Final exam practice 2

UCLA: Math 3B, Winter 2019

Instructor: Noah White
Date:

- This exam has 7 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____

ID number: ____

Discussion section (please circle):

Day/TA	Louis	Matthew
Tuesday	1A	1C
Thursday	1B	1D

Question	Points	Score
1	12	
2	12	
3	12	
4	10	
5	12	
6	14	
7	8	
Total:	80	

Questions 1 and 2 are multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following four pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) The function $f(x) = e^{x+\cos x}$ is

   A. Always increasing.
   B. Always decreasing.
   C. Always concave up.
   D. Always concave down.

   (b) (2 points) The function $f(x) = e^{x} - e^{-x}$ is

   A. Always increasing.
   B. Always decreasing.
   C. Always concave down.
   D. Always concave up.

   (c) (2 points) The function $f(x) = \frac{4x^3-x}{x^3+x}$ has a

   A. Horizontal asymptote at $y = 4$.
   B. Vertical asymptote at $x = 1$.
   C. Slanted asymptote with slope -1.
   D. Slanted asymptote with slope 1.

(d) (2 points) The function $f(x) = \frac{2x^2-x}{x+1}$ has a

A. Horizontal asymptote at $y = 1$.
B. No vertical asymptotes.
C. A vertical asymptote at $x = 0$.
D. Slanted asymptote with slope 2.

(e) (2 points) The function $f(x) = e^{\sin x}$ has a critical point at

A. $x = \pi/2$
B. $x = 0$
C. $x = \pi$
D. $x = -\pi$

(f) (2 points) The function $f(x) = x - \ln(x^4)$ has a

A. minimum at $x = 4$.
B. maximum at $x = 4$.
C. minimum $x = 1$.
D. maximum $x = 1$.

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) The definite integral $\int_1^e \frac{1}{x}\,\mathrm{d}x$ has a value of

   A. e.

   B. 0.

   C. 1.

   D. π.

   (b) (2 points) The definite integral $\int_{\pi/2}^{\pi} x\sin x\,\mathrm{d}x$ has a value of

   A. $\pi/2$.

   B. π.

   C. $\pi - 1$**.**

   D. $-\pi$.

   (c) (2 points) The solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = 4y$ when $y(0) = 2$ has

   A. $y(1) = 3e^{-2}$

   B. $y(0.5) = 2e$.

   C. $y(1) = 3e$.

   D. $y(0.5) = 2e^2$**.**

(d) (2 points) The solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-y}(2t+1)$ when $y(0) = 0$ has

A. $y(\pi) = \ln 2$

B. $y(-\pi/2) = 0$.

C. $y(4) = \ln 2$.

D. $y(1) = \ln 3$.

(e) (2 points) The differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = (1-y)\left(\frac{y}{9} - 1\right)$ has a

A. stable equilibrium at $y = 9$.

B. unstable equilibrium at $y = 1/9$.

C. stable equilibrium at $y = 1$.

D. unstable equilibrium at $y = 3$.

(f) (2 points) The differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - (y-3)^3$ has a

A. stable equilibrium at $y = -2$.

B. unstable equilibrium at $y = -1$.

C. stable equilibrium at $y = 4$.

D. unstable equilibrium at $y = 4$.

3. Let $f(x) = \frac{1}{e^{2x}+1}$. Note that $f'(x) = -\frac{2e^{2x}}{(e^{2x}+1)^2}$ and $f''(x) = \frac{4e^{2x}(e^{2x}-1)}{(e^{2x}+1)^3}$.

   (a) (2 points) Does $f(x)$ cross the x and y axes? If so, where?

   Solution: No x-intercepts, y-intercept at $y = 0.5$.

   (b) (2 points) Does $f(x)$ have any horizontal asymptotes? If so what are they?

   Solution: We need to evaluate

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{1}{e^{2x}+1} = 0.$$

   and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{e^{2x}+1} = 1.$$

   So we have horizontal asymptote in the positive direction at $y = 0$ and in the negative direction at $y = 1$.

   (c) (1 point) Does $f(x)$ have any vertical asymptotes? If so what are they?

   Solution: The denominator of $f(x)$ is never zero. Hence no vertical asymptote exits.

