This week you will get practice solving separable differential equations, as well as some practice with linear models

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.2) Solve the following differential equations.
   (a) $\frac{\mathrm{d}y}{\mathrm{d}t} = 5y$
   (b) $\frac{\mathrm{d}y}{\mathrm{d}t} = -y$
   (c) $\frac{\mathrm{d}y}{\mathrm{d}x} = -3y$
   (d) $\frac{\mathrm{d}y}{\mathrm{d}x} = 0.2y$
   (e) (6.2-17) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^3$
   (f) (6.2-18) $\frac{\mathrm{d}y}{\mathrm{d}t} = y \sin t$
   (g) (6.2-20) $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{t}{y}$
   (h) (6.2-24) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y}\sqrt{1+x^2}$
   (i) (6.2-26) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\sin x}{\cos y}$
   (j) (6.2-30) $\frac{\mathrm{d}y}{\mathrm{d}t} = yt$ with $y(1) = -1$
   (k) (6.2-32) $\frac{\mathrm{d}y}{\mathrm{d}t} = e^{-y}t$ with $y(-2) = 0$
   (l) (6.2-34) $\frac{\mathrm{d}y}{\mathrm{d}t} = ty^2 + 3t^2y^2$ with $y(-1) = 2$

Solution: We begin by factorising the right hand side,

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (t + 3t^2)y^2.$$

We can now separate variables and integrate:

$$\int \frac{1}{y^2}\,\mathrm{d}y = \int t + 3t^2\,\mathrm{d}t$$

We integrate both sides using the power rule,

$$-\frac{1}{y} = \frac{1}{2}t^2 + t^3 + C$$

for an arbitrary constant C. Rearranging,

$$y(t) = -\frac{2}{t^2 + 2t^3 + C}.$$

Now we use the fact that $y(-1) = 2$:

$$2 = -\frac{2}{1 - 2 + C}$$

so $C = 0$ an the solution is

$$y(t) = -\frac{2}{t^2 + 2t^3}.$$

(m) $\frac{dy}{dx} = y \sin x + \frac{y}{(x+1)^2}$ with $y(0) = 1$

Solution: We begin by factorising the right hand side,

$$\frac{dy}{dt} = y\left(\sin x + \frac{1}{(x+1)^2}\right).$$

We can now separate variables and integrate:

$$\int \frac{1}{y}\, dy = \int \sin x + \frac{1}{(x+1)^2}\, dx$$

We integrate both sides,

$$\ln(y) = -\cos x - \frac{1}{(x+1)} + C$$

for an arbitrary constant C. Exponentiating both sides,

$$y(t) = C\exp\left(-\cos x - \frac{1}{(x+1)}\right).$$

Now we use the fact that $y(0) = 1$:

$$1 = C\exp(-1-1) = Ce^{-2}$$

so $C = e^2$ and the solution is

$$y(t) = \exp\left(2 - \cos x - \frac{1}{(x+1)}\right).$$

(n) $\frac{dy}{dx} = \frac{x}{y}e^{-x^2}$ with $y(0) = 1$

(o) $\frac{dy}{dx} = y + ye^x$ with $y(0) = e$

2. (6.2-44) Populations may exhibit seasonal growth in response to seasonal fluctuations in resource availability. A simple model accounting for seasonal fluctuations in the abundance N of a population is

$$\frac{dN}{dt} = (R + \cos t)N$$

where R is the average per-capita growth rate and t is measured in years.

(a) Assume $R = 0$ and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

Solution: When $R = 0$ the equation is $N' = N\cos t$. Using separation of variables we find the solution $N(t) = Ce^{\sin t}$ and since $N(0) = N_0$ we see that $C = N_0$. As $N \to \infty$, this fluctuates between N_0e^{-1} and N_0e.

(b) Assume $R = 1$ (more generally $R > 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

Solution: When $R = 1$ the equation is $N' = N(1 + \cos t)$. Using separation of variables we find the solution $N(t) = Ce^{t+\sin t}$ and since $N(0) = N_0$ we see that $C = N_0$. As $N \to \infty$, the t dominates the $\sin t$ and the population grows exponentially.

(c) Assume $R = -1$ (more generally $R < 0$) and find a solution to this differential that satisfies $N(0) = N_0$. What can you say about $N(t)$ at $t \to \infty$?

> **Solution:** When $R = -1$ the equation is $N' = N(-1 + \cos t)$. Using separation of variables we find the solution $N(t) = Ce^{-t+\sin t}$ and since $N(0) = N_0$ we see that $C = N_0$. As $N \to \infty$, the e^{-t} dominates the $e^{\sin t}$ and the population decreases to zero.

