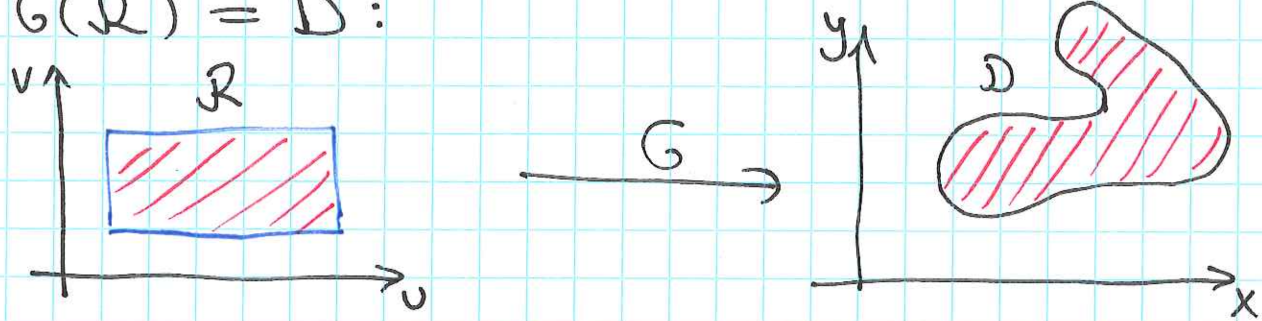
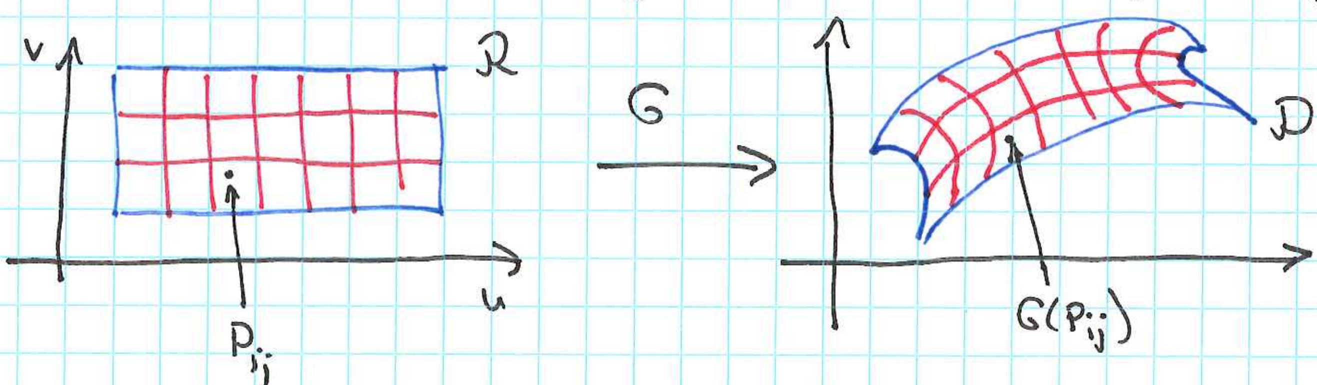
Lecture 9

- We pick up where we left off: We had a region $D \subseteq$ xy-plane and a map G and a rectangle $R \subseteq$ uv-plane so that $G(R) = D$:



- We partition R, ~~which~~ into subrectangles R_{ij}, and pick points $P_{ij} \in R_{ij}$
- This partitions $G(R) = D$ into subregions $D_{ij} = G(R_{ij})$ and gives points $G(P_{ij}) \in D_{ij}$.



- We can estimate the integral

$$\iint_D f(x\,y)\,dA = \lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f\big(G(P_{ij})\big)\,\text{Area}\big(G(R_{ij})\big)$$

- But we know that as long as Area (R_{ij}) is very small then

$$\text{Area}(G(R_{ij})) \approx J(G)(P_{ij}) \cdot \text{Area}(R_{ij})$$

(thm from lecture 7).

