This week you will get practice drawing phase lines and bifurcation diagrams as well as interpreting their results.

Homework: The homework will be due on Friday 2 December, at 2pm, the *start* of the lecture. It will consist of question 1(e) and 9.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.5) Draw phase lines, classify the equilibria, and sketch a solution satisfying the specified initial value for the equations in the following.
   (a) (6.5-2) $\frac{\mathrm{d}y}{\mathrm{d}t} = 2 - 3y$, $y(0) = 2$
   (b) (6.5-5) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-10)(20-y)$, $y(0) = 9$
   (c) (6.5-6) $\frac{\mathrm{d}y}{\mathrm{d}t} = y(y-5)(25-y)$, $y(0) = 7$
   (d) (6.5-7) $\frac{\mathrm{d}y}{\mathrm{d}t} = \sin y$, $y(0) = 0.1$
   (e) (6.5-10) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^3 - 4y$, $y(0) = 0.1$

2. (6.5-33) To account for the effect of a generalist predator (with a type II functional response) on a population, ecologists often write differential equations of the form

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0.1N\left(1 - \frac{N}{1,000}\right) - \frac{10N}{1+N}$$

   (a) Sketch the phase line for this system.
   (b) Discuss how the fate of the population depends on its initial abundance.

3. (6.5-39) Consider a population of clonally reproducing individuals consisting of two genotypes, a and A, with per capita growth rates, r_a and r_A, respectively. If N_a and N_A denote the densities of genotypes a and A, then

$$\frac{\mathrm{d}N_a}{\mathrm{d}t} = r_a N_a \qquad \frac{\mathrm{d}N_A}{\mathrm{d}t} = r_A N_A$$

   Also, let $y = \frac{N_a}{N_a + N_A}$ be the fraction of individuals in the population that are genotype a. Show that y satisfies

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (r_a - r_A)y(1-y)$$

4. (6.5-40) In the Hawk-Dove replicator equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{y}{2}(1-y)(C(1-y) - V)$$

   if the value $V > 0$ is specified, then find the range of values of C (in terms of V) that will ensure a polymorphism exists (i.e., find conditions that ensure the existence of an equilibrium $0 < y^* < 1$ that is stable).

   (Hint: you do not need to know anything about the Hawk-Dove Replicator - though it is very interesting! - all you need to know is that V is a constant and C is a parameter. A polymorphism is a stable equilibrum between zero and one.)

5. (6.5-41) Production of pigments or other protein products of a cell may depend on the activation of a gene. Suppose a gene is *autocatalytic* and produces a protein whose presence activates greater production of that protein. Let y denote the amount of the protein (say, micrograms) in the cell. A basic model for the rate of this self-activation as a function of y is

$$A(y) = \frac{ay^b}{k^b + y^b} \text{ micrograms/minute}$$

where a represents the maximal rate of protein production, $k > 0$ is a half saturation constant, and $b \geq 1$ corresponds to the number of protein molecules required to active the gene. On the other hand, proteins in the cell are likely to degrade at a rate proportional to y, say cy. Putting these two components together, we get the following differential equation model of the protein concentration dynamics:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay^b}{k^b + y^b} - cy$$

   (a) Verify that $\lim_{y\to\infty} A(y) = a$ and $A(k) = a/2$.

   (b) Verify that $y = 0$ is an equilibrium for this model and determine under what conditions it is stable.

   (Hint: the definition of autocatalytic is given in the question, it is a gene that produces a protein whose presence activate greater production of that protein.)

6. (6.5-42) Consider the model of an autocatalytic gene in Problem 41 with $b = 1$, $k > 0$, $a > 0$, and $c > 0$.

   (a) Sketch the phase line for this model when $ck > a$.

   (b) Sketch the phase line for this model when $ck < a$.

7. (6.6) Sketch the bifurcation diagrams for the equations in the following.

   (a) (6.6-7) $\frac{\mathrm{d}y}{\mathrm{d}t} = ay - y^2$

   (b) (6.6-10) $\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - ay^2$

   (c) (6.6-11) $\frac{\mathrm{d}y}{\mathrm{d}t} = \sin y + a$

   (d) (6.6-12) $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 - ay + 2$ for $a > 0$.

8. (6.6) Consider the model

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{ay^b}{k^b + y^b} - cy$$

   of an autocatalytic gene from question 5. In each of the following cases, two of the parameters are specified. Sketch a bifurcation diagram with respect to the third parameter.

   (a) (6.6-13) $k = 1$, $c = 2$ with a as the bifurcation parameter.

   (b) (6.6-14) $k = 2$, $c = 1$ with a as the bifurcation parameter.

   (c) (6.6-15) $a = 10$, $k = 1$ with c as the bifurcation parameter.

   (d) (6.6-16) $a = 10$, $c = 1$ with k^2 as the bifurcation parameter.

9. (6.6-40) Suppose the growth rate of a whale population at density N (individuals per million square kilometers of ocean), harvested at a rate h, is given by

$$\frac{\mathrm{d}N}{\mathrm{d}t} = 0.07N\left(\frac{N}{10} - 1\right)\left(1 - \frac{N}{80}\right) - h$$

   where the units of t are years.

   (a) Sketch a bifurcation diagram with respect to the parameter h as it varies over the interval $[0, 8]$.

   (b) If $h = vN$, then sketch a bifurcation diagram with respect to the parameter v as it varies over the interval $[0, 0.12]$.