This weeks problem set focuses on the ideas of linear combinations, linear dependence and bases. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework: due Friday 10 April, uploaded to Gradescope before 11:59pm: questions 3,4, and 5 below.

1. From section 1.4, problems 1, 7, 8 ($\mathsf{P}_n(F)$ is the set of polynomials of degree less than or equal to n), 11, 12, 13*.

2. From section 1.5, problems 1, $2a, c, e$, 4*, 5, 9*, 15, 18*.

   Quotient spaces: Let V be a vector space over a field $\mathbb{F}$ and W a subspace of V. For any $v \in V$, consider the set $\{v\} + W = \{v + w \mid w \in W\}$. We will denote it simply as $V + W$. Now consider the set

$$V/W = \{v + W |\, v \in V\}.$$

   We can define addition and scalar multiplication on this set by

$$(v + W) + (w + W) = (v + w) + W \qquad \text{and} \qquad \lambda\,(v + W) = \lambda v + W.$$

   It turns out this is a vector space, it is called the *quotient* of V by W. See below.

3. Let $V = \mathbb{R}^2$ and $W = \operatorname{span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$. List all the elements of V/W, making sure not to list any element twice.

Solution: Consider the set

$$\begin{aligned}\begin{pmatrix}a\\b\end{pmatrix} + W = &\left\{ \begin{pmatrix}a\\b\end{pmatrix} + \begin{pmatrix}\lambda\\\lambda\end{pmatrix} \,\middle|\, \lambda \in \mathbb{R}\right\}\\ &\left\{ \begin{pmatrix}a+\lambda\\b+\lambda\end{pmatrix} \,\middle|\, \lambda \in \mathbb{R}\right\}\\ &\left\{ \begin{pmatrix}x\\y\end{pmatrix} \,\middle|\, x - y = a - b\right\}\end{aligned}$$

So $\begin{pmatrix}a\\b\end{pmatrix} + W$ is the line $y = x + (b - a)$. Thus we can say V/W is the set of lines with slope 1. The line $y = x + b$ is the element $\begin{pmatrix}0\\b\end{pmatrix} + W$ and is clearly not equal to any other line.

Solution 2: Another way that one could say the above is that $\begin{pmatrix}a\\b\end{pmatrix} + W = \begin{pmatrix}0\\b-a\end{pmatrix} + W$. Furthermore, $\begin{pmatrix}0\\b\end{pmatrix} + W = \begin{pmatrix}0\\b'\end{pmatrix} + W$ only if $b = b'$. Thus every element in V/W is equal to $\begin{pmatrix}0\\b\end{pmatrix} + W$ for some unique b.

4.* Prove that V/W is a vector space.

Solution: We simply check each axiom one at a time.

VS1 Clearly $(v+W)+(u+W) = (v+u)+W = (u+v)+W = (u+W)+(v+W)$ since $v+u = u+v$ in V.

VS2 This holds in exactly the same way as above since $(v+u)+w = v+(u+w)$ in V.

VS3 The zero element is $0+W = W$. Indeed

$$(0+W)+(v+W) = (0+v)+W = v+W$$

Note that 0 is the zero element of V and we denote the zero element of V/W by $0+W$ or W for clarity.

VS4 The additive inverse of $v+W$ is $-v+W$. Indeed

$$(v+W)+(-v+W) = (v-v)+W = 0+W$$

since $v-v=0$ in V.

VS5 It is clear that $1\cdot(v+W) = (1v)+W = v+W$ since $1v = v$ in V.

VS6-8 These all follow in exactly the same way as above. The relations are true in V so they are true in V/W.

5. Let $\mathbb{C}[x]$ be the vector space of polynomials and let $W = \operatorname{span}\{x^{2a} \mid a \geq 0\}$.

(a) Find a set of 3 linearly independent elements of $\mathbb{C}[x]/W$.

Solution: Note that $x^a + W = W$ if a is even. Thus we can choose $x+W, x^3+W, x^5+W$. These are linearly independent since if

$$a(x+W)+b(x^3+W)+c(x^5+W) = W$$

then

$$(ax+bx^3+cx^5)+W = W$$

and thus $ax+bx^3+cx^5 \in W$, but this is impossible unless $a=b=c=0$.

(b) Find 2 nonzero elements $p, q \in \mathbb{C}[x]$ that are linearly independant and such that $p+W$ and $q+W$ are linearly dependant and nonzero. *Note: you can only receive full points for this problem if your polynomials p and q and different from everyone elses! If you understand the problem then this will be easy to ensure.*

Solution: We want two elements p, q such that they are linearly independent. The easiest way to ensure this is to pick two polynomials with different degrees. We also want $p+W$ and $q+W$ to be non-zero. That is, we need to make sure $p, q \notin W$. The easiest way to ensure this is to make sure they have, for example, a linear term. Thirdly we want to make sure $p+W$ and $q+W$ are linearly dependent. The easiest way to ensure this is to make sure $p+W = q+W$, i.e. $(p-q)+W = W$, that is we want $p-q \in W$. This means their odd degree terms should agree. Lets summarise the conditions we want p, q to have.

- They should have different degrees.
- They should have a linear term
- Their odd degree terms should be the same.

So we could pick, for example, $p = x$ and $q = 1+x$.

Note on quotient spaces: The astute reader might be worried that the addition and scalar multiplication might not be well defined. What do I mean by this? Well, it is entirely possible that $v + W = v' + W$ for two different elements $v, v' \in V$. This means we could calculate a sum in two different ways. As

$$(v + W) + (u + W) = (v + u) + W$$

or as

$$(v + W) + (u + W) = (v' + W) + (u + W) = (v' + u) + W$$

(since $v + W = v' + W$). So we need to check that $(v + u) + W = (v' + u) + W$. I will show you how to do this below. You might like to try to prove that the scalar multiplication is unambiguous for yourself.

Proof that $(v + u) + W = (v' + u) + W$*:* Note that $(v + u) + W = \{(v + u) + w \mid w \in W\}$ and $(v' + u) + W = \{(v' + u) + w \mid w \in W\}$. Also note that $v \in v + W$ since $v = v + 0$ and $0 \in W$.

Since $v + W = v' + W$ we see that $v \in v' + W$ and thus $v = v' + x$ for some $x \in W$. Now lets take an arbitrary elements $s \in (v + u) + W$, it will be of the form $s = v + u + w$. We know

$$s = v + u + w = v' + x + u + w = (v' + u) + (x + w).$$

Since $x + u \in W$ we see that $s = (v' + u) + (x + w) \in (v' + u) + W$. We have just shown that $(v + u) + W \subset (v' + u) + W$. To complete the proof we need to show the opposite containment.

We do this in almost the same way. Take an arbitrary element $t \in (v' + u) + W$. We have that $t = v' + u + w$ for some $w \in W$. Then

$$t = v' + u + w = v - x + u + w = (v + u) + (w - x) \in (v + u) + W.$$

Thus we have shown $(v' + u) + W \subset (v + u) + W$ and hence $(v + u) + W = (v' + u) + W$.