Math 3B: Lecture 13

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How to factorize polynomials

The normal method for factorizing a polynomial $p(x)$ is to find a root α and then writing

$$p(x) = q(x)(x - \alpha).$$

How to factorize polynomials

The normal method for factorizing a polynomial $p(x)$ is to find a root α and then writing

$$p(x) = q(x)(x - \alpha).$$

What if we want to devide a polynomial $p(x)$ by another polynomial $q(x)$? We want to write

$$p(x) = q(x)d(x) + r(x)$$

for a polynomial $d(x)$ (the divisor) and a remainder $r(x)$.

Long devision

We know how to do this with numbers! Long devision.

$$
\begin{array}{rr}
 & 176 \\
\cline{2-2}
34 \,\big) & 6000 \\
 & \underline{3400} \\
 & 2600 \\
 & \underline{2380} \\
 & 220 \\
 & \underline{204} \\
 & 16
\end{array}
$$

Long devision

We know how to do this with numbers! Long devision.

$$\begin{array}{r}
176\\
34\,\big)\overline{6000}\\
\underline{3400}\\
2600\\
\underline{2380}\\
220\\
\underline{204}\\
16
\end{array}$$

So $6000 = 34 \cdot 176 + 16$

Why?

Lets rewrite the equation $p(x) = q(x)d(x) + r(x)$

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

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$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

E.g.

$$\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 1} = x + \frac{1}{(x+1)^2}.$$

Why?

Lets rewrite the equation $p(x) = q(x)d(x) + r(x)$

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

E.g.

$$\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 1} = x + \frac{1}{(x+1)^2}.$$

The left hand side is difficult to integrate. The right hand side is easy!

How?

$$x+3\overline{\big)\quad x^2+5x+4}$$

So

$$\frac{x^2+5x+4}{x+3}=x+2-\frac{2}{x+3}.$$

}

How?

$$\begin{array}{r} x \\ x+3\overline{\smash{)}\quad x^2+5x+4} \end{array}$$

So

$$\frac{x^2+5x+4}{x+3} = x+2-\frac{2}{x+3}.$$

}

How?

$$\begin{array}{r@{}l} & \phantom{x^2+5{}}x \\ \cline{2-2} x+3\big) & \quad x^2+5x+4 \\ & -x^2-3x \\ \cline{2-2} \end{array}$$

So

$$\frac{x^2+5x+4}{x+3}=x+2-\frac{2}{x+3}.$$

}

How?

$$\begin{array}{r@{}l} & \phantom{x^2+5x+{}} x \\ \cline{2-2} x+3 \big) & \; x^2+5x+4 \\ & -x^2-3x \\ \cline{2-2} & \phantom{x^2+{}}2x+4 \end{array}$$

So

$$\frac{x^2+5x+4}{x+3} = x+2-\frac{2}{x+3}.$$

}

How?

$$\begin{array}{r} x+2 \\ x+3\overline{\smash{)}\;x^2+5x+4} \\ \underline{-x^2-3x\phantom{{}+4}} \\ 2x+4 \end{array}$$

So

$$\frac{x^2+5x+4}{x+3} = x+2-\frac{2}{x+3}.$$

}

How?

$$\begin{array}{rr} & x+2 \\ \cline{2-2} x+3\big) & x^2+5x+4 \\ & -\,x^2-3x \\ \cline{2-2} & 2x+4 \\ & -\,2x-6 \\ \cline{2-2} \end{array}$$

So

$$\frac{x^2+5x+4}{x+3} = x+2-\frac{2}{x+3}.$$

}

How?

$$\begin{array}{rr}
 & x + 2 \\ \cline{2-2}
x + 3 \big) & x^2 + 5x + 4 \\
 & \underline{- x^2 - 3x \phantom{{}+4}} \\
 & 2x + 4 \\
 & \underline{- 2x - 6} \\
 & - 2
\end{array}$$

