Math 3B: Lecture 13

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How to factorize polynomials

The normal method for factorizing a polynomial $p(x)$ is to find a root α and then writing

$$p(x) = q(x)(x - \alpha).$$

How to factorize polynomials

The normal method for factorizing a polynomial $p(x)$ is to find a root α and then writing

$$p(x) = q(x)(x - \alpha).$$

What if we want to devide a polynomial $p(x)$ by another polynomial $q(x)$? We want to write

$$p(x) = q(x)d(x) + r(x)$$

for a polynomial $d(x)$ (the divisor) and a remainder $r(x)$.

Long devision

We know how to do this with numbers! Long devision.

$$\begin{array}{r} 176 \\ 34\,\big)\overline{6000} \\ \underline{3400} \\ 2600 \\ \underline{2380} \\ 220 \\ \underline{204} \\ 16 \end{array}$$

Long devision

We know how to do this with numbers! Long devision.

$$\begin{array}{r} 176 \\ 34\,\overline{)\,6000} \\ \underline{3400} \\ 2600 \\ \underline{2380} \\ 220 \\ \underline{204} \\ 16 \end{array}$$

So $6000 = 34 \cdot 176 + 16$

Why?

Lets rewrite the equation $p(x) = q(x)d(x) + r(x)$

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

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E.g.

$$\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 1} = x + \frac{1}{(x+1)^2}.$$

Why?

Lets rewrite the equation $p(x) = q(x)d(x) + r(x)$

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

E.g.

$$\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 1} = x + \frac{1}{(x+1)^2}.$$

The left hand side is difficult to integrate. The right hand side is easy!

How?

$$x + 3 \overline{\smash{)}\quad x^2 + 5x + 4}$$

How?

$$\begin{array}{r l} & x \\ \cline{2-2} x+3\big) & x^2+5x+4 \end{array}$$

How?

$$
\begin{array}{r l}
 & x \\
\cline{2-2}
x+3 \big) & \;x^2 + 5x + 4 \\
 & -x^2 - 3x \\
\cline{2-2}
\end{array}
$$

How?

$$
\begin{array}{rr}
 & x \phantom{{}+4} \\
\cline{2-2}
x+3 \big) & x^2+5x+4 \\
 & -x^2-3x \phantom{{}+4} \\
\cline{2-2}
 & 2x+4
\end{array}
$$

How?

$$\begin{array}{rr}
 & x + 2 \\ \cline{2-2}
x + 3 \big) & x^2 + 5x + 4 \\
 & \underline{-\, x^2 - 3x\phantom{{}+4}} \\
 & 2x + 4
\end{array}$$

How?

$$\begin{array}{rr}
 & x+2 \\ \cline{2-2}
x+3 \big) & x^2+5x+4 \\
 & -x^2-3x \\ \cline{2-2}
 & 2x+4 \\
 & -2x-6 \\ \cline{2-2}
\end{array}$$

How?

$$
\begin{array}{rr}
 & x+2 \\
\cline{2-2}
x+3\big) & x^2+5x+4 \\
 & -\,x^2-3x\phantom{{}+4} \\
\cline{2-2}
 & 2x+4 \\
 & -\,2x-6 \\
\cline{2-2}
 & -\,2
\end{array}
$$

How?

$$\begin{array}{rl}
 & \phantom{x^2+5x+{}}x+2 \\
x+3\big) & \overline{\;x^2+5x+4} \\
 & \underline{-x^2-3x}\phantom{{}+4} \\
 & \phantom{x^2+{}}2x+4 \\
 & \underline{-2x-6} \\
 & -2
\end{array}$$

So $$\frac{x^2+5x+4}{x+3} = x+2-\frac{2}{x+3}.$$

Example 1

$$x-3\overline{\big)\quad x^3-12x^2 \qquad\qquad -42}$$

Example 1

$$\begin{array}{rl} & \qquad\qquad x^2 \\ x-3\big) & \overline{\quad x^3 - 12x^2 \qquad\qquad\qquad - 42} \end{array}$$

Example 1

$$\begin{array}{rlll}
 & x^2 & & \\ \cline{2-4}
x-3 \big) & x^3 - 12x^2 & & -42 \\
 & -x^3 + 3x^2 & & \\ \cline{2-2}
\end{array}$$

