Math 3B: Lecture 11

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Introduction

Quiz this week

- Quiz 3 questions will be drawn from problem set 5
- Notes on long division and partial fractions online

Survey

I have put a survey online:
`math.ucla.edu/~noah/3Bsurvey`

Last time

- Integration by parts

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- Polynomial long division

How to deal with rational functions?

How can we integrate something like

$$\int \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} \, \mathrm{d}x$$

or

$$\int \frac{x + 2}{x^3 - x} \, \mathrm{d}x?$$

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

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This is still not something we can integrate so we need to go further.

Partial fractions

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$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots$$

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How do we reverse this process?

Answer: partial fractions

When the denominator is $(ax + b)(cx + d) \cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_n x + b_n)$$

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we can always find constants $A_1, A_2, \ldots, n$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \ldots \frac{A_n}{a_nx+b_n}$$

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Multiplying both sides by $(x-1)(x+1)$

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x+1} + \frac{B(x-1)(x+1)}{x-1} \\ &= A(x-1) + B(x+1) \\ &= (A+B)x + (B-A) \end{aligned}$$

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$$A + B = 0 \qquad \text{and} \qquad B - A = 1$$

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So

$$-2A = 1 \quad \text{hence} \quad A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$

Repeated factors

What if $q(x)$ contains repeated factors? E.g. if $q(x) = (x-1)^2$ or $q(x) = (x-1)(x+2)^3$?

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For every factor $(ax+b)^k$ in $q(x)$, the partial fraction expansion has terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}.$$

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Side note: integrating $\frac{1}{x}$.

Recall that

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$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

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Using substitution this gives the formula

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + C.$$

Side note: integrating $\frac{1}{x^k}$.

Recall that if $k > 1$

Fact

$$\int \frac{1}{x^k}\,\mathrm{d}x = -\frac{1}{(k-1)x^{k-1}} + C$$

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Integrating rational functions $p(x)/q(x)$

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1. Express $\frac{p(x)}{q(x)}$ in the form

$$d(x) + \frac{r(x)}{q(x)}$$

   using polynomial long division.

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2. Write $\frac{r(x)}{q(x)}$ as a sum of fractions of the form

$$\frac{A}{(ax+b)^k}$$

   using partial fractions

Integrating rational functions $p(x)/q(x)$

Action plan

1. Express $\frac{p(x)}{q(x)}$ in the form
$$d(x) + \frac{r(x)}{q(x)}$$
using polynomial long division.
2. Write $\frac{r(x)}{q(x)}$ as a sum of fractions of the form
$$\frac{A}{(ax+b)^k}$$
using partial fractions
3. Integrate all these pieces seperately.

Example 1

$$I = \int \frac{x^4 - 3x^2 + 3}{x^2 - 1}\, \mathrm{d}x$$

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Solution

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Using long division and partial fractions

$$\frac{x^4 - 3x^2 + 3}{x^2 - 1} = x^2 - 2 + \frac{1}{x^2 - 1} = x^2 - 2 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

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So

$$I = \frac{1}{3}x^2 - 2x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

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$$\frac{x^3 - 2x^2 + 4x}{(x-1)^3} = 1 + \frac{x^2 + x + 1}{(x-1)^3}$$

Example 2

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$$\frac{x^3 - 2x^2 + 4x}{(x-1)^3} = 1 + \frac{x^2 + x + 1}{(x-1)^3} = 1 + \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3}$$

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So

$$I = x + \ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + C.$$