Math 3B: Lecture 18

Noah White

November 2, 2016

Last time

- Accumulated change problems

Last time

- Accumulated change problems
- Adding up a value if it is changing over time

Last time

- Accumulated change problems
- Adding up a value if it is changing over time
- Break problem into pieces

Last time

- Accumulated change problems
- Adding up a value if it is changing over time
- Break problem into pieces
- Add up pieces

Last time

- Accumulated change problems
- Adding up a value if it is changing over time
- Break problem into pieces
- Add up pieces
- Interpret as Riemann sum

Integration and applications

- Antiderivatives

Integration and applications

- Antiderivatives
- Simple accumulated change problems

Integration and applications

- Antiderivatives
- Simple accumulated change problems
- Definite integrals

Integration and applications

- Antiderivatives
- Simple accumulated change problems
- Definite integrals
- Fundamental theorem of calculus

Integration and applications

- Antiderivatives
- Simple accumulated change problems
- Definite integrals
- Fundamental theorem of calculus
- Substitution

Integration and applications

- Antiderivatives
- Simple accumulated change problems
- Definite integrals
- Fundamental theorem of calculus
- Substitution
- Integration by parts

Integration and applications

- Antiderivatives
- Simple accumulated change problems
- Definite integrals
- Fundamental theorem of calculus
- Substitution
- Integration by parts
- Partial fractions and long division

Integration and applications

- Antiderivatives
- Simple accumulated change problems
- Definite integrals
- Fundamental theorem of calculus
- Substitution
- Integration by parts
- Partial fractions and long division
- Accumulated change and Riemann sums

Differential equations (motivation)

An (ordinary) differential equation (or ODE) is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

$$x^2 y'' + xy' + x^2 y = 0$$

Differential equations (motivation)

An (ordinary) differential equation (or ODE) is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

$$x^2 y'' + xy' + x^2 y = 0$$

The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

First, second, . . . order

If a differential equation involves only first derivatives it is a first order differential equation. E.g.

First, second, . . . order

If a differential equation involves only first derivatives it is a first order differential equation. E.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$$

First, second, ... order

If a differential equation involves only first derivatives it is a first order differential equation. E.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$$

If it involves only first and second derivatives it is a second order differential equation. E.g.

$$y'' + y' = x$$

First, second, . . . order

If a differential equation involves only first derivatives it is a first order differential equation. E.g.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y$$

If it involves only first and second derivatives it is a second order differential equation. E.g.

$$y'' + y' = x$$

And so on.

Integration

We already know how to solve ODEs of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

Integration

We already know how to solve ODEs of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

We just need to find the antiderivative of $f(x)$.

Integration

We already know how to solve ODEs of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

We just need to find the antiderivative of $f(x)$.

Note

The right hand side of the equation does not have any y's.

More types of differential equations

In this course we focus on first order ODEs. You will be able to solve equations like

- $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y + 5$

More types of differential equations

In this course we focus on first order ODEs. You will be able to solve equations like

- $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y + 5$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$

More types of differential equations

In this course we focus on first order ODEs. You will be able to solve equations like

- $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y + 5$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$
- $y' = x^{-5} \sin y$

More types of differential equations

In this course we focus on first order ODEs. You will be able to solve equations like

- $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y + 5$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$
- $y' = x^{-5} \sin y$

More types of differential equations

In this course we focus on first order ODEs. You will be able to solve equations like

- $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y + 5$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$
- $y' = x^{-5} \sin y$

And you'll be able to

- draw solutions for many other ODEs

More types of differential equations

In this course we focus on first order ODEs. You will be able to solve equations like

- $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y + 5$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$
- $y' = x^{-5} \sin y$

And you'll be able to

- draw solutions for many other ODEs
- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)

More types of differential equations

In this course we focus on first order ODEs. You will be able to solve equations like

- $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y + 5$
- $\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y)$
- $y' = x^{-5} \sin y$

And you'll be able to

- draw solutions for many other ODEs
- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)
- understand how sensitive ODEs are to their parameters.

Modelling using differential equations

Question

Why are differential equations important

Modelling using differential equations

Question

Why are differential equations important

Answer

They descibe nature!