Example 1

$$
\begin{array}{r rrrr}
 & & x^2 & & \\ \cline{2-5}
x-3 \big) & x^3 & -12x^2 & & -42 \\
 & -x^3 & +3x^2 & & \\ \cline{2-3}
 & & -9x^2 & &
\end{array}
$$

Example 1

$$\begin{array}{rrrrr}
 & & x^2 & -9x & \\ \cline{2-5}
x-3\big) & x^3 & -12x^2 & & -42 \\
 & -x^3 & +3x^2 & & \\ \cline{2-3}
 & & -9x^2 & &
\end{array}$$

Example 1

$$
\begin{array}{r rrrr}
 & & x^2 & -9x & \\ \cline{2-5}
x-3\big) & x^3 & -12x^2 & & -42 \\
 & -x^3 & +3x^2 & & \\ \cline{2-3}
 & & -9x^2 & & \\
 & & 9x^2 & -27x & \\ \cline{3-4}
\end{array}
$$

Example 1

$$
\begin{array}{r|rrrr}
\multicolumn{1}{r}{} & & x^2 & -\,9x & \\
\cline{2-5}
x-3\,\big) & x^3 & -\,12x^2 & & -\,42 \\
\multicolumn{1}{r}{} & -\,x^3 & +\,3x^2 & & \\
\cline{2-3}
\multicolumn{1}{r}{} & & -\,9x^2 & & \\
\multicolumn{1}{r}{} & & 9x^2 & -\,27x & \\
\cline{3-4}
\multicolumn{1}{r}{} & & & -\,27x & -\,42
\end{array}
$$

Example 1

$$
\begin{array}{rrrrr}
 & & x^2 & -\,9x & -\,27 \\ \cline{2-5}
x-3\,\big) & x^3 & -\,12x^2 & & -\,42 \\
 & -\,x^3 & +\,3x^2 & & \\ \cline{2-3}
 & & -\,9x^2 & & \\
 & & 9x^2 & -\,27x & \\ \cline{3-4}
 & & & -\,27x & -\,42
\end{array}
$$

Example 1

$$
\begin{array}{rrrrr}
 & & x^2 & -\,9x & -\,27 \\ \cline{2-5}
x-3\big) & x^3 & -\,12x^2 & & -\,42 \\
 & -\,x^3 & +\,3x^2 & & \\ \cline{2-3}
 & & -\,9x^2 & & \\
 & & 9x^2 & -\,27x & \\ \cline{3-4}
 & & & -\,27x & -\,42 \\
 & & & 27x & -\,81 \\ \cline{4-5}
\end{array}
$$

Example 1

$$
\begin{array}{rrrrr}
 & & x^2 & -9x & -27 \\ \cline{2-5}
x-3\big) & x^3 & -12x^2 & & -42 \\
 & -x^3 & +3x^2 & & \\ \cline{2-3}
 & & -9x^2 & & \\
 & & 9x^2 & -27x & \\ \cline{3-4}
 & & & -27x & -42 \\
 & & & 27x & -81 \\ \cline{4-5}
 & & & & -123
\end{array}
$$

Example 1

$$
\begin{array}{rrrrr}
 & & x^2 & -9x & -27 \\ \cline{2-5}
x-3\big) & x^3 & -12x^2 & & -42 \\
 & -x^3 & +3x^2 & & \\ \cline{2-3}
 & & -9x^2 & & \\
 & & 9x^2 & -27x & \\ \cline{3-4}
 & & & -27x & -42 \\
 & & & 27x & -81 \\ \cline{4-5}
 & & & & -123
\end{array}
$$

So $$\frac{x^3-12x^2-42}{x-3} = x^2-9x-27-\frac{123}{x-3}.$$

Example 2

$$x^2+1\big)\overline{\quad x^3 - x^2 + x - 1}$$

Example 2

$$\begin{array}{rr} & x \\ x^2+1\big) & \overline{\quad x^3 - x^2 + x - 1} \end{array}$$

Example 2

$$
\begin{array}{rrrrr}
 & & & x & \\ \cline{2-5}
x^2+1 \big) & x^3 & -x^2 & +x & -1 \\
 & -x^3 & & -x & \\ \cline{2-4}
\end{array}
$$

