Math 3B: Lecture 6

Noah White

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Differential equations (motivation)

A differential equation is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

$$x^2 y'' + xy' + x^2 y = 0$$

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The chanllenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

Newton's second law (motivation)

The original differential equation!

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$$a = h''(t)$$

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If $h(t)$ measures the height of an object (maybe an apple?) above the earth then

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The force due to gravity is roughly $-10m$ Newtons, so

$$-10m = mh''(t)$$

Population growth (motivation)

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If $P(t)$ is the population at time t:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(t)$$

Some more examples of differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y(1-y)$$

$$y'' = \sqrt{a^2 - (y')^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(A-y)^2$$

Antiderivatives

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The solution $y = F(x)$ is called the antiderivative of $f(x)$.

Example 1

Question

What is the antiderivative of $f(x) = 2x$?

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Solution

\[F(x) = x^2 { }+ 4}+ 8}+ C} \

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$$F(x) = \frac{1}{2}e^{2x}$$

Example 4

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What is the antiderivative of $f(x) = \frac{1}{x}$ (for $x > 0$)?

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Solution

$$F(x) = \ln x$$

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What is the antiderivative of $f(x) = \frac{1}{(1+x)^2}$?

Solution

$$F(x) = -\frac{1}{1+x}$$

Example 6

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What is the antiderivative of $f(x) = 2x \cos x^2$?

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Solution

$$F(x) = \sin x^2$$

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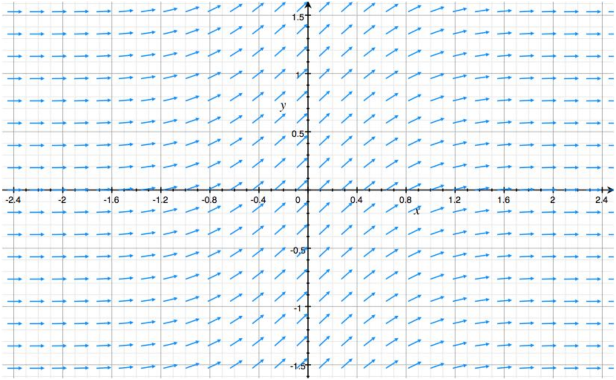
$$F(x) = 2x^{\frac{1}{2}}$$

Slope fields

In some cases it is impossible to find the antiderivative (without special functions). E.g.

$$f(x) = e^{-x^2}$$

But we can still graph the antiderivative! First we draw the slope field



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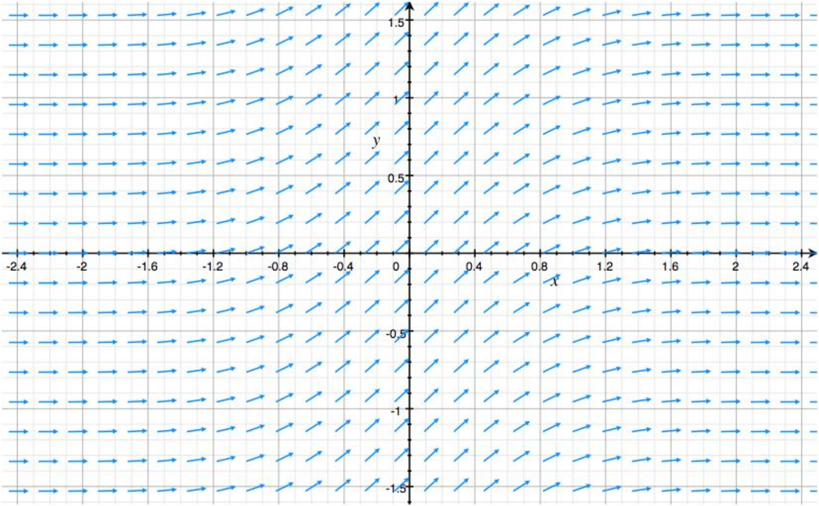