So

$$\iint_D f(x,y)\,dA = \lim_{\|P\| \to 0} \sum_{i=1}^{m} \sum_{j=1}^{n} f(G(P_{ij}))\,J(G)(P_{ij})\,\Delta u_i \Delta v_j$$

since $\Delta u_i \Delta v_j = \text{Area}(R_{ij})$.

- Thus

Thm If D is a region, $f(x,y)$ cts, $G(u,v)$ a map such that $\exists$ a rectangle R s.t. $G(R) = D$ then

$$\iint_D f(x,y)\,dA_{xy} = \iint_R f(G(u,v))\,J(G)(u,v)\,dA_{uv}$$

add subscripts to denote variables we are integrating w/ respect to

- What about more general regions?
- Suppose $\mathcal{D}_0 \subseteq$ uv-plane so that

$$G(\mathcal{D}_0) = \mathcal{D}$$

- Choose a rectangle $\mathcal{R}$ s.t. $\mathcal{D}_0 \subseteq \mathcal{R}$, let $\tilde{\mathcal{D}} = G(\mathcal{R})$. Clearly $\mathcal{D} \subseteq \hat{\mathcal{D}}$.
- Let

$$\hat{f}(x\,y) = \begin{cases} f(x\,y) & \text{if } (x\,y) \in \mathcal{D} \\ 0 & \text{o/w} \end{cases}$$

- Then

$$\iint_{\mathcal{D}} f(x\,y)\,dA_{xy} = \iint_{\hat{\mathcal{D}}} \tilde{f}(x\,y)\,dA_{xy}$$

$$= \iint_{\mathcal{R}} \hat{f}(G(u\,v))\,J(G)(u,v)\,dA_{uv}$$

(by previous theorem).

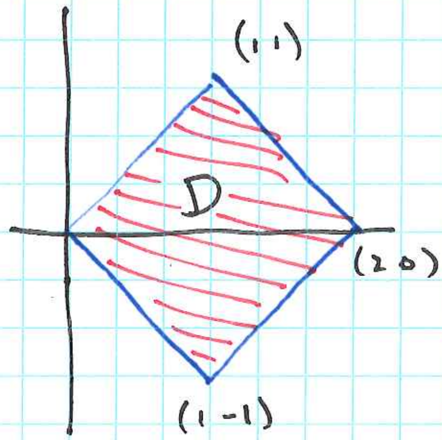
- But

$$\tilde{f}(G(u,v)) = \begin{cases} f(G(u,v)) & \text{if } (u,v) \in \mathcal{D}_0 \\ 0 & \text{o/w} \end{cases}$$

- so Thm

$$\iint_{\mathcal{D}} f(x\,y)\,dA_{xy} = \iint_{\mathcal{D}_0} f(G(u,v))\,J(G)(u,v)\,dA_{uv}.$$

Ex Calculate $\iint_D (x-y)^2 \, dA_{xy}$ where



- Note that D is given by

$$0 \le y + x \le 2$$

$$-2 \le y - x \le 0$$

- Sensible change of variables:

$$u = y + x$$

$$v = y - x$$

- Solving for x, y

$$x = \frac{1}{2}(u - v)$$

$$y = \frac{1}{2}(u + v)$$

- Set $G(u,v) = \left(\frac{1}{2}(u-v), \frac{1}{2}(u+v) \right)$

so $J(G) = \frac{1}{2} \cdot \frac{1}{2} - \left(-\frac{1}{2}\right) \cdot \frac{1}{2} = \frac{1}{2}$

- $G\left(R = [0,2] \times [-2,0]\right) = D$!

