What polynomial is that?

We now know a very large number of functions. We want to be able to answer basic questions about them, eg integrals of

$$e^{-x^2}, \ \sin(x^2),$$

but cannot do this necessarily. More specifically, we might want to approximate these things

Ex Approximate $\int_0^1 e^{-x^2}\,dx$ to 5 decimal places

To answer questions like this, we will use Taylor polynomials (next time). This time we do some related exercises.

Ex Suppose $P(x)$ is a polynomial of deg ≤ 5, and that

$$P(0) = 4$$
$$P'(0) = 3$$
$$P''(0) = 9$$
$$P^{(3)}(0) = 5$$
$$P^{(4)}(0) = 11$$
$$P^{(5)}(0) = 7$$

Can we say what $P(x)$ is?

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5$$

for some numbers $a_0, a_1, \ldots, a_5$. So

$$p'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4$$

$$p''(x) = 2a_2 + 6a_3 x + 12 a_4 x^2 + 20 a_5 x^3$$

$$p^{(3)}(x) = 6a_3 + 24 a_4 x + 60 a_5 x^2$$

$$p^{(4)}(x) = 24 a_4 + 120 a_5 x$$

$$p^{(5)}(x) = 120 a_5$$

So

$$\begin{aligned}
p(0) &= a_0 &&= 0!\,a_0 &&= 4\\
p'(0) &= a_1 &&= 1!\,a_1 &&= 3\\
p^{(2)}(0) &= 2a_2 &&= 2!\,a_2 &&= 9\\
p^{(3)}(0) &= 6a_3 &&= 3!\,a_3 &&= 5\\
p^{(4)}(0) &= 24a_4 &&= 4!\,a_4 &&= 11\\
p^{(5)}(0) &= 120a_5 &&= 5!\,a_5 &&= 7
\end{aligned}$$

from question

solving these equations gives

$$p(x) = 4 + 3x + \frac{9}{2}x^2 + \frac{5}{6}x^3 + \frac{11}{24}x^4 + \frac{7}{120}x^5$$

Ex Suppose $P(x)$ is a polynomial of deg ≤ 3

and $P(1) = 20$

$P'(1) = 2$

$P''(1) = 7$

$P^{(3)}(1) = 5$

If we set $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ then we get ~~P(0) = a~~ $P(1) = a_0 + a_1 + a_2 + a_3$ and we get similar equations for the derivatives. These are hard to ~~st solve~~ solve. Trick: let

$$P(x) = a_0 + a_1(x-1) + a_2(x-1)^2 + a_3(x-1)^3$$

Then
$$P'(x) = \cancel{a_0}\, a_1 + 2a_2(x-1) + 3a_3(x-1)^2$$
$$P''(x) = 2a_2 + 6a_3(x-1)$$
$$P'''(x) = 6a_3$$

so
$$P(1) = a_0 = 0!\,a_0 = 20$$
$$P'(1) = a_1 = 1!\,a_1 = 2$$
$$P''(1) = 2a_2 = 2!\,a_2 = 7$$
$$P'''(1) = 6a_3 = 3!\,a_3 = 5$$

so

$$P(x) = 20 + 2(x-1) + \frac{7}{2}(x-1)^2 + \frac{5}{6}(x-1)^3$$

Ex Suppose $P(x)$ is a poly. of deg ≤ 3 and we know what $P(a)$, $P'(a)$, $P''(a)$, $P'''(a)$ are. What is $P(x)$?

$$
\begin{aligned}
\text{Let} \quad P(x) &= a_0 + a_1(x-a) + a_2(x-a)^2 + a_3(x-a)^3 \\
\text{then} \quad P'(x) &= a_1 + 2a_2(x-a) + 3a_3(x-a)^2 \\
P''(x) &= 2a_2 + 6a_3(x-a) \\
P'''(x) &= 6a_3
\end{aligned}
$$

So

$$
\begin{aligned}
P(a) &= a_0 = 0!\, a_0 \\
P'(a) &= a_1 = 1!\, a_1 \\
P''(a) &= 2a_2 = 2!\, a_2 \\
P'''(a) &= 6a_2 = 3!\, a_3
\end{aligned}
$$

So

$$
P(x) = \frac{P(a)}{0!} + \frac{P'(a)}{1!}(x-a) + \frac{P''(a)}{2!}(x-a)^2 + \frac{P'''(a)}{3!}(x-a)^3
$$

Ex Suppose $P(x)$ is a poly of deg $\leq n$ and we know what $P(a), P'(a), \_\_, P^{(n)}(a)$ are. ~~Wa~~ What is $P(x)$?

We let

$$P(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \dots + a_n(x-a)^n$$

Notice that

$$P^{(k)}(x) = k!\, a_k + * \, a_{k+1}(x-a) + \dots$$

more stuff, ie higher powers of $(x-a)$.

$*$ some number I haven't bothered working out.

so $P^{(k)}(a) = k!\, a_k$. ie $a_k = \dfrac{P^{(k)}(a)}{k!}$

Thus

$$P(x) = \frac{P(a)}{0!} + \frac{P'(a)}{1!}(x-a) + \frac{P''(a)}{2!}(x-a)^2 + \dots + \frac{P^{(n)}(a)}{n!}(x-a)^n$$