Math 3B: Lecture 14

Noah White

October 24, 2016

Quiz this week

- Quiz 3 questions will be drawn from problem set 5

Quiz this week

- Quiz 3 questions will be drawn from problem set 5
- Questions 1-5

Quiz this week

- Quiz 3 questions will be drawn from problem set 5
- Questions 1-5
- New questions have been added to the problem set (these will not be on the quiz)

Quiz this week

- Quiz 3 questions will be drawn from problem set 5
- Questions 1-5
- New questions have been added to the problem set (these will not be on the quiz)
- Notes on long division and partial fractions online

Quiz this week

- Quiz 3 questions will be drawn from problem set 5
- Questions 1-5
- New questions have been added to the problem set (these will not be on the quiz)
- Notes on long division and partial fractions online
- Solutions to requested problems are coming!

Survey

I have put a survey online:
math.ucla.edu/~noah/survey

Last time

- More integration by parts examples

Last time

- More integration by parts examples
- Polynomial long division

How to deal with rational functions?

How can we integrate something like

$$\int \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} \, \mathrm{d}x$$

or

$$\int \frac{x + 2}{x^3 - x} \, \mathrm{d}x?$$

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

using polynomial long division.

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

using polynomial long division.

This is still not something we can integrate so we need to go further.

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots$$

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots = \frac{P(x)}{Q(x)}$$

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots = \frac{P(x)}{Q(x)}$$

How do we reverse this process?

Answer: partial fractions

When the denominator is $(ax+b)(cx+d)\cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$$

When the denominator is $(ax+b)(cx+d)\cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$$

- If the degree of $Q(x)$ is larger than the degree of $P(x)$

When the denominator is $(ax+b)(cx+d)\cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_nx + b_n)$$

- If the degree of $Q(x)$ is larger than the degree of $P(x)$
- $Q(x)$ has no repeated factors. E.g. $Q(x) = (x-1)(x+2)$ but not $Q(x) = (x-1)^2(x+2)$, then

When the denominator is $(ax+b)(cx+d)\cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n)$$

- If the degree of $Q(x)$ is larger than the degree of $P(x)$
- $Q(x)$ has no repeated factors. E.g. $Q(x) = (x-1)(x+2)$ but not $Q(x) = (x-1)^2(x+2)$, then

When the denominator is $(ax + b)(cx + d) \cdots$

We want to rewrite $\frac{P(x)}{Q(x)}$ as a sum. Let

$$Q(x) = (a_1 x + b_1)(a_2 x + b_2) \cdots (a_n x + b_n)$$

- If the degree of $Q(x)$ is larger than the degree of $P(x)$
- $Q(x)$ has no repeated factors. E.g. $Q(x) = (x - 1)(x + 2)$ but not $Q(x) = (x - 1)^2 (x + 2)$, then

we can always find constants $A_1, A_2, \ldots, n$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \ldots \frac{A_n}{a_n x + b_n}$$

Example 1

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

Example 1

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

Multiplying both sides by $(x-1)(x+1)$

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x+1} + \frac{B(x-1)(x+1)}{x-1} \\ &= A(x-1) + B(x+1) \\ &= (A+B)x + (B-A) \end{aligned}$$

Example 1

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

Multiplying both sides by $(x-1)(x+1)$

$$\begin{aligned} 0x + 1 &= \frac{A(x-1)(x+1)}{x+1} + \frac{B(x-1)(x+1)}{x-1} \\ &= A(x-1) + B(x+1) \\ &= (A+B)x + (B-A) \end{aligned}$$

Comparing coefficients

$$A + B = 0 \qquad \text{and} \qquad B - A = 1$$

Example 1

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x+1} + \frac{B}{x-1}$$

Multiplying both sides by $(x-1)(x+1)$

$$\begin{aligned} 0x + 1 &= \frac{A(x-1)(x+1)}{x+1} + \frac{B(x-1)(x+1)}{x-1} \\ &= A(x-1) + B(x+1) \\ &= (A+B)x + (B-A) \end{aligned}$$

Comparing coefficients

$$A + B = 0 \qquad \text{and} \qquad B - A = 1$$

So

$$-2A = 1 \quad \text{hence} \quad A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$

Example 2

$$\frac{x-3}{x^2+3x-4} = \frac{x-3}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

