Final exam practice 1

UCLA: Math 3B, Fall 2018

Instructor: Noah White
Date:

- This exam has 7 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____________________

ID number: ____________________

Discussion section (please circle):

Day/TA	Ben	Ryan
Tuesday	1A	1C
Thursday	1B	1D

Question	Points	Score
1	12	
2	12	
3	12	
4	10	
5	12	
6	14	
7	8	
Total:	80	

Questions 1 and 2 are multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following four pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) The function $f(x) = e^{x+\sin x}$ is

   A. Always increasing.
   B. Always decreasing.
   C. Always concave up.
   D. Always concave down.

   (b) (2 points) The function $f(x) = e^{-x^2}$ is

   A. Always increasing.
   B. Always decreasing.
   C. Always concave up.
   D. None of the above.

   (c) (2 points) The function $f(x) = \frac{4x-x^2}{1+x^2}$ has a

   A. Horizontal asymptote at $y = -1$.
   B. Vertical asymptote at $x = 1$.
   C. Slanted asymptote with slope -1.
   D. Slanted asymptote with slope 1.

(d) (2 points) The function $f(x) = \frac{6e^x+5x^2}{2x-2e^x}$ has a

A. Horizontal asymptote at $y = 1$.

B. No horizontal asymptotes.

C. Horizontal asymptote at $y = 0$.

D. Horizontal asymptote at $y = -3$.

(e) (2 points) The function $f(x) = \cos x$ has a critical point at

A. $x = 0$

B. $x = \pi/2$

C. $x = -\pi/2$

D. $x = 1$

(f) (2 points) The function $f(x) = e^{x-1} + e^{3-x}$ has a

A. minimum at $x = 2$.

B. maximum at $x = 2$.

C. minimum $x = 0$.

D. maximum $x = 0$.

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) The definite integral $\int_{-\pi}^{\pi} \cos x \, \mathrm{d}x$ has a value of

   A. 2.
   B. 1.
   C. 0.
   D. π.

   (b) (2 points) The definite integral $\int_{1}^{e} x \ln x \, \mathrm{d}x$ has a value of

   A. 0.
   B. $\frac{1}{2}e^2$.
   C. $\frac{1}{4}(e^2+1)$**.**
   D. e.

   (c) (2 points) The solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = -2y$ when $y(0) = 3$ has

   A. $y(1) = 2e^2$
   B. $y(0.5) = 3e$.
   C. $y(1) = 3e$.
   D. $y(0.5) = 3e^{-1}$**.**

(d) (2 points) The solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = e^y \cos t$ when $y(0) = 0$ has

A. $y(\pi) = \ln 2$

B. $y(-\pi/2) = 0$.

C. $y(1) = \ln 2$.

D. $y(-\pi/2) = -\ln 2$**.**

(e) (2 points) The differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = (1 - y)\left(\frac{y}{9} - 1\right)$ has a

A. stable equilibrium at $y = 9$**.**

B. unstable equilibrium at $y = 1/9$.

C. stable equilibrium at $y = 1$.

D. unstable equilibrium at $y = 3$.

(f) (2 points) The differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = 3 - \sqrt{5 + y^2}$ has a

A. stable equilibrium at $y = -2$.

B. unstable equilibrium at $y = -1$.

C. stable equilibrium at $y = 2$**.**

D. unstable equilibrium at $y = 0$.

3. Let $f(x) = \frac{e^x+1}{e^{2x}-1}$. Note that $f'(x) = -\frac{e^x}{(e^x-1)^2}$ and $f''(x) = \frac{e^x(e^x+1)}{(e^x-1)^3}$.

(a) (2 points) Does $f(x)$ cross the x and y axes? If so, where?

Solution: No x-intercepts, no y-intercept.

(b) (2 points) Does $f(x)$ have any horizontal asymptotes? If so what are they?

Solution: We need to evaluate

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{e^x+1}{e^{2x}-1} = 0.$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{e^x+1}{e^{2x}-1} = -1.$$

So we have horizontal asymptote in the positive direction at $y = 0$ and in the negative direction at $y = -1$.

(c) (1 point) Does $f(x)$ have any vertical asymptotes? If so what are they?

