Lecture 16: Applications of integration + accumulated change.

Start with an example:

Ex 1 (5.8 example 1 in text).

- Medical clinic opens with 300 patients
- Clinic gains new patients at a rate of

  10 patients / month
- The proportion of patients who remain with the clinic ~~after~~ t months after they first join is

$$e^{-t/20}$$

Question: How many patients does the clinic have after 15 months?

Solution: For the first 300 patients, after 15 months,

$$300e^{-15/20} = 300e^{-3/4} \approx 141.7$$

remain with the clinic.

When it comes to analysing how many of the patients gained per month remain with the

Clinic, we have a problem:

- patients are being gained continuosly
- how do we work out how many months each of these patients have been at the clinic when the clinic is 15 months old?
- We need to do this so we can apply the formula $e^{-t/20}$.

(Easier) problem: Lets make the problem easier.

- Assume the new patients are not being gained continuosly, but rather the clinic gains 10 patients at the start of every month.

When the clinic is 0 months old:

gain: 300 + 10 patients

at 15 mons: $300e^{-15/20} + 10e^{-15/20}$ of these remain

When the clinic is 1 month old:

gain: 10 patients

at 15 months: $10e^{-14/20}$ of these remain

14 comes from the fact that the patients

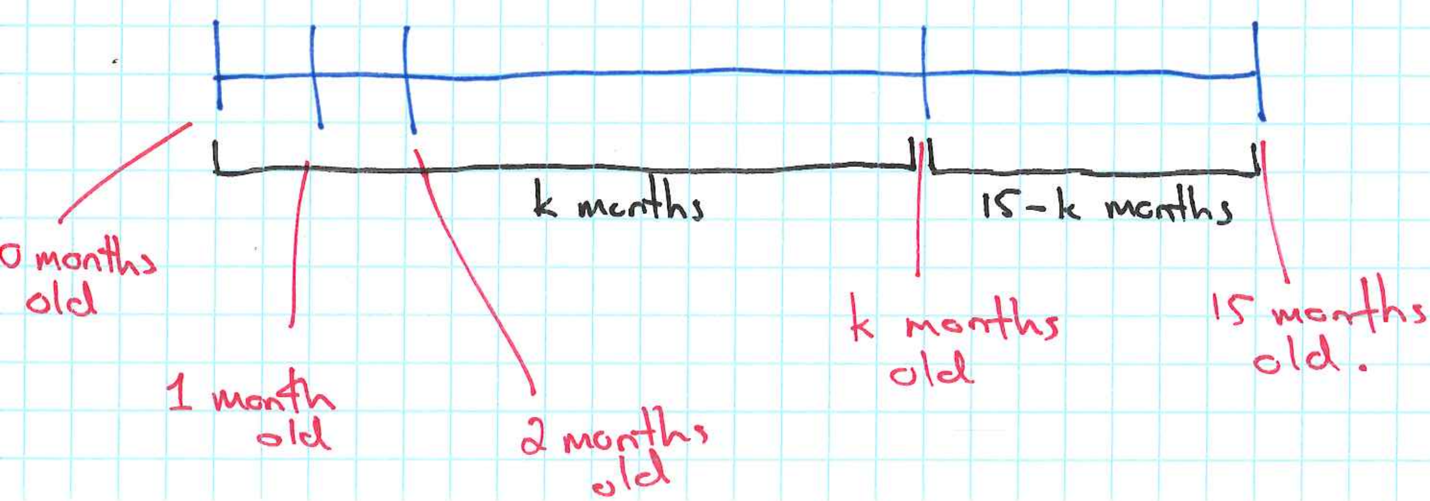
joining the clinic when it is 1 month old will have been with the clinic for 14 month when it turns 15 months old.

<u>When the</u> clinic is k months old:

gain: 10 patients

at 15 mons: $10e^{-(15-k)/20}$ of these remain

As above, the 15-k comes from the fact, this is the number of months these patients, who joined when the ~~clin~~ clinic was k months old, will have been with the clinic on it's 15 month anniversary.



So the total number of patients remaining is simply the sum of all these contributions

$$\underbrace{300e^{-15/20} + 10e^{-15/20}}_{\text{joined 0 months}} + \underbrace{10e^{-14/20}}_{\text{joined 1 month}} + \dots + \underbrace{10e^{-(15-k)/20}}_{\text{joined k months}} + \dots + \underbrace{10e^{-1/20}}_{\text{joined 14 months}}$$

$$= 300e^{-3/4} + \sum_{k=0}^{14} 10e^{-(15-k)/20}$$

(Easier) problem #2:

We could get a better appox. by assuming 5 new patients start at the beginning of every half month.

- ~~Insted~~ Instead of splitting the time period of 15 months into 15 (1-month long) pieces we split it into 30 ($\frac{1}{2}$-month long) pieces.

