This week on the problem set you will get practice at calculating integrals using substitution and integration by parts.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

Homework: The second homework will be due on Monday 4 February, at 8am, the *start* of the lecture. It will consist of questions:

6 and 8

1. (5.3) Express the limits as definite integrals of the form $\int_0^1 f(x)\,\mathrm{d}x$.
   (a) (5.3.1) $\lim_{n\to\infty}\sum_{i=1}^n \frac{i}{n^2}$
   (b) (5.3.5) $\lim_{n\to\infty}\sum_{i=1}^n \left(1-\frac{i^2}{n^2}\right)\frac{1}{n}$
   (c) (5.3.6) $\lim_{n\to\infty}\sum_{i=1}^n \sin\left(\frac{\pi i}{n}-\pi\right)\frac{\pi}{n}$

Solution: We know that if the integral ranges from $x=0$ to $x=1$ then $\Delta x = \frac{1}{n}$ and $x_i = \frac{i}{n}$. Using the definition of the definite integral

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^n f\left(\frac{i}{n}\right)\frac{1}{n}.\end{aligned}$$

So we need that $f\left(\frac{k}{n}\right)\frac{1}{n} = \sin\left(\frac{\pi i}{n}-\pi\right)\frac{\pi}{n}$ i.e. that

$$f\left(\frac{k}{n}\right) = \sin\left(\frac{\pi i}{n}-\pi\right)\pi$$

which is obviously achieved if $f(x) = \pi\sin(\pi x-\pi)$. That means the definite integral corresponding to the Riemann sum is

$$\int_0^1 \pi\sin(\pi x-\pi)\,\mathrm{d}x.$$

2. (5.3) Express the definite integrals as limits of Riemann sums.
   (a) (5.3.8) $\int_{-1}^1 (x^2-x)\,\mathrm{d}x$

Solution: We know that if the integral ranges from $x=-1$ to $x=1$ then $\Delta x = \frac{2}{n}$ and $x_i = \frac{2i}{n}-1$. Using the definition of the definite integral

$$\begin{aligned}\int_{-1}^1 f(x)\,\mathrm{d}x &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i)\Delta x\\ &= \lim_{n\to\infty}\sum_{i=1}^n f\left(\frac{2i}{n}-1\right)\frac{2}{n}.\end{aligned}$$

Thus, since $f(x) = x^2 - x = x(x-1)$ we get that

$$\int_{-1}^{1} x^2 - x \, \mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{2i}{n} - 1\right)\left(\frac{2i}{n} - 2\right)\frac{2}{n}.$$

(b) (5.3.9) $\int_0^1 e^x \, \mathrm{d}x$

(c) (5.3.11) $\int_{-1}^{1} |x| \, \mathrm{d}x$

3. (5.5) Calculate the following integrals using substitution.

(a) (5.5.12) $\int \frac{x}{\sqrt{x^2+1}} \, \mathrm{d}x$

(b) (5.5.14) $\int \sin^3 t \cos t \, \mathrm{d}t$

(c) (5.5.16) $\int \frac{z^3}{\sqrt{z^4+12}} \, \mathrm{d}z$

(d) (5.5.19) $\int_1^2 \frac{e^{1/x}}{x^2} \, \mathrm{d}x$

(e) (5.5.23) $\int_1^2 x\sqrt{x-1} \, \mathrm{d}x$

(f) (5.5.24) $\int_0^2 (e^x - e^{-x})^2 dx$

4. (5.5-30) Suppose an environmental study indicates that the ozone level, L, in the air above a major metropolitan center is changing at a rate modeled by the function

$$L'(t) = \frac{0.24 - 0.03t}{\sqrt{36 + 16t - t^2}}$$

parts per million per hour (ppm/h) t hours after 7:00 A.M.

(a) Express the ozone level $L(t)$ as a function of t if L is 4 ppm at 7:00 A.M.

