This week you will get practice drawing and understanding slope fields, making qualitative statements about solutions using them and some practice applying Euler's method.

Homework: The homework will be due on Friday 1 MarchDecember, at 8am, the *start* of the lecture. It will consist of questions

Problem set 8, question 3 and problems set 9, question 7.

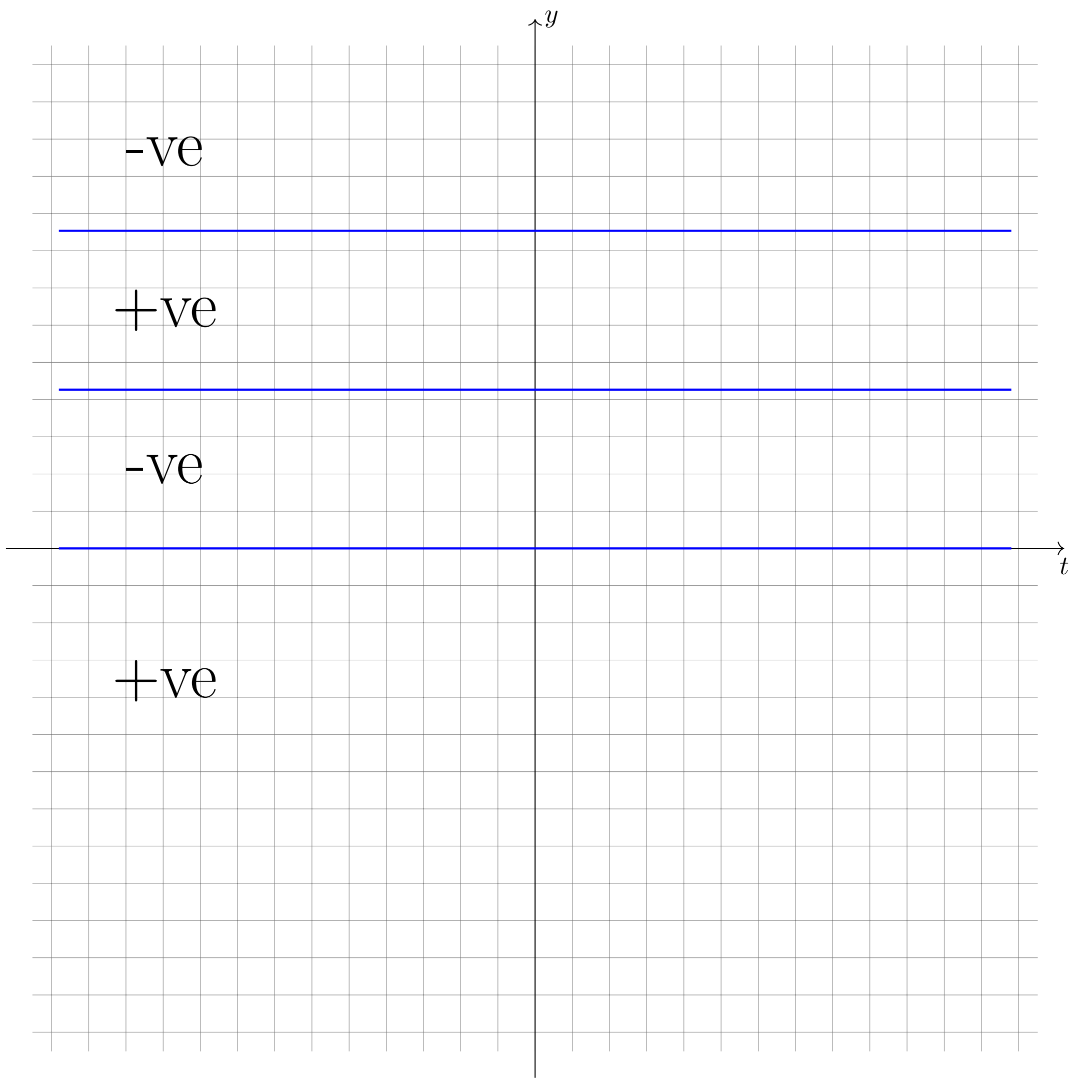
*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.4) Sketch the slope fields and a few solutions for the differential equations given
   (a) (6.4.12) $\frac{dy}{dt} = y(4-y)(y-2)$