- Thus

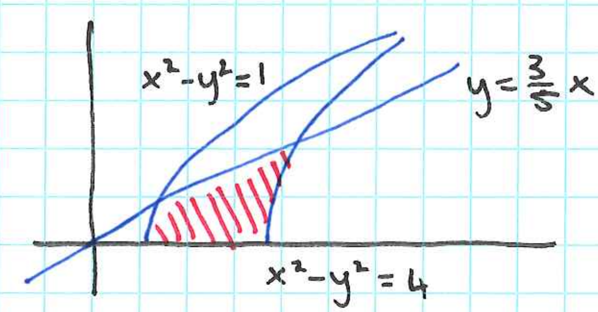
$$\iint_D (x-y)^2\, dA_{xy} = \iint_R (-\cancel{2}v)^2 \cdot \frac{1}{2}\, dA_{uv}$$

$$= \int_0^2 \int_{-2}^0 \frac{1}{2}\cancel{2}v^2\, dv\, du$$

$$= \int_0^2 \left[\frac{1}{6} v^3 \right]_{-2}^0 du$$

$$= \int_0^2 \frac{8}{6}\, du = \frac{8}{3}$$

Ex $\iint_D e^{x^2-y^2}\, dA_{xy}$ where D is



A not so obvious change of variables:

$$u = x^2 - y^2$$
$$v = x + y$$

Solving for x, y

$$x = \frac{1}{2}\left(v + \frac{u}{v}\right)$$

$$y = \frac{1}{2}\left(v - \frac{u}{v}\right)$$

- So

$$J(G) = \frac{1}{2v}\left(\frac{1}{2} + \frac{1}{2}\frac{u}{v^2}\right) + \frac{1}{2v}\left(\frac{1}{2} - \frac{1}{2}\frac{u}{v^2}\right)$$

$$= \frac{1}{2v}$$

- We see that

$u = 1$ is sent to $x^2 - y^2 = 1$

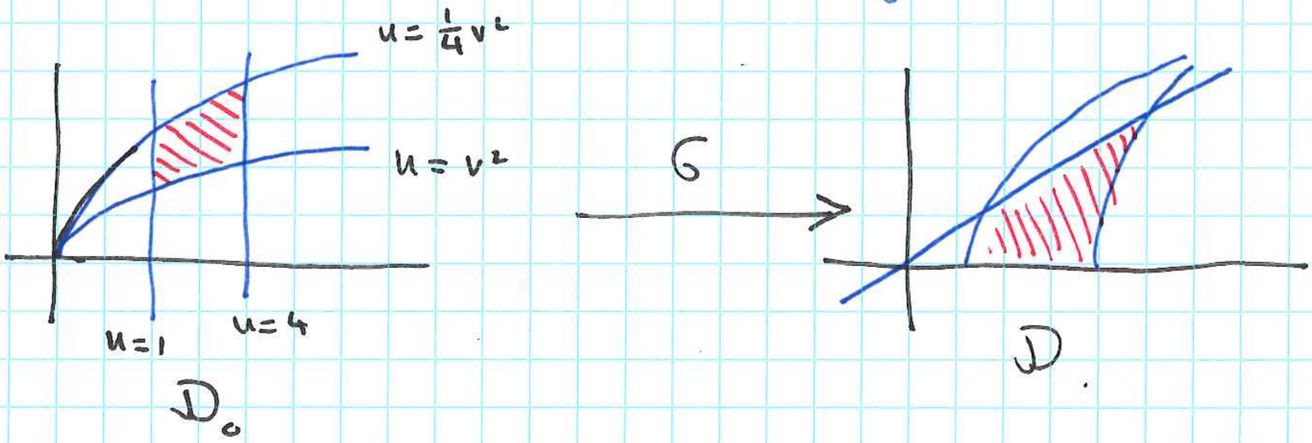
$u = 4$ is sent to $x^2 - y^2 = 4$

not so obviously

$u = v^2$ is sent to $y = 0$

$u = \frac{1}{4}v^2$ is sent to $y = \frac{3}{5}x$

(note to find these sub in $y = \dots$ $x = \dots$ into)



- So $G(D_0) = D$ and D_0 is vertically simple!

$$D_0 = \{(u,v) \mid 1 \le u \le 4,\ \sqrt{u} \le v \le 2\sqrt{u}\}$$

- Thus

$$\iint_{D} e^{x^2-y^2}\, dA_{xy} = \iint_{D_0} e^{u} \frac{1}{2v}\, dA_{uv}$$

$$= \int_1^4 \int_{\sqrt{u}}^{2\sqrt{u}} \frac{e^u}{2v}\, \cancel{dA_{uv}}\, dv\, du$$