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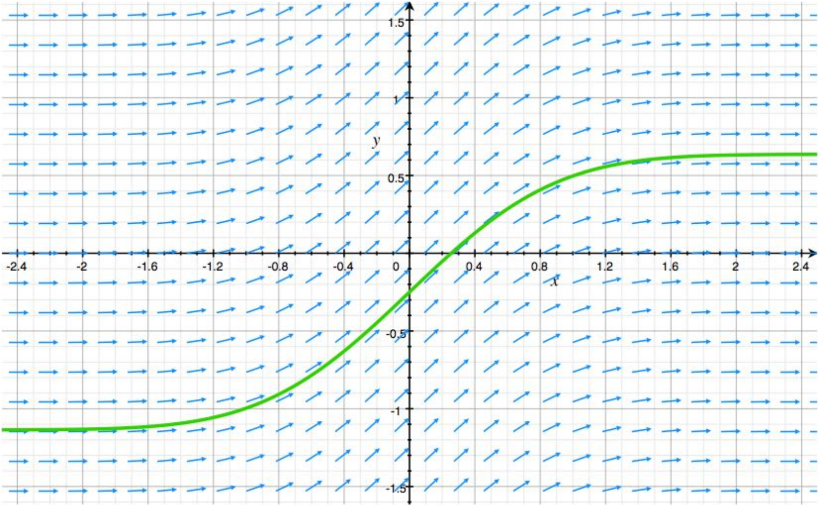
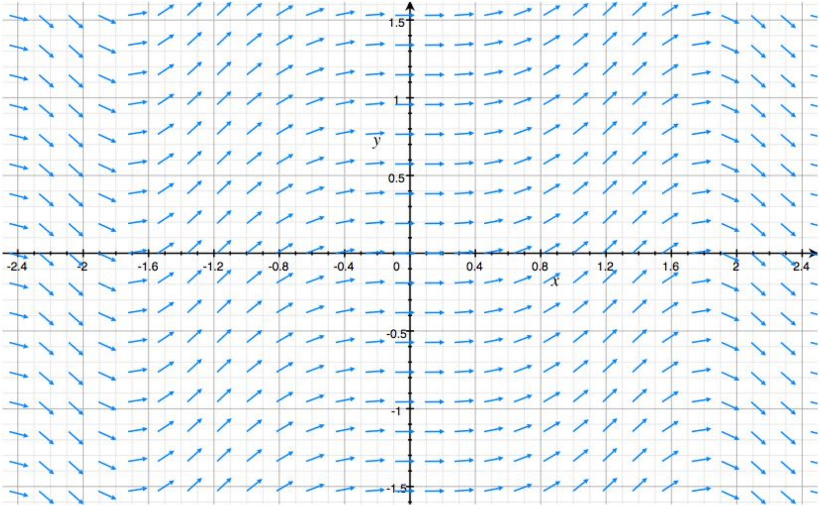


Example 2

$$f(x) = \sin(x^2)$$



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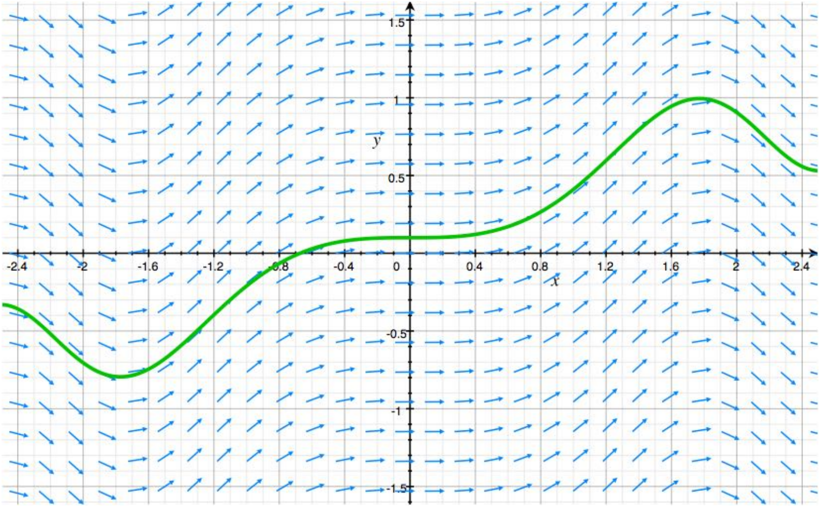

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$$v(t) = -10t + C$$

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$$t = \sqrt{67} \sim 8.2$$

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These problems involve finding the area under some curve.

