This weeks problem set focuses isomorphisms and coordinate vectors and the matrices associated to linear transformations. It will be quite a large problem set, and because of the way we will be covering it in class, don't worry if you can't do some of the problems until after next Friday. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

1. From section 2.4, problems 1, $2a, c, e$, 3, 7, 14, 15^*, 17^*, $24^{*,\dagger}$.
2. From section 2.2, problems 1, $2a, c, f$, 10, $11^\dagger$, 12^*, $14^\dagger$, 16.
3. From section 2.3, problems 1, $2a$, 3, 12, 16, $17^\dagger$, 16.

4.* Let V be a finite dimensional vector space over $\mathbb{F}$ and $B\{v_1, \dots, v_n\}$ a basis. Let W be another vector space and $w_1, \dots, w_n$ a collection of elements. Show that there is a unique linear map such that $T(v_i) = w_i$.

Solution: Since B is a basis, if we have any vector $v \in V$ we can write

$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n.$$

Define $T(v) = \lambda_1 T(v_1) + \ldots + \lambda_n T(v_n) = \lambda_1 w_1 + \cdots + \lambda_n w_n$. We can check that it is linear by supposing that $u, v \in V$ and that

$$u = \mu_1 v_1 + \ldots + \mu_n v_n$$
$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n.$$

Then we have $u + v = v = (\mu_1 + \lambda_1)v_1 + \ldots + (\mu_n + \lambda_n)v_n$, so

$$\begin{aligned} T(u+v) &= (\mu_1 + \lambda_1)w_1 + \ldots + (\mu_n + \lambda_n)w_n, \text{ but} \\ T(u) + T(v) &= \mu_1 w_1 + \ldots + \mu_n w_n + \lambda_1 w_1 + \cdots + \lambda_n w_n. \end{aligned}$$

Thus $T(u + v) = T(u) + T(v)$. Now lets suppose $\mu \in \mathbb{F}$, then $\mu v = \mu\lambda_1 v_1 + \cdots + \mu\lambda_n v_n$, so

$$\begin{aligned} T(\mu v) &= \mu\lambda_1 w_1 + \cdots + \mu\lambda_n w_n, \text{ but} \\ \mu T(v) &= \mu(\lambda_1 w_1 + \cdots + \lambda_n w_n). \end{aligned}$$

Thus $T(\mu v) = \mu T(v)$ and so T is linear.

To see that T is unique, suppose that $SLV \longrightarrow W$ is another linear map such that $S(v_i) = w_i$. Now let $v \in V$ such that $v = \lambda_1 v_1 + \cdots + \lambda_n v_n$. Then by linearity

$$\begin{aligned} S(v) &= S(\lambda_1 v_1 + \cdots + \lambda_n v_n) \\ &= \lambda_1 S(v_1) + \cdots + \lambda_n S(v_n) \\ &= \lambda_1 w_1 + \cdots + \lambda_n w_n = T(v). \end{aligned}$$

Thus $S = T$.

There are mathematical objects called $\mathfrak{sl}_2$-*representations* which are important in quantum mechanics and beautiful objects in their own right. We won't define what they are exactly**, but their are vector spaces that come packaged with a certain pair of linear maps. The next questions give an example.

5.† Let $V = \mathbb{C}[x, y]$ be the vector space of polynomials in two variables. So we have $x^2 - 2xy^2 + 1 \in V$ for example. Define two linear maps $E, F : V \longrightarrow V$ where

$$E(p) = x\frac{\partial p}{\partial y} \text{ and } F(p) = y\frac{\partial p}{\partial x}$$

(a) Find a formula for $H := EF - FE$.

Solution: We just calculate what H does to a polynomial, using the chain rule:

$$\begin{aligned} H(p) &= EF(p) - FE(p) \\ &= x\frac{\partial}{\partial y}y\frac{\partial p}{\partial x} - y\frac{\partial}{\partial x}x\frac{\partial p}{\partial y} \\ &= x\frac{\partial p}{\partial x} + xy\frac{\partial^2 p}{\partial y \partial x} - y\frac{\partial p}{\partial y} - xy\frac{\partial^2 p}{\partial y \partial x} \\ &= x\frac{\partial p}{\partial x} - y\frac{\partial p}{\partial y}. \end{aligned}$$

(b) A subspace $U \subset V$ is called a *subrepresentation* if $E(U) \subset U$ and $F(U) \subset U$. Let $V(n) = \operatorname{span}\{\, x^{n-a}y^a \mid 0 \le a \le n \,\}$, this is the space of *homogeneous polynomials of degree* n, i.e. every term on the polynomial has degree n. Show that $V(n)$ is a subrepresentation, for any $n \ge 0$.

Solution: Note that $E(x^{n-a}y^a) = ax^{n-a+1}y^{a-1} \in V(n)$ and $F(x^{n-a}y^a) = (n-a)x^{n-a-1}y^{a+1} \in V(n)$. Thus, since an arbitrary element $p \in V(n)$ is simply a linear combination of these, we have that $E(p), F(p) \in V(n)$ and hence it is a subrepresentation.

(c) With the basis $x^n, x^{n-1}y, x^{n-2}y^2, \ldots, y^n$, determine the matrix corresponding to the linear maps E, H, F restricted to the subspaces $V(n)$.

Solution: We will do H first. Note that $H(x^{n-a}y^a) = (n-2a)x^{n-a}y^a$. Hence the matrix for H is diagonal with the (i,i)-entry being $n - 2(i-1)$.

Now $E(x^{n-a}y^a) = ax^{n-a+1}y^{a-1}$ and so the matrix for E is zero everywhere, apart from the $(i, i+1)$-entry which is i.

