Midterm 2 practice

UCLA: Math 170A, Fall 2017

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Date:
Version: practice

- This exam has 5 questions, for a total of 50 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
Total:	50	

1. Each of the five parts below is worth 2 points.

   For this problem only: Please provide short answers (you can add short explanations if you find it appropriate).

   (a) (2 points) If X is a Binomial random variable with parameters 6 and 0.2 what is $\mathbb{E}(X)$?

   Solution: $\mathbb{E}(X) = 6 \times 0.2 = 1.2$.

   (b) (2 points) Let X be Bernoulli random variable with parameter 0.5 and Y be a Binomial random variable with parameters 3 and 0.5. If X and Y are independent what is the range (set of all possible values) of $X + Y$?

   Solution: X can have values 0 and 1, and Y can have values 0, 1, 2, or 3. Since they are independent any combination is possible, so the range of $X + Y$ are numbers 0, 1, 2, 3, 4.

   (c) (2 points) For random variables X and Y you know that $\text{cov}(X, Y) = 2$, $\mathbb{E}(XY) = 6$ and $\mathbb{E}(X) = 4$. What is $\mathbb{E}(Y)$?

   Solution: From $\text{cov}(X, Y) = \mathbb{E}(XY) - (\mathbb{E}X)(\mathbb{E}Y)$ we get $2 = 6 - 4\mathbb{E}Y$ so $\mathbb{E}Y = 1$.

   (d) (2 points) Let X be a random variable with expectation 0 and variance 1. Compute $\mathbb{E}(X^2 + X)$.

   Solution: Since $\mathbb{E}(X^2) = \text{var}(X) + (\mathbb{E}X)^2 = 1 + 0^2 = 1$ we have $\mathbb{E}(X^2 + X) = \mathbb{E}(X^2) + \mathbb{E}(X) = 1 + 0 = 1$.

   (e) (2 points) If random variables X and Y are independent express $\text{var}(X - Y)$ in terms of $\text{var}(X)$ and $\text{var}(Y)$.

   Solution:

   $$\text{var}(X - Y) = \text{var}(X + (-Y)) = \text{var}(X) + \text{var}(-Y) = \text{var}(X) + \text{var}(Y).$$

2. (10 points) In a certain game the first contestant chooses either number 2 or number 3 with equal probabilities. If the number chosen is $k \in \{2, 3\}$ the second contestant will choose an integer number between (and including) 1 and k uniformly at random. Let X be the number chosen by the first contestant and Y be the number chosen by the second contestant.

   - Find the range of the pair (X, Y) and it's joint probability mass function $p_{X,Y}$.
   - Find $\mathbb{E}(Y)$.

Solution:

- If $X = 2$, then Y can be either 1 or 2, and if $X = 3$ then Y can be either 1 or 2 or 3. So possible values for (X, Y) are
$$(2, 1), (2, 2), (3, 1), (3, 2), (3, 3).$$
If (k, l) is one of these pairs then
$$\mathbb{P}(X = k, Y = l) = \mathbb{P}(X = k)\mathbb{P}(Y = l|X = k) = \frac{1}{2k}.$$
- Given $X = 2$ variable Y is uniform on $\{1, 2\}$ so $\mathbb{E}(Y|X = 2) = 1 \cdot 1/2 + 2 \cdot 1/2 = 3/2$. Given $X = 3$ variable Y is uniform on $\{1, 2, 3\}$ so $\mathbb{E}(Y|X = 3) = 1 \cdot 1/3 + 2 \cdot 1/3 + 3 \cdot 1/3 = 2$. Therefore
$$\mathbb{E}(Y) = \mathbb{P}(X = 2)\mathbb{E}(Y|X = 2) + \mathbb{P}(X = 3)\mathbb{E}(Y|X = 3) = \frac{1}{2} \cdot \frac{3}{2} + \frac{1}{2} \cdot 2 = \frac{7}{4}.$$

3. (10 points) There are $2n$ boxes each of which contains one ball which is either red, blue or green (with equal probabilities). The color of the ball in any box is independent of the colors of balls in other boxes. Let X be the total number of red balls in these $2n$ boxes. Let A be the event there are no green balls in the first n boxes. Let B be the event that last n boxes contain only green balls. Given the event $A \cap B$ find the conditional probability mass function of X.

