This week on the problem set you will get practice at integration by parts, polynomial long division, using the partial fractions method and applying these to integrals. There are lots and lots of questions! You don't need to do all of them, only enough to convince yourself you are comfortable with them!

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (5.8-20) Analysts speculate that patients will enter a new clinic at a rate of $300 + 100\sin\frac{\pi t}{6}$ individuals per month. Moreover, the likelihood an individual is in the clinic $t$ months later is $e^{-t}$. Find the number of patients in the clinic one year from now.

2. (5.8-21) A patient receives a continuous drug infusion at a rate of 10 mg/h. Studies have shown that $t$ hours after injection, the fraction of drug remaining in a patient's body is $e^{-2t}$. If the patient initially has 5 mg of drug in her bloodstream, then what is the amount of drug in the patient's bloodstream 24 hours later?

**Solution:** First we can look at the 5 mg of the drug that is already in the patients bloodstream. After 24 hours, the question states that there will be

$$5e^{-48} \text{ mg}$$

left.

Now we estimate the amount of drug in the bloodstream due to the continuous infusion by dividing the time period $t = 0$ to $t = 24$ into $n$ evenly sized pieces. Let $\Delta t = \frac{24}{n}$ be the length of each of these subintervals. Let $t_k$ be the starting time of the $k^{\text{th}}$ interval, so that $t_k = \frac{24k}{n}$.
If we concentrate on the $k^{\text{th}}$ interval, then during this time, $10\Delta t$ mg of drug will have been injected into the patient. If we assume that the interval is small enough (i.e. $n$ is large) then at time $t = 24$, this amount of the drug will have been in the patients bloodstream for $24 - t_k$ hours. Using the formula from the question, this means there will be

$$10\Delta t e^{-2(24-t_k)} \text{ mg}$$

of the drug left in the system at $t = 24$. Adding all these contributions together and letting $n \to \infty$, we get that the total amount of the drug in the patients bloodstream at $t = 24$ is

$$D = 5e^{-48} + \lim_{n\to\infty}\sum_{k=1}^{n} 10\Delta t e^{-2(24-t_k)} = 5e^{-48} + \lim_{n\to\infty}\Delta t\sum_{k=1}^{n} 10e^{-2(24-t_k)}$$

We recognise this as a Riemann sum and thus

$$D = 5e^{-48} + \int_0^{24} 10e^{-2(24-t)}\,\mathrm{d}t = 5e^{-48} + \left[5e^{-2(24-t)}\right]_0^{24} = 5$$

3. (5.8-24) The administrators of a town estimate that the fraction of people who will still be residing in the town $t$ years from now is given by the function $S(t) = e^{-0.04t}$. The current population is $20,000$ people and new people are arriving at a rate of 500 per year.
   (a) What will be the population size 10 years from now?
   (b) What will be the population size 100 years from now?

**Solution:** Here is the solution for (a), the solution for (b) is entirely similar.

First we can look at the 20,000 people currently living in the town. After 10 years, the question states that there will be

$$20,000e^{-0.4}$$

left.

Now we estimate the amount of people that remain in the town, of those that arrive during the time period $t = 0$ to $t = 10$. First we divide this interval into $n$ evenly sized pieces. Let $\Delta t = \frac{10}{n}$ be the length of each of these subintervals. Let $t_k$ be the endpoint of the $k^{\text{th}}$ interval, so that $t_k = \frac{10k}{n}$.

If we concentrate on the $k^{\text{th}}$ interval, then during this time, $500\Delta t$ people arrive in the town. If we assume that the interval is small enough (i.e. $n$ is large) then at time $t = 10$, this amount of people will have been living in the town for $10 - t_k$ years. Using the formula from the question, this means there will be

$$500\Delta t e^{-0.04(10-t_k)}$$

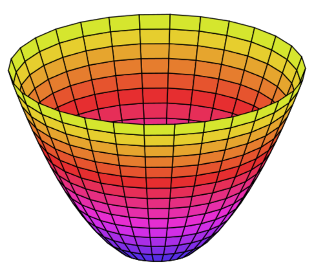
of these people left at $t = 10$. Adding all these contributions together and letting $n \to \infty$, we get that the total amount of the drug in the patients bloodstream at $t = 10$ is

$$D = 20,000e^{-0.4} + \lim_{n\to\infty} \sum_{k=1}^{n} 500\Delta t e^{-0.04(10-t_k)} = 20,000e^{-0.4} + \lim_{n\to\infty} \Delta t \sum_{k=1}^{n} 500e^{-0.04(10-t_k)}$$

