THE MONODROMY OF REAL BETHE VECTORS

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ABSTRACT. This note gives a bijection between the dual equivalence cylindrical grown diagrams that label the fibre of Speyer's cover of $\overline{M}_{0,n+1}$ and the highest weight vectors in certain crystals. This bijection is shown to be equivarient for the action of the cactus group J_n.

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1. INTRODUCTION

1.1. **Gaudin Hamiltonians.** The Hamiltonians for the Gaudin model are a set of n commuting operators depending on distinct complex parameters $z_1, z_2, \ldots, z_n$ acting on a tensor product of irreducible representations of $\mathfrak{gl}_r$. The problem considered in this paper is to describe the *Galois* or *monodromy group* of these operators. We make this more exact below.

Let $\mathfrak{gl}_r$ be the lie algebra of $r \times r$ matrices and e_{ij} the matrix with a 1 in the (i,j)-entry and 0 everywhere else. For $z = (z_1, z_2, \ldots, z_n)$ a set of distinct complex parameters the *Gaudin Hamiltonians* are

$$H_a(z) = \sum_{b \neq a} \frac{\Omega_{ab}}{z_a - z_b} \quad \textit{where} \quad \Omega_{ab} = \sum_{i,j} e_{ij}^{(a)} e_{ji}^{(b)}, \tag{1.1}$$

for $a = 1, 2, \ldots, n$. We consider these either as operators in $U(\mathfrak{gl}_r)^{\otimes n}$ or as operators acting on $L(\lambda_\bullet) = L(\lambda_1) \otimes L(\lambda_2) \otimes \cdots \otimes L(\lambda)_n$ for an n-tuple of partitions $\lambda_\bullet = (\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)})$ with at most r rows. For an operator $X \in U(\mathfrak{gl}_r)$,

$$X^{(a)} = 1 \otimes \cdots \otimes 1 \otimes X \otimes 1 \otimes \cdots \otimes 1,$$

where X is placed in the a^{th} factor.

The main problem in the study of these operators is to produce a complete family of simultaneous eigenvectors (whenever the operators are diagonalisable) ... . We consider the problem of understanding how these eigenvalues change as we vary the parameter z.

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It is well known (see ...) these operators commute with the action of $\mathfrak{gl}_r$ on $L(\lambda_\bullet)$. In particular this implies the Hamiltonians preserve weight spaces and singular vectors, hence it is enough to understand the action of the $H_a(z)$ on the space $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$ for some partition μ. Let $G(\lambda_\bullet; z)_\mu$ be the commutative subalgebra of $\mathrm{End}(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$ generated by the operators (1.1).

When the partitions $\lambda^{(s)}$ are all simply (1) (i.e. $L(\lambda^{(s)})$ is simply the vector representation) the Hamiltonians (1.1) always form a maximal commutative subalgebra, with simple spectrum when they are diagonalisable. This is not always true for more general $\lambda_\bullet$.

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1.2. **The main result.** In [FFR94] a maximal commutative subalgebra $B(\lambda_\bullet; z)_\mu$ containing $G(\lambda_\bullet; z)_\mu$ is constructed. This algebra is called *the Bethe algebra*. The affine group $\mathrm{Aff}_1 \simeq \mathbb{C} \times \mathbb{C}$ on $\mathbb{C}^n$ is the group which acts by simultaneous scaling and translation on each coordinate, i.e for two scalars $\alpha, \beta \in \mathbb{C}$

$$(\alpha, \beta) \cdot z = (\alpha z_1 + \beta, \alpha z_2 + \beta, \dots, \alpha z_n + \beta).$$

Since the denominators in (1.1) are all of the form $z_a - z_b$, the algebras are invariant under this action, $G(\lambda_\bullet; z)_\mu = G(\lambda_\bullet; \alpha z + \beta)_\mu$. It is also known $B(\lambda_\bullet; z)_\mu = B(\lambda_\bullet; \alpha z + \beta)_\mu$ (see [CFR10, Proposition 1]).

If $X_n = \{z \in \mathbb{C}^n \mid z_a \neq z_b \text{ for } a \neq b\}$, our parameter space becomes X_n / Aff_1 which we identify with $M_{0,n+1}(\mathbb{C})$, the moduli space of *irreducible* genus 0 curves with $n+1$ marked points. We obtain in this way a family of algebras $B(\lambda_\bullet)_\mu$ over $M_{0,n+1}(\mathbb{C})$. We denote the spectrum of this family by

$$\pi : \mathcal{B}(\lambda_\bullet)_\mu \stackrel{\text{def}}{=} \mathrm{Spec}\, B(\lambda_\bullet)_\mu \longrightarrow M_{0,n+1}(\mathbb{C}).$$

This morphism π is finite (i.e. it is a finite ramified covering space) and is our main object of study. Our aim will be to say something about the Galois theory of this map. In fact there is an action of the Symmetric group S_n on $\mathcal{B}(\lambda_\bullet)_\mu$ so that π is equivariant with respect to the action of S_n on $M_{0,n+1}$ by permuting the first n marked points. We will keep track of this action throughout.

In [HK06] it was shown the *cactus group* J_n acts on the crystal $\mathsf{B}(\lambda_\bullet) = \mathsf{B}(\lambda^{(1)}) \otimes \mathsf{B}(\lambda^{(2)}) \otimes \cdots \otimes \mathsf{B}(\lambda^{(n)})$. In fact this action preserves weight spaces and singular vectors so restricts to an action on $\mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$. The following is the main theorem of the paper. It was conjectured in [Ryb14] (attributed to Etingof). The statement was also conjectured independently by Gordon and Brochier which is where the author first learnt of the statement.

Theorem 1.2. *For generic $z \in M_{0,n+1}$ there exists a map $PJ_n \longrightarrow \mathrm{Gal}(\pi)$ from the pure cactus group to the Galois group of π such that there exists a naturally defined bijection*

$$\pi^{-1}(z) = \mathcal{B}(\lambda_\bullet)_\mu(z) \xrightarrow{\sim} \mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}},$$

equivariant for the induced action of PJ_n on $\pi^{-1}(z)$.

think of a better section title

1.3. **Connections to other work.** The reason such a theorem is interesting is that it relates Kazhdan-Lusztig cells to the galois theory of Calogero-Moser space. Bonnafe and Rouquier in [BR13] conjecture and provide evidence for a close link between the geometry of Calogero-Moser space and the Kazhdan-Lusztig theory of the associated Coxeter group. In particular it is conjectured the Kazhdan-Lusztig cells are produced as the orbits of a Galois group action.

When $r = n$, $\lambda^{(s)} = (1) = \square$ and $\mu = (1, 1, \dots, 1)$ we can identify $\mathtt{B}(\lambda_\bullet)^{\mathsf{sing}}_{\mu} = [\mathtt{B}^{\otimes n}]^{\mathsf{sing}}_{(1,1\dots,1)}$ with the symmetric group S_n. The orbits of PJ_n on S_n are exactly the (left) Kazhdan-Lusztig cells. In type A, the family $\mathcal{B}(\square^n)_{(1,1,\dots,1)}$ is a slice of Calogero-Moser space. In this context Theorem 1.2 provides some evidence that the Kazhdan-Lusztig cells can in fact be recovered from the Galois theory of Calogero-Moser space.

fix this terminology to something more concrete

1.4. Outline.

1.5. Acknowledgements.

2. Preliminaries

2.1. Notation. We collect here some notation used throughout the paper. The set $\{1, 2, \dots, n\}$ for some positive integer n will be denoted $[n]$. To avoid confusion capital letters from the start of the alphabet, $A, B, C, \dots$, will denote three element sets of integers. Capital letters Q, R, S, T will be used to denote tableaux.

The set of partitions will be denoted $\mathtt{Part}$. For $\lambda \in \mathtt{Part}$ we denote the set of semistandard and standard λ-tableaux by $\mathtt{SSYT}(\lambda)$ or $\mathtt{SYT}(\lambda)$ respectively. We will use the *Schützenberger involution* many times. It will play differing roles depending on whether we consider it as a involution of semistandard tableaux or standard tableaux. To make this distinction more obvious we will use the notation ξ only for semistandard tableaux and the notation $\mathtt{evac}$ when applying the involution to standard tableaux.

We will use $\square$ to denote the partition (1) and $\Lambda = \Lambda_{r,d}$ the rectangular partition with r rows and $d - r$ columns. We will often be concerned with sequences of partitions, when we want to denote the sequence of n partitions all equal to $\square$ we will write $(\square^n)$.

2.2. Equivariant fundamental groups and monodromy. We briefly recall some facts about equivariant fundamental groups and the monodromy groups they produce. We follow the definition originally given by Rhodes in [Rho66]. It is possible to formulate the definitions in many languages, some of which apply in much broader generality (i.e the language of orbifolds or stacks) but since we only deal with a reasonably simple situation we will stick with the more explicit language.

Definition 2.1. Let X be a topological space and G a discrete group acting on X. Let $b \in X$ be a basepoint. The *equivarient fundamental group* $\pi_1^G(X, b)$ has elements (α, g) where $g \in G$ and α is a (homotopy class of a) path from b to $g \cdot b$. The group structure is defined by

$$(\alpha, g) \cdot (\beta, h) = (\alpha \cdot g(\beta), gh).$$

Here we use the usual composition of paths and $g(\beta)$ to denote the g-translate of the path β.

From the definition we can see the set of elements of the form $(\alpha, 1)$ forms a subgroup of $\pi_1^G(X,b)$ isomorphic to $\pi_1(X,b)$. In fact if we consider the projection $(\alpha, g) \mapsto g$ we obtain a surjective group homomorphism from $\pi_1^G(X,b)$ to G. The kernel of this homomorphism is the above described subgroup so we have an exact sequence

$$1 \longrightarrow \pi_1(X,b) \longrightarrow \pi_1^G(X,b) \longrightarrow G \longrightarrow 1.$$

Definition 2.2. If $f : S \longrightarrow B$ is a G-equivarient topological covering we define an action of the group $\pi_1^G(X,b)$ on the fibre $f^{-1}(b)$. If $p \in f^{-1}(b)$ and $(\alpha, g) \in \pi_1^G(X,b)$ then denote by $\tilde{\alpha}$ the unique lift of α to S such that $\tilde{\alpha}(0) = p$. Since f is G-equivarient $\tilde{\alpha}(1) \in f^{-1}(g \cdot b)$. Define $(\alpha, g) \cdot p = g^{-1} \cdot \tilde{\alpha}(1)$. We call this the *$G$-equivarient monodromy action* of $\pi_1^G(B,b)$ on $f^{-1}(b)$. The image of $\pi_1^G(B,b)$ in $S_{f^{-1}(b)}$, the symmetric group on the fibre is the *equivariant monodromy group* and is denoted $M_B^G(f;b)$.

2.3. **Tilling of $\overline{M}_{0,k}(\mathbb{R})$ by associahedra.** In this section we will descibe a *CW*-structure on the set of real points $\overline{M}_{0,n+1}(\mathbb{R})$. This has been investigaved in [Dev99], [Kap93], and [DJS03]. The notation and results we use here comes mainly from [Dev99].

2.3.1. *The associahedron.* The associahedron is a CW-complex which forms the basic unit of the CW-complex $\overline{M}_{0,k}(\mathbb{R})$. It can in fact we realised as a lattice polytope however we will not need this fact.

Definition 2.3. An i-partial bracketing of the set $\{1, 2, \ldots, n\}$ is an arrangement of these numbers with i pairs of balanced brackets, we require that we have no trivial brackets, i.e. all pairs of backets must contain at least two subunits (a letter, or another pair of balanced brackets). Two partial bracketings are equivalent if we can obtain one from another by reversing the contents of one or more pairs of brackets.

So 13(42(57))6 is a valid partial bracketing but situations like $\ldots(4)\ldots$ or $\ldots((\ldots))\ldots$ are not allowed. The bracketings (1(432)5) and (5(432)1) are equivalent but the bracketing (1(423)5) is not equivalent to either of the former.

Definition 2.4. Fix an ordering for the integers $1, 2, \ldots, k+2$. For $k \geq 1$, the k-associahedra is a CW-complex with

- i-cells given by a $(k-i+1)$-bracketing of $1, 2, \ldots, k+2$ in the fixed ordering chosen with a single pair of brackets enclosing the entire expression.
- The $i+1$-cell labelled by the $k-i$-bracketing P is attached to the i-cell labelled by the $(k-i+1)$-bracketing Q if and only if P can be obtained from Q by removing a matching pair of brackets.

2.3.2. *Stratification of $\overline{M}_{0,k}(\mathbb{R})$.* We can define a stratification on $\overline{M}_{0,k}(\mathbb{R})$ by subspaces $\overline{M}_{0,k}(\mathbb{R}) = M_1 \supset M_2 \supset \ldots \supset M_{k-2} = \varnothing$ where M_i is the set of stable curves with at least i irreducible components. The strata then give a cell decomposition indexed by partial bracketings.

We can assign to each stable curve a partial bracketing. Suppose C is a stable curve and L is an irreducible component of C, for each node $s \in L$ define the stable curve $C_{L,s}$, as the connected component of $(C \setminus L) \cup \{s\}$ containing the point s. If $C_{L,s}$ has r marked points, we add a marked point labelled $r+1$ at s.

For a stable curve C in the stratum $M_i \backslash M_{i+1}$ with k marked points define an i-partial braketing by the following inductive method. Start at the k^{th} marked point and call this irreducible component L. Trace around the curve, each time a special point is encountered, do one of the following:

(i) if the special point is a marked point, record its label; or
(ii) if the special point is a node, $s \in L$, record the partial bracketing defined by the stable curve $C_{L,s}$.

Finally, surround this expression in a single pair of brackets. Intuitively we should think about a partial brackeing as representing the collision of marked points.

2.3.3. *Circular orderings.* A circular ordering of the integers $\{1, 2, \ldots, k\}$ is an element of S_k/D_k. That is, we imagine ordering the integers on a circle and identify orderings which coincide upon rotation or reflection. The orderings $(1, 2, 3, 4)$, $(4, 1, 2, 3)$ and $(4, 3, 2, 1)$ all represent the same circular ordering but are distinct from $(1, 3, 2, 4)$.

For each circular order $s \in S_k/D_k$, let $\Delta_s \subseteq \overline{M}_{0,k}(\mathbb{R})$ be the subspace of curves with a partial bracketing that (once the brackets are removed) represents s. For example, Δ_{id} is the closure of the set of irreducible curves projectively equivalent to a curve $C \in M_{0,k}(\mathbb{R})$ with marked points $z_1 < z_2 < \ldots < z_n$. Each Δ_s is a sub-CW-complex of $\overline{M}_{0,k}(\mathbb{R})$ and is isomorphic to the associahedron.

2.3.4. *The fundamental group.* The *cactus group*, J_n, is the group with generators s_{pq} for $1 \leq p < q \leq n$ and relations

(i) $s_{pq}^2 = 1$
(ii) $s_{pq}s_{kl} = s_{kl}s_{pq}$ if the intervals $[p, q]$ and $[k, l]$ are disjoint.
(iii) $s_{pq}s_{kl} = s_{uv}s_{pq}$ if $[k, l] \subseteq [p, q]$, where $v = \hat{s}_{pq}(k)$ and $v = \hat{s}_{pq}(l)$,

where $\hat{s}_{pq}$ is the permutation that reverses the order of the interval $[p, q]$. This also provides a maps the the symmetric group S_n and we call the kernel the *pure cactus group* PJ_n.

Lemma 2.5. *The cactus group J_n is generated by the elements s_{1q} for $2 \leq q \leq n$.*

Proof. By the relations for J_n given in Section 2.3.4 we have

$$s_{pq} = s_{1q}s_{1(q-p+1)}s_{1q}.$$

Since the elements s_{pq} generate J_n the Lemma follows. □

In [HK06] it is shown $\pi_1(\overline{M}_{0,n+1}(\mathbb{R})) = PJ_n$. The space $\overline{M}_{0,n+1}$ also has an action of S_n by permuting the first n marked points. This leaves the real points stable. The equivariant fundamental group is $\pi_1^{S_n}(\overline{M}_{0,n+1}(\mathbb{R})) = J_n$. The equivariant loop in $\overline{M}_{0,n+1}(\mathbb{R})$ corresponding to $s_{pq} \in J_n$ is $(\alpha, \hat{s}_{pq})$, where α is the path from C passing (transversally) through the wall swapping the labels $p, \ldots, q$ to the point $\hat{s}_{pq} \cdot C$.

2.4. **Growth diagrams.** In this section we recall the notion of a growth diagram which will give an interpretation of jue de taquin slides for standard tableaux using combinatorial objects built on subsets of the lattice $\mathbb{Z}^2$. When we draw this lattice we will depict the second coordinate as increasing northward on the vertical axis and (perhaps counter intuitively) we depict the first coordinate as increasing *westward* on the horizontal axis. This means the southeast to northwest diagonal passing through the point $(0,k)$ will consist of exactly the pairs (i,j) such that $j-i=k$. This choice is made in order to be consistent with the notation in [Spe14].

