Review lecture Math 3B

10.3.16

1. Graphing using ~~calculus~~ calculus:

You should ask:

- where are the x/y-intercepts
- Are there horizonal asymptotes
- —"— vertical asymptotes
- —"— slanted asymptotes
- where is $f(x)$ pos/neg
- —"— $f'(x)$ —"—
- —"— $f''(x)$ —"—

Collect as much of this information as you can/need and use it to sketch the graph.

Example $f(x) = {}^{2}\cancel{4} \ln(e^x + 1) - x$

x-int: $f(x) = 0$

$$\ln(e^x + 1) = \frac{1}{2}x$$

$$e^x + 1 = e^{\frac{1}{2}x}$$

$$(e^x + 1)^2 = e^x$$

$$e^{2x} + e^x + 1 = 0$$

substitute $u = e^x$

$$u^2 + u + 1 = 0$$

No solutions! $\Longrightarrow$ no x-int.

y-int: $f(0) = 2\ln(e^0 + 1) - 0$

$$= 2\ln(2) \sim 1.4.$$

hor. asymp. $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} 2\ln(e^x + 1) - x$

Note: $\ln\left(e^{2\ln(e^x+1) - x}\right)$

$$= \ln\left(e^{-x} e^{\ln(e^x+1)^2}\right)$$

$$= \ln\left(e^{-x} \cancel{e\ln}(e^x + 1)^2\right)$$

So $\lim\limits_{x \to \infty} f(x) = \lim\limits_{x \to \infty} \ln\left(e^{-x}(e^x + 1)^2\right)$

$$= \ln\left(\lim_{x \to \infty}\left(e^x + 2 + e^{-x}\right)\right)$$

$$= \infty.$$

No horizontal asymptotes!

vert asymp. log "blows up" at zero

so:

when is $e^x + 1 = 0$?

A: never!

No vertical asymptotes!

Slanted asymptotes

$$f'(x) = \frac{e^x - 1}{e^x + 1}$$

$$\lim_{x \to \infty} f'(x) = 1$$

$$\lim_{x \to -\infty} f'(x) = -1.$$

$$b^+ = \lim_{x \to \infty} \left(f(x) - x \right) = \lim_{x \to \infty} 2\ln(e^x + 1) - 2x$$

Same trick

$$f(x) - x = \ln\left(e^{2\ln(e^x+1) - 2x} \right)$$

$$= \ln\left(e^{-2x} e^{\ln(e^x+1)^2} \right)$$

$$= \ln\left(1 + 2e^{-x} + e^{-2x} \right)$$

so

$$b^+ = \lim_{x \to \infty} \ln(1 + 2e^{-x} + e^{-2x}) = 0.$$

$$b^- = \lim_{x \to -\infty} \left(f(x) + x \right) = \lim_{x \to -\infty} 2\ln\left(e^x + 1\right)$$

$$= 0$$

So slanted asymptotes:

- $y = x$ as $x \to \infty$
- $y = -x$ as $x \to -\infty$.

