Midterm 2 practice 2

UCLA: Math 32B, Spring 2018

Instructor: Noah White
Date: May 2018

- This exam has 5 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section (please circle):

Day/TA	Ryan	Eli	Khang
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	9	
2	8	
3	7	
4	5	
5	11	
Total:	40	

Here are some formulas that you may find useful as some point in the exam.

$$\int \cos^2 x \, dx = \frac{1}{2}(x + \cos x \sin x)$$

$$\int \sin^2 x \, dx = \frac{1}{2}(x - \cos x \sin x)$$

$$\int \sin x \cos x \, dx = \frac{1}{2}\sin^2 x$$

Spherical coordinates are given by

$$\begin{aligned} x(\rho, \theta, \phi) &= \rho \cos\theta \sin\phi \\ y(\rho, \theta, \phi) &= \rho \sin\theta \sin\phi \\ z(\rho, \theta, \phi) &= \rho \cos\phi \end{aligned}$$

The Jacobian for the change of coordinates is $J = \rho^2 \sin\phi$.

1. Let $\mathcal{E}$ be the solid region defined by

$$x^2 + y^2 + z^2 \le a, \quad x, y, z \ge 0,$$

for a fixed constant $a > 0$. Suppose the region has a constant mass density of $\delta(x, y, z) = 1$.

(a) (2 points) Express the total mass of $\mathcal{E}$ as an iterated integral.

Total mass $= \iiint_{\mathcal{E}} 1\,dV$, in spherical coords:

$0 \le \rho \le \sqrt{a} \qquad 0 \le \theta, \phi \le \frac{\pi}{2}$ so

$$\iiint_{\mathcal{E}} 1\,dV = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi.$$

(b) (2 points) Find the total mass of $\mathcal{E}$.

Total mass = total volume $= \frac{1}{8}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{6}\pi a^{3/2}$

or

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \left(\int_0^{\frac{\pi}{2}} d\theta\right)\left(\int_0^{\frac{\pi}{2}} \sin\phi \, d\phi\right)\left(\int_0^{\sqrt{a}} \rho^2 \, d\rho\right)$$

$$= \frac{\pi}{2} \cdot \left(-\cos\phi\right)\Big|_0^{\frac{\pi}{2}} \cdot \frac{1}{3}\rho^3\Big|_0^{\sqrt{a}}$$

$$= \frac{\pi}{2} \cdot 1 \cdot \frac{1}{3} a^{3/2} = \frac{1}{6}\pi a^{3/2}.$$

(c) (3 points) Express the coordinates of the center of mass of $\mathcal{E}$ as an iterated triple integral.

$$(x_{CM}\ y_{CM}\ z_{CM}) = \frac{1}{\text{Mass } \mathcal{E}} \iiint_{\mathcal{E}} (x\ y\ z)\, dV$$

$$= \frac{6}{a\pi a^{3/2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\rho^2\sin\phi\ d\rho\, d\theta\, d\phi$$

(d) (2 points) Find the z coordinate of the center of mass.

$$z_{CM} = \frac{6}{\pi a^{3/2}} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{a}} \rho^3 \sin\phi\cos\phi\ d\rho\, d\theta\, d\phi$$

$$= \frac{6}{\pi a^{3/2}} \left(\int_0^{\frac{\pi}{2}} d\theta\right) \cdot \left(\int_0^{\frac{\pi}{2}} \sin\phi\cos\phi\, d\phi\right)\left(\int_0^{\sqrt{a}} \rho^3\, d\rho\right)$$

$$= \frac{6}{\pi a^{3/2}} \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} a^2 = \frac{3}{8}\sqrt{a}$$

2. Consider the helix $\mathcal{C}$, given by the parameterisation

$$\mathbf{r}(t) = \left(\cos t, \sin t, \frac{1}{2\pi}t\right) \quad t \in [0, 4\pi],$$

so that $\mathcal{C}$ is oriented with the z coordinate increasing.