Solution: The function $L(t)$ expressing the ozone level at time t will be an antiderivative of $L'(t)$. That is

$$L(t) = \int \frac{0.24 - 0.03t}{\sqrt{36 + 16t - t^2}} \, \mathrm{d}t.$$

We will use the substitution $u = 36 + 16t - t^2$. Thus $u' = 2(8 - t)$. Note that $0.24 - 0.03t = 0.03(8 - t)$. Thus

$$\begin{aligned} L(t) &= \int \frac{0.03}{\sqrt{36 + 16t - t^2}} \cdot \frac{1}{2} 2(8-t) \, \mathrm{d}t \\ &= 0.03 \int \frac{1}{2\sqrt{u}} \, \mathrm{d}u \\ &= 0.03\sqrt{u} + C \\ &= 0.03\sqrt{36 + 16t - t^2} + C \end{aligned}$$

To find the constant C we simply solve the equation $L(0) = 4$, that is,

$$\begin{aligned} 0.03\sqrt{36 + 16 \cdot 0 - 0^2} + C &= 4 \\ 0.03\sqrt{36} + C &= \\ 0.18 + C &= \\ C &= 4 - 0.18 = 3.82. \end{aligned}$$

Thus

$$L(t) = 0.03\sqrt{36 + 16t - t^2} + 3.82.$$

(b) Find the time between 7:00 A.M. and 7:00 P.M. when the highest level of ozone occurs. What is the highest level? *(Note: part b has been changed slightly from what is written in the textbook.)*

> **Solution:** First we find the critical points by setting $L'(t) = 0$. This happens when $t = 8$, i.e. at 3pm. Using the first derivative test we know this is a maximum. Thus the highest level of ozone is
> $$L(8) = 0.03\sqrt{26 + 16 \cdot 8 - 64} - 3.82 = 0.09\sqrt{10} - 3.82 = 4.10\text{ppm}$$

5. The circle $x^2 + (y+1)^2 = 4$ has area 4π. What is the area of the portion of the circle lying above the x axis?

You may use the fact that

$$\int \sqrt{1-t^2}\,\mathrm{d}t = \frac{1}{2}\left(t\sqrt{1-t^2} + \sin^{-1} t\right) + C.$$

> **Solution:** We first draw a picture so that we can visualise the area we would like to find.
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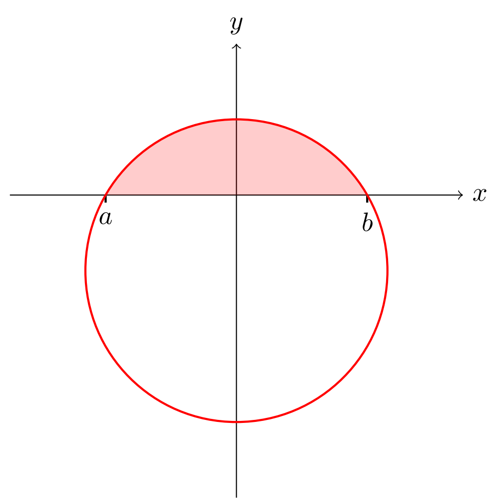
> 
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>
> We want to find the shaded area. The circle is given by the equation $x^2 + (y+1)^2 = 4$, which means the function that describes the top half semicircle is
> $$y = \sqrt{4 - x^2} - 1$$
> and the area is given by the integral
> $$A = \int_a^b \sqrt{4 - x^2} - 1\,\mathrm{d}x.$$
> Here a and b are the x-intercepts of the semicircle. We can find these by setting $y = 0$ and solving for x:
> $$\begin{aligned} 0 &= \sqrt{4 - x^2} - 1 \\ 1 &= \sqrt{4 - x^2} \\ 1 &= 4 - x^2 \\ x^2 &= 4 - 1 = 3 \\ x &= \pm\sqrt{3}. \end{aligned}$$

Thus $a = -\sqrt{3}$ and $b = \sqrt{3}$. Thus

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4 - x^2} - 1 \, \mathrm{d}x$$

which we can separate,

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4 - x^2} \, \mathrm{d}x - \int_{-\sqrt{3}}^{\sqrt{3}} \mathrm{d}x$$

and factor out the 4,

$$= \int_{-\sqrt{3}}^{\sqrt{3}} 2\sqrt{1 - \left(\frac{x}{2}\right)^2} \, \mathrm{d}x - \int_{-\sqrt{3}}^{\sqrt{3}} \mathrm{d}x.$$

