Final practice 1

UCLA: Math 115A, Winter 2020

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Date:
Version: practice

- This exam has 8 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Question 2 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. In each of the following questions, fill in the blanks to complete the statement of the definition or theorem.

(a) (2 points) *Definition:* Let V be a vector space. A subset B is called a basis of V if B is

- **linearly independent** and
- is a **spanning** set.

(b) (2 points) *Definition:* Let V and W be vector spaces over a field $\mathbb{F}$. A function $T : V \longrightarrow W$ is a linear map if

- $T(v + w) = T(v) + T(w)$ for every $v, w \in V$, and
- $T(\lambda v) = \lambda T(v)$ for every $\lambda \in \mathbb{F}$ and $v \in V$.

(c) (2 points) *Theorem:* Suppose $\dim V = n$. If $T : V \longrightarrow V$ is a linear map with n distinct eigenvalues, then T is **diagonalisable**.

(d) (2 points) *Definition:* A basis B for a vector space V equipped with an inner product is called orthonormal if

- $||v|| = 1$ for every $v \in B$, and
- $\langle v, w \rangle = 0$ for every $v, w \in B$.

(e) (2 points) *Definition:* The Frobenius inner product is an inner product defined on the vector space $\mathrm{Mat}_{m\times n}(\mathbb{F})$ and is defined by the formula

$$\langle A, B \rangle = \mathrm{tr}(B^* A).$$

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) The subset
   $$V_\lambda = \{\, p \in \mathbb{R}[x] \mid p'(\lambda) = 0 \,\}$$
   is a subspace

   A. for any choice of $\lambda \in \mathbb{R}$.
   B. only for $\lambda = 0$.
   C. whenever $\lambda > 0$.
   D. for no choice of λ.

   (b) (2 points) As a subset of $\mathrm{Mat}_{2\times 2}(\mathbb{C})$, the set
   $$\left\{ \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\}$$

   A. is a spanning set.
   B. is linear independent.
   C. is neither spanning nor linearly independent.
   D. is a basis.

   (c) (2 points) Suppose $n \geq 2$. Consider the function $T : \mathrm{Mat}_{n\times n}(\mathbb{C}) \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{C})$ given by $T(M) = M^t$, the transpose matrix. Which of the following is *not* true.

   A. T is a linear map.
   B. $T^2 = -\,\mathrm{id}$.
   C. T has eigenvalues 1 and -1.
   D. T has an eigenspace of dimension $\frac{1}{2}n(n+1)$.

   (d) (2 points) Consider again, the map T given above. Which of the following is true.

   A. T is diagonalisable, only when $n = 2$.
   B. T is diagonalisable for any n.
   C. T is never diagonalisable.
   D. The characteristic polynomial for T does not split.

   (e) (2 points) Which of the following pairs of matrices are orthogonal in $\mathrm{Mat}_{2\times 2}(\mathbb{C})$ equipped with the Frobenius inner product?

   A. $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$
   B. $\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \begin{pmatrix} i & -1 \\ 1 & i \end{pmatrix}$
   C. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$
   D. $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$

3. Consider the vector space $V = P_2(\mathbb{R})$ with its standard ordered basis

$$\beta = \{1, x, x^2\}$$

and the linear maps

$$T : P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ T(f) = f(1) + f(-1)x + f(0)x^2$$
$$S : P_2(\mathbb{R}) \to P_2(\mathbb{R}),\ S(ax^2 + bx + c) = cx^2 + bx + a.$$

(a) (2 points) What is $[T]_\beta$ and $[S]_\beta$? Show that

$$[TS]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) (6 points) Compute $[(TS)^{-1}]_\beta$.

(c) (2 points) What is $(TS)^{-1}(x^2 + x + 1)$?

