Final exam (practice)

UCLA: Math 31B, Spring 2017

Instructor: Noah White
Date:

- This exam has 8 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section: ______________________________

Question	Points	Score
1	12	
2	12	
3	8	
4	11	
5	9	
6	10	
7	8	
8	10	
Total:	80	

Questions 1 and 2 are multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following four pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

Question 2.

Part	Converges	Diverges
(a)		
(b)		
(c)		
(d)		
(e)		
(f)		

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) Find A, B and C so that

   $$\frac{4}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

   A. $A = 1, B = 1, C = 2$
   B. $A = 1, B = 1, C = 1$
   C. $A = 1, B = -1, C = 2$
   D. $A = 3, B = 0, C = 1$

   (b) (2 points) Calculate the integral $\int_1^e \ln x \, dx$.
   A. $e - 1$
   B. e
   C. 0
   D. 1

   (c) (2 points) Find an alternative expression for $\cos\left(\tan^{-1} x\right)$
   A. $\frac{1}{\sqrt{1+x^2}}$
   B. $\frac{x}{\sqrt{1-x^2}}$
   C. $\frac{1}{\sqrt{1-x^2}}$
   D. $\frac{\sqrt{1+x^2}}{x}$

(d) (2 points) The third Taylor polynomial of $\ln(x+1)$ at 0 is

A. $x - x^2 + 2x^3$
B. $1 + x + x^2 + x^3$
C. $1 - x + 2x^2$
D. $x - \frac{1}{2}x^2 + \frac{1}{3}x^3$

(e) (2 points) Find the limit $\lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$.

A. e^{-1}
B. 1
C. 0
D. e^2

(f) (2 points) Calculate the improper integral $\int_1^\infty e^{-3x}\ dx$.

A. $(3e^3)^{-1}$**.**
B. e^{-1}.
C. e.
D. $3e$.

2. In each of the following questions, analyse the integral, sequence or series below and determine whether it converges or diverges. No partial points will be given.

   (a) (2 points)

$$\int_1^\infty \frac{1}{x^2\sqrt{ln(x)}}\,dx$$

.

$\surd$ **Converges**

◯ Diverges

   (b) (2 points) The sequence (a_n) where

$$a_n = \frac{n\cos(\pi n)}{n+1}.$$

◯ Converges

$\surd$ **Diverges**

   (c) (2 points) The sequence (a_n) where

$$a_n = (-1)^n \frac{1}{1+\ln n}.$$

$\surd$ **Converges**

◯ Diverges

(d) (2 points) The series

$$\sum_{n=1}^{\infty} \frac{n\sqrt{n}}{n^4 + \ln n}$$

√ **Converges**
◯ Diverges

(e) (2 points) The series

$$\sum_{n=3}^{\infty} \frac{e^{4n}}{(n-2)!}$$

√ **Converges**
◯ Diverges

(f) (2 points) The series $\sum_{n=1}^{\infty} a_n$ where the partial sums are given by

$$S_N = \frac{\ln n + 1}{\ln n}$$

√ **Converges**
◯ Diverges

3. (8 points) Calculate the following antiderivative.

$$\int \frac{4x(x^2-2x+4)}{x^4-16}\,dx$$

Solution: If

$$\frac{4x(x^2-2x+4)}{x^4-16} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{Cx+D}{x^2+4},$$

then

$$4x(x^2-2x+4) = A(x+2)(x^2+4) + B(x-2)(x^2+4) + (Cx+D)(x-2)(x+2).$$

Let $x = 2$ to see $8 \cdot 4 = 4 \cdot 8 \cdot A$ and $A = 1$.

Let $x = -2$ to see $-8 \cdot 12 = -4 \cdot 8 \cdot B$ and $B = 3$.

Let $x = 0$ to see $0 = 8A - 8B - 4D$ so $D = 2(A - B) = -4$.

