This weeks problem set provides some review questions in the lead up to the second midterm. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 4: due Monday 25 November: questions 3 and 5 below.

1. From section 5.2, problems 1, $3a, d, e$, 8, 9, 10, 11, 18^*, 19, $20^\dagger$.
2. From section 6.1, problems 1, 2, 3, 4, 8^*, 9, 12, 16, 17^*, 23, 29.
3. Let $T : V \longrightarrow V$ be a diagonalisable linear operator. Let $C(T) \subseteq \mathrm{Hom}(V, V)$ be the set of all linear maps that commute with T. I.e

$$C(T) = \{S \in \mathrm{Hom}(V, V) \mid S \circ T = T \circ S\}.$$

(a) If T has $n = \dim V$ distinct eigenvalues, show that any $S \in C(T)$ is diagonalisable.

Solution: Since T is diagonalisable, there exits a basis B of eigenvectors. Let $v \in B$ and suppose λ be the eigenvalue for v. Now suppose that $S \in C(T)$. Consider

$$T(S(v)) = S(T(v)) = S(\lambda v) = \lambda S(v).$$

Thus $S(v)$ is a λ-eigenvector for T. If E_λ is the λ-eigenspace for T then since T has n distinct eigenvalues, the sum of its geometric multiplicities is n, and so each eigenspace is one dimensional. Thus $\{v, S(v)\} \subset E_\lambda$ is linearly dependent. I.e there exits some $\mu \in \mathbb{F}$ such that $S(v) = \mu v$. Hence each element of the basis B is an eigenvector for S, so it is a basis of eigenvectors of S. Thus S is diagonalisable.

(b) Describe explicitly $C(T)$ in the case $T = x\frac{d}{dx} : \mathbb{C}_1[x] \longrightarrow \mathbb{C}_1[x]$.

Solution: The operator T has a basis of eigenvectors $\{1, x\}$ with eigenvalues 0. By part a, if $S \in C(T)$ then this must also be a basis of eigenvectors for S. $C(T)$ consists of linear operators S given by

$$S(1) = a \text{ and } S(x) = bx$$

for any choice $a, b \in \mathbb{F}$.

(c) Show that part (a) does not necessarily hold if T does not have n distinct eigenvalues.

Solution: If $T = \mathrm{id}$ then it is diagonalisable but does not have n distinct eigenvalues. The condition that $S \circ \mathrm{id} = \mathrm{id} \circ S$ reduces to $S = S$ so there is no condition on S. Thus

$$C(T) = C(\mathrm{id}) = \mathrm{Hom}(V, V).$$

There exist non-diagonalisable linear operators on any vector space so part (a) does not hold. For an example of a non-diagonalisable linear operator fix a basis $B = \{v_1, \ldots, v_n\}$ of V and consider the linear operator defined by $S(v_1) = v_2$ and $S(v_i) = 0$ for $i > 1$. The matrix of T is lower triangular with zeros on the diagonal, so the characteristic polynomials is t^n. Thus the only eigenvalue is 0, with an algebraic multiplicity of n. To find the geometric multiplicity lets solve the equation

$$0 = S(v) = S(\lambda_1 v_1 + \ldots + \lambda_n v_n) = \lambda_1 S(v_1) = \lambda_1 v_2$$

Thus we must have that $\lambda_1 = 0$ and a basis for the 0-eigenspace is $\{v_2, v_3, \ldots, v_n\}$. Hence the geometric multiplicity is $n - 1$ which does not match the algebraic multiplicity and so the operator is not diagonalisable.

4. Suppose U, W are subspaces of a finite dimensional vector space V and that $U + W = V$. Show that $U \oplus W = V$ if and only if $\dim U + \dim W = \dim V$.

Solution: Suppose that $U \oplus W = V$, thus $U \cap W = \{0\}$. Let B be a basis of U and C be a basis of W. Clearly $B \cap C = \emptyset$, so $\#B \cup C = \#B + \#C$. We claim that $B \cup C$ is a basis of V. It is spanning since $U + W = W$ and we can prove linear independence by using $U \cap W = \{0\}$. Thus $\dim V = \dim U + \dim W$.

Conversely suppose that $\dim V = \dim U + \dim W$. Let B be a basis of U and C be a basis of W. Since $U + W = V$, we have that $\operatorname{span} B \cup C = V$. Now suppose that $0 \neq v \in U \cap W$. Then we can write

$$v = \lambda_1 u_1 + \cdots + \lambda_m u_m$$
$$v = \mu_1 w_1 + \cdots + \mu_n w_n$$

For some $\lambda_i, \mu_i \in \mathbb{F}$ and $u_i \in B$ and $w_i \in C$. By taking the difference of the two equations we see that $B \cup C$ is linearly dependent. Thus $\dim V < \dim U + \dim W$. But this is a contradiction!

The previous question, motivates the following definition.

Definition: If U_i, for $1 \leq i \leq k$, are subspaces of a vector space V, then we say $V = U_1 \oplus U_2 \ldots \oplus U_k$ if $V = U_1 + U_2 + \ldots + U_k$, i.e. every vector $v \in V$ can be written as a sum $v = \sum_{i=1}^k u_i$ with $u \in U_i$, and $\dim V = \sum_{i=1}^k \dim U_i$.

5.* Suppose that V is a finite dimensional vector space over $\mathbb{F}$ and $T : V \longrightarrow V$ is a linear operator, with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$$

if and only if T is diagonalisable.

Solution: Suppose first that

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}.$$

Let $b_i = b_{\lambda_i}$ be the geometric multiplicity of λ_i, i.e. $\dim E_{\lambda_i} = b_i$. Since we have a direct sum we have

$$b_1 + \cdots + b_k = n.$$

(see below for a careful explanation of this fact). The characteristic polynomial of T is

$$p_T(t) = q(t)(t - \lambda_1)^{a_1} \cdots (t - \lambda_k)^{a_k}$$

for some polynomial $q(t)$. If we can show the degree of q is zero, then $p_T(t)$ splits. We know that

$$n = b_1 + \cdots + b_k \leq a_1 + \cdots + a_k \leq n$$

thus $a_1 + \cdots + a_k = n$. But $a_1 + \cdots + a_k + \deg q = n$ so $\deg q = 0$. Thus $p_T(t)$ splits.

Now to see that the algebraic and geometric multiplicities are equal, consider the equality

$$b_1 + \cdots + b_k = n = a_1 + \cdots + a_k$$

along with the fact that $b_i \leq a_i$ for each i. But if $b_i < a_i$ then we would have that

$$b_1 + \cdots + b_k < a_1 + \cdots + a_k$$

which is a contradiction. Thus $b_i = a_i$ for all i. Hence T is diagonalisable.

Now suppose that T is diagonalisable. There exists a basis of eigenvectors, i.e. every vector in V can be written as the sum of eigenvectors, thus

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}.$$

We know that $a_{\lambda_i} = b_{\lambda_i} = \dim E_{\lambda_i}$. Since the characteristic polynomial splits we have that $\dim V = \sum a_{\lambda_i} = \sum b_{\lambda_i} = \sum \dim E_{\lambda_i}$.