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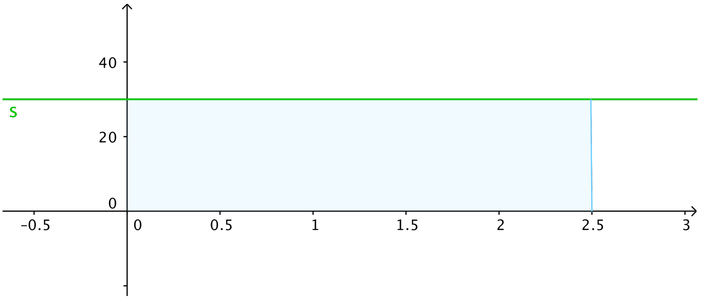
Solution

We model the car's speed using the function $s(t) = 30$. So we can see that the area under this curve



is the distance travelled (75 miles)

Example 2

If a car accellerates for 20 seconds at a rate of $2m/s^2$ and then decelerates for 30 seconds at a rate of $1m/s^2$, how far has it travelled?

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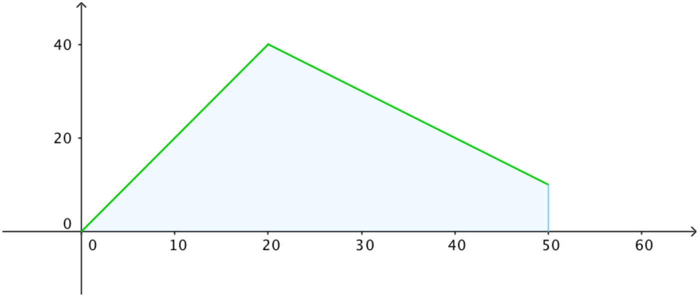
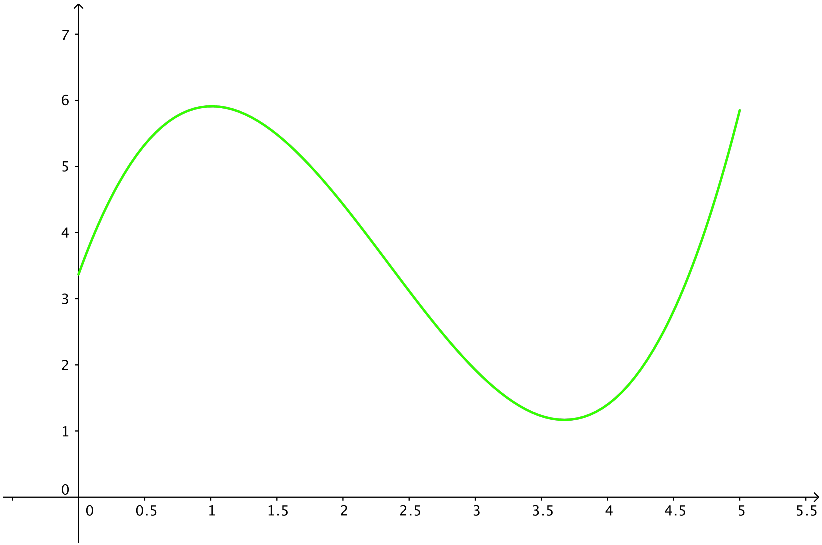
Solution

The car's speed is given by $s(t) = 2t$ when $0 \leq t \leq 20$ and $s(t) = 60 - t$ when $20 \leq t \leq 50$. So the graph looks like



More complicated areas

How do we calculate the area under more complicated curves?



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