<u>1^{st} derivative</u>

$$f'(x) = \frac{e^x - 1}{e^x + 1}$$

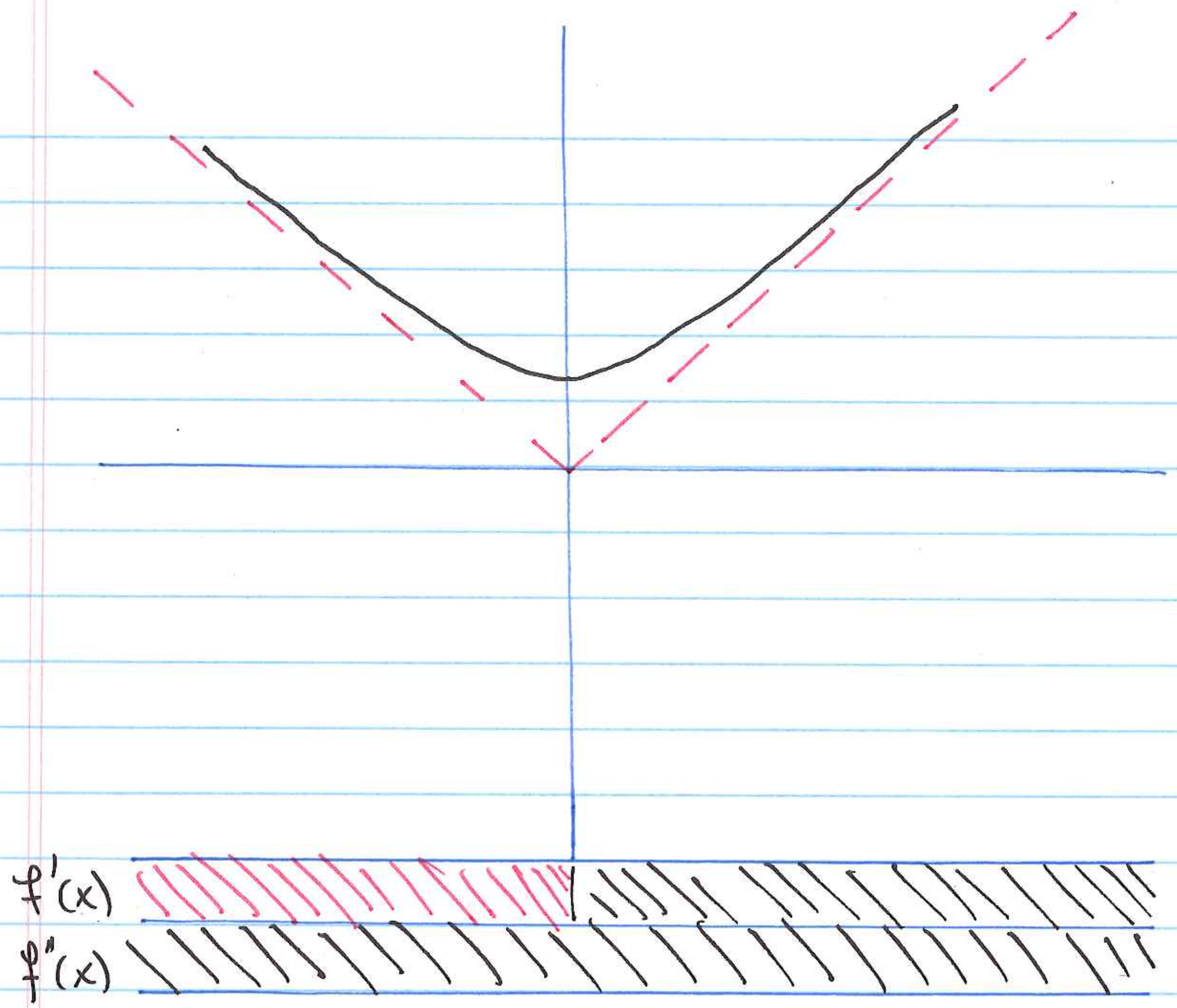
+ve : $x < 0$

-ve : $x > 0$

<u>2^{nd} derivative</u>

$$f''(x) = \frac{2e^x}{(e^x + 1)^2}$$

Always positive!



Example $f(x) = \dfrac{x^2 - 4}{x^2 - 9}$

x/y-int $\quad f(0) = \dfrac{4}{9} \sim \dfrac{1}{2} \quad \leftarrow$ y-int.

$$f(x) = 0$$

$$\frac{x^2-4}{x^2-9} = 0$$

$$x^2 - 4 = 0$$

$$\Rightarrow x = \pm 2 \leftarrow x\text{-int.}$$

hor. asymp.

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \frac{x^2-4}{x^2-9}$$

$$\overset{L'H}{=} \lim_{x\to\pm\infty} \frac{2x}{2x}$$

$$= 1$$

hor. asymp. at $y = 1$.

vert. asymp.

$$\lim_{x\to 3^\pm} f(x) = \pm\infty$$

$$\lim_{x\to -3^\pm} f(x) = \mp\infty$$

slanted asympt. None!