Remark Polar coords are given by

$$G(r\ \theta) = (r\cos\theta\ ,\ r\sin\theta)$$

So $J(G) = r\cos^2\theta + r\sin^2\theta = r$, so if $G(D_0) = D$

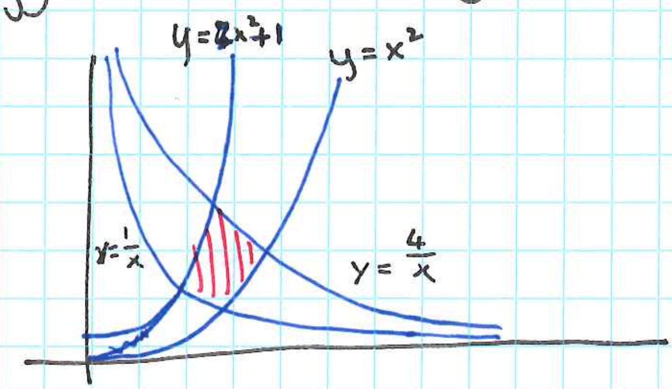
then

$$\iint_{D} f(x\ y)\, dA_{xy} = \iint_{D_0} f(r\cos\theta,\ r\sin\theta)\ \underline{\underline{r}}\ dA_{r\theta}$$

Remark Sometimes we don't need G if we already have G^{-1}:

Thm $\quad J(G^{-1})^{-1} = J(G)$

Ex $\iint_{D} xy^2 + 2x^3y\, dA_{xy}$ where



- We want to try

$$u = y - x^2$$
$$v = xy$$

since D is $0 \leq y - x^2 \leq 1$, $1 \leq xy \leq 4$.

- But solving this for x, y is very hard!
- Instead: $G^{-1}(u\ v) = (y - x^2, xy)$ by def

so $J(G) = J(G^{-1})^{-1} = \left(-2x^2 - y\right)^{-1}$

$$= \frac{-1}{2x^2 + y}$$

Thus

$$\iint_D xy^2 + 2x^3y \, dA_{xy} = \iint_R (xy^2 + 2x^3y) \frac{-1}{y + 2x^3} \, dA_{uv}$$

