Midterm 2

UCLA: Math 170A, Fall 2017

Instructor: Noah White
Date: 20 November 2017
Version: 1

- This exam has 4 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- All final answers should be exact values. Decimal approximations will not be given credit.
- Indicate your final answer clearly.
- Full points will only be awarded for solutions with correct working.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
Total:	20	

Question 1 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

Here are some formulas you might find helpful at various points in the exam.

- $\displaystyle\sum_{k=1}^{n} k = \frac{1}{2}n(n+1).$
- $\displaystyle\sum_{k=0}^{\infty} a^k = \frac{1}{1-a}.$
- $\displaystyle\sum_{k=0}^{\infty} \frac{a^k}{k!} = e^a.$
- If $X \sim \text{Poisson}(\lambda)$ then $p_X(k) = e^{-\lambda}\dfrac{\lambda^k}{k!}$.

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (1 point) Let X be a geometric random variable with parameter 0.5. Then

   A. $p_X(k) = (0.5)^{k-1}$

   B. $p_X(k) = (0.5)^k$**.**

   C. $p_X(k) = e^{0.5}\frac{(0.5)^k}{k!}$.

   D. $p_X(k) = 0.5$.

   (b) (1 point) Three fair, six sided dice are rolled. Let X and Y be the sum of the fist two and second two dice respecively. The joint PMF $p_{X,Y}(3, 10)$ is equal to

   A. 1/36.

   B. $1/6^3$.

   C. 1/2.

   D. 0.

   (c) (1 point) If $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(2)$ are independent and satisfy $\text{var}(X - Y) = 5$, then

   A. $\lambda = 1$.

   B. $\lambda = 3$**.**

   C. $\lambda = 7$.

   D. $\lambda = 5$.

(d) (1 point) Two random variables X and Y both have range $0, 1, 2$. Their joint PMF is given by

$$p_{X,Y}(x, y) = \frac{1}{3}y(x-1)e^x$$

Calculate $\mathbb{E}X$.

A. 1
B. e
C. $2e^2$
D. $\frac{1}{3}e$

(e) (1 point) The two random variables in part (d) ...(*Hint: what is* $p_{X,Y}(2, 2)$)

A. are both Poisson random variables.
B. are independent.
C. are not independent.
D. have the same PMF.

2. Suppose that we have n children at a daycare center, they are numbered 1 through n. There are n toys, also labelled 1 through n that are distributed, one to each child, randomly. Any child is equally likely to receive any toy.

   Child k will be happy with toys 1 through k but unhappy with toys $k+1$ through n. I.e. The first child will only be happy if they receive toy 1, whereas the second child will be happy only with toys 1 or 2.

   (a) (2 points) What is the probability that child k will be happy? *Hint: Don't overthink it! It should be a reasonably simple counting exercise.*

   Solution: How many ways are there to assign the toys overall? Any permutation will do so $n!$. How many ways are there to give out toys so the kth child will be happy? Fist there are k ways to give a toy to that child and then $(n-1)!$ ways to give toys to the remaining children. So $k(n-1)!$. Thus the probability is

$$\frac{k(n-1)!}{n!} = \frac{k}{n}.$$

   (b) (3 points) Let S be the number of children that are happy. What is the expected value of S? Simplify your answer (no summations should be in the final answer).

   Solution: Let X_k be the random variable which takes the value 1 if the kth child is happy and 0 otherwise. Then

$$X \sim \text{Bernoulli}(\frac{k}{n}) \text{ and } S = X_1 + X_2 + \ldots + X_n.$$

   Thus $\mathbb{E}X_k = k/n$. Since expectation is linear

$$\begin{aligned}\mathbb{E}S &= \mathbb{E}X_1 + \mathbb{E}X_2 + \ldots + \mathbb{E}X_n \\ &= \sum_{k=1}^{n} \frac{k}{n} = \frac{1}{2}(n+1)\end{aligned}$$

   Where we have used the formula on page 2.