From the hint, $C = 0$. So

$$\begin{aligned}\int \frac{4x(x^2-2x+4)}{x^4-16}\,dx &= \int \frac{1}{x-2} + \frac{3}{x+2} - \frac{4}{x^2+4}\,dx\\ &= \ln|x-2| + 3\ln|x+2| - 2\arctan\left(\frac{x}{2}\right) + c.\end{aligned}$$

4. (a) (2 points) Is there a function $f(x)$ such that the 4-th Taylor polynomial centered at 0 is given by

$$T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}?$$

If so, give an example. No justification is required.

(b) (2 points) Is there a function $f(x)$ such that the 4-th Taylor polynomial centered at 0 is given by

$$T_4(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!}?$$

If so, give an example. No justification is required.

(c) (2 points) Is there a function $f(x)$ such that the 3-rd Taylor polynomial centered at 0 is given by

$$T_3(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}?$$

If so, give an example. No justification is required.

(d) (5 points) Let $f(x) = \cos x + \sin x$ and $T_n(x)$ be the n-th Taylor polynomial of $f(x)$ centered at 0. Find an n such that

$$|T_n(2) - f(2)| \leq \frac{1}{10^{123}}.$$

Solution:

(a) Yes, $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$.

(b) Yes, $f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!}$.

(c) No, since $T_3(x)$ cannot have degree 4.

(d) Note that $|f^{(n)}(x)| = |\cos x \pm \sin x| \leq 2$ for all x and so

$$|T_n(2) - f(2)| \leq \frac{2 \cdot 2^{n+1}}{(n+1)!}.$$

We would like to pick an n so that this fraction is less than 10^{-123}. Notice that

$$2\frac{2^{n+1}}{(n+1)!} = 2\frac{2 \cdot 2 \cdot \ldots \cdot 2}{1 \cdot 2 \cdot \ldots \cdot (n+1)} \leq \frac{2^{n-18}}{20 \cdot 21 \cdot \ldots \cdot (n+1)}$$

Since $2^{19} < 19!$. And this is less than

$$\frac{2^{n-18}}{20^{n-18}} = \frac{1}{10^{n-18}}.$$

In summary we have found that

$$|T_n(2) - f(2)| \leq \frac{2 \cdot 2^{n+1}}{(n+1)!} \leq \frac{1}{10^{n-18}}.$$

Thus we just need to make sure that $n - 18 \geq 123$. Taking $n = 141$ will do.

5. (a) (5 points) Use u-substitution(s) to calculate the following antiderivative.

$$\int \frac{2\ln(-\ln x)}{x \ln x} \; dx$$

(b) (4 points) Verify whether the following improper integral converges or diverges. If it converges, calculate what it converges to.

$$\int_{\frac{1}{e}}^{1} \frac{2\ln(-\ln x)}{x \ln x} \; dx$$

Solution:

(a) Let $u = \ln x$. Then $du = \frac{1}{x}\; dx$, so

$$\int \frac{2\ln(-\ln x)}{x \ln x} \; dx = \int \frac{2\ln(-u)}{u} \; du.$$

Let $w = \ln(-u)$. Then $dw = \frac{1}{u}\; du$ so

$$\int \frac{2\ln(-u)}{u} \; du = \int 2w \; dw = w^2 + c$$

and

$$\int \frac{2\ln(-\ln x)}{x \ln x} \; dx = \ln(-\ln x)^2 + c.$$

(b)

$$\begin{aligned}\int_{\frac{1}{e}}^{1} \frac{2\ln(-\ln x)}{x \ln x} \; dx &= \lim_{S \to 1-} \int_{\frac{1}{e}}^{S} \frac{2\ln(-\ln x)}{x \ln x} \; dx \\ &= \lim_{S \to 1-} \Big[\ln(-\ln x)^2 \Big]_{\frac{1}{e}}^{S} = \lim_{S \to 1-} \Big[\ln(-\ln S)^2 \Big] = \infty\end{aligned}$$

The integral diverges.

6. (a) (4 points) Calculate the 4^{th} Taylor polynomial centered at $x = 0$ for $\ln(3x + 4)$.
   (b) (6 points) Calculate the Taylor series centered at 0 for $\ln(3x + 4)$.