Example 2

$$\frac{x-3}{x^2+3x-4} = \frac{x-3}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

Multiplying both sides by $(x+4)(x-1)$

$$x-3 = A(x-1) + B(x+4) = (A+B)x - A + 4B.$$

Example 2

$$\frac{x-3}{x^2+3x-4} = \frac{x-3}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

Multiplying both sides by $(x+4)(x-1)$

$$x-3 = A(x-1) + B(x+4) = (A+B)x - A + 4B.$$

Comparing coeffitients

$$A + B = 1 \qquad \text{and} \qquad -A + 4B = -3$$

So

Example 2

$$\frac{x-3}{x^2+3x-4} = \frac{x-3}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

Multiplying both sides by $(x+4)(x-1)$

$$x-3 = A(x-1) + B(x+4) = (A+B)x - A + 4B.$$

Comparing coeffitients

$$A + B = 1 \qquad \text{and} \qquad -A + 4B = -3$$

So

$$A = \frac{7}{5} \quad \text{and} \quad B = -\frac{2}{5}.$$

Example 3

$$\frac{6}{x^3 - 8x^2 + 19x - 12} = \frac{6}{(x-1)(x-3)(x-4)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x-4}$$

Example 3

$$\frac{6}{x^3 - 8x^2 + 19x - 12} = \frac{6}{(x-1)(x-3)(x-4)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x-4}$$

Multiplying on both sides by $(x-1)(x-3)(x-4)$

$$\begin{aligned} 6 &= A(x-3)(x-4) + B(x-1)(x-4) + C(x-1)(x-3) \\ &= (A+B+C)x^2 - (7A+5B+4C)x + 12A + 4B + 3C. \end{aligned}$$

Example 3

$$\frac{6}{x^3 - 8x^2 + 19x - 12} = \frac{6}{(x-1)(x-3)(x-4)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x-4}$$

Multiplying on both sides by $(x-1)(x-3)(x-4)$

$$\begin{aligned} 6 &= A(x-3)(x-4) + B(x-1)(x-4) + C(x-1)(x-3) \\ &= (A+B+C)x^2 - (7A+5B+4C)x + 12A + 4B + 3C. \end{aligned}$$

Comparing coefficients

$$A + B + C = 0, \quad 7A + 5B + 4C = 0 \quad \text{and} \quad 12A + 4B + 3C = 6$$

Example 3

$$\frac{6}{x^3-8x^2+19x-12} = \frac{6}{(x-1)(x-3)(x-4)} = \frac{A}{x-1}+\frac{B}{x-3}+\frac{C}{x-4}$$

Multiplying on both sides by $(x-1)(x-3)(x-4)$

$$\begin{aligned} 6 &= A(x-3)(x-4) + B(x-1)(x-4) + C(x-1)(x-3) \\ &= (A+B+C)x^2 - (7A+5B+4C)x + 12A+4B+3C. \end{aligned}$$

Comparing coefficients

$$A+B+C=0, \quad 7A+5B+4C=0 \quad \text{and} \quad 12A+4B+3C=6$$

So

$$A=1, \quad B=-3 \quad \text{and} \quad C=2.$$

Repeated factors

What if $q(x)$ contains repeated factors? E.g. if $q(x) = (x-1)^2$ or $q(x) = (x-1)(x+2)^3$?

Repeated factors

What if $q(x)$ contains repeated factors? E.g. if $q(x) = (x-1)^2$ or $q(x) = (x-1)(x+2)^3$?

For every factor $(ax+b)^k$ in $q(x)$, the partial fraction expansion has terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_k}{(ax+b)^k}.$$

Example 1

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Example 1

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Multiplying both sides by $(x-1)^2$

$$\begin{aligned} x &= A(x-1) + B \\ &= Ax + (B - A) \end{aligned}$$

Example 1

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Multiplying both sides by $(x-1)^2$

$$\begin{aligned} x &= A(x-1) + B \\ &= Ax + (B - A) \end{aligned}$$

Comparing coefficients

$$A = 1 \qquad \text{and} \qquad B - A = 0$$

Example 1

$$\frac{x}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Multiplying both sides by $(x-1)^2$

$$\begin{aligned} x &= A(x-1) + B \\ &= Ax + (B - A) \end{aligned}$$

Comparing coefficients

$$A = 1 \qquad \text{and} \qquad B - A = 0$$

So

$$A = 1 \quad \text{and} \quad B = 1.$$

Example 2

$$\frac{x^2+x+1}{x^3-3x^2+3x-1} = \frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Example 2

$$\frac{x^2+x+1}{x^3-3x^2+3x-1} = \frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Multiplying both sides by $(x-1)^3$

$$\begin{aligned} x^2+x+1 &= A(x-1)^2 + B(x-1) + C \\ &= Ax^2 + (-2A+B)x + A - B + C. \end{aligned}$$