Solution: If $A \cap B$ happens, red balls can only appear in the first n boxes, so conditioned on $A \cap B$ the number of red balls can only be $0, 1, 2, \ldots, n$. For $0 \leq k \leq n$ we compute

$$\mathbb{P}(X = k | A \cap B) = \frac{\mathbb{P}(\{X = k\} \cap A \cap B)}{\mathbb{P}(A \cap B)}.$$

The event $\{X = k\} \cap A \cap B$ if the event that exactly k of the first n boxes contain a red ball, other $n - k$ of the first n boxes contain blue balls and the last n boxes contain green ball. The probability of this is

$$\mathbb{P}(\{X = k\} \cap A \cap B) = \binom{n}{k}(1/3)^k(1/3)^{n-k}(1/3)^n = \binom{n}{k}(1/3)^{2n}.$$

Event $A \cap B$ is the event that first n boxes all have either red or blue ball and the last n boxes have green balls. The probability that any of the first n boxes contains either red or blue ball is $2/3$ so $\mathbb{P}(A \cap B) = (2/3)^n(1/3)^n = 2^n/3^{2n}$.

Then

$$\mathbb{P}(X = k | A \cap B) = \frac{\binom{n}{k}(1/3)^{2n}}{2^n/3^{2n}} = \binom{n}{k}\frac{1}{2^n}.$$

Alternatively, you can observe that red balls can only appear in the first n boxes. Given that none of these boxes contain green balls, the probability that each box contains red is $1/2$. By independence, the number of red balls is then Binomial with parameters n and $1/2$.

4. (10 points) Let X be a Poisson random variable with parameter 1. Let Y be a Geometric random variable with parameter $1/2$. If X and Y are independent compute the probability $\mathbb{P}(X = Y)$. Simplify your answer.

Solution:

$$\mathbb{P}(X=Y) = \sum_{k=1}^{\infty} \mathbb{P}(X=k, Y=k) = \sum_{k=1}^{\infty} \mathbb{P}(X=k)\mathbb{P}(Y=k) = \sum_{k=1}^{\infty} \frac{e^{-1}}{k!}\frac{1}{2^k}$$
$$= e^{-1}\sum_{k=1}^{\infty} \frac{2^{-k}}{k!} = e^{-1}(e^{1/2}-1) = e^{-1/2} - e^{-1}.$$

The sum above starts from 1 not from 0, since geometric random variable can only have values $1, 2, \ldots$. Then we have to be careful in removing the first term of the Taylor expansion in the last step.

5. (10 points) There are n couples coming to your dinner. These $2n$ people will be seated around a circular table which has exactly $2n$ seats in a uniformly random order. A couple will be happy if the partners are sitting next to each other or if there is exactly one person sitting between them.

   - What is the probability that the first couple is happy?
   - Compute the expected number of happy couples.

Solution:

- The first partner in the couple can sit in any of the $2n$ positions. But then there are only 4 positions for the 2nd partner. So they can be seated in $8n$ ways that they are happy. In total they can be seated in $2n(2n-1)$ ways. So the probability they are happy is

$$\frac{8n}{2n(2n-1)} = \frac{4}{2n-1}.$$

- Let X_i be the indicator variable of the event that i-th couple is happy. Then $\mathbb{E}(X_i) = \mathbb{P}(X_i = 1) = 4/(2n-1)$. The total number of happy couples is

$$X = X_1 + X_2 + \cdots + X_n$$

and the expectation

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_n) = \frac{4n}{2n-1}.$$

You can observe that this works for $n \geq 3$. For $n = 1, 2$ all the couple will be happy, so the answers for the first part are 1 and for the second part 1 (for $n = 1$) and 2 (for $n = 2$).

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