Solution:

(a) Notice that

$$f^{(n)}(x) = (-1)^{n-1} \frac{3^n (n-1)!}{(3x+4)^n}.$$

So

$$f^{(n)}(0) = (-1)^{n-1} \left(\frac{3}{4}\right)^n (n-1)!.$$

Thus

$$\frac{f^{(n)}(0)}{n!} = \frac{(-1)^{n-1}}{n} \left(\frac{3}{4}\right)^n$$

Hence

$$T_4(x) = \ln 4 + \frac{3}{4}x - \frac{9}{32}x^2 + \frac{27}{192}x^3 - \frac{81}{1024}x^4.$$

(b) Using the above

$$T(x) = \ln 4 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \left(\frac{3}{4}\right)^n x^n$$

7. (a) (4 points) Using the geometric series, obtain a power series with center $c = 0$ for the function
$$\frac{1}{x+1}.$$
(b) (4 points) Using the properties of derivatives and integrals of power series, find a power series with center $c = 0$ for
$$\ln(1+x^2).$$
Be sure to state the radius of convergence in both cases carefully, and to justify your choice of integration constant if necessary.

Solution:

(a) The geometric series says that
$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \quad \text{as long as } |x| < 1.$$
Subbing in $-x$, we get that
$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \quad \text{as long as } |-x| = |x| < 1.$$

(b) Notice that
$$\frac{d}{dx}\ln(1+x^2) = \frac{2x}{1+x^2}.$$
From part a) we get that
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \quad \text{as long as } \left|x^2\right| < 1.$$
which is the same as saying that $|x| < 1$. Now we simply multiply by $2x$ to get
$$\frac{2x}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1} \quad \text{as long as } |x| < 1.$$
To get the power series for $\ln(1+x^2)$ we use the fact that we can integrate to get
$$\begin{aligned}\ln(1+x^2) &= \int \sum_{n=0}^{\infty} (-1)^n 2x^{2n+1}\, dx \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n 2}{2n+2} x^{2n+2} \\ &= C + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2(n+1)}\end{aligned}$$
For some unknown constant of integration. Using the fact that when $x = 0$, $\ln(1+x^2) = 0$ we see that $C = 0$. Since integrating a power series does not change its interval of convergence, we get that whenever $|x| < 1$,
$$\ln(1+x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{2(n+1)}.$$

8. (a) (3 points) Does the series
$$\sum_{n=0}^{\infty} \frac{1}{4n^2-1}$$
converge? Justify your answer carefully.

(b) (7 points) Evaluate the series above. *Hint: use the definition of the series as a limit of partial sums. Calculate the partial sum by first using partial fractions.*

Solution:

(a) Let $a_n = 1/(4n^2-1)$ and $b_n = n^2$. Then
$$\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \frac{1}{4}$$
and since $\sum b_n$ converges (by the p-test), the limit comparison test shows that the series in the question also converges.

(b) First notice (by partial fractions) that
$$\frac{1}{4n^2-1} = \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)}.$$
Now the definition of the infinite series says that
$$\sum_{n=0}^{\infty} \frac{1}{4n^2-1} = \lim_{N\to\infty} S_N \quad \text{where} \quad S_N = \sum_{n=0}^{N} \frac{1}{4n^2-1}$$
Now
$$\begin{aligned} S_N &= \sum_{n=0}^{N} \frac{1}{2(2n-1)} - \frac{1}{2(2n+1)} \\ &= -\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \frac{1}{6} + \frac{1}{6} - \frac{1}{10} + \ldots + \frac{1}{2(2N-1)} - \frac{1}{2(2N+1)} \\ &= -\frac{1}{2} - \frac{1}{2(2N+1)}. \end{aligned}$$
where we see that we get lots of cancellation between neighbouring terms. Now we can simply take the limit
$$\sum_{n=0}^{\infty} \frac{1}{4n^2-1} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} -\frac{1}{2} - \frac{1}{2(2N+1)} = -\frac{1}{2}.$$

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