Lecture 10

1. Density

- As pointed out before, if a quantity has a density given by $\delta(x\ y\ z)$ unit/area
  The integral
  $$\iiint_{\mathcal{E}} \delta(x\ y\ z)\, dV$$
  tries to answer "what is the total amount of quantity in $\mathcal{E}$".

- Examples

  * Mass A ball $x^2+y^2+z^2 \leq 4$ has density
  $$\delta(x\ y z) = e^{5-x^2-y^2-z^2} \quad kg/m^3$$
  at the point $(x\ y\ z)$
  $$\text{Total mass} = \iiint_{\mathcal{E}} e^{5-x^2-y^2-z^2}\, dV.$$

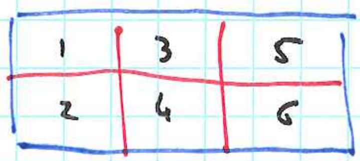
  * A crowd is gathered, in a rectangular plaza, ~~width~~ $R = [0,100] \times [0,50]$. The denisty ~~the~~ at $(x,y) \in R$ is
  $$\delta(x\ y) = 5 - \frac{1}{10} y \quad \text{people}/m^3$$

Total number of people in R =

$$\iint_R 5 - \frac{1}{10}y \, dA$$

2. <u>Average density/value.</u>

- If we subdivide the rectange R above in 6 regions



And the density is const. in each one then

Total people = $\sum \delta_i A_i$ where

δ_i = deniity in Region ~~R~~ i

A_i = area of region i.

Average density = total ~~densi~~ people / total area

- Similarly if $\delta(x,y,z)$ is the density of a quantity in a region E then

total quantity = $\iiint_E \delta(x\ y\ z)\, dV$

Average ~~quantity~~ density:

$$\overline{\delta} = \text{total quantity} \Big/ \text{total volume}$$

$$= \iiint_E \delta(x\,y\,z)\,dV \Big/ \iiint_E 1\,dV.$$

Note $\iiint_E \delta(x\,y\,z)\,dV = \overline{\delta}\cdot \mathrm{Vol}(E).$

3. Centroid of region

- The centroid ~~is the~~ $(\bar{x}, \bar{y}, \bar{z})$ of a region ($I \subseteq \mathbb{R}$, $D \subseteq \mathbb{R}^2$ or $E \subseteq \mathbb{R}^3$) is the average of all coords.

- Eg if $I \subseteq \mathbb{R}$, $I = [a, b]$ then

$$\bar{x} = \int_a^b x\,dx \Big/ \int_a^b dx$$

$$= \tfrac{1}{2}(b^2 - a^2) \big/ (b - a)$$

$$= \tfrac{1}{2}(b + a)$$

- In 2D

$$(\bar{x}, \bar{y}) = \frac{1}{\iint_D dA} \iint_D (x, y)\, dA$$

$$= \left(\frac{\iint_D x\, dA}{\iint_D dA}, \frac{\iint_D y\, dA}{\iint_D dA} \right).$$

Ex Centroid of $D = \{ x^2 + y^2 \leq r^2 \}$.

(intuitively: $(\bar{x}\ \bar{y}) = (0, 0)$).

Note $\iint_D dA = \pi r^2$

$$\bar{x} = \iint_D x\, dA = \int_0^{2\pi} \int_0^r r^2 \cos\theta\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3} r^3 \cos\theta\, d\theta = 0$$

$$\bar{y} = \iint_D y\, dA = \int_0^{2\pi} \int_0^r r^2 \sin\theta\, dr\, d\theta = 0.$$

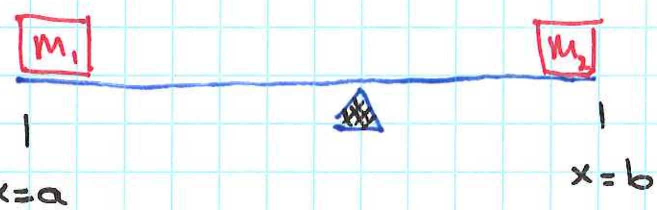
so $(\bar{x}\ \bar{y}) = (0, 0)$.

- In 3D

$$(\bar{x} \ \bar{y} \ \bar{z}) = \frac{1}{\iiint_{\mathcal{E}} dA} \iiint_{\mathcal{E}} (x, y, z) \, dV.$$

4. Centre of mass.

- Suppose we have a plank with two masses:

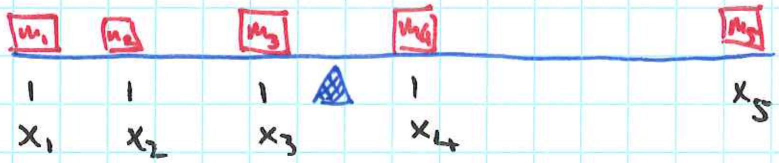


at what point should we place the fulcrum in order to balance the masses?

Answer: find the weighted average

$$\frac{m_1 a + m_2 b}{\cancel{\Sigma} m_1 + m_2}$$

- What about many masses?

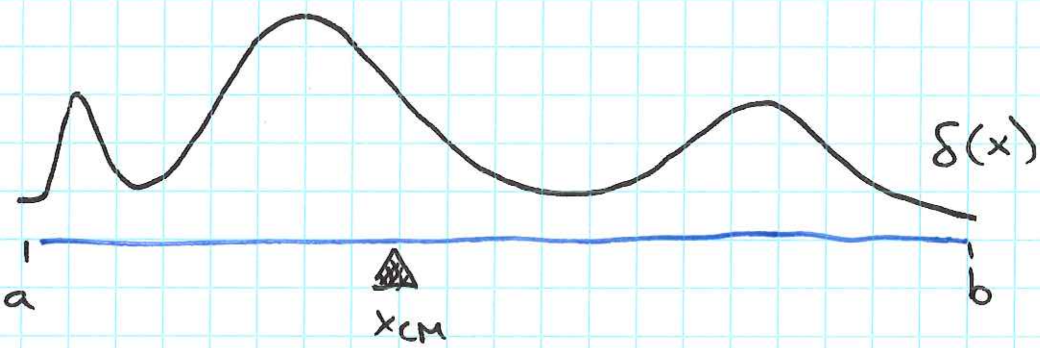


Again, weighted average:

$$x_{CM} = \frac{\sum m_i x_i}{\sum m_i}$$

We call this the center of mass

— What if we have a contiuous distribution of mass, given by a density $\delta(x)$



$$x_{CM} = \cancel{\frac{1}{b-a}\int_a^b x\,\delta(x)\,dx}$$

$$= \cancel{\frac{1}{\int_a^b \delta(x)\,dx}\int x\,\delta(x)\,dx}$$

$$= \frac{\int_a^b x\,\delta(x)\,dx}{\int_a^b \delta(x)\,dx}$$

- Similarly in 2D and 3D

$$(x_{CM}\ y_{CM}) = \frac{\iint_D (x,y)\,\delta(x\,y)\,dA}{\iint_D \delta(x\,y)\,dA}$$

$$(x_{CM}\ y_{CM}\ z_{CM}) = \frac{\iiint_E (x,y,z)\,\delta(x\,y\,z)\,dV}{\iiint_E \delta(x\,y\,z)\,dV}$$