We recognise this as a Riemann sum and thus

$$D = 20,000e^{-0.4} + \int_0^{10} 500e^{-0.04(10-t)}\, \mathrm{d}t = 20,000e^{-0.4} + \Big[12,500e^{-0.04(10-t)}\Big]_0^{10}$$

4. If we rotate the graph of $y = x^2$ around the $y$-axis, we obtain a 3D shape that looks like a bowl (see the picture below). If we take the part of this bowl that exists below $y = h$, how much volume does it contain?



*Thanks to Wikipedia for the picture!*

**Solution:** First we estimate the volume by dividing the bowl shape into $n$ horizontal slices of even thickness $\Delta y = \frac{h}{n}$. If we concentrate on the $k^{\text{th}}$ slice we can calculate first, its volume. If $n$ is large then we may assume this slice is approximately a cylinder.

The base of this slice is at $y = y_k = k\frac{h}{n}$. As pictured in the extra diagrams provided, we can take a cross section through the middle of the bowl. If $r$ is the radius of the slice, then the point where the base of the slice meets the parabola has coordinates $(r, y_k)$. Since this point lies on the parabola,

$r = \sqrt{y_k}$. Thus the volume of the $k^{\text{th}}$ slice is

$$\pi\left(\sqrt{y_k}\right)^2 \Delta y = \pi y_k \Delta y.$$

To get the accurate volume we add these contributions up for each slice and take the limit as $n \to \infty$:

$$V = \lim_{n\to\infty} \Delta y \sum_{k=1}^{n} \pi y_k,$$

which we recognise as a Riemann sum and can therefor interpret as the integral

$$V = \int_0^h \pi y \, \mathrm{d}y = \left[\frac{1}{2}\pi y^2\right]_0^h = \frac{1}{2}\pi h^2.$$

5. (5.6-28) A bucket weighing 75 lb when filled and 10 lb when empty is pulled up the side of a 100 ft building. How much more work in foot-pounds is done in pulling up the full bucket than the empty bucket?

   *Hint: this doesn't involve a Riemann sum, it is just applying the definition of "work".*

6. (5.6-29) A 20-ft rope weighing 0.4 lb/ft hangs over the edge of a building 100 ft high. How much work is done in pulling the rope to the top of the building? Assume that the top of the rope is flush with the top of the building, and the lower end of the rope is swinging freely.

   *Warning: the answer in the textbook is given in slightly strange units! See the text for an explanation of the units they are using. You don't need to know this though.*

**Solution:** First we divide the rope into $n$ subintervals. Let $y$ be a variable denoting the number feet below the top of the building. So $y = 0$ is the top of the building, $y = 20$ is the bottom of the rope, and $y = 100$ is the ground. Let $\Delta y$ be the length of each little subinterval of rope (so $\Delta y = 20/n$).

The top of the $k^{\text{th}}$ subinterval lies at $y = y_k = k\Delta y$. So this subinterval needs to be pulled up $y_k$ ft. The subinterval of rope weighs $0.4\Delta y$ lb. In foot-pounds, the work needed to lift an object is its weight in pounds, multiplied by how far we need to lift it in feet. Thus to lift the $k^{\text{th}}$ subinterval of rope we need

$$0.4 y_k \Delta y$$

foot-pounds of work.

The total amount of work is then given by adding each of these contributions up and taking the limit as $n \to \infty$.

$$\lim_{n\to\infty} \sum_{k=1}^{n} 0.4 y_k \Delta y.$$

We recognise this as a Riemann sum and the corresponding integral is

$$\int_0^{20} 0.4y \, \mathrm{d}y = \left[0.2y^2\right]_0^{20} = 80.$$

7. A helicopter rescues a sailor from a shipwreck out at sea. The rescue is achieved by lowering a rope (of weight 0.1 kg/m) to the sailor and then pulling the sailor up. If the Helicopter is hovering at 100 m above the water and the sailor weighs 75 kg, how much work is done pulling the sailor to safety?

