This week on the problem set you will get practice at calculating integrals using substitution, integration by parts and using partial fractions. Many of these are routine but some are quite difficult!

Homework: The homework will be due on Friday 21 October, at 2pm, the *start* of the lecture. It will consist of questions:

$$4(m),\ 5,\ \text{and } 6.$$

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (5.5) Calculate the following integrals using substitution.
   (a) (12) $\int \frac{x}{\sqrt{x^2+1}}\ \mathrm{d}x$
   (b) (14) $\int \sin^3 t \cos t\ \mathrm{d}t$
   (c) (16) $\int \frac{z^3}{\sqrt{z^4+12}}\ \mathrm{d}z$
   (d) (19) $\int_1^2 \frac{e^{1/x}}{x^2}\ \mathrm{d}x$
   (e) (23) $\int_1^2 x\sqrt{x-1}\ \mathrm{d}x$
   (f) (24) $\int_0^2 (e^x - e^{-x})^2 dx$
2. (5.6) Calculate the following integrals using integration by parts.
   (a) (2) $\int e^t \sin t\ \mathrm{d}t$
   (b) (6) $\int x^2 \ln x\ \mathrm{d}x$
   (c) (9) $\int \sin x \cos x\ \mathrm{d}x$
   (d) (14) $\int_0^\pi x \sin x\ \mathrm{d}x$
   (e) (16) $\int_1^e x^3 \ln x\ \mathrm{d}x$
3. (5.6) Calculate the following integrals using partial fractions.
   (a) (17) $\int \frac{\mathrm{d}N}{N(1000-N)}$
   (b) (19) $\int \frac{x}{x(1000-x)}\ \mathrm{d}x$
   (c) (20) $\int \frac{(x+1)\ \mathrm{d}x}{(x+2)(x+3)}$
   (d) (22) $\int \frac{4}{(x+1)(x+2)(x+3)}\ \mathrm{d}x$
4. Use any method to evaluate the following integrals.
   (a) $\int x\sqrt{x+1}\ \mathrm{d}x$
   (b) $\int_1^2 \frac{t}{(t^2+1)\ln(t^2+1)}\ \mathrm{d}t$
   (c) $\int \frac{\ln x}{x^5}\ \mathrm{d}x$
   (d) $\int \ln x\ \mathrm{d}x$
   (e) $\int (\ln x)^2\ \mathrm{d}x$
   (f) $\int_1^e (\ln x)^3\ \mathrm{d}x$
   (g) $\int e^{6x} \sin(e^{3x})\ \mathrm{d}x$
   (h) $\int_0^1 \frac{t^3 e^{t^2}}{(t^2+1)^2}\ \mathrm{d}t$
   (i) $\int \frac{2x+3}{z^2-9}\ \mathrm{d}z$

(j) $\int \frac{x^2+x-1}{(x^2-1)}\,\mathrm{d}x$

(k) $\int \frac{e^x}{(e^x-1)(e^x+3)}\,\mathrm{d}x$

(l) $\int_0^{\pi/3} e^t \sin t\,\mathrm{d}t$

(m) $\int e^{\sqrt{x}}\,\mathrm{d}x$

Solution: We will first use a substitution $t = \sqrt{x}$. Thus $t' = \frac{1}{2}x^{-\frac{1}{2}}$. So

$$\begin{aligned}\int e^{\sqrt{x}}\,\mathrm{d}x &= \int e^{\sqrt{x}} \cdot 2\sqrt{x} \cdot \frac{1}{2\sqrt{x}}\,\mathrm{d}x \\ &= 2\int te^t\,\mathrm{d}t.\end{aligned}$$

Now we can use integration by parts. Setting $u = t$ and $v' = e^t$, then $u' = 1$ and $v = e^t$ so

$$\begin{aligned}\int e^{\sqrt{x}}\,\mathrm{d}x &= 2te^t - 2\int e^t \\ &= 2te^t - 2e^t + C \\ &= 2(t-1)e^t + C \\ &= 2(\sqrt{x}-1)e^{\sqrt{x}} + C.\end{aligned}$$

(n) $\int \frac{1}{\cos x}\,\mathrm{d}x$ (quite challenging)

(o) $\int (\sin x)^2\,\mathrm{d}x$ (quite challenging)

5. (5.5-30) Suppose an environmental study indicates that the ozone level, L, in the air above a major metropolitan center is changing at a rate modeled by the function

$$L'(t) = \frac{0.24 - 0.03t}{\sqrt{36 + 16t - t^2}}$$

parts per million per hour (ppm/h) t hours after 7:00 A.M.

(a) Express the ozone level $L(t)$ as a function of t if L is 4 ppm at 7:00 A.M.

Solution: The function $L(t)$ expressing the ozone level at time t will be an antiderivative of $L'(t)$. That is

$$L(t) = \int \frac{0.24 - 0.03t}{\sqrt{36 + 16t - t^2}}\,\mathrm{d}t.$$

We will use the substitution $u = 36 + 16t - t^2$. Thus $u' = 2(8-t)$. Note that $0.24 - 0.03t = 0.03(8-t)$. Thus

$$\begin{aligned}L(t) &= \int \frac{0.03}{\sqrt{36 + 16t - t^2}} \cdot \frac{1}{2}2(8-t)\,\mathrm{d}t \\ &= 0.03\int \frac{1}{2\sqrt{u}}\,\mathrm{d}u \\ &= 0.03\sqrt{u} + C \\ &= 0.03\sqrt{36 + 16t - t^2} + C\end{aligned}$$

To find the constant C we simply solve the equation $L(0) = 4$, that is,

$$\begin{aligned} 0.03\sqrt{36 + 16 \cdot 0 - 0^2} + C &= 4 \\ 0.03\sqrt{36} + C &= \\ 0.18 + C &= \\ C &= 4 - 0.18 = 3.82. \end{aligned}$$

