This problem set concentrates on practice with antiderivatives. You will get lots of practice finding simple antiderivatives as well as finding antiderivatives graphically using slope fields. You will also get some practice with problems involving accumulated change (i.e. the area under curves) and definite integrals.

1. (4.3-33) The Statue of Liberty is 92 meters high, including the 46 meter pedestal upon which it stands. How far from the base should an individual stand to ensure that the view angle, θ, is maximized?

   (Hint: See the textbook for a picture.)

2. (4.3-35) An oil spill has fouled 200 miles of Pacific shoreline. The oil company responsible has been given fourteen days to clean up the shoreline, after which a fine will be levied in the amount of $\$10,000$/day. The local cleanup crew can scrub five miles of beach per day at a cost of \$500/day. Additional crews can be brought in at a cost of $\$18,000$, plus \$800/day for each crew. Determine how many additional crews should be brought in to minimize the total cost to the company and how much the cleanup will cost.

3. (4.3-33) Consider a cylindrical cell with radius r and height $r/2$. Assume that the cell gains energy at a rate proportional to its surface area (i.e., nutrients diffusing in from outside of the cell) and that the cell loses energy at a rate proportional to its volume (i.e., all parts of the cell are using energy). If the cell is trying to maximize its net gain of energy, determine the optimal value of r. Note: Your final expression will depend on your proportionality constants.

4. (5.1) Find the general antiderivative of the functions
   (a) (5.1-2) $f(x) = 4$
   (b) (5.1-8) $f(t) = 4t + 4t^2$
   (c) (5.2-10) $f(x) = \frac{1}{2x}$
   (d) (5.1-14) $f(x) = 4\sin(5x)$
   (e) (5.1-16) $f(x) = 14e^x$
   (f) (5.1-22) $f(u) = 6u + 3\cos u$

5. Find the antiderivatives of the following functions
   (a) $f(x) = 2x + 4$
   (b) $p(z) = 5 - 4z$
   (c) $q(x) = x^2 + 1$
   (d) $f(t) = t^3 - 3t^2 + 2$
   (e) $f(x) = x - 3\cos x$
   (f) $q(c) = \sin c - \cos c$
   (g) $g(u) = e^u - \frac{1}{u}$ (for $x > 0$)
   (h) $f(x) = \frac{1}{2x-4}$ (for $x > 2$)

6. Find the antiderivatives of the following functions. These are trickier and may require some reversing of the chain rule.
   (a) $f(x) = 2xe^{x^2}$
   (b) $g(x) = 3x^2\sin(x^3)$
   (c) $f(t) = \frac{t}{t^2+1}$
   (d) $p(u) = \sin(-u)$
   (e) $f(x) = \sin x\,(\cos x)^3$
   (f) $h(z) = z^2(z^3 - 1)^{\frac{13}{21}}$

(g) $f(x) = x\sin(x^2 - x) - \frac{1}{2}\sin(x^2 - x)$

(h) $f(x) = \frac{x}{\sqrt{x^2+1}}$

7. Sketch the slope field for the following differential equations and thus sketch the antiderivative with the given initial value.

   (a) $\frac{\mathrm{d}f}{\mathrm{d}x} = e^{x^2}$ where $f(0) = 0$

   (b) $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{e^x}{x}$ where $f(-1) = 1$

   (c) $\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{x}{1+e^x}$ where $f(0) = 1$

