This week on the problem set you will get practice manipulating random variables and their probability mass functions. Especially challenging questions, or questions that are not appropriate for an exam, are indicated with one or more asterisks.

Homework: The first homework will be due on Friday 3 November, at 12pm, the *start* of the lecture. It will consist of questions:

$$3 \text{ and } 4.$$

1. From the textbook, chapter 2, problems 13, 16, 21, 22, 23.

2. From the supplementary problems, chapter 2, problem 3, 5.

3. Recall Problem 5 from Problem Set 2: A burglar found your keychain with n keys and he is trying to find one of the two keys which open your house door (whenever he tries a wrong key he discards it). Find the expectation and the variance of the number of tries he has to do to find a key which opens the door. Simplify your answers, they should be closed expressions, not summations. *Hint: you will need to know some formulas for the sum of the first n integers, etc, you can look these up and simply state them.*

4. Every hour, a clock tower rings either once or twice. Once with probability $1/3$ and twice with probability $2/3$. If you're listening the clock for n hours (you hear the bell ringing on n occasions) let X be the number of times you heard the bell ring. Find the expectation and the variance of X.

5. Recall Problem 8 from Problem Set 4: we consider the PMF given by the expression

$$p(k) = \frac{\binom{a}{k}\binom{b}{n-k}}{\binom{a+b}{n}},$$

for $k = 0, 1, 2, \ldots, n$. Find the expectation of a random variable with this PMF. Simplify your answer. Hint: take a look at the derivation of the expectation of Binomial random variable using binomial probabilities (not indicator functions).

6. 1. Show that for any positive integer n we have

$$\frac{1}{n(n+1)(n+2)} = \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

2. Use part 1. to find a formula for the sum

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)}.$$

3. Let X be a random variable with the range $\{1, 2, 3, \ldots\}$ and the probability mass function $p_X(n) = \frac{C}{n(n+1)(n+2)}$, where C is some constant. Find the value of C for which this is a well defined probability mass function.

4. Show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ and using this compute $\mathbb{E}(X)$.

5. Show that the variance of X does not exist.

7. There are n people coming to your birthday party. The probability that the i-th guest brings you a birthday present is p_i independently of all other guests. What is the expected number of presents you get? (Hint: use indicator functions)

8. You are moving a token from the lower left to the upper right corner of an 8×8 chessboard (you can move one step to the left or one step up).

   (a) How many such paths are there?

(b) Assume that any such trajectory of the token is equally likely. Denote by X the step in which the token reaches the right edge of the chessboard (so it is no longer able to move to the right). For example, if the token is moved straight to the right as far as it can go (and then up to the final position), then it will reach the right edge in 7 steps, so in this case $X = 7$. Find the range of the random variable X and it's probability mass function. Find the expectation and the variance of X.