2.4.1. *Growth diagrams.* If $\mathbb{I} \subset \mathbb{Z}^2_+ = \left\{(i,j) \in \mathbb{Z}^2 \,\middle|\, j-i \geq 0\right\}$, a *growth diagram* on $\mathbb{I}$ is a map $\gamma : \mathbb{I} \longrightarrow \texttt{Part}$ obeying the following rules:

(i) If $j-i=k\geq 0$ then γ_{ij} is a partition of k.
(ii) Suppose $(i,j) \in \mathbb{I}$. Then if $(i-1,j)$ (respectively $(i,j+1)$) is in $\mathbb{I}$ then $\gamma_{ij} \subset \gamma_{(i-1)j}$ (respectively $\gamma_{ij} \subset \gamma_{i(j+1)}$).
(iii) If $(i,j),(i-1,j),(i,j+1)$ and $(i-1,j+1) \in \mathbb{I}$ and $\gamma_{(i-1)(j+1)} \backslash \gamma_{ij}$ consists of two boxes that *do not* share an edge then $\gamma_{(i-1)j} \neq \gamma_{i(j+1)}$.

In view of condition i, condition ii means that if we move one step north or one step east in $\mathbb{I}$, we add a single box. Condition iii means that if we have an entire square in $\mathbb{I}$, and if there are two possible ways to go from γ_{ij} to $\gamma_{(i-1)(j+1)}$ by adding boxes then the two paths around the square should be these two different ways.

A *path* through $\mathbb{I} \subset \mathbb{Z}^2_+$ is a connected series of steps from one vertex to another using only northward and eastward moves (i.e. only ever increasing j and decreasing i and thus $j-i$ is a strictly increasing function on the path). Given a growth diagram γ in $\mathbb{I}$, every path determines a standard tableau.

Given a rectangular region in a growth diagram, conditions i and iii mean that the entire rectangular region is determined by specifying the tableaux along any path from its bottom left corner to the top right.

2.4.2. *Schützenberger involution in growth diagrams.* We now explain how growth diagrams encode the jue de taquin slides on standard tableaux. Let $\mathbb{I}$ be a rectangular region in $\mathbb{Z}^2_+$ only one step tall. Say $\mathbb{I} = \{(i,j) \,|\, i = r, r+1 \ldots, s \text{ and } j = t, t+1\}$, so that (s,t) is the bottom left hand corner and $t-s \geq 0$. Let γ be a growth diagram on $\mathbb{I}$ and set $\mu = \gamma_{st}, \nu = \gamma_{s(t+1)}, \rho = \gamma_{rt}$ and $\lambda = \gamma_{r(t+1)}$. These are the four partitions at the corners of $\mathbb{I}$.

Let T be the $\lambda\backslash\nu$-tableau given by the top edge of $\mathbb{I}$ and S the $\rho\backslash\mu$-tableau given by the bottom edge. See Figure 1. The partition μ determines an addable inside node for T denoted by $\circ$ and similarly the partition λ determines an addable outside node for S denoted $*$.

Proposition 2.6. *The tableau S is the result of the reverse slid of T into $\circ$ and T is the result of the forward slide of S into $*$.*

Proof. See [Sta99], Proposition A1.2.7. $\square$

We now explain how this relates to the Schützenberger involution for standard tableaux. Suppose $T \in \texttt{SYT}(\lambda\backslash\mu)$ where μ is a partition of k and $\lambda\backslash\mu$ has l boxes.

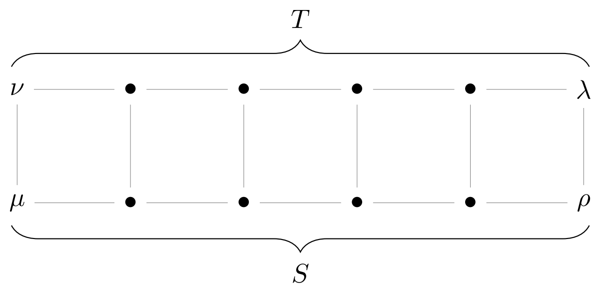
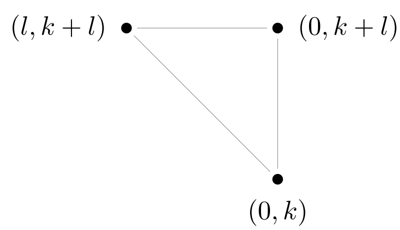


FIGURE 1. The jue de taquin growth diagram

Let

$$\mathbb{I} = \left\{ (i,j) \in \mathbb{Z}_+^2 \,\middle|\, 0 \leq i \leq l, k \leq j \leq k+l, \text{ and } i+k \leq j \right\}.$$

That is, $\mathbb{I}$ is a triangle with vertices $(0,k), (l,k+l)$ and $(0,k+l)$ as depicted below.



Note that given a triangular region $\mathbb{I}$ as above, condition iii means that any growth diagram γ on $\mathbb{I}$ can be computed recursively if we know γ either of the horizontal or vertical sides of $\mathbb{I}$. Define the growth diagram γ_T on $\mathbb{I}$ by setting $\gamma_T(r, k+r) = \mu$ for any $0 \leq r \leq l$. That is, on the diagonal edge of $\mathbb{I}$, γ_T is of constant value μ. We set the sequence of partitions

$$\gamma_T(l, k+l) \subset \gamma_T(l-1, k+l) \subset \ldots \subset \gamma_T(0, k+l)$$

on the horizontal edge of $\mathbb{I}$ so they determine the standard tableau T. By the observation above, this determines γ_T on all of $\mathbb{I}$. As an immediate consequence of the definition and of Proposition 2.6 we obtain the following corollary.

Corollary 2.7. *Let γ_T be the Schützenberger growth diagram of a standard tableau $T \in \mathtt{SYT}(\lambda\backslash\mu)$. The standard tableau determined by the sequence of partitions*

$$\gamma_T(0, k) \subset \gamma_T(0, k+1) \subset \ldots \subset \gamma_T(0, k+l)$$

along the vertical edge of $\mathbb{I}$ is $\mathtt{evac}(T)$, the Schützenberger involution of T.

This explains why the Schützenberger involution is in fact an involution, at least in the case of standard tableaux.

2.5. **Dual equivalence classes.** We now describe Haiman's notion of dual equivalence. This is an equivalence relation on skew tableaux dual to slide equivalence in the sense that it preserves the Q-symbol of a word.

Definition 2.8. Two semistandard skew tableaux, T and T' of the same shape, are called *dual equivalent*, denoted $T \sim_D T'$, if for any meaningful sequence of slides applied to both T and T' we obtain tableaux of the same shape.

2.5.1. *Dual equivalence is local.* We have the following proposition which tells us that dual equivalence is a *local* operation. That is, we can replace a subtableau with a dual equivalent one and the resulting tableau will be dual equivalent to the original one.

Proposition 2.9 (Lemma 2.1 in [Hai92])**.** *Suppose X, Y, S and T are semistandard tableaux such that $X \cup T \cup Y$ and $X \cup S \cup Y$ are semistandard tableaux. If $S \sim_D T$ then $X \cup T \cup Y \sim_D X \cup S \cup Y$.*

A skew-shape is called *antinormal* if it has a unique bottom right corner. That is, if it is the south-eastern part of a rectangle. We have the following important properties of dual equivalence which we will use later.

Theorem 2.10. *Dual equivalence has the following properties.*

(i) All tableaux of a given normal or antinormal shape are dual equivalent.
(ii) The intersection of any slide equivalence class and any dual equivalence class is a unique tableaux.
(iii) Two words are dual equivalent if and only if their Q-symbols agree.

Proof. Properties i, ii and iii are Proposition 2.14, Theorem 2.13 and Theorem 2.12 in [Hai92] respectively. □

2.5.2. *Shuffling dual equivalence classes.* Given a rectangular growth diagram let S_1 and S_2 denote the standard tableaux defined by the western edge and the northern edge respectively and let T_1 and T_2 denote the standard tableaux defined by the southern and eastern edges respectively.

Proposition 2.11 (Proposition 7.6 in [Spe14])**.** *The dual equivalence classes of T_1 and T_2 remain unchanged if we replace either (or both) S_1 or S_2 by dual equivalent tableaux.*

Let δ_1 and δ_2 be dual equivalence classes such that the shape of δ_2 extends the shape of δ_1. Choose representatives S_1 and S_2 for δ_1 and δ_2 respectively. Construct the unique rectangular growth diagram with western and northern edges given by S_1 and S_2 respectively. Define $\varepsilon_1(\delta_1, \delta_2)$ and $\varepsilon_2(\delta_1, \delta_2)$ to be the dual equivalence classes of the southern and eastern edges respectively. Proposition 2.11 implies that ε_1 and ε_2 are independent of the representatives S_1 and S_2 chosen.

Definition 2.12. If δ_1 and δ_2 are dual equivalence classes such that the shape of δ_2 extends the shape of δ_1 we say $\varepsilon_1(\delta_1, \delta_2)$ and $\varepsilon_2(\delta_1, \delta_2)$ are the dual equivalence classes given by *shuffling* δ_1 and δ_2.

2.5.3. *Dual equivalence growth diagrams.* We now introduce a notion of dual equivalence for growth diagrams. First we fix an function $m : \mathbb{Z} \longrightarrow \mathbb{Z}_{>0}$ which we call the *interval*. We define several auxiliary functions using m; a function $\hat{m} : \mathbb{Z} \longrightarrow \mathbb{Z}$

$$\hat{m}(i) \overset{\text{def}}{=} \begin{cases} 1 + \sum_{k=1}^{i-1} m(k) & \text{if } i > 0 \\ 1 - \sum_{k=i}^{0} m(k) & \text{if } i \leq 0, \end{cases}$$

a function $\bar{m} : \mathbb{Z}_+^2 \longrightarrow \mathbb{Z}_+^2$

$$\bar{m}(i,j) \overset{\text{def}}{=} (\hat{m}(i), \hat{m}(j)),$$

and a function $m_s : \mathbb{Z}_+^2 \longrightarrow \mathbb{N}$

$$m_s(i,j) \overset{\text{def}}{=} \hat{m}(j) - \hat{m}(i) = \sum_{k=i}^{j-1} m(k).$$

In particular $m_s(i,i) = 0$, $m_s(i,i+1) = m(i)$ and if m is simply the constant function 1 then $m_s(i,j) = j - i$.

Consider the graph with vertices $\mathbb{Z}_+^2$ and edges a_{ij} between (i,j) and $(i-1,j)$ and edges b_{ij} between vertices (i,j) and $(i,j+1)$. If we embed $\mathbb{Z}_+^2 \subset \mathbb{R}^2$, then we can think of these as simply the horizontal and vertical unit intervals between the points of $\mathbb{Z}_+^2$. If $\mathbb{I} \subset \mathbb{Z}_+^2$, we call an edge of $\mathbb{Z}_+^2$ *internal* to $\mathbb{I}$ if both of its endpoints are in $\mathbb{I}$.

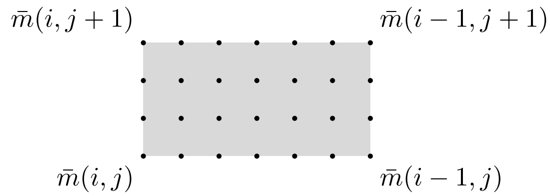
Definition 2.13. A *dual equivalence growth diagram* in $\mathbb{I}$ with interval m is a map $\gamma : \mathbb{I} \longrightarrow \mathtt{Part}$ as well as an assignment of a dual equivalence class α_{ij} (respectively β_{ij}) to every edge a_{ij} (respectively b_{ij}) internal to $\mathbb{I}$, obeying the following rules.

(i) If $(i,j) \in \mathbb{I}$ then γ_{ij} is a partition of $m_s(i,j)$.
(ii) If a_{ij} (respectively b_{ij}) is internal to $\mathbb{I}$ then $\gamma_{ij} \subset \gamma_{(i-1)j}$ (respectively $\gamma_{ij} \subset \gamma_{i(j+1)}$).
(iii) If $a_{ij}, b_{(i-1)j}, b_{ij}$ and $a_{i(j+1)} \in \mathbb{I}$ then $\varepsilon_1(\beta_{ij}, \alpha_{i(j+1)}) = \alpha_{ij}$ and $\varepsilon_2(\beta_{ij}, \alpha_{i(j+1)}) = \beta_{(i-1)j}$.

We should think of the interval m as defining how many boxes we are allowed to add with each step though the lattice. Indeed if we wish to move one step east from (i,j), m_s increases by $m(i-1)$ and if we wish to move one step north, m_s increases by $m(j)$. Thus the partition γ_{ii} is the empty partition and $\gamma_{i(i+1)}$ is a partition of $m(i)$.

This means when m is the constant function 1 our definition coincides with that for a ordinary growth diagram for $\mathbb{I}$. This is clear since in this case $m_s(i,j) = j - i$, condition ii remains unchanged and since we are only adding a single box with each step there is only a single dual equivalence class. Certain classes of dual equivalence growth diagrams will be the central combinatorial objects which we study.

We can also think of dual equivalence growth diagrams as equivalence classes of certain growth diagrams. Let $\tilde{\gamma}$ be a growth diagram on $\tilde{\mathbb{I}} \subset \mathbb{Z}_+^2$. We say $\tilde{\mathbb{I}}$ is *adapted* to an interval $m : \mathbb{Z} \longrightarrow \mathbb{Z}_{>0}$ if it has the following property: If $\tilde{\mathbb{I}}$ contains each of the four vertices



then $\tilde{\mathbb{I}}$ contains each vertex inside (and on the boundary of) the rectangular they region bound.

Definition 2.14. If $\tilde{\mathbb{I}}$ is adapted to m, the *reduction modulo* m of a growth diagram $\tilde{\gamma}$ is defined to be the map $\gamma : \mathbb{I} \longrightarrow \texttt{Part}$ for

$$\mathbb{I} = \left\{ (i, j) \in \mathbb{Z}_+^2 \,\middle|\, (\hat{m}(i), \hat{m}(j)) \in \tilde{\mathbb{I}} \right\},$$

given by $\gamma = \tilde{\gamma} \circ \bar{m}$, along with the set of dual equivalence classes

- α_{ij}, the dual equivalence class defined by the horizontal path from $\bar{m}(i, j)$ to $\bar{m}(i-1, j)$ and
- β_{ij}, the dual equivalence class of the tableaux defined by the vertical path from $\bar{m}(i, j)$ to $\bar{m}(i-1, j)$.

Proposition 2.15. *The map* $\gamma : \mathbb{I} \longrightarrow \texttt{Part}$ *along with the choice of* α_{ij} *and* β_{ij} *define a dual equivalence growth diagram on* $\mathbb{I}$.

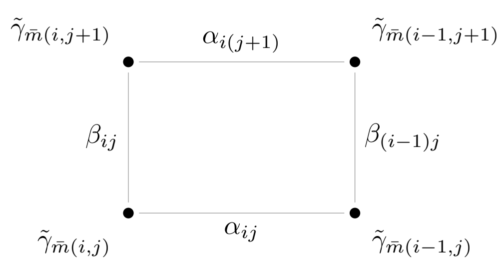
Proof. We must check the conditions in Definition 2.13. For condition i note that $\gamma_{ij} = \tilde{\gamma}_{\hat{m}(i), \hat{m}(j)}$ so $|\gamma_{ij}| = \hat{m}(j) - \hat{m}(i)$ which is $m_s(i, j)$ by definition.

We have a path in $\tilde{\mathbb{I}}$ from $\bar{m}(i, j)$ to $\bar{m}(i-1, j)$ so

$$\gamma_{ij} = \tilde{\gamma}_{\bar{m}(i,j)} \subset \tilde{\gamma}_{\bar{m}(i-1,j)} = \gamma_{(i-1)j}.$$

Similarly $\gamma_{ij} \subset \gamma_{i(j+1)}$.

To see condition iii, note that $\alpha_{ij}, \beta_{(i-1)j}, \beta_{ij}$ and $\alpha_{i(j+1)}$ are defined as the dual equivalence classes coming from the four sides of a rectangular growth diagram:



The fact that this portion of $\tilde{\mathbb{I}}$ forms a rectangular growth diagram is given by the requirement that $\tilde{\mathbb{I}}$ is adapted to m. By definition, this means $(\beta_{ij}, \alpha_{i(j+1)})$ is the shuffle of $(\alpha_{ij}, \beta_{(i-1)j})$ as required. ☐

3. Speyer's flat family

In this section we describe the main geometrical tool that will be used to prove Theorem 1.2. In a series of papers [MTV09b], [MTV09c], [MTV09a], Mukhin, Tarasov and Varchenko describe a tight relationship between Bethe algebras and

Schubert calculus. Speyer [Spe14] constructs a flat family over the projective variety $\overline{M}_{0,k}(\mathbb{C})$. In Section 4 this flat family will be related to the spectrum of the Bethe algebras. Speyer's family admits an explicit combinatorial description allowing us to calculate the monodromy action.

3.1. **Osculating flags.** In this section we recall some definitions and facts from Schubert calculus. All the Grassmanians we consider will be defined relative to some genus 0 smooth curve C. Choose a very ample line bundle $\mathcal{L}$ on C of degree $d-1$. We have the Veronese embedding

$$\varepsilon : C \longrightarrow \mathbb{P}H^0(C,\mathcal{L})^*.$$

A point p is sent to ε the hyperplane of sections vanishing at p. Let

$$\mathrm{Gr}(r,d)_C \overset{\text{def}}{=} \mathrm{Gr}(r, H^0(C,\mathcal{L})).$$

We can also define the r^{th} *associated curve* $\varepsilon_r : C \longrightarrow \mathrm{Gr}(r,d)_C$ which sends a point p to the space of sections vanishing to order at least $d-r$ at p. That is

$$\varepsilon_r(p) \overset{\text{def}}{=} H^0(C, \mathcal{I}_p^{d-r} \otimes \mathcal{L}) \subset H^0(C,\mathcal{L}),$$

Here $\mathcal{I}_p$ is the ideal sheaf of the point p. With this notation $\varepsilon = \varepsilon_1$.