**Solution:** First we account for the sailor's mass. We assume the acceleration due to gravity is 10. Thus the force needed to lift the sailor is $F = 10 \cdot 75$. We need to lift the sailor 100 metres so the worked needed is $100 \cdot 10 \cdot 75 = 75000$ Joules.

To account for the rope we divide it into $n$ subintervals. Let $y$ be a variable denoting the number meters below the helicopter. So $y = 0$ is the height of the helicopter, $y = 100$ is the bottom of the rope, where the sailor is. Let $\Delta y$ be the length of each little subinterval of rope (so $\Delta y = 100/n$).

The top of the $k^{\text{th}}$ subinterval lies at $y = y_k = k\Delta y$. So this subinterval needs to be pulled up $y_k$ m. The subinterval of rope weighs $0.1\Delta y$ kg. The force needed to lift this mass is $10 \cdot 0.1\Delta y = \Delta y$ Newtons. Thus to lift the $k^{\text{th}}$ subinterval of rope we need

$$y_k \Delta y$$

Joules of work.

The total amount of work is then given by adding each of these contributions up and taking the limit as $n \to \infty$, and adding the contribution coming from the sailor.

$$75000 + \lim_{n\to\infty} \sum_{k=1}^{n} y_k \Delta y.$$

We recognise this as a Riemann sum and the corresponding integral is

$$75000 + \int_0^{100} y \, \mathrm{d}y = 75000 + \left[0.5y^2\right]_0^{100} = 80000.$$

8. Various radioactive materials are used in medical diagnostic techniques. A company which produces these radioactive materials would like to store its waste materials in a special storage facility which can hold up to 100 kg of radioactive materials. The radioactive waste has a half-life of 5 months. This means, if we start with $M$ kg of waste, after $t$ months, only

$$Me^{-0.2t\ln(2)} \text{ kg}$$

will remain.

(a) If the company adds waste to the storage facility at a rate of $a$ kg per month, write a function for the amount of material in the storage facility $x$ months after the company opens the storage facility.

**Solution:** Let $t = 0$ be the time when the storage facility is opened. First we approximate the problem by dividing the time period into $n$ pieces. Thus each time period has length $\frac{x}{n}$ months. If these time periods are small enough (i.e. if $n$ is large enough) then we can assume all of the waste is added at the start of each interval. The $k^{\text{th}}$ interval, has a start time of $t_k = k\frac{x}{n}$. With this assumption, $a\frac{x}{n}$ kg of waste is added at time $t = t_k$ for each $k = 1, 2, \ldots, n$. The amount of time remaining is $x - t_k$ so the amount of surviving material is

$$a\frac{x}{n}e^{-0.2(x-t_k)\ln(2)} \text{ kg}.$$

Adding up the contributions from each time interval gives

$$\sum_{k=1}^{n} a\frac{x}{n}e^{-0.2(x-t_k)\ln(2)}.$$

Taking the limit as $n \to \infty$ gives us an expression $W(x)$ for the amount of waste at time $x$,

$$W(x) = \lim_{n\to\infty} \frac{x}{n} \sum_{k=1}^{n} ae^{-0.2(x-t_k)\ln(2)}$$

which we can reinterpret as a Riemann sum and thus an integral,

$$\begin{aligned} W(x) &= \int_0^x ae^{-0.2(x-t)\ln(2)}\,\mathrm{d}t \\ &= \left[\frac{5a}{\ln(2)} e^{-0.2(x-t)\ln(2)}\right]_0^x \\ &= \frac{5a}{\ln(2)} - \frac{5a}{\ln(2)} e^{-0.2x\ln(2)} \\ &= \frac{5a}{\ln(2)}\left(1 - e^{-0.2x\ln(2)}\right). \end{aligned}$$

(b) What is the maximum rate at which the company can add waste, so they never exceed the capacity of the storage facility?

**Solution:** As $x$ grows, the function $e^{-0.2x\ln(2)}$ is decreasing and thus $W(x)$ is increasing (it's derivative is never negative). Furthermore $\lim_{x\to\infty} W(x) = 5a/\ln(2)$. Thus we will never have more than $5a/\ln(2)$ kg of waste in the storage facility but the amount of waste will approach this number. Thus in order for the company ever to exceed 100 kg of waste, we need that

$$\frac{5a}{\ln(2)} \leq 100.$$

Rearranging, we get that

$$a \leq 20\ln(2) \approx 13.86.$$

So the maximum rate the company can add waste to the storage facility is 13.86 kg per month.

