This week you will get practice applying linear models to real world phenomena.

*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

1. (6.3-38) At midnight the coroner was called to the scene of the brutal murder of Casper Cooly. The coroner arrived and noted that the air temperature was 70° F and Cooly's body temperature was 85° F. At 2a.m., she noted that the body had cooled to 76° F. The police arrested Cooly's business partner Tatum Twit and charged her with the murder. She has an eyewitness who said she left the theater at 11p.m. Does her alibi help?

2. (*Note: this question is a challenge! It would be too difficult for an exam*) A cylindrical water tank, 2 meters in diameter and 5 meters tall, has a small hole in its base of radius 0.05 meters. From the *Bernoulli principle* in fluid dynamics one can derive the fact that if the tank is filled to a level of h meters then the water is flowing out of the hole at a rate of

$$A\sqrt{2gh} \text{ m}^3/\text{s}$$

where A is the area (in meters squared) of the hole and g is acceleration due to gravity (you may assume $g = 10$ m/s^2). Rainwater is caught by a guttering system and is pouring into the tank at a constant rate of I m^3/s.

(a) Write a differential equation that describes the change in the volume of water (in m$^3/s$) held by the tank, over time.

Solution: The hole has a radius of 0.05 m so it's area is $A = 0.0025\pi = \pi/400$ m^2. Furthermore, if $V(t)$ is the volume at time t and $h(t)$ is the height of the water at time t then $V(t) = \pi h(t)$ (since the tank has radius 1 m). Thus by the formula given in the question, water is flowing out of the hole at a rate of

$$\frac{\pi}{400}\sqrt{20h(t)} = \frac{\pi}{400}\sqrt{\frac{20}{\pi}}V^{\frac{1}{2}} \text{ m}^3/\text{s}.$$

Thus the total rate of change is given by the rate flowing in, minus the rate flowing out, so

$$\frac{\mathrm{d}V}{\mathrm{d}t} = I - \frac{\pi}{400}\sqrt{\frac{20}{\pi}}V^{\frac{1}{2}} = I - \frac{\sqrt{\pi}}{40\sqrt{5}}V^{\frac{1}{2}}.$$

(b) Find the equilibrium solution for this equation (leave your answer in terms of I and π).

Solution: The equilibrium solution occurs when $\mathrm{d}V/\mathrm{d}t = 0$. I.e. when

$$\begin{aligned} 0 &= I - \frac{\sqrt{\pi}}{40\sqrt{5}}V^{\frac{1}{2}} \\ \frac{\sqrt{\pi}}{40\sqrt{5}}V^{\frac{1}{2}} &= I \\ V^{\frac{1}{2}} &= \frac{40\sqrt{5}}{\sqrt{\pi}}I \\ V &= \frac{8000I^2}{\pi}. \end{aligned}$$

(c) If the tank is initially filled up to the 3 meter mark, describe how the volume of the tank behaves over the long term, for different values of I.

Solution: If the tank is initially full to the 3 meter mark, then it contains 3π m^3 of water. Thus if

$$3\pi = V = \frac{8000I^2}{\pi}$$

i.e. if

$$I = \sqrt{\frac{3\pi^2}{8000}} \approx 0.06 \text{ m}^3/\text{s}$$

then the volume of the water neither increases or decreases over time. Note that the equilibrium solution is $V = 3\pi \approx 9.2$.

If $I > 0.06$ then the rate of change in the volume is positive and thus the volume of water in the tank increases and approaches the equilibrium. If the equilibrium is greater than 5π, that is

$$\frac{8000I^2}{\pi} > 5\pi$$

so if $I > \pi/40$, then the tank eventually overflows. If $I < 0.06$ then the water in the tank decreases and approaches the equilibrium from above.

(d) Solve the differential equation assuming that $I = 0$ (i.e. it is not raining).

