This week you will practice writing differential equations modelling real world phenomena as well as understanding population models. You will also get practice solving separable differential equations.

Homework: The homework will be due on Friday 9 November, at 8am, the *start* of the lecture. It will consist of questions:

3 and 7.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.1) Write a differential equation to model the situations described below. Do not try to solve.
   (a) (6.1-1) The number of bacteria in a culture grows at a rate that is proportional to the number of bacteria present.

   Solution: Lets say that the rate is a times the number of bacteria (i.e. this is the constant of proportionality). Let $P(t)$ be the number of bacteria at time t. Then the rate at which the number grows is $\frac{\mathrm{d}P}{\mathrm{d}t}$. So the fact that the rate is a times the number of bacteria is just expressed as
   $$\frac{\mathrm{d}P}{\mathrm{d}t} = aP$$
   .

   (b) (6.1-2) A sample of radium decays at a rate that is proportional to the amount of radium present in the sample.

   Solution: This is similar to the first example. However the sample is decaying. Let $N(t)$ be the amount of radium present, and λ the proportion. Since it is decaying, the rate should be negative so
   $$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$

   (c) (6.1-5) According to Benjamin Gompertz (1779-1865) the growth rate of a population is proportional to the number of individuals present, where the factor of proportionality is an exponentially decreasing function of time.

   Solution: This is similar to the first example. However now, the proportionality a changes over time. In particular a is exponenitally decreasing. Thus $a = Ae^{-kt}$ for some A and some k. Thus
   $$\frac{\mathrm{d}P}{\mathrm{d}t} = aP = Ae^{-kt}P.$$

   (d) (6.1-7) The rate at which an epidemic spreads through a community of P susceptible people is proportional to the product of the number of people y who have caught the disease and the number $P - y$ who have not.

   (e) (6.1-8) The rate at which people are implicated in a government scandal is proportional to the product of the number N of people already implicated and the number of people involved who have not yet been implicated.

2. (6.1) A population model is given by
   $$\frac{\mathrm{d}P}{\mathrm{d}t} = P(100 - P).$$

(a) (6.1-9) For what values is the population at equilibrium?

> **Solution:** The population is at equilibrium when the right hand side is zero. I.e. either $P = 0$ or $P = 100$.

(b) (6.1-10) For what values is $\frac{dP}{dt} > 0$?

> **Solution:** When $0 < P < 100$.

(c) (6.1-11) For what values is $\frac{dP}{dt} < 0$?

> **Solution:** When $P > 100$ or when $P < 0$ (however this does not make physical sense).

(d) (6.1-12) Describe how the fate of the population depends on the initial density.

> **Solution:** If $P(0) = 0$ or 100 then the population will stay at that level forever. If $0 < P(0) < 100$ then the population will increase towards and approach 100. If $P(0) > 100$ then the population decreases to and approaches 100.

3. (6.1) A population model is given by

$$\frac{dP}{dt} = P(P-1)(100-P).$$

(a) (6.1-13) For what values is the population at equilibrium?

> **Solution:** The equilibrium solutions of the model occur when $P' = 0$. That is, when $P = 0, 1, 100$.

(b) (6.1-14) For what values is $\frac{dP}{dt} > 0$?

> **Solution:** By testing points we see that $P' > 0$ when $1 < P < 100$.

(c) (6.1-15) For what values is $\frac{dP}{dt} < 0$?

> **Solution:** By testing points we see that $P' < 0$ when $0 < P < 1$ and when $P > 100$.

(d) (6.1-16) Describe how the fate of the population depends on the initial density.

> **Solution:** Using the above information we see that if $P(0) < 1$ the population eventually dies out. When $1 < P(0)$ the population eventually stabilises at $P = 100$.

4. (6.1) Radioactive decay: Certain types of atoms (e.g. carbon-14, xenon-133, lead-210, etc.) are inherently unstable. They exhibit random transitions to a different atom while emitting radiation in the process. Based on experimental evidence, Rutherford found in the early 20th century that the number, N, of atoms in a radioactive substance can be described by the equation

$$\frac{dN}{dt} = -\lambda N$$

where t is measured in years and $\lambda > 0$ is known as the *decay constant*. The decay constant is found experimentally by measuring the half life, τ of the radioactive substance (i.e. the time it takes for half of the substance to decay). Use this information in the following problems.