(a) (4 points) Compute the length of $\mathcal{C}$.

$$\text{length} = \int_{\mathcal{C}} 1 \, ds = \int_0^{4\pi} 1 \cdot |\underline{r}'(t)| \, dt$$

$$\underline{r}'(t) = \left(-\sin t, \cos t, \frac{1}{2\pi}\cancel{t}\right)$$

$$|\underline{r}'(t)| = \sqrt{1 + \frac{1}{4\pi^2}}$$

$$\text{length} = \int_0^{4\pi} \sqrt{1 + \frac{1}{4\pi^2}} \, dt = 4\pi\sqrt{1 + \frac{1}{4\pi^2}}$$

$$= 2\sqrt{4\pi^2 + 1}$$

(b) (4 points) Compute the work done by the field

$$\mathbf{F}(x, y, z) = \langle z^2, 2yz^2, 2z(x + y^2) - e^z \rangle$$

on a particle constrained to move on the curve $\mathcal{C}$.

Note that $\underline{F}$ is defined on $\mathbb{R}^2$ (simply connected) and $\operatorname{curl}(\underline{F}) = 0$ so $\underline{F}$ is conservative.

Suppose $\nabla f = \underline{F}$ then

$$\begin{aligned} f &= xz^2 + \alpha(y\ z) \\ &= y^2z^2 + \beta(x\ z) \\ &= z^2(x + y^2) - e^z + \gamma(x\ y) \end{aligned}$$

So we can take $f = z^2(x+y^2) - e^z$

and thus

$$\int_{\mathcal{C}} \underline{F} \cdot d\underline{r} = f(\underline{r}(4\pi)) - f(\underline{r}(0))$$

$$= 5 - e^2$$

3. For this question consider the vector field

$$\mathbf{F}(x,y) = \frac{1}{r^2}\langle y(r^2-1), x(r^2+1)\rangle,$$

where $r = \sqrt{x^2+y^2}$. This vector field is defined everywhere apart from the origin.

(a) (4 points) Is $\mathbf{F}$ conservative on the domain described above? Justify your answer.

We will check $\int_{\mathcal{C}} \underline{F}\cdot d\underline{r}$ where $\mathcal{C}$ is the unit circle with ⟲ orientation.

$\underline{r}(t) = (\cos t, \sin t)$ $t = [0\ 2\pi]$

$\underline{r}'(t) = (-\sin t, \cos t)$

when $(x,y)\in\mathcal{C}$, $r=1$ so $\underline{F} = \langle 0, 2x\rangle = \langle 0, 2\cos t\rangle$

so $\int_{\mathcal{C}} \underline{F}\cdot d\underline{r} = \int_0^{2\pi} \underline{F}\cdot\underline{r}'(t)\,dt = \int_0^{2\pi} 2\cos^2 t\,dt$

$= (t + \cos t \sin t)\Big|_0^{2\pi}$

$= 2\pi \neq 0$

so $\underline{F}$ is <u>not</u> conservative.

(b) (1 point) Give a domain on which $\mathbf{F}$ is conservative.