Note that

$$\int_{-\sqrt{3}}^{\sqrt{3}} \mathrm{d}x = 2\sqrt{3}. \tag{1}$$

We can solve the first part of A by using the substitution $u = \frac{x}{2}$, so $u' = \frac{1}{2}$. Note that when $x = \pm\sqrt{3}$ then $u = \pm\frac{\sqrt{3}}{2}$. This means

$$2\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{1 - \left(\frac{x}{2}\right)^2} \, \mathrm{d}x = 4\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \sqrt{1 - u^2} \, \mathrm{d}x$$

now we can apply the antiderivative given in the question,

$$= 4\left[\frac{1}{2}u\sqrt{1 - u^2} + \frac{1}{2}\sin^{-1} u\right]_{-\sqrt{3}/2}^{\sqrt{3}/2}$$

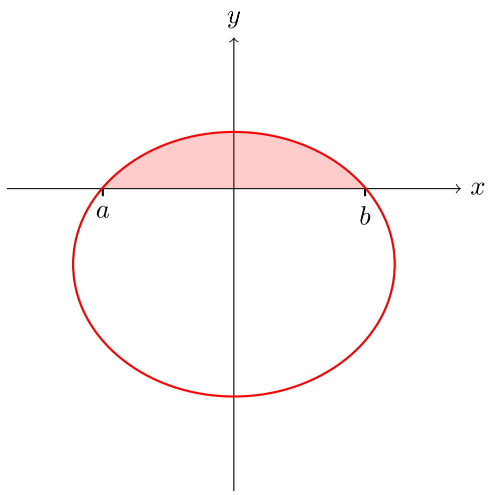
noting that $\sin^{-1}\left(\pm\frac{\sqrt{3}}{2}\right) = \pm\frac{pi}{3}$, we get

$$\begin{aligned} &= \sqrt{3} \cdot \frac{1}{2} + 2 \cdot \frac{\pi}{3} - \left(-\sqrt{3}\right) \cdot \frac{1}{2} - 2 \cdot \left(-\frac{\pi}{3}\right) \\ &= \sqrt{3} + \frac{4\pi}{3}. \end{aligned} \tag{2}$$

Since $A = (4) - (3)$ we have $A = \frac{4\pi}{3} - \sqrt{3}$.

6. Consider the ellipse $x^2 + 3(y + 1)^2 = 4$. What is the area of the portion of the ellipse lying above the x axis?

Solution: We first draw a picture so that we can visualise the area we would like to find.



We want to find the shaded area. The circle is given by the equation $x^2 + 3(y+1)^2 = 4$, which means the function that describes the top half semicircle is

$$y = \sqrt{\frac{4-x^2}{3}} - 1$$

and the area is given by the integral

$$A = \int_a^b \sqrt{\frac{4-x^2}{3}} - 1 \, \mathrm{d}x.$$

Here a and b are the x-intercepts of the semicircle. We can find these by setting $y = 0$ and solving for x:

$$\begin{aligned} 0 &= \sqrt{\frac{4-x^2}{3}} - 1 \\ 1 &= \sqrt{\frac{4-x^2}{3}} \\ 3 &= 4 - x^2 \\ x^2 &= 1 \\ x &= \pm 1. \end{aligned}$$

Thus $a = -1$ and $b = 1$. Thus

$$A = \int_{-1}^{1} \sqrt{\frac{4-x^2}{3}} - 1 \, \mathrm{d}x$$

which we can separate,

$$= \int_{-1}^{1} \sqrt{\frac{4-x^2}{3}} \, \mathrm{d}x - \int_{-1}^{1} \mathrm{d}x$$

and factor out the $4/3$,

$$= \int_{-1}^{1} \frac{2}{\sqrt{3}} \sqrt{1 - \left(\frac{x}{2}\right)^2} \, \mathrm{d}x - \int_{-1}^{1} \mathrm{d}x.$$

