Math 3B: Lecture 9

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January 30, 2019

Differential equations (motivation)

A differential equation is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

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The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

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The force due to gravity is roughly $-10m$ Newtons, so

$$-10m = mh''(t)$$

Population growth (motivation)

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If $P(t)$ is the population at time t:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(t)$$

Some more examples of differential equations

$$\frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = y(1 - y)$$

$$y'' = \sqrt{a^2 - (y')^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(A - y)^2$$

Antiderivatives

We have been solving differential equations of the form

$$\frac{dy}{dx} = f(x).$$

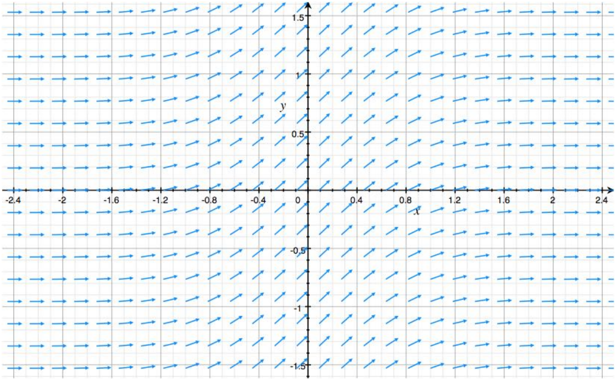
Any antiderivative $y = F(x)$ of $f(x)$ is a solutions to this differential equation!

Slope fields

In some cases it is impossible to find the antiderivative (without special functions). E.g.

$$f(x) = e^{-x^2}$$

But we can still (approximately) graph the antiderivative! First we draw the slope field



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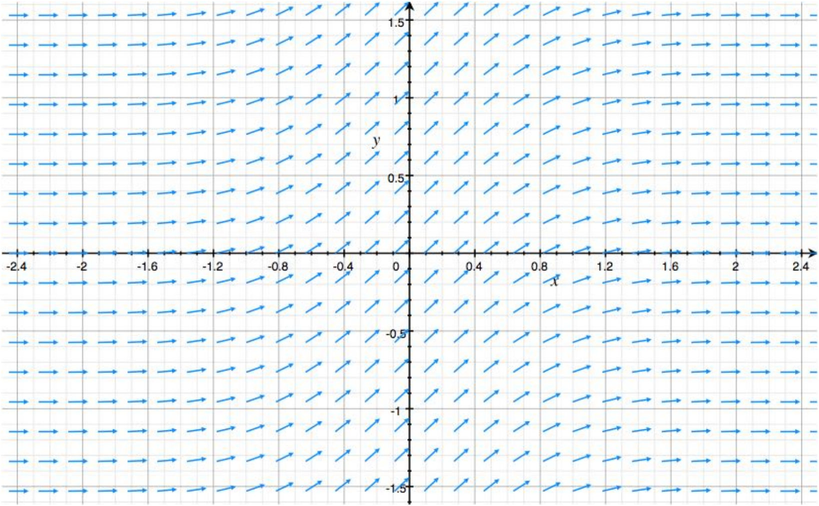
4. Draw a small arrow with slope $f(x)$ and the point (x, y)
5. Do this for a grid of points on the xy-plane.