Solution: The denominator of $f(x)$ is zero when $e^{2x} = 1$, i.e. when $x = 0$, thus a vertical asymptote at $x = 0$.

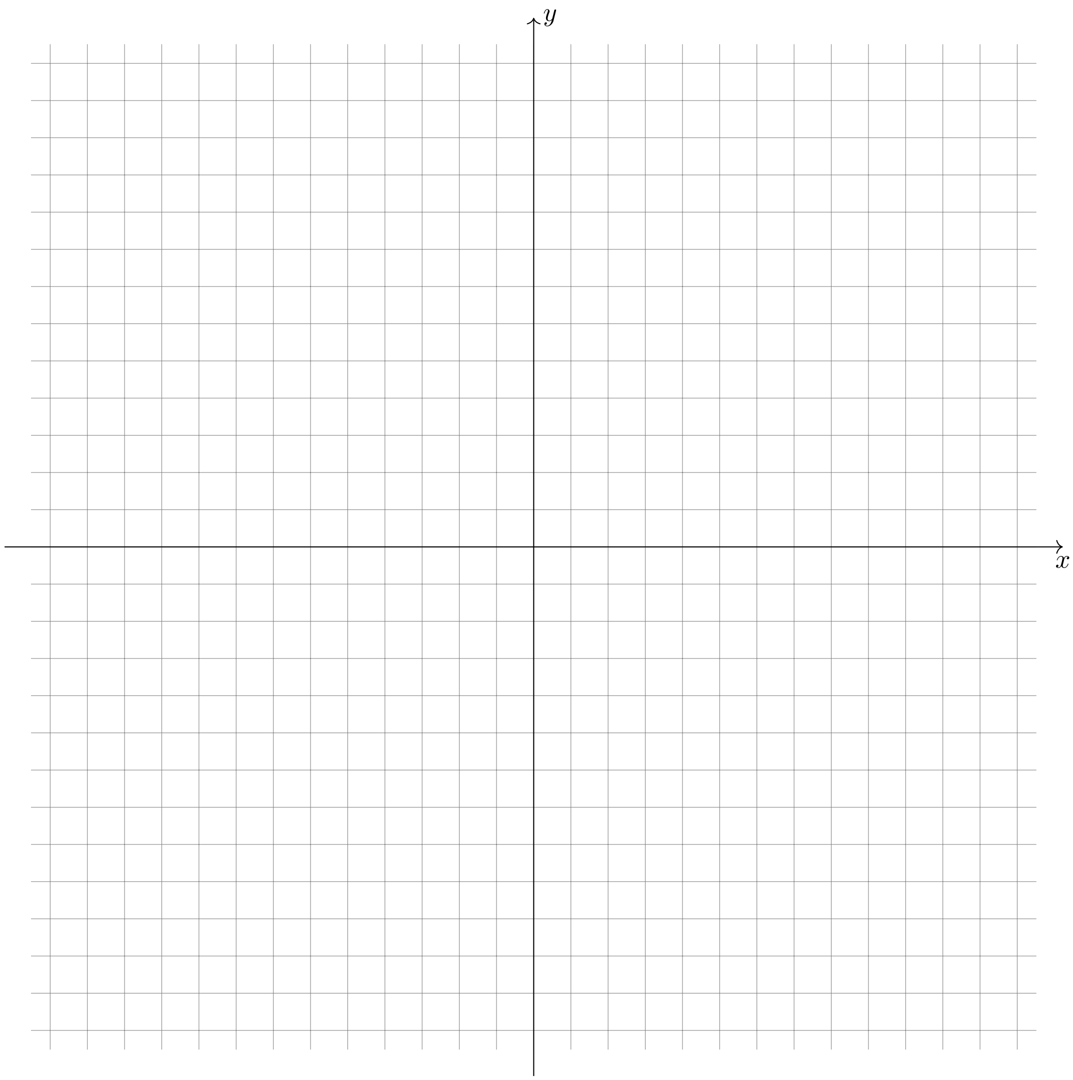
(d) (2 points) For what x is the first derivative $f'(x)$ positive?

