

This weeks problem set provides some review questions in the lead up to the second midterm. A question marked with a  $\dagger$  is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a  $*$  is especially important.

**Homework 4:** due Friday 24 May: questions 3 and 5 below.

1. From section 5.2, problems 1, 3a, d, e, 8, 9, 10, 11, 18\*, 19, 20 $\dagger$ .
2. From section 6.1, problems 1, 2, 3, 4, 8\*, 9, 12, 16, 17\*, 23, 29.
3. Let  $T : V \longrightarrow V$  be a diagonalisable linear operator. Let  $C(T) \subseteq \text{Hom}(V, V)$  be the set of all linear maps that commute with  $T$ . I.e

$$C(T) = \{S \in \text{Hom}(V, V) \mid S \circ T = T \circ S\}.$$

- (a) If  $T$  has  $n = \dim V$  distinct eigenvalues, show that any  $S \in C(T)$  is diagonalisable.
  - (b) Describe explicitly  $C(T)$  in the case  $T = x \frac{d}{dx} : \mathbb{C}_1[x] \longrightarrow \mathbb{C}_1[x]$ .
  - (c) Show that part (a) does not necessarily hold if  $T$  does not have  $n$  distinct eigenvalues.
4. Suppose  $U, W$  are subspaces of a finite dimensional vector space  $V$  and that  $U + W = V$ . Show that  $U \oplus W = V$  if and only if  $\dim U + \dim W = \dim V$ .

The previous question, motivates the following definition.

**Definition:** If  $U_i$ , for  $1 \leq i \leq k$ , are subspaces of a vector space  $V$ , then we say  $V = U_1 \oplus U_2 \dots \oplus U_k$  if  $V = U_1 + U_2 + \dots + U_k$ , i.e. every vector  $v \in V$  can be written as a sum  $v = \sum_{i=1}^k u_i$  with  $u \in U_i$ , and  $\dim V = \sum_{i=1}^k \dim U_i$ .

- 5\* Suppose that  $V$  is a finite dimensional vector space over  $\mathbb{F}$  and  $T : V \longrightarrow V$  is a linear operator, with distinct eigenvalues  $\lambda_1, \dots, \lambda_k$ . Prove that

$$V = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \dots \oplus E_{\lambda_k}$$

if and only if  $T$  is diagonalisable.