

Midterm 2 practice

UCLA: Math 32B, Fall 2019

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Version: practice

- This exam has 4 questions, for a total of 35 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: _____

ID number: _____

Question	Points	Score
1	10	
2	8	
3	8	
4	9	
Total:	35	

1. (a) (5 points) Let \mathcal{D} be the region in the xy -plane above the x -axis and below the curve $y = 1 - x^2$. Compute the integrals

$$I_1 = \frac{1}{A} \iint_{\mathcal{D}} x \, dA \text{ and } I_2 = \frac{1}{A} \iint_{\mathcal{D}} y \, dA$$

where A is the area of \mathcal{D} .

Solution: We describe \mathcal{D} as a vertically simple region

$$\mathcal{D} = \{ (x, y) \mid -1 \leq x \leq 1, 0 \leq y \leq 1 - x^2 \}$$

We first compute the area

$$A = \iint_{\mathcal{D}} 1 \, dA = \int_{-1}^1 \int_0^{1-x^2} 1 \, dy \, dx = \int_{-1}^1 1 - x^2 \, dx = 2 - 2/3 = 4/3.$$

Now we compute

$$\iint_{\mathcal{D}} x \, dA = \int_{-1}^1 \int_0^{1-x^2} x \, dy \, dx = \int_{-1}^1 (1 - x^2)x \, dx = 0.$$

Now we compute

$$\iint_{\mathcal{D}} y \, dA = \int_{-1}^1 \int_0^{1-x^2} y \, dy \, dx = \int_{-1}^1 \frac{1}{2}(1 - x^2)^2 \, dx = 16/30.$$

So

$$I_1 = 0 \text{ and } I_2 = 2/5.$$

- (b) (5 points) Parametrize the paraboloid and find the normal vector for this parametrisation.

$$x^2 + y^2 = 2z, \quad 0 \leq z \leq 1.$$

Solution: We can express $z = \frac{1}{2}(x^2 + y^2)$ so we get an easy parametrisation in this case

$$G(x, y) = (x, y, \frac{1}{2}(x^2 + y^2))$$

But we need to understand the domain for (x, y) . When $0 \leq z \leq 1$ we see that

$$0 \leq \frac{1}{2}(x^2 + y^2) \leq 1$$

so we can see that (x, y) should be contained in $\mathcal{D} \subset \mathbb{R}^2$ where \mathcal{D} is the disk of radius $\sqrt{2}$ centred at the origin.

To find the normal vector, we first find two tangent vectors

$$T_x(x, y) = \langle 1, 0, x \rangle$$

$$T_y(x, y) = \langle 0, 1, y \rangle$$

Thus we get

$$N(x, y) = T_x \times T_y = \langle -x, -y, 1 \rangle.$$

2. (8 points) Consider the region \mathcal{E} given by

$$0 \leq z \leq (y - x^2)^2, \quad x^2 \leq y \leq x.$$

Use the change of variables

$$x = u, y = v + u^2, z = uv^2,$$

to evaluate

$$\iiint_{\mathcal{E}} \frac{1}{y - x^2} dV.$$

Solution: First we describe \mathcal{E} in the form

$$\mathcal{E} = \{ (x, y, z) \mid (x, y) \in \mathcal{D} \text{ and } 0 \leq z \leq (y - x^2)^2 \}$$

where

$$\mathcal{D} = \{ (x, y) \mid 0 \leq x \leq 1 \text{ and } x^2 \leq y \leq x \}.$$

Our next job is to figure out which region in uvw -space is mapped to \mathcal{E} when we apply $G(u, v, w) = (u, v + u^2, uv^2)$. We can use the inequalities given, in terms of u, v, w .

$$0 \leq uv^2 \leq v^2, u^2 \leq v + u^2 \leq u.$$

We can manipulate these to

$$0 \leq w \leq 1, \text{ and } 0 \leq v \leq u - u^2.$$

Thus if we take

$$\mathcal{E}' = \{ (u, v, w) \mid (u, v) \in \mathcal{D}' \text{ and } 0 \leq w \leq 1 \}$$

where

$$\mathcal{D}' = \{ (u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq u - u^2 \}.$$

