Power series as functions.

Last time we saw that a power series

$$\sum_{n=0}^{\infty} a_n (x-c)^n$$

has a radius of convergence R, ie it converges when $x \in (c-R, c+R)$. Thus we can define a function

$$F : (c-R, c+R) \longrightarrow \mathbb{R}$$

$$F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

Ex We saw that $\sum_{n=0}^{\infty} x^n$ has a radius of conv. of 1. so is defined on $(-1,1)$. We also saw (Geometric series) that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad (\text{when } |x|<1).$$

This is the power series representation of $\frac{1}{1-x}$ around zero.

Ex How can we find a power series for

$$\frac{1}{8+x^3}$$

around zero?

Notice that

$$\frac{1}{8+x^3} = \frac{1}{8} \cdot \frac{1}{1-\left(-\frac{x^3}{8}\right)}$$

$$= \frac{1}{8} F\left(-\frac{x^3}{8}\right)$$

where $F(x) = \frac{1}{1-x}$.

But we have a power series for F! so

$$\frac{1}{8+x^3} = \frac{1}{8} \sum_{n=0}^{\infty} \left(-\frac{x^3}{8}\right)^{\cancel{3}n}$$ (as long as $\left|-\frac{x^3}{8}\right| < 1$).

$$= \frac{1}{8} \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{8^n}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{8^{n+1}}$$

as long as $|-x^3/8| < 1$ i.e. $|x| < 2$

Radius of conv.

This is great! We can write down any old power series and get a function! E.g.

$$\sum_{n=0}^{\infty} \frac{\ln(n)}{n+1} \cdot \sin(n)\, x^n$$

What function is it? Who knows, probably a completely new function!

Now that we have new ways to write functions, how do we differentiate/integrate them?

Thm Let $F(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ with $R > 0$ then F is differentiable on $(c-R, c+R)$ and

$$F'(x) = \sum_{n=0}^{\infty} a_n \cdot n \cdot (x-c)^{n-1}$$

$$\int F(x)dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x-c)^{n+1} + C$$

We can differentiate and integrate term by term!

Ex Suppose we want to find a power series for $\frac{1}{(1-x)^2}$. Notice that

$$\frac{1}{(1-x)^2} = \frac{d}{dx} \frac{1}{1-x}$$

$$= \frac{d}{dx} \sum_{n=0}^{\infty} x^n \qquad (|x| < 1)$$

$$= \sum_{n=1}^{\infty} n x^{n-1}$$

$$= \sum_{m=0}^{\infty} (m+1) x^m \qquad (\text{let } m = n-1).$$

We can also use power series to solve differential equations!

Ex Solve the DE $\frac{df}{dx} = f$ where $f(0) = 1$.

First, assume that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

then $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{m=0}^{\infty} (m+1) a_{m+1} x^m$

so

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n$$

Thus: $n \cdot a_n = a_{n-1}$ ie $a_n = \frac{a_{n-1}}{n}$

$$a_{n-1} = \frac{a_{n-2}}{n-1}$$

$$\vdots$$

$$a_1 = \frac{a_0}{1}$$

plugging these in gives

$$a_n = \frac{a_{n-1}}{n} = \frac{a_{n-2}}{n(n-1)} = \frac{a_{n-3}}{n(n-1)(n-2)} = \dots = \frac{a_0}{n!}$$

Thus $f(x) = \sum_{n=0}^{\infty} \frac{a_0}{n!} x^n$

But we also know $f(0)=1$, but $f(0)=a_0$ so $a_0=1$

i.e.

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n .$$

<u>Note</u> We know that the unique function that satisfies $f'=f$ and $f(0)=1$ is $f(x)=e^x$

so we have just shown

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

One of the most important expressions in all of mathematics!