9. (5.8-32) Determine the length of a rectangular trench you can dig with the energy gained from eating one Milky Way bar (270 Cal). Assume that you convert the energy gained from the food with 10% efficiency and that the trench is 1 meter wide and 1 meter deep. Assume the density of soil is $1,000\ \mathrm{kg/m^3}$.

10. Use any method to evaluate the following integrals.
   (a) $\int x\sqrt{x+1}\,\mathrm{d}x$
   (b) $\int_1^2 \frac{t}{(t^2+1)\ln(t^2+1)}\,\mathrm{d}t$
   (c) $\int \frac{\ln x}{x^5}\,\mathrm{d}x$
   (d) $\int \ln x\,\mathrm{d}x$
   (e) $\int (\ln x)^2\,\mathrm{d}x$
   (f) $\int_1^e (\ln x)^3\,\mathrm{d}x$
   (g) $\int e^{6x}\sin(e^{3x})\,\mathrm{d}x$
   (h) $\int_0^1 \frac{t^3e^{t^2}}{(t^2+1)^2}\,\mathrm{d}t$
   (i) $\int \frac{2z+3}{z^2-9}\,\mathrm{d}z$
   (j) $\int \frac{x^2+x-1}{(x^2-1)}\,\mathrm{d}x$ (*Hint: see partial fraction on PS6*)

(k) $\int \frac{e^x}{(e^x-1)(e^x+3)}\,\mathrm{d}x$ (*Hint: see partial fraction on PS6*)

(l) $\int_0^{\pi/3} e^t \sin t\,\mathrm{d}t$

**Solution:** We use integration by parts with $u = \sin t$ and $dv = e^t dt$. Then $du = \cos t dt$ and $v = e^t$. Thus

$$I = \int e^t \sin t\,\mathrm{d}t = e^t \sin t - \int e^t \cos t dt.$$

We need to use integration by parts again, this time with $p = \cos t$ and $dq = e^t dt$, so $dp = -\sin t dt$ and $q = e^t$, thus

$$I = e^t \sin t - \left(e^t \cos t + \int e^t \sin t dt\right) = e^t(\sin t - \cos t) - I$$

Thus

$$I = \frac{1}{2}e^t(\sin t - \cos t)$$

is an antiderivative.

Now we need to evaluate this at the end points so we see that

$$\begin{aligned}
\int_0^{\pi/3} e^t \sin t\,\mathrm{d}t &= \left[\frac{1}{2}e^t(\sin t - \cos t)\right]_0^{\pi/3} \\
&= \left(\frac{1}{2}e^{\pi/3}(\sin \pi/3 - \cos \pi/3)\right) - \left(\frac{1}{2}e^0(\sin 0 - \cos 0)\right) \\
&= \left(\frac{1}{2}e^{\pi/3}(\frac{\sqrt{3}}{2} - \frac{1}{2})\right) - \left(\frac{1}{2}(0-1)\right) \\
&= \frac{1}{4}e^{\pi/3}(\sqrt{3}-1) + \frac{1}{2}.
\end{aligned}$$

(m) $\int e^{\sqrt{x}}\,\mathrm{d}x$

**Solution:** We will first use a substitution $t = \sqrt{x}$. Thus $t' = \frac{1}{2}x^{-\frac{1}{2}}$. So

$$\begin{aligned}
\int e^{\sqrt{x}}\,\mathrm{d}x &= \int e^{\sqrt{x}} \cdot 2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\,\mathrm{d}x \\
&= 2\int te^t\,\mathrm{d}t.
\end{aligned}$$

Now we can use integration by parts. Setting $u = t$ and $v' = e^t$, then $u' = 1$ and $v = e^t$ so

$$\begin{aligned}
\int e^{\sqrt{x}}\,\mathrm{d}x &= 2te^t - 2\int e^t \\
&= 2te^t - 2e^t + C \\
&= 2(t-1)e^t + C \\
&= 2(\sqrt{x}-1)e^{\sqrt{x}} + C.
\end{aligned}$$

(n) $\int \frac{1}{\cos x}\,\mathrm{d}x$ (quite challenging)

(o) $\int (\sin x)^2\,\mathrm{d}x$ (quite challenging)