Solution: The denominator is always positive, as is the numerator. Thus $f'(x)$ is negative for all x.

(e) (2 points) For what x is the second derivative $f''(x)$ positive?

Solution: The denominator is positive when $(e^x - 1)^3 > 0$, i.e when $e^x > 0$. The numerator is always positive thus, $f''(x) > 0$ when $x > 0$.

(f) (3 points) On the graph provided, sketch $f(x)$



4. A new nature reserve is being planned. The planning officials estimate that there will be a steady stream of deer immigrating to the reserve, at a rate of 300 deer per year. They also estimate that in the early years, the growth in population due to births will be roughly exponential; this means, for every one deer in the reserve at a given point, t years later there will be

$$0.2e^{0.5t}$$

deer. Initially the reserve contains no deer at all.

(a) (6 points) Write a Riemann sum which represents the total number of deer in the reserve in 10 years. Be sure to define any symbols that you use (e.g. t_k, Δt, etc).

Solution: Let $\Delta t = 10/n$ and $t_k = k\Delta t$.

$$T = \lim_{n\to\infty} \sum_{k=1}^{n} 60e^{0.5(10-t_k)}\Delta t$$

(b) (4 points) Use an integral to evaluate the Riemann sum above.

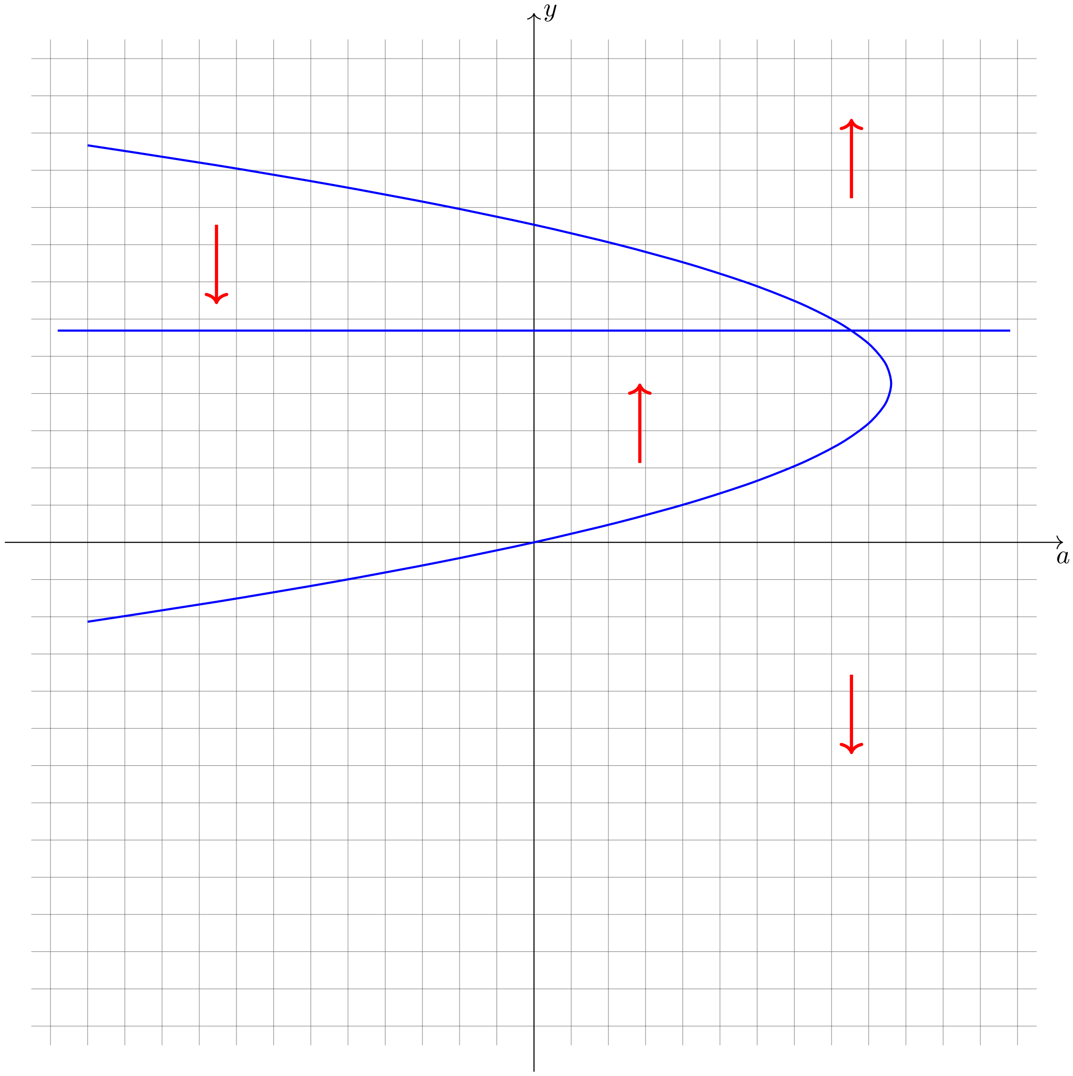
Solution:

$$\begin{aligned} W &= \int_0^{10} 60e^{0.5(10-t)}\,\mathrm{d}t \\ &= \left[-120e^{0.5(10-t)}\right]_0^{10} \\ &= -120 - \left(-120e^5\right) \\ &\approx 17690 \end{aligned}$$

5. In this question we will investigate the behaviour of the solutions of

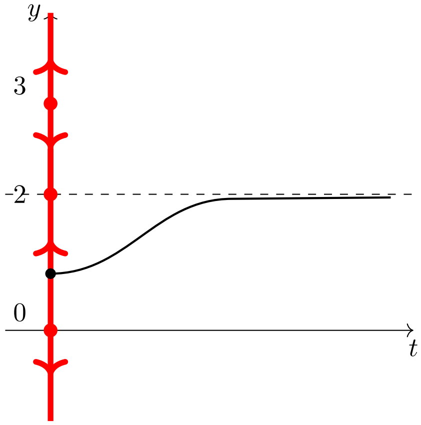
$$\frac{\mathrm{d}y}{\mathrm{d}t} = (a + y^2 - 3y)(y - 2)$$

   (a) (4 points) Draw a bifurcation diagram for this equation with parameter a. Make sure to label the regions of your diagram with up/down arrows according to the direction of the derivative.



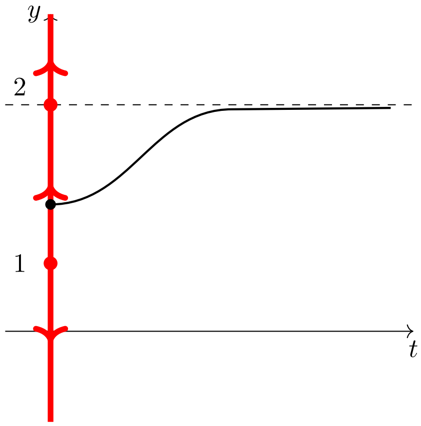
(b) (2 points) Draw a phase diagram when $a = 0$ and sketch the solution if $y(0) = 1$.

Solution:



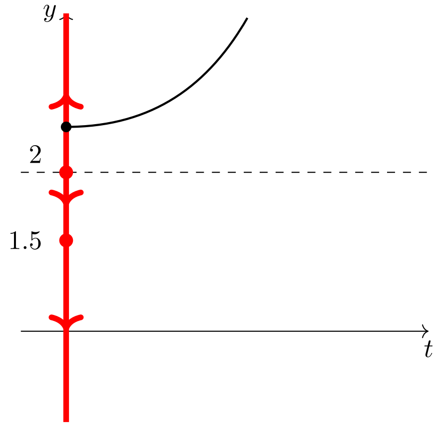
(c) (2 points) Draw a phase diagram when $a = 2$ and sketch the solution if $y(0) = 1.5$.

Solution:



(d) (2 points) Draw a phase diagram when $a = 2.25$ and sketch the solution if $y(0) = 3$.

Solution:



(e) (2 points) The differential equation above has an a equilibrium solution of $y = 2$ for any value of a. For what a is this equilibrium stable?

Solution: It is stable before $a = -y^2 + 3y$ intersects with $y = 2$, i.e. when $a = 2$. That is, the equilibrium is stable when $a < 2$.

