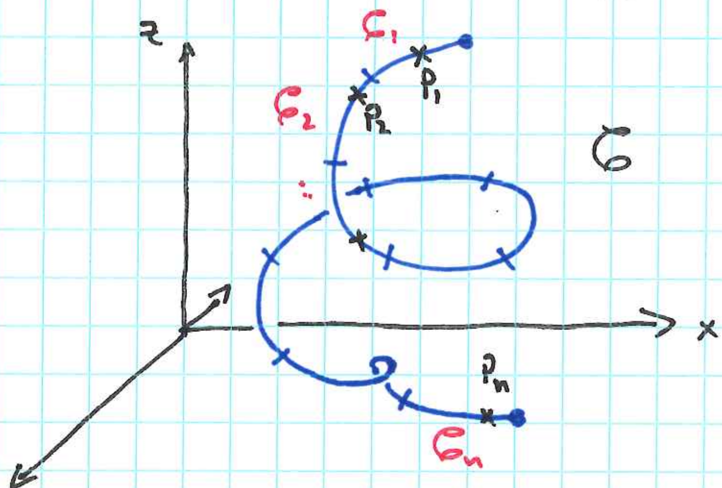
Lecture 13

1. Scalar line integrals

- Suppose $\mathcal{C} \subseteq \mathbb{R}^3$ is a curve, $f(x\,y\,z)$ a $f^{\underline{n}}$
- We would like to integrate f over $\mathcal{C}$
- As usual we use Riemann sums



- Partition $\mathcal{C}$ into n pieces $\mathcal{C}_1, \ldots, \mathcal{C}_n$ and choose points $P_i \in \mathcal{C}_i$. Call this partition $\mathcal{P}$ and let $\|\mathcal{P}\| = \max\{\Delta s_i\}$ where Δs_i = length of $\mathcal{C}_i$

Def

$$\int_{\mathcal{C}} f(x\,y\,z)\,ds = \lim_{\|\mathcal{P}\| \to 0} \sum_{i=1}^{n} f(P_i)\,\Delta s_i$$

Rmk

$$\int_{\mathcal{C}} 1\, ds = \text{length}(\mathcal{C})$$

2. Parametrisations

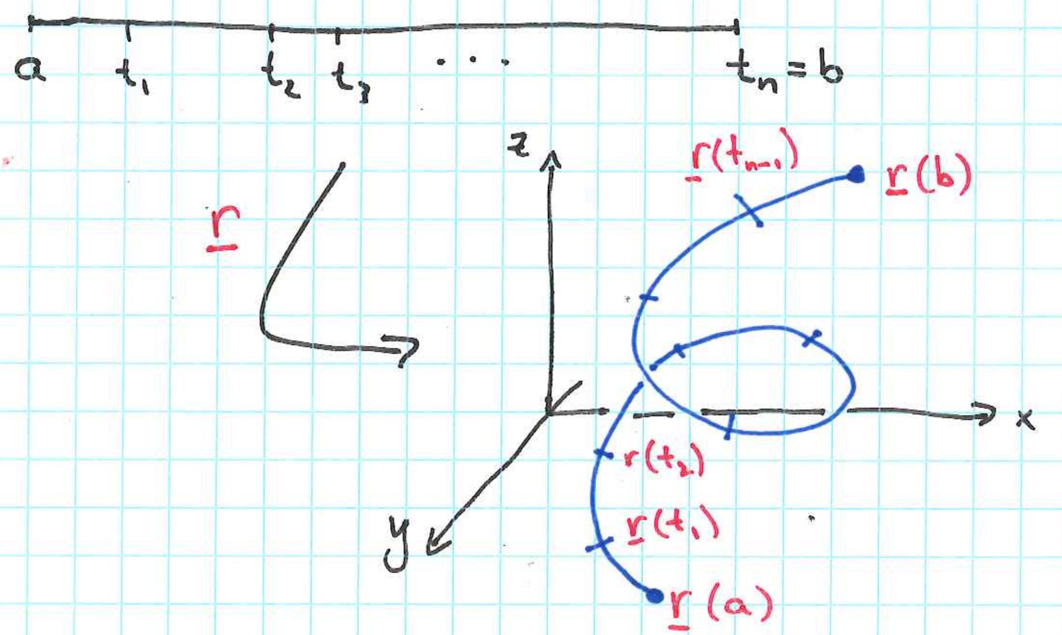
- Let $\underline{r}(t) = (x(t), y(t), z(t))$ be a parametrization of the curve $\mathcal{C}$

$$\underline{r} : [a,b] \longrightarrow \mathbb{R}^3$$

- We can find

$$a = t_0 < t_1 < \_\_ < t_n = b$$

such that $\mathcal{C}_i = \underline{r}([t_{i-1}, t_i])$



- Using the length formula from 32A:

$$\text{length}(\mathcal{C}_i) = \Delta s_i = \int_{t_{i-1}}^{t_i} |\underline{r}'(t)|\, dt$$

- If $\mathcal{C}_i$ is very small then

$$\Delta s_i \approx |\underline{r}'(Q_i)|\, \Delta t_i \qquad \text{where } \underline{r}(Q_i) = P_i$$

- Thus

$$\int_{\mathcal{C}} f(x\,y\,z)\, ds = \cancel{\sum} \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\underline{r}(Q_i))\, |\underline{r}'(Q_i)|\, \Delta t_i$$

Thm If $\underline{r}(t)$ is a parametrization of $\mathcal{C}$ for $a \leq t \leq b$ and f, $\underline{r}'$ are cts then

$$\int_{\mathcal{C}} f(x\,y\,z)\, ds = \int_a^b f(\underline{r}(t))\, |\underline{r}'(t)|\, dt$$

Steps to solve a line integral:

1. Find a parametrisation of $\mathcal{C}$
2. Write line integral as regular 1D integral
3. Solve.

Ex $\mathcal{C}$ is the ~~cir~~ semicircle of radius 1, center $(0,0)$ in the upper half plane. Find

$$\int_{\mathcal{C}} y \, ds$$

- First we parametrise

$$\underline{r}(t) = \langle \cos(t), \sin(t) \rangle \qquad t \in [0, \pi]$$

- Thus

$$\underline{r}'(t) = \langle -\sin t, \cos t \rangle$$

$$|\underline{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

and $\quad y = \sin(t)$

- $$\int_{\mathcal{C}} y \, ds = \int_0^{\pi} \sin t \cdot 1 \cdot dt = 2.$$