3. Suppose that a tree drops X seeds. Assume that X is a Poisson random variable with parameter 2. Each seed becomes a new tree with probability 0.5.

   (a) (1 point) What is the expected number of seeds dropped?

   Solution: Since $X \sim \text{Poisson}(2)$, we have $\mathbb{E}X = 2$.

   (b) (3 points) Find the PMF of the random variable S that is the number of new trees. Simplify your answer (no summations should be in the final answer).

   Solution: We want to compute $p_S(k)$ for $k = 0, 1, 2, \ldots$. We first compute $p_{S|X}(k|n)$. Note that if we condition against $X = n$, then S is a binomial random variable with parameters n and 0.5. Thus
$$p_{S|X}(k|n) = \binom{n}{k}\frac{1}{2^n}$$
if $n \geq k$ and $p_{S|X}(k|n) = 0$ otherwise. We also know that $p_X(n) = e^{-2}\frac{2^n}{n!}$. Thus using the formula for total probability
$$\begin{aligned}
p_S(k) &= \sum_{n=0}^{\infty} p_X(n) p_{S|X}(k|n) \\
&= \sum_{n=k}^{\infty} e^{-2}\frac{2^n}{n!}\binom{n}{k}\frac{1}{2^n} \\
&= e^{-2}\sum_{n=k}^{\infty}\binom{n}{k}\frac{1}{n!} \\
&= e^{-2}\sum_{n=k}^{\infty}\frac{n!}{(n-k)!k!}\frac{1}{n!} \\
&= e^{-2}\frac{1}{k!}\sum_{n=k}^{\infty}\frac{1}{(n-k)!} \\
&= e^{-2}\frac{1}{k!}e = e^{-1}\frac{1}{k!}.
\end{aligned}$$
In particular S is a Poisson random variable with parameter 1.

   (c) (1 point) What is the expected number of new trees? *Hint: what kind of random variable is S?*

   Solution: Since S is a Poisson random variable with parameter 1, $\mathbb{E}S = 1$.

4. A factory has a machine that prints circuit boards. Each circuit board has a probability of p of jamming the machine. Once the machine jams, it destoys the circuit board that jammed it and stops working.

   Each circuit board contains a circuit with 6 components. In order to be working, at least 5 of these compoents must be functional. Each component is functional with probability a, independently.

   (a) (1 point) What is the expected number of circuit boards printed? (working or not, not including the destoyed board)? *Hint: be careful, the number of circuit boards is not quite a geometric random variable, but it is close.*

   Solution: Let P be the number of circuit boards printed. Then since one is destroyed, $P+1$ is a geometric random variable with parameter p. Thus $\mathbb{E}P + 1 = \frac{1}{p}$ so

   $$\mathbb{E}P = \frac{1}{p} - 1.$$

   (b) (4 points) What is the expected number of working circuit boards? Simplify your answer (no summations should be in the final answer).

   Solution: First lets figure out the probability that any given board works. We have 6 independant components, each working with probability a and we want 5 or 6 to work, so the probability is

   $$u := \binom{6}{5} a^5(1-a) + \binom{6}{6} a^6 = 6a^5 - 5a^6.$$

   Now lets figure out the conditional expectation $\mathbb{E}(W|P=n)$, where W is the number of working boards. If X_i is the random variable that is 1 if the ith printed board works and 0 otherwise then $X_i \sim \text{Bernoulli}(u)$. If we condition against $P = n$ then $W \sim X_1 + \ldots + X_n$ and W is a binomial rv with parameters n and u. thus

   $$\mathbb{E}(W|P=n) = nu.$$

   Using the formula for total expectation we get

   $$\begin{aligned}
   \mathbb{E}W &= \sum_{n=0}^{\infty} p_P(n)\mathbb{E}(W|P=n) \\
   &= \sum_{n=0}^{\infty} unp_P(n) \\
   &= u\mathbb{E}P = u\left(\frac{1}{p} - 1\right) = \left(6a^5 - 5a^6\right)\left(\frac{1}{p} - 1\right).
   \end{aligned}$$

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