(a) (6.1-18) Find a solution to the decay equation assuming that $N(0) = N_0$.

Solution: Using separation of variables we get

$$N(t) = Ce^{-\lambda t}$$

Using the intitial value gives $C = N_0$ so

$$N(t) = N_0 e^{-\lambda t}$$

(b) (6.1-19) For xenon-133, the half-life is 5 days. Find λ. Assume t is measured in days.

Solution: We know that after 5 days we will have half the amount left, i.e. $N(5) = N_0/2$ this we have the equation

$$\frac{N_0}{2} = N_0 e^{-\lambda 5}$$

We can cancel the N_0 and rearrange (by taking a log) to get

$$\lambda = \frac{\ln 2}{5}.$$

(c) (6.1-20) For carbon-14 the half life is $5,568$ years. Find the decay constant λ, assuming t is measured in years.

Solution: Very similar to the above, we get $\lambda = \frac{\ln 2}{5568}$

(d) (6.1-21) How old is a piece of human bone which contains just 60% of the amount of carbon-14 expected in a sample of bone from a living person, assuming the half life of carbon-14 is $5,568$ years?

Solution: We know that it obeys the equation $N(t) = N_0 e^{-\lambda t}$ where λ is as given above. The question is telling us to find t such that $N(t) = 0.6N_0$, we get an equation

$$0.6N_0 = N_0 e^{-\frac{\ln 2}{5568} t}.$$

We can rearrange this and solve for t (the N_0 cancels)

$$t = -\frac{5568 \ln 0.6}{\ln 2}$$

(e) (6.1-22) The Dead Sea Scrolls were written on parchment at about 100 B.C. What percentage of carbon-14 originally contained in the parchment remained when the scrolls were discovered in 1947?

Solution: If originally there was N_0 cabon-14 then after 2047 years there would be

$$N(2047) = N_0 e^{-\frac{\ln 2}{5568} 2047}$$

left. As a percentage of N_0 this would be $100e^{-\frac{\ln 2}{5568} 2047} \approx 77.5\%$.

5. (6.1-30) Hyperthyroidism is caused by a new growth of tumor-like cells that secrete thyroid hormones in excess to the normal hormones. If left untreated, a hyperthyroid individual can exhibit extreme weight loss, anorexia, muscle weakness, heart disease intolerance to stress, and eventually death. The most successful and least invasive treatment option is radioactive iodine-131 therapy.

   This involves the injection of a small amount of radioactivity into the body. For the type of hyperthyroidism called Graves' disease, it is usual for about $40-80\%$ of the administered activity to concentrate in the thyroid gland. For functioning adenomas ("hot nodules"), the uptake is closer to $20-30\%$. Excess iodine-131 is excreted rapidly by the kidneys. The quantity of radioiodine used to treat hyperthyroidism is not enough to injure any tissue except the thyroid tissue, which slowly shrinks over a matter of weeks to months. Radioactive iodine is either swallowed in a capsule or sipped in solution through a straw. A typical dose is $5-15$ millicures. The half-life of iodine-131 is 8 days.

   (a) Suppose that it takes 48 hours for a shipment of iodine-131 to reach a hospital. How much of the initial amount shipped is left once it arrives at the hospital?

   Solution: After t days the fraction of the substance that will remain is

   $$\left(\frac{1}{2}\right)^{t/8} = e^{-t(\ln 2)/8}$$

   Thus after 2 days there will be $e^{-0.25\ln 2}$ of it remaining.

   (b) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left one week later?

   Solution:

   $$3e^{-\frac{7}{8}\ln 2} \text{ millicures}$$

   (c) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left 30 days later?

   Solution:

   $$3e^{-\frac{15}{4}\ln 2} \text{ millicures}$$

6. (6.2) Solve the following differential equations.

