Exponentials

- Let $b > 0$ be a real number. We want to define a function, $f : \mathbb{R} \longrightarrow \mathbb{R}$

$$f(x) = b^x$$

- If x is a whole, positive number then obviously

$$b^x = \underbrace{b \cdot b \cdot \_\_\_ \cdot b}_{x \text{ times}}.$$

- If x is a whole negative ~~nbr~~ ~~ne~~ number then

$$b^x = \frac{1}{b^{-x}}$$

- If $x = p/q$ is a rational number then

$$b^x = \sqrt[q]{b^p}$$

- What if x is irrational? Later we will find a much better definition. For ~~nu~~ now, we will settle for the following: our calculater gives us a number! (for those interested: we can ~~appro~~ approximate x by rational numbers

$$x_1, x_2, \ldots, \xrightarrow{\text{tend to}} x$$

then $b^{x_1}, b^{x_2}, \ldots, \xrightarrow{\text{tends to}} b^x$

— e.g.

$$\frac{3}{1}, \frac{22}{7}, \frac{333}{106}, \ldots \xrightarrow{\text{tends to}} \pi$$

so

$$2^3, \sqrt[7]{2^{22}}, \sqrt[106]{2^{333}}, \ldots \xrightarrow{\text{tends to}} 2^\pi$$

Properties

* $b^x b^y = b^{x+y}$
* $b^{-x} = \frac{1}{b^x}$
* $b^0 = 1$
* $(b^x)^y = b^{xy}$

Derivative Lets investigate the derivative of our new function $f(x) = b^x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{b^{x+h} - b^x}{h} = \lim_{h \to 0} \frac{b^x b^h - b^x}{h}$$

$$= b^x \underbrace{\left(\lim_{h \to 0} \frac{b^h - 1}{h} \right)}$$ call this number m_b.

Note that m_b does not depend on x, so whatever it is,

$$\frac{d}{dx} b^x = m_b b^x$$

Ex Find the derivative of $3^{\sin x - x}$:

Chain rule:

$$\frac{d}{dx} 3^{\sin x - x} = (\cos x - 1)\, m_3 \cdot 3^{\sin x - x}$$

In fact, (we will see this later) m_b is increasing with b, and there is a unique number e such that $m_e = 1$. $e = 2.718\ldots$

$$\boxed{\frac{d}{dx} e^x = e^x}$$

Logarithms

- For a number $b > 0$, the lograrithm with base b of x,

$$\log_b x$$

answers the question:

"to what power do we raise b in order to obtain x?".

- i.e. if $a = \log_b x$ then $b^a = x$

or

if $b^a = x$ then $\log_b x = a$.

Ex $\log_2 4 = 2$ since $2^2 = 4$

$\log_4 2 = \frac{1}{2}$ since $4^{\frac{1}{2}} = 2$

$\log_b 1 = 0$ since $b^0 = 1$ for any b.

- By design $\log_b(x)$ is the inverse function to b^x.

$$b^{\log_b(x)} = x = \log_b(b^x)$$

- Note that b^x is always positive so,

$$\log_b(x)$$

does not make sense if $x \leq 0$

<u>Def</u> The function $\log_b : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ def, by

$$\log_b(x) = \log_b x$$

is the logarithm function.

We give a special name to the log of base e:

$$\log_e x = \ln x.$$

<u>Rmk</u> * Since $b^x b^y = b^{x+y}$

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

* Since $b^{-x} = \frac{1}{b}$

$$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$$

* $\log_b x = \dfrac{\log_a x}{\log_a b}$ in particular $\log_b x = \dfrac{\ln x}{\ln b}$

Thm

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

proof: If $y = \ln x$ then $e^y = x$ diff'ing

$$\frac{d}{dx} e^y = \frac{d}{dx} x$$

$$e^y \frac{dy}{dx} = 1$$

$$x \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{x}$$

Now let us determine m_b from the prev lecture.

If
$$y = b^x$$
take natural log's

see assside below.

$$\ln y = \ln(b^x) = x\ln(b)$$
differentiating
$$\frac{1}{y}\frac{dy}{dx} = \ln(b)$$
$$\frac{dy}{dx} = \ln(b)\, b^x$$
So $m_b = \ln(b)$.

Asside $\log_b(a^x) = x\log_b(a)$ since...

$$\log_b(a^x) = \frac{\log_a(a^x)}{\log_a(b)} \quad \text{(change of base)}$$
$$= \frac{x}{\log_a(b)} \quad \text{(def.)}$$
$$= x\left(\frac{\log_b(b)}{\log_b(a)}\right)^{-1} \quad \text{(change of base)}$$
$$= x\left(\frac{1}{\log_b(a)}\right)^{-1} = x\log_b(a).$$