Example 1

$$f(x) = e^{-x^2} \text{ with } F(0) = -0.25$$



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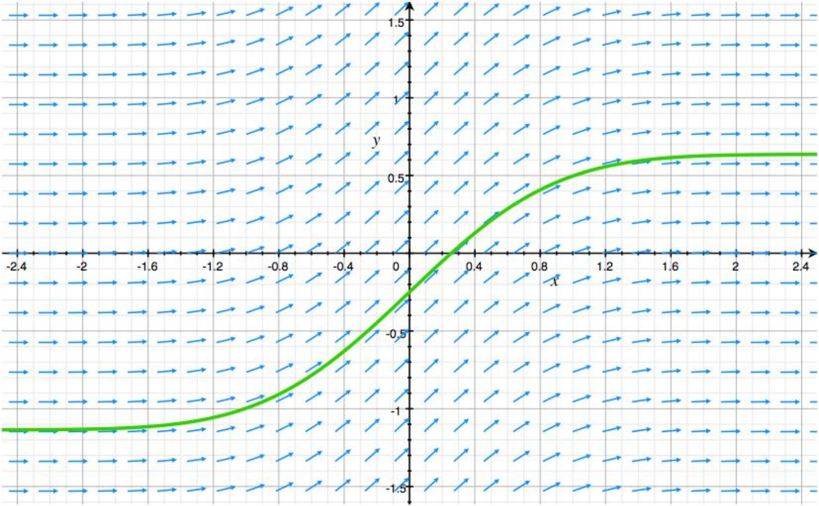
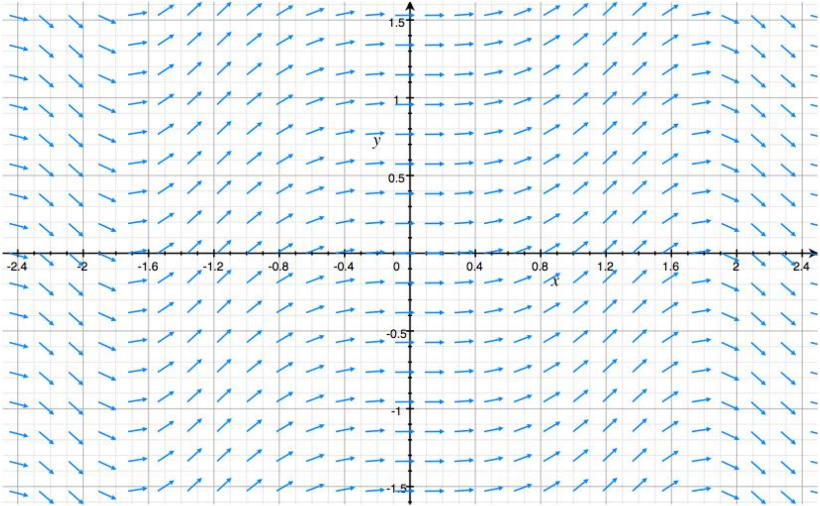


Example 2

$$f(x) = \sin(x^2) \text{ with } F(0) = 0.1$$



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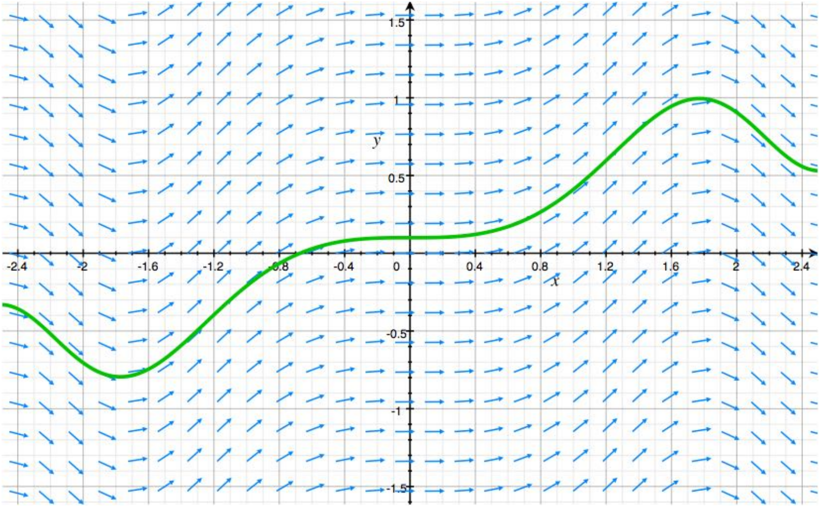

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These problems involve finding the area under some curve.

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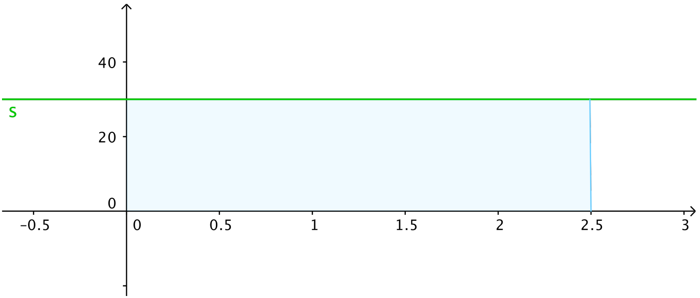
Solution

We model the car's speed using the function $s(t) = 30$. So we can see that the area under this curve



is the distance travelled (75 miles)

Example 2

If a car accellerates for 20 seconds at a rate of $2m/s^2$ and then decelerates for 30 seconds at a rate of $1m/s^2$, how far has it travelled?

