Midterm 2 practice 2

UCLA: Math 32B, Fall 2019

Instructor: Noah White
Date:

- This exam has 5 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	9	
2	8	
3	7	
4	5	
5	11	
Total:	40	

Here are some formulas that you may find useful as some point in the exam.

$$\int \cos^2 x \, \mathrm{d}x = \frac{1}{2}\left(x + \cos x \sin x\right)$$

$$\int \sin^2 x \, \mathrm{d}x = \frac{1}{2}\left(x - \cos x \sin x\right)$$

$$\int \sin x \cos x \, \mathrm{d}x = \frac{1}{2}\sin^2 x$$

Spherical coordinates are given by

$$\begin{aligned} x(\rho, \theta, \phi) &= \rho \cos\theta \sin\phi \\ y(\rho, \theta, \phi) &= \rho \sin\theta \sin\phi \\ z(\rho, \theta, \phi) &= \rho \cos\phi \end{aligned}$$

The Jacobian for the change of coordinates is $J = \rho^2 \sin\phi$.

1. Let $\mathcal{E}$ be the solid region defined by

$$x^2 + y^2 + z^2 \leq a, \quad x, y, z \geq 0,$$

for a fixed constant $a > 0$. Suppose the region has a constant mass density of $\delta(x, y, z) = 1$.

(a) (2 points) Express the total mass of $\mathcal{E}$ as an iterated integral.

Solution:

(b) (2 points) Find the total mass of $\mathcal{E}$.

Solution:

(c) (3 points) Express the coordinates of the center of mass of $\mathcal{E}$ as an iterated triple integral.

Solution:

(d) (2 points) Find the z coordinate of the center of mass.

Solution:

2. Consider the helix $\mathcal{C}$, given by the parameterisation

$$\mathbf{r}(t) = \left(\cos t, \sin t, \frac{1}{2\pi} t\right) \quad t \in [0, 4\pi],$$

so that $\mathcal{C}$ is oriented with the z coordinate increasing.

(a) (4 points) Compute the length of $\mathcal{C}$.

Solution:

(b) (4 points) Compute the work done by the field

$$\mathbf{F}(x, y, z) = \langle z^2, 2yz^2, 2z(x + y^2) - e^z \rangle$$

on a particle constrained to move on the curve $\mathcal{C}$.

Solution:

3. For this question consider the vector field

$$\mathbf{F}(x, y) = \frac{1}{r^2}\langle y(r^2 - 1), x(r^2 + 1)\rangle,$$

where $r = \sqrt{x^2 + y^2}$. This vector field is defined everywhere apart from the origin.

(a) (4 points) Is $\mathbf{F}$ conservative on the domain described above? Justify your answer.

Solution:

(b) (1 point) Give a domain on which $\mathbf{F}$ is conservative.

Solution:

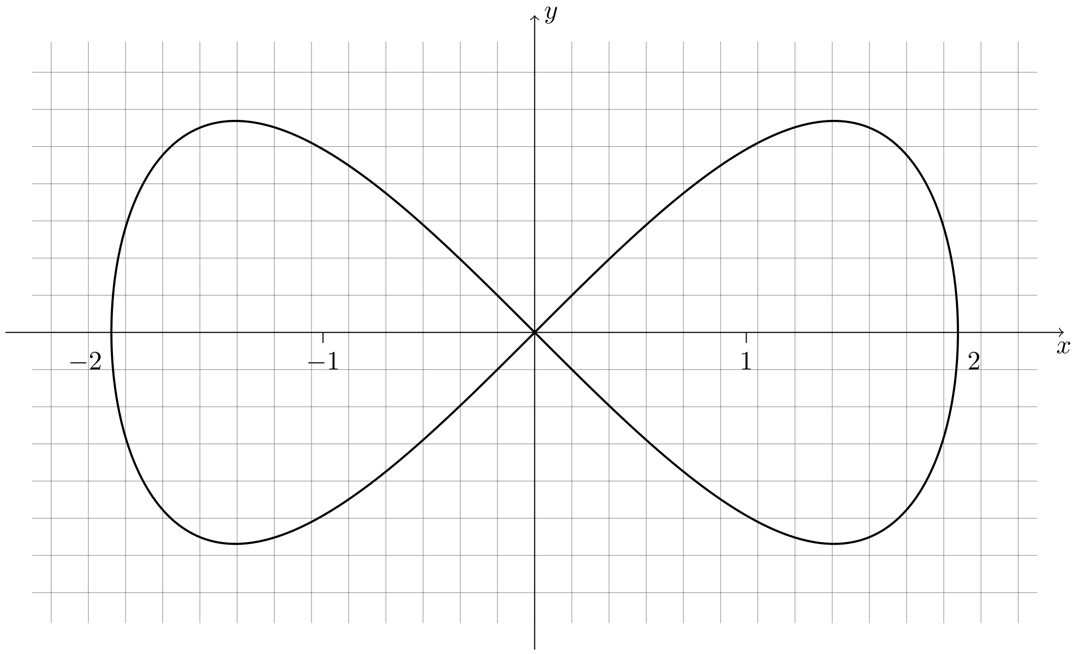
(c) (2 points) Calculate the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r}$$

where $\mathcal{C}$ is the ellipse $\frac{(x-4)^2}{2} + y^2 = 1$, oriented in the counter clockwise direction.

Solution:

4. In this question assume that $\mathbf{E}$ is a vector field defined on the whole plane, apart from the points $(\pm 1, 0)$. Suppose that $\nabla \times \mathbf{E} = 0$. The function $\mathbf{r}(t) = (2\cos t, \sin 2t)$ for $t \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ defines the curve $\mathcal{C}$ on the graph below



(a) (1 point) Indicate on the above graph, the orientation of the curve.

(b) (4 points) Let $\mathcal{A}$ and $\mathcal{B}$ be the circles, radius $\frac{1}{2}$, and center $(1, 0)$ and $(-1, 0)$ respectively, both oriented counter clockwise. Suppose that

$$\int_{\mathcal{A}} \mathbf{E} \cdot \mathrm{d}\mathbf{r} = 2 \quad \text{and} \quad \int_{\mathcal{B}} \mathbf{E} \cdot \mathrm{d}\mathbf{r} = 1.$$

What is $\int_{\mathcal{C}} \mathbf{E} \cdot \mathrm{d}\mathbf{r}$? Justify your answer.

5. The *hyperboloid* is Noah's favorite surface. It is given by the equation $x^2 + y^2 - z^2 = 1$.

   (a) (3 points) Find a parameterisation

   $$G(s,\theta) = (x(s,\theta), y(s,\theta), z(s,\theta)) \quad (s,\theta) \in \mathbb{R} \times [0, 2\pi]$$

   for the hyperboloid. *Hint: Let $z = s$.*

   Solution:

   (b) (5 points) Express the surface area of the hyperboloid between $z = a$ and $z = -a$ as an iterated integral.

   Solution:

(extra working room for part (b))

Solution:

(c) (3 points) Calculate the surface area. You may use the formula $\int \sqrt{1+x^2}\ dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)$.

Solution:

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