6. A saltwater pool has a volume of $100,000$ liters. Fresh water is being pumped in using a hose at 1000 L/h (liters per hour) and is draining out through an outlet in the bottom of the pool at the same rate. You should assume that the fresh and saltwater mix immediately.

   (a) (3 points) Write a differential equation modelling the total amount of salt (in grams) in the pool.

   Solution: There is no salt entering the pool so the rate of salt in is zero. Let $y(t)$ be the amount of salt at time t. Since 1000 L of water are draining out, and since this is 0.01 times the total volume of the pool, the rate of salt leaving the pool is $-0.01y$. Thus the differential equation is

   $$\frac{\mathrm{d}y}{\mathrm{d}t} = -0.01y.$$

   (b) (2 points) If the initial concentration of salt is 3 g/L, solve the differential equation.

   Solution: The general solution is

   $$y(t) = Ce^{-0.01t}.$$

   Initially there are $100,000 \cdot 3 = 300,000$ grams of salt in the pool. Thus solving for C:

   $$y(t) = 300,000e^{-0.01t}$$

(c) (3 points) How long does it take to fall below 0.1 g/L?

Solution: We are looking for the time when the amount of salt in the pool is $100,000 \cdot 0.1 = 10,000$ grams. This amounts to solving

$$10,000 = 300,000e^{-0.01t}.$$

Rearranging

$$-0.01t = \ln\left(\frac{10,000}{300,000}\right)$$

and so

$$t = -100\ln\left(\frac{1}{30}\right) = 100\ln(30) \approx 340.1.$$

(d) (3 points) Now suppose that the water coming from the hose is saltwater containing a g/L of salt. Write a differential equation modelling the amount of salt in the pool over time and solve it. (You should still assume that the initial concentration of salt in the pool is 3 g/L).

Solution: Now we have an unknown rate of salt coming into the pool. Let a g/L be the salinity of the water entering the pool through the hose. Thus the rate in is $1000a$ g/h. Thus our differential equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1000a - 0.01y.$$

The solution to this equation is

$$y(t) = 100,000a - Ce^{-0.01t}.$$

To solve for C we use the fact that $y(0) = 300,000$ and thus

$$300,000 = 100,000a - C$$

and so $C = 100,000a - 300,000$. Thus the solution is

$$y(t) = 100,000a - (100,000a - 300,000)e^{-0.01t}.$$

(e) (3 points) What concentration of salt should the water coming out of the hose have (i.e. what should a be) in order for the pool to achieve a concentration of 3.4 g/L in 12 hours?

Solution: We would like $y(12) = 100,000 \cdot 3.4 = 340,000$. Thus we need to solve

$$\begin{aligned} 340,000 &= 100,000a - (100,000a - 300,000)e^{-0.12} \\ 340,000 - 300,000e^{-0.12} &= 100,000a - 100,000ae^{-0.12} \end{aligned}$$

i.e.

$$a = \frac{340,000 - 300,000e^{-0.12}}{100,000a - 100,000e^{-0.12}} \approx 6.5$$

7. (8 points) What point (x, y) on the parabola $y = x^2$ is closest to the point $(16, 0.5)$?

Solution: Let (x, y) be a point on the parabola (so that $y = x^2$). The distance between the point $(1, 2)$ and (x, y) is

$$\sqrt{(x-16)^2 + (y-0.5)^2}$$

and we would like to minimize this. However, we might as well minimize the square of this function, i.e.

$$d = (x-16)^2 + (y-0.5)^2.$$

Note that $y = x^2$ so

$$\begin{aligned} d &= (x-16)^2 + (x^2-0.5)^2 \\ &= x^2 - 32x + 16^2 + x^4 - x^2 + 0.25 \\ &= x^4 - 32x + 16^2 + 0.25. \end{aligned}$$

To minimize this we find the critical points. So

$$d' = 4x^3 - 32 = 0$$

means that $x = 2$. To check this is a minimum we look at $d'' = 12x^2$ which is always positive so we indeed have a minimum. Thus the closest point on $y = x^2$ is $(2, 4)$.

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