Thus

$$L(t) = 0.03\sqrt{36 + 16t - t^2} + 3.82.$$

(b) Find the time between 7:00 A.M. and 7:00 P.M. when the highest level of ozone occurs. What is the highest level? *(Note: part b has been changed slightly from what is written in the textbook.)*

Solution: First we find the critical points by setting $L'(t) = 0$. This happens when $t = 8$, i.e. at 3pm. Using the first derivative test we know this is a maximum. Thus the highest level of ozone is

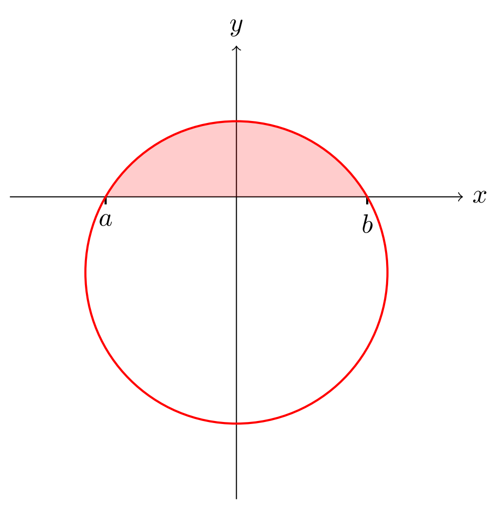
$$L(8) = 0.03\sqrt{26 + 16 \cdot 8 - 64} - 3.82 = 0.09\sqrt{10} - 3.82 = 4.10\text{ppm}$$

6. The circle $x^2 + (y + 1)^2 = 4$ has area 4π. What is the area of the portion of the circle lying above the x axis?

You may use the fact that

$$\int \sqrt{1 - t^2}\, \mathrm{d}t = \frac{1}{2}\left(t\sqrt{1 - t^2} + \sin^{-1} t\right) + C.$$

Solution: We first draw a picture so that we can visualise the area we would like to find.



We want to find the shaded area. The circle is given by the equation $x^2 + (y + 1)^2 = 4$, which means the function that describes the top half semicircle is

$$y = \sqrt{4 - x^2} - 1$$

and the area is given by the integral

$$A = \int_a^b \sqrt{4-x^2} - 1 \, \mathrm{d}x.$$

Here a and b are the x-intercepts of the semicircle. We can find these by setting $y = 0$ and solving for x:

$$\begin{aligned} 0 &= \sqrt{4-x^2} - 1 \\ 1 &= \sqrt{4-x^2} \\ 1 &= 4 - x^2 \\ x^2 &= 4 - 1 = 3 \\ x &= \pm\sqrt{3}. \end{aligned}$$

Thus $a = -\sqrt{3}$ and $b = \sqrt{3}$. Thus

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2} - 1 \, \mathrm{d}x$$

which we can separate,

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4-x^2} \, \mathrm{d}x - \int_{-\sqrt{3}}^{\sqrt{3}} \mathrm{d}x$$

and factor out the 4,

$$= \int_{-\sqrt{3}}^{\sqrt{3}} 2\sqrt{1 - \left(\frac{x}{2}\right)^2} \, \mathrm{d}x - \int_{-\sqrt{3}}^{\sqrt{3}} \mathrm{d}x.$$

Note that

$$\int_{-\sqrt{3}}^{\sqrt{3}} \mathrm{d}x = 2\sqrt{3}. \tag{1}$$

We can solve the first part of A by using the substitution $u = \frac{x}{2}$, so $u' = \frac{1}{2}$. Note that when $x = \pm\sqrt{3}$ then $u = \pm\frac{\sqrt{3}}{2}$. This means

$$2\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{1 - \left(\frac{x}{2}\right)^2} \, \mathrm{d}x = 4\int_{-\sqrt{3}/2}^{\sqrt{3}/2} \sqrt{1-u^2} \, \mathrm{d}x$$

now we can apply the antiderivative given in the question,

$$= 4\left[\frac{1}{2}u\sqrt{1-u^2} + \frac{1}{2}\sin^{-1} u\right]_{-\sqrt{3}/2}^{\sqrt{3}/2}$$

noting that $\sin^{-1}\left(\pm\frac{\sqrt{3}}{2}\right) = \pm\frac{pi}{3}$, we get

$$\begin{aligned} &= \sqrt{3} \cdot \frac{1}{2} + 2 \cdot \frac{\pi}{3} - \left(-\sqrt{3}\right) \cdot \frac{1}{2} - 2 \cdot \left(-\frac{\pi}{3}\right) \\ &= \sqrt{3} + \frac{4\pi}{3}. \end{aligned} \tag{2}$$

Since $A = (2) - (1)$ we have $A = \frac{4\pi}{3} - \sqrt{3}$.

7. (5.6-36) Assume that after t hours on the job, a factory worker can produce $100te^{0.5t}$ units per hour. How many units does the worker produce during the first 3 hours?

8. (5.6-38) An actuary measures the probability that a person in a certain population will die at age x by the formula

$$P(x) = \lambda^2 x e^{-\lambda x}$$

where λ is a parameter such that $0 < \lambda < e$.

   (a) For a given λ, find the maximum value of $P(x)$.

   (b) Sketch the graph of $P(x)$.

   (c) Find the area under the probability curve $y = P(x)$ for $0 \leq x \leq 100$, and interpret your result.

9. (5.6-39) A population P, grows at the rate

$$P'(t) = 5(t+1)\ln\sqrt{t+1}$$

thousand individuals per year at time t (in years). By how much does the population change during the eighth year?