1. At Cincinnati Zoo, Bibi, a female hippo gives birth to a baby hippo called Fiona. Her father is called Henry and he is unfortunately sick. It is important for his treatment that we know some information about a partiular gene. For hippos the gene can either be A or a, and Henry comes from a line of hippos where it is known that the gene pair aa occurs with probability p and the gene pair Aa with probabilty $1-p$.

   We can't take DNA from Henry - he is super angry about being sick, but we can take some DNA from Fiona and Bibi. Bibi has the gene pair AA. Given that Fiona has the gene pair Aa, what is the probabiltiy that Henry has the gene pair aa?

Solution: We will condition everything against the event that Bibi has gene pair AA, that means our sample space only consists of outcomes where this is true.

Let F_{Aa} be the event that Fiona has the gene pair Aa. Let H_{aa} and H_{Aa} be the events that Henry has gene pair aa or Aa repectively. We want to calculate $\mathbb{P}(H_{aa}|F_{Aa})$. To do this we will use Baye's law.

$$\mathbb{P}(H_{aa}|F_{Aa}) = \frac{\mathbb{P}(F_{Aa}|H_{aa})\mathbb{P}(H_{aa})}{\mathbb{P}(F_{Aa}|H_{aa})\mathbb{P}(H_{aa}) + \mathbb{P}(F_{Aa}|H_{Aa})\mathbb{P}(H_{Aa})}.$$

Here we have used that H_{aa} and H_{Aa} partition the sample space. Now

$$\mathbb{P}(H_{aa}) = p$$
$$\mathbb{P}(H_{Aa}) = 1-p$$

are given in the question. We just need to calculate $\mathbb{P}(F_{Aa}|H_{aa})$, the probability that Fiona has gene type Aa given that her father has gene type aa. For this we look at the Punnett square:

	aa	
A	Aa	Aa
A	Aa	Aa

so $\mathbb{P}(F_{Aa}|H_{aa}) = 1$. Similarly for $\mathbb{P}(F_{Aa}|H_{Aa})$:

	Aa	
A	AA	Aa
A	AA	Aa

so $\mathbb{P}(F_{Aa}|H_{Aa}) = 1/2$. Thus

$$\mathbb{P}(H_{aa}|F_{Aa}) = 1 \cdot p1 \cdot p + \frac{1}{2} \cdot (1-p) = \frac{2p}{1+p}.$$

2. Suppose we are collecting coupons. We get a coupon each time we go shopping. There are n different types of coupons to collect and we are equally likely to get any particular coupon and each shopping trip is independant from any other. *Later in the course we will be interested in the question: on avergae, how many shopping trips will it take before we have collected at least one of each different type of coupon.*

   For now, calculate the probability that after k shopping trips, we have collected

   1. at least one coupon of type i,
   2. at least one coupon of type i or type j,
   3. at least one coupon of type i and at least one of type j.

Solution: Let A_i be the event that we collect at least one coupon of type i.

1. We want to calculate $\mathbb{P}(A_i)$. Instead we will calculate $\mathbb{P}(A_i^c)$ and then $\mathbb{P}(A_i) = 1 - \mathbb{P}(A_i^c)$. A_i^c is the event that we do not receive a coupon of type i at all. Thus

$$\mathbb{P}(A_i^c) = \frac{(n-1)^k}{n^k} \text{ and } \mathbb{P}(A_i) = 1 - \frac{(n-1)^k}{n^k} = \frac{n^k - (n-1)^k}{n^k}.$$

2. We want to calculate $\mathbb{P}(A_i \cup A_j)$. We can use the fact that $(A_i \cup A_j)^c = A_i^c \cap A_j^c$, so

$$\mathbb{P}(A_i \cup A_j) = 1 - \mathbb{P}(A_i^c \cap A_j^c)$$

$\mathbb{P}(A_i^c \cap A_j^c)$ is the probability that we do not receive a coupon of type i or of type j. So

$$\mathbb{P}(A_i \cup A_j) = 1 - \frac{(n-2)^k}{n^k} = \frac{n^k - (n-2)^k}{n^k}.$$

3. We want to calculate $\mathbb{P}(A_i \cap A_j)$. To do this we rearrange the inclusion exclusion principle

$$\mathbb{P}(A_i \cup A_j) = \mathbb{P}(A_i) + \mathbb{P}(A_j) - \mathbb{P}(A_i \cap A_j)$$

to get

$$\begin{aligned}\mathbb{P}(A_i \cap A_j) &= \mathbb{P}(A_i) + \mathbb{P}(A_j) - \mathbb{P}(A_i \cup A_j) \\ &= \frac{n^k - (n-1)^k}{n^k} + \frac{n^k - (n-1)^k}{n^k} - \frac{n^k - (n-2)^k}{n^k} \\ &= \frac{n^k - 2(n-1)^k - (n-2)^k}{n^k}.\end{aligned}$$

3. Let $n \geq 4$ be a positive integer. We are given a deck of n cards labeled with numbers $1, 2, \ldots, n$, and we select uniformly at random two cards from this deck. Each of these two selected cards is destroyed with probability $1/2$, independently of the other card. What is the probability that neither card number 1 nor card number 2 get destroyed?

