This week on the problem set you will get practice thinking about potential functions and calculating line integrals.

Homework: The second homework will be due on Monday 11 May and will consist of 3, 4, 5, and 6 below. *Numbers in parentheses indicate the question has been taken from the textbook:

J. Rogawski, C. Adams, *Calculus, Multivariable*, 3rd Ed., W. H. Freeman & Company,

and refer to the section and question number in the textbook.

1. (Section 17.1) Questions 13 – 17, 22, 26, 28, 29, 38, 42, 44, 47, 52, 56*. (Use the following translations 4th $\mapsto$ 3rd editions: 47 $\mapsto$ 45, 52 $\mapsto$ 50, 56 $\mapsto$ 54, otherwise the questions are the same).

2. (Section 17.2) 3, 10, 12, 13, 21, 24, 28, 43, 44, 46, 47, 54, 55, 57, 63, 64, 67. (Use the following translations 4th $\mapsto$ 3rd editions: 43 $\mapsto$ 41, 44 $\mapsto$ 42, 46 $\mapsto$ 44, 47 $\mapsto$ 45, 54 $\mapsto$ 52, 55 $\mapsto$ 53, 57 $\mapsto$ 55, otherwise the questions are the same).

3. A parameterized curve $\mathbf{r} : [a, b] \to \mathbb{R}^2$ (the codomain could be $\mathbb{R}^3$ as well) is a *flow line* for the vector field $\mathbf{F}$ if for all $t \in (a, b)$ we have that $\mathbf{F}(\mathbf{r}(t)) = \mathbf{r}'(t)$. A flow line for a vector field is the path that a particle would follow if the vector field was a velocity vector field for a fluid.
   (a) Consider the vector field $\mathbf{F}(x, y) = \langle x, y\rangle$. Find a flow line for $\mathbf{F}$ (note that a vector field will have many different flow lines).
   (b) Find a collection of flow lines for $\mathbf{F}$ so that every point (x, y) is contained in exactly one of the flow lines in the collection.
   (c) Consider the vector field $\mathbf{G}(x, y) = \left\langle \dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}} \right\rangle$. Find a collection of flow lines for $\mathbf{G}$ so that every point (x, y) is contained in exactly one of the flow lines in the collection.
   (d) What do the flow lines look like in $\mathbb{R}^2$ for the vector fields $\mathbf{F}$ and $\mathbf{G}$? Relate how the flow lines are similar and different to how the vector fields $\mathbf{F}$ and $\mathbf{G}$ are similar and different.
   (e) A particle is dropped into the plane at the point $(-1, 1)$ at time $t = 0$. If the particle is located at (x, y) in the plane its velocity vector is $(1, 2x)$. What is the position of the particle at time $t = 3$?

4. Consider the vector field $\mathbf{F} = \left\langle \dfrac{1 - y}{x^2 + (y - 1)^2} + \dfrac{-y}{x^2 + y^2}, \dfrac{x}{x^2 + (y - 1)^2} + \dfrac{x}{x^2 + y^2} \right\rangle$
   (a) Show that the curl of $\mathbf{F}$ is zero.
   (b) Show that $\mathbf{F}$ is not conservative on the largest domain on which it is defined.
   (c) Show that $\mathbf{F}$ *is* conservative on the right half plane and find a potential function.

   Hint: For all of these, it will be useful to think of $\mathbf{F}$ as the sum of two more familiar vector fields.

5. A vector field $\mathbf{F} = \langle F_1, F_2\rangle$ is called *holomorphic* if it satisfies the *Cauchy-Riemann* equations:
$$\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y} \text{ and } \frac{\partial F_2}{\partial x} = -\frac{\partial F_1}{\partial y}.$$
   For example, the vector field $\mathbf{G} = \langle x + p, y + q\rangle$ is holomorphic for any fixed $p, q \in \mathbb{R}$, but $\langle x^2, F_2\rangle$ is not holomorphic, no matter what F_2 is! The purpose of this question is to convince you that holomorphic vector fields satisfy some spooky properties. (If you want to know more about this, take Math 132).
   (a) Give an example of a holomophic vector field $\mathbf{H} = \langle H_1, H_2\rangle$ (other than simple multiples or additions of the above examples). This is in general quite difficult, so to get to started, look for a vector field where
$$H_1 = x^2 + ay^2 + y$$
   You will need to work out an appropriate value of a. Include verification that your example is in fact holomorphic.

(b) Let C be the unit circle centred at $(a, b) \in \mathbb{R}^2$ oriented counter clockwise. An amazing fact is that if $\mathbf{F}$ is holomorphic then

$$F_1(a,b) = \frac{1}{2\pi}\oint_C F_1\, ds \quad \text{and} \quad F_2(a,b) = \frac{1}{2\pi}\oint_C F_2\, ds$$

Notice the right hand side only involves values of $\mathbf{F}$ on C, and yet, somehow this knows about the value of $\mathbf{F}$ inside C! Verify that the formulas hold for the example $\mathbf{G}$ and your example $\mathbf{H}$ from part (a).

(c) Maybe you aren't yet convinced that there is something strange going on with holomorphic vector fields. Well it turns out the values of $\mathbf{F}$ on a circle know even a little more, namely

$$\frac{\partial F_1}{\partial x}(a,b) = \frac{1}{2\pi}\oint_C (x-a)F_1 + (y-b)F_2\, ds \quad \text{and} \quad \frac{\partial F_2}{\partial x}(a,b) = \frac{1}{2\pi}\oint_C (x-a)F_2 - (y-b)F_1\, ds$$

Verify these formulas as well. Note the above aren't as symmetrical as the previous formulas. You will most likely need to use some formulas for the antiderivative of things like $\sin^n t$. You can look these up.

(d) What are the analogous formulas for the y-derivatives of F_1 and F_2. In general, we will have formulas for all x and y derivatives but they become very complicated, the language of complex analysis allows us to understand this phenomenon is a clearer way.

6. Let $\mathcal{C}$ be the portion of the curve defined by the intersection of the surfaces $y = x^2$ and $x = y + z$ where $z \geq 0$. Take the orientation to be away from the origin. Let

$$\mathbf{F} = \left\langle yz + \frac{1}{\sqrt{1-x^3}},\, xz - \frac{1}{\sqrt{1-y^3}},\, xy \right\rangle$$

.

(a) What is curl($\mathbf{F}$)?

(b) Parameterise the curve and write out the integral of $\mathbf{F}$ as a single integral. There is no need to evaluate this integral.

(c) Calculate $\displaystyle\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. *Hint: the integral in the previous question is very difficult so you should find another way to compute this.*

*The questions marked with an asterisk are more difficult or are of a form that would not appear on an exam. Nonetheless they are worth thinking about as they often test understanding at a deeper conceptual level.