   (d) $\frac{\mathrm{d}f}{\mathrm{d}x} = \sqrt{2 + \sin x}$ where $f(2) = 1$

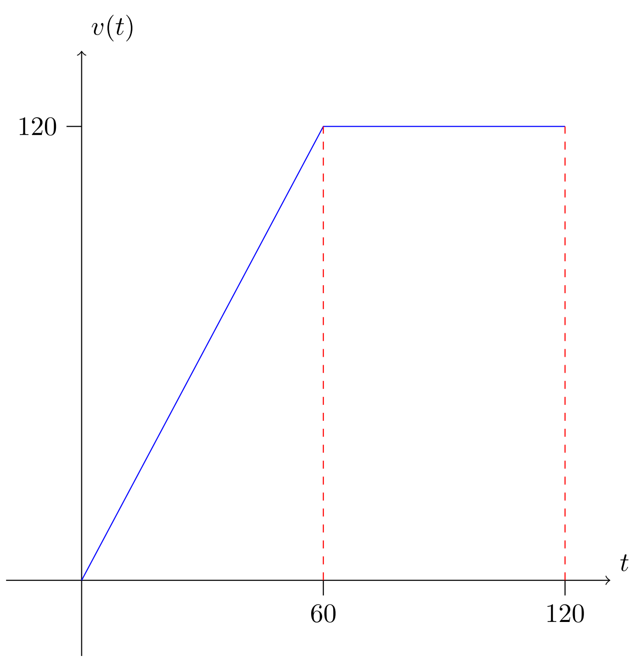
   (e) $\frac{\mathrm{d}f}{\mathrm{d}x} = \sqrt{x}\sin x$ where $f(1) = 0$

8. A rocket starts from a stationary position and accelerates at a rate of $2m/s^2$ for 60 seconds and then travels at a constant speed (i.e. its acceleration is zero) for another 60 seconds. How far has the rocket travelled (assuming it is always moving in a straight line).

Solution: The velocity of the rocket is given by

$$v(t) = \begin{cases} 2t & \text{if } 0 \le t \le 60, \\ 120 & \text{if } 60 \le t \le 120. \end{cases}$$

The graph of the function is



We are asked to find the distance the rocket has travelled which is the accumulated change in the velocity of the 120 seconds the rocket has been airborne. Thus we need to find the area under the graph. We have split this into a triangle and a rectangle, for which the areas are 3600 and 7200 respectively. Thus the total distance travelled is 10800 meters.

9. Rainwater fills a tank at a rate of 4 gallons per hour. If it rains for 6 hours how much rain does the tank collect?

> **Solution:** Rain is being collected for 6 hours and every hour the tank collects 4 gallons so the total rain collected is $4 \cdot 6 = 24$ gallons.

10. A hydroelectric dam produces 2 Megawatts of power for every million litres of water which flows through it. Over a period of 4 hours, you measure $r(t)$ million litres per hour flowing through the turbines. If

$$r(t) = \sqrt{4-(x-2)^2} + 5,$$

how many Megawatts were produced in those 4 hours?

