This weeks problem set provides practice with diagonalisable operators and the basic properties of inner products. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 5: is optional and due Saturday March 14: questions 22 from Section 6.2 (see 3 below) and question 2 below.

1. From section 6.2, problems 1, 2b, g, i, k, 5*, 6, 7, 9, 13*, 17*, 22.

2. Let V be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

   (a) Fix $y \in V$ and suppose $\langle x, y \rangle = 0$ for all $x \in V$. Show that $y = 0$.

   > **Solution:** Let $B = \{v_1, \ldots, v_n\}$ be an orthonormal basis for V. Then
   >
   > $$y = \sum_{i=1}^{n} \langle y, v_i \rangle v_i = 0$$
   >
   > since $\langle y, v_i \rangle = 0$ for all i.

   (b) Let $T : V \longrightarrow V$ be a linear map such that $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all pairs $x, y \in V$ (we call such a map an *isometry*). Prove that T is an isomorphism.

   > **Solution:** First we show that T is injective. Suppose that $T(x) = T(y)$. One one hand we have
   >
   > $$\|T(x)\| = \langle T(x), T(x) \rangle = \langle x, x \rangle.$$
   >
   > On the other hand,
   >
   > $$\|T(x)\| = \langle T(x), T(x) \rangle = \langle T(x), T(y) \rangle = \langle x, y \rangle.$$
   >
   > Thus $\langle x, x \rangle = \langle x, y \rangle$, i.e. $\langle 0, x - y \rangle = 0$. Thus by the above, $x - y = 0$ or $x = y$.
   > Now, since T is an injective map from V to V, it must be surjective by the dimension theorem. Thus it is an isomorphism.

   (c) † Find all isometries $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that have $\det T = 1$.

   > **Solution:** The map T will be given by a matrix
   >
   > $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
   >
   > From the fact that it is an isometry we see that $\|T(e_i)\| = \|e_i\| = 1$ for $i = 1, 2$. Thus
   >
   > $$\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} a \\ c \end{pmatrix} \right\| = a^2 + c^2 = 1$$
   >
   > and similarly $b^2 + d^2 = 1$. This means, both columns of the matrix are points on the unit circle. I.e. for some choice of $\theta, \psi \in [0, 2\pi)$ then we have that
   >
   > $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\psi \\ \sin\theta & \sin\psi \end{pmatrix}$$

Additionally we have that $\langle T(e_1), T(e_2) \rangle = \langle e_1, e_2 \rangle = 0$. I.e the two columns of the matrix are at right angles to each other, so $\psi = \theta \pm \pi/2$ (modulo 2π). Alternatively this can be seen since

$$0 = \langle T(e_1), T(e_2) \rangle = \langle \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \rangle = \cos\theta\cos\psi + \sin\theta\sin\phi = \cos(\theta - \psi)$$

Which means that $\theta - \psi = \pm\frac{\pi}{2} + 2n\pi$ for $n \in \mathbb{Z}$.

The condition that the determinant is 1 is that $ad - bc = 1$ which translates to

$$1 = \cos\theta\sin\psi - \cos\psi\sin\theta = \sin(\theta - \psi)$$

hence $\theta - \psi = \pi/2$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\theta + \pi/2 \\ \sin\theta & \sin\theta + \pi/2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some choice of $\theta \in [0, 2\pi)$.

3. (22 from 6.2) Let $V = \mathcal{C}([0,1], \mathbb{R})$ be the space of real valued, continuous functions on the interval $[0,1]$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$. Let W be the subspace spanned by the linearly independent set $\{t, \sqrt{t}\}$.

(a) Find an orthonormal basis for W.

(b) Let $h(t) = t^2$. Use the orthonormal basis obtained in (a) to obtain the "best" (closest) approximation of h in W.

Solution: *Solution to 22 from 6.2.* The question asks us to consider the vector space $\mathcal{C}([0,1], \mathbb{R})$ of continuous functions on $[0,1]$ into $\mathbb{R}$ with inner product, $\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt$, and to use the Gram Schmidt process to find an orthonormal basis for the subspace $\text{span}\{t, \sqrt{t}\}$.

Part a). We use the Gram-Schmidt process to define first an orthogonal basis $\{f_1, f_2\}$. Set $f_1 = t$. Then

$$f_2 = \sqrt{t} - \frac{\langle \sqrt{t}, t \rangle}{||t||^2} t.$$

To get an explicit expression we do some calculations.

$$||t||^2 = \int_0^1 t^2\, dt = \frac{1}{3}.$$

$$\langle \sqrt{t}, t \rangle = \int_0^1 t^{3/2}\, dt = \frac{2}{5}.$$

$$||\sqrt{t}||^2 = \int_0^1 t\, dt = \frac{1}{2}.$$

Putting this together we get

$$f_2 = \sqrt{t} - \frac{6}{5}t.$$

To get an orthonormal basis, we need to normalise, so we need to calculate

$$||f_2||^2 = \int_0^1 \left(\sqrt{t} - \frac{6}{5}t\right)^2 dt$$

$$= \int_0^1 t - \frac{12}{5}t^{3/2} + \frac{36}{25}t^2 \, dt$$

$$= \frac{1}{2} - \frac{24}{25} + \frac{36}{75} = \frac{1}{50}$$

We already know that $||f_1|| = \frac{1}{\sqrt{3}}$ and now we also know $||f_2|| = \frac{1}{5\sqrt{2}}$ thus, an orthonormal basis is

$$\{g_1 = \sqrt{3}t, g_2 = \sqrt{2}\left(5\sqrt{t} - 6t\right)\}.$$

Part b). We want to project t^2 onto W. The result will be

$$\langle t^2, g_1 \rangle g_1 + \langle t^2, g_2 \rangle g_2.$$

We calculate the coefficients.

$$\langle t^2, g_1 \rangle = \int_0^1 \sqrt{3}t^3 \, dt = \frac{\sqrt{3}}{4}.$$

$$\langle t^2, g_2 \rangle = \int_0^1 \sqrt{2}\left(5t^{5/2} - 6t^3\right) dt$$

$$= \sqrt{2}\left(\frac{10}{7} - \frac{3}{2}\right)$$

$$= -\frac{\sqrt{2}}{14}.$$

Thus, the best approximation is

$$\frac{3}{4}t - \frac{1}{7}\left(5\sqrt{t} - 6t\right) = \frac{45}{28}t - \frac{5}{7}\sqrt{t}.$$