

This week you will practice writing differential equations modelling real world phenomena as well as understanding population models. You will also get practice solving separable differential equations.

**Homework:** The homework will be due on Friday 9 November, at 8am, the *start* of the lecture. It will consist of questions:

3 and 7.

\*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.1) Write a differential equation to model the situations described below. Do not try to solve.

- (a) (6.1-1) The number of bacteria in a culture grows at a rate that is proportional to the number of bacteria present.

**Solution:** Lets say that the rate is  $a$  times the number of bacteria (i.e. this is the constant of proportionality). Let  $P(t)$  be the number of bacteria at time  $t$ . Then the rate at which the number grows is  $\frac{dP}{dt}$ . So the fact that the rate is  $a$  times the number of bacteria is just expressed as

$$\frac{dP}{dt} = aP$$

- (b) (6.1-2) A sample of radium decays at a rate that is proportional to the amount of radium present in the sample.

**Solution:** This is similar to the first example. However the sample is decaying. Let  $N(t)$  be the amount of radium present, and  $\lambda$  the proportion. Since it is decaying, the rate should be negative so

$$\frac{dN}{dt} = -\lambda N$$

- (c) (6.1-5) According to Benjamin Gompertz (1779-1865) the growth rate of a population is proportional to the number of individuals present, where the factor of proportionality is an exponentially decreasing function of time.

**Solution:** This is similar to the first example. However now, the proportionality  $a$  changes over time. In particular  $a$  is exponentially decreasing. Thus  $a = Ae^{-kt}$  for some  $A$  and some  $k$ . Thus

$$\frac{dP}{dt} = aP = Ae^{-kt}P.$$

- (d) (6.1-7) The rate at which an epidemic spreads through a community of  $P$  susceptible people is proportional to the product of the number of people  $y$  who have caught the disease and the number  $P - y$  who have not.
- (e) (6.1-8) The rate at which people are implicated in a government scandal is proportional to the product of the number  $N$  of people already implicated and the number of people involved who have not yet been implicated.

2. (6.1) A population model is given by

$$\frac{dP}{dt} = P(100 - P).$$

- (a) (6.1-9) For what values is the population at equilibrium?

**Solution:** The population is at equilibrium when the right hand side is zero. I.e. either  $P = 0$  or  $P = 100$ .

- (b) (6.1-10) For what values is  $\frac{dP}{dt} > 0$ ?

**Solution:** When  $0 < P < 100$ .

- (c) (6.1-11) For what values is  $\frac{dP}{dt} < 0$ ?

**Solution:** When  $P > 100$  or when  $P < 0$  (however this does not make physical sense).

- (d) (6.1-12) Describe how the fate of the population depends on the initial density.

**Solution:** If  $P(0) = 0$  or  $100$  then the population will stay at that level forever. If  $0 < P(0) < 100$  then the population will increase towards and approach  $100$ . If  $P(0) > 100$  then the population decreases to and approaches  $100$ .

3. (6.1) A population model is given by

$$\frac{dP}{dt} = P(P - 1)(100 - P).$$

- (a) (6.1-13) For what values is the population at equilibrium?

**Solution:** The equilibrium solutions of the model occur when  $P' = 0$ . That is, when  $P = 0, 1, 100$ .

- (b) (6.1-14) For what values is  $\frac{dP}{dt} > 0$ ?

**Solution:** By testing points we see that  $P' > 0$  when  $1 < P < 100$ .

- (c) (6.1-15) For what values is  $\frac{dP}{dt} < 0$ ?

**Solution:** By testing points we see that  $P' < 0$  when  $0 < P < 1$  and when  $P > 100$ .

- (d) (6.1-16) Describe how the fate of the population depends on the initial density.

**Solution:** Using the above information we see that if  $P(0) < 1$  the population eventually dies out. When  $1 < P(0)$  the population eventually stabilises at  $P = 100$ .

4. (6.1) Radioactive decay: Certain types of atoms (e.g. carbon-14, xenon-133, lead-210, etc.) are inherently unstable. They exhibit random transitions to a different atom while emitting radiation in the process. Based on experimental evidence, Rutherford found in the early 20th century that the number,  $N$ , of atoms in a radioactive substance can be described by the equation

$$\frac{dN}{dt} = -\lambda N$$

where  $t$  is measured in years and  $\lambda > 0$  is known as the *decay constant*. The decay constant is found experimentally by measuring the half life,  $\tau$  of the radioactive substance (i.e. the time it takes for half of the substance to decay). Use this information in the following problems.

- (a) (6.1-18) Find a solution to the decay equation assuming that  $N(0) = N_0$ .