At the start of 0^{th} $\frac{1}{2}$ month:

gained: 300 + 5 patients

at 15 mons: $300e^{-15/20} + 5e^{-15/20}$ remain.

At the start of the k^{th} $\frac{1}{2}$ month:

gained : 5 ~~[illegible]~~ patients

at 15 months: $5\cancel{}e^{-(15-t_k)/20}$ of these remain

Where: t_k is the number of months elapsed after k $\frac{1}{2}$ months, i.e. $t_k = k/2$.

Adding all of these contributions

$$300e^{-3/4} + \sum_{k=0}^{29} \cancel{}5\, e^{-(15-t_k)/20}$$

Actual problem

We can get the actual answer by taking better and better approximations:

- Split the 15 month period into n subintervals
- Let $n \longrightarrow \infty$.

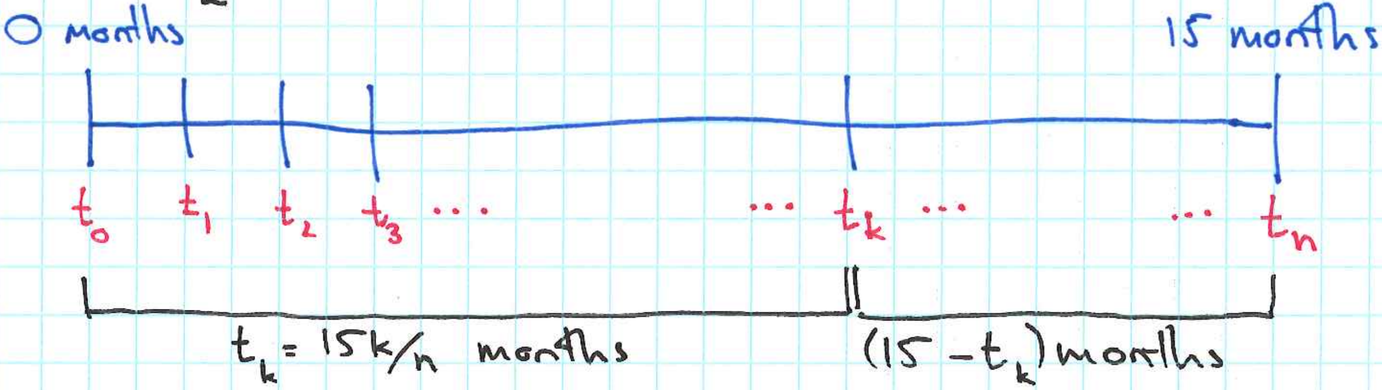
If we split into n intervals:

- Each time period is $15/n$ months long.
- Gain $10 \cdot \frac{15}{n} = 150/n$ patients at the ~~stat~~ start of every time period.

- Let t_k be the time when k of these intervals have elapsed. i.e.

$$t_k = k \cdot \frac{15}{n}.$$

~~At t_k~~ .



At t_0:

gain: $300 + \frac{150}{n}$ patients

at 15 mons: $300e^{-15/20} + \frac{150}{n}e^{-15/n}$ remain

At t_k:

gain: $\frac{150}{n}$ patients

at 15 mon's: $\frac{150}{n}e^{-(15-t_k)/20}$ remain.

Adding the contributions up

$$300e^{-3/4} + \sum_{k=0}^{n-1} \frac{15}{n} \cdot 10\, e^{-(15-t_k)/20}$$

And taking a limit, the number of patients at 15 mons is

$$P = \lim_{n \to \infty} \left(300 e^{-3/4} + \sum_{k=0}^{n-1} \frac{15}{n} \cdot 10 e^{-(15-t_k)/20} \right)$$

$$= 300 e^{-3/4} + \lim_{n \to \infty} \left(\frac{15}{n} \sum_{k=0}^{n-1} 10 e^{-(15-t_k)/20} \right)$$

Where have we seen this before ...?

Reminder:

$$\int_a^b f(t)\,dt := \lim_{n \to \infty} \left(\frac{b-a}{n} \sum_{k=0}^{n-1} f(t_k) \right)$$

where $t_k = a + k \cdot \frac{b-a}{n}$

Applying this to $\lim_{n \to \infty} \frac{15}{n} \sum_{k=0}^{n-1} 10 e^{-(15-t_k)/20}$

with

- $b = 15$
- $a = 0$
- $f(t) = 10 e^{-(15-t)/20}$

gives

$$P = 300e^{-3/4} + \int_0^{15} 10e^{-(15-t)/20}\, dt$$

Which we can solve to get an actual numerical answer!

