

This week you will get practice drawing and understanding slope fields, making qualitative statements about solutions using them and some practice applying Euler's method.

\*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.4.33) Consider the differential equation

$$\frac{dy}{dt} = \frac{1}{t}$$

- (a) verify that  $y(t) = \ln t$  is a solution to this differential equation satisfying  $y(1) = 0$ .  
 (b) Use Euler's method to approximate  $y(2) = \ln 2$  with  $h = 0.5$ .

2. (6.5) Draw phase lines, classify the equilibria, and sketch a solution satisfying the specified initial value for the equations in the following.

- (a) (6.5-2)  $\frac{dy}{dt} = 2 - 3y$ ,  $y(0) = 2$   
 (b) (6.5-5)  $\frac{dy}{dt} = y(y - 10)(20 - y)$ ,  $y(0) = 9$   
 (c) (6.5-6)  $\frac{dy}{dt} = y(y - 5)(25 - y)$ ,  $y(0) = 7$   
 (d) (6.5-7)  $\frac{dy}{dt} = \sin y$ ,  $y(0) = 0.1$   
 (e) (6.5-10)  $\frac{dy}{dt} = y^3 - 4y$ ,  $y(0) = 0.1$

3. (6.5-33) To account for the effect of a generalist predator (with a type II functional response) on a population, ecologists often write differential equations of the form

$$\frac{dN}{dt} = 0.1N \left( 1 - \frac{N}{1,000} \right) - \frac{10N}{1 + N}$$

- (a) Sketch the phase line for this system.  
 (b) Discuss how the fate of the population depends on its initial abundance.

*Hint: don't worry about what the first sentence means, you don't need to know where the differential equation comes from.*

4. (6.5-39) Consider a population of clonally reproducing individuals consisting of two genotypes,  $a$  and  $A$ , with per capita growth rates,  $r_a$  and  $r_A$ , respectively. If  $N_a$  and  $N_A$  denote the densities of genotypes  $a$  and  $A$ , then

$$\frac{dN_a}{dt} = r_a N_a \quad \frac{dN_A}{dt} = r_A N_A$$

Also, let  $y = \frac{N_a}{N_a + N_A}$  be the fraction of individuals in the population that are genotype  $a$ . Show that  $y$  satisfies

$$\frac{dy}{dt} = (r_a - r_A)y(1 - y)$$

5. (6.5-40) In the Hawk-Dove replicator equation

$$\frac{dy}{dt} = \frac{y}{2}(1 - y)(C(1 - y) - V)$$

if the value  $V > 0$  is specified, then find the range of values of  $C$  (in terms of  $V$ ) that will ensure a polymorphism exists (i.e., find conditions that ensure the existence of an equilibrium  $0 < y^* < 1$  that is stable).

*(Hint: you do not need to know anything about the Hawk-Dove Replicator - though it is very interesting! - all you need to know is that  $V$  is a constant and  $C$  is a parameter. A polymorphism is a stable equilibrium between zero and one.)*

6. (6.5-41) Production of pigments or other protein products of a cell may depend on the activation of a gene. Suppose a gene is *autocatalytic* and produces a protein whose presence activates greater production of that protein. Let  $y$  denote the amount of the protein (say, micrograms) in the cell. A basic model for the rate of this self-activation as a function of  $y$  is

$$A(y) = \frac{ay^b}{k^b + y^b} \text{ micrograms/minute}$$

where  $a$  represents the maximal rate of protein production,  $k > 0$  is a “half saturation” constant, and  $b \geq 1$  corresponds to the number of protein molecules required to active the gene. On the other hand, proteins in the cell are likely to degrade at a rate proportional to  $y$ , say  $cy$ . Putting these two components together, we get the following differential equation model of the protein concentration dynamics:

$$\frac{dy}{dt} = \frac{ay^b}{k^b + y^b} - cy$$

- (a) Verify that  $\lim_{y \rightarrow \infty} A(y) = a$  and  $A(k) = a/2$ .
- (b) Verify that  $y = 0$  is an equilibrium for this model and determine under what conditions it is stable. (*Hint: the definition of autocatalytic is given in the question, it is a gene that produces a protein whose presence activate greater production of that protein.*)
7. (6.5-42) Consider the model of an autocatalytic gene in Problem 41 with  $b = 1$ ,  $k > 0$ ,  $a > 0$ , and  $c > 0$ .
- (a) Sketch the phase line for this model when  $ck > a$ .
- (b) Sketch the phase line for this model when  $ck < a$ .