Example 2

$$\begin{array}{r|rrrr}
 & & & x & \\
\cline{2-5}
x^2+1) & x^3 & -x^2 & +x & -1 \\
 & -x^3 & & -x & \\
\cline{2-4}
 & & -x^2 & & -1
\end{array}$$

Example 2

$$
\begin{array}{rrrrr}
 & & & x & -\,1 \\ \cline{2-5}
x^2+1\big) & x^3 & -\,x^2 & +\,x & -\,1 \\
 & -\,x^3 & & -\,x & \\ \cline{2-4}
 & & -\,x^2 & & -\,1
\end{array}
$$

Example 2

$$\begin{array}{rrrrr}
 & & & x & -\,1 \\ \cline{2-5}
x^2+1\big) & x^3 & -\,x^2 & +\,x & -\,1 \\
 & -\,x^3 & & -\,x & \\ \cline{2-4}
 & & -\,x^2 & & -\,1 \\
 & & x^2 & & +\,1 \\ \cline{3-5}
\end{array}$$

Example 2

$$
\begin{array}{rrrrr}
 & & & x & -1 \\ \cline{2-5}
x^2+1\big) & x^3 & -x^2 & +x & -1 \\
 & -x^3 & & -x & \\ \cline{2-4}
 & & -x^2 & & -1 \\
 & & x^2 & & +1 \\ \cline{3-5}
 & & & & 0
\end{array}
$$

Example 2

$$
\begin{array}{rrrrr}
 & & & x & -\,1 \\ \cline{2-5}
x^2+1\big) & x^3 & -\,x^2 & +\,x & -\,1 \\
 & -\,x^3 & & -\,x & \\ \cline{2-4}
 & & -\,x^2 & & -\,1 \\
 & & x^2 & & +\,1 \\ \cline{3-5}
 & & & & 0
\end{array}
$$

So $$\frac{x^3-x^2+x-1}{x^2+1}=x-1.$$

Example 3

$$x^2+x+1\overline{\big)\quad x^3 \qquad\qquad\qquad -1}$$

Example 3

$$\begin{array}{rl} & \qquad\qquad\qquad\quad x \\ x^2+x+1 \big) & \overline{\quad x^3 \qquad\qquad\qquad -1} \end{array}$$

Example 3

$$
\begin{array}{rllll}
 & & & x & \\
\cline{2-5}
x^2+x+1\big) & x^3 & & & -1 \\
 & -x^3 & -x^2 & -x & \\
\cline{2-4}
\end{array}
$$

Example 3

$$\begin{array}{rrrrr} & & & x & \\ \cline{2-5} x^2+x+1\big) & x^3 & & & -1 \\ & -x^3 & -x^2 & -x & \\ \cline{2-4} & & -x^2 & -x & -1 \end{array}$$

Example 3

$$
\begin{array}{rrrrr}
 & & & x & -\,1 \\ \cline{2-5}
x^2+x+1\big) & x^3 & & & -\,1 \\
 & -\,x^3 & -\,x^2 & -\,x & \\ \cline{2-4}
 & & -\,x^2 & -\,x & -\,1
\end{array}
$$

Example 3

$$
\begin{array}{r@{\,}r@{\,}r@{\,}r@{\,}r}
 & & & x & -\,1 \\ \cline{2-5}
x^2+x+1\big) & x^3 & & & -\,1 \\
 & -\,x^3 & -\,x^2 & -\,x & \\ \cline{2-4}
 & & -\,x^2 & -\,x & -\,1 \\
 & & x^2 & +\,x & +\,1 \\ \cline{3-5}
\end{array}
$$

Example 3

$$\begin{array}{rrrrr}
 & & & x & -1 \\ \cline{2-5}
x^2+x+1\big) & x^3 & & & -1 \\
 & -x^3 & -x^2 & -x & \\ \cline{2-4}
 & & -x^2 & -x & -1 \\
 & & x^2 & +x & +1 \\ \cline{3-5}
 & & & & 0
\end{array}$$

Example 3

$$
\begin{array}{rrrrr}
 & & & x & -1 \\ \cline{2-5}
x^2+x+1 \big) & x^3 & & & -1 \\
 & -x^3 & -x^2 & -x & \\ \cline{2-4}
 & & -x^2 & -x & -1 \\
 & & x^2 & +x & +1 \\ \cline{3-5}
 & & & & 0
\end{array}
$$