Solution: If $I = 0$ then the equation we would like to solve is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = -\frac{\sqrt{\pi}}{40\sqrt{5}}V^{\frac{1}{2}}.$$

Separating variables and integrating we get

$$\int V^{-\frac{1}{2}}\,\mathrm{d}V = \int -\frac{\sqrt{\pi}}{40\sqrt{5}}\,\mathrm{d}t$$

The right hand side is just the integral of a constant and the left hand side is the integral of a square root so we can use the power law to get

$$2V^{\frac{1}{2}} = -\frac{\sqrt{\pi}}{40\sqrt{5}}t + C.$$

Initially we have that $V(0) = 3\pi$ so

$$2\sqrt{3\pi} = C.$$

Putting this into the above solution and solving for V we get

$$V(t) = \left(\sqrt{3\pi} - \frac{\sqrt{\pi}}{80\sqrt{5}}t\right)^2.$$

(e) Under the above assumptions, how long would it take for the tank to drain? Here we will declare that the tank is drained once it contains less than 0.001 m^3 of water.

Solution: In the case $I = 0$, the derivative is always negative, so V is always decreasing. Thus we just want to know when $V(t) = 0.001$. We simple put this into our solution found above and solve for t:

$$\begin{aligned}
0.001 &= \left(\sqrt{3\pi} - \frac{\sqrt{\pi}}{80\sqrt{5}}t\right)^2 \\
\sqrt{0.001} &= \sqrt{3\pi} - \frac{\sqrt{\pi}}{80\sqrt{5}}t \\
\sqrt{0.001} - \sqrt{3\pi} &= -\frac{\sqrt{\pi}}{80\sqrt{5}}t \\
-\frac{80\sqrt{5}}{\sqrt{\pi}}\sqrt{0.001} + \frac{80\sqrt{5}}{\sqrt{\pi}}\sqrt{3\pi} &= t.
\end{aligned}$$

Using a calculator we obtain $t \approx 307$ seconds (5 minutes and 7 seconds).

(f) Solve the differential equation assuming that $I = 0.5$ but leave the answer as an implicit function (do not try to solve for $V(t)$).

Solution: We begin by separating the variables and integrating,

$$\int \frac{1}{0.5 - \frac{\sqrt{\pi}}{40\sqrt{5}}V^{\frac{1}{2}}}\,\mathrm{d}V = \int \mathrm{d}t.$$

The integral on the left can be rearranged to

$$2\int \frac{1}{1 - \frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}}\,\mathrm{d}V.$$

Now we use the substitution $u = \frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}$, with this choice we have that

$$\frac{\mathrm{d}u}{\mathrm{d}V} = \frac{\sqrt{\pi}}{40\sqrt{5}}V^{-\frac{1}{2}}.$$

Now we apply the substitution:

$$\begin{aligned}
2\int \frac{1}{1 - \frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}}\,\mathrm{d}V &= 2\int \frac{\frac{40\sqrt{5}}{\sqrt{\pi}}V^{\frac{1}{2}}}{1 - \frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}}\left(\frac{\sqrt{\pi}}{40\sqrt{5}}V^{-\frac{1}{2}}\right)\,\mathrm{d}V \\
&= \frac{8000}{\pi}\int \frac{\frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}}{1 - \frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}}\left(\frac{\sqrt{\pi}}{40\sqrt{5}}V^{-\frac{1}{2}}\right)\,\mathrm{d}V \\
&= \frac{8000}{\pi}\int \frac{u}{1-u}\,\mathrm{d}u.
\end{aligned}$$

Note that we can use polynomial long division to rewrite

$$\begin{aligned}
\int \frac{u}{1-u}\,\mathrm{d}u &= \int \frac{1}{1-u} - 1\,\mathrm{d}u \\
&= -\ln(1-u) - u.
\end{aligned}$$

Thus

$$\begin{aligned}
2\int \frac{1}{1 - \frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}}\,\mathrm{d}V &= \frac{8000}{\pi}\left(-\ln(1-u) - u\right) + C \\
&= \frac{8000}{\pi}\left(-\ln\left(1 - \frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}\right) - \frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}\right) + C
\end{aligned}$$

We can now equate this will the right hand side of the equation above to obtain

$$\frac{8000}{\pi}\left(-\ln\left(1-\frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}\right)-\frac{\sqrt{\pi}}{20\sqrt{5}}V^{\frac{1}{2}}\right)=t+C$$

for an arbitrary constant C. To find the value of this constant we use the fact that $V(0)=3\pi$.

