PARTIAL FRACTIONS AND POLYNOMIAL LONG DIVISION

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The basic aim of this note is to describe how to break rational functions into pieces. For example

$$\frac{x^2-2x+3}{x^2-1} = 1 + \frac{1}{x-1} - \frac{3}{x+1}.$$

The point is that we don't know how to integrate the left hand side, but integrating the right hand side is easy! We will break this problem down into pieces.

In general we have two polynomials $p(x)$ and $q(x)$. Lets say $p(x)$ has *degree* m (this means the largest power of x is x^m) and $q(x)$ has degree n. So that

$$p(x) = p_m x^m + p_{m-1}x^{m-1} + \ldots + p_1 x + p_0$$

and

$$q(x) = q_n x^n + q_{n-1}x^{n-1} + \ldots + q_1 x + q_0$$

for some numbers p_i and q_i. We want to be able to write

$$\frac{p(x)}{q(x)} = d(x) + \frac{A_1}{(a_1x+b_1)^{c_i}} + \frac{A_2}{(a_2x+b_2)^{c_2}} + \ldots + \frac{A_n}{(a_nx+b_n)^{c_n}}$$

for some numbers A_i and c_i and some polynomial $r(x)$. The numbers a_i and b_i come from factorising $q(x)$:

$$q(x) = (a_1x+b_1)(a_2x+b_2)\cdots(a_nx+b_n).$$

How are we going to find $b(x)$ and the A_i's?

Remark 1. Factorising $q(x)$ into linear factors, as above, might not always be possible (though it is if we use complex numbers). There is a way to deal with this however we will sweep this under the rug for now and assume that $q(x)$ can be neatly factored into linear factors.

If the above general explanation doesn't quite make sense, here are some examples of how we would like to rewrite rational functions:

$$\frac{x-3}{x^2+3x-4} = \frac{7}{5(x+4)} - \frac{2}{5(x-1)} \tag{1}$$

$$\frac{6}{x^3-8x^2+19x-12} = \frac{1}{x-1} - \frac{3}{x-3} + \frac{2}{x-4} \tag{2}$$

$$\frac{x^2+x+1}{x^3-3x^2+3x-1} = \frac{1}{x-1} + \frac{3}{(x-1)^2} + \frac{3}{(x-1)^3} \tag{3}$$

$$\frac{x^4-3x^3+12x-9}{x^2-7x+12} = x^2+4x+16+\frac{103}{x-4} - \frac{27}{x-3} \tag{4}$$

$$\frac{x^4+4x^2+x-5}{x^3+2x^2-7x+4} = x-2+\frac{311}{25\,(x+4)} + \frac{64}{25\,(x-1)} + \frac{1}{5\,(x-1)^2} \tag{5}$$

Hopefully, you will agree that the right hand side of each expression look far easier to integrate than the left hand side!

1. Polynomial long division

The first step in achieving out aim is to rewrite $p(x)/q(x)$ in the form

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}$$

where $d(x)$ and $r(x)$ are polynomials and the degree of $r(x)$ is less than the degree of $q(x)$. For example, the first step in examples (4) and (5) above would be

$$\frac{x^4-3x^3+12x-9}{x^2-7x+12} = x^2+4x+16+\frac{76x-201}{x^2-7x+12}$$

$$\frac{x^4+4x^2+x-5}{x^3+2x^2-7x+4} = x-2+\frac{15x^2-17x+3}{x^3+2x^2-7x+4}.$$

This is achieved by *polynomial long division.* We call $d(x)$ the *divisor* and $r(x)$ the *remainder.*

Polynomial long division works exactly like normal long division:

$$\begin{array}{r|lllll}
 & & x^2 & +4x & +16 \\ \hline
x^2-7x+12 & x^4-3x^3 & & +12x & -9 \\
 & -x^4+7x^3 & -12x^2 & & \\ \hline
 & 4x^3 & -12x^2 & +12x & \\
 & -4x^3 & +28x^2 & -48x & \\ \hline
 & & 16x^2 & -36x & -9 \\
 & & -16x^2 & +112x & -192 \\ \hline
 & & & 76x & -201
\end{array}$$

Lets annotate this step by step:

$$
\begin{array}{rrrrrr}
 \cline{2-6}
x^2-7x+12 \big) & x^4 & -3x^3 & & +12x & -9
\end{array}
$$

First we take the leading term of the numerator $x^4 - 3x^3 + 12x - 9$ and divide it by the leading term of the denominator. So x^4 divided by x^2 is x^2.

$$
\begin{array}{rrrrrr}
 & & & x^2 & & \\ \cline{2-6}
x^2-7x+12 \big) & x^4 & -3x^3 & & +12x & -9
\end{array}
$$