Note that

$$\int_{-1}^{1} \mathrm{d}x = 2. \tag{3}$$

We can solve the first part of A by using the substitution $u = \frac{x}{2}$, so $u' = \frac{1}{2}$. Note that when $x = \pm 1$ then $u = \pm\frac{1}{2}$. This means

$$\frac{2}{\sqrt{3}} \int_{-1}^{1} \sqrt{1 - \left(\frac{x}{2}\right)^2}\, \mathrm{d}x = \frac{4}{\sqrt{3}} \int_{-1/2}^{1/2} \sqrt{1-u^2}\, \mathrm{d}u$$

now we can apply the antiderivative given in the question,

$$= \frac{4}{\sqrt{3}} \left[\frac{1}{2} u\sqrt{1-u^2} + \frac{1}{2}\sin^{-1} u\right]_{-1/2}^{1/2}$$

noting that $\sin^{-1}\left(\pm\frac{1}{2}\right) = \pm\frac{\pi}{6}$, we get

$$\begin{aligned} &= \frac{4}{\sqrt{3}} \left(\frac{1}{4}\cdot\frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\pi}{6} - \left(-\frac{1}{4}\right)\cdot\frac{\sqrt{3}}{2} - \frac{1}{2}\cdot\left(-\frac{\pi}{6}\right)\right) \\ &= 1 + \frac{2\pi}{3\sqrt{3}}. \end{aligned} \tag{4}$$

Since $A = (4) - (3)$ we have $A = \frac{2\pi}{3\sqrt{3}} - 1$.

7. (5.6) Calculate the following integrals using integration by parts.
   (a) (2) $\int e^t \sin t \,\mathrm{d}t$
   (b) (6) $\int x^2 \ln x \,\mathrm{d}x$
   (c) (9) $\int \sin x \cos x \,\mathrm{d}x$
   (d) (14) $\int_0^{\pi} x \sin x \,\mathrm{d}x$
   (e) (16) $\int_1^{e} x^3 \ln x \,\mathrm{d}x$

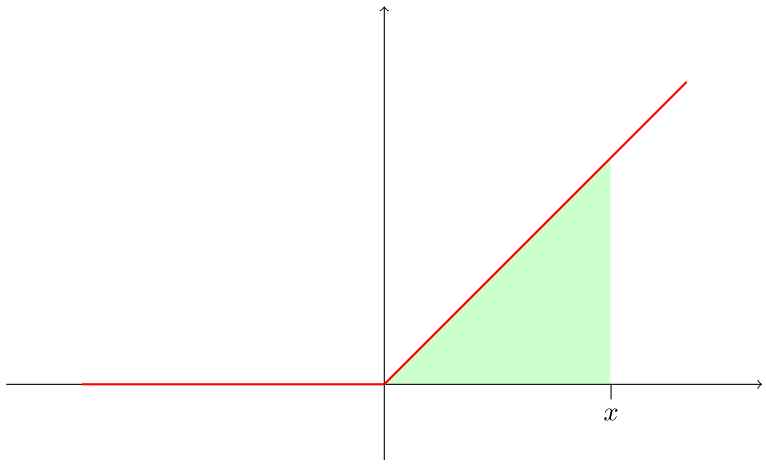
8. Use the fundamental theorem of calculus and the interpretation of the definite integral as an area to find a formula for the general antiderivative of the function $f(x) = \max\{0, x\}$.

Solution: The fundamental theorem of calculus says that for any constant a, the function

$$F(x) = \int_a^x \max\{0, t\}\, \mathrm{d}t$$

will be an antiderivative of $f(x) = \max\{0, x\}$.

Let us choose $a = 0$ for simplicity. To evaluate this integral we consider two cases. First when $x \geq 0$. In this case we can interpret the integral as an area.



