This weeks problem set focuses on examples of vector spaces and subspaces. It is expecially important to become comfortable with these since they will be used throughout the course to test our understanding. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

1. From section 1.2, problems 1, 4, 8, 10, 11, 15, 16.

2. From section 1.3, problems 1, 6, 7, 8a, c, d, 11, 15, 20*, 23, 24*, 30†, 31*.

3.* Define the symmetric difference of two sets to be $S \uplus T = (S \cup T) \setminus (S \cap T)$. Show that the power set $\mathcal{P}(S)$ is a vector space over $\mathbb{Z}_2$ with addition given by $\uplus$.

Solution: First we need to define scalar multiplication. The field $\mathbb{Z}_2$ only has two elements, [0] and [1]. As we will see in a moment, we are forced to make the definition

$$[0] \cdot A = \emptyset \quad \text{and} \quad [1] \cdot A = A.$$

VS1 We can easily see that

$$A \uplus B = (A \cup B) \setminus (A \cap B) = (B \cup A) \setminus (B \cap A) = B \uplus A$$

VS2 This can be shown by writing out the identity and using the fact that $(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$ and the same expression with $\cup$ replaced by $\cap$, but this is extremely tedious. A better, if not totally rigorous way, is to draw both sets using Venn diagrams.

VS3 The zero element is $\emptyset$. Indeed,

$$A \uplus \emptyset = (A \cup \emptyset) \setminus (A \cap emptyset) = A \setminus \emptyset = A.$$

VS4 The additive inverse to A is A itself! Indeed,

$$A \uplus A = (A \cup A) \setminus (A \cap A) = A \setminus A = \emptyset.$$

VS5 This is true by definition.

VS6-8 These just need to be checked in every case, it is tedious, but not difficult.

4.† For the prime 5, fill in the following tables of sums and products of elements in $\mathbb{Z}_5$. In addition, find the multiplicative inverses of each element.

+	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[1]	[2]	[3]	[4]
[1]	[1]	[2]	[3]	[4]	[0]
[2]	[2]	[3]	[4]	[0]	[1]
[3]	[3]	[4]	[0]	[1]	[2]
[4]	[4]	[0]	[1]	[2]	[3]

·	[0]	[1]	[2]	[3]	[4]
[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]
[2]	[0]	[2]	[4]	[1]	[3]
[3]	[0]	[3]	[1]	[4]	[2]
[4]	[0]	[4]	[3]	[2]	[1]

5. Convince yourself that

(a) $\mathbb{C}$ is a vector space over $\mathbb{R}$

Solution: We define addition in $\mathbb{C}$ and scalar multiplication by an element of $\mathbb{R}$ as usual. It is very easy to see that this satisfies all the axioms of a vector space.

(b) $\mathbb{R}$ is a vector space over $\mathbb{Q}$

Solution: As above.