Solution: The answer clearly depends on which (if any) of cards 1 and 2 were selected initially. Let A_0 be the event that neither card 1 or 2 was selected initially; A_1 be the event that the card 1 was selected initially, but 2 wasn't, let A_2 be the event that the card 2 was selected initially but 1 wasn't, and let A_3 be the event that both cards 1 and 2 were selected initially. If B is the event that neither the card 1 nor 2 was destroyed then clearly we have

$$\mathbb{P}(B|A_0) = 1,\ \mathbb{P}(B|A_1) = 1/2,\ \mathbb{P}(B|A_2) = 1/2,\ \mathbb{P}(B|A_3) = 1/4.$$

Moreover, by the law of uniform probability

$$\mathbb{P}(A_0) = \frac{\binom{n-2}{2}}{\binom{n}{2}} = \frac{(n-2)(n-3)}{n(n-1)},\ \mathbb{P}(A_1) = \mathbb{P}(A_2) = \frac{\binom{n-2}{1}}{\binom{n}{2}} = \frac{2(n-2)}{n(n-1)},\ \mathbb{P}(A_3) = \frac{1}{\binom{n}{2}}.$$

Therefore,

$$\begin{aligned}\mathbb{P}(B) &= \mathbb{P}(B|A_0)\mathbb{P}(A_0) + \mathbb{P}(B|A_1)\mathbb{P}(A_1) + \mathbb{P}(B|A_2)\mathbb{P}(A_2) + \mathbb{P}(B|A_3)\mathbb{P}(A_3) \\ &= \frac{(n-2)(n-3)}{n(n-1)} + \frac{2(n-2)}{n(n-1)} + \frac{1}{2n(n-1)} = \frac{(n-2)(n-1) + 1/2}{n(n-1)}.\end{aligned}$$

4. There are two stores in your neighborhood, store A and B. Store A sells four items, priced at \$1, \$2, \$3 and \$4 respectively. Store B sells five times, priced at \$1, \$2, \$3, \$4 and \$5 respectively. You select one of these two stores at random with equal probabilities and go there. You select three different items from that store uniformly at random and buy them. Given that you spent \$8, what is the conditional probability that you selected store A.

Solution: Let A be the even that you chose store A and B the event that you chose store B, and C the event that you spent \$8. If you went to store A then you could spend \$8 only if you bought 1st, 3rd and 4th product, which has probability $\mathbb{P}(C|A) = 1/\binom{4}{3} = 1/4$. If you went to store B then you could spend \$8 if you bought 1st, 3rd and 4th product, or 1st, 2nd and 5th which has probability $\mathbb{P}(C|A) = 2/\binom{5}{3} = 1/5$. Since $\mathbb{P}(A) = \mathbb{P}(B) = 1/2$ we have by Bayes formula

$$\mathbb{P}(A|C) = \frac{\mathbb{P}(C|A)\mathbb{P}(A)}{\mathbb{P}(C|A)\mathbb{P}(A) + \mathbb{P}(C|B)\mathbb{P}(B)} = \frac{1/4}{1/4 + 1/5} = \frac{5}{9}$$

5. A company has 3 departments and there are 50 people in each department. How many ways can a committee of 12 people be formed so that there are exactly 4 people from each department? (The order of the 12 people does not matter.)

Solution: We simply need to choose 4 people from each department. Since we need to do this for each of the 3 departments, the answer is

$$\binom{50}{4}^3$$

6. A store sells three colors of balloons: yellow (50%), blue (20%) and purple (30%). The balloons are either filled with air or helium gas. We know that

- 10% of the gas balloons are yellow.
- 40% of the gas balloons are blue.
- 20% of the purple balloons are gas balloons.

Find the proportion of gas balloons.

Solution: The important thing to notice is that a proportion is the same as a probability. For example if Y is the event that we choose a yellow balloon randomly from the store then $\mathbb{P}(Y) = 0.5$. Let B, P and G be the events we choose a blue, purple or gass filled balloon. We would like to calculate $\mathbb{P}(G)$.

The question gives us the probabilities

- $\mathbb{P}(Y|G) = 0.1$
- $\mathbb{P}(B|G) = 0.4$, and
- $\mathbb{P}(G|P) = 0.2$ (note which event is being conditioned on).

So we should try an use the law of total probability:

$$\mathbb{P}(G) = \mathbb{P}(Y)\mathbb{P}(G|Y) + \mathbb{P}(B)\mathbb{P}(G|B) + \mathbb{P}(P)\mathbb{P}(G|P)$$

But then we need to know $\mathbb{P}(G|Y)$ and $\mathbb{P}(G|B)$, which we havent been given. If we use the information we have been given, and let $\mathbb{P}(G) = x$ then

$$x = \frac{1}{2}\mathbb{P}(G|Y) + \frac{1}{5}\mathbb{P}(G|B) + \frac{3}{10} \cdot \frac{1}{5}. \tag{1}$$

Now lets try and figure out $\mathbb{P}(G|Y)$. Using Bayes' law,

$$\mathbb{P}(G|Y) = \frac{\mathbb{P}(Y|G)\mathbb{P}(G)}{\mathbb{P}(Y)} = \frac{0.1x}{0.5} = \frac{1}{5}x. \tag{2}$$

Similarly for $\mathbb{P}(G|B)$:

$$\mathbb{P}(G|B) = \frac{\mathbb{P}(B|G)\mathbb{P}(G)}{\mathbb{P}(B)} = \frac{0.4x}{0.2} = 2x. \tag{3}$$

So plugging the information from (2) and (3) into the equation (1) we get

$$x = \frac{1}{10}x + \frac{2}{5}x + \frac{3}{50}$$

solving gives

$$\frac{1}{2}x = \frac{3}{50} \text{ i.e. } x = \frac{3}{25}.$$

Solution: Someone pointed out a much nicer way to solve this problem: you can notice that the 50% of the gas balloons are either yellow or blue. That means 50% of the gas balloons must be purple. So lets find out what percentage of the total balloons are both gas and purple. Well 30% of the total balloons are purple and 20% of these are gass, i.e. 6% (that is 20% of 30%) of the total balloons are purle and gas. We know this makes up half of the gas balloons, so the 12% of the total number of balloons are gas.