Solution:

(a) We have

$$T(1) = 1 + x + x^2$$
$$T(x) = 1 - x$$
$$T(x^2) = 1 + x$$

Hence

$$[T]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

Also clearly

$$[S]_\beta = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Using $[TS]_\beta = [T]_\beta [S]_\beta$ we get

$$[TS]_\beta = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) We have $[(TS)^{-1}]_\beta = [TS]_\beta^{-1}$. We hence have to invert $[TS]_\beta$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Substract 2nd from 1st

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Norm 2nd

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Substract 2nd from 1st

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Substract 3nd from 1st

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

Hence

$$[(TS)^{-1}]_\beta = \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) We know

$$\begin{aligned} [(TS)^{-1}(x^2+x+1)]_\beta &= [(TS)^{-1}]_\beta[(x^2+x+1)]_\beta \\ &= \begin{bmatrix} 1/2 & 1/2 & -1 \\ 1/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

These are the coordinates of $x^2 = (TS)^{-1}(x^2+x+1)$.

4. Consider the matrix

$$A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \in M_{3\times 3}(\mathbb{R}),$$

   (a) (2 points) Compute the characteristic polynomial of A and determine the eigenvalues and their algebraic multiplicity.

   (b) (6 points) Is A is diagonalizable? If yes, compute a basis β of eigenvectors of A.

   (c) (2 points) Compute $[\mathrm{L}_A]_\beta$, where the L_A is the linear transformation given by

$$\mathrm{L}_A : \mathbb{R}^3 \to \mathbb{R}^3,\ v \mapsto Av.$$

Solution:

(a) We compute

$$\det(A - tI_3) = \det \begin{bmatrix} -t & 1 & -2 \\ 1 & -t & -2 \\ 0 & 0 & -1-t \end{bmatrix} = (t^2 - 1)(-1 - t) = -(t+1)^2(t-1)$$

(b) The eigenvalues are the roots of the characteristic polynomial and hence $\lambda = 1, -1$ with multiplicity 1 and 2 respectively.

(c) We compute

$$\mathrm{N}(A - 1I_3) = \mathrm{Span}((1,1,0)^t)$$
$$\mathrm{N}(A - (-1)I_3) = \mathrm{Span}((-1,1,0)^t, (2,0,1)^t)$$

with our favorite algorithm (Wolfram Alpha). Hence

$$\beta = \{(1,1,0)^t, (-1,1,0)^t, (2,0,1)^t\}$$

is a basis of eigenvectors.

(d) A diagonal matrix with entries $1, -1, -1$ in an appropriate order.

5. Consider the vector space $V = \mathbb{R}^4$ with its standard inner product. Consider the linearly independent subset
$$S = \{w_1 = (1,0,1,0), w_2 = (1,1,1,1), w_3 = (2,2,0,2)\}.$$
   (a) (6 points) Apply the Gram-Schmidt orthogonalization algorithm to S to compute an orthogonal basis β' of $\mathrm{Span}(S)$.
   (b) (2 points) Use your result to compute an orthonormal basis β of $\mathrm{Span}(S)$.
   (c) (2 points) Let $x = (1,2,3,2) \in \mathrm{Span}(S)$. Compute the coordinate vector $[x]_\beta$.

Solution:

(a)

$$\begin{aligned}
v_1 &= w_1 \\
&= (1,0,1,0) \\
v_2 &= w_2 - \frac{\langle w_2, v_1\rangle}{\langle v_1, v_1\rangle} v_1 = (1,1,1,1) - \frac{2}{2}(1,0,1,0) \\
&= (0,1,0,1) \\
v_3 &= w_3 - \frac{\langle w_3, v_1\rangle}{\langle v_1, v_1\rangle} v_1 - \frac{\langle w_3, v_2\rangle}{\langle v_2, v_2\rangle} v_2 = (2,2,0,2) - \frac{2}{2}(1,0,1,0) - \frac{4}{2}(0,1,0,1) \\
&= (1,0,-1,0)
\end{aligned}$$

(b)

$$\begin{aligned}
u_1 &= \frac{1}{||v_1||} v_1 = \frac{1}{\sqrt{2}}(1,0,1,0) \\
u_2 &= \frac{1}{||v_2||} v_2 = \frac{1}{\sqrt{2}}(0,1,0,1) \\
u_3 &= \frac{1}{||v_3||} v_3 = \frac{1}{\sqrt{2}}(1,0,-1,0)
\end{aligned}$$

(c)

$$\begin{aligned}
\langle x, v_1\rangle &= \frac{4}{\sqrt{2}} \\
\langle x, v_2\rangle &= \frac{4}{\sqrt{2}} \\
\langle x, v_3\rangle &= \frac{-2}{\sqrt{2}}
\end{aligned}$$

Hence $[x]_\beta = \frac{1}{\sqrt{2}}(4,4,-2)$.

