# Midterm 2 practice 1

## UCLA: Math 115A, Winter 2020

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*Date:*
*Version: practice*

- This exam has 4 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

| Question | Points | Score |
|---|---|---|
| 1 | 5 | |
| 2 | 5 | |
| 3 | 5 | |
| 4 | 5 | |
| Total: | 20 | |

Question 1 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

*Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!*

**Question 1.**

| *Part* | A | B | C | D |
|---|---|---|---|---|
| (a) | | | | |
| (b) | | | | |
| (c) | | | | |
| (d) | | | | |
| (e) | | | | |

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (1 point) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be the linear map given by $T(a, b, c) = (a + b - c, b + c)$. Consider the bases $B = \{(1, 1, 1), (1, 0, -1), (1, 0, 1)\}$ and $C = \{(2, 1), (1, 2)\}$. The matrix $[T]_B^C$ is

   A. $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

   **B.** $\frac{1}{3}\begin{pmatrix} 0 & 5 & -1 \\ 3 & -4 & 2 \end{pmatrix}$

   C. $\begin{pmatrix} 0 & -2 & 2 \\ 1 & -1 & 2 \end{pmatrix}$

   D. $\frac{1}{5}\begin{pmatrix} 0 & -2 & 2 \\ 1 & 0 & 1 \end{pmatrix}$

   (b) (1 point) For an arbitrary linear combination $T : V \longrightarrow W$ and bases $B = \{v_1, \ldots, v_n\}, C = \{w_1, \ldots, w_m\}$ of $V$ and $W$, the $i^{\text{th}}$ column of the matrix $[T]_B^C$ is

   A. $T(v_i)$

   B. $[T(v_i)]_B$

   C. $[w_i]_C$

   **D.** $[T(v_i)]_C$

   (c) (1 point) Consider the linear map $T : \mathbb{R}_1[x] \longrightarrow \mathbb{R}_1[x]$ given by $T(a + bx) = (a + b) + (a - b)x$. Which of the following is a true statement.

   A. $T$ has an eigenvalue of 2.

   **B. $T$ is diagonalizable.**

   C. The only eigenvalue of $T$ is $-1$.

   D. $T$ has infinitely many eigenvalues.

(d) (1 point) If $\dim V = 6$, and $W$ is a subspace such that $\dim W = 3$ then

A. $\dim V/W = 1$

B. $\dim V/W = 2$

**C.** $\dim V/W = 3$

D. $\dim V/W = 4$

(e) (1 point) The linear map $T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ defined by $T(a + bx + cx^2) = (a + c) - (b - c)x + bx^2$ is invertible. What is $T^{-1}(1 - x + x^2)$?

A. $1 + x + x^2$

B. $2x - x^2$

**C.** $1 + x$

D. $x^2$

2. (a) (2 points) Define what it means for two vector space $V$ and $W$ to be isomorphic.

**Solution:** $V$ and $W$ are isomorphic if there exists an invertible linear map $T : V \longrightarrow W$.

(b) (3 points) Prove that, if $V$ and $W$ are finite dimensional, then $V$ and $W$ are isomorphic if and only if $\dim V = \dim W$.

**Solution:** Suppose that $V$ and $W$ are isomorphic and $T : V \longrightarrow W$ is an isomorphism. Let $B = \{v_1, \ldots, v_n\}$ be a basis for $V$. The consider $T(B) = \{w_1, \ldots, w_n\}$ where $w_i = T(v_i)$. Since $T$ is surjective this set spans $W$. Suppose we have a linear combination

$$\begin{aligned} 0 &= \lambda_1 w_1 + \ldots + \lambda_n w_n \\ &= \lambda_1 T(v_1) + \ldots + \lambda_n T(v_n) \\ &= T(\lambda_1 v_1 + \ldots + \lambda_n v_n) \end{aligned}$$

Thus $\lambda_1 v_1 + \ldots + \lambda_n v_n \in \ker T$, but since $T$ is injective, this means that $\lambda_1 v_1 + \ldots + \lambda_n v_n = 0$, and since $B$ is a basis we must have $\lambda_i = 0$ and hence $T(B)$ is linearly independent and thus is a basis. Thus $\dim W = \#T(B) = \#B = \dim V$.

