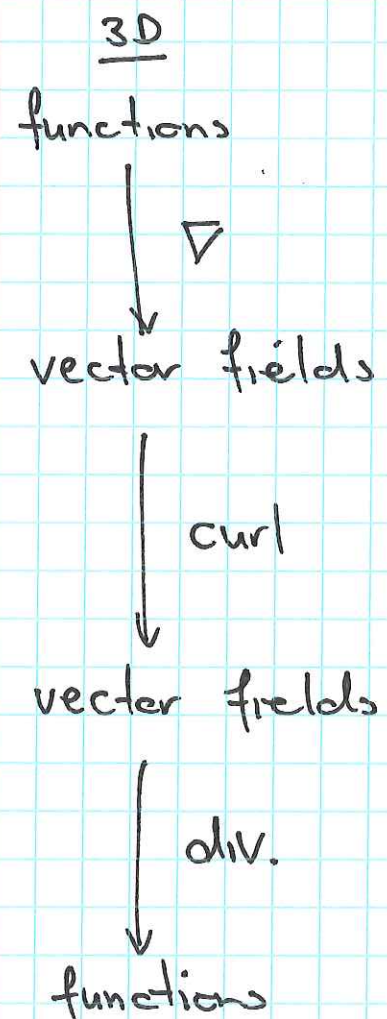
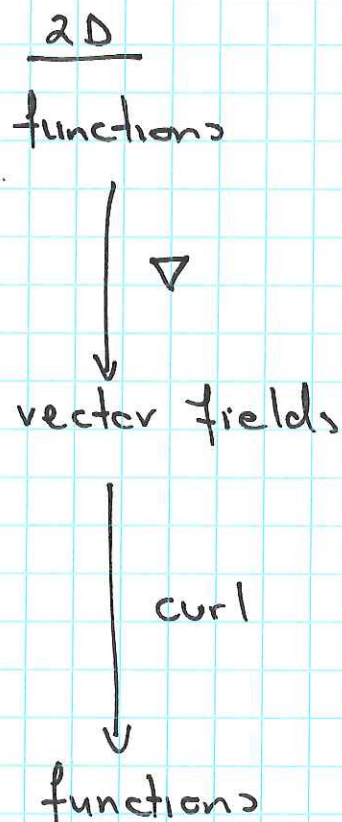
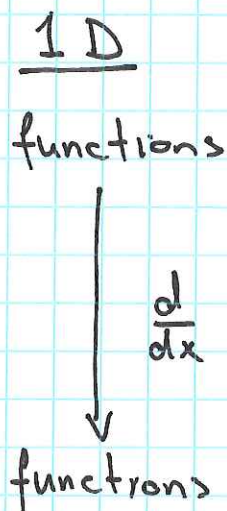


## Lecture 21

- We are now at a point in the course where we have defined all the pieces and we can try and ~~st~~ say something about their relationships
- This course has basically been about various types of derivative



- One aim is to say something about the antiderivatives of these operations.

Eg. \* When can we find an  $\varphi$  st  $\nabla\varphi = \underline{F}$  for a given vector field

\* When can we find an  $\underline{A}$  st  $\text{curl}(\underline{A}) = \underline{F}$  for a given  $\underline{F}$ ?

- We have given some hint, but a full explanation is beyond this course (this is the realm of "topology" and something called "cohomology".)

- Another aim is to generalise the following theorems:

Thm (FTC) If  $\frac{d}{dx}f(x) = g(x)$  then

$$\int_a^b g(x) dx = f(b) - f(a)$$

Thm (FTC for conservative v.f's) If  $\nabla\varphi = \underline{F}$

then 
$$\int_C \underline{F} \cdot d\underline{r} = \varphi(Q) - \varphi(P).$$



- We want statements ~~like~~ like: If  $\text{curl}(\underline{A}) = \underline{F}$  then  $\iint_S \underline{F} \cdot d\underline{S} = \dots \underline{A} \dots ?$
- We will answer this question.

## Green's Theorem

- For now we stick to 2D world.
- If  $D \subseteq \mathbb{R}^2$  is a region of 2D space denote by  $\partial D$  its boundary
- We define an orientation on the closed curve  $\partial D$  by
  - \* If we walk around the boundary in the direction of orientation the region is on our left
- This is called the boundary orientation
- ~~We say  $\underline{F}$  have a vector potential  $\underline{A}$  if  $\text{curl}(\underline{A})$~~
- Recall  $\text{curl}(\underline{F}) = \partial_x F_2 - \partial_y F_1$ .

Thm If  $\underline{F}$  is defined on  $D$ , then

$$\oint_{\partial D} \underline{F} \cdot d\underline{r} = \iint_D \text{curl}(\underline{F}) dA$$

where  $\partial D$  is equipped w/ the boundary orientation.

### Area

- The area of  $D$  is

$$\iint_D 1 dA$$

- If we find  $\underline{F}$  s.t.  $\text{curl}(\underline{F}) = 1$  then by Green's theorem

$$\oint_{\partial D} \underline{F} \cdot d\underline{r} = \text{Area}(D).$$

- Eg.  $\langle 0, x \rangle$ ,  $\langle -y, 0 \rangle$ ,  $\frac{1}{2} \langle -y, x \rangle$ .

### Flux and Green's theorem

- Let  $\underline{F}^\perp = \langle -F_2, +F_1 \rangle$  be the vector field orthogonal to  $\underline{F}$ .

Prop  $\text{curl}(\underline{F}^\perp) = \text{div}(\underline{F})$

proof  $\text{curl}(\underline{F}^\perp) = \frac{\partial}{\partial x} F_1 - \frac{\partial}{\partial y} (-F_2) = \text{div}(\underline{F})$



Thm (GT for flux)

$$\oint_{\partial D} (\underline{F} \cdot \underline{n}) ds = \iint_D \operatorname{div}(\underline{F}) dA$$

proof:

$$\begin{aligned} \oint_{\partial D} (\underline{F} \cdot \underline{n}) ds &= \iint_D \oint_{\partial D} \underline{F}^\perp \cdot d\underline{r} \\ &= \iint_D \operatorname{curl}(\underline{F}^\perp) dA \\ &= \iint_D \operatorname{div}(\underline{F}) dA \quad \square. \end{aligned}$$