Definition 3.1. The flag $\mathcal{F}_\bullet(p)$ defined by $\mathcal{F}_i(p) = \varepsilon_i(p)$ is called the *osculating flag* at p.

Example 3.2. We can make this concrete by considering the case $C = \mathbb{P}^1 = \mathbb{P}\mathbb{C}^2$. Fix the standard homogeneous coordinates $[x : y]$ on $\mathbb{P}^1$. Choose the line bundle $\mathcal{O}_{\mathbb{P}^1}(d-1)$. Then $H^0(\mathbb{P}^1, \mathcal{O}(d-1)) = \mathbb{C}[x,y]_{d-1}$, the homogeneous polynomials of degree $d-1$. If we work in the affine patch where $y \neq 0$ then we identify identify this with $\mathbb{C}_d[u]$, the space of polynomials of degree *strictly less than* d (u is the coordinate on this patch). The Grassmanian $\mathrm{Gr}(r,d)_{\mathbb{P}^1}$ is then the set of r-dimensional subspaces of $\mathbb{C}_d[u]$.

The map ε_r sends the point $[b : 1] \in \mathbb{P}^1$ to $(x-b)^d\mathbb{C}_{d-r}[x]$ and the osculating flag $\mathcal{F}_\bullet(b)$ is

$$(x-b)^{d-1}\mathbb{C}_0[x] \subset (x-b)^{d-2}\mathbb{C}_1[x] \subset \ldots \subset (x-b)\mathbb{C}_{(d-1)}[x] \subset \mathbb{C}_d[x].$$

The flag $\mathcal{F}_\bullet(\infty)$ is

$$\mathbb{C}_0[x] \subset \mathbb{C}_1[x] \subset \ldots \subset \mathbb{C}_{(d-1)}[x] \subset \mathbb{C}_d[x].$$

Remark 3.3. When we are in the situation of Example 3.2 we will drop the subscript $\mathbb{P}^1$. That is, we will write $\mathrm{Gr}(r,d)$ instead of $\mathrm{Gr}(r,d)_{\mathbb{P}^1}$.

Suppose we have pairs $(C,\mathcal{L})$ and $(D,\mathcal{K})$ of curves and very ample line bundles as well as an isomorphism $\phi : C \longrightarrow D$ such that $\phi_*\mathcal{L} \cong \mathcal{K}$. We would like some relation between the Grassmanian and osculating flags on each curve. It is important to note that it is not possible to choose a canonical isomorphism between $\phi_*\mathcal{L}$ and $\mathcal{K}$, however we have the following fact.

Lemma 3.4. *Let $\mathcal{E}$ be an invertible $\mathcal{O}_X$-module for a projective $\mathbb{C}$-scheme X. Then* $\mathrm{End}(\mathcal{E}) \cong \mathbb{C}$.

Proof. Note that $\mathrm{End}(\mathcal{O}_x) \cong \mathbb{C}$. The lemma follows from the fact that $\mathcal{O}_X \cong \mathcal{E} \otimes \mathcal{E}^*$ and the hom-tensor adjunction formula:

$$\begin{aligned} \mathbb{C} \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X) \cong & \mathrm{Hom}(\mathcal{E} \otimes \mathcal{E}^*, \mathcal{O}_x) \\ & \mathrm{Hom}(\mathcal{E}, \mathcal{H}om(\mathcal{E}^*, \mathcal{O}_X)) \\ & \mathrm{Hom}(\mathcal{E}, \mathcal{E}). \end{aligned}$$

□

This means the isomorphism $\phi_* \mathcal{L} \cong \mathcal{K}$ is unique up to scalar multiple. By noting $H^0(C, -) = H^0(D, \phi_* -)$ we have a canonical induced isomorphism

$$\phi_1 : \mathbb{P}H^0(C, \mathcal{L}) \longrightarrow \mathbb{P}H^0(D, \mathcal{K}),$$

as well as canonical isomorphisms

$$\phi_r : \mathrm{Gr}(r, d)_C \longrightarrow \mathrm{Gr}(r, d)_D$$

for any r.

Lemma 3.5. *The diagram*

$$\begin{tikzcd} C \arrow[r, "\varepsilon_r"] \arrow[d, "\phi"] & \mathrm{Gr}(r, d)_C \arrow[d, "\phi_r"] \\ D \arrow[r, "\varepsilon_r"] & \mathrm{Gr}(r, d)_D \end{tikzcd}$$

commutes. In particular $\mathcal{F}_i(\phi(p)) = \phi_r(\mathcal{F}_i(p))$.

Proof. Choose an isomorphism $\psi : \phi_* \mathcal{L} \longrightarrow \mathcal{K}$ and thus an isomorphism

$$H^0(C, \mathcal{L}) \longrightarrow H^0(D, \mathcal{K})$$

which we also denote by ψ. Choose $p \in C$ and let $q = \phi(p)$. We need to show that the image of $H^0(C, \mathcal{I}_p^{d-r} \otimes \mathcal{L})$ under ψ is $H^0(D, \mathcal{I}_q^{d-r} \otimes \mathcal{K})$. In other words we would like to find an isomorphism making the diagram

$$\begin{tikzcd} H^0(C, \mathcal{I}_p^{d-r} \otimes \mathcal{L}) \arrow[r, hook] \arrow[d, dashed] & H^0(C, \mathcal{L}) \arrow[d, "\psi"] \\ H^0(D, \mathcal{I}_q^{d-r} \otimes \mathcal{K}) \arrow[r, hook] & H^0(D, \mathcal{K}). \end{tikzcd}$$

To prove this, note that the commutative diagram of schemes

$$\begin{tikzcd} \{p\} \arrow[r, "i"] \arrow[d] & C \arrow[d, "\phi"] \\ \{q\} \arrow[r, "j"] & D \end{tikzcd}$$

has an associated commutative diagram of sheaves on D

$$\begin{tikzcd} \phi_* \mathcal{O}_C \arrow[r, "\phi_* i^\#"] \arrow[d, "(\phi^\#)^{-1}"] & \phi_* i_* \mathcal{O}_{\{p\}} \arrow[d] \\ \mathcal{O}_D \arrow[r, "j^\#"] & j_* \mathcal{O}_{\{q\}}. \end{tikzcd}$$

Since i and j are closed immersions and since ϕ_* is exact, $\phi_* i^\#$ and $j^\#$ are surjective. The horizontal arrows are isomorphisms. We can extend this diagram, uniquely, to an isomorphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & \phi_*\mathcal{I}_p \arrow[r] \arrow[d, dashed] & \phi_*\mathcal{O}_C \arrow[r, "\phi_* i^\#"] \arrow[d, "(\phi^\#)^{-1}"] & \phi_* i_*\mathcal{O}_{\{p\}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_q \arrow[r] & \mathcal{O}_D \arrow[r, "j^\#"] & j_*\mathcal{O}_{\{q\}} \arrow[r] & 0.
\end{tikzcd}$$

We denote the dashed isomorphism θ. By tensoring the top row by $\phi_*\mathcal{I}_p$, the bottom row by $\mathcal{I}_q$ and the morphisms by θ we obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \phi_*\mathcal{I}_p^2 \arrow[r] \arrow[d, "\theta^2"] & \phi_*\mathcal{I}_p \arrow[r] \arrow[d, "\theta"] & \phi_*\mathcal{O}_C \arrow[d, "(\phi^\#)^{-1}"] \\
0 \arrow[r] & \mathcal{I}_q^2 \arrow[r] & \mathcal{I}_q \arrow[r] & \mathcal{O}_D.
\end{tikzcd}$$

Continuing in this fashion and tensoring with $\psi : \phi_*\mathcal{L} \longrightarrow \mathcal{K}$ we obtain the desired diagram

$$\begin{tikzcd}
0 \arrow[r] & \phi_*\left(\mathcal{I}_p^{d-r} \otimes \mathcal{L}\right) \arrow[r] \arrow[d, "\theta^{d-r}"] & \phi_*\mathcal{L} \arrow[d, "\psi"] \\
0 \arrow[r] & \mathcal{I}_q^{d-r} \arrow[r] & \mathcal{O}_D.
\end{tikzcd}$$

All that is left to do is take global sections, noting that $H^0(D, \phi_* -) = H^0(C, -)$. $\square$

3.2. **Schubert intersections.** For a partition λ, with at most r rows and $d-r$ columns and a point $p \in C$ we will denote the Schubert variety corresponding to the osculating flag $\mathcal{F}_\bullet(p)$ by $\Omega(\lambda; p)_C$. Similarly the corresponding Schubert cell will be denoted $\Omega^\circ(\lambda; p)_C$. Let λ^{c} be the partition *complementary* to λ for $\mathrm{Gr}(r, d)_C$. This means

$$\lambda^{\mathsf{c}} = (d - r - \lambda_r, d - r - \lambda_{r-1}, \dots, d - r - \lambda_1).$$

Lemma 3.6. *The image of $\Omega(\lambda, p)_{\mathcal{L}}$ under ϕ_d is $\Omega(\lambda, \phi(p))_{\mathcal{K}}$.*

Proof. Choose an isomorphism $\psi : \phi_*\mathcal{L} \longrightarrow \mathcal{K}$ and thus an isomorphism

$$H^0(C, \mathcal{L}) \longrightarrow H^0(D, \mathcal{K})$$

which we also denote by ψ. As noted in Section 3.1 and choice of ψ induces the same isomorphism ϕ_d. If $\mathcal{F}$ is a flag in $H^0(C, \mathcal{L})$ then for any subspace $V \subset H^0(C, \mathcal{L})$

$$\dim V \cap \mathcal{F}_i = \dim \psi(V) \cap \psi(\mathcal{F}_i).$$

This means the image of $\Omega(\lambda, \mathcal{F})_{\mathcal{L}}$ under ϕ_d is $\Omega(\lambda, \psi(\mathcal{F}))$. Here $\psi(\mathcal{F})$ is the flag defined by $\psi(\mathcal{F})_i = \psi(\mathcal{F}_i)$. By Lemma 3.5 and the definition of $\mathcal{F}(p)$ we have $\psi(\mathcal{F}(p)) = \mathcal{F}(q)$. $\square$

Let $\lambda_\bullet$ be a sequence of k partitions, each with at most r rows and at most $d-r$ columns. We will be concerned with the family $\Omega(\lambda_\bullet) \longrightarrow M_{0,k}$ whose fibre over a point C with marked points $z = (z_1, z_2, \dots, z_k)$ is

$$\Omega(\lambda_\bullet; z) \stackrel{\text{def}}{=} \Omega(\lambda^{(1)}; z_1) \cap \Omega(\lambda^{(2)}; z_2) \cap \ldots \cap \Omega(\lambda^{(k)}; z_k).$$

3.3. **Speyer's compactification.** In this section, we fix the following data

- positive integers d, r and k such that $d \le r$, and
- a sequence of partitions $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that each λ_i has at most r rows and $d-r$ columns and and $k \le r(d-r)$.

We recall the construction of Speyer's flat families $\mathcal{G}(r,d)$, and $\mathcal{S}(\lambda_\bullet)$ from [Spe14].

3.3.1. *The construction.* If A is a three elements subset of $[k]$, fix a curve C_T isomorphic to $\mathbb{P}^1$ with three points marked by the elements of A. Since A consists of exactly three elements, this choice is unique up to projective equivalence. We write $\mathrm{Gr}(r,d)_A$ for $\mathrm{Gr}(r,d)_{C_A}$. For a curve $C \in M_{0,k}$ with marked points $(z_1, \ldots, z_k)$ let $\phi_A(C) : \mathbb{P}^1 \longrightarrow C_A$ be the unique isomorphism sending $z_a \in \mathbb{P}^1$ to the point on C_A marked by $a \in A$. In this way we obtain a morphism

$$\phi_A : M_{0,k} \times \mathbb{P}^1 \longrightarrow M_{0,k} \times C_A.$$

Applying the Grassmanian construction to these curves we obtain a morphism (we will abuse notation and also call this ϕ_A)

$$\phi_A : M_{0,k} \times \mathrm{Gr}(r,d)_{\mathbb{P}^1} \longrightarrow M_{0,k} \times \mathrm{Gr}(r,d)_A.$$

Definition 3.7 ([Spe14])**.** The family $\mathcal{G}(r,d)$ is the closure of the image of the embedding defined by the above morphisms

$$\begin{aligned} M_{0,k} \times \mathrm{Gr}(r,d) &\hookrightarrow \overline{M}_{0,k} \times \textstyle\prod_A \mathrm{Gr}(r,d)_A \\ (C, X) &\longmapsto (C, \phi_A(C,X)). \end{aligned}$$

The family $\mathcal{S}(\lambda_\bullet) \subseteq \mathcal{G}(r,d)$ is $\mathcal{G}(r,d) \cap \bigcap_{a\in A} \Omega(\lambda^{(a)}, a)_A$.

Theorem 3.8 ([Spe14, Theorem 1.1])**.** *The family $\mathcal{G}(r,d)$ and its subfamily $\mathcal{S}(\lambda_\bullet)$ have the following properties:*

(i) $\mathcal{G}(r,d)$ and $\mathcal{S}(\lambda_\bullet)$ are Cohen-Macaulay and flat over $\overline{M}_{0,k}$.
(ii) $\mathcal{G}(r,d)$ is reduced.
(iii) $\mathcal{G}(r,d)$ is isomorphic to $M_{0,k} \times \mathrm{Gr}(r,d)$ over $M_{0,k}$.
(iv) $\mathcal{S}(\lambda_\bullet)$ is isomorphic to $\Omega(\lambda_\bullet)$ over $M_{0,k}$.
(v) If a representative of $C \in M_{0,k}$ has marked points $z_1, z_2, \ldots z_k \in \mathbb{P}^1$ then the fibre of $\mathcal{S}(\lambda_\bullet)$ over C is isomorphic to $\bigcap \Omega(\lambda_i, z_i)$

Over the real points $\overline{M}_{0,k}(\mathbb{R})$ of $\overline{M}_{0,k}$ we have the

Theorem 3.9 ([Spe14, Theorems 1.2, 1.3, 1.4])**.** *If $|\lambda_\bullet| = \sum |\lambda_i| = r(d-r)$, then the fibre of $\mathcal{S}(\lambda_\bullet)$ over $C \in \overline{M}_{0,k}(\mathbb{R})$ is a reduced union of real points.*

3.3.2. *The fibre.* We will also want an explicit description of the fibres of the family $\mathcal{S}(\lambda_\bullet)$ so let us recall this from [Spe14]. Fix $C \in \overline{M}_{0,k}$, a not necessarily irreducible curve and denote its irreducible components $C_1, C_2, \ldots C_l$. Fix an irreducible component C_i and let $A \subseteq [k]$ be a three element subset. If $d_1, \ldots, d_e$ are the nodes lying on C_i we say that $v(A) = C_i$ if the points marked by A lie on three separate connected components of $C \setminus \{d_1, \ldots, d_e\}$.

Similarly to in the previous section by consider the isomorphism $\phi_{i,A} : C_i \longrightarrow C_A$ given by the following properties.

(i) If $a \in A$ marks a point lying on C_i, $\phi_{i,A}$ sends that point to the point marked by A in C_A.
(ii) If $a \in A$ lies on a different irreducible component then there is a unique node $d \in C_i$ via which a is path connected to C_i. The point d is sent by $\phi_{i,A}$ to the point marked by a in C_A.

This morphism is uniquely determined since $v(A) = C_i$. By considering the corresponding isomorphisms $\mathrm{Gr}(r,d)_{C_i} \longrightarrow \mathrm{Gr}(r,d)_A$ we obtain an embedding

$$\mathrm{Gr}(r,d)_{C_i} \hookrightarrow \textstyle\prod_{v(A)=C_i} \mathrm{Gr}(r,d)_A.$$

We will identify $\mathrm{Gr}(r,d)_{C_i}$ with its image. Speyer shows the projection from $\mathcal{G}(r,d)$ into $\prod_{v(A)=C_i} \mathrm{Gr}(r,d)_A$ lands inside $\mathrm{Gr}(r,d)_{C_i}$. In this way we can think of the fibre $\mathcal{G}(r,d)(C)$ as a subvariety of $\prod_i \mathrm{Gr}(r,d)_{C_i}$.

Definition 3.10. A *node labelling* for C is a function ν which assigns to every pair (C_i, d) of an irreducible component and a node $d \in C_j$ a partition $\nu(C_j, x)$ such that if $d \in C_i \cap C_j$ then $\nu(C_i, d)^{\mathsf{c}} = \nu(C_j, d)$. We denote the set of node labellings by $\mathtt{N}_C$

Theorem 3.11 ([Spe14, Section 3, proof of Theorem 1.2])**.** *Let $C \in \overline{M}_{0,k}$ be a stable curve with irredicible components $C_1, C_2, \dots, C_l$. Let D_i be the set of nodes on C_i and P_i the set of marked points. The fibres of $\mathcal{G}(r,d)$ and $\mathcal{S}(\lambda_\bullet)$ over C are*

$$\mathcal{G}(r,d)(C) = \bigcup_{\nu \in \mathtt{N}_C} \prod_i \bigcap_{d \in D_i} \Omega(\nu(C_i,d),d)_{C_i}, \tag{3.12}$$

$$\mathcal{S}(\lambda_\bullet)(C) = \bigcup_{\nu \in \mathtt{N}_C} \prod_i \left(\bigcap_{d \in D_i} \Omega(\nu(C_i,d),d)_{C_i} \cap \bigcap_{p \in P_i} \Omega(\lambda^{(p)},p)_{C_i} \right). \tag{3.13}$$

3.4. **Speyer's labelling of the fibre.** We now consider the family $\mathcal{S}(\lambda_\bullet)(\mathbb{R}) \longrightarrow \overline{M}_{0,k}(\mathbb{R})$ which by Theorem 3.9 is a topological covering of degree $c^{\beta}_{\lambda_\bullet}$. Here β is the rectangle partition with r rows and $d-r$ columns.