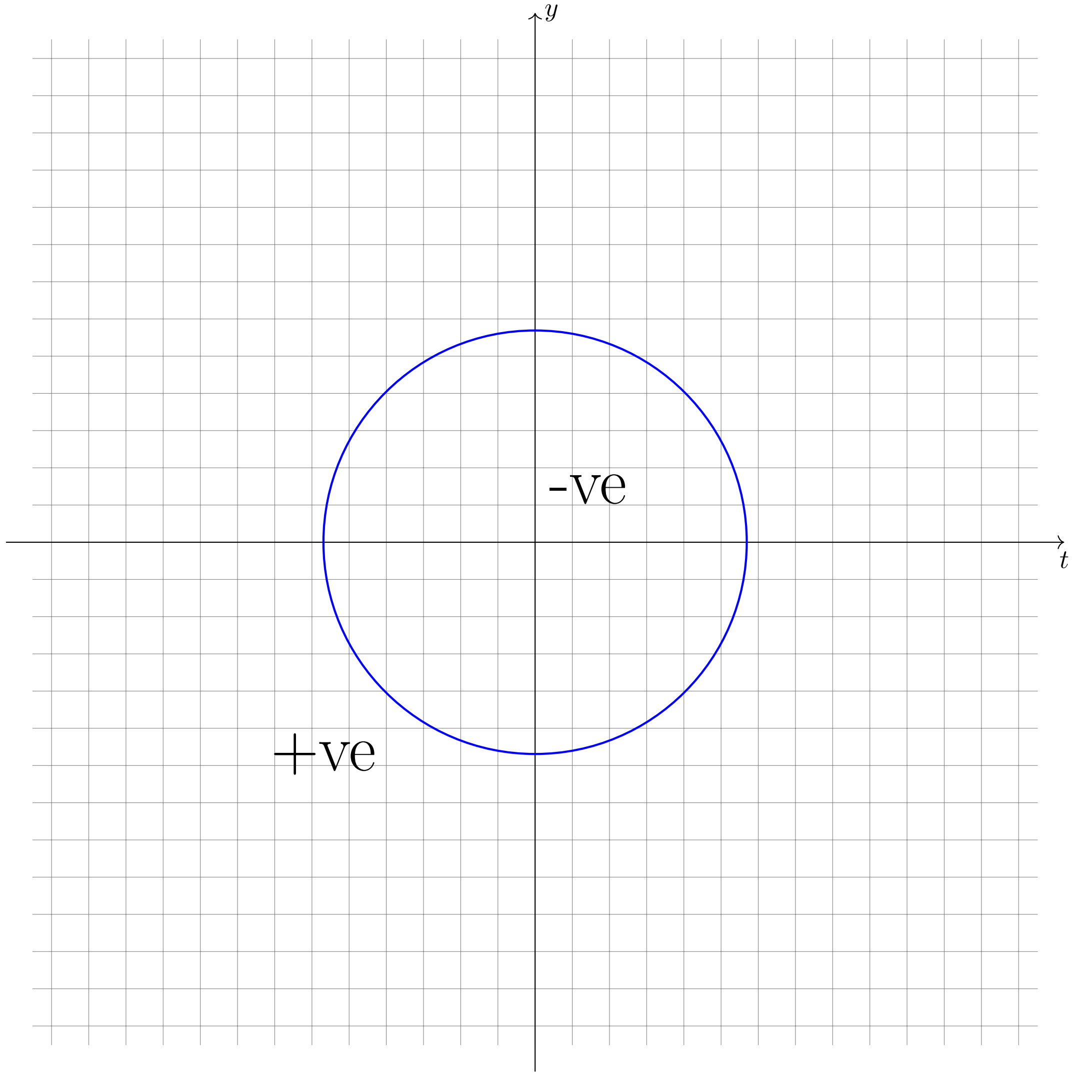
Solution:



   (b) (6.4.14) $\frac{dy}{dt} = t^2 - y$

(c) (6.4.16) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + t^2 - 1$

Solution:



(d) (6.4.17) $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{y}{t}$

Hint: feel free to use technology, just make sure you know how to draw a solution if you are given a slope field.