**Solution:** Using separation of variables we get

$$N(t) = Ce^{-\lambda t}$$

Using the initial value gives  $C = N_0$  so

$$N(t) = N_0 e^{-\lambda t}$$

- (b) (6.1-19) For xenon-133, the half-life is 5 days. Find  $\lambda$ . Assume  $t$  is measured in days.

**Solution:** We know that after 5 days we will have half the amount left, i.e.  $N(5) = N_0/2$  this we have the equation

$$\frac{N_0}{2} = N_0 e^{-\lambda 5}$$

We can cancel the  $N_0$  and rearrange (by taking a log) to get

$$\lambda = \frac{\ln 2}{5}.$$

- (c) (6.1-20) For carbon-14 the half life is 5,568 years. Find the decay constant  $\lambda$ , assuming  $t$  is measured in years.

**Solution:** Very similar to the above, we get  $\lambda = \frac{\ln 2}{5568}$

- (d) (6.1-21) How old is a piece of human bone which contains just 60% of the amount of carbon-14 expected in a sample of bone from a living person, assuming the half life of carbon-14 is 5,568 years?

**Solution:** We know that it obeys the equation  $N(t) = N_0 e^{-\lambda t}$  where  $\lambda$  is as given above. The question is telling us to find  $t$  such that  $N(t) = 0.6N_0$ , we get an equation

$$0.6N_0 = N_0 e^{-\frac{\ln 2}{5568} t}.$$

We can rearrange this and solve for  $t$  (the  $N_0$  cancels)

$$t = -\frac{5568 \ln 0.6}{\ln 2}$$

- (e) (6.1-22) The Dead Sea Scrolls were written on parchment at about 100 B.C. What percentage of carbon-14 originally contained in the parchment remained when the scrolls were discovered in 1947?

**Solution:** If originally there was  $N_0$  carbon-14 then after 2047 years there would be

$$N(2047) = N_0 e^{-\frac{\ln 2}{5568} 2047}$$

left. As a percentage of  $N_0$  this would be  $100e^{-\frac{\ln 2}{5568} 2047} \approx 77.5\%$ .

5. (6.1-30) Hyperthyroidism is caused by a new growth of tumor-like cells that secrete thyroid hormones in excess to the normal hormones. If left untreated, a hyperthyroid individual can exhibit extreme weight loss, anorexia, muscle weakness, heart disease intolerance to stress, and eventually death. The most successful and least invasive treatment option is radioactive iodine-131 therapy.

This involves the injection of a small amount of radioactivity into the body. For the type of hyperthyroidism called Graves' disease, it is usual for about 40 – 80% of the administered activity to concentrate in the thyroid gland. For functioning adenomas ("hot nodules"), the uptake is closer to 20 – 30%. Excess iodine-131 is excreted rapidly by the kidneys. The quantity of radioiodine used to treat hyperthyroidism is not enough to injure any tissue except the thyroid tissue, which slowly shrinks over a matter of weeks to months. Radioactive iodine is either swallowed in a capsule or sipped in solution through a straw. A typical dose is 5 – 15 millicuries. The half-life of iodine-131 is 8 days.

- (a) Suppose that it takes 48 hours for a shipment of iodine-131 to reach a hospital. How much of the initial amount shipped is left once it arrives at the hospital?

**Solution:** After  $t$  days the fraction of the substance that will remain is

$$\left(\frac{1}{2}\right)^{t/8} = e^{-t(\ln 2)/8}$$

Thus after 2 days there will be  $e^{-0.25 \ln 2}$  of it remaining.

- (b) Suppose a patient is given a dosage of 10 millicuries of which 30% concentrates in the thyroid gland. How much is left one week later?

**Solution:**

$$3e^{-\frac{7}{8} \ln 2} \text{ millicuries}$$

- (c) Suppose a patient is given a dosage of 10 millicuries of which 30% concentrates in the thyroid gland. How much is left 30 days later?

**Solution:**

$$3e^{-\frac{15}{4} \ln 2} \text{ millicuries}$$

6. (6.2) Solve the following differential equations.