Recall from Section 2.3 that $\overline{M}_{0,k}(\mathbb{R})$ is tiled by associahedra. This tiling is indexed by circular orderings of the set $[k]$, that is elements of S_k/D_k. We can lift the cellular structure to a tiling by associahedra of $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$ and the aim of this section will be to explain Speyer's combinatorial description of this CW-complex structure.

For now, we will just consider $\mathcal{S} = \mathcal{S}(\lambda_\bullet)$ where $\lambda^{(s)} = \square$ for each s and $k = r(d-r)$. Choose a circular ordering $s = (s(1), s(2), \dots, s(k))$ and let θ be an associahedron of $\mathcal{S}(\mathbb{R})$ lying above the associahedron corresponding to s in $\overline{M}_{0,k}(\mathbb{R})$. The associahedron has facets labeled by non-adjacent pairs (i,j) where $i<j$. The facet θ_{ij} of θ lies over stable curves that generically have two components, one containing (in order) the labels $s(i), s(i+1), \dots, s(j-1)$ and the other containing the labels $s(j), s(j+1), \dots, s(i-1)$. Such a stable curve is depicted in Figure 2.

Fix a generic point x in θ_{ij}. Theorem 3.11 tells us that the map ν assigns a partition to either side of the node of the stable curve at x. Let γ_{ij} be the partition assigned to the side of the node *away from* component labelled $s(i), s(i+1), \dots, s(j-$

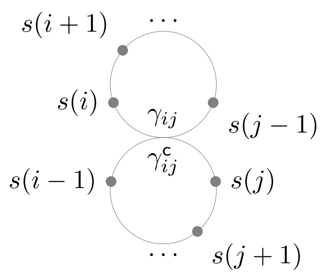


FIGURE 2. A stable curve in θ_{ij}.

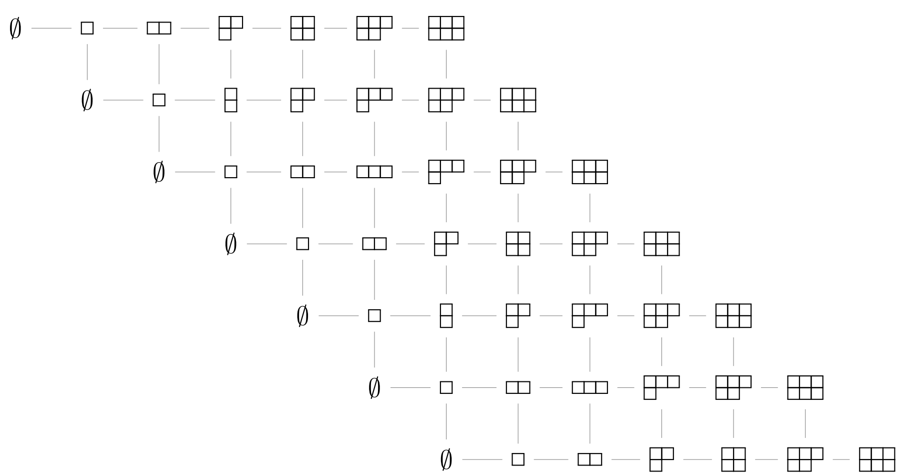


FIGURE 3. An example of a growth diagram for $r = 2$ and $d = 5$. We can take the bottom left corner as $(1, 1)$.

1). Again see Figure 2 for a depiction of this situation. In fact, γ_{ij} does not depend on x (see [Spe14, Lemma 7.1]).

3.4.1. *Cylindrical growth diagrams.* Let $\mathbb{I} = \left\{ (i,j) \in \mathbb{Z}^2 \,\middle|\, 0 \leq j - i \leq k \right\}$.

Definition 3.14. A growth diagram on $\mathbb{I}$ is a *cylindrical growth diagram* for (r, d) if it satisfies the condition that $\gamma_{i(i+k)} = \Lambda$.

The reason for the adjective cylindrical is that these growth diagrams will turn out to be periodic along the north-west diagonal. An example of (part of) a cylindrical growth diagram for $r = 2$ and $d = 5$ is given in Figure 3. As one can see from the diagram, the bottom row is repeated at the top. This is in fact a general phenomenon and is why these diagrams are given the adjective cylindrical.

Remark 3.15. A path though γ, as defined in Section 2.4 from a node (i, i) to a node $(j, j + k)$ (i.e. nodes lying on the left and right edges of the diagram) completely defines all of the partitions lying in the rectangular region the path spans. In the case of cylindrical diagrams the extra condition that $\gamma_{i(i+k)}$ is the rectangular shape means, in fact, such a path determines the entire cylindrical growth diagram completely.

Proposition 3.16 ([Spe14, Lemma 6.4 and Theorem 6.5])**.** *For an associahedron $\theta \subset \mathcal{S}(\lambda_\bullet)(\mathbb{R})$ the map γ is a cylindrical growth diagram. The associahedra which tile $\mathcal{S}(\mathbb{R})$ are labelled by pairs (s, γ) of a circular ordering s and a cylindrical growth diagram γ.*

Note that the cylindrical growth diagram depends on the particular representative of the circular ordering $s \in S_k/D_k$ that we choose. If we choose another representative the cylindrical growth diagram shifted or we take the mirror image (this comes from the action of D_k).

3.4.2. *Wall crossing.* The only thing that remains is to give a discription of how these associahedra are joined together. Fix an associahedra θ in $\mathcal{S}(\mathbb{R})$ labelled by (s, γ). Let $(\hat{s}, \hat{\gamma})$ be the labelling of the associahedra $\hat{\theta}$ joined to θ by the facet θ_{pq}. Using the description of $\overline{M}_{0,k}(\mathbb{R})$ the circular ordering $\hat{s}$ is obtained from s by reversing the order of $s(p), s(p+1), \ldots, s(q-1)$.

Proposition 3.17 (Proposition 6.7 [Spe14])**.** *The cylindrical growth diagram $\hat{\gamma}$ is given by*

Add extra areas where it is preserved (also for decgd's)

$$\hat{\gamma}_{ij} = \begin{cases} \gamma_{ij} & \text{if } i \leq p \leq q \leq j \\ \gamma_{(p+q-j)(p+q-i)} & \text{if } p \leq i \leq j \leq q \end{cases}. \tag{3.18}$$

Proof. We will not repeat the proof here except to say that since the partitions γ_{ij} are constant along the relevant divisor of $\overline{M}_{0,k}(\mathbb{R})$, the partitions do not change, only their indexation. Thinking about how the indexation changes allows one to write the conditions in (3.18). ☐

Note that $\hat{\gamma}$ can be determined recursively by the information given in (3.18). The pairs (i, j) which appear in (3.18) are those for which θ_{ij} intersects θ_{pq}. The rule means when we cross a wall we flip certain triangles and leave others fixed. This is depicted in Figure 4. The red triangles are flipped about the axis shown, green triangles are fixed, and other areas are computed recursively (or using the cyclicity properties of the diagram).

3.4.3. *Dual equivalence cylindical growth diagrams.* We now describe the dual equivalence version of this object. As before we fix $r \geq d$. However now we choose $k \leq r(d-r)$ and a sequence of partitions $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

$$|\lambda_\bullet| \stackrel{\text{def}}{=} \sum_{i=1}^{k} |\lambda_i| = r(d-r).$$

It will be convenient to always take our indices, when referring to this sequence, modulo k. That is, by λ_l we will always mean $\lambda_{(l \mod k)}$. With this convention we set $m(i) = |\lambda_i|$.

Definition 3.19. A *dual equivalence cylindrical growth diagram* (or decgd for short) of shape $\lambda_\bullet$ is a dual equivalence growth diagram γ on $\mathbb{I}_k$ with $\gamma_{i(i+k)} = R(d, r)$, and such that $\gamma_{i(i+1)} = \lambda_i$.

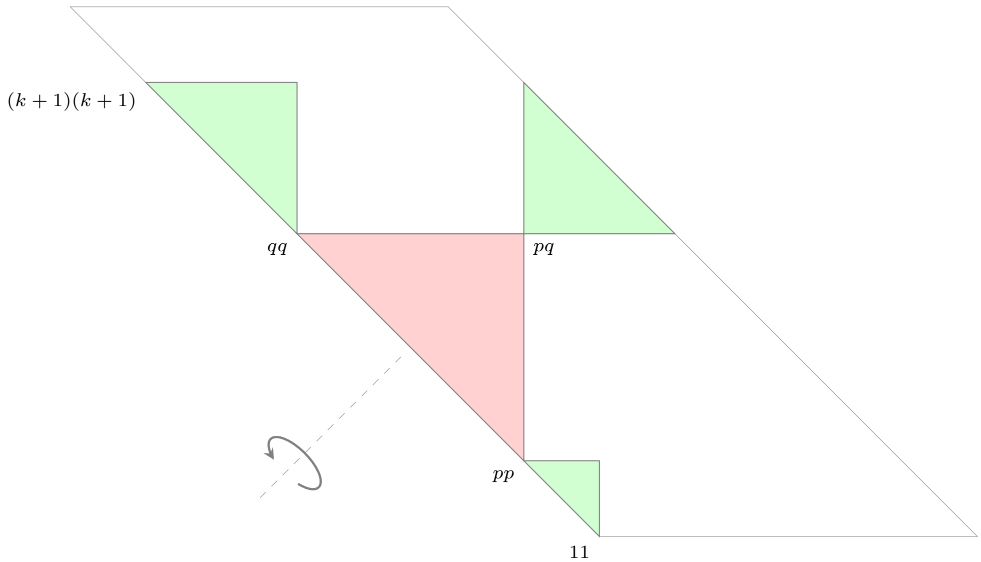


FIGURE 4. Crossing walls and flipping triangles.

As an example we can take $r = 5$ and $d = 2$ again. If we choose

$$\lambda_\bullet = \left(\ydiagram{2,1}, \ydiagram{1}, \ydiagram{2}\right)$$

then Figure 5 gives an example of a decgd of shape $\lambda_\bullet$. Note that since shapes with only a single box, as well as shapes of normal or antinormal shape (c.f. Theorem 2.10 (i)) only have a single dual equivalence class we do not indicate the dual equivalence class for edges that correspond to such a shape.

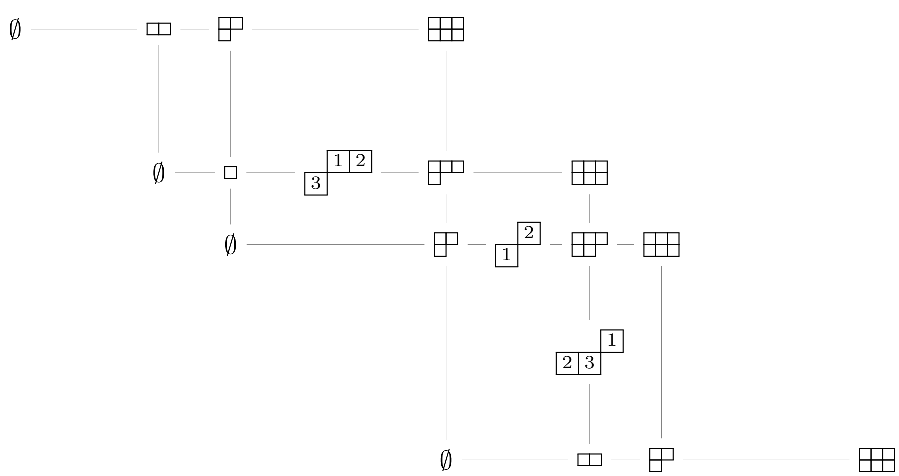


FIGURE 5. An example of a dual equivalence cylindrical growth diagram for $r = 5$ and $d = 2$.

3.4.4. *Reduction of cylindrical growth diagrams.* The reason for the strange choice of layout in Figure 5 is the following. If we superimpose the diagram on top of Figure 3 we can see that it was simply obtained by forgetting certain nodes in Figure 3 but remember the dual equivalence classes defined by the paths between nodes that we kept. In fact it is the reduction modulo m of the cylindrical growth diagram from Figure 3.

As above fix $r \geq d$ and a sequence of partitions $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ for $k \leq r(d-r)$ and such that $|\lambda_\bullet| = r(d-r)$, also set $\tilde{k} = r(d-r)$.

Lemma 3.20. *The set $\mathbb{I}_{\tilde{k}}$ is adapted to $m(i) = |\lambda_i|$ and*

$$\left\{(i,j) \in \mathbb{Z}_+^2 \,\middle|\, \bar{m}(i,j) \in \mathbb{I}_{\tilde{k}}\right\} = \mathbb{I}_k.$$

That is, the reduction modulo m of a cylindrical growth diagram on $\mathbb{I}_{\tilde{k}}$ for (d,r) is a decgd on $\mathbb{I}_k$ for (d,r) of shape

$$\gamma_{\bar{m}(1,2)}, \gamma_{\bar{m}(2,3)}, \ldots, \gamma_{\bar{m}(k,k+1)}.$$

Proof. By overlapping $\mathbb{I}_{\tilde{k}}$ with any rectangular region we see that the only way for the rectangle to ne be contained in $\mathbb{I}_{\tilde{k}}$ completely is if one of its corners is not contained in $\mathbb{I}_{\tilde{k}}$. Hence $\mathbb{I}_{\tilde{k}}$ is adapted to m (and in fact to any interval).

Suppose that $(i,j) \in \mathbb{I}_k$, thus $j - i \leq k$. We would like to show $\bar{m}(i,j)]in\mathbb{I}_{\tilde{k}}$, that is we would like to show $\hat{m}(j) - \hat{m}(i) = m_s(i,j) \leq \tilde{k}$. But

$$m_s(i,j) = \sum_{l=i}^{j-1} m(l) = \sum_{l=i}^{j-1} \left|\lambda_{(l \mod k)}\right|,$$

and since $j - i \leq k$ each λ_l occurs at most once in the above sum. By the assumption that $|\lambda_\bullet| = \tilde{k}$ we have $m_s(i,j) \leq \tilde{k}$ as required.

No consider $(i,j) \in \mathbb{Z}_+^2$ such that $m_s(i,j) \leq \tilde{k}$. If $j - i > k$ then by the pigeonhole principle each λ_l must occur at least once, and some λ_l must occur twice in the sum

$$m_s(i,j) = \sum_{l=i}^{j-1} \left|\lambda_{(l \mod k)}\right|,$$

contradicting the fact that $m_s(i,j) \leq \tilde{k}$. This proves the second claim.

The only thing left to check to ensure that the reduction modulo m of a cylindrical growth diagram on $\mathbb{I}_{\tilde{k}}$ is a decgd on $\mathbb{I}_k$ is that for any i, $\bar{m}(i, i+k) = (j, j+\tilde{k})$ for some j. Equivalently we check that $m_s(i, i+k) = \tilde{k}$. This is straightforward since the sum

$$m_s(i, i+k) = \sum_{l=i}^{i+k-1} \left|\lambda_{(l \mod k)}\right|$$

contains one of each λ_l appearing in $\lambda_\bullet$ and by assumption $|\lambda_\bullet| = \tilde{k}$. □

The key result we will need is the following.

Proposition 3.21 (Proposition 8.1 in [Spe14])**.** *Every decgd on $\mathbb{I}_k$ is the restriction of a cylindrical growth diagram on $\mathbb{I}_{\tilde{k}}$ and the number of decgd's of shape $\lambda_\bullet$ is $c^{\beta}_{\lambda_1, \lambda_2, \ldots, \lambda_k}$.*

We denote the set of decgd's of shape $\lambda_\bullet$ by $\texttt{decgd}(\lambda_\bullet)$.

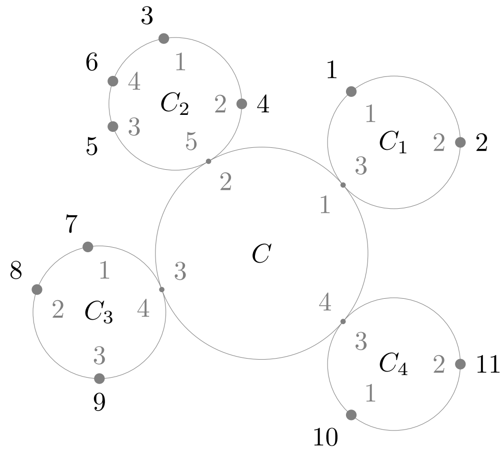


FIGURE 6. A generic point in $\overline{M}_{0,4} \times \prod_{i=1}^{k} \overline{M}_{0,|\lambda_i|+1}$ when $|\lambda_2| = 4$, $|\lambda_3| = 3$ and $|\lambda_1| = |\lambda_4| = 2$. The original label of each marked point is shown in grey on the inside of each curve.