$R = [0\ 1] \times [1\ 4]$

$$= \iint_R - \frac{xy(y + 2x^3)}{y + 2x^3} \, dA_{uv}$$

Here we get lucky twice

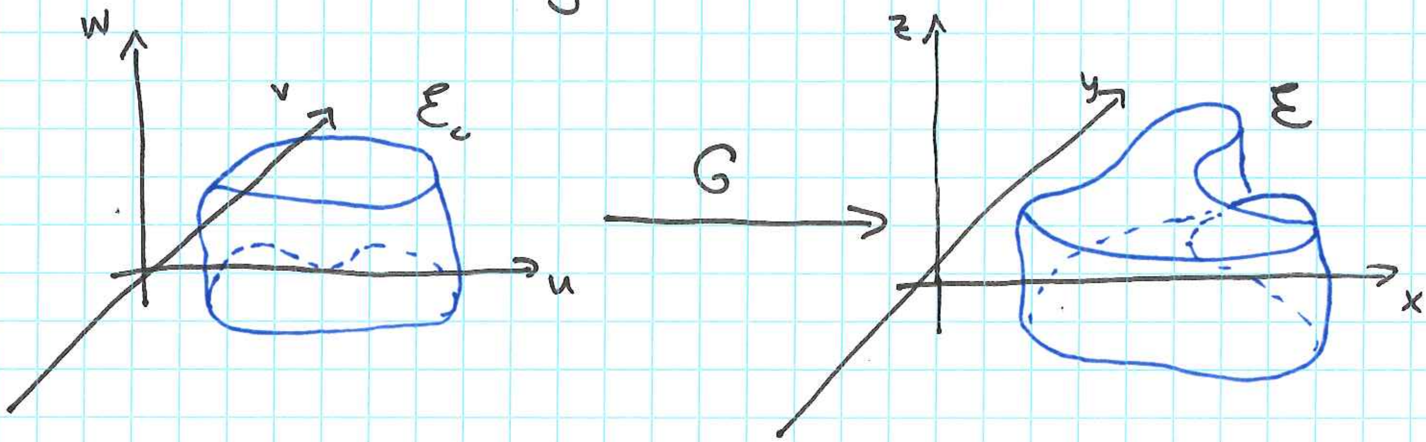
$$= \iint_R - xy \, dA_{uv}$$

$$= \iint_R - v \, dA_{uv}$$

$$= \int_0^1 \int_1^4 -v \, dv \, du$$

2. 3D change of variables

- This is basically the same as 2D



where

$$G(u,v,w) = \Big(x(u\ v\ w) \quad y(u\ v\ w) \quad w(u\ v\ w) \Big)$$

Def The Jacobian of G is

$$J(G) = \det \begin{pmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{pmatrix}$$

Thm If $G(\mathcal{E}_0) = \mathcal{E}$ then

$$\iiint_{\mathcal{E}} f(x\ y\ z)\, dV_{xyz} = \iiint_{\mathcal{E}_0} f(G(u\ v\ w))\, J(G)(u\ v\ w)\, dV_{uvw}.$$

The two most important changes of coordinates in 3D are

Ex Spherical

$x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho\cos\phi$

where

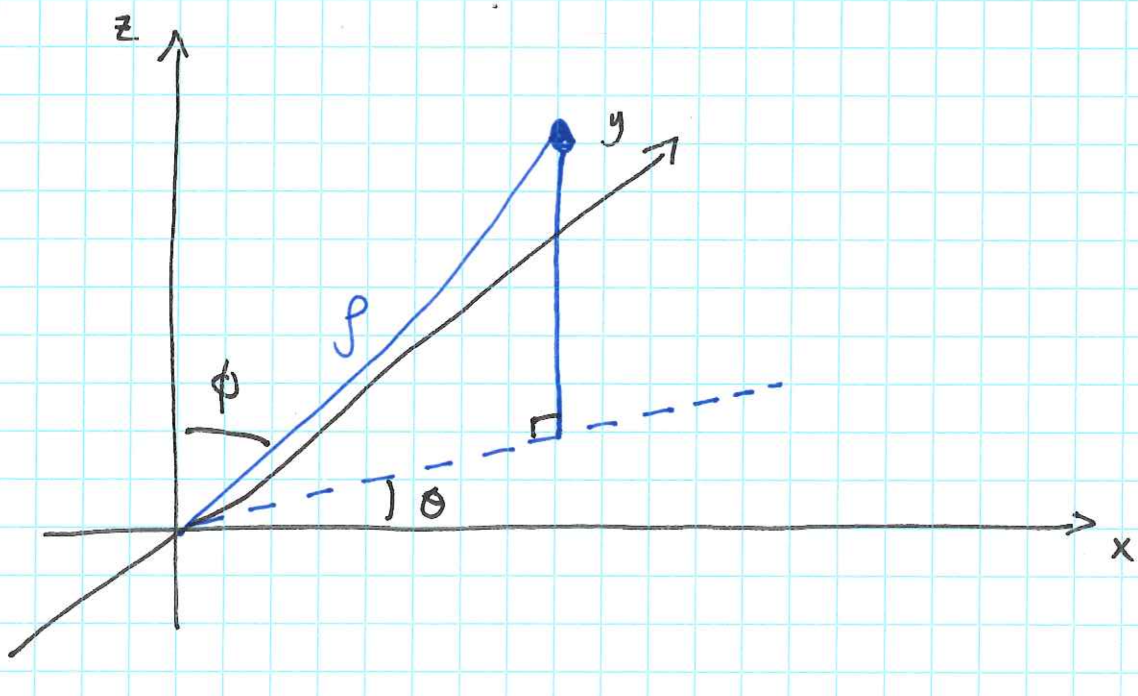
ρ = distance from origin

θ = angle on xy-plane, anticlockwise from x-axis

ϕ = angle from the verical (z-axis).

ie, sphere of radius 4 would be

$\rho = 4, \quad \theta = [0, 2\pi], \quad \phi = [0, \pi]$.