   (a) $\frac{\mathrm{d}y}{\mathrm{d}t} = 5y$

   (b) $\frac{\mathrm{d}y}{\mathrm{d}t} = -y$

   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y$

   (d) $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.2y$

   (e) (6.2-17) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^3$

   (f) (6.2-18) $\frac{\mathrm{d}y}{\mathrm{d}t} = y\sin t$

   (g) (6.2-20) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y}$

   (h) (6.2-24) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}\sqrt{1+x^2}$

   (i) (6.2-26) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x}{\cos y}$

   (j) (6.2-30) $\frac{\mathrm{d}y}{\mathrm{d}t} = yt$ with $y(1) = -1$

   (k) (6.2-32) $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-y}t$ with $y(-2) = 0$

(l) (6.2-34) $\frac{\mathrm{d}y}{\mathrm{d}t} = ty^2 + 3t^2y^2$ with $y(-1) = 2$

Solution: We begin by factorising the right hand side,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (t + 3t^2)y^2.$$

We can now separate variables and integrate:

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int t + 3t^2\,\mathrm{d}t$$

We integrate both sides using the power rule,

$$-\frac{1}{y} = \frac{1}{2}t^2 + t^3 + C$$

for an arbitrary constant C. Rearranging,

$$y(t) = -\frac{2}{t^2 + 2t^3 + C}.$$

Now we use the fact that $y(-1) = 2$:

$$2 = -\frac{2}{1 - 2 + C}$$

so $C = 0$ an the solution is

$$y(t) = -\frac{2}{t^2 + 2t^3}.$$

(m) $\frac{\mathrm{d}y}{\mathrm{d}x} = y\sin x + \frac{y}{(x+1)^2}$ with $y(0) = 1$

Solution: We begin by factorising the right hand side,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = y\left(\sin x + \frac{1}{(x+1)^2}\right).$$

We can now separate variables and integrate:

$$\int \frac{1}{y}\,\mathrm{d}y = \int \sin x + \frac{1}{(x+1)^2}\,\mathrm{d}x$$

We integrate both sides,

$$\ln(y) = -\cos x - \frac{1}{(x+1)} + C$$

for an arbitrary constant C. Exponentiating both sides,

$$y(t) = C\exp\left(-\cos x - \frac{1}{(x+1)}\right).$$

Now we use the fact that $y(0) = 1$:

$$1 = C\exp(-1 - 1) = Ce^{-2}$$

so $C = e^2$ and the solution is

$$y(t) = \exp\left(2 - \cos x - \frac{1}{(x+1)}\right).$$

(n) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}e^{-x^2}$ with $y(0) = 1$

(o) $\frac{\mathrm{d}y}{\mathrm{d}x} = y + ye^x$ with $y(0) = e$

7. (6.2-44) Populations may exhibit seasonal growth in response to seasonal fluctuations in resource availability. A simple model accounting for seasonal fluctuations in the abundance N of a population is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = (R + \cos t)N$$

where R is the average per-capita growth rate and t is measured in years.

(a) Assume $R = 0$ and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

> **Solution:** When $R = 0$ the equation is $N' = N \cos t$. Using separation of variables we find the solution $N(t) = Ce^{\sin t}$ and since $N(0) = N_0$ we see that $C = N_0$. As $N \to \infty$, this fluctuates between N_0e^{-1} and N_0e.

(b) Assume $R = 1$ (more generally $R > 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

> **Solution:** When $R = 1$ the equation is $N' = N(1 + \cos t)$. Using separation of variables we find the solution $N(t) = Ce^{t+\sin t}$ and since $N(0) = N_0$ we see that $C = N_0$. As $N \to \infty$, the t dominates the $\sin t$ and the population grows exponentially.

(c) Assume $R = -1$ (more generally $R < 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

> **Solution:** When $R = -1$ the equation is $N' = N(-1 + \cos t)$. Using separation of variables we find the solution $N(t) = Ce^{-t+\sin t}$ and since $N(0) = N_0$ we see that $C = N_0$. As $N \to \infty$, the e^{-t} dominates the $e^{\sin t}$ and the population decreases to zero.