3. (6.3-25) In 1990 the gross domestic product (GDP) of the United States was \$5,464 billion. Suppose the growth rate from 1989 to 1990 was 5.08%. Predict the GDP in 2003.

   (Hint: You should assume that the percentage growth rate is constant - not very realistic!)

4. (6.3-28) According to the Department of Health and Human Services, the annual growth rate in the number of marriages per year in 1990 in the United States was 9.8% and there were 2,448,000 marriages that year. How many marriages will there be in 2004 if the annual growth rate in the number of marriages per year is constant?

> **Solution:** Let $M(t)$ be the number of marriages per year at time
>
> $$t$$
>
> . If the growth rate in marriages is 9.8% then the number of marriages is modelled by
>
> $$\frac{dM}{dt} = 0.098M$$
>
> So $M(t) = Ce^{0.098t}$ and using the initial condition, $M(0) = 2448000$ gives $M(t) = 2448000e^{0.047t}$. The number of marriages in 2004 is $M(14) \approx 9,653,000$.

5. (6.3-30) Calculate the infusion rate in milligrams per hour required to maintain a long-term drug concentration of 50 mg/L (i.e., the rate of change of drug in the body equals zero when the concentration is 50 mg/L). Assume that the half-life of the drug is 3.2 hours and that the patient has 5 L of blood.

6. (6.3-31) Calculate the infusion rate in milligrams per hour required to maintain a desired drug concentration of 2 mg/L. Assume the patient has 5.6 L of blood and the half-life of the drug is 2.7 hours.

> **Solution:** The amount of drug (in mg) in the body $y(t)$ at time t will obey a differential equation of the form
>
> $$\frac{\mathrm{d}y}{\mathrm{d}t} = \text{ rate in } - \text{ rate out }.$$
>
> If the drug is being infused at a rate of a mg/h then this is the rate in. If the drug has a half-life of 2.7 hours, this means, after t hours, the fraction of the drug that is left in the body is given by
>
> $$\left(\frac{1}{2}\right)^{\frac{t}{2.7}} = e^{-\frac{\ln 2}{2.7}t}.$$
>
> Thus, in the absence of any infusion, the drug is being expelled by the body at a rate of
>
> $$\frac{\mathrm{d}}{\mathrm{d}t}e^{-\frac{\ln 2}{2.7}t} = -\frac{\ln 2}{2.7}e^{-\frac{\ln 2}{2.7}t} = -\frac{\ln 2}{2.7}(\text{current level of drug}).$$

Thus if $y(t)$ is the current level of drug in the body, then at time t the drug is being expelled at a rate of $-\frac{\ln 2}{2.7}y(t)$ mg/h. This is the rate out. Our differential equation becomes

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a - \frac{\ln 2}{2.7}y.$$

Over the long term, the solution of this equation will approach the equilibrium solution $y(t) = \frac{2.7a}{\ln 2}$. Over the long term we would like the concentration of the drug to be 2 mg/L, since the patient has 5.6 L of blood, that means we would like there to be 11.2 mg of drug in the body in the long term. I.e. we want

$$\frac{2.7a}{\ln 2} = 11.2$$

Rearranging, we get

$$a = \frac{11.2 \ln 2}{2.7} \approx 2.88 \text{ mg/h}.$$

7. (6.3-34) A drug is given at an infusion rate of 50 mg/h. The drug concentration value determined at 3 hours after the start of the infusion is 8 mg/L. Assuming the patient has 5 L of blood, estimate the half-life of this drug.

Solution: The rate in is 50 and the rate out is given by the half life of λ. Using the formula we learnt, this gives

$$\frac{dy}{dt} = 50 - \frac{\ln 2}{\lambda}y$$

where $y(t)$ is the amount of drug in the system at time t. Solving the ODE gives

$$y(t) = \frac{50\lambda}{\ln 2} - Ce^{-t(\ln 2)/\lambda}$$

Using the initial condition $y(0) = 0$ we find that $C = \frac{50\lambda}{\ln 2}$. Thus

$$y(t) = \frac{50\lambda}{\ln 2}\left(1 - e^{-t(\ln 2)/\lambda}\right).$$

To find λ we use $y(3) = 8 \cdot 5 = 40$. Plugging this in produces

$$40 = \frac{50\lambda}{\ln 2}\left(1 - e^{-3(\ln 2)/\lambda}\right).$$

This is not something we can solve using normal methods, so we could just plug it into a computer to get an approximate value: $\lambda \approx 1.07$.

8. (6.3-37) After one hydrodynamic experiment, a tank contains 300 L of a dye solution with a dye concentration of 2 g/L. To prepare for the next experiment, the tank is to be rinsed with water flowing in at a rate of 2 L/min, with the well-stirred solution flowing out at the same rate. Write an equation that describes the amount of dye in the container. Be sure to identify variables and their units.