2. (6.4) Sketch the slope fields and the solution passing through the specified point for the differential equations given

   (a) (6.4.19) $\frac{\mathrm{d}y}{\mathrm{d}t} = t^2 - y^2, (t, y) = (0, 0)$

   (b) (6.4.20) $\frac{\mathrm{d}y}{\mathrm{d}t} = 1.5y(1-y), (t, y) = (0, 0.1)$

   (c) (6.4.21) $\frac{\mathrm{d}y}{\mathrm{d}t} = \sqrt{\frac{t}{y}}, (t, y) = (4, 1)$

   (d) (6.4.22) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2\sqrt{t}, (t, y) = (9, -1)$

3. (6.4.33) Consider the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{t}$$

(a) verify that $y(t) = \ln t$ is a solution to this differential equation satisfying $y(1) = 0$.

> **Solution:** Clearly, if $y(t) = \ln t$ then $y(1) = \ln 1 = 0$ so the initial condition is satisfied. Now we only need to check that is satisfies the given differential equation. The left hand side is $y'(t) = 1/t$. And the right hand side is the same thing, so it is indeed a solution.

(b) Use Euler's method to approximate $y(2) = \ln 2$ with $h = 0.5$.

> **Solution:** We use Euler's method to approximate $\ln 2$ by incrememting twice.

n	t_n	y_n	$y_{n+1} = y_n + \frac{h}{t_n}$
0	1	0	$y_1 = 0 + 0.5/1$
1	1.5	0.5	$y_2 = 0.5 + 0.5/1.5$
2	2	7/6	

> So $\ln 2 \approx \frac{7}{6}$.

4. (6.4.37) A population subject to seasonal fluctuations can be described by the logistic equation with an oscillating carrying capacity. Consider, for example,

$$\frac{\mathrm{d}P}{\mathrm{d}t} = P\left(1 - \frac{1}{100 + 50\sin 2\pi t}\right)$$

Although it is difficult to solve this differential equation, it is easy to obtain a qualitative understanding.

(a) Sketch a slope field over the region $0 \le t \le 5$ and $0 \le P \le 200$.

(b) Sketch solutions that satisfy $P(0) = 0$, $P(0) = 10$, and $P(0) = 200$.

(c) Use technology to obtain a better rendition of the slope field and solutions.

(d) Comment on your solutions and compare to your work using different methods.

5. (6.4.40) A population, in the absence of harvesting, exhibits the following growth

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N\left(\frac{N}{100} - 1\right)\left(1 - \frac{N}{1000}\right)$$

where N is abundance and t is time in years.

(a) Write an equation that corresponds to harvesting the population at a rate of 0.5% per day.

> **Solution:** This means that 0.005 of the population is beign removed per day, so over a year, the total fraction of the population being removed is $365 \cdot 0.005 = 1.825 = 365/200$ so the DE becomes

$$\frac{\mathrm{d}N}{\mathrm{d}t} = N\left(\frac{N}{100} - 1\right)\left(1 - \frac{N}{1000}\right) - 1.825N \tag{1}$$