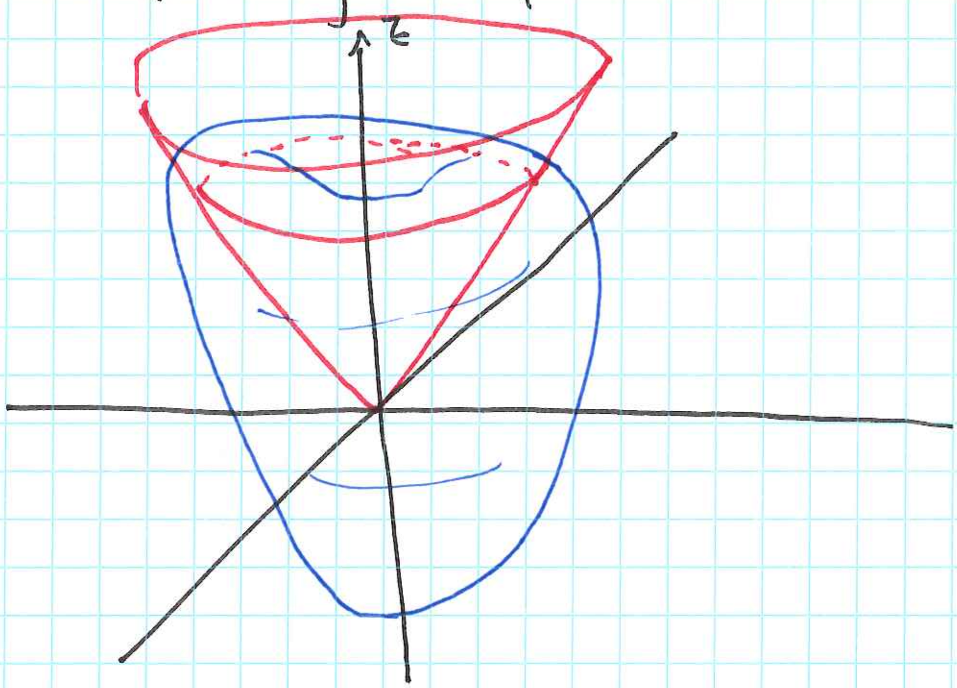
Exercise $J(G)$ for spherical coords is

$$J(G) = \rho^2 \sin\phi$$

Ex Complete the volume between the cone

$$x^2 + y^2 = z^2$$

and the surface $\rho = 1 + \phi$.



- The volume in the cone is given by

$$0 \le \phi \le \frac{\pi}{4}$$

$$0 \le \theta \le 2\pi$$

$$0 \le \rho < \infty .$$

- The volume in the surface is

$$0 \le \phi \le \pi$$

$$0 \le \theta \le 2\pi$$

$$0 \le \rho \le 1 + \phi$$

So the intersection is

$$0 \leq \phi \leq \pi/4$$

$$0 \leq \theta \leq 2\pi$$

$$0 \leq \rho \leq 1 + \phi$$

Thus

$$\iiint_{\mathcal{E}} dV_{xyz} = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{1+\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \left[\frac{1}{3} \rho^3 \sin\phi \right]_0^{1+\phi} d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{3} (1+\phi) \sin\phi \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} \Big[\sin\phi - (1+\phi)\cos\phi \Big]_0^{\pi/4} d\theta$$

$$= \frac{2\pi}{3} \left(1 - \frac{\pi}{4\sqrt{2}} \right).$$

Ex Cylindrical (really just polar).

$x = r\cos\theta$, $y = r\sin\theta$ $z = z$

$G(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$

where

r = distance from the x-axis

θ = angle on the xy-plane from x-axis

z = height above xy-plane

So $J(G) = r$.