1st der. $f'(x) = -\dfrac{10x}{(x^2-9)^2}$

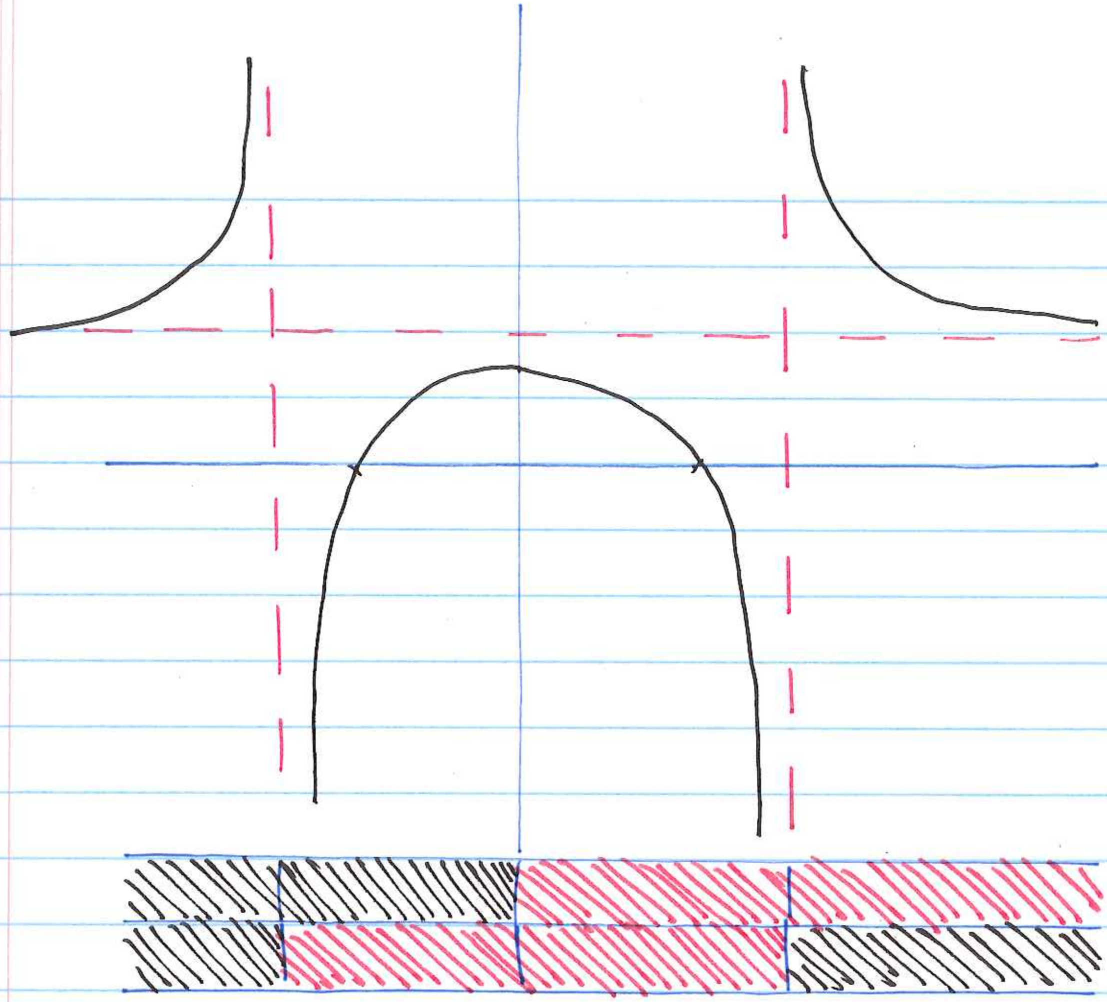
+ve : $x < 0$

-ve : $x > 0$

2nd der $f''(x) = \dfrac{30(x^2+3)}{(x^2-9)^3}$

+ve : $|x| > 3$

-ve : $|x| < 3$



2. Maxima and minima

Recipe for finding max/min

1. find crit. points by solving
   $f'(x) = 0$ and
   $f'(x) = \text{undef.}$
2. test crit points using
   - 2^{nd} der test
   - 1^{st} der test
3. evalutate max/min points (i.e. find y-vals) to find global max/min.

Example Harvesting rates.

The population of fin whales grows at a rate of

$$G(x) = 0.08x\left(1 - \frac{x}{500\,000}\right)$$

(individuals per year), x is the current pop.

A harvesting rate H (also individuals/year) is called "sustainable" (in the long term) if there exists an x such that

$$H = G(x).$$

Q What is the max sustainable harvesting rate?

A We want to maximise

$$H = G(x).$$

$$G'(x) = 0.08\left(1 - \frac{x}{250\,000}\right).$$

The crit points:

$$\begin{aligned} 0 &= G'(x) \\ &= 0.08\left(1 - \frac{x}{250000}\right) \end{aligned}$$

$$\frac{x}{250\,000} = 1$$

$$x = 250\,000$$

$$G''(x) = -\frac{0.08}{250\,000} < 0$$

So we have a ~~minimum~~ maximum.

$$\begin{aligned} G(250\,000) &= 0.08 \times 250000\left(1 - \frac{1}{2}\right) \\ &= 10000 \end{aligned}$$

Is the max sustainable harvesting rate.

Example

A campaign manager for the Democratic candidate running for the NH senate seat needs to decide how to spend \$10 mil. in TV and radio ~~ads~~ advertising money. The data analytics firm employed by the campaign has experimentally detetermined that in NH's 1^{st} and 2^{nd} congressional districts, \$x mil spent would result in

$$R_1(x) = 1860 \ln(1+x)$$
$$R_2(x) = 790 \ln(1+x)$$

undecided voters switching to your candidate.

Q How much money should you spend in each district?

A We will spend all of the money so if we spend \$x mil in district 1, the total increase will be

$$R(x) = R_1(x) + R_2(10 - x).$$

$$= 1860 \ln(1+x) + 790 \ln(1+10-x)$$

We want to maximise this function.
Note: domain = $[0, 10]$.

Find the critical points

$$R'(x) = \frac{1860}{1+x} - \frac{790}{1+10-x}$$

setting equal to zero:

$$(11-x)1860 = (1+x)790$$

$$19670 = 2650x$$

$$x = \frac{1967}{265} \sim 7.46$$

We have critical points at

$$x = 0, 10, \frac{1967}{265}$$

Using 1st der test: $x = 0, 10$ are minimums.

$$R''(x) = -\frac{1860}{(1+x)^2} - \frac{790}{(11-x)^2} < 0$$

So $x = \frac{1976}{268}$ is a max!

Spend \$7.46 mil in district 1,
\$2.54 mil in district 2.

~~Mat~~ Max $= R(7.46) \sim 4970$

$\sim 0.5\%$ of Registered voters.