Similarly, $F(x^{n-a}y^a) = (n-a)x^{n-a-1}y^{a+1}$ and so the matrix for F is zero everywhere, apart from the $(i+1, i)$-entry which is $n - i + 1$.

Examples for $n = 3$ are

$$H = \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ and } F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

6.† Another example of an $\mathfrak{sl}_2$ representation is given by $W = \mathbb{C}^2$ and where E' and F' are the linear transformations given by left multiplication by the matrices

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Find an isomorphism $\theta : V(1) \longrightarrow W$ so that $\theta E = E'\theta$ and $\theta F = F'\theta$ as linear maps $V(1) \longrightarrow W$.

Solution: The isomorphism θ will be determined by the values of $\theta(x)$ and $\theta(y)$. So lets set

$$\theta(x) = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \theta(x) = \begin{pmatrix} c \\ d \end{pmatrix}.$$

In order for $\theta E = E'\theta$ and $\theta F = F'\theta$, we must have that four things hold:

$$\theta E(x) = E'\theta(x) \tag{1}$$
$$\theta E(y) = E'\theta(y) \tag{2}$$
$$\theta F(x) = F'\theta(x) \tag{3}$$
$$\theta F(y) = F'\theta(y). \tag{4}$$

Notice that $E(x) = 0$, thus $\theta E(x) = 0$ and by 1 we must have that

$$0 = E'\theta(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

Thus $b = 0$. We do something similar and notice that $F(y) = 0$ so by we have

$$0 = F'\theta(y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 0 \\ c \end{pmatrix}$$

so $c = 0$. Now we only need to check 2 and 3. Notice that $E(y) = x$ so $\theta E(y) = \theta(x)$. By 2,

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} d \\ c \end{pmatrix}$$

Which gives us that $a = d$ and $b = c$. Checking 3 gives us the same result. So now we have that

$$\theta(x) = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } \theta(x) = \begin{pmatrix} c \\ d \end{pmatrix}.$$

We just need to pick an a which ensures this will be an isomorphism. It is clear that picking any $a \neq 0$ is fine.

7.† Show that there is no, nonzero, linear map $\theta : V(n) \longrightarrow V(m)$ so that $E\theta = \theta E$ and $F\theta = \theta F$ whenever $n \neq m$. *Hint: if such a map does exist, where does x^n get sent? Now use that $H\theta = \theta H$. This is pretty hard, let me know if you need more hints*

Solution: We will give a brief sketch. Consider $\theta(x^n)$. We know that $E(x^n) = 0$ so $0 = \theta(E(x^n)) = E\theta(x^n)$. I.e. we must have that $E\theta(x^n) = 0$. If

$$\theta(x^n) = \lambda_0 x^m + \lambda_1 x^{m-1} y + \ldots + \lambda_m y^m$$

then

$$E\theta(x^n) = 0 + \lambda_1 x^m + \ldots + m\lambda_m x y^{m-1}$$

The only way for this to be zero is if $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 0$. Thus $\theta(x^n) = \lambda x^m$ for some $\lambda \in \mathbb{C}$. Now lets use the fact that $H\theta = \theta H$ (this is true since $(EF - FE)H = H(EF - FE)$).

Observe that $H(x^n) = nx^n$, thus $\theta H(x^n) = \lambda n x^m$. On the other hand $H\theta(x^n) = H(\lambda x^m) = \lambda m x^m$. Hence $\lambda n x^m = \lambda m x^n$. The only way this is possible, is if either $m = n$, or if $\lambda = 0$.

If $m \neq n$ then $\lambda = 0$, so $\theta(x^n) = 0$. Now comes the somewhat challending part. How can we figure out that this means that $\theta(x^{n-a}y^a) = 0$? We consider $F^a(x^n)$. This is the result of applying F to the element x^n, a times. The result is $F^a(x^n) = \frac{n!}{(n-a)!}x^{n-a}y^a$. But we know that $\theta F^a = F^a\theta$ so we must have that $\theta(F^a(x^n)) = F^a(\theta(x^n)) = F^a(0) = 0$. But since θ is linear

$$\theta(x^{n-a}y^a) = \frac{(n-a)!}{n!}F^a(\theta(x^n)) = 0.$$

** Ok, if you really want to know exactly what they are here is the definition: An $\mathfrak{sl}_2$-representation is a vector space V with two linear maps $E, F : V \longrightarrow V$ such that

$$E^2F - 2EFE + FE^2 = -2E$$

and the same equation with the E's and F's swapped. There is a much more intuitive definition but one would need to know some more abstract algebra. If you are really keen, try and find more $\mathfrak{sl}_2$ representations and show me!

The results in the following three exercises are called the first, second, and third isomorphism theorems respectively. The exercises are difficult (but only really because they involve unwrapping lots of definitions) but the interested student should attempt them. They turn up over and over again in algebra and there are analogues for different algebraic objects like *groups*, *rings* and *modules*. The area of mathematics that explains why the same types of theorems appear over and over again is called *category theory*.

8.† Suppose $T : V \longrightarrow W$ is a linear map between two vector spaces over $\mathbb{F}$. Prove that

$$V/\ker T \cong \operatorname{im} T.$$

The space on the left is the *quotient space* which you met in the first problem set.

9.† Let V be a vector space over $\mathbb{F}$ and let U, W be subspaces of V. Prove that

$$(U + W)/U \cong W/(U \cap W).$$

10.† Let V be a vector space over $\mathbb{F}$ and let $U \subseteq W$ be subspaces of V. Prove that W/U is a subspace of V/U. Furthermore prove that

$$(V/U)/(W/U) \cong V/W.$$