$x > 0$

(c) (2 points) Calculate the line integral

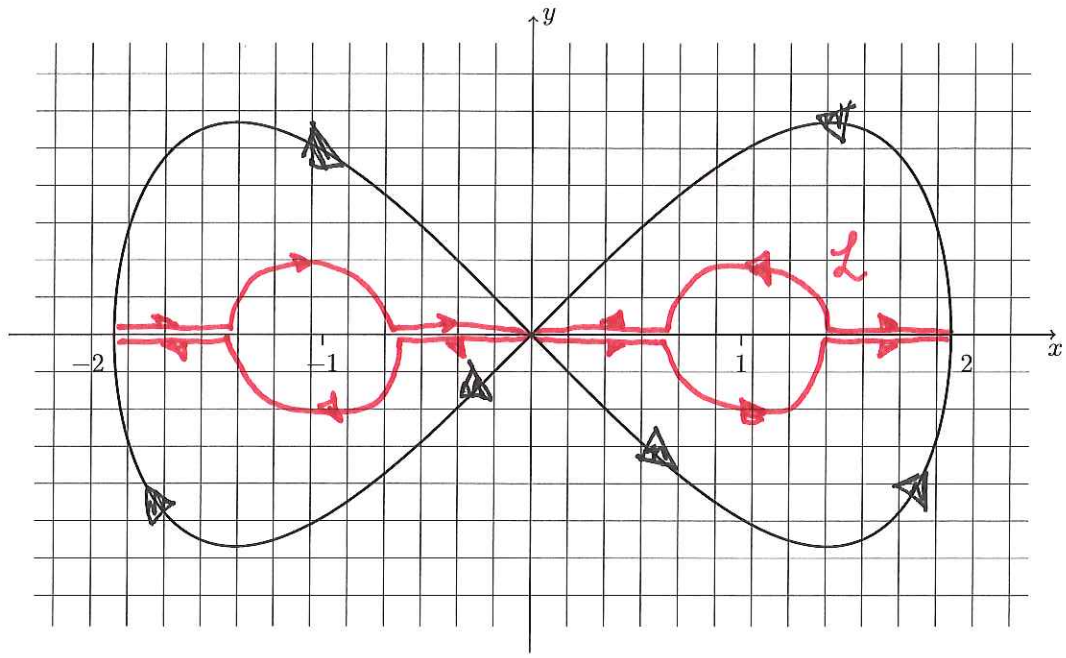
$$\int_C \mathbf{F} \cdot \mathbf{dr}$$

where C is the ellipse $\frac{(x-4)^2}{2} + y^2 = 1$, oriented in the counter clockwise direction.

$\mathcal{C}$ is entirely within $\{x > 0\}$ and since $\underline{F}$ is conservative on this domain

$$\oint_{\mathcal{C}} \underline{F} \cdot d\underline{r} = 0 .$$

4. In this question assume that $\mathbf{E}$ is a vector field defined on the whole plane, apart from the points $(\pm 1, 0)$. The function $\mathbf{r}(t) = (2\cos t, \sin 2t)$ for $t \in [-\frac{\pi}{2}, \frac{3\pi}{2}]$ defines the curve $\mathcal{C}$ on the graph below



(a) (1 point) Indicate on the above graph, the orientation of the curve.

(b) (4 points) Let $\mathcal{A}$ and $\mathcal{B}$ be the circles, radius $\frac{1}{2}$, and center $(1,0)$ and $(-1,0)$ respectively, both oriented counter clockwise. Suppose that

$$\int_{\mathcal{A}} \mathbf{E} \cdot \mathrm{d}\mathbf{r} = 2 \quad \text{and} \quad \int_{\mathcal{B}} \mathbf{E} \cdot \mathrm{d}\mathbf{r} = 1.$$

What is $\int_{\mathcal{C}} \mathbf{E} \cdot \mathrm{d}\mathbf{r}$? Justify your answer.

* $\underline{E}$ is conservative when restricted to any quadrant.
* By path ind. $\int_{\mathcal{C}} \underline{E} \cdot d\underline{r} = \int_{\mathcal{L}} \underline{E} \cdot d\underline{r}$ where $\mathcal{L}$ is shown in red above
* The straight line segments of the path cancel so

$$\int_{\mathcal{L}} \underline{E} \cdot d\underline{r} = \int_{\mathcal{A}} \underline{E} \cdot d\underline{r} + \int_{-\mathcal{B}} \underline{E} \cdot d\underline{r}$$

opposite orientation!!

$$= \int_{\mathcal{A}} \underline{E} \cdot d\underline{r} - \int_{\mathcal{B}} \underline{E}\, d\underline{r}$$

