This weeks problem set provides some review questions in the lead up to the second midterm. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 3: due Tuesday 26 May: questions 3 and 6 below.

1. From section 5.2, problems 1, $3a, d, e$, 8, 9, 10, 11, 18^*, 19, $20^\dagger$.

2. From section 6.1, problems 1, 2, 3, 4, 8^*, 9, 12, 16, 17^*, 23, 29.

3. We say that two linear operators S and T *commute* if $S \circ T = T \circ S$. Let $T : V \longrightarrow V$ be a diagonalisable linear operator. Define
$$C(T) = \{S \in \mathrm{Hom}(V, V) \mid S \text{ and } T \text{ commute}\}.$$
   (a) If T has $n = \dim V$ distinct eigenvalues, show that any $S \in C(T)$ is diagonalisable.
   (b) Describe explicitly $C(T)$ in the case $T = x\frac{d}{dx} : \mathbb{C}_1[x] \longrightarrow \mathbb{C}_1[x]$.
   (c) Show that part (a) does not necessarily hold if T does not have n distinct eigenvalues.

4. Suppose U, W are subspaces of a finite dimensional vector space V and that $U + W = V$. Show that $U \oplus W = V$ if and only if $\dim U + \dim W = \dim V$.

   The previous question, motivates the following definition.

   Definition: If U_i, for $1 \le i \le k$, are subspaces of a vector space V, then we say $V = U_1 \oplus U_2 \ldots \oplus U_k$ if $V = U_1 + U_2 + \ldots + U_k$, i.e. every vector $v \in V$ can be written as a sum $v = \sum_{i=1}^k u_i$ with $u \in U_i$, and $\dim V = \sum_{i=1}^k \dim U_i$.

5.* Suppose that V is a finite dimensional vector space over $\mathbb{F}$ and $T : V \longrightarrow V$ is a linear operator, with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that
$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$$
   if and only if T is diagonalisable.

6.* Let V be a vector space and $\mathcal{A} \subset \mathrm{Hom}(V, V)$ a subset such that every $X \in \mathcal{A}$ is diagonalisable. We say $\mathcal{A}$ is diagonalisable if there exists a basis B of V such that B is an eigenbasis for all $X \in \mathcal{A}$.
   (a) Show that $\mathcal{A}$ is diagonalisable if and only if for every pair of elements $X, Y \in \mathcal{A}$, we have $X \circ Y = Y \circ X$.
   (b) Give an example of a set $\mathcal{A}$ that is *not* diagonalisable. Every element of $\mathcal{A}$ must be diagonalisable, it must contain at least two elements.

7.* Suppose (A, B) is a pair of linear operators on V. We say $v \in V$ is a (λ, μ)-eigenvector if
$$A(v) = \lambda v \text{ and } B(v) = \mu v$$
   Let $E_{(\lambda,\mu)}$ be the set of all (λ, μ)-eigenvectors. We say (A, B) has *simple spectrum* if $\{A, B\}$ is diagonalisable (see previous question) and all the $E_{(\lambda,\mu)}$ are one dimensional (or zero).

   Now consider the linear maps $S_{12}, S_{23}, S_{13} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ given by
$$S_{12}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} b \\ a \\ c \end{pmatrix}, \quad S_{23}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ c \\ b \end{pmatrix}, \text{ and } S_{13}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}$$
   And let $H_1 = uS_{12} + S_{13}$ and $H_2 = (1 - u)S_{12} + S_{23}$ for any $u \in \mathbb{R}$.
   (a) Show that $\{H_1, H_2\}$ is diagonalisable.
   (b) Show that (H_1, H_2) has simple spectrum.