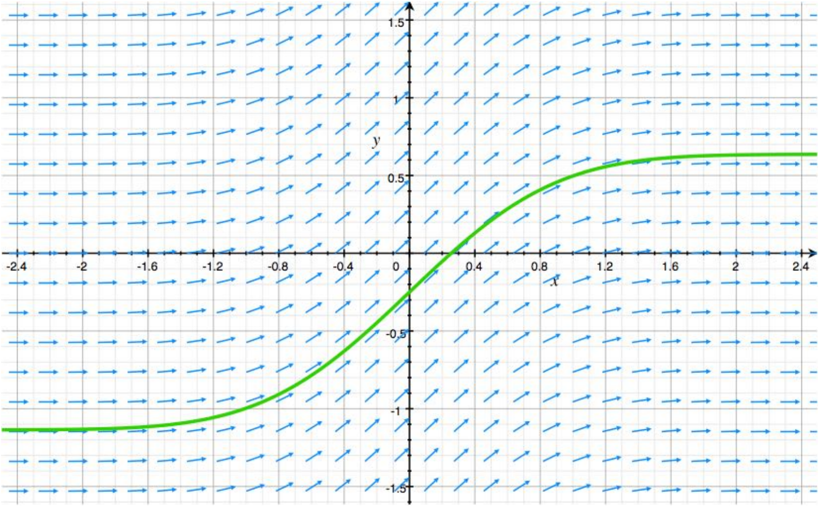
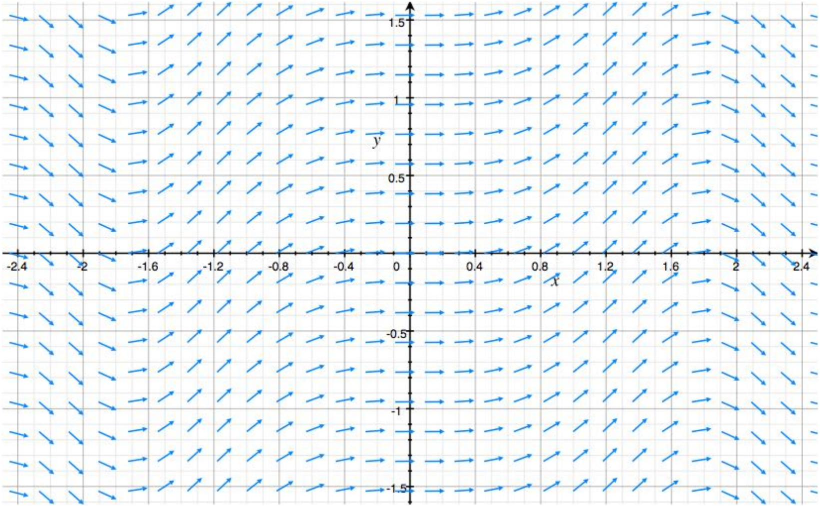


Example 2

$$f(x) = \sin(x^2) \text{ with } F(0) = 0.1$$



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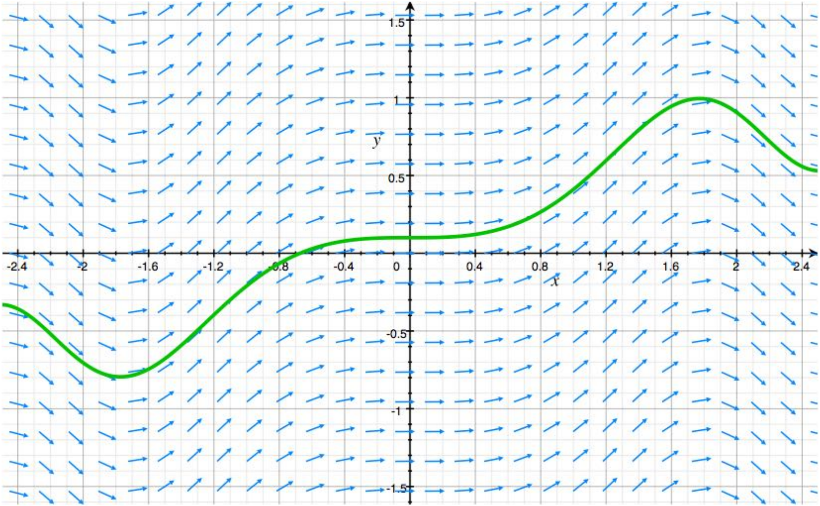

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These problems involve finding the area under some curve.