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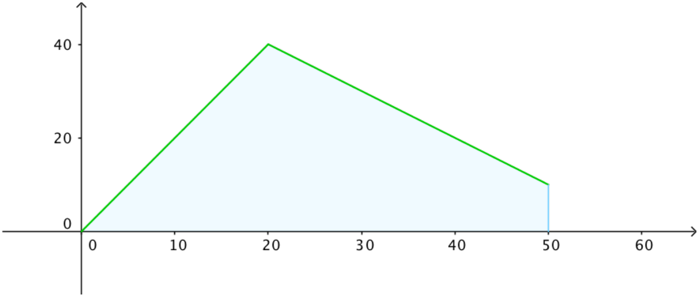
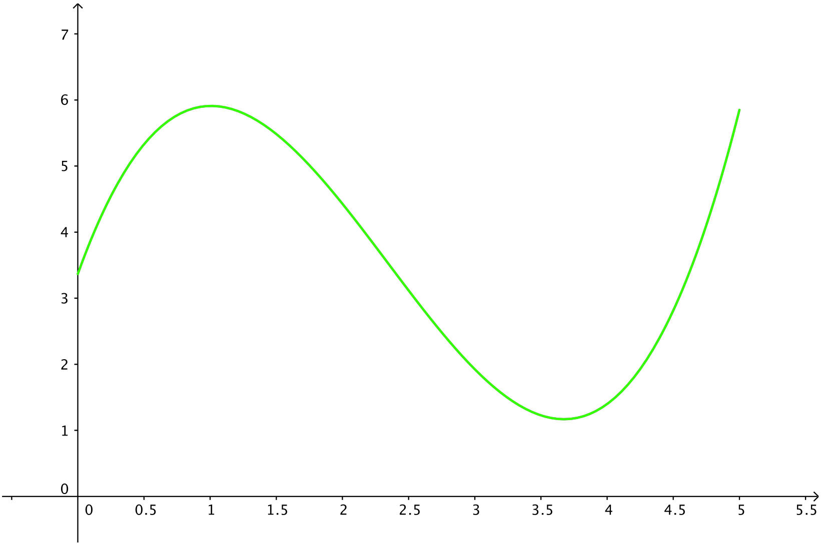
Solution

The car's speed is given by $s(t) = 2t$ when $0 \leq t \leq 20$ and $s(t) = 60 - t$ when $20 \leq t \leq 50$. So the graph looks like



More complicated rates of change

Suppose we have a car whose speed is descibed by the following curve. How far has it travelled in this time?



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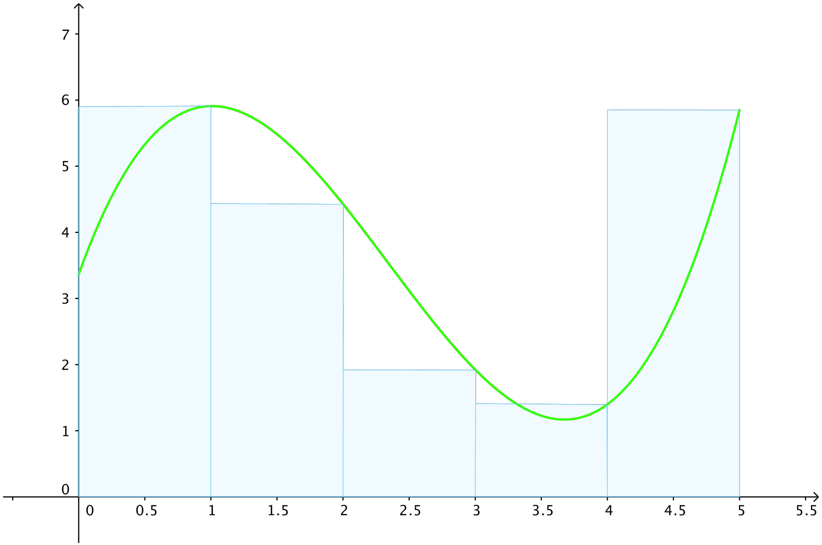

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- Answer: area under $f(t)$ between a and b.