

This weeks problem set provides some review questions in the lead up to the second midterm. A question marked with a \dagger is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a $*$ is especially important.

Homework 3: due Tuesday 26 May: questions 3 and 6 below.

1. From section 5.2, problems 1, 3a, d, e, 8, 9, 10, 11, 18*, 19, 20 \dagger .
2. From section 6.1, problems 1, 2, 3, 4, 8*, 9, 12, 16, 17*, 23, 29.
3. We say that two linear operators S and T *commute* if $S \circ T = T \circ S$. Let $T : V \rightarrow V$ be a diagonalisable linear operator. Define

$$C(T) = \{S \in \text{Hom}(V, V) \mid S \text{ and } T \text{ commute}\}.$$

- (a) If T has $n = \dim V$ distinct eigenvalues, show that any $S \in C(T)$ is diagonalisable.

Solution: Since T is diagonalisable, there exists a basis B of eigenvectors. Let $v \in B$ and suppose λ be the eigenvalue for v . Now suppose that $S \in C(T)$. Consider

$$T(S(v)) = S(T(v)) = S(\lambda v) = \lambda S(v).$$

Thus $S(v)$ is a λ -eigenvector for T . If E_λ is the λ -eigenspace for T then since T has n distinct eigenvalues, the sum of its geometric multiplicities is n , and so each eigenspace is one dimensional. Thus $\{v, S(v)\} \subset E_\lambda$ is linearly dependent. I.e. there exists some $\mu \in \mathbb{F}$ such that $S(v) = \mu v$. Hence each element of the basis B is an eigenvector for S , so it is a basis of eigenvectors of S . Thus S is diagonalisable.

- (b) Describe explicitly $C(T)$ in the case $T = x \frac{d}{dx} : \mathbb{C}_1[x] \rightarrow \mathbb{C}_1[x]$.

Solution: The operator T has a basis of eigenvectors $\{1, x\}$ with eigenvalues 0. By part a, if $S \in C(T)$ then this must also be a basis of eigenvectors for S . $C(T)$ consists of linear operators S given by

$$S(1) = a \text{ and } S(x) = bx$$

for any choice $a, b \in \mathbb{F}$.

- (c) Show that part (a) does not necessarily hold if T does not have n distinct eigenvalues.

Solution: If $T = \text{id}$ then it is diagonalisable but does not have n distinct eigenvalues. The condition that $S \circ \text{id} = \text{id} \circ S$ reduces to $S = S$ so there is no condition on S . Thus

$$C(T) = C(\text{id}) = \text{Hom}(V, V).$$

There exist non-diagonalisable linear operators on any vector space so part (a) does not hold. For an example of a non-diagonalisable linear operator fix a basis $B = \{v_1, \dots, v_n\}$ of V and consider the linear operator defined by $S(v_1) = v_2$ and $S(v_i) = 0$ for $i > 1$. The matrix of T is lower triangular with zeros on the diagonal, so the characteristic polynomial is t^n . Thus the only eigenvalue is 0, with an algebraic multiplicity of n . To find the geometric multiplicity let's solve the equation

$$0 = S(v) = S(\lambda_1 v_1 + \dots + \lambda_n v_n) = \lambda_1 S(v_1) = \lambda_1 v_2$$

Thus we must have that $\lambda_1 = 0$ and a basis for the 0-eigenspace is $\{v_2, v_3, \dots, v_n\}$. Hence the geometric multiplicity is $n - 1$ which does not match the algebraic multiplicity and so the operator is not diagonalisable.

4. Suppose U, W are subspaces of a finite dimensional vector space V and that $U + W = V$. Show that $U \oplus W = V$ if and only if $\dim U + \dim W = \dim V$.

Solution: Suppose that $U \oplus W = V$, thus $U \cap W = \{0\}$. Let B be a basis of U and C be a basis of W . Clearly $B \cap C = \emptyset$, so $\#B \cup C = \#B + \#C$. We claim that $B \cup C$ is a basis of V . It is spanning since $U + W = V$ and we can prove linear independence by using $U \cap W = \{0\}$. Thus $\dim V = \dim U + \dim W$.

Conversely suppose that $\dim V = \dim U + \dim W$. Let B be a basis of U and C be a basis of W . Since $U + W = V$, we have that $\text{span } B \cup C = V$. Now suppose that $0 \neq v \in U \cap W$. Then we can write

$$v = \lambda_1 u_1 + \cdots + \lambda_m u_m$$

$$v = \mu_1 w_1 + \cdots + \mu_n w_n$$

For some $\lambda_i, \mu_i \in \mathbb{F}$ and $u_i \in B$ and $w_i \in C$. By taking the difference of the two equations we see that $B \cup C$ is linearly dependent. Thus $\dim V < \dim U + \dim W$. But this is a contradiction!

The previous question, motivates the following definition.

