Ratio + Root tests

Two more convergence tests!

Thm (Ratio test)

Assume the existance of

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

* If $\rho < 1$ then $\sum a_n$ is absolutely convergent
* If $\rho > 1$ then $\sum a_n$ diverges.

Rmk If $\rho = 1$ we cannot use the ratio test to determine convergence

Ex $\sum \frac{r^n}{n!}$ for any r.

$$\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{r^{n+1}}{(n+1)!} \cdot \frac{n!}{r^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{r}{n+1} \right|$$

$$= 0$$

So $\sum \frac{r^n}{n!}$ converges absolutely.

Thm (Root test)

Assume the existance of

$$L = \lim_{n \to \infty} \sqrt[n]{|a_n|}$$

* If $L < 1$ then $\sum a_n$ converges absolutely
* If $L > 1$ then $\sum a_n$ diverges.

Ex $\sum_{0}^{\infty} \frac{n^{1-3n}}{4^{2n}}$.

$$L = \lim_{n \to \infty} \left(\frac{n^{1-3n}}{4^{2n}} \right)^{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^{\frac{1}{n}-3}}{4^2}$$

$$= \lim_{n \to \infty} \frac{n^{\frac{1}{n}}}{4^2 \cdot n^3}$$

(using L'H $n^{\frac{1}{n}} \to 1$)

$$= 0.$$

so converges absolutely.