Math 3B: Lecture 14

Noah White

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Last time

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Integration and applications

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- Partial fractions and long division
- Accumulated change and Riemann sums

Differential equations (motivation)

An (ordinary) differential equation (or ODE) is an equation that involves derivatives of an unknown function.

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or

$$x^2 y'' + xy' + x^2 y = 0$$

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The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

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And so on.

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Note

The right hand side of the equation does not have any y's.

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- draw solutions for many other ODEs
- classify the behaviour of many ODEs (e.g. does the solution go to zero or infinity?)
- understand how sensitive ODEs are to their parameters.

Initial value problems

- If we try to solve the differential equation

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we get (by integrating)

$$y(t) = t^3 + \cos t + C.$$

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- The extra piece of data we need is called an "initial value".
- E.g. $y(0) = 2$.
- Then we see that $y(0) = 1 + C$, so $C = 1$.

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- If we want to draw the graph of $y(t)$ then we look at $g(0, 1)$.
- If this is positive we go up, negative we go down!

Modelling using differential equations

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- The goal is to write down a function $y(t)$ that describes something we are interested in (e.g. population/mass/etc)
- as some other variable changes (usually time)
- We can't do this directly, but we can write down an ODE that y satisfies instead.

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- Number of deaths is proportional to the total number of people. So

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The total change in population at time t is

$$\begin{aligned}\frac{\mathrm{d}N}{\mathrm{d}t} &= \text{ births at } t - \text{ deaths at } t\\ &= bN(t) - dN(t)\\ &= (b-d)N(t).\end{aligned}$$

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In real life we would determine b and d experimentally. Let $r = b - d$. the instinsic growth rate. So our model is

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

and we know $N(0) = 100$.

Behaviour of solutions

$$\frac{\mathrm{d}N}{\mathrm{d}t} = rN.$$

Case 1: $r = 0$

The population never grows or shrinks, it always stays the same (so $N(t) = 100$ for all t).

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The population is increasing indefinitely.

Case 3: $r < 0$
The population is decreasing indefinitely.

Solution to a simple ODE

Theorem

For any constant a, if y is a solution to the ODE

$$\frac{\mathrm{d}y}{\mathrm{d}x} = ay$$

then y is given by

$$y = Ce^{ax}$$

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Next time

We will see why, but for now we can verify it is actually a solution:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x} Ce^{ax} = C \frac{\mathrm{d}}{\mathrm{d}x} e^{a}x = Cae^{ax} = ay.$$

Back to example 1

We know our population model was governed by

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$$100 = Ce^{(b-d)} \quad \text{so} \quad C = 100e^{(d-b)}.$$

Logistic growth

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Where $K = r/k$.

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$$\begin{aligned}
\frac{\mathrm{d}N}{\mathrm{d}t} &= bN - (d + kN)N \\
&= (b - d - kN)N = (r - kN)N \\
&= r\left(1 - \frac{kN}{r}\right) N = r\left(1 - \frac{N}{K}\right) N
\end{aligned}$$

Where $K = r/k$.

Logistic growth

The equation

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

is called the Logistic equation and K is the carrying capacity.

Behaviour of logistic growth

Assume that $r > 0$ and $K > 0$.

$$\frac{\mathrm{d}N}{\mathrm{d}t} = r\left(1 - \frac{N}{K}\right)N$$

Case 1. $N(0) = 0$

In this case the growth rate is 0 initially, so N(t) does not increase or decrease, so remains 0.

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Key takeaway

Both $N(t) = 0$ and $N(t) = K$ are solutions to the ODE. They are called equalibrium solutions.

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Case 3. $0 \leq N(0) \leq K$

In this case, N is initially increasing and so becomes more positive, slowing down as it gets close to K.

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Case 4. $N(0) \geq K$

In this case N is initially decreasing but decreases slower and slower as it gets close to K.