3.4.5. *Another realisation of* $\mathcal{S}(\lambda_\bullet)$. Before we are able to describe the CW-structure for more general $\lambda_\bullet$, we realise $\mathcal{S}(\lambda_\bullet)$ as a subvariety of $\mathcal{S}(\square^{\tilde{k}})$, in fact, the real points will be a CW-subcomplex. Here, $\tilde{k} = \sum_{i=1}^{k} |\lambda_i|$. We should think of obtaining $\mathcal{S}(\lambda_\bullet)$ inside $\mathcal{S}(\square^{\tilde{k}})$ by colliding the first $|\lambda_1|$ marked points in such a way to obtain λ_1, the next $|\lambda_2|$ to obtain λ_2, and so on.

For our purposes, we will take $\overline{M}_{0,2}$ to be a single point. With this convention, we have an embedding $\overline{M}_{0,k} \times \prod_{i=1}^{k} \overline{M}_{0,|\lambda_i|+1}$ into $\overline{M}_{0,\tilde{k}}$ by sending the tuple $(C, C^{(1)}, C^{(2)}, \ldots, C^{(k)})$ to the stable curve obtained by the following process:

- If $|\lambda_i| \geq 2$ the last marked point of C_i is glued to the i^{th} marked point of C.
- The l^{th} marked point of these C_i is renumbered $l + \sum_{j=1}^{i-1} |\lambda_j|$, for $1 \leq l \leq |\lambda_i|$.
- If $|\lambda_i| = 1$ then the i^{th} marked point of C is renumbered $\sum_{j=1}^{i} |\lambda_j|$.

This is an example of a *clutching map* as described in Section 3 of [Knu83] where it is also shown that this is in fact a closed embedding. Figure 6 shows generically what such a stable curve looks like.

Restrict the family $\mathcal{S}(\square^{\tilde{k}})$ to $\overline{M}_{0,k} \times \prod_{i=1}^{k} \overline{M}_{0,|\lambda_i|+1}$ and let $\mathcal{Y}$ be the connected components where the k central nodes are labelled by $\lambda_1, \lambda_2, \ldots, \lambda_k$. As families over $\overline{M}_{0,k} \times \prod_{i=1}^{k} \overline{M}_{0,|\lambda_i|+1}$, $\mathcal{Y}$ is isomorphic to $\mathcal{S}(\lambda_\bullet) \times \prod_{i=1}^{k} \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^C)$.

3.4.6. *The CW-structure for more general* $\mathcal{S}(\lambda_\bullet)$. To label the maximal faces of the CW-structure of $\mathcal{S}(\lambda_\bullet)$ we make use of dual equivalence classes and dual equivalence cylindrical growth diagrams. See Sections 2.5 and 3.4.1 for the definitions and main properties.

Theorem 3.22. *Theorem 8.2 of [Spe14] The maximal faces of the CW-structure on $\mathcal{S}(\lambda_\bullet)$ are labelled by pairs (s,γ) of a circular ordering s and a decgd γ of shape $(\lambda_{s(1)},\lambda_{s(2)}\dots,\lambda_{s(k)})$.*

We leave it to the reader to consult Speyer for a rigorous proof of this fact however we will comment on how the results of Section 3.4.5 allow us to make this statement and produce the dual equivalence classes. We use the notation of Section 3.4.5.

Let θ be an associahedron in $\mathcal{S}(\lambda_\bullet)$. We consider the embedding

$$\theta\times\prod_{i=1}^k\mathcal{S}(\Box^{|\lambda_i|},\lambda_i^C)(\mathbb{R})\longhookrightarrow\mathcal{S}(\Box^{\tilde{k}})(\mathbb{R}).$$

Since this is an embedding of CW-complexes $\theta\times\prod_{i=1}^k\mathcal{S}(\Box^{|\lambda_i|},\lambda_i^C)(\mathbb{R})$ must be contained in some maximal face $\tilde{\theta}$ of $\mathcal{S}(\Box^{\tilde{k}})(\mathbb{R})$. Let $(\tilde{s},\tilde{\gamma})$ be the circular order (of $\tilde{k}$) and cylindrical growth diagram labelling $\tilde{\theta}$ as per Proposition 3.16. Let τ be the unique order preserving bijection

$$\left\{\tilde{s}(k_1^i)\,\middle|\,1\le i\le k\right\}\longrightarrow\{1,2,\dots,k\}\,,\quad\text{where } k_1^i=1+\sum_{j=1}^{i-1}\left|\lambda^{(j)}\right|,$$

then $s(i)=\tau\circ\tilde{s}(k_1^i)$. Let $m(i)=\left|\lambda_{s(i)}\right|$, then γ is the reduction of $\tilde{\gamma}$ modulo m.

Proposition 3.23. *Suppose we have two neighbouring maximal faces of $\mathcal{S}(\lambda_\bullet)$ labelled by the pairs (s,γ) and $(\hat{s},\hat{\gamma})$ where $\hat{s}$ is obtained from s by reversing $s(p),s(p+1),\dots,s(q-1)$. The dual equivalence cylindrical growth diagram $\hat{\gamma}$ is given by*

$$\hat{\gamma}_{ij}=\begin{cases}\gamma_{ij} & \text{if } i\le p\le q\le j\\ \gamma_{(p+q-j)(p+q-i)} & \text{if } p\le i\le j\le q\end{cases},\tag{3.24}$$

with the dual equivalence classes α_{ij} and β_{ij} being similarly flipped inside the triangle south-west of the node $(p,q+1)$.

3.5. **The S_k-action.** Given a permutation $\sigma\in S_k$ and a subset $A\subset[k]$ we use the notation

$$\sigma A\stackrel{\text{def}}{=}\{\sigma(a)\,|\,a\in A\}\,.$$

It is clear that in this way σ defines a permutation of the set of three element sets $A\subset[k]$. The permutation σ also induces an isomorphism $C_A\to C_{\sigma A}$ (sending marked points to marked points) and hence an isomorphism $\mathrm{Gr}(d,r)_A\to\mathrm{Gr}(d,r)_{\sigma A}$. In order to keep our notation tidy we will use σ to denote all of these isomorphisms, the context should make it clear which we a referring to. Since the constructions were functorial, diagrams of the type

$$\begin{tikzcd}\mathrm{Gr}(r,d)_A \arrow[rr, "\sigma"] & & \mathrm{Gr}(r,d)_{\sigma A}\\ & \mathrm{Gr}(r,d)_{\mathbb{P}^1} \arrow[ul, "\phi_A"] \arrow[ur, "\phi_{\sigma A}"'] &\end{tikzcd}\tag{3.25}$$

commute. Let S_k act on $\overline{M}_{0,k}$ by permuting marked points. The above discussion means that we have an action of S_k on the trivial family

$$\overline{M}_{0,k} \times \prod_{A \in \binom{[k]}{3}} \mathrm{Gr}(d,r)_A.$$

Proposition 3.26. *The variety $\mathcal{G}(d,r)$ is stable under the action of S_k and the variety $\mathcal{S}(\lambda_\bullet)$ is sent isomorphically onto $\mathcal{S}(\sigma \cdot \lambda_\bullet)$. In particular the stabiliser of $\lambda_\bullet$ acts on $\mathcal{S}(\lambda_\bullet)$.*

Proof. Recall $\mathcal{G}(r,d)$ is defined as the closure of the image of the embedding

$$\begin{aligned} M_{0,k} \times \mathrm{Gr}(r,d) &\hookrightarrow \overline{M}_{0,k} \times \textstyle\prod_A \mathrm{Gr}(r,d)_A \\ (C,X) &\longmapsto (C, \phi_A(C,X)). \end{aligned}$$

By the commutativity of (3.25), the S_k-action preserves the image of $\mathrm{Gr}(r,d)_{\mathbb{P}^1}$. Thus the action of S_k also preserves the closure.

By Lemma 3.6, for any pair $a \in A$ the isomorphism σ sends $\Omega(\lambda^{(a)}, a)_A$ to the Schubert variety $\Omega(\lambda^{(a)}, \sigma(a))_{\sigma A}$. Which means

$$\begin{aligned} \sigma\mathcal{S}(\lambda_\bullet) &= \sigma\left(\mathcal{G}(r,d) \cap \bigcap_{a\in A} \Omega(\lambda^{(a)}, a)_A\right) \\ &= \mathcal{G}(r,d) \cap \bigcap_{a\in A} \Omega(\lambda^{(a)}, \sigma(t))_{\sigma A} \\ &= \mathcal{G}(r,d) \cap \bigcap_{a\in A} \Omega(\lambda^{(\sigma^{-1}(a))}, a)_A. \end{aligned}$$ ☐

3.6. **Acting on the ν labelling.** We will also need some finer information on exactly what the orbits in $\mathcal{S}(\lambda_\bullet)$ look like. This will be helpful when it comes to detirmining what the S_k action does to the cylindrical growth diagram indexing a face of $\mathcal{S}(\mathbb{R})$.

Consider the fibre $\mathcal{G}(r,d)(C)$ for some stable curve $C \in \overline{M}_{0,k}$. Using the description of the fibre from Theorem 3.11, we have

$$\mathcal{G}(r,d)(C) = \bigcup_{\nu \in \mathtt{N}_C} \prod_i \bigcap_{d\in D_i} \Omega(\nu(C_i,d),d)_{C_i},$$

where $\mathtt{N}_C$ is the set of node labellings for C, C_i the irreducible components of C and D_i the set of nodes on the component C_i. The action of S_k on $\overline{M}_{0,k}$ permutes marked points, thus C and its image σC are the same curve simply with different marked points. That is, there is an isomorphism $C \to \sigma C$ which we can take to be the identity morphism, which sends the point marked by a to the point marked by $\sigma(a)$. In this way we identify the irreducible components C_i and σC_i and if d is a node in C_i, we also have a node $d \in \sigma C_i$. In this way identify the node labellings both curves, $\mathtt{N}_C = \mathtt{N}_{\sigma C}$.

Lemma 3.27. *The S_k action on $\mathcal{G}(d,r)$ "perserves the ν-component of the fibre". More precisely, if we fix a $\nu \in \mathtt{N}_C$ the image of*

$$\prod_i \bigcap_{d\in D_i} \Omega(\nu(C_i,d),d)_{C_i}$$

under the action of σ *is*

$$\prod_i \bigcap_{d \in \sigma D_i} \Omega(\nu(\sigma C_i, d), d)_{\sigma C_i}$$

Proof. Again by the commutativity of (3.25), the Grassmanian $\mathrm{Gr}(r,d)_{C_i}$ is sent isomorphically onto $\mathrm{Gr}(r,d)_{\sigma C_i}$. Lemma 3.6 then tells us that $\Omega(\nu(C_i,d),d)_{C_i}$ is mapped onto $\Omega(\nu(\sigma C_i,d),d)_{\sigma C_i}$. □

3.7. **The S_k-action on $\mathcal{S}(\mathbb{R})$.** We can now describe how the S_k-action effects the labelling of the fibre by cylindrical growth diagrams.

Proposition 3.28. *If* θ *is a maximal face in* $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$ *labelled by* (s,γ) *then* $\sigma\theta = \hat{\theta}$ *is the maximal face labelled by* $(\sigma \cdot s, \gamma)$*.*

Proof. We first restrict to the case that $r = r(d-r)$ and $\lambda^{(i)} = (1)$ for all i. It is clear by the action of S_k on $\overline{M}_{0,k}(\mathbb{R})$ that $\sigma \cdot s$ is the circular ordering labelling $\hat{\theta}$. Recall the cylindrical growth diagram $\hat{\gamma}$ of $\hat{\theta}$ is determined by considering the ν-labelling of a point on each of its facets. As shown in Lemma 3.27 this ν-labelling is preserved, so $\hat{\gamma} = \gamma$.

Now consider the general case for arbitrary $\lambda_\bullet$. We use the notation from Section 3.4.5. Choose a permutation $\tilde{\sigma} \in S_{\tilde{k}}$ such that this section needs to exist

$$\theta \times \prod_{i=1}^{k} \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^C)(\mathbb{R}) \subset \tilde{\theta}$$

is sent to

$$\sigma \cdot \theta \times \prod_{i=1}^{k} \mathcal{S}(\square^{|\lambda_i|}, \lambda_i^C)(\mathbb{R}) \subset \tilde{\sigma} \cdot \tilde{\theta}.$$

Here $\tilde{\theta}$ is a choice of associahedron in $\mathcal{S}(\square^{\tilde{k}})(\mathbb{R})$. If $\tilde{\gamma}$ in the cylindrical growth diagram labelling $\tilde{\theta}$ then by the first part $\tilde{\gamma}$ is also the cylindrical growth diagram labelling $\tilde{\sigma} \cdot \gamma$. But the decgd labelling $\sigma \cdot \theta$ is by definition the reduction of this cylindrical growth diagram, which by assumption is γ. □

Let us now take $k = n+1$ and consider the action of S_n which permutes the first n marked points. As noted in Proposition 3.26 there is an S_n-action on $\bigsqcup_{\lambda_\bullet} \mathcal{S}(\lambda_\bullet)(\mathbb{R})$ where we range over all sequence of appropriate partitions. This is a covering of $\overline{M}_{0,n+1}(\mathbb{R})$. Using the above result we can say exactly what the S_n-equivariant monodromy action of J_n is.

Given a point $C \in M_{0,n+1}(\mathbb{R})$, Theorem 3.22 tells us that we can identify the fibre with $\texttt{decgd}(\lambda_\bullet)$. We could potentially do this in a number of ways but fix one by choosing for the associahedron containing C a representation $s = (s(1), s(2), \ldots, s(n+1))$ of the corresponding circular ordering. Let γ and $\hat{\gamma}$ be as above.

Corollary 3.29. *The equivariant monodromy action of* J_n *on* $\texttt{decgd}(\lambda_\bullet)$ *is given by* $s_{1q} \cdot \gamma = \hat{\gamma}$*.*

Proof. Recall from Section 2.3.4, the equivariant loop via which s_{1q} acts is $(\alpha, \hat{s}_{1q})$ where α is a path from C to $\hat{s}_{1q} \cdot C$ passing through the face which swaps the market points $s(1), s(2), \dots, s(q)$. We lift α to $\tilde{\alpha}$ the unique path in $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$ starting at the point over C labelled γ. Proposition 3.23 shows that the point over $\hat{s}_{1q} \cdot C$ at the end of $\tilde{\alpha}$ is labelled by $\hat{\gamma}$. Now Proposition 3.28 tells us that acting by $\hat{s}_{1q}$ does not change the decgd. Hence $s_{1q} \cdot \gamma = \hat{\gamma}$. □

Of course, certain connected components of $\bigsqcup_{\lambda_\bullet} \mathcal{S}(\lambda_\bullet)(\mathbb{R})$ are invariant under the action of S_n. For example when all the $\lambda^{(i)}$ are equal $\mathcal{S}(\lambda_\bullet)$ is left invariant. The above Corollary also describes the equivariant monodromy of these families.

4. Bethe algebras

4.1. **Definition of Bethe algebras.** In this section we define the Bethe algebras and state some of their main properties. Let $\mathfrak{gl}_r[t] := \mathfrak{gl}_r \otimes \mathbb{C}[t]$ be the *current algebra* of polynomials with coefficients in $\mathfrak{gl}_r$. For a formal variable u and an element $x \in \mathfrak{gl}_r$ define the generating function

$$x(u) := \sum_{s=0}^{\infty} (x \otimes t^s) u^{-s-1}.$$

This is a useful accounting device, for example, it allows us to define an automorphism ρ_a of $\mathfrak{gl}_r[t]$ for every complex number $a \in \mathbb{C}$ by the assignment,

$$x(u) \longmapsto x(u-a),$$

for every element $x \in \mathfrak{gl}_r$. What this means is we map xt^s to $x(a+t)^s$, the coefficient of u^{-s-1} in $x(u-a)$ which we expand using the formula for the geometric series. For a $\mathfrak{gl}_r[t]$-module M and a complex number $a \in \mathbb{C}$, we define the *evaluation module* $M(a)$ as the pullback over the map ρ_a.

In a similar fashion we define the *evaluation* morphism $\mathrm{ev} \colon \mathfrak{gl}_r[t] \longrightarrow \mathfrak{gl}_r$ by the assignment $x(u) \mapsto xu^{-1}$, which basically means we send t to zero and x to itself. Then any $\mathfrak{gl}_r$-module can be made into $\mathfrak{gl}_r[t]$-module by pullback. Given a $\mathfrak{gl}_r$-module N, as a $\mathfrak{gl}_r[t]$-module, t acts by zero. Hence on $N(a)$, xt^s acts by $a^s x$.