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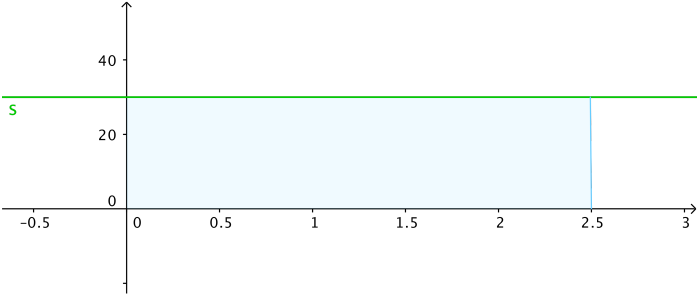
Solution

We model the car's speed using the function $s(t) = 30$. So we can see that the area under this curve



is the distance travelled (75 miles)

Example 2

If a car accellerates for 20 seconds at a rate of $2m/s^2$ and then decelerates for 30 seconds at a rate of $1m/s^2$, how far has it travelled?

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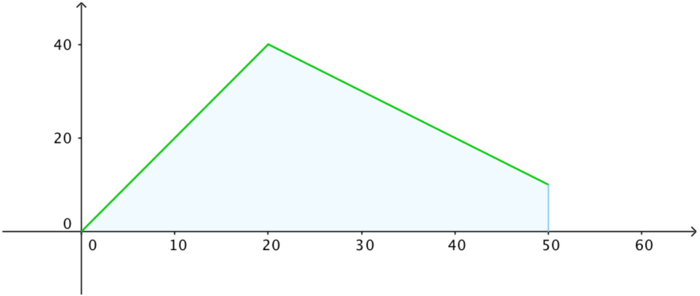
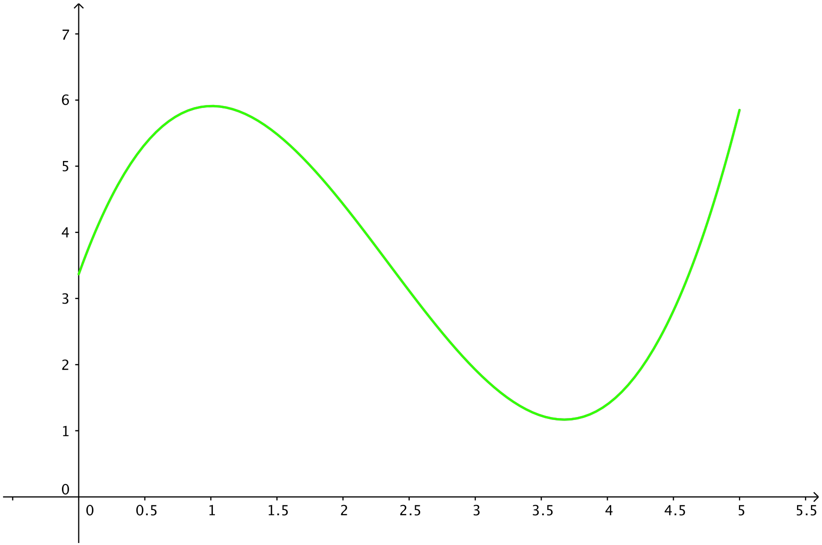
Solution

The car's speed is given by $s(t) = 2t$ when $0 \leq t \leq 20$ and $s(t) = 60 - t$ when $20 \leq t \leq 50$. So the graph looks like



More complicated rates of change

Suppose we have a car whose speed is descibed by the following curve. How far has it travelled in this time?