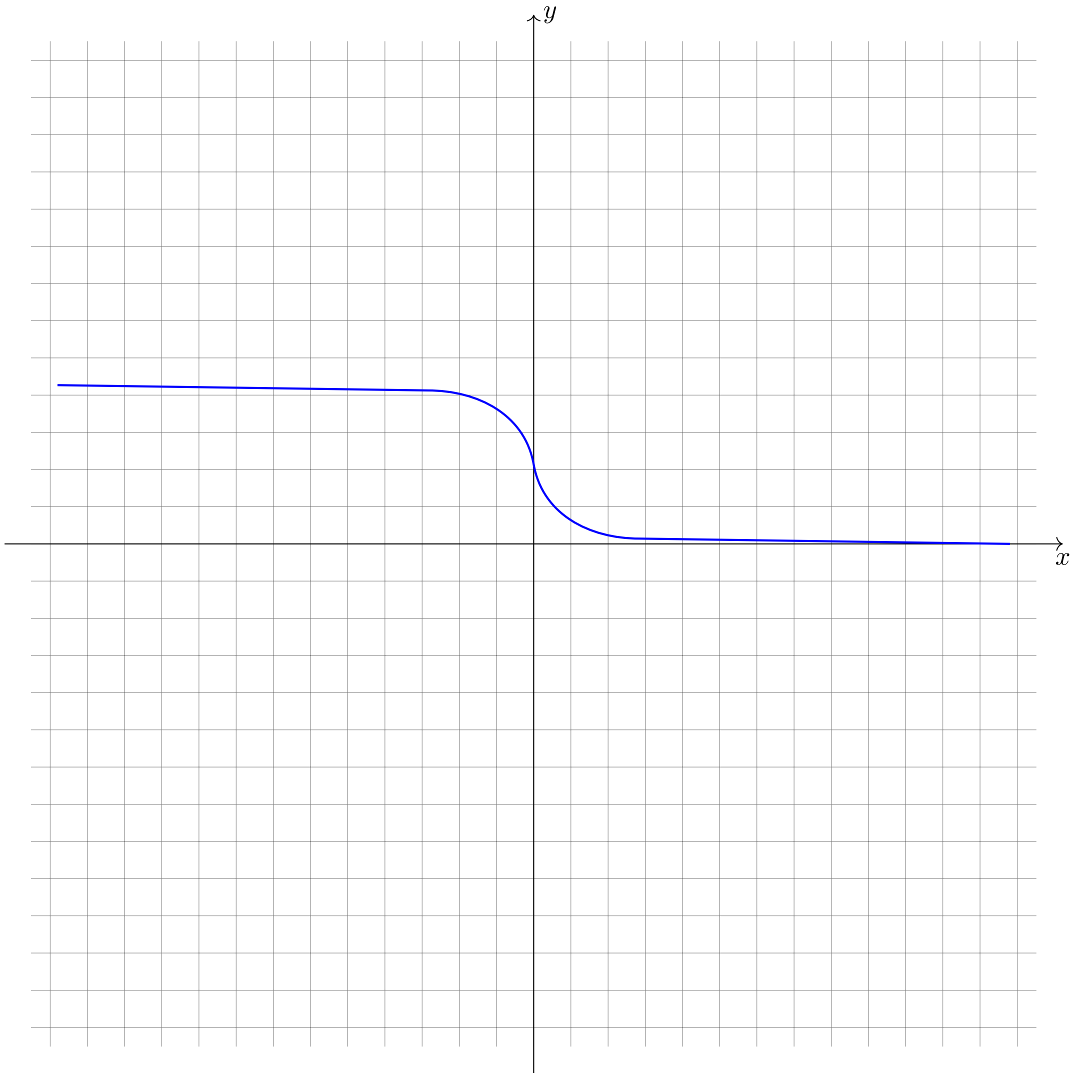
   (d) (2 points) For what x is the first derivative $f'(x)$ positive?

   Solution: The denominator is always positive, as is the numerator. Thus $f'(x)$ is negative for all x.

(e) (2 points) For what x is the second derivative $f''(x)$ positive?

> **Solution:** The denominator is always positive. The numerator is positive when $e^{2x} - 1 > 0$, i.e when $e^{2x} > 1$, thus, $f''(x) > 0$ when $x > 0$.

(f) (3 points) On the graph provided, sketch $f(x)$



4. A new invasive species of fish has recently been found of the coast of Australia in the Great Barrier Reef. Government researchers determine that the species of fish are migrating to the Reef at a rate of 30,000 individuals per year. They also estimate that the growth in population due to births will be roughly exponential; this means, for every one fish of this species at the Reef, t years later there will be

$$e^{0.5t}$$

fish. Initially the Reef contained no individuals from this species.

(a) (6 points) Write a Riemann sum which represents the total number of fish from this species in the Reef in 10 years. Be sure to define any symbols that you use (e.g. t_k, Δt, etc).

