

# Final practice 1

## UCLA: Math 115A, Fall 2019

*Instructor:* Noah White

*Date:*

*Version:* practice

- This exam has 8 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: \_\_\_\_\_

ID number: \_\_\_\_\_

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Question 2 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

*Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!*

**Question 2.**

<i>Part</i>	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. In each of the following questions, fill in the blanks to complete the statement of the definition or theorem.

(a) (2 points) *Definition:* Let  $V$  be a vector space. A subset  $B$  is called a basis of  $V$  if  $B$  is

- linearly independent and
- is a spanning set.

(b) (2 points) *Definition:* Let  $V$  and  $W$  be vector spaces over a field  $\mathbb{F}$ . A function  $T : V \rightarrow W$  is a linear map if

- $T(v + w) = T(v) + T(w)$  for every  $v, w \in V$ , and
- $T(\lambda v) = \lambda T(v)$  for every  $\lambda \in \mathbb{F}$  and  $v \in V$ .

(c) (2 points) *Theorem:* Suppose  $\dim V = n$ . If  $T : V \rightarrow V$  is a linear map with  $n$  distinct eigenvalues, then  $T$  is diagonalisable.

(d) (2 points) *Definition:* A basis  $B$  for a vector space  $V$  equipped with an inner product is called orthonormal if

- $\|v\| = 1$  for every  $v \in B$ , and
- $\langle v, w \rangle = 0$  for every  $v, w \in B$ .

(e) (2 points) *Definition:* The Frobenius inner product is an inner product defined on the vector space  $\text{Mat}_{m \times n}(\mathbb{F})$  and is defined by the formula

$$\langle A, B \rangle = \text{tr}(B^* A).$$

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) The subset

$$V_\lambda = \{ p \in \mathbb{R}[x] \mid p'(\lambda) = 0 \}$$

is a subspace

**A. for any choice of  $\lambda \in \mathbb{R}$ .**

B. only for  $\lambda = 0$ .

C. whenever  $\lambda > 0$ .

D. for no choice of  $\lambda$ .

(b) (2 points) As a subset of  $\text{Mat}_{2 \times 2}(\mathbb{C})$ , the set

$$\left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\}$$

A. is a spanning set.

**B. is linear independent.**

C. is neither spanning nor linearly independent.

D. is a basis.

(c) (2 points) Suppose  $n \geq 2$ . Consider the function  $T : \text{Mat}_{n \times n}(\mathbb{C}) \rightarrow \text{Mat}_{n \times n}(\mathbb{C})$  given by  $T(M) = M^t$ , the transpose matrix. Which of the following is *not* true.

A.  $T$  is a linear map.

**B.  $T^2 = -\text{id}$ .**

C.  $T$  has eigenvalues 1 and  $-1$ .

D.  $T$  has an eigenspace of dimension  $\frac{1}{2}n(n+1)$ .

(d) (2 points) Consider again, the map  $T$  given above. Which of the following is true.

A.  $T$  is diagonalisable, only when  $n = 2$ .

**B.  $T$  is diagonalisable for any  $n$ .**

C.  $T$  is never diagonalisable.

D. The characteristic polynomial for  $T$  does not split.

(e) (2 points) Which of the following pairs of matrices are orthogonal in  $\text{Mat}_{2 \times 2}(\mathbb{C})$  equipped with the Frobenius inner product?

A.  $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$

**B.  $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$**

C.  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$

D.  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

3. Consider the vector space  $V = P_2(\mathbb{R})$  with its standard ordered basis

$$\beta = \{1, x, x^2\}$$

and the linear maps

$$T : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R}), \quad T(f) = f(1) + f(-1)x + f(0)x^2$$

$$S : P_2(\mathbb{R}) \rightarrow P_2(\mathbb{R}), \quad S(ax^2 + bx + c) = cx^2 + bx + a.$$

(a) (2 points) What is  $[T]_\beta$  and  $[S]_\beta$ ? Show that

$$[TS]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (6 points) Compute  $[(TS)^{-1}]_\beta$ .

(c) (2 points) What is  $(TS)^{-1}(x^2 + x + 1)$ ?