> **Solution:**
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> The graph of the rate of flow is
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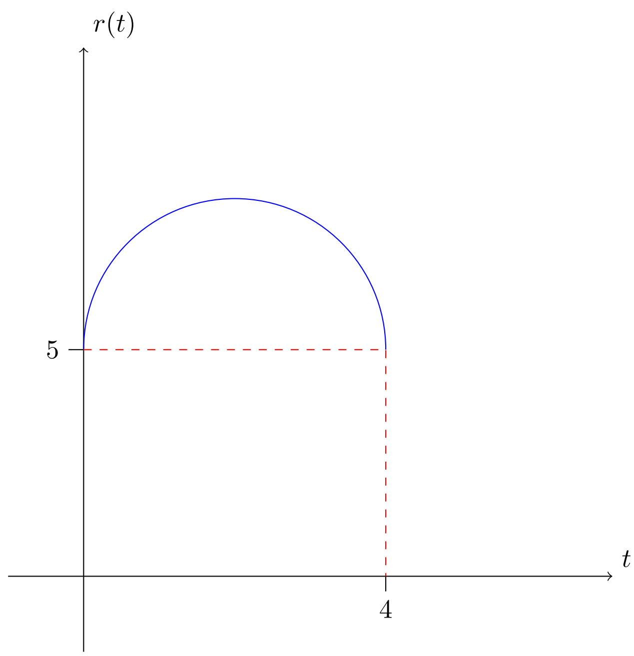
> 
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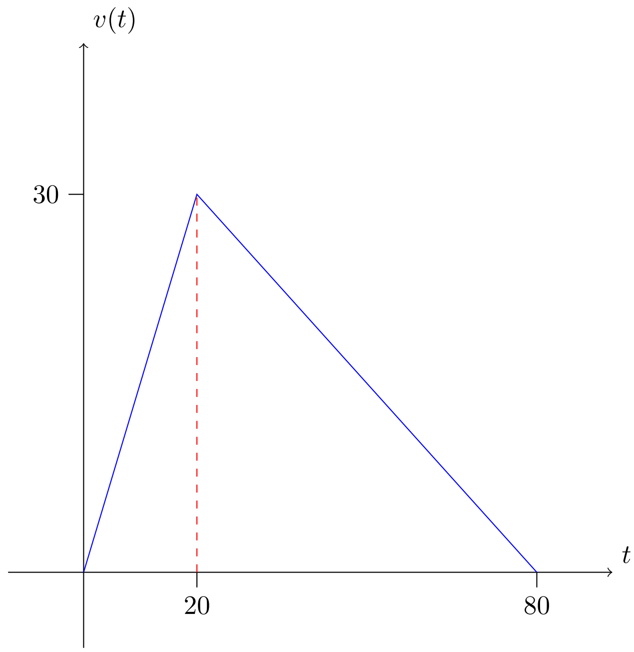
> It is a semicircle with radius 2 centred at $(2, 5)$. Thus we split the area under the graph into a rectangle with area 20 and a semicircle with area 2π. So the total amount of water that flowed through in the 4 hours is $20 + 2\pi$ million litres. Thus the total number of Megawatts produced is twice this number, i.e. $40 + 4\pi$ megawatts.

11. A car starts at a stationary position and accelerates at $1.5m/s^2$ for 20 seconds and then decelerates at $0.5m/s^2$ until coming to a standstill. How far did the car travel?

Solution: The car starts from a stationary position so if $v(t)$ is the velocity of the car at time t then $v(0) = 0$. Thus the velocity of the car is given by

$$v(t) = \begin{cases} 1.5t & \text{if } 0 \leq t \leq 20, \\ 30 - 0.5(t-20) & \text{if } t \geq 20. \end{cases}$$

Thus the graph of the velocity is



The car comes to a standstill when the velocity is zero again so we need to solve the equation

$$\begin{aligned} 30 - 0.5(t-20) &= 0 \\ 30 - 0.5t + 10 &= \\ 40 &= 0.5t \\ t &= 80. \end{aligned}$$

We can divide the area under the graph into two triangles, the left with area $0.5 \cdot 20 \cdot 30 = 300$ and the triangle on the right with area $0.5 \cdot 60 \cdot 30 = 900$ so the total distance travelled is $300 + 900 = 1200$ meters.

12. Rain adds 5 litres of water per minute to your rain tank. You check the rain tank and find that it currently has 200 litres of water in it. After 20 minutes you notice that the tank has had a hole in the bottom this whole time! Water is able to flow out of the hole at a rate of 10 litres per minute.

    You immediately begin repairing the hole. During this time, water is able to drain out at a rate of $10 - \frac{1}{3}t$ litres per minute, where t is the time elapsed since you begun the repair. It takes you 30 minutes.

    How much water is in the tank, 1 hour after you checked the level of the tank?

