# Math 3B: Lecture 6

Noah White

January 22, 2017

# Introduction

Last time

- Examples of maxima/minima problems

Quiz 3

# Introduction

Last time

- Examples of maxima/minima problems
- Maximising functions with constaints

Quiz 3

# Introduction

Last time

- Examples of maxima/minima problems
- Maximising functions with constaints
- Competing economies

Quiz 3

# Introduction

Last time

- Examples of maxima/minima problems
- Maximising functions with constaints
- Competing economies

Quiz 3

- Arrive on time

# Introduction

Last time

- Examples of maxima/minima problems
- Maximising functions with constaints
- Competing economies

Quiz 3

- Arrive on time
- You will have 10 mins

# Introduction

Last time

- Examples of maxima/minima problems
- Maximising functions with constaints
- Competing economies

Quiz 3

- Arrive on time
- You will have 10 mins
- Look at questions 1-8 on PS3

# Differential equations (motivation)

A differential equation is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

$$x^2 y'' + xy' + x^2 y = 0$$

# Differential equations (motivation)

A differential equation is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

$$x^2 y'' + xy' + x^2 y = 0$$

The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

# Newton's second law (motivation)

The original differential equation!

$$F = ma$$

# Newton's second law (motivation)

The original differential equation!

$$F = ma$$

If $h(t)$ measures the height of an object (maybe an apple?) above the earth then

$$a = h''(t)$$

# Newton's second law (motivation)

The original differential equation!

$$F = ma$$

If $h(t)$ measures the height of an object (maybe an apple?) above the earth then

$$a = h''(t)$$

The force due to gravity is roughly $-10m$ Newtons, so

$$-10m = mh''(t)$$

# Population growth (motivation)

A very rudimentary (but sometimes surprisingly accurate) way to model population growth is

# Population growth (motivation)

A very rudimentary (but sometimes surprisingly accurate) way to model population growth is

## Assumption

*The rate of growth of a population is proportional to its current size*

# Population growth (motivation)

A very rudimentary (but sometimes surprisingly accurate) way to model population growth is

Assumption

*The rate of growth of a population is proportional to its current size*

# Population growth (motivation)

A very rudimentary (but sometimes surprisingly accurate) way to model population growth is

**Assumption**

*The rate of growth of a population is proportional to its current size*

If $P(t)$ is the population at time $t$:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(t)$$

# Some more examples of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(1-y)$$

$$y'' = \sqrt{a^2 - (y')^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(A-y)^2$$

# Antiderivatives

We will be concentrating on solving differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

# Antiderivatives

We will be concentrating on solving differential equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

A solution $y = F(x)$ is called an antiderivative of $f(x)$.

# Example 1

**Question**

What is the antiderivative of $f(x) = 2x$?

# Example 1

## Question

What is the antiderivative of $f(x) = 2x$?

## Solution

$$F(x) = x^2$$

# Example 1

## Question

What is the antiderivative of $f(x) = 2x$?

## Solution

$$F(x) = x^2 + 4$$

# Example 1

## Question

What is the antiderivative of $f(x) = 2x$?

## Solution

$$F(x) = x^2 + 8$$

# Example 1

## Question

What is the antiderivative of $f(x) = 2x$?

## Solution

$$F(x) = x^2 + C$$

# Example 2

## Question

What is the antiderivative of $f(x) = x^3 + 4x - 1$?

# Example 2

## Question

What is the antiderivative of $f(x) = x^3 + 4x - 1$?

## Solution

$$F(x) = \frac{1}{4}x^4$$

# Example 2

## Question

What is the antiderivative of $f(x) = x^3 + 4x - 1$?

## Solution

$$F(x) = \frac{1}{4}x^4 + 2x^2$$

# Example 2

## Question

What is the antiderivative of $f(x) = x^3 + 4x - 1$?

## Solution

$$F(x) = \frac{1}{4}x^4 + 2x^2 - x$$

# Example 2

## Question

What is the antiderivative of $f(x) = x^3 + 4x - 1$?

## Solution

$$F(x) = \frac{1}{4}x^4 + 2x^2 - x + C$$

# Example 3

## Question

What is the antiderivative of $f(x) = e^{2x}$?

# Example 3

## Question

What is the antiderivative of $f(x) = e^{2x}$?

## Solution

$$F(x) = \frac{1}{2}e^{2x}$$

## Example 4

**Question**

What is the antiderivative of $f(x) = \frac{1}{x}$ (for $x > 0$)?