Example 2

$$\frac{x^2+x+1}{x^3-3x^2+3x-1} = \frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Multiplying both sides by $(x-1)^3$

$$\begin{aligned} x^2+x+1 &= A(x-1)^2 + B(x-1) + C \\ &= Ax^2 + (-2A+B)x + A - B + C. \end{aligned}$$

Comparing coefficients

$$\begin{aligned} A &= 1 \\ -2A+B &= 1 \\ A-B+C &= 1. \end{aligned}$$

Example 2

$$\frac{x^2+x+1}{x^3-3x^2+3x-1} = \frac{x^2+x+1}{(x-1)^3} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$$

Multiplying both sides by $(x-1)^3$

$$\begin{aligned} x^2+x+1 &= A(x-1)^2 + B(x-1) + C \\ &= Ax^2 + (-2A+B)x + A - B + C. \end{aligned}$$

Comparing coefficients

$$\begin{aligned} A &= 1 \\ -2A+B &= 1 \\ A-B+C &= 1. \end{aligned}$$

So

$$A = 1, \quad B = 3 \quad \text{and} \quad C = 3.$$

Example 3

$$\frac{15x^2 - 17x + 3}{x^3 + 2x^2 - 7x + 4} = \frac{15x^2 - 17x + 3}{(x+4)(x-1)^2} = \frac{A}{x+4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Example 3

$$\frac{15x^2 - 17x + 3}{x^3 + 2x^2 - 7x + 4} = \frac{15x^2 - 17x + 3}{(x+4)(x-1)^2} = \frac{A}{x+4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiplying both sides by $(x+4)(x-1)^2$

$$\begin{aligned} 15x^2 - 17x + 3 &= A(x-1)^2 + B(x-1)(x+4) + C(x+4) \\ &= (A+B)x^2 + (-2A + 3B + C)x + A + 4B + 4C. \end{aligned}$$

Example 3

$$\frac{15x^2 - 17x + 3}{x^3 + 2x^2 - 7x + 4} = \frac{15x^2 - 17x + 3}{(x+4)(x-1)^2} = \frac{A}{x+4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiplying both sides by $(x+4)(x-1)^2$

$$\begin{aligned} 15x^2 - 17x + 3 &= A(x-1)^2 + B(x-1)(x+4) + C(x+4) \\ &= (A+B)x^2 + (-2A + 3B + C)x + A + 4B + 4C. \end{aligned}$$

Comparing coefficients

$$\begin{aligned} A + B &= 15 \\ -2A + 3B + C &= -17 \\ A + 4B + 4C &= 3. \end{aligned}$$

Example 3

$$\frac{15x^2 - 17x + 3}{x^3 + 2x^2 - 7x + 4} = \frac{15x^2 - 17x + 3}{(x+4)(x-1)^2} = \frac{A}{x+4} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Multiplying both sides by $(x+4)(x-1)^2$

$$\begin{aligned} 15x^2 - 17x + 3 &= A(x-1)^2 + B(x-1)(x+4) + C(x+4) \\ &= (A+B)x^2 + (-2A + 3B + C)x + A + 4B + 4C. \end{aligned}$$

Comparing coefficients

$$\begin{aligned} A + B &= 15 \\ -2A + 3B + C &= -17 \\ A + 4B + 4C &= 3. \end{aligned}$$

So

$$A = \frac{311}{25}, \quad B = \frac{64}{25} \quad \text{and} \quad C = \frac{1}{5}.$$

Side note: integrating $\frac{1}{x}$.

Recall that

Fact

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

Side note: integrating $\frac{1}{x}$.

Recall that

Fact

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

Side note: integrating $\frac{1}{x}$.

Recall that

Fact

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C$$

Using substitution this gives the formula

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln|ax+b| + C.$$

Side note: integrating $\frac{1}{x^k}$.