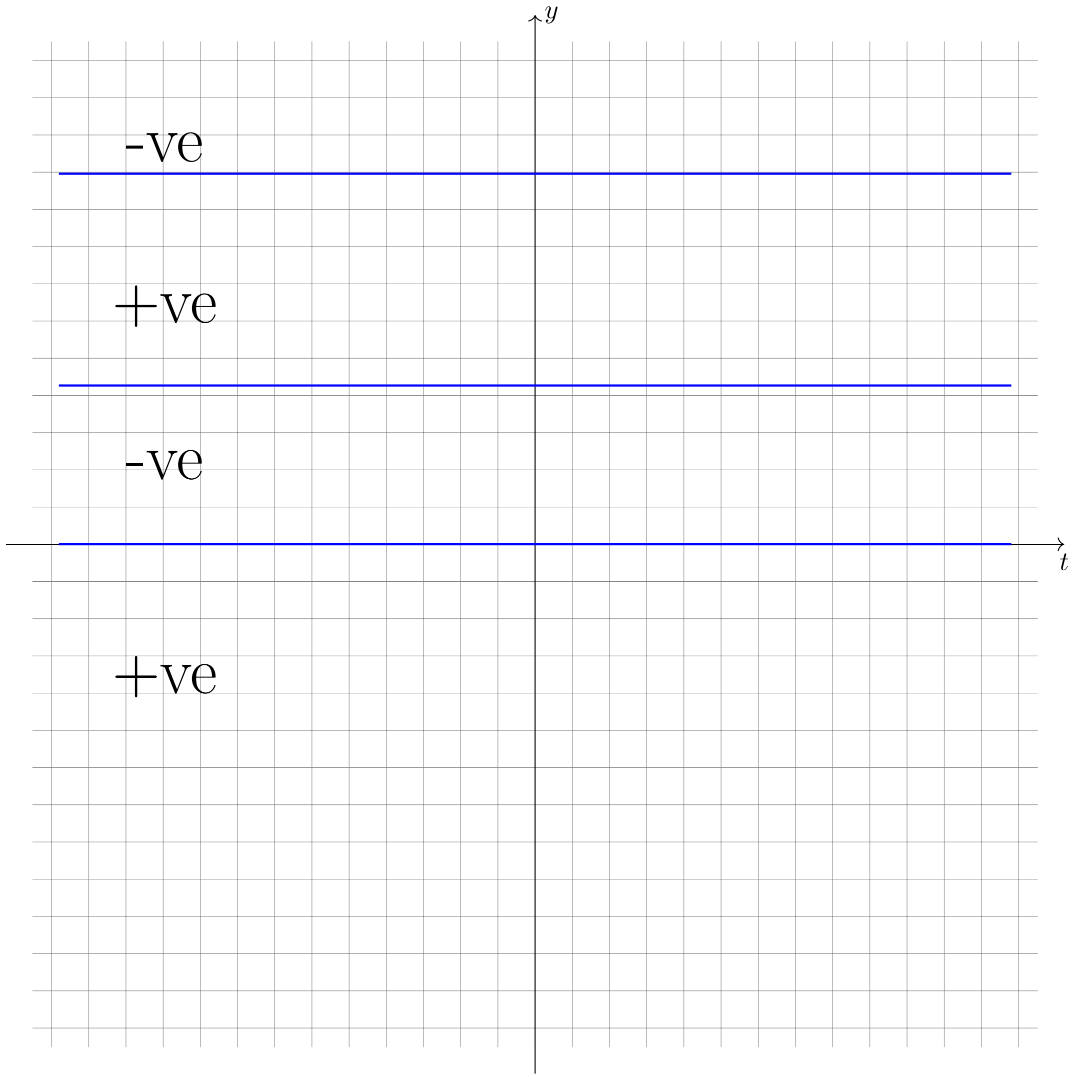
$$= N\left[\left(\frac{N}{100} - 1\right)\left(1 - \frac{N}{1000}\right) - \frac{365}{200}\right] \tag{2}$$

$$= N\left(-\frac{N^2}{100000} + \frac{N}{100} + \frac{N}{1000} - 1 - \frac{365}{200}\right) \tag{3}$$

$$= -\frac{N}{100000}\left(N^2 - 1100N + 282500\right) \tag{4}$$

(b) Sketch the slope field for the differential equation you found in part a; by sketching solutions, describe how the fate of the population depends on its initial abundance.