Modelling using differential equations

Question

Why are differential equations important

Answer

They descibe nature!

Modelling using differential equations

Question

Why are differential equations important

Answer

They descibe nature!

Today we will see an number of real world situations

- The goal is to write down a function $y(t)$ that describes something we are interested in (e.g. population/mass/etc)

Modelling using differential equations

Question

Why are differential equations important

Answer

They descibe nature!

Today we will see an number of real world situations

- The goal is to write down a function $y(t)$ that describes something we are interested in (e.g. population/mass/etc)
- as some other variable changes (usually time)

Modelling using differential equations

Question

Why are differential equations important

Answer

They descibe nature!

Today we will see an number of real world situations

- The goal is to write down a function $y(t)$ that describes something we are interested in (e.g. population/mass/etc)
- as some other variable changes (usually time)
- We can't do this directly, but we can write down an ODE that y satisfies instead.

Example 1

The senario

Want to find a function $N(t)$, descibing the sie of a population at time t (in years). Assume

Example 1

The senario

Want to find a function $N(t)$, descibing the sie of a population at time t (in years). Assume

- The population is 100 people when $t = 0$

Example 1

The senario

Want to find a function $N(t)$, descibing the sie of a population at time t (in years). Assume

- The population is 100 people when $t = 0$
- No immigration or emmigration

Example 1

The senario

Want to find a function $N(t)$, descibing the sie of a population at time t (in years). Assume

- The population is 100 people when $t = 0$
- No immigration or emmigration
- Number of births is proportional to the total number of people. So

$$bN(t) \text{ births per year, for some } b$$

Example 1

The senario

Want to find a function $N(t)$, descibing the sie of a population at time t (in years). Assume

- The population is 100 people when $t = 0$
- No immigration or emmigration
- Number of births is proportional to the total number of people. So

  $$bN(t) \text{ births per year, for some } b$$

- Number of deaths is proportional to the total number of people. So

  $$dN(t) \text{ deaths per year, for some } d$$

Example 1

The total change in population at time t is

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= \text{ births at } t - \text{ deaths at } t \\ &= bN(t) - dN(t) \\ &= (b-d)N(t).\end{aligned}$$

Example 1

The total change in population at time t is

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= \text{ births at } t - \text{ deaths at } t \\ &= bN(t) - dN(t) \\ &= (b-d)N(t).\end{aligned}$$

In real life we would determine b and d experimentally.

Example 1

The total change in population at time t is

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= \text{ births at } t - \text{ deaths at } t\\ &= bN(t) - dN(t)\\ &= (b-d)N(t).\end{aligned}$$

In real life we would determine b and d experimentally. Let $r = b - d$. the instinsic growth rate. So our model is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

and we know $N(0) = 100$.

Behaviour of solutions

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

Case 1: $r = 0$

The population never grows or shrinks, it always stays the same (so $N(t) = 100$ for all t).

Behaviour of solutions

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

Case 1: $r = 0$
The population never grows or shrinks, it always stays the same (so $N(t) = 100$ for all t).

Case 2: $r > 0$
The population is increasing indefinitely.

Behaviour of solutions

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

Case 1: $r = 0$

The population never grows or shrinks, it always stays the same (so $N(t) = 100$ for all t).

Case 2: $r > 0$

The population is increasing indefinitely.

Case 3: $r < 0$

The population is decreasing indefinitely.

Solution to a simple ODE

Theorem

For any constant a, if y is a solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay$$

then y is given by

$$y = Ce^{ax}$$

for some constant C.

Solution to a simple ODE

Theorem

For any constant a, if y is a solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay$$

then y is given by

$$y = Ce^{ax}$$

for some constant C.

Next time

We will see why, but for now we can verify it is actually a solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}Ce^{ax} = C\frac{\mathrm{d}}{\mathrm{d}x}e^{a}x = Cae^{ax} = ay.$$

Back to example 1

We know our population model was governed by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b - d)N.$$

Back to example 1

We know our population model was governed by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b - d)N.$$

So the general solution is

$$N(t) = Ce^{(b-d)t}$$

for some constant C.