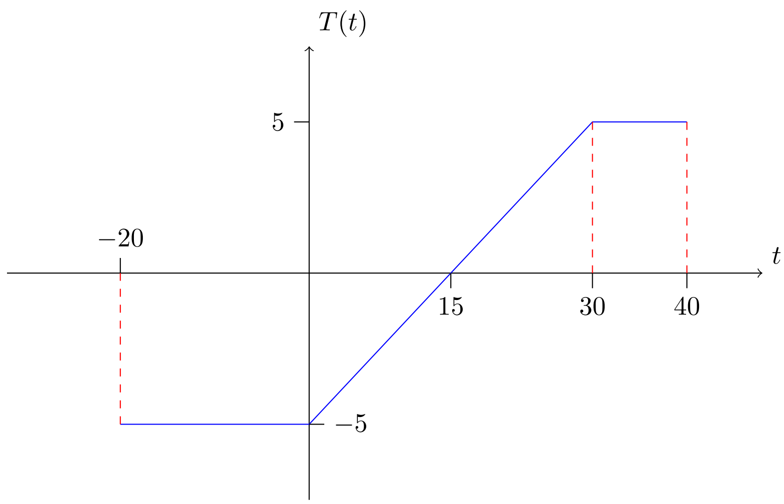
Solution: Lets say that you start fixing the hole at $t = 0$. That means you first checked the tank level at $t = -20$ and one hour after this would be $t = 40$. The rate of water coming into the tank due to rain is constant and is given by $I(t) = 5$. The rate of water flowing out of the tank due to the hole is given by

$$O(t) = \begin{cases} 10 & \text{if } -20 \le t \le 0 \\ 10 - \frac{1}{3}t & \text{if } 0 \le t \le 30 \\ 0 & \text{if } 30 \le t \le 40. \end{cases}$$

The total water flowing in is therefore $T(t) = I(t) - O(t)$ which is

$$T(t) = \begin{cases} -5 & \text{if } -20 \le t \le 0 \\ -5 + \frac{1}{3}t & \text{if } 0 \le t \le 30 \\ 5 & \text{if } 30 \le t \le 40. \end{cases}$$

A graph of this is given below.



Thus we can split the area under the graph up into four regions. Each of these areas is easy to calculate and remembering that the area under the graph is negative, the total area is

$$-100 - \frac{75}{2} + \frac{75}{2} + 50 = -50$$

This represents the total change in the amount of water in the tank in litres. Since we started with 200 litres there must now be 150 litres left.

13. Evaluate the definite integrals (hint: it might be best to do questions 1 and 2 first)

(a) $\displaystyle\int_1^2 2x + 4 \,\mathrm{d}x$

(b) $\displaystyle\int_0^{10} 5 - 4z \,\mathrm{d}z$

(c) $\displaystyle\int_{-1}^1 x^2 + 1 \,\mathrm{d}x$

(d) $\displaystyle\int_{-1}^{0} t^3 - 3t^2 + 2 \, \mathrm{d}t$

(e) $\displaystyle\int_{-\pi}^{2\pi} x - 3\cos x \, \mathrm{d}x$

(f) $\displaystyle\int_{0}^{\pi} \sin c - \cos c \, \mathrm{d}c$

(g) $\displaystyle\int_{1}^{3} e^u - \frac{1}{u} \, \mathrm{d}u$

(h) $\displaystyle\int_{-1}^{1} \frac{1}{2x-4} \, \mathrm{d}x$

(i) $\displaystyle\int_{0}^{2} 2xe^{x^2} \, \mathrm{d}x$

(j) $\displaystyle\int_{0}^{\sqrt[3]{\pi}} 3x^2 \sin(x^3) \, \mathrm{d}x$

(k) $\displaystyle\int_{-1}^{1} \frac{t}{t^2+1} \, \mathrm{d}t$

(l) $\displaystyle\int_{-\pi}^{0} \sin(-u) \, \mathrm{d}u$

(m) $\displaystyle\int_{\pi/2}^{\pi} \sin x \, (\cos x)^3 \, \mathrm{d}x$

(n) $\displaystyle\int_{1}^{2} z^2 (z^3 - 1)^{\frac{13}{21}} \, \mathrm{d}z$

(o) $\displaystyle\int_{4}^{5} x\sin(x^2 - x) - \frac{1}{2}\sin(x^2 - x) \, \mathrm{d}x$

(p) $\displaystyle\int_{0}^{1} \frac{x}{\sqrt{x^2+1}} \, \mathrm{d}x$