Lecture 1

1. Functions

- Mathematics was revolutionised by the observation

  "To understand a geometric space we should study functions on it"

- geometric spaces: $\mathbb{R}^1$, $\mathbb{R}^2$, ..., curves in $\mathbb{R}^3$,

  a circle $= \{(x\ y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

  a sphere $= \{(x\ y\ z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$

  twisted cubic $= \{(x\ y\ z) \in \mathbb{R}^3 \mid \cancel{xz = y^2 =}\ x - z^3,\ y - z^2\}$

  etc.

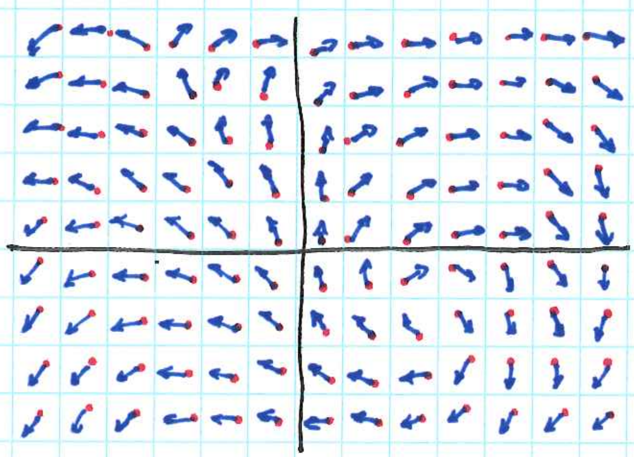
- functions assign, to ever point of our space a

  number, vector, other things (functions / lines / ...)

  We concentrate on

- Often we call this a "field" (scalar or vector field).

A vector field in $\mathbb{R}^2$

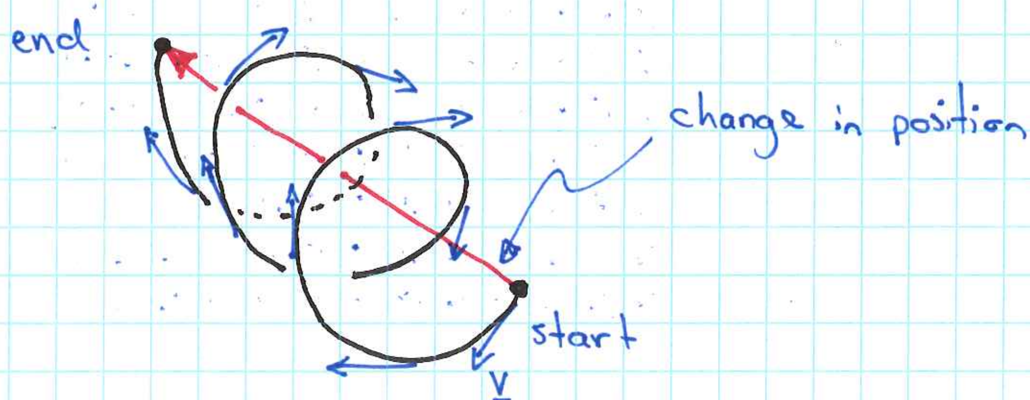


Examples

- temperature. We can assign to ever point on earth its temp (scalar field)
- wind direction/velocity (vector field)
- force due to gravity (vecter field)
- density (kg/m^3) at each point in the earth. (scalar field).

2. Integration

- What does it mean to integrate a function over a geometric space?
- Integrals answer questions like:
  * Given a solid body and its density at every point, what is its mass?
  * Given a probability density function/distribution, what is the probability the out come of the experiment is in a given range?
  * Given a particle's velocity (a vector in $\mathbb{R}^3$) over a period of time, how has its position changed?