Solution: Let $\Delta t = 10/n$ and $t_k = k\Delta t$.

$$T = \lim_{n\to\infty} \sum_{k=1}^{n} 30,000 e^{0.5(10-t_k)} \Delta t$$

(b) (4 points) Use an integral to evaluate the Riemann sum above.

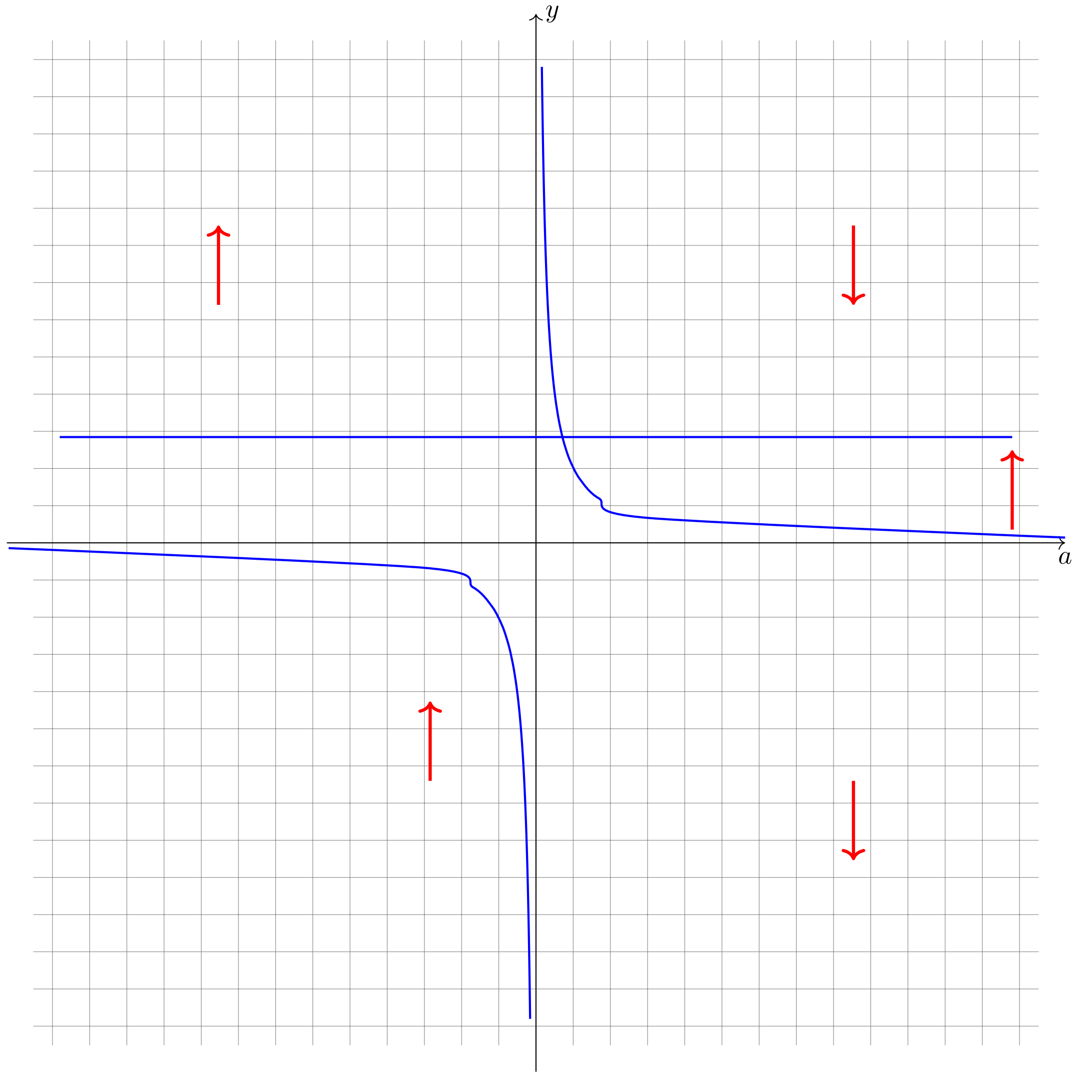
Solution:

$$\begin{aligned} W &= \int_0^{10} 30,000 e^{0.5(10-t)}\, \mathrm{d}t \\ &= \Big[-60,000 e^{0.5(10-t)}\Big]_0^{10} \\ &= -60,000 - \big(-60,000 e^5\big) \\ &\approx 8,905,000 \end{aligned}$$