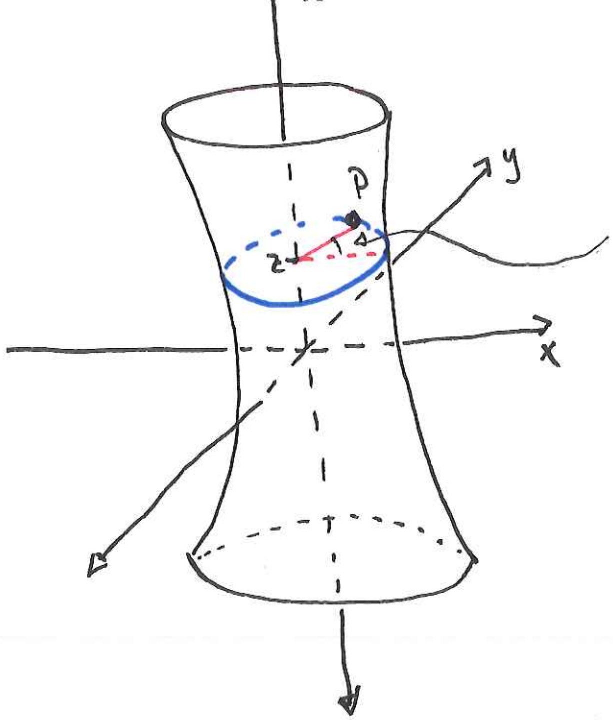
$$= 2 - 1 = 1.$$

5. The *hyperboloid* is Noah's favorite surface. It is given by the equation $x^2 + y^2 - z^2 = 1$.

(a) (3 points) Find a parameterisation

$$G(s,\theta) = (x(s,\theta), y(s,\theta), z(s,\theta)) \quad (s,\theta) \in \mathbb{R} \times [0, 2\pi]$$

for the hyperboloid. *Hint: Let $z = s$.*



when ~~z=P~~ $z = s$, P lies on the circle $x^2+y^2 = s^2+1$ so

θ $P = G(s,\theta) = \left(\sqrt{s^2+1}\cos\theta, \sqrt{s^2+1}\sin\theta, s\right)$

(b) (5 points) Express the surface area of the hyperboloid between $z = a$ and $z = -a$ as an iterated integral.

$$S.A. = \iint_S 1 \cdot dS = \iint_{\mathcal{D}} \|\underline{N}(s\theta)\| \, dA_{s\theta} \qquad \mathcal{D} = [-a, a] \times [0\ 2\pi]$$

$$\underline{T}_s = \left(\frac{s}{\sqrt{s^2+1}}\cos\theta, \frac{s}{\sqrt{s^2+1}}\sin\theta, 1\right)$$

$$\underline{T}_\theta = \left(-\sqrt{s^2+1}\sin\theta, \sqrt{s^2+1}\cos\theta, 0\right)$$

$$\underline{N} = \left(-\sqrt{s^2+1}\cos\theta, -\sqrt{s^2+1}\sin\theta, s\right)$$

$$\|\underline{N}\| = \sqrt{2s^2+1}$$

$$S.A. = \int_{-a}^{a}\int_0^{2\pi} \sqrt{2s^2+1}\, d\theta\, ds$$

(extra working room for part (b))

(c) (3 points) Calculate the surface area. You may use the formula $\int \sqrt{1+x^2}\,dx = \frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)$.

Make the sub $x = \sqrt{2}\,s$, $dx = \sqrt{2}\,ds$

$$S.A. = \int_{-a}^{a}\int_{0}^{2\pi} \sqrt{2s^2+1}\,d\theta\,ds = \frac{2\pi}{\sqrt{2}}\int_{-\sqrt{2}a}^{\sqrt{2}a} \sqrt{x^2+1}\,dx$$

$$= \cancel{2\pi}\sqrt{2}\pi\left(\frac{1}{2}x\sqrt{1+x^2} + \frac{1}{2}\ln\left(x+\sqrt{1+x^2}\right)\right)\Bigg|_{-\sqrt{2}a}^{\sqrt{2}a}$$

$$= \sqrt{2}\pi\left(\frac{\sqrt{2}}{2}a\sqrt{1+2a^2} + \frac{\sqrt{2}}{2}a\sqrt{1+2a^2} + \frac{1}{2}\ln\left(\sqrt{2}a+\sqrt{1+2a^2}\right) - \frac{1}{2}\ln\left(-\sqrt{2}a+\sqrt{1+2a^2}\right)\right)$$

$$= \sqrt{2}\pi\left(\frac{2}{\cancel{\sqrt{2}}}\sqrt{2}\,a\sqrt{1+2a^2} + \frac{1}{2}\ln\left(\frac{\sqrt{1+2a^2}+\sqrt{2}a}{\sqrt{1+2a^2}-\sqrt{2}a}\right)\right)$$

$$= \pi\left[2a\sqrt{1+2a^2} + \frac{1}{\sqrt{2}}\ln\left(\frac{\sqrt{1+2a^2}+\sqrt{2}a}{\sqrt{1+2a^2}-\sqrt{2}a}\right)\right]$$

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