

This weeks problem set focuses on the ideas of bases and linear transformations. A question marked with a  $\dagger$  is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a  $*$  is especially important.

**Homework 2:** due Monday 4 Feb: question 18 and 19 from section 2.1. Note that the notation textbook uses  $N(T)$  instead of  $\ker T$  and  $R(T)$  instead of  $\operatorname{im} T$ . The questions are typed out below.

1. From section 2.1, problems 15, 17, 18, 19, 24, 26\*, 28, 31 $\dagger$ , 40\*.
2. From section 2.1, problems 1, 2, 5, 6, 9\*, 14, 14b.
- 3\* Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$  and  $B\{v_1, \dots, v_n\}$  a basis. Let  $W$  be another vector space and  $w_1, \dots, w_n$  a collection of elements. Show that there is a unique linear map such that  $T(v_i) = w_i$ .
- 4\* Let  $V$  and  $W$  be vector spaces over  $\mathbb{F}$ . Define  $\operatorname{Hom}(V, W)$  to be the set of linear maps from  $V$  to  $W$ .
  - (a) Show that  $\operatorname{Hom}(V, W)$  is itself a vector space.
  - (b) If  $V$  is finite dimensional and  $B$  is a basis for  $V$ , construct a basis for  $V^* = \operatorname{Hom}(V, \mathbb{F})$ . The vector space  $V^*$  is called the *dual space* to  $V$ .
- 5\* Let  $T : V \longrightarrow W$  be an injective linear map. Show that, if we consider  $T$ , instead, as a linear map  $V \longrightarrow \operatorname{im} T$  (just restrict what we consider to be the codomain), then it defines an isomorphism and shows that  $V \cong \operatorname{im} T$ .
- 6\* Let  $V$  and  $W$  be vector spaces over  $\mathbb{F}$ . Define the set

$$V \times W = \{ (v, w) \mid v \in V \text{ and } w \in W \}.$$

This is called the *product* of the vector spaces.

- (a) Show that  $V \times W$  is a vector space.
  - (b) Define a map  $\iota_V : V \rightarrow V \times W$  by  $\iota_V(v) = (v, 0)$ . Show that  $\iota_V$  is an injective linear map. Note that we can define a similar map  $\iota_W$ .
  - (c) If  $U \subset V$  is a subspace, show that  $U \times W$  is a subspace of  $V \times W$ .
  - (d) Show that  $V \times W = (V \times \{0\}) \oplus (\{0\} \times W)$ .

Note that we can consider  $V \times \{0\}$  as a copy of  $V$  in  $V \times W$ . For this reason, often mathematicians write  $V \oplus W$  instead of  $V \times W$  and call it the external direct product. Though this is a little confusing so we won't talk about it in this way in this class.
- 7\* (2.1.18) Give an example of a linear transformation  $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$  such that  $\ker T = \operatorname{im} T$ .
  - 8\* (2.1.19) Give an example of distinct linear transformations  $T$  and  $U$  such that  $\ker T = \ker U$  and  $\operatorname{im} T = \operatorname{im} U$ .