Infinite positive series

For now we will only deal with series where $\sum a_n$, s.t. $a_n > 0$ for all n.

This means $S_{N+1} > S_N$

i.e (S_N) is a monotonially increasing seq:

Thm If $a_n > 0$ then either

* (S_N) is bounded above and $\sum a_n$ converges
* (S_N) is unbounded and $\sum a_n$ diverges.

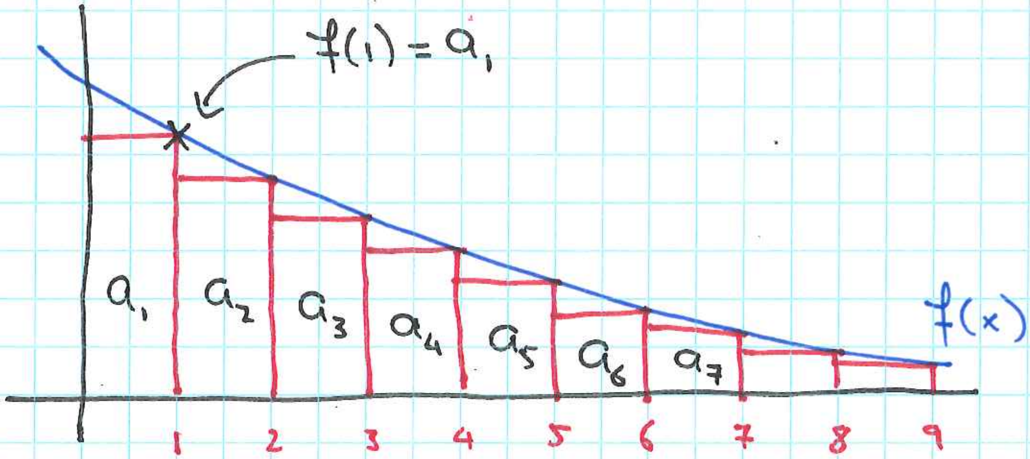
How can we tell whether a series converges / diverges?

Thm (integral test)

If $a_n = f(n)$ for $f(x) \geq 0$ and decreasing (i.e. $f'(x) < 0$). Then

* If $\int_a^\infty f(x)dx$ converges, so does $\sum_{n \geq a} a_n$
* If $\int_a^\infty f(x)dx$ diverges, so does $\sum_{n \geq a} a_n$

proof:



Ex (Harmonic series again)

$$\int_1^{\infty} \frac{1}{x}\,dx \text{ diverges} \implies \sum_{n\geq 1} \frac{1}{n} \text{ diverges}$$

If $p>1$:

$$\int_1^{\infty} \frac{1}{x^p}\,dx \text{ converges} \implies \sum_{n\geq 1} \frac{1}{n^p} \text{ converges}$$

If $0<p\leq 1$:

$$\int_1^{\infty} \frac{1}{x^p}\,dx \text{ diverges} \implies \sum_{n\geq 1} \frac{1}{n^p} \text{ diverges.}$$

Ex Does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\ln n}$ converge?

Consider $f(x)=\dfrac{1}{x\ln x}$.

1. $f(n)=\dfrac{1}{n\ln n}$

2. $f(x) > 0$

3. $f'(x) = -\frac{1}{(1+\ln x)^2} < 0$

Now $\int_2^\infty \frac{1}{x \ln x}\, dx = \lim_{R\to\infty} \int_2^R \frac{1}{x \ln x}\, dx$

$= \lim_{R\to\infty} \ln \ln x \Big|_2^R$

$= \lim_{R\to\infty} \ln \ln R - \ln \ln 2$

$= \infty$

$\therefore \sum \frac{1}{n \ln n}$ diverges.

Thm (Direct comparison test)

Suppose we want to know if $\sum a_n$ converges:

* If we can find (b_n) so that

  $b_n \geq a_n \geq 0$ and $\sum b_n$ converges

  then $\sum a_n$ converges.

* If we can find (b_n) so that

  $a_n \geq b_n \geq 0$ and $\sum b_n$ diverges

  then $\sum a_n$ diverges.

Ex $\sum_{n \geq 0} \frac{1}{\sqrt{n^3+3n+2}}$

We have
$$n^3+3n+2 \geq n^3$$
$$\sqrt{n^3+3n+2} \geq \sqrt{n^3}$$
$$0 \leq \frac{1}{\sqrt{n^3+3n+2}} \leq \frac{1}{\sqrt{n^3}}$$

but $\sum_{n \geq 1} \frac{1}{\sqrt{n^3}}$ converges, $\therefore$ so does $\sum_{n \geq 0} \frac{1}{\sqrt{n^3+3n+2}}$.

Thm (limit comparison test)

Suppose (a_n) and (b_n) are sequences and
$$\lim_{n \to \infty} \frac{a_n}{b_n} = L$$

* If $L > 0$ then
  $\sum a_n$ converges if and only if $\sum b_n$ converges

* If $L = \infty$ and $\sum a_n$ converges then $\sum b_n$ does

* If $L = 0$ and $\sum b_n$ converges, so does $\sum a_n$.

Ex Suppose we want to know about

$$\sum \frac{1}{\sqrt[5]{n^4+6n+3}} = \sum a_n$$

we can compare this with the simpler series

$$\sum \frac{1}{\sqrt[5]{n^4}} = \sum b_n$$

Now

$$\lim_{n\to\infty} \frac{a_n}{b_n} =$$

$$= \lim_{n\to\infty} \frac{\sqrt[5]{n^4+6n+3}}{\sqrt[5]{n^4}} \cdot \frac{\sqrt[5]{\frac{1}{n^4}}}{\sqrt[5]{\frac{1}{n^4}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[5]{1+\frac{6}{n^3}+\frac{3}{n^4}}}{1}$$

$$= 1 > 0$$

and since $\sum \frac{1}{\sqrt[5]{n^4}} = \sum \frac{1}{n^{4/5}}$ diverges

so does $\sum \frac{1}{\sqrt[5]{n^4+6n+3}}$.