So

$$\frac{x^2 + 5x + 4}{x + 3} = x + 2 - \frac{2}{x + 3}.$$

}

How?

$$
\begin{array}{rr}
 & x+2 \\
x+3\big) & \overline{\;x^2+5x+4} \\
 & \underline{-x^2-3x} \\
 & 2x+4 \\
 & \underline{-2x-6} \\
 & -2
\end{array}
$$

{ So

$$\frac{x^2+5x+4}{x+3} = x+2-\frac{2}{x+3}.$$

}

Example 1

$$x-3\overline{)\quad x^3-12x^2 \qquad\qquad -42}$$

So

$$\frac{x^3-12x^2-42}{x-3}=x^2-9x-27-\frac{123}{x-3}.$$

}

Example 1

$$\begin{array}{r} x^2 \phantom{{}-12x^2 \qquad\qquad - 42} \\ x-3\overline{\smash{)}\quad x^3 - 12x^2 \qquad\qquad - 42} \end{array}$$

So

$$\frac{x^3 - 12x^2 - 42}{x-3} = x^2 - 9x - 27 - \frac{123}{x-3}.$$

}

Example 1

$$\begin{array}{rllll} & & x^2 & & \\ \cline{2-5} x-3\big) & x^3 & -12x^2 & & -42 \\ & -x^3 & +3x^2 & & \\ \cline{2-3} \end{array}$$

So

$$\frac{x^3-12x^2-42}{x-3} = x^2-9x-27-\frac{123}{x-3}.$$

}

Example 1

$$\begin{array}{r l}
 & x^2 \\ \cline{2-2}
x-3\big) & x^3 - 12x^2 \qquad\qquad - 42 \\
 & -x^3 + 3x^2 \\ \cline{2-2}
 & -9x^2
\end{array}$$

So

$$\frac{x^3 - 12x^2 - 42}{x-3} = x^2 - 9x - 27 - \frac{123}{x-3}.$$

}

Example 1

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & -9x & \\ \cline{2-5}
x-3 & x^3 & -12x^2 & & -42 \\
\multicolumn{1}{r}{} & -x^3 & +3x^2 & & \\ \cline{2-3}
\multicolumn{1}{r}{} & & -9x^2 & &
\end{array}
$$

So

$$\frac{x^3 - 12x^2 - 42}{x-3} = x^2 - 9x - 27 - \frac{123}{x-3}.$$

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Example 1

$$\begin{array}{rrrrr}
 & & x^2 & -9x & \\ \cline{2-5}
x-3\big) & x^3 & -12x^2 & & -42 \\
 & -x^3 & +3x^2 & & \\ \cline{2-3}
 & & -9x^2 & & \\
 & & 9x^2 & -27x & \\ \cline{3-4}
\end{array}$$

So

$$\frac{x^3-12x^2-42}{x-3} = x^2-9x-27-\frac{123}{x-3}.$$

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 & & x^2 & -9x & \\ \cline{2-5}
x-3\big) & x^3 & -12x^2 & & -42 \\
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 & & 9x^2 & -27x & \\ \cline{3-4}
 & & & -27x & -42
\end{array}
$$

So

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\begin{array}{rrrrr}
 & & x^2 & -9x & -27 \\
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x-3\big) & x^3 & -12x^2 & & -42 \\
 & -x^3 & +3x^2 & & \\
\cline{2-3}
 & & -9x^2 & & \\
 & & 9x^2 & -27x & \\
\cline{3-4}
 & & & -27x & -42
\end{array}
$$

So

$$\frac{x^3-12x^2-42}{x-3} = x^2 - 9x - 27 - \frac{123}{x-3}.$$

}

Example 1

$$
\begin{array}{rrrrr}
 & & x^2 & -\,9x & -\,27 \\ \cline{2-5}
x-3\big) & x^3 & -\,12x^2 & & -\,42 \\
 & -\,x^3 & +\,3x^2 & & \\ \cline{2-3}
 & & -\,9x^2 & & \\
 & & 9x^2 & -\,27x & \\ \cline{3-4}
 & & & -\,27x & -\,42 \\
 & & & 27x & -\,81 \\ \cline{4-5}
\end{array}
$$

