This weeks problem set provides practice with diagonalisable operators and the basic properties of inner products. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 4: is due Friday June 5: questions $2, 4, 5$ below.

1. From section 6.2, problems 1, $2b, g, i, k$, 5^*, 6, 7, 9, 13^*, 17^*, 22.

2. Let V be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

   (a) Fix $y \in V$ and suppose $\langle x, y\rangle = 0$ for all $x \in V$. Show that $y = 0$.

   Solution: Let $B = \{v_1, \ldots, v_n\}$ be an orthonormal basis for V. Then

   $$y = \sum_{i=1}^{n} \langle y, v_i\rangle v_i = 0$$

   since $\langle y, v_i\rangle = 0$ for all i.

   (b) Let $T : V \longrightarrow V$ be a linear map such that $\langle T(x), T(y)\rangle = \langle x, y\rangle$ for all pairs $x, y \in V$ (we call such a map a *metric* map). Prove that T is an isomorphism.

   Solution: First we show that T is injective. Suppose that $T(x) = T(y)$. One one hand we have

   $$||T(x)|| = \langle T(x), T(x)\rangle = \langle x, x\rangle.$$

   On the other hand,

   $$||T(x)|| = \langle T(x), T(x)\rangle = \langle T(x), T(y)\rangle = \langle x, y\rangle.$$

   Thus $\langle x, x\rangle = \langle x, y\rangle$, i.e. $\langle 0, x - y\rangle = 0$. Thus by the above, $x - y = 0$ or $x = y$.
   Now, since T is an injective map from V to V, it must be surjective by the dimension theorem. Thus it is an isomorphism.

   (c) † Find all metric maps $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that have $\det T = 1$.

   Solution: The map T will be given by a matrix

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

   From the fact that it is an isometry we see that $|||T(e_i)||| = |||e_i||| = 1$ for $i = 1, 2$. Thus

   $$\left\| \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\| \right| = \left\| \left| \begin{pmatrix} a \\ c \end{pmatrix} \right\| \right| = a^2 + c^2 = 1$$

   and similarly $b^2 + d^2 = 1$. This means, both columns of the matrix are points on the unit circle. I.e. for some choice of $\theta, \psi \in [0, 2\pi)$ then we have that

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\psi \\ \sin\theta & \sin\psi \end{pmatrix}$$

   Additionally we have that $\langle T(e_1), T(e_2)\rangle = \langle e_1, e_2\rangle = 0$. I.e the two columns of the matrix are at right angles to each other, so $\psi = \theta \pm \pi/2$ (modulo 2π). Alternatively this can be seen since

   $$0 = \langle T(e_1), T(e_2)\rangle = \langle \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \rangle = \cos\theta\cos\psi + \sin\theta\sin\phi = \cos(\theta - \psi)$$

Which means that $\theta - \psi = \pm\frac{\pi}{2} + 2n\pi$ for $n \in \mathbb{Z}$.
The condition that the determinant is 1 is that $ad - bc = 1$ which translates to

$$1 = \cos\theta \sin\psi - \cos\psi \sin\theta = \sin(\theta - \psi)$$

hence $\theta - \psi = \pi/2$. Thus

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \cos\theta + \pi/2 \\ \sin\theta & \sin\theta + \pi/2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some choice of $\theta \in [0, 2\pi)$.

3. (22 from 6.2) Let $V = \mathcal{C}([0,1],\mathbb{R})$ be the space of real valued, continuous functions on the interval $[0,1]$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)\ dt$. Let W be the subspace spanned by the linearly independent set $\{t, \sqrt{t}\}$.
   (a) Find an orthonormal basis for W.
   (b) Let $h(t) = t^2$. Use the orthonormal basis obtained in (a) to obtain the "best" (closest) approximation of h in W.

Solution: *Solution to 22 from 6.2.* The question asks us to consider the vector space $\mathcal{C}([0,1],\mathbb{R})$ of continuous functions on $[0,1]$ into $\mathbb{R}$ with inner product, $\langle f, g\rangle = \int_0^1 f(t)g(t)\ dt$, and to use the Gram Schmidt process to find an orthonormal basis for the subspace $\mathrm{span}\{t, \sqrt{t}\}$.

Part a). We use the Gram-Schmidt process to define first an orthogonal basis $\{f_1, f_2\}$. Set $f_1 = t$. Then

$$f_2 = \sqrt{t} - \frac{\langle\sqrt{t}, t\rangle}{||t||^2}t.$$

To get an explicit expression we do some calculations.

$$||t||^2 = \int_0^1 t^2\ dt = \frac{1}{3}.$$

$$\langle\sqrt{t}, t\rangle = \int_0^1 t^{3/2}\ dt = \frac{2}{5}.$$

$$||\sqrt{t}||^2 = \int_0^1 t\ dt = \frac{1}{2}.$$

Putting this together we get

$$f_2 = \sqrt{t} - \frac{6}{5}t.$$

To get an orthonormal basis, we need to normalise, so we need to calculate

$$\begin{aligned} ||f_2||^2 &= \int_0^1 \left(\sqrt{t} - \frac{6}{5}t\right)^2\ dt \\ &= \int_0^1 t - \frac{12}{5}t^{3/2} + \frac{36}{25}t^2\ dt \\ &= \frac{1}{2} - \frac{24}{25} + \frac{36}{75} = \frac{1}{50} \end{aligned}$$

