This weeks problem set provides some review questions in the lead up to the second midterm. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 4: due Monday 2 February: questions 3 and 5 below.

1. From section 5.2, problems 1, $3a, d, e$, 8, 9, 10, 11, 18^*, 19, $20^\dagger$.

2. From section 6.1, problems 1, 2, 3, 4, 8^*, 9, 12, 16, 17^*, 23, 29.

3. Let $T : V \longrightarrow V$ be a diagonalisable linear operator. Let $C(T) \subseteq \mathrm{Hom}(V,V)$ be the set of all linear maps that commute with T. I.e

$$C(T) = \{S \in \mathrm{Hom}(V,V) \mid S \circ T = T \circ S\}.$$

   (a) If T has $n = \dim V$ distinct eigenvalues, show that any $S \in C(T)$ is diagonalisable.

   (b) Describe explicitly $C(T)$ in the case $T = x\frac{d}{dx} : \mathbb{C}_1[x] \longrightarrow \mathbb{C}_1[x]$.

   (c) Show that part (a) does not necessarily hold if T does not have n distinct eigenvalues.

4. Suppose U, W are subspaces of a finite dimensional vector space V and that $U + W = V$. Show that $U \oplus W = V$ if and only if $\dim U + \dim W = \dim V$.

   The previous question, motivates the following definition.

   Definition: If U_i, for $1 \leq i \leq k$, are subspaces of a vector space V, then we say $V = U_1 \oplus U_2 \ldots \oplus U_k$ if $V = U_1 + U_2 + \ldots + U_k$, i.e. every vector $v \in V$ can be written as a sum $v = \sum_{i=1}^k u_i$ with $u \in U_i$, and $\dim V = \sum_{i=1}^k \dim U_i$.

5.* Suppose that V is a finite dimensional vector space over $\mathbb{F}$ and $T : V \longrightarrow V$ is a linear operator, with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$. Prove that

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k}$$

   if and only if T is diagonalisable.