So

$$\frac{x^3 - 12x^2 - 42}{x-3} = x^2 - 9x - 27 - \frac{123}{x-3}.$$

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Example 1

$$\begin{array}{r rrrr}
 & & x^2 & -9x & -27 \\ \cline{2-5}
x-3\big) & x^3 & -12x^2 & & -42 \\
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 & & -9x^2 & & \\
 & & 9x^2 & -27x & \\ \cline{3-4}
 & & & -27x & -42 \\
 & & & 27x & -81 \\ \cline{4-5}
 & & & & -123
\end{array}$$

So

$$\frac{x^3-12x^2-42}{x-3} = x^2-9x-27-\frac{123}{x-3}.$$

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Example 1

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\begin{array}{rrrrr}
 & & x^2 & -\,9x & -\,27 \\ \cline{2-5}
x-3\big) & x^3 & -\,12x^2 & & -\,42 \\
 & -\,x^3 & +\,3x^2 & & \\ \cline{2-3}
 & & -\,9x^2 & & \\
 & & 9x^2 & -\,27x & \\ \cline{3-4}
 & & & -\,27x & -\,42 \\
 & & & 27x & -\,81 \\ \cline{4-5}
 & & & & -\,123
\end{array}
$$

{ So

$$\frac{x^3-12x^2-42}{x-3}=x^2-9x-27-\frac{123}{x-3}.$$

}

Example 2

$$x^2+1\overline{\smash{)}\quad x^3-x^2+x-1}$$

So

$$\frac{x^3-x^2+x-1}{x^2+1}=x-1.$$

}

Example 2

$$\begin{array}{rr} & x \\ x^2+1 \big) & \overline{\quad x^3 - x^2 + x - 1} \end{array}$$

So

$$\frac{x^3 - x^2 + x - 1}{x^2 + 1} = x - 1.$$

}

Example 2

$$\begin{array}{rllll}
 & & & x & \\ \cline{2-5}
x^2+1\big) & x^3 & -\,x^2 & +\,x & -\,1 \\
 & -\,x^3 & & -\,x & \\ \cline{2-4}
\end{array}$$

So

$$\frac{x^3 - x^2 + x - 1}{x^2 + 1} = x - 1.$$

}

Example 2

$$\begin{array}{rrrrr}
 & & & x & \\ \cline{2-5}
x^2+1\big) & x^3 & -x^2 & +x & -1 \\
 & -x^3 & & -x & \\ \cline{2-4}
 & & -x^2 & & -1
\end{array}$$

So

$$\frac{x^3-x^2+x-1}{x^2+1} = x - 1.$$

}

Example 2

$$\begin{array}{rrrrr}
 & & & x & -1 \\ \cline{2-5}
x^2+1) & x^3 & -x^2 & +x & -1 \\
 & -x^3 & & -x & \\ \cline{2-4}
 & & -x^2 & & -1
\end{array}$$

So

$$\frac{x^3 - x^2 + x - 1}{x^2 + 1} = x - 1.$$

}

Example 2

$$
\begin{array}{rrrrr}
 & & & x & -\,1 \\ \cline{2-5}
x^2+1\big) & x^3 & -\,x^2 & +\,x & -\,1 \\
 & -\,x^3 & & -\,x & \\ \cline{2-4}
 & & -\,x^2 & & -\,1 \\
 & & x^2 & & +\,1 \\ \cline{3-5}
\end{array}
$$