So $$\frac{x^3-1}{x^2+x+1} = x-1.$$

Example 4

$$3x - 1\overline{\smash{\big)}\quad 2x^3 \;\; - 4x^2 \qquad\qquad\quad + 1}$$

Example 4

$$\begin{array}{rl} & \tfrac{2}{3}x^2 \\ \cline{2-2} 3x-1\big) & 2x^3 \quad - 4x^2 \qquad\qquad + 1 \end{array}$$

Example 4

$$\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & & \\ \cline{2-5}
3x-1\big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
\end{array}$$

Example 4

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & & \\ \cline{2-5}
3x-1 \big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{10}{3}x^2 & &
\end{array}
$$

Example 4

$$
\begin{array}{rllll}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & \\
\cline{2-5}
3x-1\big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
 & & -\frac{10}{3}x^2 & &
\end{array}
$$

Example 4

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & \\
\cline{2-5}
3x-1\big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\
\cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\
\cline{3-4}
\end{array}
$$

Example 4

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & \\ \cline{2-5}
3x-1 \big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\ \cline{3-4}
 & & & -\frac{10}{9}x & +1
\end{array}
$$

Example 4

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} \\ \cline{2-5}
3x-1\big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\ \cline{3-4}
 & & & -\frac{10}{9}x & +1
\end{array}
$$

Example 4

$$
\begin{array}{r|rrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} \\ \cline{2-5}
3x-1 & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\ \cline{3-4}
 & & & -\frac{10}{9}x & +1 \\
 & & & \frac{10}{9}x & -\frac{10}{27} \\ \cline{4-5}
\end{array}
$$

Example 4

$$
\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} \\ \cline{2-5}
3x-1 \big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\ \cline{3-4}
 & & & -\frac{10}{9}x & +1 \\
 & & & \frac{10}{9}x & -\frac{10}{27} \\ \cline{4-5}
 & & & & \frac{17}{27}
\end{array}
$$

Example 4

$$\begin{array}{rrrrr}
 & & \frac{2}{3}x^2 & -\frac{10}{9}x & -\frac{10}{27} \\ \cline{2-5}
3x-1\big) & 2x^3 & -4x^2 & & +1 \\
 & -2x^3 & +\frac{2}{3}x^2 & & \\ \cline{2-3}
 & & -\frac{10}{3}x^2 & & \\
 & & \frac{10}{3}x^2 & -\frac{10}{9}x & \\ \cline{3-4}
 & & & -\frac{10}{9}x & +1 \\
 & & & \frac{10}{9}x & -\frac{10}{27} \\ \cline{4-5}
 & & & & \frac{17}{27}
\end{array}$$

So $$\frac{2x^3-4x^2+1}{3x-1} = \frac{2}{3}\left(x^2-\frac{5}{3}x-\frac{5}{9}\right) - \frac{17}{27(3x-1)}.$$

Example 5

$$x^2 - 2x + 5 \overline{\smash{)}\quad x^4 \qquad\qquad\quad - x^2 \quad + x \quad - 4}$$

Example 5

$$\begin{array}{rl} & x^2 \\ x^2-2x+5 & \hspace{-0.5em}\overline{\big)\; x^4 \qquad\qquad - x^2 \quad + x \quad - 4} \end{array}$$

Example 5

$$
\begin{array}{rlllll}
 & & & x^2 & & \\ \cline{2-6}
x^2-2x+5 \big) & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\ \cline{2-4}
\end{array}
$$

Example 5

$$
\begin{array}{rlllll}
 & & & x^2 & & \\ \cline{2-6}
x^2-2x+5\big) & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\ \cline{2-4}
 & & 2x^3 & -6x^2 & +x &
\end{array}
$$

Example 5

$$\begin{array}{rrrrrr}
 & & & x^2 & +\,2x & \\ \cline{2-6}
x^2-2x+5\big) & x^4 & & -\,x^2 & +\,x & -\,4 \\
 & -\,x^4 & +\,2x^3 & -\,5x^2 & & \\ \cline{2-4}
 & & 2x^3 & -\,6x^2 & +\,x &
\end{array}$$