Solution: To sketch the solution, we need to know where the right hand side is zero. One obvious place is when $N = 0$. Two others are provided by solving the quadratic equation. Approximately, the roots are $N \approx 409, 691$. Thus, the slope field looks like:



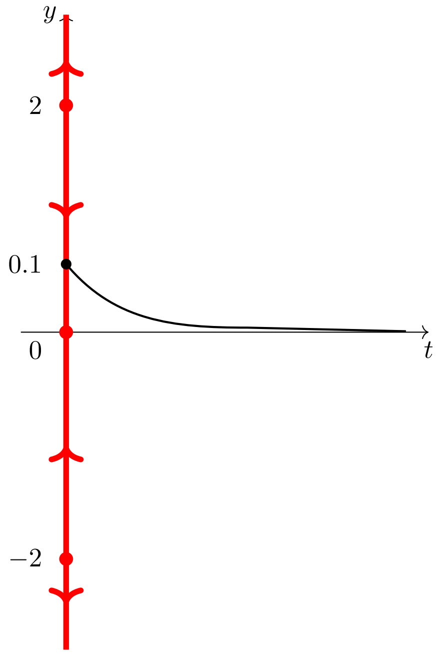
From this we can see that if the initial abundance is between 0 and 409, the population will eventually go extinct. If it is larger than 409 then the population will stabalize at 691.

6. (6.5) Draw phase lines, classify the equilibria, and sketch a solution satisfying the specified initial value for the equations in the following.
   (a) (6.5-2) $\frac{dy}{dt} = 2 - 3y$, $y(0) = 2$
   (b) (6.5-5) $\frac{dy}{dt} = y(y-10)(20-y)$, $y(0) = 9$
   (c) (6.5-6) $\frac{dy}{dt} = y(y-5)(25-y)$, $y(0) = 7$
   (d) (6.5-7) $\frac{dy}{dt} = \sin y$, $y(0) = 0.1$
   (e) (6.5-10) $\frac{dy}{dt} = y^3 - 4y$, $y(0) = 0.1$

Solution: The equilibrium solutions will be found when $y^3 - 4y = 0$. I.e. when $y = 0$ or when $y = \pm 2$. We see that

- when $y > 2$, then $y' > 0$,
- when $0 < y < 2$, then $y' < 0$,
- when $-2 < y < 0$, then $y' > 0$, and
- when $y < -2$, then $y' < 0$.

Thus the phase diagram is,



7. (6.5-33) To account for the effect of a generalist predator (with a type II functional response) on a population, ecologists often write differential equations of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0.1N\left(1 - \frac{N}{1,000}\right) - \frac{10N}{1+N}$$

(a) Sketch the phase line for this system.

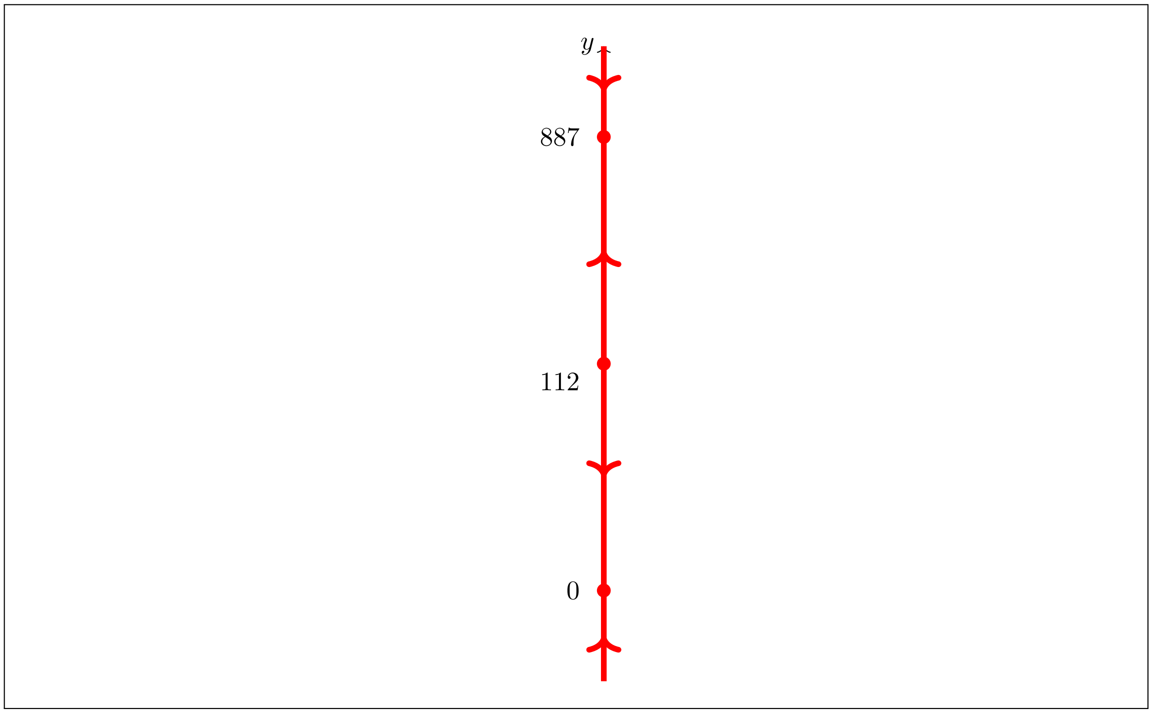
Solution: We can factorise the right hand side as