Definition: If U_i , for $1 \leq i \leq k$, are subspaces of a vector space V , then we say $V = U_1 \oplus U_2 \cdots \oplus U_k$ if $V = U_1 + U_2 + \cdots + U_k$, i.e. every vector $v \in V$ can be written as a sum $v = \sum_{i=1}^k u_i$ with $u_i \in U_i$, and $\dim V = \sum_{i=1}^k \dim U_i$.

- 5* Suppose that V is a finite dimensional vector space over \mathbb{F} and $T : V \rightarrow V$ is a linear operator, with distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Prove that

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$$

if and only if T is diagonalisable.

Solution: Suppose first that

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}.$$

Let $b_i = b_{\lambda_i}$ be the geometric multiplicity of λ_i , i.e. $\dim E_{\lambda_i} = b_i$. Since we have a direct sum we have

$$b_1 + \cdots + b_k = n.$$

(see below for a careful explanation of this fact). The characteristic polynomial of T is

$$p_T(t) = q(t)(t - \lambda_1)^{a_1} \cdots (t - \lambda_k)^{a_k}$$

for some polynomial $q(t)$. If we can show the degree of q is zero, then $p_T(t)$ splits. We know that

$$n = b_1 + \cdots + b_k \leq a_1 + \cdots + a_k \leq n$$

thus $a_1 + \cdots + a_k = n$. But $a_1 + \cdots + a_k + \deg q = n$ so $\deg q = 0$. Thus $p_T(t)$ splits.

Now to see that the algebraic and geometric multiplicities are equal, consider the equality

$$b_1 + \cdots + b_k = n = a_1 + \cdots + a_k$$

along with the fact that $b_i \leq a_i$ for each i . But if $b_i < a_i$ then we would have that

$$b_1 + \cdots + b_k < a_1 + \cdots + a_k$$

which is a contradiction. Thus $b_i = a_i$ for all i . Hence T is diagonalisable.

Now suppose that T is diagonalisable. There exists a basis of eigenvectors, i.e. every vector in V can be written as the sum of eigenvectors, thus

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}.$$

We know that $a_{\lambda_i} = b_{\lambda_i} = \dim E_{\lambda_i}$. Since the characteristic polynomial splits we have that $\dim V = \sum a_{\lambda_i} = \sum b_{\lambda_i} = \sum \dim E_{\lambda_i}$.

6* Let V be a vector space and $\mathcal{A} \subset \text{Hom}(V, V)$ a subset such that every $X \in \mathcal{A}$ is diagonalisable. We say \mathcal{A} is diagonalisable if there exists a basis B of V such that B is an eigenbasis for all $X \in \mathcal{A}$.

- (a) Show that if \mathcal{A} is diagonalisable then for every pair of elements $X, Y \in \mathcal{A}$, we have $X \circ Y = Y \circ X$. (In fact, the other direction is true as well, but a little more difficult to prove.)

Solution: Suppose that \mathcal{A} is diagonalisable. Thus there is a basis B that is an eigenbasis for every $X \in \mathcal{A}$. This means that $[X]_B^B$ is diagonal. So for any two elements $X, Y \in \mathcal{A}$ we have that $[Y \circ X]_B^B = [Y]_B^B [X]_B^B = [X]_B^B [Y]_B^B = [X \circ Y]_B^B$ (since diagonal matrices commute). Hence $X \circ Y = Y \circ X$.

- (b) Give an example of a set \mathcal{A} that is *not* diagonalisable. Every element of \mathcal{A} must be diagonalisable, it must contain at least two elements.

Solution: Let $V = \mathbb{R}^2$ and consider the two linear operators $X(v) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} v$ and $Y(v) = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} v$. These are both diagonalisable (both have two distinct eigenvalues 0 and 1), but an eigenbasis of X must be of the form

$$\left\{ \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ b \end{pmatrix} \right\}$$

for some $a, b \neq 0$, while an eigenbasis of Y must be of the form

$$\left\{ \begin{pmatrix} a \\ 0 \end{pmatrix}, \begin{pmatrix} -b \\ b \end{pmatrix} \right\}$$

For $a, b \neq 0$. Clearly the only choice that makes these coincide is $b = 0$ so the set $\{X, Y\}$ is not diagonalisable.

7* Suppose (A, B) is a pair of linear operators on V . We say $v \in V$ is a (λ, μ) -eigenvector if

$$A(v) = \lambda v \text{ and } B(v) = \mu v$$

Let $E_{(\lambda, \mu)}$ be the set of all (λ, μ) -eigenvectors. We say (A, B) has *simple spectrum* if $\{A, B\}$ is diagonalisable (see previous question) and all the $E_{(\lambda, \mu)}$ are one dimensional (or zero).

Now consider the linear maps $S_{12}, S_{23}, S_{13} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$S_{12} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}, \quad S_{23} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ c \\ b \end{pmatrix}, \quad \text{and } S_{13} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$$

And let $H_1 = uS_{12} + S_{13}$ and $H_2 = (1 - u)S_{12} + S_{23}$ for any $u \in \mathbb{R}$.

- (a) Show that $\{H_1, H_2\}$ is diagonalisable.
- (b) Show that (H_1, H_2) has simple spectrum.