Math 3B: Lecture 9

Noah White

February 1, 2017

Introduction

Last time

- Accumulated change

This time

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- Accumulated change
- Area under a curve

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- Accumulated change
- Area under a curve
- The definite integral

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- The fundamental theorem of calculus

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- The FTC
- Substitution

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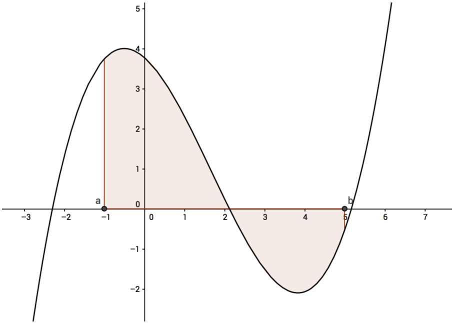
Grade	Range
A	26+
B	21-25
C	14-20

Defintion

The definite integral of a function $f(x)$ is defined to be

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty}\sum_{k=1}^{n} f(a+k\Delta x)\Delta x$$



Properties of definite integrals

Zero area

$$\int_a^a f(x)\,\mathrm{d}x = 0$$

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Adding areas

$$\int_a^c f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_b^c f(x)\,\mathrm{d}x$$

More properties of definite integrals

Additivity

$$\int_a^b f(x) + g(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x + \int_a^b g(x)\,\mathrm{d}x$$

More properties of definite integrals

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Lineararity (scalars factor out)

$$\int_a^b \alpha f(x)\,\mathrm{d}x = \alpha \int_a^b f(x)\,\mathrm{d}x$$

The fundamental theorem of calculus

Theorem

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_a^x f(t)\, \mathrm{d}t = f(x)$$

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That is, $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is an antiderivative of $f(x)$!

Note

The fundamental theorem of calculus

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- $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is a function of x.

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Note

- $F(x) = \int_a^x f(t)\,\mathrm{d}t$ is a function of x.
- every input x produces a number as an output.

Why is the FTC true?

We want to show that $\frac{\mathrm{d}}{\mathrm{d}x}F(x) = f(x)$.

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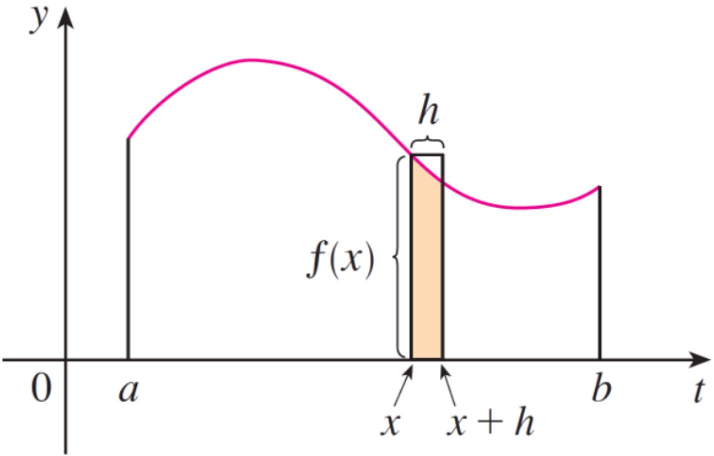
When h is very small $\displaystyle\int_x^{x+h} f(t) \approx$



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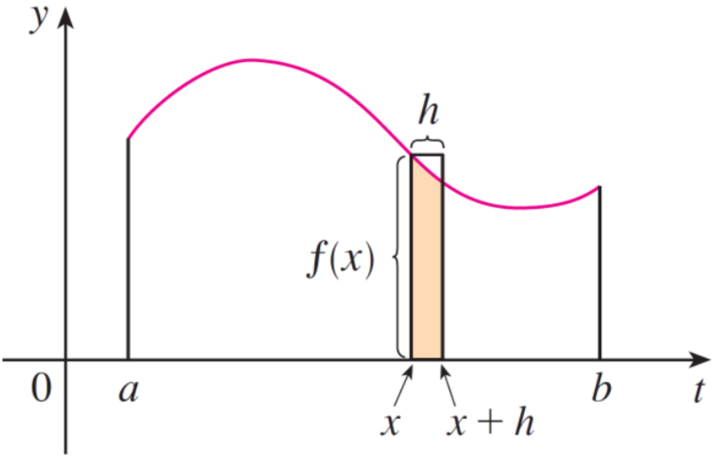

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&= f(x)
\end{aligned}$$

A consequence (corrollary)

Corollary

For any antiderivative $F(x)$ of $f(x)$

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

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Why?