So

$$\frac{x^3 - x^2 + x - 1}{x^2 + 1} = x - 1.$$

}

Example 2

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & & x & -1 \\ \cline{2-5}
x^2+1 & x^3 & -x^2 & +x & -1 \\
\multicolumn{1}{r}{} & -x^3 & & -x & \\ \cline{2-4}
\multicolumn{1}{r}{} & & -x^2 & & -1 \\
\multicolumn{1}{r}{} & & x^2 & & +1 \\ \cline{3-5}
\multicolumn{1}{r}{} & & & & 0
\end{array}$$

So

$$\frac{x^3 - x^2 + x - 1}{x^2 + 1} = x - 1.$$

}

Example 2

$$\begin{array}{rrrrr}
 & & & x & -1 \\ \cline{2-5}
x^2+1\big) & x^3 & -x^2 & +x & -1 \\
 & -x^3 & & -x & \\ \cline{2-4}
 & & -x^2 & & -1 \\
 & & x^2 & & +1 \\ \cline{3-5}
 & & & & 0
\end{array}$$

{ So

$$\frac{x^3-x^2+x-1}{x^2+1}=x-1.$$

}

Example 3

$$x^2+x+1\overline{\smash{)}\quad x^3 \qquad\qquad\qquad -1}$$

So

$$\frac{x^3-1}{x^2+x+1}=x-1.$$

}

Example 3

$$\begin{array}{r@{}l}
 & x \phantom{{}-1} \\ \cline{2-2}
x^2+x+1 \big) & \; x^3 \qquad\qquad\quad -1
\end{array}$$

So

$$\frac{x^3-1}{x^2+x+1} = x-1.$$

}

Example 3

$$\begin{array}{rl} & \phantom{-x^3 - x^2 - {}} x \\ x^2+x+1 \big) & \overline{x^3 \qquad\qquad\quad -1} \\ & \underline{-x^3 - x^2 - x} \end{array}$$

So

$$\frac{x^3-1}{x^2+x+1} = x - 1.$$

}

Example 3

$$
\begin{array}{rl}
 & \qquad\qquad\qquad\qquad x \\
\cline{2-2}
x^2+x+1\big) & \quad x^3 \qquad\qquad\qquad -1 \\
 & -x^3 - x^2 - x \\
\cline{2-2}
 & \qquad\quad -x^2 - x - 1
\end{array}
$$

So

$$\frac{x^3-1}{x^2+x+1} = x-1.$$

}

Example 3

$$\begin{array}{r@{}r}
 & x - 1 \\
\cline{2-2}
x^2 + x + 1 \big) & x^3 \qquad\qquad\quad - 1 \\
 & - x^3 - x^2 - x \quad\;\; \\
\cline{2-2}
 & - x^2 - x - 1
\end{array}$$

So

$$\frac{x^3 - 1}{x^2 + x + 1} = x - 1.$$

}

Example 3

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & & x & -1 \\ \cline{2-5}
x^2+x+1 & x^3 & & & -1 \\
 & -x^3 & -x^2 & -x & \\ \cline{2-4}
 & & -x^2 & -x & -1 \\
 & & x^2 & +x & +1 \\ \cline{3-5}
\end{array}
$$

So

$$\frac{x^3-1}{x^2+x+1} = x-1.$$

}

Example 3

$$\begin{array}{rl} & \; x - 1 \\ x^2+x+1 \big) \!\!\!\! & \overline{\;x^3 \qquad\qquad\quad - 1} \\ & \underline{-x^3 - x^2 - x\phantom{{}-1}} \\ & -x^2 - x - 1 \\ & \phantom{-x^3 -{}}\underline{x^2 + x + 1} \\ & \phantom{-x^3 - x^2 - x - {}}0 \end{array}$$

So

$$\frac{x^3 - 1}{x^2 + x + 1} = x - 1.$$

}

Example 3

$$
\begin{array}{rr}
 & x - 1 \\ \cline{2-2}
x^2 + x + 1\big) & x^3 \qquad\qquad\qquad - 1 \\
 & -\, x^3 - x^2 - x \\ \cline{2-2}
 & -\, x^2 - x - 1 \\
 & x^2 + x + 1 \\ \cline{2-2}
 & 0
\end{array}
$$

