Lecture 24

Boundaries of 3D regions

- If $E \subseteq \mathbb{R}^3$ is a region in three dimensional space, its boundary, ∂E, is a closed surface.
- Recall a surface ~~S~~ is closed if $\partial S = \phi$. so $\partial\partial E = 0$.
- We give the surface ∂E an ~~ori~~ orientation called the boundary orientation by choosing the normal vector pointing away from E.
- Points on the boundary are "limiting points". i.e. if we are inside the region E and we start moving, and we reach a point where we would exit the region if we kept going, that means we are on the boundary.

Ex The ball $\{x^2+y^2+z^2 \leq 1\}$ has boundary the sphere $\{x^2+y^2+z^2=1\}$ with normal pointing away from the origin.

Thm (Divergence theorem)

Let $\mathcal{E}$ be a three dimensional region in $\mathbb{R}^3$ ~~and~~ with smooth boundary $\partial\mathcal{E}$. Then

$$\iint_{\partial\mathcal{E}} \underline{F}\cdot d\underline{S} = \iiint_{\mathcal{E}} \operatorname{div}(\underline{F})\,dV$$

Rmk - This thm is useful in both directions

- If $\operatorname{div}(\underline{F}) = 0$, then

$$\iint_{\partial\mathcal{E}} \underline{F}\cdot d\underline{S} = \iiint_{\mathcal{E}} 0\,dV = 0.$$

- Be careful though! Let $\underline{F} = \frac{1}{r^3}\langle x, y, z\rangle$ where $r = x^2 + y^2 + z^2$. Then $\operatorname{div}(\underline{F}) = 0$ but if S = sphere of radius 1 then

$$\iint_S \underline{F}\cdot d\underline{S} = \iint_S \frac{1}{r^3}\langle x, y, z\rangle \cdot d\underline{S}$$

$$= \int_0^{\pi}\int_0^{2\pi} \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle \cdot \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle \sin\phi\, d\theta\, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \sin\phi\, d\theta\, d\phi$$

$$= 2\pi\Big[-\cos\phi\Big]_0^{\pi} = 4\pi \neq 0$$

So what went wrong? Well $\underline{F}$ is not defined at all points of $\mathcal{E}$ (i.e. at $(0\,0\,0)$.)

Summary

- We have learnt about various types of integrals

$$\int_a^b f(x)\,dx \,, \quad \int_C \underline{F}\,d\underline{r} \,, \quad \iint_S \underline{F}\cdot dr \,, \quad \iiint_E f\,dV$$

- Four main theorems of the course
  * FTC (for conservative vector fields)
  * Greens theorem
  * Stokes' theorem
  * Divergence theorem
- These theorem summarise the interation of integrals over regions + their boundaries using the various "derivatives" we have learnt:

1D

functions

$\Big\downarrow \dfrac{d}{dx}$

functions

FTC

If $F = \dfrac{df}{dx}$ then

$$\int_a^b f\,dx = F(b) - F(a)$$

2D

functions

$\Big\downarrow \nabla$

vector fields

$\Big\downarrow$ curl

functions

FTC

If $\underline{F} = \nabla f$ then

$$\int_C \underline{F}\cdot d\underline{r} = f(Q) - f(P)$$

Green's

If $f = \text{curl}(\underline{F})$ then

$$\iint_D f\,dA = \int_{\partial D} \underline{F}\cdot d\underline{r}$$

3D

$$\begin{tikzcd}
\text{functions} \arrow[d, "\nabla"] \\
\text{vector fields} \arrow[d, "\text{curl}"] \\
\text{vector fields} \arrow[d, "\text{div}"] \\
\text{functions}
\end{tikzcd}$$

FTC

If $\underline{F} = \nabla \varphi$ then

$$\int_{C} \underline{F} \cdot d\underline{r} = \varphi(Q) - \varphi(P)$$

Stokes'

If $\underline{F} = \text{curl}(\underline{A})$

$$\iint_S \underline{F} \cdot d\underline{S} = \int_{\partial S} \underline{A} \cdot dr$$

Divergence.

If $f = \text{div}(\underline{A})$

$$\iiint_E f \, dV = \iint_{\partial E} \underline{A} \cdot d\underline{S}$$

4D

$$\begin{tikzcd}
\text{functions} \arrow[d, "?"] \\
? \text{ (vector fields)} \arrow[d, "?"] \\
? \arrow[d, "?"] \\
? \text{ (vector fields)} \arrow[d, "?"] \\
\text{functions}
\end{tikzcd}$$