$$= 300e^{-3/4} + \left[200e^{-(15-t)/20}\right]_0^{15}$$

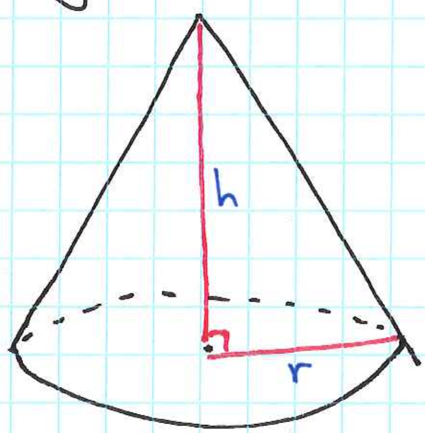
$$= 300e^{-3/4} + 200 - 200e^{-3/4}$$

$$= 100\left(e^{-3/4} + 2\right) \approx 247.2$$

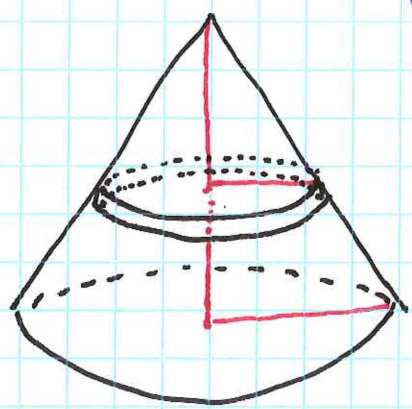
Phew!

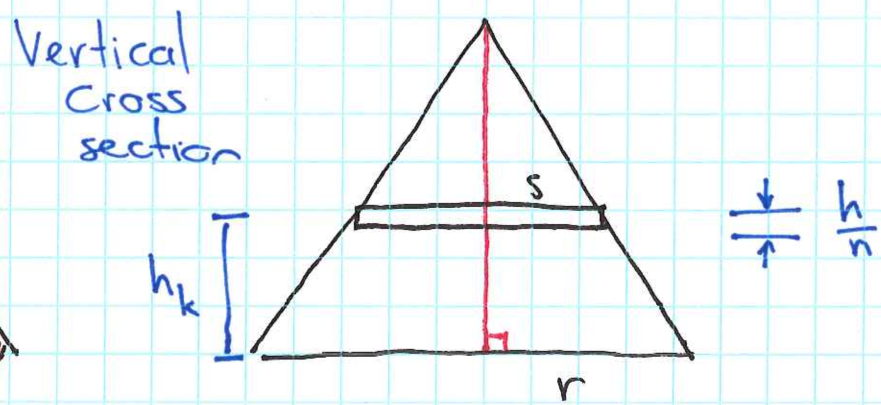
Lets try another. They'll get easier...

Ex 2 What is the volume of a cone with height h and base radius r.



Solution: We will slice the cone horitontally into n slices. As long as n is very big, these slices are thin ane are appox. squashed cylinders:





Consider the k^{th} slice:

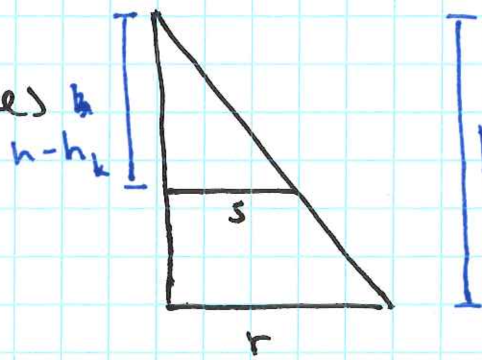
- We count the bottom slice as slice 0.
- We have n slices, so each slice is h/n thick.
- If h_k is the distance the slice is

from the bottom of the cone then

$$h_k = k \cdot \frac{h}{n}$$

- We want to ~~ct~~ calculate the volume of this slice (and the add them up).
- So we need to know s:

Use similar triangles



so $\frac{s}{r} = \frac{h - h_k}{h}$

i.e $s = \frac{r}{h}(h - h_k)$

<u>Volume of slice k</u>: $\underbrace{\text{thickness}}_{h/n} \times \underbrace{\text{area of circlular base}}_{\pi s^2}$

$$= \frac{h}{n} \cdot \pi \cdot \frac{r^2}{h^2}(h - h_k)^2 =$$

<u>Adding these up</u>

$$\text{Volume} = \lim_{n \to \infty} \sum_{k=0}^{n-1} \frac{h}{n} \pi \frac{r^2}{h^2}(h - h_k)^2$$

$$= \int_0^h \pi \frac{r^2}{h^2}(h - x)^2 \, dx$$

$$= \frac{\pi r^2}{h^2} \left[-\frac{1}{3} (h - x)^3 \right]_0^h$$

$$= \frac{\pi r^2}{h^2} \left(-\frac{1}{3} 0^2 + \frac{1}{3} h^3 \right)$$

$$= \frac{1}{3} \pi r^2 h \qquad \square.$$