

This week's problem set provides practice with diagonalisable operators and the basic properties of inner products. A question marked with a  $\dagger$  is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a  $*$  is especially important.

**Homework 4:** is due Friday June 5: questions 2, 4, 5 below.

1. From section 6.2, problems 1, 2b, g, i, k, 5\*, 6, 7, 9, 13\*, 17\*, 22.
2. Let  $V$  be a finite dimensional inner product space over  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$ .
  - (a) Fix  $y \in V$  and suppose  $\langle x, y \rangle = 0$  for all  $x \in V$ . Show that  $y = 0$ .
  - (b) Let  $T : V \rightarrow V$  be a linear map such that  $\langle T(x), T(y) \rangle = \langle x, y \rangle$  for all pairs  $x, y \in V$  (we call such a map a *metric* map). Prove that  $T$  is an isomorphism.
  - (c)  $\dagger$  Find all metric maps  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  that have  $\det T = 1$ .
3. (22 from 6.2) Let  $V = \mathcal{C}([0, 1], \mathbb{R})$  be the space of real valued, continuous functions on the interval  $[0, 1]$  with the inner product  $\langle f, g \rangle = \int_0^1 f(t)g(t) dt$ . Let  $W$  be the subspace spanned by the linearly independent set  $\{t, \sqrt{t}\}$ .
  - (a) Find an orthonormal basis for  $W$ .
  - (b) Let  $h(t) = t^2$ . Use the orthonormal basis obtained in (a) to obtain the “best” (closest) approximation of  $h$  in  $W$ .
4. Let  $V$  be an inner product space and let  $r : V \rightarrow V^*$  be the map  $r(x) = \varphi_x := \langle x, - \rangle$ . In class we showed that if  $V$  is finite dimensional then  $r$  is an isomorphism.
  - (a) Assume that  $V$  is infinite dimensional. Prove that  $r$  is injective.
  - (b) Let  $V = \mathbb{R}[x]$  and let  $W = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{R}\}$  be the vector space of all infinite sequences. Show that the map  $f : V^* \rightarrow W$  given by  $f(\varphi) = (\varphi(x^n))_{n \geq 0}$ .
  - (c) Use this to demonstrate that  $r$  is not necessarily surjective, i.e. find an element  $\varphi \in V^*$  such that  $\varphi \neq r(p)$  for any  $p \in \mathbb{R}[x]$ .
5. Let  $V$  be a finite dimensional inner product space. For any  $T : V \rightarrow V$  define  $\check{T} : V^* \rightarrow V^*$  by  $\check{T}(\phi) = \phi \circ T$ . Furthermore for any  $X : V^* \rightarrow V^*$  define  $X^\perp : V \rightarrow V$  by  $X^\perp = r^{-1} \circ X \circ r$ . Prove that  $T^* = \check{T}^\perp$ .