- (a)  $\frac{dy}{dt} = 5y$
- (b)  $\frac{dy}{dt} = -y$
- (c)  $\frac{dy}{dx} = -3y$
- (d)  $\frac{dy}{dx} = 0.2y$
- (e) (6.2-17)  $\frac{dy}{dt} = y^3$
- (f) (6.2-18)  $\frac{dy}{dt} = y \sin t$
- (g) (6.2-20)  $\frac{dy}{dt} = \frac{t}{y}$
- (h) (6.2-24)  $\frac{dy}{dx} = \frac{x}{y} \sqrt{1+x^2}$
- (i) (6.2-26)  $\frac{dy}{dx} = \frac{\sin x}{\cos y}$
- (j) (6.2-30)  $\frac{dy}{dt} = yt$  with  $y(1) = -1$
- (k) (6.2-32)  $\frac{dy}{dt} = e^{-y}t$  with  $y(-2) = 0$

- (l) (6.2-34)
- $\frac{dy}{dt} = ty^2 + 3t^2y^2$
- with
- $y(-1) = 2$

**Solution:** We begin by factorising the right hand side,

$$\frac{dy}{dt} = (t + 3t^2)y^2.$$

We can now separate variables and integrate:

$$\int \frac{1}{y^2} dy = \int t + 3t^2 dt$$

We integrate both sides using the power rule,

$$-\frac{1}{y} = \frac{1}{2}t^2 + t^3 + C$$

for an arbitrary constant  $C$ . Rearranging,

$$y(t) = -\frac{2}{t^2 + 2t^3 + C}.$$

Now we use the fact that  $y(-1) = 2$ :

$$2 = -\frac{2}{1 - 2 + C}$$

so  $C = 0$  and the solution is

$$y(t) = -\frac{2}{t^2 + 2t^3}.$$

- (m)
- $\frac{dy}{dx} = y \sin x + \frac{y}{(x+1)^2}$
- with
- $y(0) = 1$

**Solution:** We begin by factorising the right hand side,

$$\frac{dy}{dt} = y \left( \sin x + \frac{1}{(x+1)^2} \right).$$

We can now separate variables and integrate:

$$\int \frac{1}{y} dy = \int \sin x + \frac{1}{(x+1)^2} dx$$

We integrate both sides,

$$\ln(y) = -\cos x - \frac{1}{(x+1)} + C$$

for an arbitrary constant  $C$ . Exponentiating both sides,

$$y(t) = C \exp \left( -\cos x - \frac{1}{(x+1)} \right).$$

Now we use the fact that  $y(0) = 1$ :

$$1 = C \exp(-1 - 1) = C e^{-2}$$

so  $C = e^2$  and the solution is

$$y(t) = \exp \left( 2 - \cos x - \frac{1}{(x+1)} \right).$$

- (n)  $\frac{dy}{dx} = \frac{x}{y}e^{-x^2}$  with  $y(0) = 1$   
(o)  $\frac{dy}{dx} = y + ye^x$  with  $y(0) = e$

7. (6.2-44) Populations may exhibit seasonal growth in response to seasonal fluctuations in resource availability. A simple model accounting for seasonal fluctuations in the abundance  $N$  of a population is

$$\frac{dN}{dt} = (R + \cos t)N$$

where  $R$  is the average per-capita growth rate and  $t$  is measured in years.

- (a) Assume  $R = 0$  and find a solution to this differential that satisfies  $N(0) = N_0$ . What can you say about  $N(t)$  at  $t \rightarrow \infty$ ?

**Solution:** When  $R = 0$  the equation is  $N' = N \cos t$ . Using separation of variables we find the solution  $N(t) = Ce^{\sin t}$  and since  $N(0) = N_0$  we see that  $C = N_0$ . As  $N \rightarrow \infty$ , this fluctuates between  $N_0e^{-1}$  and  $N_0e$ .

- (b) Assume  $R = 1$  (more generally  $R > 0$ ) and find a solution to this differential that satisfies  $N(0) = N_0$ . What can you say about  $N(t)$  at  $t \rightarrow \infty$ ?

**Solution:** When  $R = 1$  the equation is  $N' = N(1 + \cos t)$ . Using separation of variables we find the solution  $N(t) = Ce^{t + \sin t}$  and since  $N(0) = N_0$  we see that  $C = N_0$ . As  $N \rightarrow \infty$ , the  $t$  dominates the  $\sin t$  and the population grows exponentially.

- (c) Assume  $R = -1$  (more generally  $R < 0$ ) and find a solution to this differential that satisfies  $N(0) = N_0$ . What can you say about  $N(t)$  at  $t \rightarrow \infty$ ?

**Solution:** When  $R = -1$  the equation is  $N' = N(-1 + \cos t)$ . Using separation of variables we find the solution  $N(t) = Ce^{-t + \sin t}$  and since  $N(0) = N_0$  we see that  $C = N_0$ . As  $N \rightarrow \infty$ , the  $e^{-t}$  dominates the  $e^{\sin t}$  and the population decreases to zero.