Recall that if $k > 1$

Fact

$$\int \frac{1}{x^k}\,\mathrm{d}x = -\frac{1}{(k-1)x^{k-1}} + C$$

Side note: integrating $\frac{1}{x^k}$.

Recall that if $k > 1$

Fact

$$\int \frac{1}{x^k} \, \mathrm{d}x = -\frac{1}{(k-1)x^{k-1}} + C$$

Side note: integrating $\frac{1}{x^k}$.

Recall that if $k > 1$

Fact

$$\int \frac{1}{x^k}\,\mathrm{d}x = -\frac{1}{(k-1)x^{k-1}} + C$$

Using substitution this gives the formula

$$\int \frac{1}{(ax+b)^k}\,\mathrm{d}x = -\frac{1}{a(k-1)(ax+c)^{k-1}} + C.$$

Integrating rational functions $p(x)/q(x)$

Action plan

Integrating rational functions $p(x)/q(x)$

Action plan

1. Express $\frac{p(x)}{q(x)}$ in the form

$$d(x) + \frac{r(x)}{q(x)}$$

   using polynomial long division.

Integrating rational functions $p(x)/q(x)$

Action plan

1. Express $\frac{p(x)}{q(x)}$ in the form

$$d(x) + \frac{r(x)}{q(x)}$$

   using polynomial long division.

2. Write $\frac{r(x)}{q(x)}$ as a sum of fractions of the form

$$\frac{A}{(ax+b)^k}$$

   using partial fractions

Integrating rational functions $p(x)/q(x)$

Action plan

1. Express $\frac{p(x)}{q(x)}$ in the form
$$d(x) + \frac{r(x)}{q(x)}$$
using polynomial long division.
2. Write $\frac{r(x)}{q(x)}$ as a sum of fractions of the form
$$\frac{A}{(ax+b)^k}$$
using partial fractions
3. Integrate all these pieces seperately.

Example 1

$$I = \int \frac{x^4 - 3x^2 + 3}{x^2 - 1}\,\mathrm{d}x$$

Example 1

$$I = \int \frac{x^4 - 3x^2 + 3}{x^2 - 1} \, \mathrm{d}x$$

Solution

Using long division

$$\frac{x^4 - 3x^2 + 3}{x^2 - 1} = x^2 - 2 + \frac{1}{x^2 - 1}$$

Example 1

$$I = \int \frac{x^4 - 3x^2 + 3}{x^2 - 1} \,\mathrm{d}x$$

Solution

Using long division and partial fractions

$$\frac{x^4 - 3x^2 + 3}{x^2 - 1} = x^2 - 2 + \frac{1}{x^2 - 1} = x^2 - 2 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

Example 1

$$I = \int \frac{x^4 - 3x^2 + 3}{x^2 - 1} \, \mathrm{d}x$$

Solution

Using long division and partial fractions

$$\frac{x^4 - 3x^2 + 3}{x^2 - 1} = x^2 - 2 + \frac{1}{x^2 - 1} = x^2 - 2 + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}$$

So

$$I = \frac{1}{3}x^2 - 2x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.$$

Example 2

$$I = \int \frac{x^3 - 2x^2 + 4x}{(x-1)^3} \, \mathrm{d}x$$

Example 2

$$I = \int \frac{x^3 - 2x^2 + 4x}{(x-1)^3}\,\mathrm{d}x$$

Solution

Using long division

$$\frac{x^3 - 2x^2 + 4x}{(x-1)^3} = 1 + \frac{x^2 + x + 1}{(x-1)^3}$$

Example 2

$$I = \int \frac{x^3 - 2x^2 + 4x}{(x-1)^3}\,\mathrm{d}x$$

Solution

Using long division and partial fractions

$$\frac{x^3 - 2x^2 + 4x}{(x-1)^3} = 1 + \frac{x^2 + x + 1}{(x-1)^3} = 1 + \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3}$$

Example 2

$$I = \int \frac{x^3 - 2x^2 + 4x}{(x-1)^3} \, \mathrm{d}x$$

Solution

Using long division and partial fractions

$$\frac{x^3 - 2x^2 + 4x}{(x-1)^3} = 1 + \frac{x^2 + x + 1}{(x-1)^3} = 1 + \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3}$$

So

$$I = x + \ln|x-1| - \frac{3}{x-1} - \frac{3}{2(x-1)^2} + C.$$