Example 5

$$
\begin{array}{rrrrrr}
 & & & x^2 & +2x & \\ \cline{2-6}
x^2-2x+5\big) & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\ \cline{2-4}
 & & 2x^3 & -6x^2 & +x & \\
 & & -2x^3 & +4x^2 & -10x & \\ \cline{3-5}
\end{array}
$$

Example 5

$$
\begin{array}{rrrrrr}
 & & & x^2 & +2x & \\
\cline{2-6}
x^2-2x+5\big) & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\
\cline{2-4}
 & & 2x^3 & -6x^2 & +x & \\
 & & -2x^3 & +4x^2 & -10x & \\
\cline{3-5}
 & & & -2x^2 & -9x & -4
\end{array}
$$

Example 5

$$
\begin{array}{rrrrrr}
 & & & x^2 & +\,2x & -\,2 \\ \cline{2-6}
x^2-2x+5\big) & x^4 & & -\,x^2 & +\,x & -\,4 \\
 & -\,x^4 & +\,2x^3 & -\,5x^2 & & \\ \cline{2-4}
 & & 2x^3 & -\,6x^2 & +\,x & \\
 & & -\,2x^3 & +\,4x^2 & -\,10x & \\ \cline{3-5}
 & & & -\,2x^2 & -\,9x & -\,4
\end{array}
$$

Example 5

$$
\begin{array}{rrrrrr}
 & & & x^2 & +2x & -2 \\
\cline{2-6}
x^2-2x+5 \big) & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\
\cline{2-4}
 & & 2x^3 & -6x^2 & +x & \\
 & & -2x^3 & +4x^2 & -10x & \\
\cline{3-5}
 & & & -2x^2 & -9x & -4 \\
 & & & 2x^2 & -4x & +10 \\
\cline{4-6}
\end{array}
$$

Example 5

$$
\begin{array}{rrrrrr}
 & & & x^2 & +2x & -2 \\
\cline{2-6}
x^2-2x+5\big) & x^4 & & -x^2 & +x & -4 \\
 & -x^4 & +2x^3 & -5x^2 & & \\
\cline{2-4}
 & & 2x^3 & -6x^2 & +x & \\
 & & -2x^3 & +4x^2 & -10x & \\
\cline{3-5}
 & & & -2x^2 & -9x & -4 \\
 & & & 2x^2 & -4x & +10 \\
\cline{4-6}
 & & & & -13x & +6
\end{array}
$$

Example 5

$$
\begin{array}{rrrrrr}
 & & & x^2 & +\,2x & -\,2 \\ \cline{2-6}
x^2-2x+5\,\big) & x^4 & & -\,x^2 & +\,x & -\,4 \\
 & -\,x^4 & +\,2x^3 & -\,5x^2 & & \\ \cline{2-4}
 & & 2x^3 & -\,6x^2 & +\,x & \\
 & & -\,2x^3 & +\,4x^2 & -\,10x & \\ \cline{3-5}
 & & & -\,2x^2 & -\,9x & -\,4 \\
 & & & 2x^2 & -\,4x & +\,10 \\ \cline{4-6}
 & & & & -\,13x & +\,6
\end{array}
$$

So $$\frac{x^4 - x^2 + x - 4}{x^2 - 2x + 5} = x^2 + 2x - 2 + \frac{-13x + 6}{x^2 - 2x + 5}.$$

How to deal with rational functions?

How can we integrate something like

$$\int \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8}\,\mathrm{d}x$$

or

$$\int \frac{x+2}{x^3 - x}\,\mathrm{d}x?$$

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

using polynomial long division.

Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

using polynomial long division.

This is still not something we can integrate so we need to go further.

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots$$

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots = \frac{P(x)}{Q(x)}$$

Partial fractions

When we are faced with a sum of the form

$$\frac{1}{x+1} + \frac{3}{2-3x} + \cdots = \frac{P(x)}{Q(x)}$$

How do we reverse this process?

Answer: partial fractions

When the denominator is $(ax+b)(cx+d)\cdots$

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we can always find constants $A_1, A_2, \dots, n$ so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots \frac{A_n}{a_nx + b_n}$$

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So

$$-2A = 1 \quad \text{hence} \quad A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$