This weeks problem set focuses on the concept of a change of basis matrix. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 2: due Friday May 8: questions 2, 5 and 6 below.

1. From section 2.5, problems 1, $2a, c$, $3a, c$, 5, 7, 10^*, 13^*.

2.* Let V be a finite dimensional vector space and W a subspace. Show that V and $W \times V/W$ are isomorphic by finding an explicit isomorphism (rather than simply computing the dimensions).

Solution: Let $B = \{ v_1, \ldots, v_n \}$ be a basis of V so that $\{ v_1, \ldots, v_k \}$ is a basis for W. Now define a map $\phi : V \longrightarrow W \times V/W$ by

$$\phi(v_i) = \begin{cases} (v_i, 0) & \text{if } 0 \leq i \leq k \\ (0, v_i + W) & \text{otherwise.} \end{cases}$$

We claim this is an isomorphism. Indeed, for every element $(w, v + W) \in W \times V/W$ we can express this as

$$(w, v + W) = \lambda_1(v_1, 0) + \ldots + \lambda_k(v_k, 0) + \mu_{k+1}(0, v_{k+1}) + \ldots + \mu_n(0, v_n + W)$$

where $w = \lambda_1 v_1 + \ldots + \lambda_k v_k$ and $v = \mu_1 v_1 + \ldots + \mu_n v_n$. So ϕ is surjective.
Now if

$$\lambda_1(v_1, 0) + \ldots + \lambda_k(v_k, 0) + \lambda_{k+1}(0, v_{k+1}) + \ldots + \lambda_n(0, v_n + W) = 0$$

then $\lambda_1 v_1 + \ldots + \lambda_k v_k = 0$ and $\lambda_{k+1} v_{k+1} + \ldots + \lambda_n v_n \in W$. Thus $\lambda_i = 0$ for all i and ϕ is injective and thus an isormorphism.

3.* Let V be a finite dimensional vector space and W a subspace. Show that $\dim(V/W) = \dim V - \dim W$. *Hint: consider a basis of W and extend it to V. Now find a basis for V/W. You can also prove it using the dimension theorem.*

Solution: Let $B = \{ v_1, \ldots, v_n \}$ be a basis of V so that $\{ v_1, \ldots, v_k \}$ is a basis for W. Then $\{ v_{k+1} + W, \ldots, v_n + W \}$ is a basis for V/W. Hence

$$\dim(V/W) = n - k = \dim V - \dim W.$$

4.* Let $T : V \longrightarrow W$ be a linear map.

(a) Show that $\operatorname{im} T$ and $V/\ker T$ are isomorphic.

Solution: Define a map $\phi : V/\ker T \longrightarrow \operatorname{im} T$ by $\phi(v + \ker T) = T(v)$. We must first check that this is well defined. I.e. if $v + \ker T = v' + \ker T$ then we should check that $\phi(v + \ker T) = \phi(v' + \ker T)$. This translates to checking that $T(v) = T(v')$ if $v - v' \in \ker T$. In this situation, $T(v - v') = 0$ so $T(v) - T(v') = 0$ by linearity, so ϕ is well defined.
To check this is an isomorphism, note first of all that it is surjective. Now suppose that $\phi(v+\ker T) = \phi(v'+\ker T)$. This means $T(v-v') = 0$ so $v-v' \in \ker T$, i.e $v+\ker T = v'+\ker T$ so ϕ is injective and thus an isomorphism.

(b) Use this (and the previous exercise) to give an alternative proof of the dimension theorem.

> **Solution:** Note first that $\dim(V/\ker T) = \dim V - \dim\ker T$. Thus, using the previous part, we see that $\dim V - \dim\ker T = \dim\operatorname{im} T$ which is the dimension theorem.

5. A differential operator on $\mathbb{R}_n[x]$ is a linear combination of expressions of the form $x^a \frac{d^b}{dx^b}$ where $a - b \leq 0$ (otherwise the degree would potentially increase!) and $b \leq n$. We can consider a differential operator as a linear map $\mathbb{R}_n[x] \longrightarrow \mathbb{R}_n[x]$.