Now we need to find the Jacobian:

$$J(G) = \det \begin{pmatrix} 1 & 0 & 0 \\ 2u & 1 & 0 \\ 0 & 2vw & v^2 \end{pmatrix} = v^2$$

This is always positive! Thus

$$\begin{aligned} \iiint_{\mathcal{E}} \frac{1}{y - x^2} dV &= \iiint_{\mathcal{E}'} \frac{1}{v} \|J(G)\| dV_{uvw} \\ &= \iiint_{\mathcal{E}'} v dV_{uvw} \\ &= \iint_{\mathcal{D}'} \int_0^1 v dw dA_{uv} \\ &= \int_0^1 \int_0^{u-u^2} \int_0^1 v dw dv du \\ &= \int_0^1 \int_0^{u-u^2} v dv du \\ &= \int_0^1 \frac{1}{2} (u - u^2) du = 1/60 \end{aligned}$$

3. Let \mathbf{F} be the vector field on \mathbb{R}^3 given by

$$\mathbf{F}(x, y, z) = (y \cos z - yze^x, x \cos z - ze^x, -xy \sin z - ye^x).$$

(a) (4 points) Show that \mathbf{F} is conservative.

Solution: Our vector field is defined on a simply connected domain. This means being conservative is equivalent to having curl zero. So we simply check this:

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle \partial_x, \partial_y, \partial_z \rangle \times \langle y \cos z - yze^x, x \cos z - ze^x, -xy \sin z - ye^x \rangle \\ &= \langle -x \sin z - e^x - (-x \sin z - e^x), -y \sin z - ye^x - (-y \sin z - ye^x), \cos z - ze^x - (\cos z - ze^x) \rangle = 0.\end{aligned}$$

(b) (4 points) Find a potential function for \mathbf{F} .

Solution: We need a function f such that

$$\partial_x f = y \cos z - yze^x$$

$$\partial_y f = x \cos z - ze^x$$

$$\partial_z f = -xy \sin z - ye^x$$

This means we get three conditions

$$f = xy \cos z - yze^x + \alpha(y, z)$$

$$f = xy \cos z - yze^x + \beta(x, z)$$

$$f = xy \cos z - yze^x + \gamma(x, y)$$

We can simply let $\alpha = \beta = \gamma = 0$ and take

$$f = xy \cos z - yze^x.$$

4. In this question we will calculate the surface area of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + z^2 = 1$.

(a) (4 points) Find a parameterisation of the ellipsoid given above.

Solution: Here we can take our idea from spherical coordinates:

$$G'(\theta, \phi) = (\cos \theta \sin \phi, \sin \theta \sin \phi, \cos \phi).$$

This obviously doesn't work unless $a = 1$, so we should adjust for this. We notice that if we multiply the first two coordinates by a we do get something that works:

$$G(\theta, \phi) = (a \cos \theta \sin \phi, a \sin \theta \sin \phi, \cos \phi).$$

Now also thinking about spherical coordinates gives us the fact that we should let $(\theta, \phi) \in [0, 2\pi] \times [0, \pi]$.

(b) (5 points) Find the normal vector to this parameterisation and its length.

Solution: We find the two tangent vectors

$$T_\theta(\theta, \phi) = \langle -a \sin \theta \sin \phi, a \cos \theta \sin \phi, 0 \rangle$$

$$T_\phi(\theta, \phi) = \langle a \cos \theta \cos \phi, a \sin \theta \cos \phi, -\sin \phi \rangle$$

Thus we get

$$N(\theta, \phi) = T_\theta \times T_\phi = \langle -a \cos \theta \sin^2 \phi, -a \sin \theta \sin^2 \phi, -a^2 \sin \phi \cos \phi \rangle$$

and

$$\|N\|^2 = a^2 \sin^4 \phi + a^4 \sin^2 \phi \cos^2 \phi$$

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