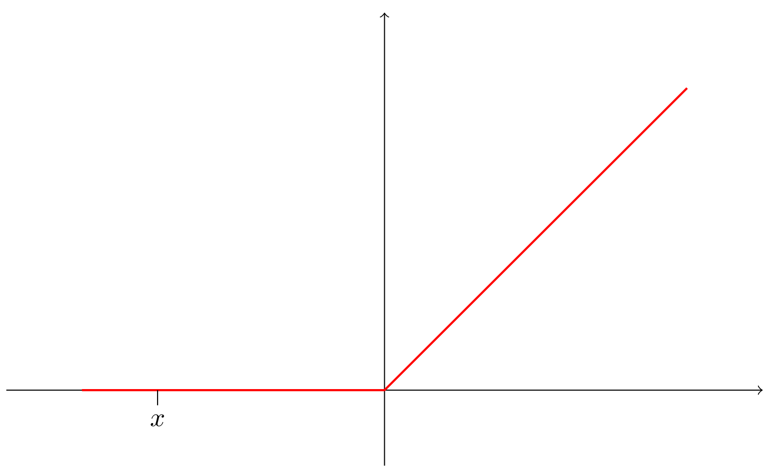
This is a triangle with height x and base length x so the area is $\frac{1}{2}x^2$. I.e. $F(x) = \frac{1}{2}x^2$ when $x \geq 0$. Now consider the case when $x < 0$. We cannot use the area interpretation of the integral

$$F(x) = \int_0^x \max\{0, t\} \, \mathrm{d}t$$

since we are going from right to left, however we can use the property of definite integrals that says we can swap the limits at the expense of a minus sign:

$$F(x) = \int_0^x \max\{0, t\} \, \mathrm{d}t = -\int_0^x \max\{0, t\} \, \mathrm{d}t.$$

The integral on the right is now one we can evaluate using the same area interpretation:



The area now is obviously zero, thus $F(x) = 0$ when $x < 0$. Summarising, the general antiderivative is thus and shift by a constant of what we have found above:

$$F(x) = \begin{cases} \frac{1}{2}x^2 + C & \text{if } x \geq 0 \\ C & \text{if } x < 0 \end{cases}$$

$$= \frac{1}{2}x \cdot \max\{0, x\} + C.$$

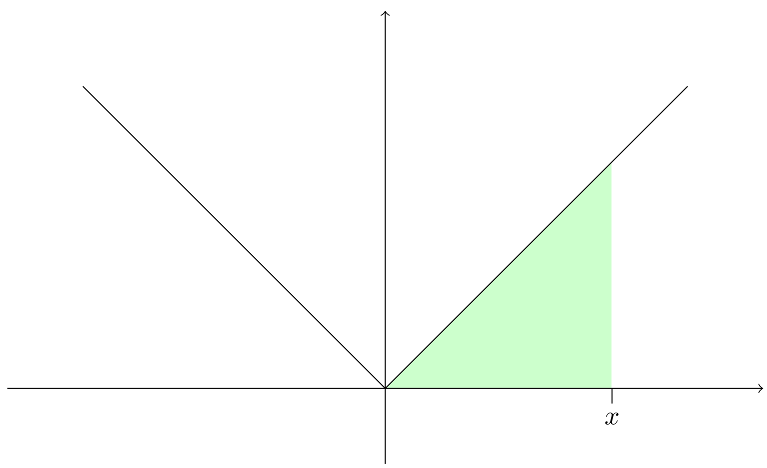
9. Use the fundamental theorem of calculus and the interpretation of the definite integral as an area to find a formula for the general antiderivative of the function $f(x) = |x|$.

Solution: The fundamental theorem of calculus says that for any constant a, the function

$$F(x) = \int_a^x |t|\ \mathrm{d}t$$

will be an antiderivative of $f(x) = |x|$.

Let us choose $a = 0$ for simplicity. To evaluate this integral we consider two cases. First when $x \geq 0$. In this case we can interpret the integral as an area.



This is a triangle with height x and base length x so the area is $\frac{1}{2}x^2$. I.e. $F(x) = \frac{1}{2}x^2$ when $x \geq 0$.