If ∂ is differentiation with respect to u then we can define the following noncommutative determinant by expansion along the first column,

$$\mathcal{D}^{\mathcal{B}} := \det \begin{pmatrix} \partial - e_{11}(u) & -e_{21}(u) & \cdots & -e_{r1}(u) \\ -e_{12}(u) & \partial - e_{22}(u) & \cdots & -e_{r2}(u) \\ \vdots & \vdots & \ddots & \vdots \\ -e_{1r}(u) & -e_{2r}(u) & \cdots & \partial - e_{rr}(u) \end{pmatrix}$$

where e_{ij} are the standard generators for $\mathfrak{gl}_r$. The determinant $\mathcal{D}^{\mathcal{B}}$ must be of the form

$$\mathcal{D}^{\mathcal{B}} = \partial^r + \sum_{i=1}^{r} B_i(u) \partial^{r-i},$$

for some power series with coefficients $B_{is} \in \mathfrak{gl}_r[t]$,

$$B_i(u) = \sum_{s=i}^{\infty} B_{is} u^{-s}.$$

Definition 4.1. The *universal Bethe algebra* is the subalgebra $\mathcal{B}$, of $U(\mathfrak{gl}_r[t])$ generated by the coefficients B_{is}. For a $\mathcal{B}$-module M, we call the image of $\mathcal{B}$ in $\mathrm{End}(M)$ the *Bethe algebra associated to* M.

By [MTV06, Propositions 8.2 and 8.3] the universal Bethe algebra is a commutative subalgebra of $U(\mathfrak{gl}_r[t])$ and commutes with the action of $\mathfrak{gl}_r \subset \mathfrak{gl}_N[t]$. As a result, for any $\mathfrak{gl}_r[t]$-module M, and any weight λ, the subspaces M_λ, M^{sing} and $M_\lambda^{\mathrm{sing}} \subset M$ are $\mathcal{B}$-submodules.

Let $\lambda_\bullet = (\lambda^{(i)})_{i=1}^n$ be a sequence of partitions with at most r rows, and $L(\lambda^{(i)})$ the irreducible representation of $\mathfrak{gl}_r$ with highest weight $\lambda^{(i)}$. As a special case of Definition 4.1, we denote the Bethe algebra associated to

$$L(\lambda_\bullet; z)_\mu = [L(\lambda^{(1)})(z_1) \otimes L(\lambda^{(2)})(z_2) \otimes \cdots \otimes L(\lambda^{(n)})(z_n)]_\mu^{\mathrm{sing}}$$

by $B(\lambda_\bullet; z)_\mu$ for $z = (z_1, z_2, \ldots, z_n) \in X_n$. If σ is a permutation fixing $\lambda_\bullet$ under the natural action, we can define an automorphism of $L(\lambda_\bullet)$ by simply permuting the tensor factors. We denote this automorphism by σ as well.

Lemma 4.2. *The Bethe algebras $B(\lambda_\bullet; z)_\mu$ are invariant under two separate group actions. Let $S_n^{\lambda_\bullet} \subseteq S_n$ be the stabiliser of $\lambda_\bullet$ under the natural permutation action.*

(i) The group Aff_1*. Suppose $\alpha, \beta \in$, then $B(\lambda_\bullet; \alpha z + \beta)_\mu = B(\lambda_\bullet; z)_\mu$ as subalgebras of* $\mathrm{End}(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$*.*

(ii) The group $S_n^{\lambda_\bullet}$. Let σ fix $\lambda_\bullet$, then $\sigma B(\lambda_\bullet; \sigma\cdot z)_\mu \sigma^{-1} = B(\lambda_\bullet; z)_\mu$ as subalgebras of $\mathrm{End}(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$*.*

This statement needs to cleaned up

Proof. Property i is proved for example in [Ryb14, Proposition 1]. Property ii ... ∎

Fill in proof once statement is cleaned up

By Lemma 4.2 (i) we can consider the family of Bethe algebras over the variety $M_{0,n+1}(\mathbb{C})$. Lemma 4.2 (ii) then says we have an action of the symmetric group on this vector bundle. The following theorem summarises the main facts about the Bethe algebra we will use.

Theorem 4.3 ([MTV09b], Corollary 6.3)**.** *Suppose $z_1, z_2, \ldots, z_n$ are distinct real numbers and μ is a partition of n (with at most r rows). In this case the Bethe algebra $B(\lambda_\bullet; z)_\mu$ has simple spectrum. That is, its joint eigenspaces are all one dimensional. Furthermore $B(\lambda_\bullet; z)_\mu$ has dimension $c_{\lambda_\bullet}^\mu$.*

4.2. **The MTV isomorphism.** In this section we recall the definition of the MTV isomorphism. Let $\mathrm{Gr}(r, d)$ be the Grassmanian of r-planes in the d-dimensional vector space $\mathbb{C}_d[u]$. Mukhin, Tarasov and Varchenko [MTV09b] define the *MTV-isomorphism* $\theta : \mathcal{B}(\lambda_\bullet)_\mu \longrightarrow \Omega(\lambda_\bullet; \mu^c)$ in the following way. Let $\chi \in \mathcal{B}(\lambda_\bullet; z)_\mu$, be an element of the fibre over z. The we think of χ as a map $\chi : B(\lambda_\bullet; z)_\mu \longrightarrow \mathbb{C}$. Let $b_{is} = \chi(B_{is})$ be the image of the generators. Define

$$b_i(u) = \sum_{s=i}^{\infty} b_{is} u^{-s}.$$

Consider the differential operator

$$\mathcal{D}^\chi = \partial^r + \sum_{i=1}^r b_i(u)\partial^{r-i},$$

which is just the result of the differential operator $\mathcal{D}^\mathcal{B}$ acting on the eigenspace of $L(\lambda_\bullet)_\mu^{\mathsf{sing}}$ corresponding to χ. Define $\theta(\chi)$ to be the kernel of $\mathcal{D}^\chi$ acting on $\mathbb{C}_d[u]$.

Theorem 4.4 ([MTV09b]). *The subspace $\theta(\chi) \subseteq \mathbb{C}_d[u]$ has dimension r and is contained in the Schubert intersection $\Omega(\lambda_\bullet, \mu^{\mathsf{c}}; z, \infty)$. Moreover it defines an isomorphism of families over $\overline{M}_{0,n+1}$ of the varieties $\mathcal{B}(\lambda_\bullet)_\mu$ and $\Omega(\lambda_\bullet, \mu^{\mathsf{c}})$.*

4.3. **Crystals.** We recall briefly how the crystals of the irreducible $\mathfrak{gl}_r$-modules are realised using semistandard tableaux. First the crystal $B = B(\square)$ of the vector representation is

$$\boxed{1} \xrightarrow{\ 1\ } \boxed{2} \xrightarrow{\ 2\ } \cdots \xrightarrow{r-1} \boxed{r}.$$

Using the tensor product rule for crystals we have a description of the crystal for $B^{\otimes n}$ as the set of words of length n in the letters $1, 2, \ldots, r$, we identify the element $\boxed{i_1} \otimes \boxed{i_2} \otimes \ldots \otimes \boxed{i_n} \in B^{\otimes n}$ with the word $i_1 i_2 \cdots i_n$. As is well known, the irreducible $L(\lambda)$ embeds into the tensor power of vector representations $V^{\otimes n}$ where $n = |\lambda|$. We can thus realise the crystal $B(\lambda)$ as an appropriate connected component of $B^{\otimes n}$.

To describe this explicitly and in terms of semistandard tableaux, we will use the *RSK-correspondence*

$$\mathtt{RSK} : \mathtt{words}(n) \longrightarrow \bigsqcup_{|\lambda|=n} \mathtt{SSYT}(\lambda) \times \mathtt{SYT}(\lambda).$$

We use $\mathtt{P}$ and $\mathtt{Q}$ to denote projection onto the first and second factors (the P and Q-symbols of the word). The connected components (and thus irreducible constituents) of $B^{\otimes n}$ can be identified by the fact $u, v \in B^{\otimes n}$ lie in the same irreducible crystal (on the same connected component) if and only if $\mathtt{Q} \circ \mathtt{RSK}(u) = \mathtt{Q} \circ \mathtt{RSK}(v)$.

4.3.1. *Coboundary structure.* The category of (finite dimensional) $\mathfrak{gl}_r$-modules is a *braided monoidal category*. In [HK06], Henriques and Kamnitzer show this structure does not descend to the category of crystals and in fact the category of crystals cannot be given a braiding. Instead, Henriques and Kamnitzer show that the category of crystals satisfies the axioms of a *coboundary category*.

A coboundary monoidal category is a monoidal category with a *commuter* $\sigma_{XY} : X \otimes Y \longrightarrow Y \otimes X$, an isomorphism natural in both arguments that satisfies a certain condition (similar to the hexagon condition for braided monoidal categories). The important fact for us is that the cactus group acts on n-fold tensor products in a coboundary category. For objects $A_1, A_2, \ldots, A_n$, for $1 \le p < q \le n$ set

$$\sigma_{pq} = \mathrm{id}^{\otimes(p-1)} \otimes \sigma^c_{A_p \otimes \cdots \otimes A_{q-1}, A_q} \otimes \mathrm{id}^{\otimes(n-q)}.$$

Then the generators s_{pq} of the cactus group act in the following way. First set $s_{p(p+1)} = \sigma_{p(p+1)}$ and inductively $s_{pq} = s_{(p+1)q} \circ \sigma_{pq}$. We should think of s_{pq} as swapping the order of $A_p, A_{p+1}, \ldots, A_q$.

In the case $\mathfrak{g} = \mathfrak{gl}_n$ Henriques and Kamnitzer give a simple description of the commutor for the category of crystals. Let $\xi : \mathtt{SSYT}(\lambda, n) \longrightarrow \mathtt{SSYT}(\lambda, n)$ be the *Schützenberger involution*. For two $\mathfrak{gl}_n$-crystals, A, B define the following map

$$\sigma_{A,B} : A \otimes B \longrightarrow B \otimes A, \quad \text{where } \sigma_{A,B}(a \otimes b) = \xi(\xi(b) \otimes \xi(a)). \tag{4.5}$$

This defines the cactus commutor on the category of $\mathfrak{gl}_n$-crystals.

4.4. **Galois actions for finite maps.** In this section we recall the notion of the Galois group for a finite map. This was defined by Harris in [Har79]. If $\pi : Y \longrightarrow X$ is a dominant morphism between varieties of equal dimension, we say it has *degree* d is the associated field extension $K(X) \hookrightarrow K(Y)$ has degree d. For a generic point $x \in X$ the fibre therefor consists of d reduced points.

If L is the Galois closure of the extension of $K(X) \hookrightarrow K(Y)$ then the Galois group $\mathrm{Gal}(L/K(X))$ acts on the fibre $\pi^{-1}(x)$.

Definition 4.6. The image of $\mathrm{Gal}(L/K(X))$ in $S_{\pi^{-1}(x)} \simeq S_d$, the group of permutations of the fibre, is called the *Galois group of* π and is denoted $\mathrm{Gal}(\pi)$ or $\mathrm{Gal}(\pi; x)$ if we wish to emphasise the basepoint.

Remark 4.7. The definition of the Galois group $\mathrm{Gal}(\pi; x)$ depends only on the field extension. The Galois group is thus a birational invariant of *pi*. Let $\pi' : X' \longrightarrow Y'$ be another degree d morphism and suppose we have birational maps making the following diagram commute,

$$\begin{tikzcd} X \arrow[r, dashed, "g"] \arrow[d, "\pi"] & X' \arrow[d, "\pi'"] \\ Y \arrow[r, dashed, "f"] & Y'. \end{tikzcd}$$

If f is defined on x and g identifies the fibres over x and $f(x)$, then the galois groups are equal $\mathrm{Gal}(\pi; x) = \mathrm{Gal}(\pi'; f(x))$.

We can always find a dense open subset $U \subseteq Y$ over which π is unramified. Restricting we obtain a topological covering map $\pi|_{\pi^{-1}(U)}$. If $x \in U$ we can consider the *monodromy group* $M_U(\pi; x) \subseteq S_{\pi^{-1}(x)}$. The following theorem relates the Galois group to the monodromy group.

Proposition 4.8 ([Har79, Section I.2])**.** *For any U as above the monodromy group equals the Galois group, $M_U(\pi; x) = \mathrm{Gal}(\pi; x)$. In particular the monodromy group does not depend on the open neighbourhood chosen to define it.*

Remark 4.7 and Proposition 4.8 are the key tools we need to calculate (part of) the Galois group of the spectrum of the Bethe algebras.

4.5. **The main theorem.** The symmetric group S_k acts in the obvious way on sequences of partitions $\lambda_\bullet$. Denote the stabiliser of $\lambda_\bullet$ by $S_k^{\lambda_\bullet}$. Recall by Proposition 3.26 we have an action of $S_k^{\lambda_\bullet}$ on $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$ which preserves the labelling of the fibres by dual equivalence cylindrical growth diagrams.

In this section we will identify the equivarient monodromy of this $S_k^{\lambda_\bullet}$ action on $\mathcal{S}(\lambda_\bullet)(\mathbb{R})$ with an action coming from representation theory. Set $n + 1 = k$. Let $\lambda_\bullet = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and let μ be a partition such that $|\mu| = |\lambda_\bullet|$. Let $S_n \subset S_k$ be identified with the subgroup permuting the first n letters.

4.5.1. *The equivarient fundamental group of $\overline{M}_{0,n+1}(\mathbb{R})$.* As noted in Section 3.5, the action of S_n does not preserve $\mathcal{S}(\lambda_\bullet;\mu)(\mathbb{R})$ for arbitrary $\lambda_\bullet$. Instead $\sigma \in S_n$ sends $\mathcal{S}(\lambda_\bullet,\mu^{\mathsf{c}})(\mathbb{R})$ isomorphically onto $\mathcal{S}(\sigma\cdot\lambda_\bullet,\mu^{\mathsf{c}})(\mathbb{R})$. However we can restrict to those permutations that do fix $\lambda_\bullet$. We denote the stabiliser of $\lambda_\bullet$ by $S_n^{\lambda_\bullet}$ and define

$$J_n^{\lambda_\bullet} \stackrel{\text{def}}{=} \pi_1^{S_n^{\lambda_\bullet}}(\overline{M}_{0,n+1}(\mathbb{R})).$$

In the case when all $\lambda^{(i)}$ are the same (for example when all λ_i are the partition $\square$) then $S_n^{\lambda_\bullet} = S_n$. We are now in a position to state the main theorem.

Theorem 4.9. *There is a $J_n^{\lambda_\bullet}$ equivarient bijection* $\mathtt{decgd}(\lambda_\bullet,\mu) \longrightarrow [B(\lambda_\bullet)]_\mu^{\mathrm{sing}}$ *where the action on* $\mathtt{decgd}(\lambda_\bullet,\mu)$ *is given by the equivariant monodromy of the covering $\mathcal{S}(\lambda_\bullet,\mu^{\mathsf{c}})(\mathbb{R}) \to \overline{M}_{0,n+1}(\mathbb{R})$ and the action on $[B(\lambda_\bullet)]_\mu^{\mathrm{sing}}$ is that described in Section 4.3.1.*

The $J_n^{\lambda_\bullet}$-equivarient bijection is actually given by the RSK correspondence. We will prove this theorem in two stages. First we will prove the special case mentioned above when $\lambda^{(i)} = \square$ for all i. In this case $B(\lambda_i) = B$ the crystal associated to the vector representation and $B(\lambda_\bullet) = B^{\otimes n}$.

4.5.2. *Proof of Theorem 1.2.* Before we move on to the proof of Theorem 4.9, we will explain how using this result we can prove Theorem 1.2. The four varieties we have been investigating and their relationship is summarised in (4.10).

$$\begin{tikzcd}
\mathcal{B}(\lambda_\bullet)_\mu \arrow[r, "\theta"] \arrow[d, "\pi"] & \Omega(\lambda_\bullet,\mu^{\mathsf{c}}) \arrow[r, hook, "\iota"] \arrow[d, "\rho"] & \mathcal{S}(\lambda_\bullet,\mu^{\mathsf{c}}) \arrow[d, "\eta"] & \mathcal{S}(\lambda_\bullet,\mu^{\mathsf{c}})(\mathbb{R}) \arrow[l, hook'] \arrow[d, "\eta|_{\mathbb{R}}"] \\
M_{0,n+1}(\mathbb{C}) \arrow[r, "="] & M_{0,n+1}(\mathbb{C}) \arrow[r, hook] & \overline{M}_{0,n+1}(\mathbb{C}) & \overline{M}_{0,n+1}(\mathbb{R}) \arrow[l, hook']
\end{tikzcd} \tag{4.10}$$

The first claim of Theorem 1.2 is that there is a homomorphism $PJ_n \to \mathrm{Gal}(\pi;z)$ for some generic point $z \in M_{0,n+1}$. The group $\mathrm{Gal}(\pi;z)$ is a subgroup of $S_{\pi^{-1}(z)}$. The morphism $\iota\circ\theta$ identifies the sets $\pi^{-1}(z)$ and $\eta^{-1}(z)$. With this identification fixed, $\mathrm{Gal}(\pi;z) = \mathrm{Gal}(\eta;z)$ by Remark 4.7.

To calculate the monodromy group of $\mathcal{S}(\lambda_\bullet,\mu^{\mathsf{c}})$ choose a dense open subset $z \in U \subseteq \overline{M}_{0,n+1}(\mathbb{C})$ over which η is unramified. We can choose U so that is contains $\overline{M}_{0,n+1}(\mathbb{R})$ by Theorem 3.9. Let $M_{\mathbb{R}} \subseteq S_{\eta^{-1}(z)}$ be the monodromy group of $\eta|_{\mathbb{R}}$. Since $PJ_n = \pi_1(\overline{M}_{0,n+1}(\mathbb{R});z)$, by definition we have a surjective homomorphism $PJ_n \to M_{\mathbb{R}}$. The inclusion of the real points $\mathcal{S}(\lambda_\bullet,\mu^{\mathsf{c}})(\mathbb{R})$ induces an inclusion $M_{\mathbb{R}} \hookrightarrow M_U(\eta;z)$. Proposition 4.8 now implies that $M_U(\eta;z) = \mathrm{Gal}(\eta;z)$. Hence we have a homomorphism from PJ_n onto the subgroup $M_{\mathbb{R}} \subseteq \mathrm{Gal}(\pi;z)$.

