Taylor polynomials

Ex What if we pretend that e^x is a polynomial of deg 3. Can we use last lectures technique to compute it?

Let $f(x) = e^x$ that $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

Note that

$$f(0) = e^0 = 1 \quad \text{and} \quad f^{(k)}(0) = e^0 = 1$$

But

$$\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 & f(0) &= 0!\,a_0 = 1 \\
f'(x) &= a_1 + 2a_2 x + 3a_3 x^2 & f'(0) &= 1!\,a_1 = 1 \\
f''(x) &= 2a_2 + 6a_3 x & f''(0) &= 2!\,a_2 = 1 \\
f'''(x) &= 6a_3 & f'''(0) &= 3!\,a_3 = 1
\end{aligned}$$

So

$$f(x) = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3$$

$$= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

What if we pretent that e^x is a poly of deg n?

Again, if $f(x)=e^x$ then

$$f^{(k)}(x)=e^x \text{ so } f^{(k)}(0)=1$$

k^{th} coeficient $\Rightarrow$ $f^{(k)}(0)/k!$ thus

$$f(x)=1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+\dots+\frac{1}{n!}x^n$$

This is the n^{th} Taylor polynomial of e^x

Def For any function $f(x)$, the n^{th} Taylor polynomial $T_n(x)$ expanded at a is

$$T_n(x)=\sum_{k=0}^{n}\frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$=f(a)+\frac{f'(a)}{1!}(x-a)+\frac{f''(a)}{2!}(x-a)^2+\dots+\frac{f^{(n)}(a)}{n!}(x-a)^n$$

Rmk * If $f(x)$ is a polynomial of deg $\leq n$ then

$$T_n(x)=f(x)$$

* for other functions $T_n(x)$ is a good approx. of $f(x)$ for x near a.

But how good of an approximation is it?

Let $R_n(x) = f(x) - T_n(x)$ and

$$E_n(x) = |R_n(x)|$$

* $R_n(x)$ is the remainder because $f(x) = T_n(x) + R_n(x)$
* $E_n(x)$ is the error, how far is $f(x)$ from $T_n(x)$ for a given x?

If we want to answer q's like

"Approximate $\ln(2)$ to 4 deci places"

we need to

1. Calculate $T_n(x)$ for $\ln(2x)$ say for $x=1$
2. Figure out for what n

$$E_n(x) \leq 0.0001$$

Thm Suppose $f(x)$ is a function and $T_n(x)$ the Taylor poly at a.

If $|f^{(n+1)}(u)| \leq K$ for all u between a, x

then
$$E_n(x) \leq \frac{K|x-a|^{n+1}}{(n+1)!}$$

This theorem says, if we can find a suitable K, then our error is only so big.

Ex $f(x) = e^x$, $T_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

Estimate $E_4(-1)$.

From the theorem:

$$\underline{\underline{\text{If}}} \quad |e^u| \leq K \quad \text{for } \cancel{u \in [0,1]} \; u \in [-1,0]$$

$f^{(5)}(u)$

$$\underline{\underline{\text{then}}} \quad E_n(-1) \leq \frac{K\,|(-1)-0|^5}{5!} = \frac{K}{120}$$

We could choose ~~it~~ $K = 1$, thus

$$E_n(-1) \leq \frac{1}{120}$$

Ex Calculate $\cos\left(\frac{3}{4}\right)$ to two decimal places.

To do this we let $T_n(x)$ be the Taylor poly ~~of~~ of $\cos x$ at 0. We want to find an n such that $E_n(x) \leq 0.01$.

The theorem says

$$\underline{\underline{\text{If}}} \quad |\pm\cos(u)| \leq K \quad \text{for} \quad u \in [0, \tfrac{3}{4}]$$

$$\underline{\underline{\text{then}}} \qquad E_n(\tfrac{3}{4}) \leq \frac{K\left|\frac{3}{4} - 0\right|^{n+1}}{(n+1)!}$$

We could choose $K = 1$ thus

$$E_n\left(\frac{3}{4}\right) \leq \frac{1}{(n+1)!}\left(\frac{3}{4}\right)^{n+1}$$

for $n=1$ $\quad \frac{1}{(n+1)!}\left(\frac{3}{4}\right)^{n+1} = \frac{1}{2}\left(\frac{3}{4}\right)^2 \sim 0.28$

$n=2$ $\quad \frac{1}{6}\left(\frac{3}{4}\right)^3 \sim 0.07$

$n=3$ $\quad \frac{1}{24}\left(\frac{3}{4}\right)^4 \sim 0.013$

$n=4$ $\quad \frac{1}{120}\left(\frac{3}{4}\right)^5 \sim 0.001 \leq 0.01$ ✓✓

so we can approx. $\cos\left(\frac{3}{4}\right)$ by $T_4\left(\frac{3}{4}\right)$.