Now we multiply the result, x^2 by -1 and the denominator $x^2 - 7x + 12$ and place it underneath,

$$
\begin{array}{rrrrrr}
 & & & x^2 & & \\ \cline{2-6}
x^2-7x+12 \big) & x^4 & -3x^3 & & +12x & -9 \\
 & -x^4 & +7x^3 & -12x^2 & & \\ \cline{2-4}
\end{array}
$$

We add the two polynomials together to get

$$
\begin{array}{rrrrrr}
 & & & x^2 & & \\ \cline{2-6}
x^2-7x+12 \big) & x^4 & -3x^3 & & +12x & -9 \\
 & -x^4 & +7x^3 & -12x^2 & & \\ \cline{2-4}
 & & 4x^3 & -12x^2 & +12x &
\end{array}
$$

and repeat the process. Divide the leading term $4x^3$ by x^2 to obtain $4x$,

$$
\begin{array}{rrrrrr}
 & & & x^2 & +4x & \\ \cline{2-6}
x^2-7x+12 \big) & x^4 & -3x^3 & & +12x & -9 \\
 & -x^4 & +7x^3 & -12x^2 & & \\ \cline{2-4}
 & & 4x^3 & -12x^2 & +12x &
\end{array}
$$

multiply the result by -1 and the denominator and add the resulting polynomial,

$$
\begin{array}{rrrrrr}
 & & & x^2 & +4x & \\ \cline{2-6}
x^2-7x+12 \big) & x^4 & -3x^3 & & +12x & -9 \\
 & -x^4 & +7x^3 & -12x^2 & & \\ \cline{2-4}
 & & 4x^3 & -12x^2 & +12x & \\
 & & -4x^3 & +28x^2 & -48x & \\ \cline{3-5}
 & & & 16x^2 & -36x & -9
\end{array}
$$

Once more we repeat the process,

$$
\begin{array}{rrrrrr}
 & & & x^2 & +4x & +16 \\ \cline{2-6}
x^2-7x+12 \big) & x^4 & -3x^3 & & +12x & -9 \\
 & -x^4 & +7x^3 & -12x^2 & & \\ \cline{2-4}
 & & 4x^3 & -12x^2 & +12x & \\
 & & -4x^3 & +28x^2 & -48x & \\ \cline{3-5}
 & & & 16x^2 & -36x & -9 \\
 & & & -16x^2 & +112x & -192 \\ \cline{4-6}
 & & & & 76x & -201
\end{array}
$$

Now we are at a point were we cannot repeat the process anymore (since $76x$ is not divisible by x^2) so we halt. The divisor is $d(x) = x^2 + 4x + 16$ and the remainder is $r(x) = 76x - 201$.

2. Partial fractions, distinct factors

Now we just need to deal with the case when $p(x)$ has degree less than the degree of $q(x)$ as in examples (1), (2) and (3). First we will deal with the case where

$$q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$$

and the factors are all distinct. That means we allow $q(x) = (x - 1)(x - 2)(x - 3)$ but not $q(x) = (x - 1)^2(x - 3)$. In this case it is always true that we can find constants $A_1, A_2, \dots, A_n$ such that

$$\frac{p(x)}{q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n}.$$

To find these constants we simply multiply out and *compare coefficients*. We illustrate this with an example.

Example 2.

$$\frac{x-3}{x^2+3x-4} = \frac{x-3}{(x+4)(x-1)} = \frac{A}{x+4} + \frac{B}{x-1}$$

If we multiply this equation on both sides by $(x + 4)(x - 1)$ we obtain

$$x - 3 = A(x - 1) + B(x + 4) = (A + B)x - A + 4B.$$

Since we are comparing two polynomials, the coefficients of every power of x must be equal. Explicitly, looking at the coefficient of x gives $A + B = 1$ and looking at the constant term gives $4B - A = -3$. These are simultaneous equations which we can solve to get

$$A = \frac{7}{5} \quad \text{and} \quad B = -\frac{2}{5}.$$

Example 3.

$$\frac{6}{x^3 - 8x^2 + 19x - 12} = \frac{6}{(x-1)(x-3)(x-4)} = \frac{A}{x-1} + \frac{B}{x-3} + \frac{C}{x-4}$$

If we multiply this equation on both sides by $(x - 1)(x - 3)(x - 4)$ we obtain

$$\begin{aligned} 6 &= A(x-3)(x-4) + B(x-1)(x-4) + C(x-1)(x-3) \\ &= (A+B+C)x^2 - (7A+5B+4C)x + 12A + 4B + 3C. \end{aligned}$$