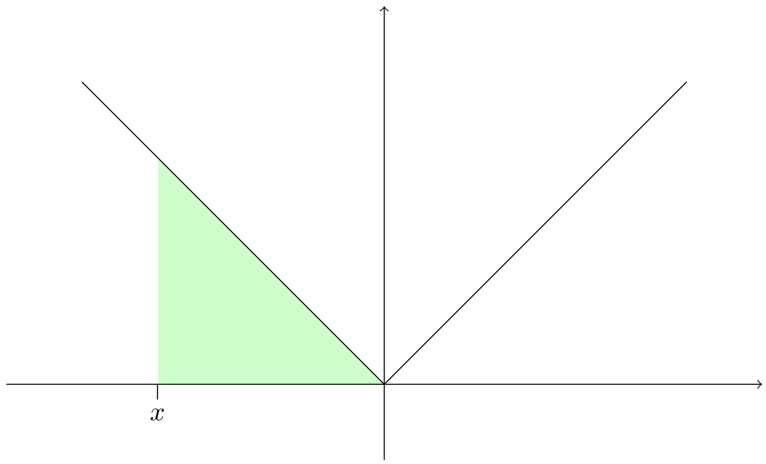
Now consider the case when $x < 0$. We cannot use the area interpretation of the integral

$$F(x) = \int_0^x |t|\ \mathrm{d}t$$

since we are going from right to left, however we can use the property of definite integrals that says we can swap the limits at the expense of a minus sign:

$$F(x) = \int_0^x |t|\ \mathrm{d}t = -\int_x^0 |t|\ \mathrm{d}t.$$

The integral on the right is now one we can evaluate using the same area interpretation:



The area again is a triangle with area $\frac{1}{2}x^2$, thus $F(x) = -\frac{1}{2}x^2$ when $x < 0$. Summarising, the general antiderivative is thus and shift by a constant of what we have found above:

$$F(x) = \begin{cases} \frac{1}{2}x^2 + C & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 + C & \text{if } x < 0 \end{cases}$$

$$= \frac{1}{2}x|x|.$$

10. Use the fundamental theorem of calculus and the interpretation of the definite integral as an area to find a formula for the general antiderivative of the function $f(x) = \frac{1}{x}$.

Solution: The fundamental theorem of calculus says that for any constant a, the function

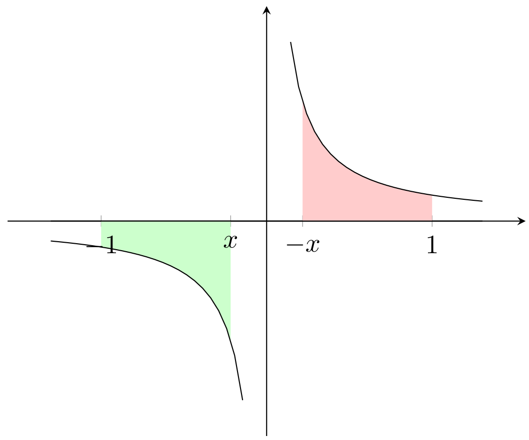
$$F(x) = \int_a^x \frac{1}{t}\,\mathrm{d}t$$

will be an antiderivative of $f(x) = \frac{1}{x}$. We know what happens when $x > 0$, in this case $F(x) = \ln x + C$. So we concentrate on $x < 0$.

Let us choose $a = -1$ for simplicity. We are interested in the area

$$F(x) = \int_{-1}^x \frac{1}{t}\,\mathrm{d}t$$

This is the green shaded area below



By symmetry, this is exactly the negative of the red area! So

$$F(x) = \int_{-1}^{x} \frac{1}{t}\,\mathrm{d}t = -\int_{-x}^{1} \frac{1}{t}\,\mathrm{d}t = [-\ln t]_{-x}^{1} = \ln(-x).$$

So, summarising,

$$F(x) = \begin{cases} \ln x + C & \text{if } x > 0 \\ \ln -x + C & \text{if } x < 0 \end{cases}$$

$$= \ln |x|.$$