Math 115 A

- We have all learnt how useful vectors and matrices are.
- In this course we will identify what parts of the structure of vectors/matrices make them so ~~the~~ useful.
- We will call any collection of objects with the same properties a vector space
- Linear algebra is the study of vector spaces

Def (very vague) A vector space is any collection of objects which we can add together and multiply by scalars.

(for now a scalar is either a real number or a complex number.)

- A much more formal and precise definition will be given later.
- Instead, lets learn some examples that we will come back to again and again.

Examples

1 (i) $\mathbb{R}^n = \{$ column vectors with n coords $\}$.

We can add:

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_n + b_n \end{pmatrix}.$$

and scalar multiply

$$\lambda \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} \lambda a_1 \\ \lambda a_2 \\ \vdots \\ \lambda a_n \end{pmatrix}$$

(ii) $\Sigma_n^n = \{$ column vectors with n coords that add to zero, e.g. (if $n=3$) $\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\}$.

If we have two vectors

$$\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \Sigma_n$$

so $a_1 + \dots + a_n = b_1 + \dots + b_n = 0$ ~~thus~~ $a_1 + b_1 + \dots + a_n + b_n = 0$

so

$$\begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} \in \Sigma_n$$

so we can add. Similarly, scalar mult work too.

2. $\text{Mat}_{m\times n}(\mathbb{C}) = \{ m \times n \text{ matrices with entries in } \mathbb{C} \}$

We can add and scalar mult matrices as usual.

3(i) $\underline{\ell} = \{$ infinite sequences $(a_0, a_1, \dots)$ with entries in $\mathbb{C} \}$.

$(a_0, a_1, \dots) + (b_0, b_1, \dots) = (a_0 + b_0, a_1 + b_1, \dots)$

(ii) $\underline{\ell_c} = \{$ infinite series with only finitely many nonzero terms $\}$.

If $(a_0, a_1, \dots), (b_0, b_1, \dots)$ only have fin. many non zero terms, so does $(a_0, a_1, \dots) + (b_0, b_1, \dots)$

(iii) $\underline{\ell_{\to 0}} = \{$ infinite series $(a_0, \dots)$ s.t. $\lim_{n \to 0} a_n = 0 \}$.

note that $\lim_{n\to 0} (a_n + b_n) = 0$ if $\lim_{n\to 0} a_n = \lim_{n \to 0} b_n = 0$.

and $\lim_{n \to 0} \lambda a_n = 0$.

4(i) $\mathcal{C}(\mathbb{R}) = \{$ real valued function, continuous, in one variable, eg x^2, $\sin(x) - e^{x^3}$, etc $\}$.

If $f(x), g(x)$ are cts then so if $f(x) + g(x)$ and $\lambda f(x)$.

(ii) $\mathbb{R}[x] = \{$ polynomials in one variable $\}$

sum of polynomials is a polynomial.

scalar multiple of a polynomial is a polynomial.

(iii) $\mathbb{R}(x) = \{$ rational functions, i.e. ~~qo~~ quotients of polynomials $\frac{p(x)}{q(x)}$ $\}$.

again, sums and scalar multiples of rational functions, are rational functions

e.g. $$\frac{x^2-1}{x} + \frac{1}{x+2} = \frac{(x^2-1)(x+2) + x}{x(x+2)}$$

Scalars + Fields

In all of these examples, the scalars were either $\mathbb{R}$ or $\mathbb{C}$. These are examples of fields

That is, a set of mathematical objects that

- we can add together
- multiply together (and order doesn't matter)
- take inverses

More formally:

Def (Fields) A *field* is a set $\mathbb{F}$ along with two binary operations

- Addition $+$
- multiplication $\cdot$

such that

(F1) $a+b=b+a$ and $ab=ba$
(commutativity)

(F2) $(a+b)+c=a+(b+c)$ and $(ab)c=a(bc)$
(associativity)

(F3) There exist special elements $0, 1 \in \mathbb{F}$

$$0+a=a \qquad 1\cdot a=a$$

(identity elements)

(F4) There exist elements $-a$ and a^{-1}

$$a+(-a)=0 \qquad aa^{-1}=1$$

(inverses)

(F5) $a(b+c)=ab+ac$ (distributivity)

Examples $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are the examples you already know. It is tedious to check all the axioms! Lets do it once for $\mathbb{Q}$:

$\mathbb{Q} = \{ \text{rational numbers } \frac{p}{q} \}$.

Addition is given by $\frac{p}{q} + \frac{s}{t} = \frac{pt+sq}{qt}$

Multiplication is given by $\frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt}$

F1: $\frac{p}{q} + \frac{s}{t} = \frac{pt+sq}{qt} = \cancel{\frac{tp+qs}{qtq}} = \frac{sq+pt}{tq} = \frac{s}{t} + \frac{p}{q}$

$$\frac{p}{q} \cdot \frac{s}{t} = \frac{ps}{qt} = \frac{sp}{tq} = \frac{s}{t} \cdot \frac{p}{q}$$

F2: $\left(\frac{p}{q} + \frac{s}{t}\right) + \frac{u}{v} = \frac{pt+sq}{qt} + \frac{u}{v} = \frac{ptv+sqv+uqt}{qtv}$

these agree.

$$\frac{p}{q} + \left(\frac{s}{t} + \frac{u}{v}\right) = \frac{p}{q} + \frac{sv+ut}{tv} = \frac{ptv+svq+utq}{qtv}$$

and so on...