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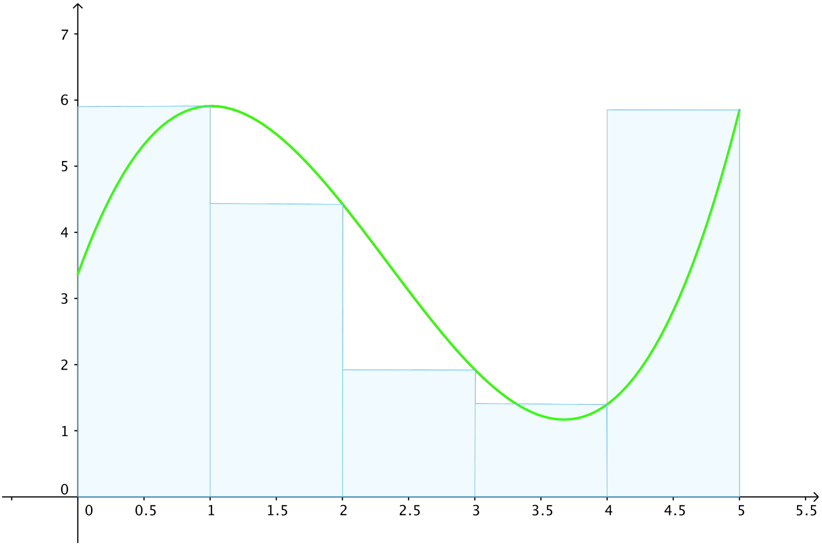

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- Suppose we know the rate of change $f(t)$, of some quantity (distance, water flow, population, etc).
- How do we find the total amount by which this changes between $t = a$ and $t = b$?
- Answer: area under $f(t)$ between a and b.

Areas under general curves

We would like to calculate the area between a function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

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(Too hard to draw, lets look at an animation)

The definite integral

Defintion

The definite integral of a function $f(x)$ is defined to be

$$\int_a^b f(x) = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(a + k\Delta x)$$

where $\Delta x = \frac{b-a}{n}$.

The fundamental theorem of calculus

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t) = f(x)$$

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That is, $F(x) = \int_a^x f(t)$ is an antiderivative of $f(x)$!

Note

The fundamental theorem of calculus

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Note

- $F(x) = \int_a^x f(t)$ is a function of x.

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Note

- $F(x) = \int_a^x f(t)$ is a function of x.
- every input x produces a number as an output.

A consequence (corrollary)

Corollary

For any antiderivative $F(x)$ of $f(x)$

$$\int_a^b f(x) = F(b) - F(a)$$

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For any antiderivative $F(x)$ of $f(x)$

$$\int_a^b f(x) = F(b) - F(a)$$

Why?

Well $F(x) = \int_a^x f(t) + C$ for some a and C. So

$$\begin{aligned} F(b) - F(a) &= \int_a^b f(t) + C - \int_a^a f(t) - C \\ &= \int_a^b f(t) \end{aligned}$$

Example 1

Question

Evaluate the definite integral

$$\int_0^1 x^2 - 4$$

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$$\int_0^1 x^2 - 4$$

Solution

An antiderivative of $x^2 - 4$ is $\frac{1}{3}x^3 - 4x$ so

$$\begin{aligned} \int_0^1 x^2 - 4 &= \frac{1}{3} \cdot 1^3 - 4 - \frac{1}{3} \cdot 0^3 + 4 \cdot 0 \\ &= \frac{1}{3} - 4 = -\frac{11}{3} \end{aligned}$$

Example 2

Question

Evaluate the definite integral

$$\int_0^{\pi} \sin x$$

Example 2

Question

Evaluate the definite integral

$$\int_0^\pi \sin x$$

Solution

An antiderivative of $\sin x$ is $-\cos x$ so

$$\begin{aligned} \int_0^\pi \sin x &= -\cos \pi + \cos 0 \\ &= -(-1) + 1 = 2 \end{aligned}$$