We already know that $||f_1|| = \frac{1}{\sqrt{3}}$ and now we also know $||f_2|| = \frac{1}{5\sqrt{2}}$ thus, an orthonormal basis is

$$\{g_1 = \sqrt{3}t, g_2 = \sqrt{2}\left(5\sqrt{t} - 6t\right)\}.$$

Part b). We want to project t^2 onto W. The result will be

$$\langle t^2, g_1\rangle g_1 + \langle t^2, g_2\rangle g_2.$$

We calculate the coefficients.

$$\begin{aligned}
\langle t^2, g_1\rangle &= \int_0^1 \sqrt{3}t^3\, dt = \frac{\sqrt{3}}{4}.\\
\langle t^2, g_2\rangle &= \int_0^1 \sqrt{2}\left(5t^{5/2} - 6t^3\right)\, dt\\
&= \sqrt{2}\left(\frac{10}{7} - \frac{3}{2}\right)\\
&= -\frac{\sqrt{2}}{14}.
\end{aligned}$$

Thus, the best approximation is

$$\frac{3}{4}t - \frac{1}{7}\left(5\sqrt{t} - 6t\right) = \frac{45}{28}t - \frac{5}{7}\sqrt{t}.$$

4. Let V be a real inner product space and let $r : V \longrightarrow V^*$ be the map $r(x) = \varphi_x := \langle -, x\rangle$. In class we showed that if V is finite dimensional then r is an isomorphism.

(a) Assume that V is infinite dimensional. Prove that r is injective.

Solution: The exact same proof works. We will show that the kernel is $\{0\}$. Suppose $r(x) = 0$, then $\varphi_x(y) = 0$ for all $y \in V$ so $\langle x, x\rangle = \varphi_x(x) = 0$ and hence $x = 0$.

(b) Let $V = \mathbb{R}[x]$ and let $W = \{(a_0, a_1, \ldots) \mid a_i \in \mathbb{R}\ \}$ be the vector space of all infinite sequences. Show that the map $f : V^* \longrightarrow W$ given by $f(\varphi) = (\varphi(x^n))_{n\geq 0}$ is an isomorphism.

Solution: First we show the map is injective. Suppose $f(\varphi) = 0$. Thus $\varphi(x^n) = 0$ for all n. But $1, x, \ldots$ is a basis of V so this means $\varphi(p) = 0$ for all $p \in V$. Hence $\varphi = 0$.
Now to see that f is surjective, let $(a_n) \in W$. Define $\varphi \in V^*$ by $\varphi(x^n) = a_n$. Then

$$\varphi(b_0 + b_1x + \cdots + b_nx^n) = b_0a_0 + b_1a_a + \cdots b_na_n$$

and so it linear. Clearly $f(\varphi) = (a_n)$. Hence f is surjective and thus an isormophism.

(c) We can define the following inner product on $\mathbb{R}[x]$

$$\langle x^i, x^j\rangle = \begin{cases} 1 & \text{if } i = j\\ 0 & \text{if } i \neq j,\end{cases}$$

and extending linearly (so $\langle 1 + x, 2 - x^2\rangle = 2$ for example). Use this to demonstrate that r is not necessarily surjective, i.e. find an element $\varphi \in V^*$ such that $\varphi \neq r(p)$ for any $p \in \mathbb{R}[x]$.

Solution: Note that if $p = a_0 + a_1 + \cdots + a_n x^n \in V$ then $\langle x^i, p\rangle = a_i$ where $a_i = 0$ if $i > n$. I.e. $\langle x^i, p\rangle$ is the coefficient of x^i in p.
Consider $\eta = f^{-1}(1, 1, \ldots)$. I.e. $\eta(p) = p(1)$. If r was surjective, there would be a polynomial $p \in V$ such that $\eta = \varphi_p = \langle -, p\rangle$. This means the coefficient of x^i in p is equal to $\eta(x^i) = 1$ for all $i \geq 0$. So $p = 1 + x + x^2 + \cdots$, which is nonsense since this is not a polynomial. Thus r is not surjective.

5. Let V be a finite dimensional inner product space. For any $T : V \longrightarrow V$ define $\check{T} : V^* \longrightarrow V^*$ by $\check{T}(\phi) = \phi \circ T$. Furthermore for any $X : V^* \longrightarrow V^*$ define $X^\perp : V \longrightarrow V$ by $X^\perp = r^{-1} \circ X \circ r$. Prove that $T^* = \check{T}^\perp$.

Solution:

$$
\begin{aligned}
\check{T}^\perp(x) &= r^{-1} \circ \check{T} \circ r(x) \\
&= r^{-1} \circ \check{T}(\phi_x) \\
&= r^{-1}(\phi_x \circ T) \\
&= r^{-1}\left(\langle T(-), x\rangle\right) \\
&= r^{-1}\left(\langle -, T^*(x)\rangle\right) \\
&= T^*(x).
\end{aligned}
$$