Now suppose that $\dim V = \dim W = n$. Let $B = \{v_1, \ldots, v_n\}$ and $C == \{w_1, \ldots, w_n\}$ be bases of $V$ and $W$ respectively. Define linear maps $T : V \longrightarrow W$ by $T(v_i) = w_i$ and $S : W \longrightarrow V$ by $S(w_i) = v_i$ respectively. It is clear that these maps are inverse and thus isomorphisms.

3. Let $T : V \longrightarrow V$ be a linear transformation for a vector space $V$ over $\mathbb{C}$. Define $T^n$ to be the linear map obtained by repeatedly applying $T$, $n$ times. E.g. $T^3(v) = T(T(T(v)))$.

   (a) (2 points) Suppose $T^n = 0$ for some $n$. Show that the only eigenvalue of $T$ is zero.

   **Solution:** Suppose that $\lambda$ is an eigenvalue of $T$. That means there is a vector $v$ such that $T(v) = \lambda v$. Thus $T^k(v) = \lambda^k v$ and hence $0 = T^n(v) = \lambda^n v$. Since $v$ is nonzero, this means $\lambda^n = 0$. Hence $\lambda = 0$.

   (b) (3 points) Prove that, if $T^2 = 0$ if and only if $\operatorname{im} T \subseteq \ker T$.

   **Solution:** Assume that $T^2 = 0$. Let $w \in \operatorname{im} T$, that is, there exists $v \in V$ such that $T(v) = w$. Then $T(w) = T(T(v)) = T^2(v) = 0$ so $w \in \ker T$ and hence $\operatorname{im} T \subseteq \ker T$.
   Now assume that $\operatorname{im} T \subseteq \ker T$. If $v \in V$ then $T(v) \in \operatorname{im} T$. Since this is contained in the kernel we must have $T(T(v)) = 0$, i.e. $T^2(v) = 0$ for all $v$. So $T^2 = 0$.

4. Let $T : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ be a linear map such that

$$[T]_B^B = \frac{1}{2}\begin{pmatrix} 2 & 6 & -2 \\ 0 & 3 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

where $B = \{1, (x-1)^2, (x+1)^2\}$

(a) (1 point) Is $T$ an isomorphism? *Hint: This should be a very easy, and quick calculation. If you are spending more than 30 seconds on it, you are missing something important.*

**Solution:** The matrix $[T]_B^B$ is easily seen to have nonzero determinant, thus is invertible, hence the linear map is an isomorphism.

(b) (2 points) Let $E = \{1, x, x^2\}$. Find the change of basis matrix $[\mathrm{id}]_E^B$.

**Solution:** We just note that

$$\begin{aligned} 1 &= 1 \cdot 1 + 0 \cdot (x-1)^2 + 0 \cdot (x+1)^2 \\ x &= 0 \cdot 1 - \frac{1}{4} \cdot (x-1)^2 + \frac{1}{4} \cdot (x+1)^2 \\ x^2 &= -1 \cdot 1 + \frac{1}{2} \cdot (x-1)^2 + \frac{1}{2} \cdot (x+1)^2 \end{aligned}$$

so we have

$$[\mathrm{id}]_E^B = \frac{1}{4}\begin{pmatrix} 4 & 0 & -4 \\ 0 & -1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

(c) (2 points) Calculate $[T]_E^E$ and $T^6(x)$.

**Solution:**

$$[T]_E^E = ([\mathrm{id}]_E^B)^{-1}[T]_B^B[\mathrm{id}]_E^B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $T(x) = x - 1$ and $T(1) = 1$ so $T^6(x) = x - 6$.

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