$$N\left(\frac{1}{10} - \frac{N}{10,000} - \frac{10N}{1+N}\right)$$

So we have an equilibrium at $N = 0$. The other factor can be put over a common denominator and expressed as

$$\frac{-99,000 + 999N - N^2}{10,000(1+N)}$$

To see where this is zero we use the quadratic equation and get $N \approx 112$ and 887. Thus we get the picture



(b) Discuss how the fate of the population depends on its initial abundance.

Solution: We can see from the phase diagram that if the population is initially between 0 and 112 then the population will eventually die out.
If the poulation is initially greater than 112 it will eventually stabilise at 887.

Hint: dont worry about what the first sentence means, you dont need to know where the differential equation comes from.

8. (6.5-39) Consider a population of clonally reproducing individuals consisting of two genotypes, a and A, with per capita growth rates, r_a and r_A, respectively. If N_a and N_A denote the densities of genotypes a and A, then

$$\frac{\mathrm{d}N_a}{\mathrm{d}t} = r_a N_a \qquad \frac{\mathrm{d}N_A}{\mathrm{d}t} = r_A N_A$$

Also, let $y = \frac{N_a}{N_a+N_A}$ be the fraction of individuals in the population that are genotype a. Show that y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (r_a - r_A)y(1-y)$$

9. (6.5-40) In the Hawk-Dove replicator equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{2}(1-y)(C(1-y)-V)$$

if the value $V > 0$ is specified, then find the range of values of C (in terms of V) that will ensure a polymorphism exists (i.e., find conditions that ensure the existence of an equilibrium $0 < y^* < 1$ that is stable).

(Hint: you do not need to know anything about the Hawk-Dove Replicator - though it is very interesting! - all you need to know is that V is a constant and C is a parameter. A polymorphism is a stable equilibrum between zero and one.)

10. (6.5-41) Production of pigments or other protein products of a cell may depend on the activation of a gene. Suppose a gene is *autocatalytic* and produces a protein whose presence activates greater production of that protein. Let y denote the amount of the protein (say, micrograms) in the cell. A basic model for the rate of this self-activation as a function of y is

$$A(y) = \frac{ay^b}{k^b + y^b} \text{ micrograms/minute}$$

where a represents the maximal rate of protein production, $k > 0$ is a half saturation constant, and $b \geq 1$ corresponds to the number of protein molecules required to active the gene. On the other hand, proteins in the cell are likely to degrade at a rate proportional to y, say cy. Putting these two components together, we get the following differential equation model of the protein concentration dynamics:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay^b}{k^b + y^b} - cy$$

(a) Verify that $\lim_{y\to\infty} A(y) = a$ and $A(k) = a/2$.

(b) Verify that $y = 0$ is an equilibrium for this model and determine under what conditions it is stable.

(Hint: the definition of autocatalytic is given in the question, it is a gene that produces a protein whose presence activate greater production of that protein.)

11. (6.5-42) Consider the model of an autocatalytic gene in Problem 41 with $b = 1$, $k > 0$, $a > 0$, and $c > 0$.

(a) Sketch the phase line for this model when $ck > a$.

(b) Sketch the phase line for this model when $ck < a$.