(a) Let $D : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ be the differential operator given by $2 - 4\frac{d}{dx} + 2x\frac{d^2}{dx^2}$. Find the matrix of D relative to the basis $\{x^2, (x-1)^2, (x+1)^2\}$. *Note: the 2 in D means multiply by 2, so $D(1) = 2$ and $D(x) = 2x - 4$.*

> **Solution:**
> $$\begin{pmatrix} 2 & -8 & 8 \\ 1 & 7 & -3 \\ -1 & 3 & -3 \end{pmatrix}$$

(b) Does the differential equation $2f - 4\frac{df}{dx} + 2x\frac{df^2}{dx^2} = 0$ have any solutions $f \in \mathbb{R}_2[x]$? *Hint: what is a solution in terms of the linear map D?*

(c) Suppose $E : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ is a differential operator and that the matrix of E, relative to the basis $\{1, x, x^2\}$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find E.

> **Solution:**
> $$E = \frac{d}{dx} - \frac{1}{2}x\frac{d^2}{dx^2}$$

6. Consider the linear map $X : \mathbb{R}_n[x] \longrightarrow \mathbb{R}_n[x]$ given by $X(p) = \frac{dp}{dx} + \frac{x^n}{n!}p(0)$. Calculate the dimension of

$$C(X) = \{\, T \in \operatorname{Hom}(\mathbb{R}_n[x], \mathbb{R}_n[x]) \,|\, T \circ X = X \circ T \,\}.$$

Hint: this will be quite tricky without involving matrices. It is also a very good idea to try $n = 1, 2, 3$ before moving on to the general statement.

> **Solution:** First we pick a useful choice of ordered basis $u_k = \frac{x^k}{k!}$ for $0 \leq k \leq n$. This means
> $$X(u_k) = \begin{cases} u_{k-1} & \text{if } k \neq 0 \\ u_n & \text{if } k = 0. \end{cases}$$
> This lets us define the matrix $Y = [X]_B^B = (y_{ij})$. We can see from the above formula that
> $$y_{ij} = \begin{cases} 1 & \text{if } j = i+1 \\ 1 & \text{if } i = n+1, j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

That is,

$$Y = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Now suppose that $T \in C(X)$, i.e. that $T \circ X = X \circ T$. If $A = [T]_B^B$ then this is equivalent to $AY = YA$. Let $A = (a_{ij})$. We have, using the formulas above,

$$(AY)_{ij} = \sum_{k=1}^{n+1} a_{ik} y_{kj} = \begin{cases} a_{i,n+1} & \text{if } j = 1, \\ a_{i,j-i} & \text{if } j > 1, \end{cases},$$

$$(YA)_{ij} = \sum_{k=1}^{n+1} y_{ik} a_{kj} = \begin{cases} a_{i+1,j} & \text{if } i < n+1, \\ a_{1j} & \text{if } i = n+1. \end{cases}.$$

Equating these we get four cases. Case one: $i < n+1, j > 1$

$$a_{i,j-1} = a_{i+1,j}$$

Case two: $i < n+1, j = 1$

$$a_{i,n+1} = a_{i+1,1}$$

Case three: $i = n+1, j > 1$

$$a_{n+1,j-1} = a_{1j}$$

Case four: $i = n+1, j = 1$

$$a_{n+1,n+1} = a_{11}.$$

This means that our matrix looks something like this:

$$A = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{n+1} \\ c_{n+1} & c_1 & c_2 & \cdots & c_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_3 & c_4 & c_5 & \cdots & c_2 \\ c_2 & c_3 & c_4 & \cdots & c_1 \end{pmatrix}.$$

So the rows of A are determined from the top row by cyclically permuting one step every row we go down. Thus A is determined by it's top row. We have an isomorphism

$$\Phi_B^B : C(X) \longrightarrow C(Y) = \{ A \in \mathrm{Mat}_{n+1 \times n+1}(\mathbb{R}) \,|\, AY = YX \circ T \}$$

so we simply compute the dimension of $C(Y)$. Let $U_k = (u_{ij})$ be the matrix where

$$u_{ij} = \begin{cases} 1 & \text{if } j - i = k \text{ or } i - j = n + 1 - k \\ 0 & \text{otherwise.} \end{cases}$$

i.e. the first row of U_k has a 1 in the position u_{1k} and zeros everywhere else, and the other rows are determined from the first by cyclically permuting. It is then clear that $\{U_k | 0 \leq k \leq n\}$ is linearly independent and spanning in $C(Y)$. Thus $\dim C(X) = \dim C(Y) = n + 1$.