6. Let $S : U \to V$ and $T : V \to W$ be linear transformations between finite dimensional vector spaces U, V and W over a field F.
   (a) (2 points) Let $v_1, v_2, \ldots, v_n \in V$ be linearly independent and $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$, such that $\lambda_i \neq 0$ for all $1 \leq i \leq n$. Show that also $\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n$ are linearly independent.
   (b) (4 points) Let $v, w \in V$. Show that $\text{span}(v, w) = \text{span}(v, v + w)$.
   Hint: Proceed in two steps: Show that for all $x \in V$,
   1. $x \in \text{span}(v, w)$ implies $x \in \text{span}(v, v + w)$ and
   2. $x \in \text{span}(v, v + w)$ implies $x \in \text{span}(v, w)$.

   (c) (4 points) Assume that $\text{R}(S) = \text{N}(T)$, i.e. the range of S is equal to the nullspace of T. Assume furthermore that S is one-to-one and T is onto. Show that
   $$\dim V = \dim U + \dim W.$$
   Hint: Use the rank-nullity formula.

Solution:

(a) Let $a_1, a_2, \ldots, a_n \in F$ such that
$$a_1\lambda_1 v_1 + a_2\lambda_2 v_2 + \cdots + a_n\lambda_n v_n = 0.$$
Since $v_1, v_2, \ldots, v_n$ are linearly independent we get
$$a_1\lambda_1 = a_2\lambda_2 = \cdots = a_n\lambda_n = 0.$$
Since the λ_i are nonzero this implies
$$a_1 = a_2 = \cdots = a_n = 0.$$
Hence $\lambda_1 v_1, \lambda_2 v_2, \ldots, \lambda_n v_n$ are linearly independent.

(b) 1.) Let $x \in \text{span}(v, w)$. Hence there are $a, b \in F$ such that
$$x = av + bw = (a - 1)v + b(v + w).$$
Hence $x \in \text{span}(v, v + w)$.
2.) Now let $x \in \text{span}(v, v + w)$. Hence there are $a, b \in F$ such that
$$x = av + b(v + w) = (a + 1)v + bw.$$
Hence $x \in \text{span}(v, w)$.
Putting 1.) and 2.) together we get $\text{span}(v, w) = \text{span}(v, v + w)$.

(c) Since S is one-to-on we have
$$\dim(U) = \text{rank}(S).$$
Since T is onto we have
$$\dim(V) = \text{nullity}(T) + \dim(W).$$
Since $\text{R}(S) = \text{N}(T)$ we have
$$\text{nullity}(T) = \text{rank}(S) = \dim(U).$$
Hence we get
$$\dim(V) = \dim(U) + \dim(W).$$

7. Let V be a finite dimensional vector space over a field F. Recall that

$$\mathcal{L}(V,V) = \{T : V \to V \mid T \text{ is a linear transformation}\}$$

denotes the vector space of linear transformations from V to V (also called linear operators on V). Fix a vector $v \in V$ and define

$$Z = \{T \in \mathcal{L}(V,V) \mid T(v) = 0\}.$$

One calls Z the *annihilator* of v in $\mathcal{L}(V,V)$.

(a) (4 points) Show that Z is a subspace of $\mathcal{L}(V,V)$.

(b) (2 points) Let $\lambda \in F$ such that $\lambda \neq 0$. Prove or disprove (by finding a counterexample) that

$$Z' = \{T \in \mathcal{L}(V,V) \mid T(v) = \lambda v\}$$

is a subspace of $\mathcal{L}(V,V)$.

(c) (2 points) Assume that $\beta = \{v_1, \dots, v_n\}$ is an ordered basis of V, such that $v_1 = v$. Let $T \in \mathcal{L}(V,V)$. Show that $T \in Z$ if and only if the first column of $A = [T]_\beta$ equals 0.

(d) (2 points) Assuming $v \neq 0$, what is $\dim(Z)$?