{ So

$$\frac{x^3 - 1}{x^2 + x + 1} = x - 1.$$

}

Example 4

$$3x - 1\,\big)\overline{\;2x^3 \quad - 4x^2 \qquad\qquad\qquad + 1}$$

So

$$\frac{2x^3 - 4x^2 + 1}{3x - 1} = \frac{2}{3}\left(x^2 - \frac{5}{3}x - \frac{5}{9}\right) - \frac{17}{27(3x-1)}.$$

}

Example 4

$$\begin{array}{r l}
 & \qquad\quad \tfrac{2}{3}x^2 \\ \cline{2-2}
3x-1\big) & \;2x^3 \quad -4x^2 \qquad\qquad\quad +1
\end{array}$$

So

$$\frac{2x^3-4x^2+1}{3x-1}=\frac{2}{3}\left(x^2-\frac{5}{3}x-\frac{5}{9}\right)-\frac{17}{27(3x-1)}.$$

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Example 4

$$\begin{array}{r l}
 & \frac{2}{3}x^2 \\
\cline{2-2}
3x-1\big) & 2x^3 \;\; - 4x^2 \qquad\qquad + 1 \\
 & -\,2x^3 \;\; + \frac{2}{3}x^2 \\
\cline{2-2}
\end{array}$$

So

$$\frac{2x^3 - 4x^2 + 1}{3x - 1} = \frac{2}{3}\left(x^2 - \frac{5}{3}x - \frac{5}{9}\right) - \frac{17}{27(3x - 1)}.$$

}

Example 4

$$
\begin{array}{r|rrrr}
 & & \frac{2}{3}x^2 & & \\ \cline{2-5}
3x-1) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{10}{3}x^2 & &
\end{array}
$$

So

$$\frac{2x^3-4x^2+1}{3x-1}=\frac{2}{3}\left(x^2-\frac{5}{3}x-\frac{5}{9}\right)-\frac{17}{27(3x-1)}.$$

}

Example 4

$$\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & \frac{2}{3}x^2 & -\frac{10}{9}x & \\
\cline{2-5}
3x-1) & 2x^3 & -4x^2 & & +1 \\
\multicolumn{1}{r}{} & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
\multicolumn{1}{r}{} & & -\frac{10}{3}x^2 & &
\end{array}$$

So

$$\frac{2x^3-4x^2+1}{3x-1} = \frac{2}{3}\left(x^2-\frac{5}{3}x-\frac{5}{9}\right)-\frac{17}{27(3x-1)}.$$

}

Example 4

$$\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & -\ \frac{10}{9}x & \\ \cline{2-5}
3x-1\big) & 2x^3 & -4x^2 & & +1 \\
 & -\ 2x^3 & +\ \frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\ \frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\ \frac{10}{9}x & \\ \cline{3-4}
\end{array}$$

So

$$\frac{2x^3-4x^2+1}{3x-1}=\frac{2}{3}\left(x^2-\frac{5}{3}x-\frac{5}{9}\right)-\frac{17}{27(3x-1)}.$$

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Example 4

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & \\
\cline{2-5}
3x-1\big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\
\cline{3-4}
 & & & -\frac{10}{9}x & +1
\end{array}
$$

So

$$\frac{2x^3-4x^2+1}{3x-1} = \frac{2}{3}\left(x^2-\frac{5}{3}x-\frac{5}{9}\right)-\frac{17}{27(3x-1)}.$$

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Example 4

$$
\begin{array}{r@{}rrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} \\
\cline{2-5}
3x-1\big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\
\cline{3-4}
 & & & -\frac{10}{9}x & +1
\end{array}
$$

So

$$\frac{2x^3-4x^2+1}{3x-1} = \frac{2}{3}\left(x^2 - \frac{5}{3}x - \frac{5}{9}\right) - \frac{17}{27(3x-1)}.$$