The second claim of Theorem 1.2 is that for real z there exists a bijection $\mathcal{B}(\lambda_\bullet;z)_\mu \to \mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$ equivariant for the action of PJ_n. The isomorphism θ identifies $\mathcal{B}(\lambda_\bullet;z)_\mu$ with $\Omega(\lambda_\bullet,\mu^{\mathsf{c}};z,\infty)$. By definition this identification is equivariant for the action of PJ_n. By Theorem 3.22 $\Omega(\lambda_\bullet,\mu^{\mathsf{c}};z,\infty)$ can be identified with $\mathtt{decgd}(()\lambda_\bullet,\mu^{\mathsf{c}})$. Now we may use Theorem 4.9 to find a bijection to $\mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$ which is equivariant with respect to J_n (and thus PJ_n).

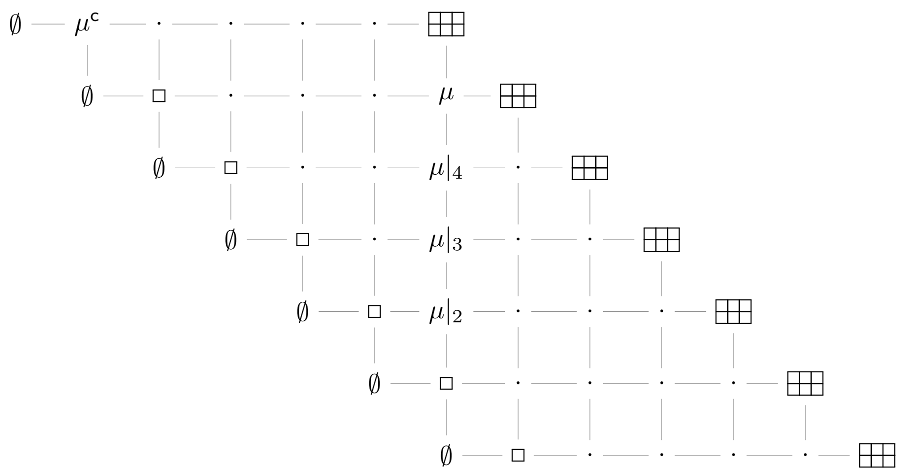


FIGURE 7. A decgd of shape $(\lambda_\bullet, \mu^{\mathsf{c}})$

4.6. **The special case $\lambda_\bullet = (\square^n)$.** We are interested in $\mathtt{decgd}(\square^n, \mu^{\mathsf{c}})$, dual equivalence growth diagrams of shape $(\lambda_\bullet, \mu^{\mathsf{c}})$. When $n = 5$ such a diagram looks something like in Figure 7. Note that the partition in position $(1, 6)$ is μ (the bottom left corner is in position $(1, 1)$). This comes from the following lemma.

Lemma 4.11. *If $\gamma \in \mathtt{decgd}(\square^n, \mu^{\mathsf{c}})$ then $\gamma_{1(n+1)} = \mu$.*

Proof. By definition $\gamma_{n(n+1)} = \mu^{\mathsf{c}}$. Consider the rectangle with corners $(n+1, n+1), (n+1, n+2), (1, n+1)$ and $(1, n+2)$. Lift this rectangular region to a growth diagram and let $\gamma_{1(n+1)} = \nu$,

We label the tableaux obtained by lifting the unique dual equivalence classes by S and T. Since our rectangle is a growth diagram, S must be the rectification of T. However the rectification of a tableaux of shape $\Lambda\backslash\mu$ has shape μ^{c}. One way to see this is to note $c^{\Lambda}_{\nu\mu^{\mathsf{c}}}$ is the number of $T \in \mathtt{SYT}(\Lambda\backslash\nu)$ slide equivalent to S, since we have produced such a T, $c^{\Lambda}_{\nu\mu^{\mathsf{c}}} > 0$. But $c^{\Lambda}_{\nu\mu^{\mathsf{c}}} = \delta_{\nu\mu}$ hence $\nu = \mu$. $\square$

We can use Lemma 4.11 to deduce there is a bijection between $\mathtt{decgd}(\square^n, \mu^{\mathsf{c}})$ and $\mathtt{SYT}(\mu)$ given by choosing any path though $\mathbb{I}$. The standard Young tableaux associated to a decgd γ is described by the growth diagram defined by the path. Fix the unique path from $(1, 1)$ to $(1, n+2)$ and denote it α. As described above use α to identify $\mathtt{decgd}(\square^n, \mu^{\mathsf{c}})$ with $\mathtt{SYT}(\mu)$ using the corresponding bijection.

There is an action of the cactus group J_n on $\mathtt{SYT}(\mu)$ by *partial Schützenberger involutions*. This action was studied by Berenstein and Kirillov [KB95]. The partial Schützenberger involution of order q on $T \in \mathtt{SYT}(\mu)$ is defined by applying the

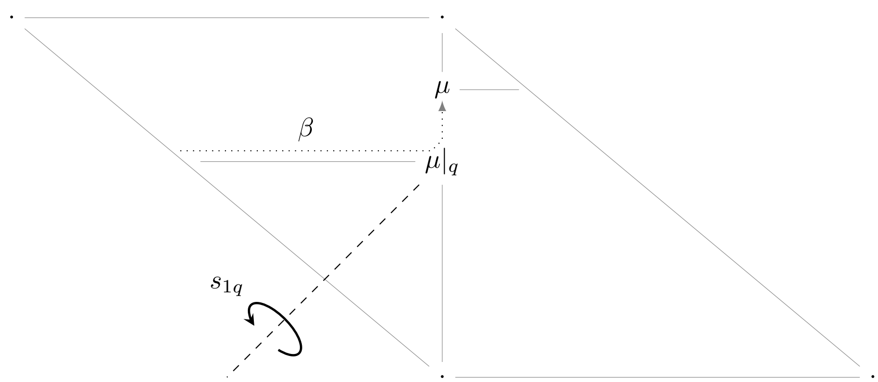


FIGURE 8. The action of s_{1q}

Schützenberger involution to the subtableau $T|_q$ and leaving the remaining entries (i.e. those in $T|_{q+1,n}$) unchanged.

Proposition 4.12. *The identification of* $\mathtt{SYT}(\mu)$ *and* $\mathtt{decgd}(\square^n, \mu^c)$ *above, identifies the action of* J_n *on both sides. More precisely if* T *is obtained using the path* α *from the decgd* γ *then the standard tableaux* $s \cdot T$ *is obtained by taking the path* α *through the decgd* $s \cdot \gamma$*, for all* $s \in J_n$*.*

Proof. We only need to show this for $s = s_{1q}$ by Lemma 2.5. Let γ be the decgd with T along the path α. Denote the shape of $T|_q$ by $\mu|_q$, so $\gamma_{1(q+1)} = \mu|_q$. Consider the triangle in γ depicted in Figure 8. Proposition 3.23 says that $s_{1q} \cdot \gamma$ will contain the same triangle, flipped about the axis shown. In particular the tableaux obtained along the path α in $s_{1q} \cdot \gamma$ is the same as the tableaux obtained along the path β from $(q+1, q+1)$ to $(1, q+1)$ and then to $(1, n+2)$ in γ. By Corollary 2.7 this is the partial Schützenberger involution $s_{1q} \cdot T$. $\square$

4.6.1. *Schützenberger involution and RSK.* If $w = x_1 x_2 \ldots x_n$ is a word in some alphabet, with $x_i \in \mathcal{A}$ then denote

$$w^* \stackrel{\text{def}}{=} x_n^* x_{n-1}^* \cdots x_1^*.$$

With this notation we have an amazing duality theorem.

Theorem 4.13. *If* $\mathtt{RSK}(w) = (P, Q)$ *then* $\mathtt{RSK}(w^*) = (\xi P, \mathtt{evac} Q)$*.*

Proof. See Section 1 of Appendix A in [Ful97] for the proof. $\square$

We also have the following proposition that gives information about the Q-symbol of subwords. Denote by $T|_{r,s}$ the skew tableaux obtained from T by throwing away boxes labelled with numbers outside the range $[r, s]$.

Proposition 4.14. *If* $w = x_1 x_2 \cdots x_n$ *is a word with* Q*-symbol* Q *and* $u = x_r x_{r+1} \cdots x_s$ *is a (contiguous) subword, then the* Q *symbol of* u *is* $\mathtt{Rect}(Q|_{r,s})$*.*

Proof. See Proposition 1 in Section 5.1 of [Ful97]. $\square$

4.6.2. *Proof of special case.* The result of this process is much easier to describe in $\mathtt{SYT}(\mu)$. If $\gamma \in \mathtt{SYT}(\mu)$ then $\hat{\gamma} = s_{1q} \cdot \gamma$ is the result of applying the Schützenberger involution to the boxes labelled $1, 2, \ldots, q$ in γ.

We will give a J_n-equivarient bijection $[B^{\otimes n}]_\mu^{\text{sing}} \longrightarrow \mathtt{SYT}(\mu)$. The bijection we will consider is $\mathtt{Q} \circ \mathtt{RSK}$, where $\mathtt{Q}$ denotes taking the Q-symbol.

Suppose first that $q = n$. Let $w = b_1 b_2 \cdots b_n \in [B^{\otimes n}]_\mu^{\text{sing}}$ be a highest weight word and $\mathtt{Q} \circ \mathtt{RSK}(w) = Q$, in this case $s_{1n} \cdot w$ is by definition

$$\xi(\xi(b_n)\xi(b_{n-1}) \cdots \xi(b_1)) = \xi(w^*).$$

Since the involution ξ does not change the Q-symbol of a word, by Theorem 4.13 we have $\mathtt{Q} \circ \mathtt{RSK}(s_{1n} \cdot w) = \mathtt{evac}Q$.

Now suppose $q < n$. In this case, note by the definition of the RSK algorithm, the position of the first q integers in the Q-symbol of a word is completely determined by the first q letters if the word. Hence by the above

$$\mathtt{Q} \circ \mathtt{RSK}(s_{1q} \cdot w) = s_{1q} \cdot \mathtt{Q} \circ \mathtt{RSK}(w).$$

4.7. **The proof of Theorem 4.9 in the general case.** We now consider the case for general $\lambda_\bullet$, let $\tilde{n} = |\lambda_\bullet|$ Our strategy will be the following, we will define embeddings $[B(\lambda_\bullet)]_\mu^{\text{sing}} \hookrightarrow [B^{\otimes n}]_\mu^{\text{sing}}$ and $\mathtt{decgd}(\lambda_\bullet, \mu) \hookrightarrow \mathtt{decgd}((1)^n, \mu)$ that are consistent in the sense that the outer squares of the following diagram commute.

$$\begin{tikzcd}
{[B(\lambda_\bullet)]_\mu^{\text{sing}}} \arrow[r, hook] \arrow[d, "s"] & \left[B^{\otimes \tilde{n}}\right]_\mu^{\text{sing}} \arrow[r, leftrightarrow] \arrow[d, "\bar{s}"] & \mathtt{decgd}((1)^{\tilde{n}}, \mu) \arrow[d, "\bar{s}"] & \mathtt{decgd}(\lambda_\bullet, \mu) \arrow[l, hook'] \arrow[d, "s"] \\
{[B(\hat{s} \cdot \lambda_\bullet)]_\mu^{\text{sing}}} \arrow[r, hook] & \left[B^{\otimes \tilde{n}}\right]_\mu^{\text{sing}} \arrow[r, leftrightarrow] & \mathtt{decgd}((1)^{\tilde{n}}, \mu) & \mathtt{decgd}(\hat{s} \cdot \lambda_\bullet, \mu) \arrow[l, hook']
\end{tikzcd} \tag{4.15}$$

Here s is an element of $J_n^{\lambda_\bullet}$, $\hat{s}$ its image in S_n and $\bar{s}$ is an element of $J_{\tilde{n}}$ defined using s which will be explained below. Both embeddings will be given by a sequence of standard tableaux $(T_\bullet)$, where T_i is a standard λ_i-tableau, we call this object a standard $\lambda_\bullet$-tableau. Note that the inner square of (4.15) commutes by Section 4.6.2.

Let us first explain the construction of $\bar{s}_{pq}$ for a generator $s_{pq} \in J_n$. The idea is that the image of $\bar{s}$ in $S_{\tilde{n}}$ acts by preserving the blocks of the first $|\lambda_1|$ letters, the next $|\lambda_2|$ letters and so on, while permuting these n blocks in the same way as $\hat{s}$. Let $m_i = \sum_{j=1}^{i-1} |\lambda_j|$ and $m_i^s = \sum_{j=0}^{i-1} \left|\lambda_{\hat{s}(j)}\right|$ for $s \in J_n$, and denote the generators of $J_{\tilde{n}}$ by $\tilde{s}_{pq}$. Then define

$$\bar{s}_{pq} = \left(\prod_{i=p}^{q} \tilde{s}_{(m_i^{s_{pq}}+1) m_{i+1}^{s_{pq}}} \right) \tilde{s}_{(m_p+1)m_q}.$$

Given a standard $\lambda_\bullet$-tableau, $T_\bullet$, we define the embedding $[B(\lambda_\bullet)]_\mu^{\text{sing}} \hookrightarrow [B^{\otimes n}]_\mu^{\text{sing}}$, denoted $\iota_{T_\bullet}$ by sending $b_1 \otimes \cdots \otimes b_n \in [B(\lambda_\bullet)]_\mu^{\text{sing}}$ to the word $w = x_1 x_2 \cdots x_{\tilde{n}} \in [B^{\otimes n}]_\mu^{\text{sing}}$ such that $x_{m_i+1} \cdots x_{m_{i+1}}$ is the unique word with the same weight as b_i and with Q-symbol T_i. We say that w having this property has $T_\bullet$ as its $\lambda_\bullet$-*partial* Q-*symbols*.

Lemma 4.16. *The left hand square of (4.15),*

$$\begin{tikzcd}
{[B(\lambda_\bullet)]^{\mathrm{sing}}_\mu} \arrow[r, hook, "\iota_{T_\bullet}"] \arrow[d, "s_{1q}"] & {\left[B^{\otimes \tilde{n}}\right]^{\mathrm{sing}}_\mu} \arrow[d, "\bar{s}_{1q}"] \\
{[B(\hat{s}_{1q} \cdot \lambda_\bullet)]^{\mathrm{sing}}_\mu} \arrow[r, hook, "\iota_{\hat{s}_{1q} \cdot T_\bullet}"] & {\left[B^{\otimes \tilde{n}}\right]^{\mathrm{sing}}_\mu}
\end{tikzcd},$$

commutes.

Proof. Let $b = b_1 \otimes \cdots \otimes b_n \in [B(\lambda_\bullet)]^{\mathrm{sing}}_\mu$. By definition $\iota_{\hat{s}_{1q} \cdot T_\bullet} \circ s_{1q}(b)$ has $\hat{s}_{1q} \cdot \lambda_\bullet$-partial Q-symbols $\hat{s}_{1q} \cdot T_\bullet$. Our first job is to show that the same is true for $\bar{s}_{1q} \circ \iota_{T_\bullet}(b)$, i.e. that $\bar{s}_{1q} \circ \iota_{T_\bullet}(b)$ lies in the same copy of $[B(\lambda_\bullet)]^{\mathrm{sing}}_\mu$.

By definition $w = \iota_{T_\bullet}(b)$ has $\lambda_\bullet$-partial Q-symbols $T_\bullet$, that is, if $w = x_1 x_2 \cdots x_{\tilde{n}}$ then $x_{m_i+1} \cdots x_{m_{i+1}}$ has Q-symbol T_i. We will use the notation $w|_{i,j}$ for the subword $x_i \cdots x_j$. Since we are now looking at the $\hat{s}_{1q} \cdot \lambda_\bullet$ partial Q-symbols. If $i > q$ then

$$\begin{aligned}
\mathtt{Q} \circ \mathtt{RSK} \left((\bar{s}_{1q} \cdot w) \,|_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}} \right) &= \mathtt{Q} \circ \mathtt{RSK} \left(w|_{m_i+1, m_{i+1}} \right) \\
&= T_i,
\end{aligned}$$

and since $\hat{s}_{1q}(i) = i$, $T_i = T_{\hat{s}_{1q}(i)}$.