Theorem The elements $0, 1 \in \mathbb{F}$ are unique, i.e. they are the only elements satisfying

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

for all a.

proof: Suppose we had another additive inverse, lets call it 0^*. Then for every a we have

$$a + 0^* = a$$

but we also have

$$a + 0 = a$$

so $\quad a + 0^* = a + 0$

by adding $-a$ to both sides of the equation we get

$$-a + a + 0^* = -a + a + 0$$

i.e $\; 0^* = 0.$

A similar method works ~~we~~ for 1. $\square$.

<u>Theorem</u> For any $a \in \mathbb{F}$

a) $0 \cdot a = 0$

b) $-1 \cdot a = -a$

<u>proof</u>: a) call $0 \cdot a =: x$. Note that $0 = 0 + 0$

so $\quad x = (0+0) \cdot a = 0 \cdot a + 0 \cdot a = x + x$

adding $-x$ to both sides:

$$0 = x$$

b) Note that $-a$ is the unique element such that $a + (-a) = 0$.

$$\begin{aligned} a + (-1 \cdot a) &= (1 \cdot a) + (-1 \cdot a) \\ &= (1 - 1) \cdot a \\ &= 0 \cdot a \\ &= 0 \end{aligned}$$

$\square$.

Examples Here are some more examples of fields.

1. $\mathbb{R}(x) = \{\text{rational functions}\}$.
   The proof is the same as for $\mathbb{Q}$.

2. Let p be a prime number. Consider the set $\mathbb{Z}_p = \{[0], [1] \ldots, [p-1]\}$.

We define addition and multiplication "mod p". This means

$[a]+[b]$ = remainder after dividing $(a+b)$ by p

$[a]\cdot[b]$ = —— " —— $(a.b)$ by p.

eg: If $p=5$, then $\mathbb{Z}_5 = \{[0][1][2][3][4]\}$ and

$$[3]+[4] = 7 = [2]$$
$$[3]\cdot[4] = 12 = [2]$$
$$[2]\cdot[3] = 6 = [1]$$

Lets check the axioms.

(F1) (F2) and (F5) are clear.

F3: The zero element is $[0]$

The one element is $[1]$

clearly $[0]+[a]=[a]$ and $[1]\cdot[a]=[a]$.

(F4): Clearly $[a]+[-a]=[0]$ so $-[a]=[a]$.
To ~~find~~ show $[a]^{-1}$ is tricky and this is where we need that p is prime. Ask me in office hours if you like, but we will skip it here since we won't use it.

Vector spaces

We fix a choice of field $\mathbb{F}$ (keep $\mathbb{R}$ or $\mathbb{C}$ in mind).

Def (vector space) A vector space is a set V with two operations

- addition $+$
- scalar multiplication $\cdot$

such that

(VS1) For all $v, w \in V$ $\quad v+w=w+v$

(VS2) For all $u\,v\,w \in V$ $\quad (u+v)+w=u+(v+w)$

(VS3) There exists an element $0 \in V$ such that
$0+v=v$ for any $v \in V$

Subspaces

Def (Subspace) A subset $W \subseteq V$ is a **subspace** if W is a vector space with addition and scalar mult inherited from V.

Example Both $\{0\}, V \subseteq V$ are subspaces.

Theorem $W \subseteq V$ is a subspace if and only if

a) W is closed under addition

b) W is closed under scalar mult, and

c) $0 \in W$.

proof "$\Longrightarrow$" If W is a subspace, then it is closed under addition and scalar mult automatically. It also has a zero element, denote it 0_W. Let $0 \in V$ be the zero element of V. For any $w \in W$ we have

$$w + 0_W = w = w + 0$$

adding $-w$, we get $0_W = 0$, so $0 \in W$.

"$\Longleftarrow$" Assume a) b) c) are true. This means VS1,2,3,5,6,7,8 are all automatically true.

(VS4) For each $v \in V$ there exists an element $-v \in V$ so that $v + (-v) = 0$

(VS5) For each $v \in V$, $1 \cdot v = v$ (where $1 \in \mathbb{F}$)