Well $F(x) = \int_a^x f(t)\,\mathrm{d}t + C$ for some a and C. So

$$\begin{aligned} F(b) - F(a) &= \int_a^b f(t)\,\mathrm{d}t + C - \int_a^a f(t)\,\mathrm{d}t - C \\ &= \int_a^b f(t)\,\mathrm{d}t \end{aligned}$$

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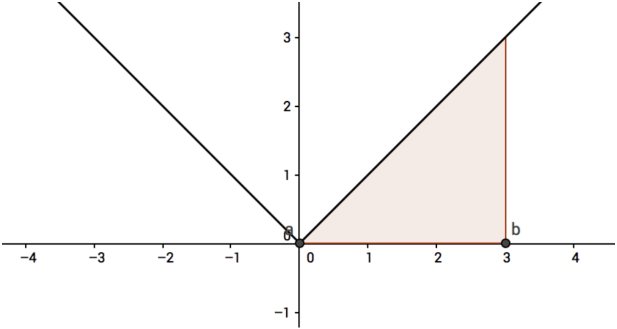
  is an antiderivative for any choice of a.
- Lets use $a = 0$.
- How should we calculate $F(x)$?

Example

Use the defintition!

$$F(x) = \int_0^x |t| \, \mathrm{d}t$$

is the area under $|t|$!



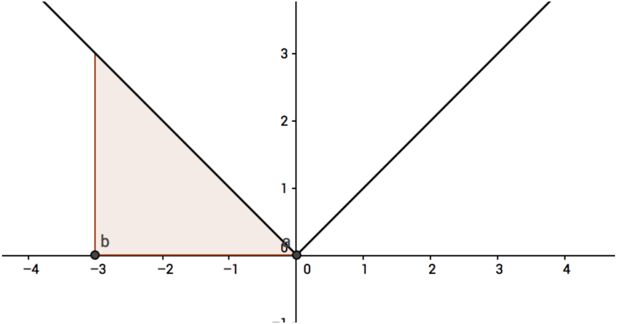
$$F(x) = \frac{1}{2}x^2 \quad \text{if } x \geq 0$$

Example

If $x \leq 0$ then

$$F(x) = \int_0^x |t| \, \mathrm{d}t = -\int_x^0 |t| \, \mathrm{d}t$$

is the negative of the area under $|t|$!



$$F(x) = -\frac{1}{2}x^2 \quad \text{if } x \leq 0$$

Example

In summary

$$F(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 & \text{if } x \leq 0 \end{cases}$$

or

$$F(x) = \frac{1}{2}x\,|x|$$

The indefinite integral

We also use the following notation for the general antiderivative:

Definition

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$$\int \sin(x) - x\,\mathrm{d}x = -\cos(x) - \frac{1}{2}x^2 + C$$

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Suppose $u = g(x)$, then

$$\int f(g(x))g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

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We use the substitution $u = x^2 + 1$, so $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$, we can write the integral

$$\int 2 \cdot \sqrt{x^2+1} \cdot 2x\,\mathrm{d}x = 2\int \sqrt{u}\,\mathrm{d}u$$

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$$\begin{aligned}\int 2\cdot\sqrt{x^2+1}\cdot 2x\,\mathrm{d}x &= 2\int\sqrt{u}\,\mathrm{d}u\\ &= 2\cdot\frac{2}{3}u^{\frac{3}{2}} + C\end{aligned}$$

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&= 2 \cdot \frac{2}{3} u^{\frac{3}{2}} + C \\
&= \frac{4}{3}(x^2+1)^{\frac{3}{2}} + C
\end{aligned}$$

Integration by substitution (definite integrals)

Substitution for definite integrals

Suppose $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}u$$

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$$\int_0^1 4x\sqrt{x^2+1}\,\mathrm{d}x = 2\int_1^2 \sqrt{u}\,\mathrm{d}u$$

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