Solution:

(a) One easily checks

$$\phi_v : \mathcal{L}(V,V) \to F, T \mapsto T(v)$$

is linear. Hence $Z = \mathrm{N}(\phi_v)$ is a subspace.

(b) Let $V = \mathbb{R}$, $v = 1$, $T_0 : \mathbb{R} \to \mathbb{R}, x \mapsto 0$ be the zero map and $\lambda = 1$. Then $T_0 \notin Z'$, since $T_0(v) = 0 \neq 1 = \lambda v$. Hence Z' is not a subspace.

(c) We know that the first column in A is given by $[T(v)]_\beta$. Assume that $T \in Z$, then $T(v) = 0$ and clearly $[T(v)]\beta = 0$. Assume that the first column of A equals zero, i.e $[T(v)]_\beta = 0$. Then clearly $T(v) = 0$.

(d) Denote by $X \subset M_{n,n}(F)$ the subspace of all matrices whose first column is zero. Then clearly $\dim Z_m = n^2 - n$. By the last part

$$[-]_\beta : Z \to X, T \mapsto [T]_\beta$$

is an isomorphism (since $[-]_\beta : \mathcal{L}(V,V) \to M_{n,n}(F)$ is). Hence $\dim Z = \dim X = n^2 - n$.

8. Let V be a finite dimensional vector space over $\mathbb{R}$ with an inner product $\langle x, y\rangle \in \mathbb{R}$ for $x, y \in V$.
   (a) (3 points) Let $\lambda \in \mathbb{R}$ with $\lambda > 0$. Show that
   $$\langle x, y\rangle' = \lambda\langle x, y\rangle, \text{ for } x, y \in V,$$
   defines an inner product on V.
   (b) (2 points) Let $T : V \to V$ be a linear operator, such that
   $$\langle T(x), T(y)\rangle = \langle x, y\rangle, \text{ for all } x, y \in V.$$
   Show that T is one-to-one.
   (c) (2 points) Recall that the *norm* of a vector $x \in V$ is defined by $||x|| = \sqrt{\langle x, x\rangle}$. Show that
   $$\langle x, y\rangle = \frac{1}{2}(||x + y||^2 - ||x||^2 - ||y||^2), \text{ for all } x, y \in V.$$
   Hence, the inner product can be recovered from the norm.
   Hint: Rewrite $\langle x + y, x + y\rangle$ using the properties of inner products. Use that $\langle x, y\rangle \in \mathbb{R}$ is a real number by assumption.
   (d) (3 points) Let $\beta = \{v_1, \dots, v_n\}$ be a basis of V. The *Gram matrix* $G \in \mathrm{M}_{n,n}(\mathbb{R})$ of the inner product $\langle -, -\rangle$ with respect to β is defined by
   $$G_{i,j} = \langle v_i, v_j\rangle.$$
   Show that G is invertible.

Solution:

(a) Show the 4 properties.

(b) Use non-degeneracy.

(c) Follow hint.

(d) We will show that if $Ga = 0$ for $a \in \ltimes$ then $a = 0$. This means the nullity of G os zero and thus it is invertible. To see that $a = 0$ note that there is an $x \in V$ such that $[x]_\beta = a = (a_1, \dots, a_n)^t$. Then

$$\begin{aligned} Ga = G[x]_\beta &= (\sum_{j=1}^n G_{1,j}a_j, \dots, \sum_{j=1}^n G_{n,j}a_j)^t \\ &= (\sum_{j=1}^n \langle v_1, v_j\rangle a_j, \dots, \sum_{j=1}^n \langle v_n, v_j\rangle a_j)^t \\ &= (\langle v_1, \sum_{j=1}^n a_j v_j\rangle, \dots, \langle v_n, \sum_{j=1}^n a_j v_j\rangle)^t \\ &= (\langle v_1, x\rangle, \dots, \langle v_n, x\rangle)^t \end{aligned}$$

Now, since $G[x]_\beta = 0$ that means that $\langle v_i, x\rangle = 0$ for all i, so that means $x \in V^\perp$. But $V^\perp = \{0\}$ so $x = 0$, and thusXS G is invertible.

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