(VS6) For every v ~~w~~ $\in V$ and every $\lambda, \mu \in \mathbb{F}$

$$(\lambda\mu) \cdot v = \lambda \cdot (\mu \cdot v)$$

(VS7) For every $v, w \in V$ and every $\lambda \in \mathbb{F}$

$$\lambda \cdot (v + w) = \lambda \cdot v + \lambda w$$

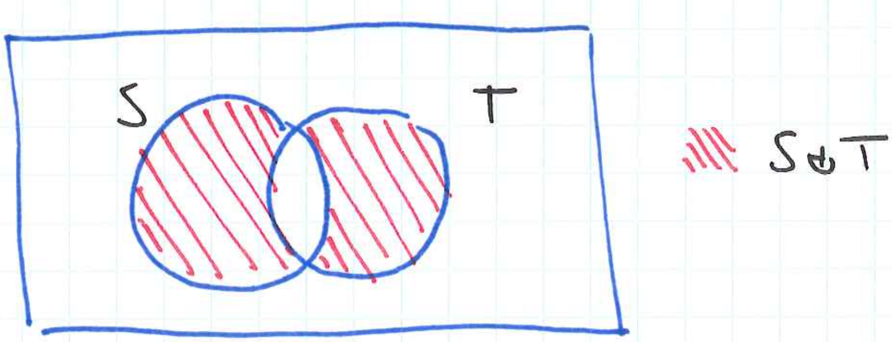
(VS8) For every $v \in V$ and every $\lambda \mu \in \mathbb{F}$

$$(\lambda + \mu) \cdot v = \lambda \cdot v + \mu \cdot w.$$

Examples It is tedious but very good practice to check the examples given above are vector spaces.

Example If ~~X~~ S and T are sets, define the symmetric difference to by

$$S \oplus T = S \cup T \setminus S \cap T$$



Claim: The set $\mathcal{P}(S)$ of subsets of S is a vector space over $\mathbb{Z}_2 = \{[0],[1]\}$ with addition given by $\oplus$ and scalar mult ~~proof~~

$$[0]\cdot A := \phi \ , \ [1]\cdot A = A$$

proof:

VS1: true since $S\cup T = T\cup S$ and $S\cap T = T\cap S$

so $A \oplus B = A\cup B \setminus A\cap B = B\cup A \setminus B\cap A = B \oplus A$.

VS2: This follows by drawing Venn diagrams.

VS3: The zero element is ϕ. Indeed

$$A \oplus \phi = A\cup\phi \setminus A\cap\phi$$
$$= A \setminus \phi = A.$$

VS4: $-A$ is given by A itself! Indeed

$$A \oplus A = A\cup A \setminus A\cap A$$
$$= A\setminus A = \phi.$$

VS5: true by definition

VS6, VS7, VS8: one just needs to check each case.

Theorem For any vector space V, any $v \in V$, $\lambda \in \mathbb{F}$

a) $0 \cdot v = 0$

b) $-1 \cdot v = -v$

c) $\lambda \cdot 0 = 0$

proof: a) by VS8

$$0 \cdot v + 0 \cdot v = (0+0) \cdot v$$
$$= 0 \cdot v$$
$$= 0 + 0 \cdot v \qquad \text{(by VS3)}$$

By adding $-(0 \cdot v)$ to both sides:

$$0 \cdot v = 0 \qquad \square.$$

b) We just need to check that

$$v + (-1 \cdot v) = 0$$

indeed

$$v + (-1 \cdot v) = (1 \cdot v) + (-1 \cdot v) \qquad \text{(by VS5)}$$
$$= (1-1) \cdot v \qquad \text{(by VS8)}$$
$$= 0 \cdot v$$
$$= 0 \qquad \text{(by prev.)}$$

c) $\lambda \cdot 0 + \lambda \cdot 0 = \lambda \cdot (0+0)$

$$= \lambda \cdot 0$$

by adding $-\lambda \cdot 0$ to both sides

$$\lambda \cdot 0 = 0 \qquad \square.$$

We just need to check, if $v \in W$ then $-v \in W$.

To see this recall from prev. theorem

$$-v = -1 \cdot v$$

Since $v \in W$ and W is closed under scalar mult we must have that $-1 \cdot v = -v \in W$ $\square$.

Examples

1 $\Sigma_n \subseteq \mathbb{R}^n$ is a subspace.

2 Let $Mat^{\circ}_{n\times m}(\mathbb{C}) = \{ M \in Mat_{n\times m}(\mathbb{C}) \mid tr(M) = 0 \}$.

$Mat^{\circ}_{n\times m}(\mathbb{C}) \subseteq Mat_{n\times m}(\mathbb{C})$ is a subspace.

3 $\underline{\ell}_c \subseteq \underline{\ell}_{\to 0} \subseteq \underline{\ell}$ are subspaces.

4 $\mathbb{R}[x] \subseteq \mathbb{R}(x) \subseteq \mathcal{C}(\mathbb{R})$ are subspaces.

Theorem If $U, W \subseteq V$ are two subspaces of a vector space then $U \cap W$ is a subspace.

proof: Both U, W are closed under addition and scalar mult. i.e. if $u, w \in U \cap W$ then

$$u + w \in U \quad \text{and} \quad u + w \in W$$

so $u + w \in U \cap W$. Hence $U \cap W$ is closed under addition. Similarly if $\lambda \in \mathbb{F}$ and $v \in$ ~~[illegible]~~ $U \cap W$ then $\lambda \cdot v \in W$ and $\lambda \cdot v \in U$. so $\lambda \cdot v \in U \cap W$.

We also know that $0 \in U$ and $0 \in W$ so $0 \in U \cap W$. Thus $U \cap W$ is a subspace $\Box$.

Warning! If U and $W \subseteq V$ are subspaces, it is (almost never) true that $U \cup W$ is a subspace. Try and come up with a counter example.