}

Example 4

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} \\ \cline{2-5}
3x-1\big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\ \cline{3-4}
 & & & -\frac{10}{9}x & +1 \\
 & & & \frac{10}{9}x & -\frac{10}{27} \\ \cline{4-5}
\end{array}
$$

So

$$\frac{2x^3 - 4x^2 + 1}{3x - 1} = \frac{2}{3}\left(x^2 - \frac{5}{3}x - \frac{5}{9}\right) - \frac{17}{27(3x-1)}.$$

}

Example 4

$$
\begin{array}{r|rrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} \\
\cline{2-5}
3x-1 & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\
\cline{3-4}
 & & & -\frac{10}{9}x & +1 \\
 & & & \frac{10}{9}x & -\frac{10}{27} \\
\cline{4-5}
 & & & & \frac{17}{27}
\end{array}
$$

So

$$\frac{2x^3-4x^2+1}{3x-1} = \frac{2}{3}\left(x^2-\frac{5}{3}x-\frac{5}{9}\right) - \frac{17}{27(3x-1)}.$$

}

Example 4

$$
\begin{array}{rllll}
 & & \frac{2}{3}x^2 & - \frac{10}{9}x & - \frac{10}{27} \\ \cline{2-5}
3x - 1 \big) & 2x^3 & - 4x^2 & & + 1 \\
 & - 2x^3 & + \frac{2}{3}x^2 & & \\ \cline{2-3}
 & & - \frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & - \frac{10}{9}x & \\ \cline{3-4}
 & & & - \frac{10}{9}x & + 1 \\
 & & & \frac{10}{9}x & - \frac{10}{27} \\ \cline{4-5}
 & & & & \frac{17}{27}
\end{array}
$$

{ So

$$\frac{2x^3 - 4x^2 + 1}{3x - 1} = \frac{2}{3}\left(x^2 - \frac{5}{3}x - \frac{5}{9}\right) - \frac{17}{27(3x - 1)}.$$

}

Example 5

$$x^2 - 2x + 5\overline{)\quad x^4 \qquad\qquad - x^2 \quad + x \quad - 4}$$

So

$$\frac{x^4 - x^2 + x - 4}{x^2 - 2x + 5} = x^2 + 2x - 2 + \frac{-13x + 6}{x^2 - 2x + 5}.$$

}

Example 5

$$\begin{array}{r} x^2 \\ x^2-2x+5\overline{\smash{)}\quad x^4 \qquad\qquad - x^2 \quad + x \quad - 4} \end{array}$$

So

$$\frac{x^4-x^2+x-4}{x^2-2x+5}=x^2+2x-2+\frac{-13x+6}{x^2-2x+5}.$$

}

Example 5

$$\begin{array}{rlllll}
 & & & x^2 & & \\
\cline{2-6}
x^2-2x+5 \big) & x^4 & & -x^2 & +x & -4 \\
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\end{array}$$

So

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How to deal with rational functions?

How can we integrate something like

$$\int \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} \, \mathrm{d}x$$

or

$$\int \frac{x + 2}{x^3 - x} \, \mathrm{d}x?$$

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

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using polynomial long division.

This is still not something we can integrate so we need to go further.

Partial fractions

When we are faced with a sum of the form

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How do we reverse this process?

Answer: partial fractions

When the denominator is $(ax + b)(cx + d) \cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$$

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we can always find constants $A_1, A_2, \ldots, n$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \ldots \frac{A_n}{a_nx + b_n}$$

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$$\frac{1}{(x-1)(x+1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

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Multiplying both sides by $(x-1)(x+1)$

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x+1} + \frac{B(x-1)(x+1)}{x-1} \\ &= A(x-1) + B(x+1) \\ &= (A+B)x + (B-A) \end{aligned}$$

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So

$$-2A = 1 \quad \text{hence} \quad A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$