

# Midterm 2 practice 3

UCLA: Math 115A, Spring 2020

*Instructor:* Noah White

*Date:*

- This exam has 4 questions, for a total of 20 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: \_\_\_\_\_

ID number: \_\_\_\_\_

**Discussion section (please circle):**

Question	Points	Score
1	5	
2	5	
3	5	
4	5	
Total:	20	

**Question 1** is multiple choice. Indicate your answers in the table below. *The following three pages will not be graded, your answers must be indicated here.*

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				

*Clarification on notation:* Let  $T : V \longrightarrow W$  be a linear map. The *kernel* of  $T$  is the same thing as the *nullspace* of  $T$ , i.e.  $\ker T = N(T)$ . Similarly the *image* of  $T$  is the same thing as the *range* of  $T$ , i.e.  $\operatorname{im} T = R(T)$ .

Note also that

$$\Sigma_n = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n \mid x_1 + x_2 + \cdots + x_n = 0 \right\}.$$

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (1 point) If  $V$  is a finite dimensional vector space, with two bases,  $B$  and  $C$  and  $T : V \rightarrow W$  is a linear map, and if  $Q$  is the matrix such that  $Q^{-1}[T]_B^B Q = [T]_C^C$  then  $Q$  equals

- A.  $[T]_B^C$
- B.  $[T]_C^B$
- C.  $[\text{id}]_B^C$
- D.  $[\text{id}]_C^B$

(b) (1 point) Let  $E = \{1, x\}$ ,  $C = \{x + 2, x + 1\}$  be bases of  $\mathbb{C}_1[x]$ . What is  $[\text{id}]_E^C$ ?

- A.  $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$
- B.  $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$
- C.  $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$
- D.  $\begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$

- (c) (1 point) Consider the linear map  $\frac{d}{dx} : \mathbb{R}[x] \rightarrow \mathbb{R}[x]$ . Which of the following is an eigenvector?
- A.  $x^2$
  - B.  $1 - x$
  - C.  $x$
  - D.  $3$

- (d) (1 point) Suppose  $T : V \rightarrow V$  is a diagonalizable linear map. Which of the following is true?
- A.  $T$  is invertible.
  - B.  $T$  has non-zero kernel.
  - C. The characteristic polynomial of  $T$  splits.
  - D.  $T$  must have a non-zero eigenvalue.

- (e) (1 point) What is the dimension of  $\text{Hom}(\mathbb{R}^2, \mathbb{R}^2)$ ? (this is the space of linear maps)
- A. 0
  - B. 2
  - C. 4
  - D. 6

2. Let  $T : V \longrightarrow W$  be a linear map between vector spaces.

(a) (2 points) Define what it means for  $T$  to be an isomorphism.

(b) (3 points) Suppose  $B$  is a basis for  $V$ , prove that if  $T(B) = \{T(v) \mid v \in B\}$  is a basis for  $W$  then  $T$  is an isomorphism.

3. Consider the linear map  $T : \Sigma_4 \longrightarrow \Sigma_4$  (see front cover), given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_3 \\ x_2 \\ x_1 \end{pmatrix}.$$

- (a) (2 points) Find the characteristic polynomial and eigenvalues of  $T$ . *Hint: recall that  $\Sigma_4$  is three dimensional! You shouldn't need to work any  $4 \times 4$  matrices!*

- (b) (2 points) For each eigenvalue, determine an eigenvector of  $T$  (note that the eigenvectors should live in  $\Sigma_4 \subset \mathbb{R}^4$ ).

- (c) (1 point) Is  $T$  diagonalisable?

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4. Consider the differential operator  $D = x \frac{d}{dx}$ , so for example  $D((x-1)^2) = 2x^2 - 2x$ . This is called the *Euler operator*.
- (a) (1 point) Consider the linear map  $D : \mathbb{C}_n[x] \rightarrow \mathbb{C}_n[x]$ , given by the Euler operator. Is this an isomorphism?
- (b) (4 points) Prove or disprove that the linear map  $D$  is diagonalisable. *Hint: you might want to first try thinking about  $n = 2$  or  $3$  before attempting to answer the question as stated, though you will need to say something about general  $n$  to get the points. Bonus: if you do this problem correctly with  $\mathbb{C}$  replaced by an arbitrary field  $\mathbb{F}$ , you will get 2 top-up points (100% max total).*

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