This weeks problem set provides practice with diagonalisable operators and the basic properties of inner products. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 4: is due Friday June 5: questions $2, 4, 5$ below.

1. From section 6.2, problems 1, 2b, g, i, k, 5*, 6, 7, 9, 13*, 17*, 22.

2. Let V be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
   (a) Fix $y \in V$ and suppose $\langle x, y\rangle = 0$ for all $x \in V$. Show that $y = 0$.
   (b) Let $T : V \longrightarrow V$ be a linear map such that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all pairs $x, y \in V$ (we call such a map a *metric* map). Prove that T is an isomorphism.
   (c) † Find all metric maps $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that have $\det T = 1$.

3. (22 from 6.2) Let $V = \mathcal{C}([0, 1], \mathbb{R})$ be the space of real valued, continuous functions on the interval $[0, 1]$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\ dt$. Let W be the subspace spanned by the linearly independent set $\{t, \sqrt{t}\}$.
   (a) Find an orthonormal basis for W.
   (b) Let $h(t) = t^2$. Use the orthonormal basis obtained in (a) to obtain the "best" (closest) approximation of h in W.

4. Let V be a real inner product space and let $r : V \longrightarrow V^*$ be the map $r(x) = \varphi_x := \langle -, x\rangle$. In class we showed that if V is finite dimensional then r is an isomorphism.
   (a) Assume that V is infinite dimensional. Prove that r is injective.
   (b) Let $V = \mathbb{R}[x]$ and let $W = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{R}\ \}$ be the vector space of all infinite sequences. Show that the map $f : V^* \longrightarrow W$ given by $f(\varphi) = (\varphi(x^n))_{n\geq 0}$ is an isomorphism.
   (c) We can define the following inner product on $\mathbb{R}[x]$

$$\langle x^i, x^j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

   and extending linearly (so $\langle 1 + x, 2 - x^2\rangle = 2$ for example). Use this to demonstrate that r is not necessarily surjective, i.e. find an element $\varphi \in V^*$ such that $\varphi \neq r(p)$ for any $p \in \mathbb{R}[x]$.

5. Let V be a finite dimensional inner product space. For any $T : V \longrightarrow V$ define $\check{T} : V^* \longrightarrow V^*$ by $\check{T}(\phi) = \phi \circ T$. Furthermore for any $X : V^* \longrightarrow V^*$ define $X^\perp : V \longrightarrow V$ by $X^\perp = r^{-1} \circ X \circ r$. Prove that $T^* = \check{T}^\perp$.