# Example 4

## Question

What is the antiderivative of $f(x) = \frac{1}{x}$ (for $x > 0$)?

## Solution

$$F(x) = \ln x$$

# Example 5

**Question**

What is the antiderivative of $f(x) = \frac{1}{(1+x)^2}$?

# Example 5

**Question**

What is the antiderivative of $f(x) = \frac{1}{(1+x)^2}$?

**Solution**

Note that $f(x) = (1+x)^{-2}$. So

$$F(x) = \quad \frac{1}{1+x}$$

# Example 5

### Question

What is the antiderivative of $f(x) = \frac{1}{(1+x)^2}$?

### Solution

Note that $f(x) = (1 + x)^{-2}$. So

$$F(x) = -\frac{1}{1 + x}$$

# Example 6

## Question

What is the antiderivative of $f(x) = 2x \cos x^2$?

# Example 6

**Question**

What is the antiderivative of $f(x) = 2x \cos x^2$?

**Solution**

$$F(x) = \sin x^2$$

## Example 7

### Question

What is the antiderivative of $f(x) = \frac{1}{\sqrt{x}}$?

# Example 7

## Question

What is the antiderivative of $f(x) = \frac{1}{\sqrt{x}}$?

## Solution

$$f(x) = x^{-\frac{1}{2}}$$

# Example 7

## Question

What is the antiderivative of $f(x) = \frac{1}{\sqrt{x}}$?

## Solution

$$f(x) = x^{-\frac{1}{2}}$$

# Example 7

## Question

What is the antiderivative of $f(x) = \frac{1}{\sqrt{x}}$?

## Solution

$$f(x) = x^{-\frac{1}{2}}$$

$$F(x) = \quad x^{\frac{1}{2}}$$

# Example 7

## Question

What is the antiderivative of $f(x) = \frac{1}{\sqrt{x}}$?

## Solution

$$f(x) = x^{-\frac{1}{2}}$$

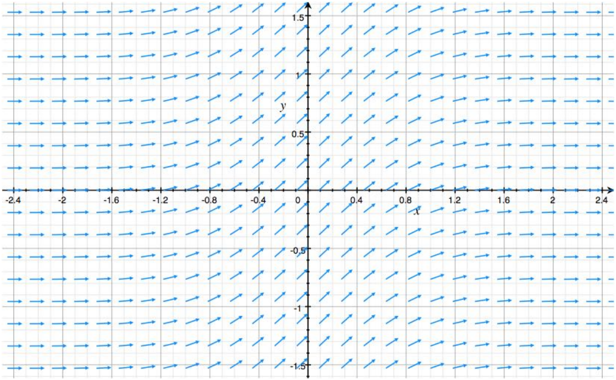
$$F(x) = 2x^{\frac{1}{2}}$$

# Slope fields

In some cases it is impossible to find the antiderivative (without special functions). E.g.

$$f(x) = e^{-x^2}$$

But we can still (approximately) graph the antiderivative! First we draw the slope field



# Slope fields (how to draw)

How to draw a slope field for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

1. Draw the $xy$-plane.

# Slope fields (how to draw)

How to draw a slope field for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

1. Draw the $xy$-plane.
2. At every point $(x, y)$ what would the slope of $y = F(x)$ be if it passed through that point?

# Slope fields (how to draw)

How to draw a slope field for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

1. Draw the $xy$-plane.
2. At every point $(x, y)$ what would the slope of $y = F(x)$ be if it passed through that point?
3. Answer given by differential equation above, slope is $f(x)$

# Slope fields (how to draw)

How to draw a slope field for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

1. Draw the $xy$-plane.
2. At every point $(x, y)$ what would the slope of $y = F(x)$ be if it passed through that point?
3. Answer given by differential equation above, slope is $f(x)$
4. Draw a small arrow with slope $f(x)$ and the point $(x, y)$