5. In this question we will investigate the behaviour of the solutions of

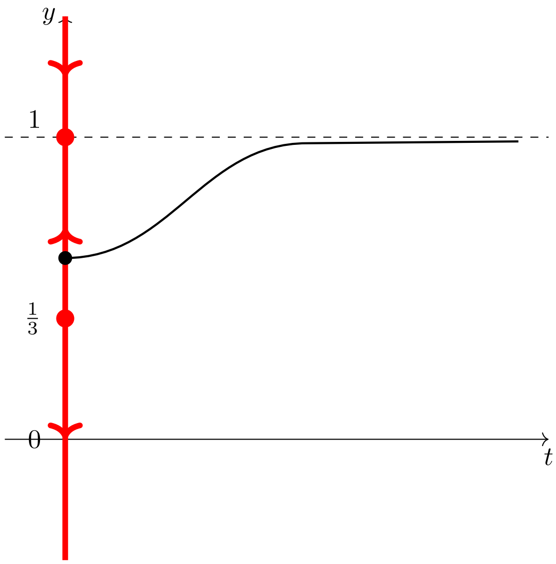
$$\frac{\mathrm{d}y}{\mathrm{d}t} = (y-1)(1-ay)$$

(a) (4 points) Draw a bifurcation diagram for this equation with parameter a. Make sure to label the regions of your diagram with up/down arrows according to the direction of the derivative.



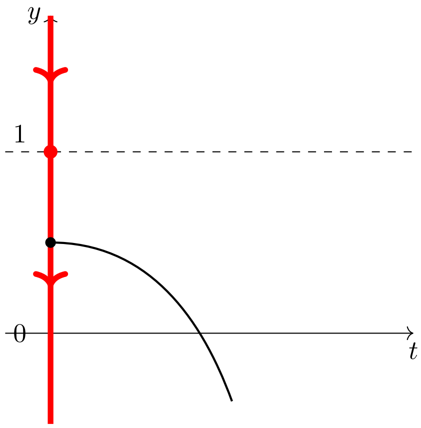
(b) (2 points) Draw a phase diagram when $a = 3$ and sketch the solution if $y(0) = 0.5$.

Solution:



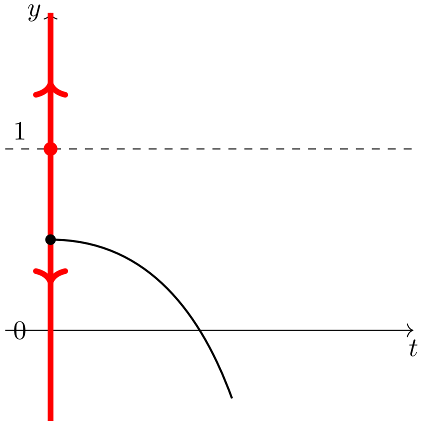
(c) (2 points) Draw a phase diagram when $a = 1$ and sketch the solution if $y(0) = 0.5$.

Solution:



(d) (2 points) Draw a phase diagram when $a = 0$ and sketch the solution if $y(0) = 0.5$.

Solution:



(e) (2 points) The differential equation above has an a equilibrium solution of $y = 1$ for any value of a. For what a is this equilibrium stable?

Solution: We can see from the picture that this equilibrium is stable to the right of the point it crosses the hyperbola and otherwise is unstable. Thus this equilibrium is stable when $a > 1$. (Note the strictly greater than!)

6. A patient is receiving a continuous injection of a drug with a half life of 5 hours. The drug is being administered at 2 mg/h.

   (a) (4 points) Write a differential equation modelling the total amount of drug $y(t)$ (in milligrams) in the patient's body.

   Solution: The rate in is 2 and the rate out, because of the half life is $\frac{\ln 2}{5}$. Thus the ODE is

   $$\frac{\mathrm{d}y}{\mathrm{d}t} = 2 - \frac{\ln 2}{5} y.$$

   (b) (3 points) Solve the differential equation. Note that initially the patient does not have any drug in their system.

   Solution: The general solution is

   $$y(t) = \frac{10}{\ln 2} - Ce^{-t\frac{\ln 2}{5}}.$$

   Since $y(0) = 0$ we can use this to find that $C = \frac{10}{\ln 2}$, thus

   $$y(t) = \frac{10}{\ln 2}\left(1 - e^{-t\frac{\ln 2}{5}}\right).$$

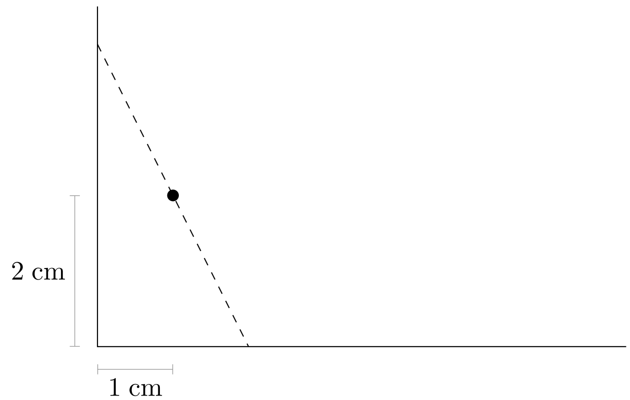
(c) (1 point) If the patient has 5 L of blood, what is the concentration of the drug on their bloodstream in the long term?