Back to example 1

We know our population model was governed by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b-d)N.$$

So the general solution is

$$N(t) = Ce^{(b-d)t}$$

for some constant C. To detirmine C, we need one extra piece of information, $N(0) = 100$.

Back to example 1

We know our population model was governed by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (b - d)N.$$

So the general solution is

$$N(t) = Ce^{(b-d)t}$$

for some constant C. To detirmine C, we need one extra piece of information, $N(0) = 100$.

$$100 = Ce^{(b-d)} \quad \text{so} \quad C = 100e^{(d-b)}.$$

Logistic growth

The previous example is a good first approximation but it is not very realistic in the long term. Usually there are constrains, e.g amount of space, food, etc.

Logistic growth

The previous example is a good first approximation but it is not very realistic in the long term. Usually there are constrains, e.g amount of space, food, etc.

- When the popluation becomes large, the death rate should increase!

Logistic growth

The previous example is a good first approximation but it is not very realistic in the long term. Usually there are constrains, e.g amount of space, food, etc.

- When the popluation becomes large, the death rate should increase!
- Lets assume this is linear, i.e. number of deaths at time t

$$(d \qquad\qquad\quad) N(t).$$

Logistic growth

The previous example is a good first approximation but it is not very realistic in the long term. Usually there are constrains, e.g amount of space, food, etc.

- When the popluation becomes large, the death rate should increase!
- Lets assume this is linear, i.e. number of deaths at time t

$$(d + kN(t))\, N(t).$$

- This means

Logistic growth

The previous example is a good first approximation but it is not very realistic in the long term. Usually there are constrains, e.g amount of space, food, etc.

- When the popluation becomes large, the death rate should increase!
- Lets assume this is linear, i.e. number of deaths at time t

$$(d + kN(t))\,N(t).$$

- This means

Logistic growth

The previous example is a good first approximation but it is not very realistic in the long term. Usually there are constrains, e.g amount of space, food, etc.

- When the popluation becomes large, the death rate should increase!
- Lets assume this is linear, i.e. number of deaths at time t

$$(d + kN(t))\, N(t).$$

- This means

$$\frac{\mathrm{d}N}{\mathrm{d}t} = bN - (d + kN)N$$

Where $K = r/k$.

Logistic growth

The previous example is a good first approximation but it is not very realistic in the long term. Usually there are constrains, e.g amount of space, food, etc.

- When the popluation becomes large, the death rate should increase!
- Lets assume this is linear, i.e. number of deaths at time t

$$(d + kN(t))\,N(t).$$

- This means

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= bN - (d + kN)N \\ &= (b - d - kN)N = (r - kN)N\end{aligned}$$

Where $K = r/k$.

Logistic growth

The previous example is a good first approximation but it is not very realistic in the long term. Usually there are constrains, e.g amount of space, food, etc.

- When the popluation becomes large, the death rate should increase!
- Lets assume this is linear, i.e. number of deaths at time t

$$(d + kN(t))\,N(t).$$

- This means

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= bN - (d + kN)N \\ &= (b - d - kN)N = (r - kN)N \\ &= r\left(1 - \frac{kN}{r}\right)N = r\left(1 - \frac{N}{K}\right)N\end{aligned}$$

Where $K = r/k$.

Logistic growth

The equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

is called the Logistic equation and K is the carrying capacity.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 1. $N(0) = 0$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains 0.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 1. $N(0) = 0$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains 0.

Case 2. $N(0) = K$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains K.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 1. $N(0) = 0$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains 0.

Case 2. $N(0) = K$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains K.

Key takeaway

Both $N(t) = 0$ and $N(t) = K$ are solutions to the ODE. They are called equalibrium solutions.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 3. $0 \leq N(0) \leq K$

In this case, N is initially increasing and so becomes more positive, slowing down as it gets close to K.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 3. $0 \leq N(0) \leq K$

In this case, N is initially increasing and so becomes more positive, slowing down as it gets close to K.

Case 4. $N(0) \geq K$

In this case N is initially decreasing but decreases slower and slower as it gets close to K.