Midterm 1 practice 1

UCLA: Math 115A, Spring 2020

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Date:
Version: practice

- This exam has 4 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____

ID number: ____

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
Total:	20	

Question 1 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (1 point) If V is a vector space over the field $\mathbb{R}$ and $v \in V$ then

$$(2-3) \cdot v + (5-3) \cdot v$$

equals

A. 1
B. v
C. 0
D. $2v$

The following two questions concern the subsets

$$A = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \middle| \; a_1 \neq \lambda \right\} \subseteq \mathbb{R}^3$$

$$B = \left\{ \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \middle| \; a_1 + a_2 + a_3 = \lambda \right\} \subseteq \mathbb{R}^3$$

for some $\lambda \in \mathbb{R}$.

(b) (1 point) Which of the following is a true statement?

A. Both A and B are subspaces regardless of the value of $\lambda \in \mathbb{R}$.
B. Only A is a subspace.
C. Both are subspaces when $\lambda = 0$.
D. Only B is a subspace when $\lambda = 0$.

(c) (1 point) When $\lambda = 0$, the subspace B has dimension

A. 1
B. 2
C. 3
D. 4

(d) (1 point) Let V be a vector space and W a subspace. Consider the quotient space V/W. Which of the following is true?

A. V/W is a subspace of V.

B. $v + W = w + W$ for any $v \in V$ and some $w \in W$.

C. $-v + W$ is the additive inverse of $v + W$.

D. $1 + W$ is the zero element.

(e) (1 point) Which of the following definitions, makes $p : V \longrightarrow V/W$ into a surjective linear map?

A. $p(v) = W$

B. $p(v) = 0 + W$

C. $p(v) = 2v + W$

D. $p(v) = 2v$

2. Give (simple) examples of all of the following situations.

(a) (2 points) A vector space V and a subspace W where $\dim W \geq 2$ and $\dim V \geq 3$.

Solution: Let $V = \mathbb{R}^3$ and W the space of vectors (a, b, c) where $c = 0$.

(b) (2 points) A basis for your example V above such that a subset of this basis is a basis for W.

Solution: Take the standard basis $\{e_1, e_2, e_3\}$. Then $\{e_1, e_2\}$ is a basis for W.

(c) (1 point) A basis for V/W.

Solution: $\{e_3 + W\}$ is a basis for V/W.

3. Consider the following maps. Prove or disprove that they are linear and find the dimension of the kernel (nullspace).
   (a) (2 points) $T : \mathbb{C}_2[x] \longrightarrow \mathbb{C}$ given by $T(p) = p(2)$ (i.e. evaluate the polynomial at two). Recall that $\mathbb{C}_n[x]$ is the set of polynomials of degree at most n.

Solution: We just need to check that $T(\lambda p + \mu q) = \lambda T(p) + \mu T(q)$. To see this note

$$\begin{aligned} T(\lambda p + \mu q) &= (\lambda p + \mu q)(2) \\ &= \lambda p(2) + \mu q(2) \\ &= \lambda T(p) + \mu T(q). \end{aligned}$$

Take an arbitrary element $p = a + bx + cx^2 \in \mathbb{C}_2[x]$. Then

$$T(p) = a + 2b + 4c.$$

So if $p \in \ker T$ then $a + 2b + 4c = 0$. I.e. the only requirement is $a = -2b - 4c$. So $\ker T = \left\{ cx^2 + bx - 2b - 4c \right\}$ which is clearly two dimensional (e.g. it has a basis $x^2 - 4$ and $x - 2$).

   (b) (3 points) $E : \mathrm{Mat}_{2\times 2}(\mathbb{R}) \longrightarrow \mathrm{Mat}_{2\times 2}(\mathbb{R})$ defined by

$$E(M) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot M - M \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Solution: Let $A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then

$$\begin{aligned} E(M+N) &= A(M+N) - (M+N)A \\ &= AM + AN - MA - NA \\ &= (AM - MA) + (AN - NA) \\ &= E(M) + E(N) \end{aligned}$$

and

$$E(\lambda M) = A(\lambda M) - (\lambda M)A = \lambda(AM - MA) = \lambda E(M).$$

Thus E is linear. We can explicitly calculate what E does to a matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

$$\begin{aligned} E(M) &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} - \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & a \\ 0 & c \end{pmatrix} \\ &= \begin{pmatrix} c & d-a \\ 0 & -c \end{pmatrix}. \end{aligned}$$

So $M \in \ker E$ iff $c = 0$ and $a = d$. Thus

$$\ker E = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \right\}$$

which is clearly two dimensional.

4. Let $\mathcal{F}(\mathbb{R}, \mathbb{R})$ be the space of all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$.

   (a) (1 point) Is the subset $\{ f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f(0) = 1 \} \subseteq \mathcal{F}(\mathbb{R}, \mathbb{R})$ a subspace? Justify your answer.

   Solution: It is not a subspace since if $f, g \in F f$ then $(f+g)(0) = f(0)+g(0) = 2$ so $f+g \notin \mathcal{F}$. I.e. $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is not closed under addition.

   (b) (4 points) Let $\mathcal{O}$ and $\mathcal{E}$ be the subspaces of odd and even functions in $\mathcal{F}(\mathbb{R}, \mathbb{R})$. Prove that $\mathcal{F}(\mathbb{R}, \mathbb{R}) = \mathcal{O} \oplus \mathcal{E}$. (Recall that a function is even if $f(-x) = f(x)$ for all $x \in \mathbb{R}$, or odd if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$).

   Solution: First we show that $\mathcal{F} = \mathcal{F}(\mathbb{R}, \mathbb{R}) = \mathcal{O} + \mathcal{E}$. I.e, for any element f we find an odd function f_- and an even function f_+ such that $f = f_- + f_+$. Consider

$$f_-(x) = \frac{1}{2}\left(f(x) - f(-x)\right) \text{ and}$$
$$f_+(x) = \frac{1}{2}\left(f(x) + f(-x)\right).$$

   The clearly $f = f_- + f_+$. It is easy to see that f_- is odd and f_+ is even.
   Now we just need to show that $\mathcal{O} \cap Ee = \{ 0 \}$. Consider a function $f \in \mathcal{O} \cap \mathcal{E}$, i.e. it is both even and odd. This means

$$f(-x) = -f(x) \text{ and } f(-x) = f(x) \text{ for all } x \in \mathbb{R}.$$

   Thus $f(x) = -f(x)$ for all $x \in \mathbb{R}$. The only way this can be true if $f(x) = 0$ for all $x \in \mathbb{R}$. I.e. $f = 0$.

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