# Slope fields (how to draw)

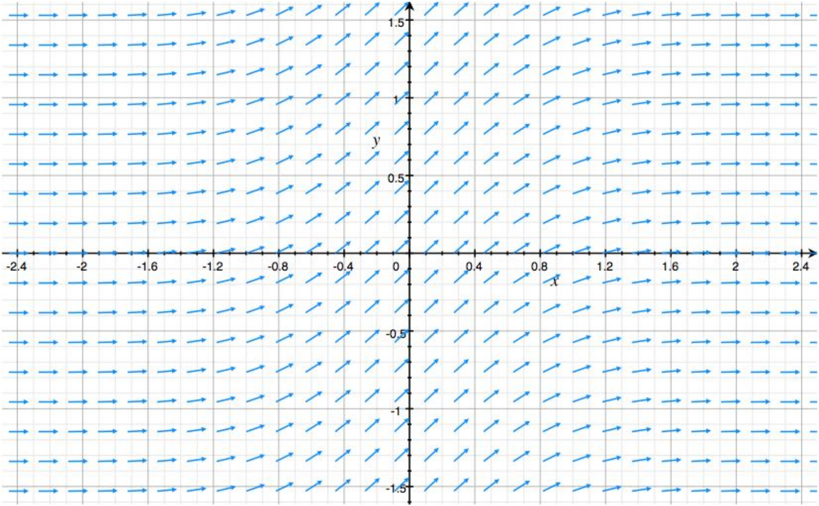
How to draw a slope field for

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

1. Draw the $xy$-plane.
2. At every point $(x, y)$ what would the slope of $y = F(x)$ be if it passed through that point?
3. Answer given by differential equation above, slope is $f(x)$
4. Draw a small arrow with slope $f(x)$ and the point $(x, y)$
5. Do this for a grid of points on the $xy$-plane.

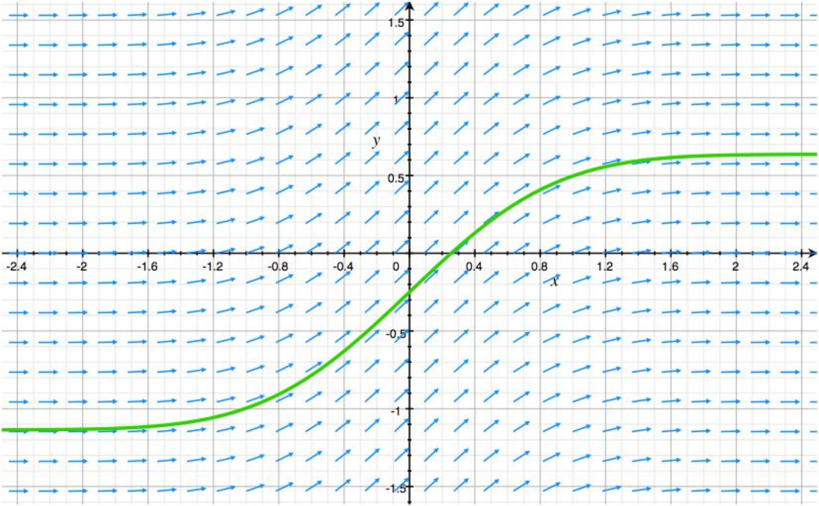
# Example 1

$$f(x) = e^{-x^2}$$



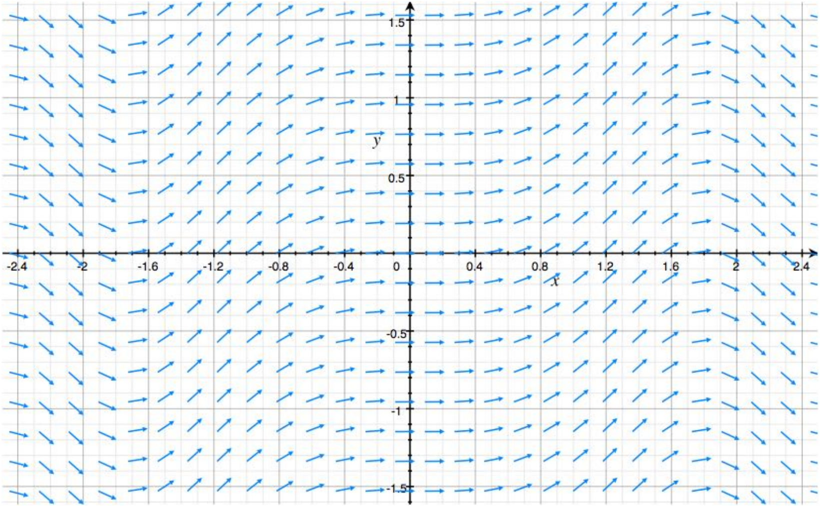
# Example 1

$$f(x) = e^{-x^2}$$



# Example 2

$$f(x) = \sin(x^2)$$



# Example 2

$$f(x) = \sin(x^2)$$