* Given the graph of a function $\mathbb{R} \to \mathbb{R}$
  what is the "area under the graph"
* Given ~~a~~ the height of a rectangular tent ~~abo~~
  at every point on the ground, what volume does
  the tent enclose

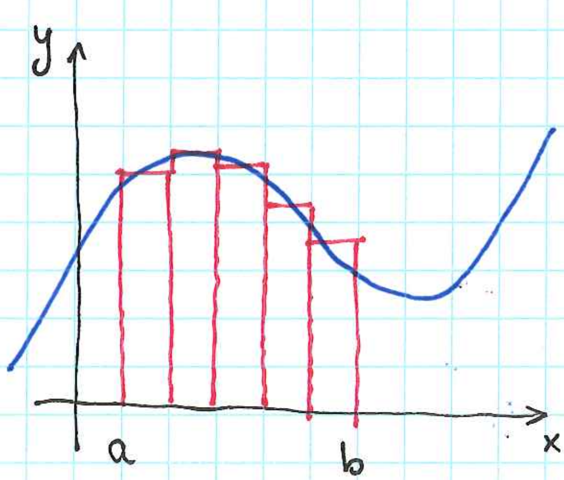
- In the last example, we can think of the height
  of the tent as a function

$$f(x,y) : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

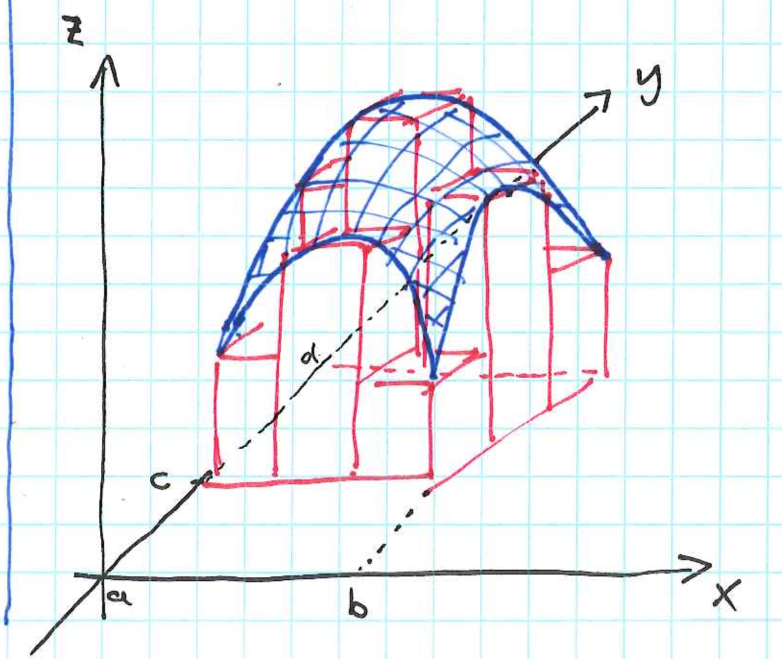
and its "graph" as a surface floating in $\mathbb{R}^3$
above (or under / through) the x-y-plane. We can
consider finding the volume under such a surface.

3. Riemann Sums $\int_a^b f(x)dx$ vs $\iint_R f(x\ y)dA$ $R=[a\ b]\times[c\ d]$

1 D



2 D

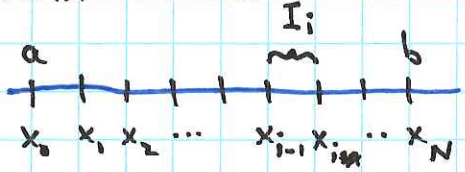


1D

* Partition interval [a b] into

$a = x_0 < x_1 < \dots < x_N = b$

so we have a line of subintervals



$I_i = [x_{i-1}, x_i]$

* Pick a sample point in each $P_i \in I_i$

* What is the area of the rectangle with

width: $\Delta x_i = x_i - x_{i-1}$

height: $f(P_i)$

A $f(P_i)\Delta x_i$

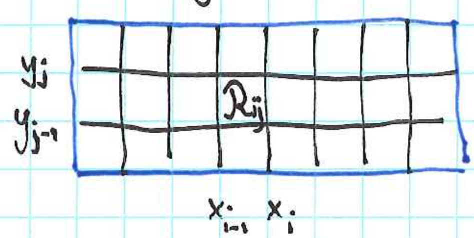
2D

* Partition [a,b] and [c d] into

$a < x_0 < x_1 < \dots < x_N = b$

$c < y_0 < y_1 < \dots < y_M = d$

so we have a $N \times M$ grid of subrectangles



$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$

* Pick a sample point in each, $P_{ij} \in R_{ij}$, let $P_{ij} = (P_{ij}^{(x)}, P_{ij}^{(y)})$