**Solution:**

(a) We have

$$T(1) = 1 + x + x^2$$

$$T(x) = 1 - x$$

$$T(x^2) = 1 + x$$

Hence

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Also clearly

$$[S]_\beta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using  $[TS]_\beta = [T]_\beta[S]_\beta$  we get

$$[TS]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) We have  $[(TS)^{-1}]_\beta = [TS]_\beta^{-1}$ . We hence have to invert  $[TS]_\beta$

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Subtract 2nd from 1st

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Norm 2nd

$$\left[ \begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Subtract 2nd from 1st

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Subtract 3rd from 1st

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right]$$

Hence

$$[(TS)^{-1}]_{\beta} = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) We know

$$\begin{aligned} [(TS)^{-1}(x^2 + x + 1)]_{\beta} &= [(TS)^{-1}]_{\beta}[(x^2 + x + 1)]_{\beta} \\ &= \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

These are the coordinates of  $x^2 = (TS)^{-1}(x^2 + x + 1)$ .

4. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \in M_{3 \times 3}(\mathbb{R}),$$

- (a) (2 points) Compute the characteristic polynomial of  $A$  and determine the eigenvalues and their algebraic multiplicity.
- (b) (6 points) Is  $A$  diagonalizable? If yes, compute a basis  $\beta$  of eigenvectors of  $A$ .
- (c) (2 points) Compute  $[L_A]_\beta$ , where the  $L_A$  is the linear transformation given by

$$L_A : \mathbb{R}^3 \rightarrow \mathbb{R}^3, v \mapsto Av.$$

**Solution:**

- (a) We compute

$$\det(A - tI_3) = \det \begin{bmatrix} -t & 1 & -2 \\ 1 & -t & -2 \\ 0 & 0 & -1-t \end{bmatrix} = (t^2 - 1)(-1 - t) = -(t + 1)^2(t - 1)$$

- (b) The eigenvalues are the roots of the characteristic polynomial and hence  $\lambda = 1, -1$  with multiplicity 1 and 2 respectively.
- (c) We compute

$$N(A - 1I_3) = \text{Span}((1, 1, 0)^t)$$

$$N(A - (-1)I_3) = \text{Span}((-1, 1, 0)^t, (2, 0, 1)^t)$$

with our favorite algorithm (Wolfram Alpha). Hence

$$\beta = \{(1, 1, 0)^t, (-1, 1, 0)^t, (2, 0, 1)^t\}$$

is a basis of eigenvectors.

- (d) A diagonal matrix with entries  $1, -1, -1$  in an appropriate order.

5. Consider the vector space  $V = \mathbb{R}^4$  with its standard inner product. Consider the linearly independent subset

$$S = \{w_1 = (1, 0, 1, 0), w_2 = (1, 1, 1, 1), w_3 = (2, 2, 0, 2)\}.$$

- (a) (6 points) Apply the Gram-Schmidt orthogonalization algorithm to  $S$  to compute an orthogonal basis  $\beta'$  of  $\text{Span}(S)$ .  
 (b) (2 points) Use your result to compute an orthonormal basis  $\beta$  of  $\text{Span}(S)$ .  
 (c) (2 points) Let  $x = (1, 2, 3, 2) \in \text{Span}(S)$ . Compute the coordinate vector  $[x]_\beta$ .

**Solution:**

(a)

$$\begin{aligned} v_1 &= w_1 \\ &= (1, 0, 1, 0) \\ v_2 &= w_2 - \frac{\langle w_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 = (1, 1, 1, 1) - \frac{2}{2}(1, 0, 1, 0) \\ &= (0, 1, 0, 1) \\ v_3 &= w_3 - \frac{\langle w_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle w_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 = (2, 2, 0, 2) - \frac{2}{2}(1, 0, 1, 0) - \frac{4}{2}(0, 1, 0, 1) \\ &= (1, 0, -1, 0) \end{aligned}$$

(b)

$$\begin{aligned} u_1 &= \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{2}}(1, 0, 1, 0) \\ u_2 &= \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{2}}(0, 1, 0, 1) \\ u_3 &= \frac{1}{\|v_3\|} v_3 = \frac{1}{\sqrt{2}}(1, 0, -1, 0) \end{aligned}$$

(c)

$$\begin{aligned} \langle x, v_1 \rangle &= \frac{4}{\sqrt{2}} \\ \langle x, v_2 \rangle &= \frac{4}{\sqrt{2}} \\ \langle x, v_3 \rangle &= \frac{-2}{\sqrt{2}} \end{aligned}$$

Hence  $[x]_\beta = \frac{1}{\sqrt{2}}(4, 4, -2)$ .