Looking at the coefficient of x^2 gives $A + B + C = 0$, at the coefficient of x gives $7A + 5B + 4C = 0$ and looking at the constant term gives $12A + 4B + 3C = 6$. These are simultaneous equations which we can solve to get

$$A = 1, \quad B = -3 \quad \text{and} \quad C = 2.$$

3. Partial fraction, distinct quadratic factors

Up to now we have covered only denominators which feature unique linear factors. When factorising a polynomial, it is also possible that we may obtain quadratic factors that we cannot factorise further, for example

$$x^3 - x^2 + x - 1 = (x^2 + 1)(x - 1).$$

Here $x^2 + 1$ cannot be broken up further into linear factors. So how do we deal with a fraction of the form

$$\frac{1}{(x^2 + 1)(x - 1)}?$$

Answer: for each quadratic factor $ax^2 + bx + c$ in the denominator, we get a summand in the partial fraction expansion of the form

$$\frac{Ax + B}{ax^2 + bx + c}.$$

Example 4.

$$\frac{1}{(x^2 + 1)(x - 1)} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1}$$

We multiply both sides by $(x^2 + 1)(x - 1)$ to obtain

$$1 = (Ax + B)(x - 1) + C(x^2 + 1).$$

We can substitute $x = 1$ to obtain $1 = 2C$, i.e. $C = \frac{1}{2}$. Putting this back into the equation above and expanding, we get

$$\begin{aligned} 1 &= (Ax + B)(x - 1) + frac12(x^2 + 1) \\ &= (A + \frac{1}{2})x^2 + (B - A)x - B + \frac{1}{2}. \end{aligned}$$

Equating coefficients we obtain $A = B = -\frac{1}{2}$. So

$$\frac{1}{(x^2 + 1)(x - 1)} = \frac{x + 1}{2(x^2 + 1)} + \frac{1}{2(x - 1)}.$$

4. Partial fractions, repeated factors

We are now left to deal with the case when $q(x)$ has repeated factors. For example when $q(x) = (x - 1)^2(x - 3)$ or $q = (x - 1)^3$. In general, for every factor of $q(x)$ of the form $(ax + b)^k$ or $(ax^2 + bx + c)^k$, the partial fraction expansion contains terms of the form

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \frac{A_3}{(ax + b)^3} + \cdots + \frac{A_k}{(ax + b)^k}.$$

or

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \frac{A_3x + B_3}{(ax^2 + bx + c)^3} + \cdots + \frac{A_kx + B_k}{(ax^2 + bx + c)^k}.$$

To find the constants A_i we follow exactly the same process as above. Some examples should make this clear.

Example 5.

$$\frac{x^2 + x + 1}{x^3 - 3x^2 + 3x - 1} = \frac{x^2 + x + 1}{(x - 1)^3} = \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{C}{(x - 1)^3}$$

We multiply both sides by $(x-1)^3$ to obtain

$$\begin{aligned} x^2+x+1 &= A(x-1)^2+B(x-1)+C \\ &= Ax^2+(-2A+B)x+A-B+C. \end{aligned}$$

Comparing coefficient gives us the simultaneous equations

$$\begin{aligned} A &= 1 \\ -2A+B &= 1 \\ A-B+C &= 1. \end{aligned}$$

Solving these gives

$$A=1, \quad B=3 \quad \text{and} \quad C=3.$$

Example 6.

$$\frac{15x^2-17x+3}{x^3+2x^2-7x+4} = \frac{15x^2-17x+3}{(x+4)(x-1)^2} = \frac{A}{x+4}+\frac{B}{x-1}+\frac{C}{(x-1)^2}$$

Multiplying both sides by $(x+4)(x-1)^2$ gives

$$\begin{aligned} 15x^2-17x+3 &= A(x-1)^2+B(x-1)(x+4)+C(x+4) \\ &= (A+B)x^2+(-2A+3B+C)x+A+4B+4C. \end{aligned}$$

Comparing coefficient gives us the simultaneous equations

$$\begin{aligned} A+B &= 15 \\ -2A+3B+C &= -17 \\ A+4B+4C &= 3. \end{aligned}$$

Solving these gives

$$A=\frac{311}{25}, \quad B=\frac{64}{25} \quad \text{and} \quad C=\frac{1}{5}.$$

5. Some useful integrals

Here we list some integrals that are useful when using partial fractions to solve integration questions.

$$\begin{aligned} \int \frac{1}{x+a}\,\mathrm{d}x &= \ln|x+a|+C \\ \int \frac{1}{(x+a)^n}\,\mathrm{d}x &= -\frac{1}{n-1}\frac{1}{(x+a)^{n-1}}+C \quad \text{if } n\neq 1 \\ \int \frac{1}{a^2+x^2}\,\mathrm{d}x &= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+C \end{aligned}$$