* What is the volume of the rectangular prism with

width: $\Delta x_i = x_i - x_{i-1}$

depth: $\Delta y_i = y_i - y_{i-1}$

height: $f(P_{ij}) = f(P_{ij}^{(x)}, P_{ij}^{(y)})$

$f(P_{ij})\Delta x_i \Delta y_i$

1D

* Estimate the total area by summing all contributions

$$S_{N\cancel{M}} = \sum_{i=1}^{N} \cancel{\sum_{j=1}^{M}} f(P_i)\,\Delta x_i$$

* Define the integral by taking a limit as the size of the partition shrinks.

2D

* Estimate the total volume by summing all contributions

$$S_{NM} = \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij})\,\Delta x_i\,\Delta y_i$$

* ditto.

Def

$$\iint_R f(x\,y)\,dA = \lim_{\|P\|\to 0} \sum_{i=1}^{N} \sum_{j=1}^{M} f(P_{ij})\,\Delta x_i\,\Delta y_i$$

where $\|P\| = \max \left\{ \Delta x_i , \Delta y_j \right\}_{i=1,j=1}^{N,M}$.

If the limit exists we say $f(x\,y)$ is integrable over R.

Thm Some properties: suppose $f(x\,y)$ and $g(x\,y)$ are integrable over R, then

$$\text{(i)} \quad \iint_R f(x\,y) + g(x\,y)\,dA = \iint_R f(x\,y)\,dA + \iint_R g(x\,y)\,dA$$

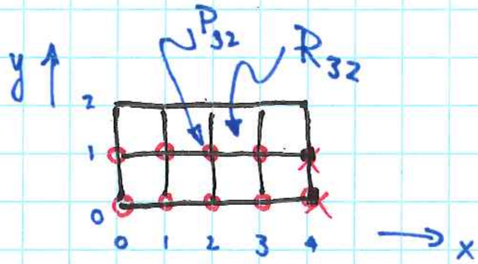
and for ~~and~~ any scalar α

$$\text{(ii)} \quad \iint_R \alpha f(x\,y)\,dA = \alpha \iint_R f(x\,y)\,dA.$$

4. Examples

Ex $\iint_R x^2+y^2 \, dA \qquad R = [0,4] \times [0, 2]$

with the partition:



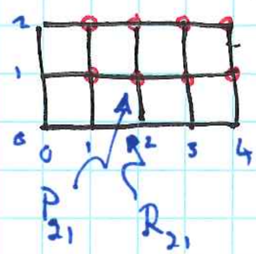
* Area $(\Delta x_i \Delta y_i)$ of each rectangle $= 1$
* Riemann sum is

$$(0^2+0^2)+(0^2+1^2)+(1^2+0^2)+(1^2+1^2)$$

$$+(2^2+0^2)+(2^2+1^2)+(3^2+0^2)+(3^2+1^2)$$

$$= 0+1+1+2+4+5+9+10 = 32$$

with the partition:



* Areas are again 1.
* Riemann sum is

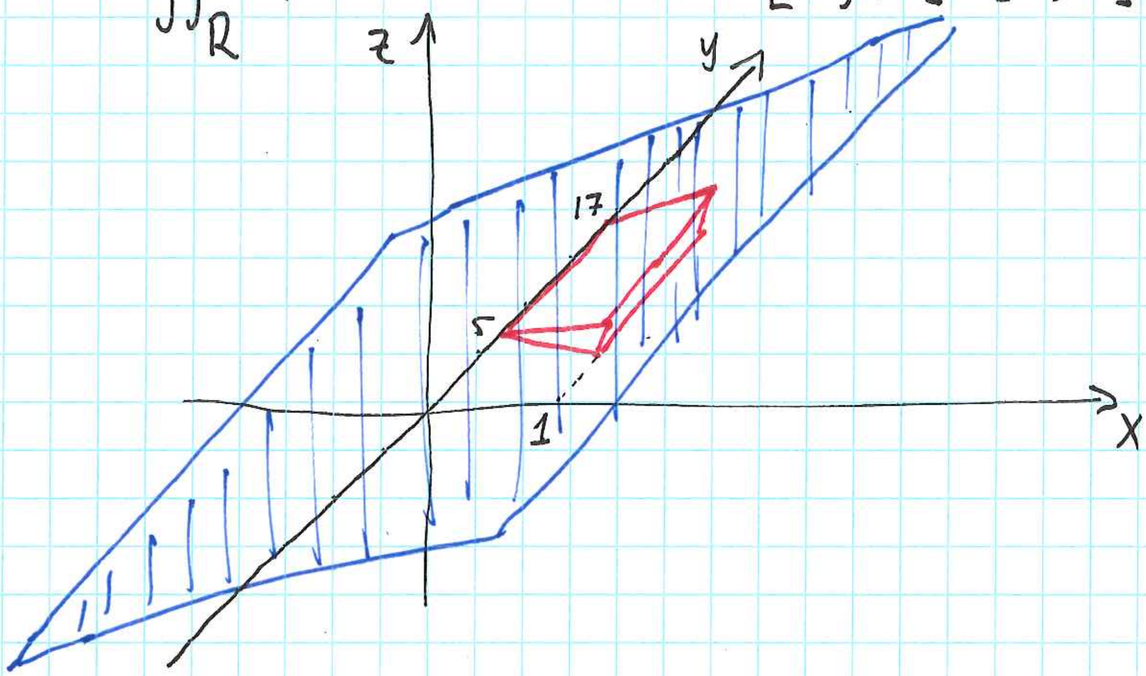
$$(1^2+1^2)+(1^2+2^2)+(2^2+1^2)+(2^2+2^2)$$

$$+(3^2+1^2)+(3^2+2^2)+(4^2+1^2)+(4^2+2^2)$$

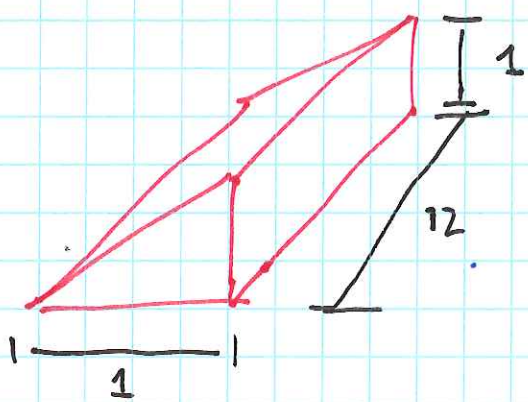
$$= 2+5+5+8+10+13+17+20 = 80$$

Ex Sometimes we can use simple geometry

$$\iint_R x \, dA \quad \text{where} \quad R = [5, 17] \times [0, 1]$$



* So we want the vol of a tri. prism



$$\iint_R x \, dA = \frac{1}{2} \cdot 1 \cdot 1 \cdot 12 = 6$$

Thm If $R = [a, b] \times [c, d]$ then

$$\iint_R 1 \, dA = \text{Area}(R) = (b-a) \cdot (d-c).$$

- Note that the integral

$$\iint_R f(x\ y)\, dA$$

is a <u>signed</u> volume. Volume under the x-y-plane counts as negative

Ex We can often argue by symmetry if the function is <u>odd</u> in one of its variables.

$$\iint_R x e^{-y^2}\, dA \qquad R = [-3,3] \times [0,1]$$

- Note $f(-x, y) = -f(x\ y)$ and R is symmetric about the x-axis
- Thus we will have equal ~~area~~ volume under and above the x-y-plane
- Thus $\displaystyle\iint_R x e^{-y^2}\, dA = 0$ //

Ex $\displaystyle\iint_R (x+y)(x^2+y^2) + 1\, dA \qquad R = [-1,1] \times [-1,1]$

$$= \iint_R x(x^2+y^2)\, dA + \iint_R y(x^2+y^2) + \iint_R dA$$

$$= 0 + 0 + 4 = 4 \ //$$