6. Let  $S : U \rightarrow V$  and  $T : V \rightarrow W$  be linear transformations between finite dimensional vector spaces  $U, V$  and  $W$  over a field  $F$ .
- (a) (2 points) Let  $v_1, v_2, \dots, v_n \in V$  be linearly independent and  $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ , such that  $\lambda_i \neq 0$  for all  $1 \leq i \leq n$ . Show that also  $\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n$  are linearly independent.
- (b) (4 points) Let  $v, w \in V$ . Show that  $\text{span}(v, w) = \text{span}(v, v + w)$ .  
*Hint:* Proceed in two steps: Show that for all  $x \in V$ ,
1.  $x \in \text{span}(v, w)$  implies  $x \in \text{span}(v, v + w)$  and
  2.  $x \in \text{span}(v, v + w)$  implies  $x \in \text{span}(v, w)$ .
- (c) (4 points) Assume that  $R(S) = N(T)$ , i.e. the range of  $S$  is equal to the nullspace of  $T$ . Assume furthermore that  $S$  is one-to-one and  $T$  is onto. Show that

$$\dim V = \dim U + \dim W.$$

*Hint:* Use the rank-nullity formula.

**Solution:**

- (a) Let  $a_1, a_2, \dots, a_n \in F$  such that

$$a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + \dots + a_n \lambda_n v_n = 0.$$

Since  $v_1, v_2, \dots, v_n$  are linearly independent we get

$$a_1 \lambda_1 = a_2 \lambda_2 = \dots = a_n \lambda_n = 0.$$

Since the  $\lambda_i$  are nonzero this implies

$$a_1 = a_2 = \dots = a_n = 0.$$

Hence  $\lambda_1 v_1, \lambda_2 v_2, \dots, \lambda_n v_n$  are linearly independent.

- (b) 1.) Let  $x \in \text{span}(v, w)$ . Hence there are  $a, b \in F$  such that

$$x = av + bw = (a - 1)v + b(v + w).$$

Hence  $x \in \text{span}(v, v + w)$ .

- 2.) Now let  $x \in \text{span}(v, v + w)$ . Hence there are  $a, b \in F$  such that

$$x = av + b(v + w) = (a + 1)v + bw.$$

Hence  $x \in \text{span}(v, w)$ .

Putting 1.) and 2.) together we get  $\text{span}(v, w) = \text{span}(v, v + w)$ .

- (c) Since  $S$  is one-to-on we have

$$\dim(U) = \text{rank}(S).$$

Since  $T$  is onto we have

$$\dim(V) = \text{nullity}(T) + \dim(W).$$

Since  $R(S) = N(T)$  we have

$$\text{nullity}(T) = \text{rank}(S) = \dim(U).$$

Hence we get

$$\dim(V) = \dim(U) + \dim(W).$$

7. Let  $V$  be a finite dimensional vector space over a field  $F$ . Recall that

$$\mathcal{L}(V, V) = \{T : V \rightarrow V \mid T \text{ is a linear transformation}\}$$

denotes the vector space of linear transformations from  $V$  to  $V$  (also called linear operators on  $V$ ). Fix a vector  $v \in V$  and define

$$Z = \{T \in \mathcal{L}(V, V) \mid T(v) = 0\}.$$

One calls  $Z$  the *annihilator* of  $v$  in  $\mathcal{L}(V, V)$ .

- (a) (4 points) Show that  $Z$  is a subspace of  $\mathcal{L}(V, V)$ .
- (b) (2 points) Let  $\lambda \in F$  such that  $\lambda \neq 0$ . Prove or disprove (by finding a counterexample) that

$$Z' = \{T \in \mathcal{L}(V, V) \mid T(v) = \lambda v\}$$

is a subspace of  $\mathcal{L}(V, V)$ .

- (c) (2 points) Assume that  $\beta = \{v_1, \dots, v_n\}$  is an ordered basis of  $V$ , such that  $v_1 = v$ . Let  $T \in \mathcal{L}(V, V)$ . Show that  $T \in Z$  if and only if the first column of  $A = [T]_\beta$  equals 0.
- (d) (2 points) Assuming  $v \neq 0$ , what is  $\dim(Z)$ ?

