Math 3B: Lecture 18

Noah White

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Linear models

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A first order ODE is linear if it is of the form

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Examples

$$\frac{\mathrm{d}y}{\mathrm{d}t} = ay, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\lambda y.$$

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- concentration of a drug in bloodstream
- pollutant in water supply

General solution

Using separation of variables, we can show that the general solution to

$$\frac{\mathrm{d}y}{\mathrm{d}t} = a - by$$

is

$$y(t) = \frac{a}{b} - Ce^{-bt}$$

where C is an arbitrary constant.

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- Thus the rate at which the drug is leaving (at time t) is given by

$$0.5\ln(2)Me^{-0.5t\ln(2)} = 0.5\ln(2)(\text{current concentration}) \ \text{ mg/h.}$$

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- Thus at time t the concentration is

$$y(t) = 28.9 - 28.9e^{-0.3t} = 28.9\left(1 - e^{-0.3t}\right)$$