This week you will practice writing differential equations modelling real world phenomena as well as understanding population models. You will also get practice solving separable differential equations.

Homework: The homework will be due on Friday 17 January, at 8am, the *start* of the lecture. It will consist of questions:

2 and 7.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.1) Write a differential equation to model the situations described below. Do not try to solve.

   (a) (6.1-1) The number of bacteria in a culture grows at a rate that is proportional to the number of bacteria present.

   (b) (6.1-2) A sample of radium decays at a rate that is proportional to the amount of radium present in the sample.

   (c) (6.1-5) According to Benjamin Gompertz (1779-1865) the growth rate of a population is proportional to the number of individuals present, where the factor of proportionality is an exponentially decreasing function of time.

   (d) (6.1-7) The rate at which an epidemic spreads through a community of P susceptible people is proportional to the product of the number of people y who have caught the disease and the number $P - y$ who have not.

   (e) (6.1-8) The rate at which people are implicated in a government scandal is proportional to the product of the number N of people already implicated and the number of people involved who have not yet been implicated.

2. (6.1) A population model is given by
$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(100 - P).$$

   (a) (6.1-9) For what values is the population at equilibrium?

   (b) (6.1-10) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$?

   (c) (6.1-11) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$?

   (d) (6.1-12) Describe how the fate of the population depends on the initial density.

3. (6.1) A population model is given by

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P(P - 1)(100 - P).$$

   (a) (6.1-13) For what values is the population at equilibrium?

   (b) (6.1-14) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} > 0$?

   (c) (6.1-15) For what values is $\frac{\mathrm{d}P}{\mathrm{d}t} < 0$?

   (d) (6.1-16) Describe how the fate of the population depends on the initial density.

4. (6.1) Radioactive decay: Certain types of atoms (e.g. carbon-14, xenon-133, lead-210, etc.) are inherently unstable. They exhibit random transitions to a different atom while emitting radiation in the process. Based on experimental evidence, Rutherford found in the early 20th century that the number, N, of atoms in a radioactive substance can be described by the equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\lambda N$$

where t is measured in years and $\lambda > 0$ is known as the *decay constant*. The decay constant is found experimentally by measuring the half life, τ of the radioactive substance (i.e. the time it takes for half of the substance to decay). Use this information in the following problems.

(a) (6.1-18) Find a solution to the decay equation assuming that $N(0) = N_0$.

(b) (6.1-19) For xenon-133, the half-life is 5 days. Find λ. Assume t is measured in days.

(c) (6.1-20) For carbon-14 the half life is $5,568$ years. Find the decay constant λ, assuming t is measured in years.

(d) (6.1-21) How old is a piece of human bone which contains just 60% of the amount of carbon-14 expected in a sample of bone from a living person, assuming the half life of carbon-14 is $5,568$ years?

(e) (6.1-22) The Dead Sea Scrolls were written on parchment at about 100 B.C. What percentage of carbon-14 originally contained in the parchment remained when the scrolls were discovered in 1947?

5. (6.1-30) Hyperthyroidism is caused by a new growth of tumor-like cells that secrete thyroid hormones in excess to the normal hormones. If left untreated, a hyperthyroid individual can exhibit extreme weight loss, anorexia, muscle weakness, heart disease intolerance to stress, and eventually death. The most successful and least invasive treatment option is radioactive iodine-131 therapy.

This involves the injection of a small amount of radioactivity into the body. For the type of hyperthyroidism called Graves disease, it is usual for about $40-80\%$ of the administered activity to concentrate in the thyroid gland. For functioning adenomas (hot nodules), the uptake is closer to $20-30\%$. Excess iodine-131 is excreted rapidly by the kidneys. The quantity of radioiodine used to treat hyperthyroidism is not enough to injure any tissue except the thyroid tissue, which slowly shrinks over a matter of weeks to months. Radioactive iodine is either swallowed in a capsule or sipped in solution through a straw. A typical dose is $5-15$ millicures. The half-life of iodine-131 is 8 days.

(a) Suppose that it takes 48 hours for a shipment of iodine-131 to reach a hospital. How much of the initial amount shipped is left once it arrives at the hospital?

(b) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left one week later?

(c) Suppose a patient is given a dosage of 10 millicures of which 30% concentrates in the thyroid gland. How much is left 30 days later?

6. (6.2) Solve the following differential equations.

(a) $\frac{dy}{dt} = 5y$

(b) $\frac{dy}{dt} = -y$

(c) $\frac{dy}{dx} = -3y$

(d) $\frac{dy}{dx} = 0.2y$

(e) (6.2-17) $\frac{dy}{dt} = y^3$

(f) (6.2-18) $\frac{dy}{dt} = y\sin t$

(g) (6.2-20) $\frac{dy}{dt} = \frac{t}{y}$

(h) (6.2-24) $\frac{dy}{dx} = \frac{x}{y}\sqrt{1+x^2}$

(i) (6.2-26) $\frac{dy}{dx} = \frac{\sin x}{\cos y}$

(j) (6.2-30) $\frac{dy}{dt} = yt$ with $y(1) = -1$

(k) (6.2-32) $\frac{dy}{dt} = e^{-y}t$ with $y(-2) = 0$

(l) (6.2-34) $\frac{dy}{dt} = ty^2 + 3t^2y^2$ with $y(-1) = 2$

(m) $\frac{dy}{dx} = y \sin x + \frac{y}{(x+1)^2}$ with $y(0) = 1$

(n) $\frac{dy}{dx} = \frac{x}{y} e^{-x^2}$ with $y(0) = 1$

(o) $\frac{dy}{dx} = y + y e^x$ with $y(0) = e$

7. (6.2-44) Populations may exhibit seasonal growth in response to seasonal fluctuations in resource availability. A simple model accounting for seasonal fluctuations in the abundance N of a population is

$$\frac{dN}{dt} = (R + \cos t)N$$

where R is the average per-capita growth rate and t is measured in years.

(a) Assume $R = 0$ and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

(b) Assume $R = 1$ (more generally $R > 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

(c) Assume $R = -1$ (more generally $R < 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?