**Solution:**

- (a) One easily checks

$$\phi_v : \mathcal{L}(V, V) \rightarrow F, T \mapsto T(v)$$

is linear. Hence  $Z = \mathcal{N}(\phi_v)$  is a subspace.

- (b) Let  $V = \mathbb{R}$ ,  $v = 1$ ,  $T_0 : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto 0$  be the zero map and  $\lambda = 1$ . Then  $T_0 \notin Z'$ , since  $T_0(v) = 0 \neq 1 = \lambda v$ . Hence  $Z'$  is not a subspace.
- (c) We know that the first column in  $A$  is given by  $[T(v)]_\beta$ . Assume that  $T \in Z$ , then  $T(v) = 0$  and clearly  $[T(v)]_\beta = 0$ . Assume that the first column of  $A$  equals zero, i.e.  $[T(v)]_\beta = 0$ . Then clearly  $T(v) = 0$ .
- (d) Denote by  $X \subset M_{n,n}(F)$  the subspace of all matrices whose first column is zero. Then clearly  $\dim X = n^2 - n$ . By the last part

$$[-]_\beta : Z \rightarrow X, T \mapsto [T]_\beta$$

is an isomorphism (since  $[-]_\beta : \mathcal{L}(V, V) \rightarrow M_{n,n}(F)$  is). Hence  $\dim Z = \dim X = n^2 - n$ .

8. Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$  with an inner product  $\langle x, y \rangle \in \mathbb{R}$  for  $x, y \in V$ .

(a) (3 points) Let  $\lambda \in \mathbb{R}$  with  $\lambda > 0$ . Show that

$$\langle x, y \rangle' = \lambda \langle x, y \rangle, \text{ for } x, y \in V,$$

defines an inner product on  $V$ .

(b) (2 points) Let  $T : V \rightarrow V$  be a linear operator, such that

$$\langle T(x), T(y) \rangle = \langle x, y \rangle, \text{ for all } x, y \in V.$$

Show that  $T$  is one-to-one.

(c) (2 points) Recall that the *norm* of a vector  $x \in V$  is defined by  $\|x\| = \sqrt{\langle x, x \rangle}$ . Show that

$$\langle x, y \rangle = \frac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2), \text{ for all } x, y \in V.$$

Hence, the inner product can be recovered from the norm.

*Hint:* Rewrite  $\langle x + y, x + y \rangle$  using the properties of inner products. Use that  $\langle x, x \rangle \in \mathbb{R}$  is a real number by assumption.

(d) (3 points) Let  $\beta = \{v_1, \dots, v_n\}$  be a basis of  $V$ . The *Gram matrix*  $G \in M_{n,n}(\mathbb{R})$  of the inner product  $\langle -, - \rangle$  with respect to  $\beta$  is defined by

$$G_{i,j} = \langle v_i, v_j \rangle.$$

Show that  $G$  is invertible.

**Solution:**

(a) Show the 4 properties.

(b) Use non-degeneracy.

(c) Follow hint.

(d) We will show that if  $Ga = 0$  for  $a \in \mathbb{R}^n$  then  $a = 0$ . This means the nullity of  $G$  is zero and thus it is invertible. To see that  $a = 0$  note that there is an  $x \in V$  such that  $[x]_\beta = a = (a_1, \dots, a_n)^t$ . Then

$$\begin{aligned} Ga = G[x]_\beta &= \left( \sum_{j=1}^n G_{1,j} a_j, \dots, \sum_{j=1}^n G_{n,j} a_j \right)^t \\ &= \left( \sum_{j=1}^n \langle v_1, v_j \rangle a_j, \dots, \sum_{j=1}^n \langle v_n, v_j \rangle a_j \right)^t \\ &= \left( \langle v_1, \sum_{j=1}^n a_j v_j \rangle, \dots, \langle v_n, \sum_{j=1}^n a_j v_j \rangle \right)^t \\ &= (\langle v_1, x \rangle, \dots, \langle v_n, x \rangle)^t \end{aligned}$$

Now, since  $G[x]_\beta = 0$  that means that  $\langle v_i, x \rangle = 0$  for all  $i$ , so that means  $x \in V^\perp$ . But  $V^\perp = \{0\}$  so  $x = 0$ , and thus  $G$  is invertible.

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