Midterm 2 practice 3

UCLA: Math 115A, Winter 2020

Instructor: Noah White
Date:

- This exam has 4 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section (please circle):

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
Total:	20	

Question 1 is multiple choice. Indicate your answers in the table below. *The following three pages will not be graded, your answers must be indicated here.*

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

Clarification on notation: Let $T : V \longrightarrow W$ be a linear map. The *kernel* of T is the same thing as the *nullspace* of T, i.e. $\ker T = \mathsf{N}(T)$. Similarly the *image* of T is the same thing as the *range* of T, i.e. $\operatorname{im} T = \mathsf{R}(T)$.

Note also that

$$\Sigma_n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \,\middle|\, x_1 + x_2 \cdots + x_n = 0 \right\}.$$

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (1 point) If V is a finite dimensional vector space, with two bases, B and C and $T : V \longrightarrow W$ is a linear map, and if Q is the matrix such that $Q^{-1}[T]_B^B Q = [T]_C^C$ then Q equals

   A. $[T]_B^C$
   B. $[T]_C^B$
   C. $[\mathrm{id}]_B^C$
   D. $[\mathrm{id}]_C^B$

   (b) (1 point) Let $E = \{1, x\}$, $C = \{x + 2, x + 1\}$ be bases of $\mathbb{C}_1[x]$. What is $[\mathrm{id}]_E^C$?

   A. $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

   B. $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$

   C. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

   D. $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

(c) (1 point) Consider the linear map $\frac{d}{dx} : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$. Which of the following is an eigenvector?

A. x^2

B. $1 - x$

C. x

D. 3

(d) (1 point) Suppose $T : V \longrightarrow V$ is a diagonalizable linear map. Which of the following is true?

A. T is invertible.

B. T has non-zero kernel.

C. The characteristic polynomial of T splits.

D. T must have a non-zero eigenvalue.

(e) (1 point) What is the dimension of $\mathrm{Hom}(\mathbb{R}^2, \mathbb{R}^2)$? (this is the space of linear maps)

A. 0

B. 2

C. 4

D. 6

2. Let $T : V \longrightarrow W$ be a linear map between vector spaces.
   (a) (2 points) Define what it means for T to be an isomorphism.

   Solution: There exists a linear map $S : W \longrightarrow V$ such that $S \circ T = \mathrm{id}_V$ and $T \circ S = \mathrm{id}_W$

   (b) (3 points) Suppose B is a basis for V, prove that if $T(B) = \{T(v) \mid mv \in B\}$ is a basis for W then T is an isomorphism.

   Solution: We will prove that T is both injective and surjective. For surjectivity, notice that $T(v) \in \mathrm{im}(T)$ for all $v \in B$. Thus $\operatorname{span} T(B) \subset \mathrm{im}(T)$. But $\operatorname{span} T(B) = W$ since it is a basis so $W \subset \mathrm{im}(T)$ and thus T is surjective. For injectivity, suppose that $T(v) = 0$ for some $v \in V$. Since B is a basis, there are vectors $v_1, \dots, v_n \in B$ and scalars $\lambda_1, \dots, \lambda_n \in \mathbb{F}$ such that $v = \sum_i \lambda_i v_i$. But this means

$$0 = T(v) = T(\sum_i \lambda_i v_i) = \sum_i \lambda_i T(v_i)$$

   by linearity. But $T(B)$ is linearly independent, so we must have that $\lambda_i = 0$ for all i and so $v = 0$. Thus $\ker T = \{0\}$ and thus T is injective.

3. Consider the linear map $T : \Sigma_4 \longrightarrow \Sigma_4$ (see front cover), given by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix}.$$

(a) (2 points) Find the characteristic polynomial and eigenvalues of T. *Hint: recall that Σ_4 is three dimensional! You shouldn't need to work any 4×4 matrices!*

Solution: To find the characteristic polynomial we will use the basis

$$B = \left\{ \alpha_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \right\}$$

of Σ_4. In this basis $T(\alpha_1) = -\alpha_3$, $T(\alpha_2) = -\alpha_2$ and $T(\alpha_3) = -\alpha_1$. Thus the matrix in this basis is

$$[T]_B^B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial is then $p_T(t) = (1+t)^2(1-t)$ and so there are two eigenvalues 1 and -1 with algebraic multiplicities 1 and 2 respectively.

(b) (2 points) For each eigenvalue, determine an eigenvector of T (note that the eigenvectors should live in $\Sigma_4 \subset \mathbb{R}^4$).

Solution: We first calculate the 1-eigenvectors of the matrix. This means solving

$$\begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

This is given by $b = 0$ and $a = -c$. That means if $v \in \Sigma_4$ is a 1-egienvector then

$$[v]^B = \begin{pmatrix} a \\ 0 \\ -a \end{pmatrix}$$

and so $E_1 = \operatorname{span}\{\alpha_1 - \alpha_3\}$. We do the same calculation for the -1-eigenvectors.

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = -\begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

Which means $a = c$. Thus if $v \in \Sigma_4$ is a -1-eigenvector then

$$[v]^B = \begin{pmatrix} a \\ b \\ a \end{pmatrix}$$

so $E_{-1} = \operatorname{span}\{\alpha_1 + \alpha_3, \alpha_2\}$.

(c) (1 point) Is T diagonalisable?

Solution: Yes. By the above, the characteristic polynomial splits and the algebraic and geometric multiplicities match.

4. Consider the differential operator $D = x\frac{d}{dx}$, so for example $D((x-1)^2) = 2x^2 - 2x$. This is called the *Euler operator.*

   (a) (1 point) Consider the linear map $D : \mathbb{C}_n[x] \longrightarrow \mathbb{C}_n[x]$, given by the Euler operator. Is this an isomorphism?

   Solution: No. It is not injective. For example $D(1) = 0$.

   (b) (4 points) Prove or disprove that the linear map D is diagonalisable. *Hint: you might want to first try thinking about $n = 2$ or 3 before attempting to answer the question as stated, though you will need to say something about general n to get the points. Bonus: if you do this problem correctly with $\mathbb{C}$ replaced by an arbitrary field $\mathbb{F}$, you will get 2 top-up points (100% max total).*

   Solution: There are some obvious eigenvectors for D. In fact, $D(x^k) = kx^k$ so x^k is an eigenvector for every $0 \leq k \leq n$. Thus D has eigenvalues $0, 1, \ldots, n$ which are all distinct so D is diagonalisable.

   If we wanted to prove this over an arbitrary field, we need to be more careful. The eigenvalues are not necessarily distinct (for example over a finite field $\mathbb{Z}_p$). But we still have that $\{1, x, \ldots, x^n\}$ is a basis of eigenvectors so D is still diagonalisable.

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