$$\frac{8000}{\pi}\left(-\ln\left(1-\frac{\pi\sqrt{3}}{20\sqrt{5}}\right)-\frac{\pi\sqrt{3}}{20\sqrt{5}}\right)=C\approx 20.5.$$

Noting also that $\frac{\sqrt{\pi}}{20\sqrt{5}}\approx 0.04$ and $8000/\pi\approx 2546.5$ we have the final relationship is given by

$$-2546.5\ln(1-0.04\sqrt{V})-101.9\sqrt{V}=t+20.5.$$

3. A river flows into a small lake and another river flows out of the lake such that the lake has a constant volume of 2000 m^3 (the rate of water flowing in equals the rate of water flowing out). The river flowing into the lake contains a pollutant present at 0.5 $\mathrm{mg/m}^3$. In this question you will model the total amount of pollutant, $y(t)$, present at time t (Note that $y(t)$ is the total amount of pollutant in the lake and not a concentration).

(a) Assume that the river flowing in, flows at a constant rate of 20 m^3/h. At what rate is the pollutant flowing into the lake (in mg/h)?

Solution: Every hour there is 0.5 milligrams of pollutant entering the lake *per meter cubed* of water. Since there are 20 m^3 of water entering the lake every hour, there is 10 g/h of pollutant entering the lake.

(b) Under the above assumption, write a differential equation describing the change in the level of pollution in the lake.

Solution: The differential equation will take the form

$$\frac{\mathrm{d}y}{\mathrm{d}t}=\text{rate in } - \text{ rate out.}$$

Thus we need to find the rate out. There are 20 m^3 flowing out every hour. At time t the concentration of pollutant in the lake is

$$\frac{y(t)}{2000}\ \mathrm{mg/m}^3.$$

Thus at time t there is

$$\frac{20y(t)}{2000}=\frac{y(t)}{100}\ \mathrm{mg/h}$$

of pollutant leaving the lake. Thus our differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t}=10-\frac{y}{100}$$

(c) Assuming that initially there is no pollutant in the lake, solve this differential equation.

Solution: We can use the general solution of the linear model to get

$$y(t) = 1000 - Ce^{-0.01t}.$$

We assume that $y(0) = 0$ to get that

$$0 = 1000 - C$$

so the final solution is

$$y(t) = 1000\left(1 - e^{-0.01t}\right)$$

(d) Now assume that there is some seasonal variability and that the river flowing in (and thus also the river flowing out), flow at a rate of $40\sin^2 t$ m^3/h. Write and solve a differential equation to model this situation, assuming there is initially no pollution in the lake.

Solution: Here we repeat the analysis above with the changed assumption. At time t, there is $40\sin^2 t$ m^3 of water entering the lake every hour. Thus there is $20\sin^2 t$ g/h of pollutant entering the lake at time t.

Now, at time t, there is $y(t)$ milligrams of pollutant in the lake and thus the concentration of pollutant is

$$\frac{y(t)}{2000}\ \text{mg/m}^3.$$

Thus there is

$$\frac{40y(t)\sin^2 t}{2000} = \frac{y(t)\sin^2 t}{50}\ \text{mg/h}$$

flowing out of the lake. The differential equation is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 20\sin^2 t - \frac{y(t)\sin^2 t}{50} = \left(20 - \frac{y}{50}\right)\sin^2 t.$$

To solve this we separate variables and integrate

$$\int \frac{50}{1000 - y}\,\mathrm{d}y = \int \sin^2 t\,\mathrm{d}t.$$

We use the hint to obtain

$$-50\ln(1000 - y) = \frac{1}{2}(t - \sin(t)\cos(t)) + C.$$

Rearranging we get that the solution is

$$y(t) = 1000 - C\exp\left(-0.01(t - \sin(t)\cos(t))\right)$$

We can use the fact that $y(0) = 0$ to get

$$C = 1000$$

so the final solution is

$$y(t) = 1000\left(1 - e^{-0.01(t - \sin(t)\cos(t))}\right).$$

(e) Compare the long term behaviour of the two solutions.

Solution: In the long term, both solution approach 1000 as the $\sin(t)\cos(t)$ term becomes insignificant.