> **Solution:** In the long term, the total amount of drug in the blood stream is $\lim_{t\to\infty} y(t) = \frac{10}{\ln 2}$. To find the concentration we divide by the total amount of blood to get (in milligrams per liter)
>
> $$\frac{2}{\ln 2}.$$

(d) (6 points) Now suppose that the drug is being administered at an unknown rate. You may assume that it still has a half life of 5 hours. After 10 hours, doctors find that the concentration of the drug is 3 mg/L. If the patient has 5 L of blood, at what rate was the drug being administered? Please give an exact value. No decimals.

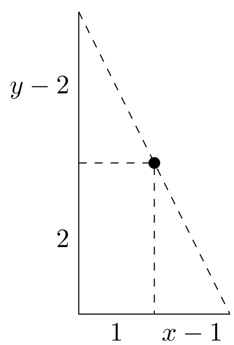
> **Solution:** If the drug is being administered at a milligrams per hout, the ODE we have now is
>
> $$\frac{\mathrm{d}y}{\mathrm{d}t} = a - \frac{\ln 2}{5} y.$$
>
> The solution is $y(t) = \frac{5a}{\ln 2} - Ce^{-t\frac{\ln 2}{5}}$. We use $y(0) = 0$ to find that $C = \frac{5a}{\ln 2}$ and our solution is
>
> $$y(t) = \frac{5a}{\ln 2}\left(1 - e^{-t\frac{\ln 2}{5}}\right).$$
>
> Now we use the fact that $y(10) = 3 \cdot 5 = 15$ so
>
> $$15 = \frac{5a}{\ln 2}\left(1 - e^{-10\frac{\ln 2}{5}}\right)$$
>
> Solving for a we get
>
> $$\begin{aligned} \frac{1}{a} &= \frac{5}{15\ln 2}\left(1 - e^{-10\frac{\ln 2}{5}}\right) \\ &= \frac{1}{3\ln 2}\left(1 - e^{-2\ln 2}\right) \\ &= \frac{1}{3\ln 2}\left(1 - e^{\ln 4^{-1}}\right) \\ &= \frac{1}{3\ln 2}\left(1 - \frac{1}{4}\right) \\ &= \frac{1}{4\ln 2}. \end{aligned}$$
>
> so $a = 4\ln 2 \approx 2.77$.

7. (8 points) A large, rectangular piece of material has a small stain in a corner, 1 cm from one edge and 2 cm from the other. We would like to make a single, straight cut to remove the stained corner. What is the area of the smallest piece we could remove?



Solution: We want to make a cut along a stright line that passes through the small stain as shown by the dashed line above. Let x be the distance from the lower left corner to where the cut starts on the bottom edge. And similarly y is the distance from the lower left corner to where the cut meets the left edge.

We have the following diagram.



Thus the area of the piece we cut off is $A = \frac{1}{2}xy$. To eliminate the dependance on y we can use the diagram above. Notice the triangles are similar so we can see that

$$\frac{y-2}{2} = \frac{1}{x-1}$$

Thus

$$y = \frac{2}{x-1} + 2.$$

Thus the area is

$$A(x) = \frac{x}{x-1} + x.$$

To minimize this function we find the derivative and its zeros (the critical points).

$$A'(x) = \frac{1}{x-1} - \frac{x}{(x-1)^2} + 1 = \frac{x(x-2)}{(x-1)^2}$$

The only way for this to be zero is if $x = 0$ or $x = 2$. It is also undefined at $x = 1$. Note that the domain of our function is $(1, \infty)$ (any other value of x does no correspond to us cutting off the corner!). We use teh open interval method: our only critical point is $x = 2$ and $A(2) = 4$. We need to compare this to

$$L = \lim_{x \to 1} A(x) = \infty \qquad M = \lim_{x \to \infty} A(x) = \infty$$

so since $A(2) < L, M$, it must be that $x = 2$ is a global minimum.

The smallest area we can cut off is 4.

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