

Math 3B: Lecture 18

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Linear models

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Examples

$$\frac{dy}{dt} = ay, \quad \frac{dy}{dt} = -\lambda y.$$

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Something could mean (for example)

- concentration of a drug in bloodstream
- pollutant in water supply

General solution

Using separation of variables, we can show that the general solution to

$$\frac{dy}{dt} = a - by$$

is

$$y(t) = \frac{a}{b} - Ce^{-bt}$$

where C is an arbitrary constant.

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$$M \left(\frac{1}{2} \right)^{t/2} = Me^{-0.5t \ln(2)} \text{ mg left}$$

- Thus the rate at which the drug is leaving (at time t) is given by

$$0.5 \ln(2) Me^{-0.5t \ln(2)} = 0.5 \ln(2)(\text{current concentration}) \text{ mg/h.}$$

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$$0 = \frac{20}{\ln(2)} - C \approx 28.9 - C$$

- Thus at time t the concentration is

$$y(t) = 28.9 - 28.9e^{-0.3t} = 28.9(1 - e^{-0.3t})$$

Newton's Law of Cooling

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$$\frac{dT}{dt} = k(A - T)$$

General solution

$$T(t) = A - Ce^{-kt}.$$

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An object takes 20 minutes to cool from 90° to 86° in a room which is 70° . At what time will it be 75° ?

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- The solution is given by

$$T(t) = 70 - Ce^{-kt}$$

- We know $T(0) = 90$ and $T(20) = 86$.
- Thus

$$90 = 70 - C \quad \text{so} \quad C = -20.$$

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- Thus

$$e^{-20k} = \frac{86 - 70}{20} = \frac{4}{5} \quad \text{so} \quad k = -\frac{1}{20} \ln \left(\frac{4}{5} \right) \approx -0.01.$$

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- The model is thus $T(t) = 70 + 20e^{-0.01t}$. We want to solve

$$75 = 70 + 20e^{-0.01t}.$$

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- To find k we use $T(20) = 86$

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$$75 = 70 + 20e^{-0.01t}.$$

- Rearranging we get $20e^{-0.01t} = 5$ i.e.

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$$-0.01t = \ln\left(\frac{1}{4}\right)$$

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$$e^{-0.01t} = \frac{1}{4}$$

- Applying a logarithm

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- So we get

$$t = -100 \ln\left(\frac{1}{4}\right) \approx 138 = 2 \text{ hours } 18 \text{ minutes.}$$