If $i < q$ and $\mathtt{Q} \circ \mathtt{RSK}(s_{1m_q} \cdot w) = Q$ then let $s_{1m_q} \cdot w = y_1 y_2 \cdots y_{\tilde{n}}$.

$$\begin{aligned}
T_i' &= \mathtt{Q} \circ \mathtt{RSK} \left((\bar{s}_{1q} \cdot w) \,|_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}} \right) \\
&= \mathtt{Q} \circ \mathtt{RSK} \left(\left(s_{(m_i^{s_{1q}}+1) m_{i+1}^{s_{1q}}} \cdot y_1 y_2 \cdots y_{\tilde{n}} \right) |_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}} \right) \\
&= \mathtt{Q} \circ \mathtt{RSK} \left(\xi(y^*_{m_{i+1}^{s_{1q}}} \cdots y^*_{m_i^{s_{1q}}+1}) \right) \\
&= \mathtt{evac} \circ \mathtt{Q} \circ \mathtt{RSK} \left(y_{m_i^{s_{1q}}+1} \cdots y_{m_{i+1}^{s_{1q}}} \right) \\
&= \mathtt{evac} \circ \mathtt{Rect} \left(\mathtt{Q} \circ \mathtt{RSK} \left(y_1 \cdots y_{\tilde{n}} \right) |_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}} \right) \\
&= \mathtt{evac} \circ \mathtt{Rect} \left(\mathtt{Q} \circ \mathtt{RSK} \left(\xi(x^*_{m_q} \cdots x^*_1) x_{m_q+1} \cdots x_{\tilde{n}} \right) |_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}} \right) \\
&= \mathtt{evac} \circ \mathtt{Rect} \left(\mathtt{Q} \circ \mathtt{RSK} \left(x^*_{m_q} \cdots x^*_1 x_{m_q+1} \cdots x_{\tilde{n}} \right) |_{m_i^{s_{1q}}+1, m_{i+1}^{s_{1q}}} \right) \\
&= \mathtt{evac} \circ \mathtt{Q} \circ \mathtt{RSK} \left(x^*_{m_{s_{1q}(i)+1}} \cdots x^*_{m_{s_{1q}(i)}+1} \right) \\
&= \mathtt{evac} \circ \mathtt{evac} \circ \mathtt{Q} \circ \mathtt{RSK} \left(x_{m_{s_{1q}(i)}+1} \cdots x_{m_{s_{1q}(i)+1}} \right) \\
&= \mathtt{Q} \circ \mathtt{RSK} \left(x_{m_{s_{1q}(i)}+1} \cdots x_{m_{s_{1q}(i)+1}} \right) \\
&= T_{\hat{s}_{1q}(i)}.
\end{aligned}$$

The important facts here are that ξ preserves Q-symbols and Proposition 4.14.

Note that the above, by Proposition 2.9 and Theorem 2.10, property iii, $\iota_{\hat{s}_{1q} \cdot T_\bullet} \circ s_{1q}(b)$ and $\bar{s}_{1q} \circ \iota_{T_\bullet}(b)$ are dual equivalent words. Since they are by definition highest weight words they are also slide equivalent and thus by Theorem 2.10, property ii, must be the same word. $\square$

Now we explain the embeddings in the right hand square of (4.15). Again let $T_\bullet$ be a standard $\lambda_\bullet$-tableaux. For $\gamma \in \mathtt{decgd}(\lambda_\bullet, \mu^c)$ we can lift this to $\mathtt{decgd}(\Box^{\tilde{n}}, \mu^c)$ by simply choosing a representative for each dual equivalence class along the path from $(1,1)$ to $(1, n+2)$. We want to do this in a controlled way. Let α_i be the dual equivalence class allocated to the edge $(1,i)-(1,i+1)$. Choose a lift S_i of α_i such that S_i is slide equivalent to T_i. Since the intersection of any slide equivalence class and dual equivalence class is a single tableaux, we have a unique choice for S_i. Let $\iota_{T_\bullet}$ be the decgd in $\mathtt{decgd}(\Box^{\tilde{n}}, \mu^c)$ defined by this choice.

Lemma 4.17. *The right hand square of (4.15),*

$$\begin{tikzcd}
\mathtt{decgd}(\lambda_\bullet, \mu^c) \arrow[r, hook, "\iota_{T_\bullet}"] \arrow[d, "s_{1q}"] & \mathtt{decgd}(\Box^{\tilde{n}}, \mu^c) \arrow[d, "\bar{s}_{1q}"] \\
\mathtt{decgd}(\hat{s}_{1q} \cdot \lambda_\bullet, \mu^c) \arrow[r, hook, "\iota_{\hat{s}_{1q} \cdot T_\bullet}"] & \mathtt{decgd}(\Box^{\tilde{n}}, \mu^c)
\end{tikzcd},$$

commutes.

Proof. By Corollary 3.29, the action of the cactus group on decgd's is simply given by the rotation of certain triangles. Let $\gamma \in \mathtt{decgd}(\lambda_\bullet, \mu^c)$. We will first calculate the tableaux defined by the growth along the path from $(0,0)$ to $(0, \tilde{n}+1)$ in $\iota_{\bar{s}_{1q} \cdot T_\bullet}(s_{1q} \cdot \gamma)$. As depicted in Figure 9 (for $q = 4$) let α_i be the dual equivalence class on the edge connecting $(0, i-1)$ and $(0, i)$, for $1 \leq i \leq n$ and β_i the dual equivalence class of of the edge connecting $(q, i-1)$ and (q, i) for $1 \leq i \leq q$. Furthermore let U_i and V_i be the unique standard tableaux of α_i and β_i respectively which is slide equivalent to T_i.

The action of s_{1q} flips the triangle about the axis shown and preserves the partitions and dual equivalence classes along the path from $(0,q)$ to $(0, n+1)$ by Proposition 3.23. By definition, $\iota_{\hat{s}_{1q} \cdot T_\bullet}(s_{1q} \cdot \gamma)$ is then constructed by lifting the appropriate dual equivalence classes to the following tableaux along the path $(0,0) - (0, \tilde{n}+1)$,

$$V_q \;\text{------}\; V_{q-1} \;\text{------}\; \cdots \;\text{------}\; V_1 \;\text{------}\; U_{q+1} \;\text{------}\; \cdots \;\text{------}\; U_n. \tag{4.18}$$

This determines $\iota_{\hat{s}_{1q} \cdot T_\bullet}(s_{1q} \cdot \gamma)$. Now we make the same calculation for the other side of the commutative diagram.

First apply $\iota_{T_\bullet}$ to γ, which means lifting the dual equivalence classes along $(0,0) - (0, n+1)$ (these are the classes α_i) to U_i. The we need to apply $\bar{s}_{1q}$. This means flipping a large triangle and several smaller triangles. Figure 10 depicts (for $q = 4$) the resulting diagram after flipping *only* the large triangle. We have only marked the dual equivalence classes on the vertical and not the actual tableaux.

Now we flip the small triangles, working right to left. The order we flip does not matter as these elements of the cactus group commute. The first triangle is easy, by Proposition 3.23 we preserve all the other small triangles as well as the entire path $(0, |\lambda^{(q)}|) - (0, \tilde{n}+1)$. We end up with $T_q = V_q$ in the first spot on the path $(0,0) - (0, \tilde{n}+1)$.

The triangles further to the right take some more thought, we will try and flip the i^{th} triangle. Flipping this triangle preserves all the small triangles to the left and the right as well as everything on the path $(0,0) - (0, \tilde{n}+1)$ except for the part

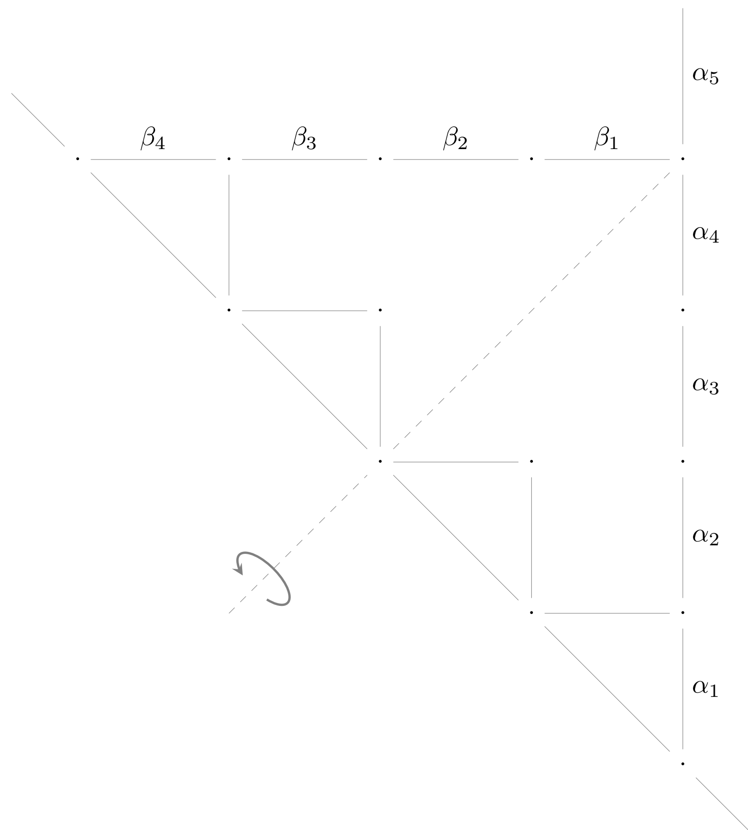


FIGURE 9. The dual equivalence classes α_i and β_i

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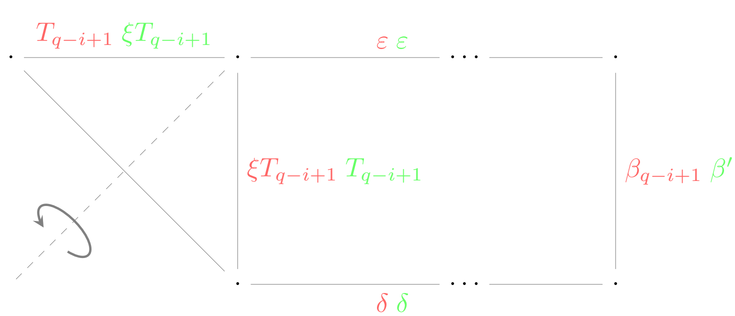
between $(0, m_i^{\hat{s}_{1q}})$ and $(0, m_{i+1}^{\hat{s}_{1q}})$. Locally we have the picture



where we have marked the dual equivalence classes and tableaux before the flip in red and after the flip in green. In fact $\beta' = \beta_{q-i+1}$. To see this denote by η the dual equivalence class of T_i, this is also the dual equivalence class of ξT_i by Theorem 2.10 (i). Thus (δ, β_{q-i+1}) is the shuffle of (η, ε). However (δ, β') is also the shuffle of (η, ε), thus $\beta' = \beta_{q-i+1}$.

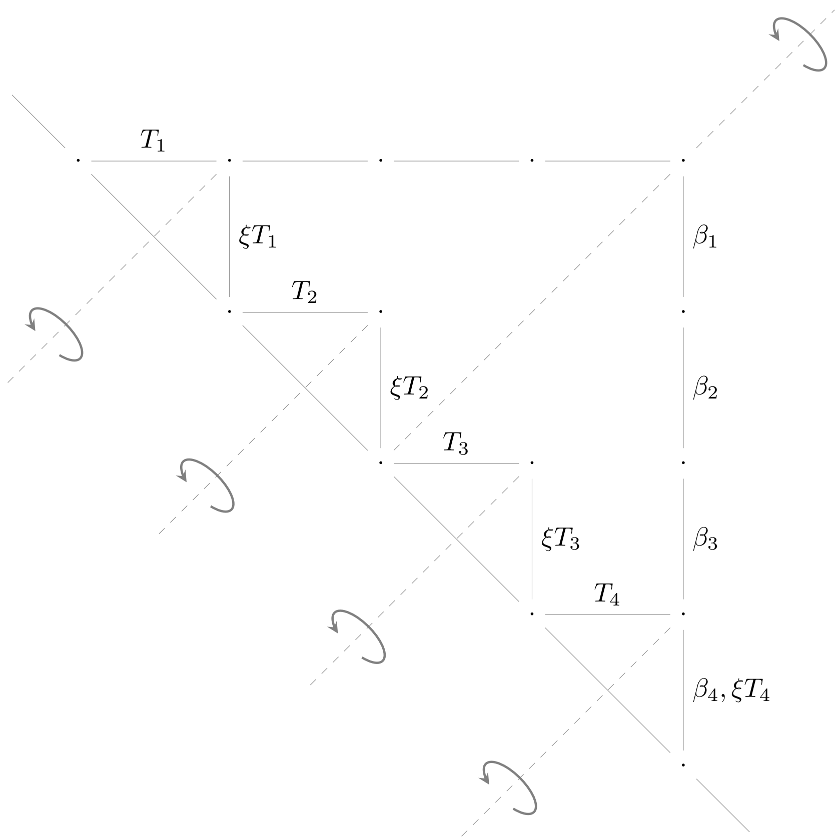


FIGURE 10. $\overline{s}_{1q}$ acting on a decgd

What we have proven now is that after flipping all the triangles, (i.e. in $\overline{s}_{1q} \cdot \iota_{T_\bullet}(\gamma)$) the dual equivalence class in the i^{th} position on the path $(0,0)-(0,m_q)$ is β_{q-i+1} and the tableaux in this position is thus the unique tableaux in β_{q-i+1} slide equivalent to T_{q-i+1}. By assumption this is V_{q-i+1}. Since we never changed anything on the path $(0,m_q)-(0,\tilde{n}+1)$, the sequence of tableaux along the path $(0,0)-(0,\tilde{n}+1)$ is

$$V_q \;\text{------}\; V_{q-1} \;\text{------}\; \cdots \;\text{------}\; V_1 \;\text{------}\; U_{q+1} \;\text{------}\; \cdots \;\text{------}\; U_n.$$

which coincides with (4.18). Hence $\iota_{\hat{s}_{1q}\cdot T_\bullet}(s_{1q}\cdot\gamma) = \overline{s}_{1q}\cdot\iota_{T_\bullet}(\gamma)$. $\square$

4.8. **Monodromy in Rybnikov's compactification.** Theorem 1.2 is conjectured by Rybnikov [Ryb14] in a significantly different form. In this section we recall the conjecture and use Theorem 1.2 to prove it. In fact it is easily seen they are equivalent.

Rybnikov constructs commutative subalgebras $A(\lambda_\bullet;z)_\mu$ of $\mathrm{End}(L(\lambda_\bullet)_\mu^{\mathsf{sing}})$ for *any* point in the compactified moduli space $\overline{M}_{0,n+1}(\mathbb{C})$. For $z \in M_{0,n+1}(\mathbb{C})$, the algebra $A(\lambda_\bullet;z)_\mu$ is simply the Bethe algebra $B(\lambda_\bullet;z)_\mu$. Rybnikov shows that for *all real* $z \in \overline{M}_{0,n+1}(\mathbb{C})$ the algebra $A(\lambda_\bullet; z)_\mu$ has simple spectrum. This means in particular if we let $A(\lambda_\bullet)_\mu$ be the corresponding flat family of algebras over $\overline{M}_{0,n+1}(\mathbb{C})$ and

$$\mathcal{A}(\lambda_\bullet)_\mu \stackrel{\text{def}}{=} \operatorname{Spec} A(\lambda_\bullet)_\mu,$$

the spectrum of these algebras, then the finite map $\mathcal{A}(\lambda_\bullet)_\mu(\mathbb{R}) \to \overline{M}_{0,n+1}(\mathbb{R})$ is a topological covering. The conjecture stated in [Ryb14] which we prove is the following.

Theorem 4.19. *For* $z \in \overline{M}_{0,n+1}(\mathbb{R})$ *the monodromy action of* PJ_n *on* $\mathcal{A}(\lambda_\bullet)_\mu(z)$ *is isomorphic to the action of* PJ_n *on* $\mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$*.*

To prove the theorem we require a lemma about the topology of $\overline{M}_{0,n+1}(\mathbb{R})$ sitting inside $\overline{M}_{0,n+1}(\mathbb{C})$. Let $U \subset \overline{M}_{0,n+1}(\mathbb{C})$ be the dense open set over which $\mathcal{A}(\lambda_\bullet)_\mu$ is unramified, U contains $\overline{M}_{0,n+1}(\mathbb{R})$. Let $U_0 = U \cap M_{0,n+1}(\mathbb{C})$.

Lemma 4.20. *Let* $x, y \in M_{0,n+1}(\mathbb{R})$*. Any path in* $\overline{M}_{0,n+1}(\mathbb{R})$ *with endpoints* x *and* y *is homotopy equivalent to a path in* U_0 *with endpoints* x *and* y*.*

Proof. Any path in $\overline{M}_{0,n+1}(\mathbb{R})$ is homotopy equivalent to another path in $\overline{M}_{0,n+1}(\mathbb{R})$ which passes transversally through codimension 1 cells only. Since U_0 is open and contains $M_{0,n+1}(\mathbb{R})$ it is enough to show we can move our path off such an intersection while remaining in an arbitrarily small neighbourhood of the intersection point.

Locally at the intersection our point is given by n marked points $z_1(t), z_2(t), \ldots, z_n(t) \in \mathbb{R}$ depending on a single parameter t, and we fix the last marked point at infinity. Assume for simplicity that $z_1(t) < z_2(t) < \ldots < z_n(t)$ for $t < 1$ and the path hits the wall at $t = 1$. Suppose we intersect the wall which swaps the order of the marked points $z_p(t), \ldots, z_q(t)$. □

this argument needs to finished.

Proof of Theorem 4.19. Fix a basepoint $z \in M_{0,\mathbb{R}}$. We identify the fibre with $\mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$ using the same process described in Section 4.5.2. Suppose we have a loop γ_s in $\overline{M}_{0,n+1}(\mathbb{R})$ given by the element $s \in PJ_n$. By Lemma 4.20 γ_s is homotopy equivalent to a loop γ_s' contained entirely in U_0. By Theorem 1.2 the monodromy action of s on the fibre is the same as the action of s on $\mathsf{B}(\lambda_\bullet)_\mu^{\mathsf{sing}}$. □

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