

# Midterm 2 practice

## UCLA: Math 3B, Fall 2016

*Instructor:* Noah White  
*Date:* Monday, November 21, 2016  
*Version:* *practice*.

- This exam has 3 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: \_\_\_\_\_

ID number: \_\_\_\_\_

Discussion section: \_\_\_\_\_

Question	Points	Score
1	14	
2	12	
3	14	
Total:	40	

1. A hemispherical hole needs to be dug in order to plant a tree. The hole should be 2 m in diameter. Recall that the volume of a sphere, radius  $r$ , is given by

$$V = \frac{4}{3}\pi r^3.$$

- (a) (1 point) What is the total volume of the hole?

**Solution:**

$$\frac{1}{2} \cdot \frac{4}{3}\pi = \frac{2}{3}\pi \text{ m}^3$$

- (b) (1 point) Suppose we divide the hole into  $n$  slices, horizontally, of constant thickness  $\Delta h$ . What is  $\Delta h$  in terms of  $n$ ?

**Solution:**

$$\Delta h = \frac{h}{n}$$

- (c) (1 point) Let  $h = 0$  be the top of the hole and  $h = 1$  be the bottom. Let  $h_k$  be the depth of the top of the  $k^{\text{th}}$  slice (where the first slice is the  $0^{\text{th}}$  slice), this  $h_0 = 0$ . What is  $h_k$  in terms of  $\Delta h$  and  $k$ ?

**Solution:**

$$h_k = k\Delta h$$

- (d) (2 points) At a depth of  $h = h_k$ , what is the radius of the hole?

**Solution:**

$$\sqrt{1 - h_k^2}$$

- (e) (2 points) If  $n$  is large enough, we can approximate each slice by a cylinder of height  $\Delta h$  and radius as in part (d). Using this approximation, what is the volume of the  $k^{\text{th}}$  slice?

**Solution:**

$$\pi(1 - h_k^2)\Delta h \text{ m}^3$$

- (f) (3 points) The work needed to lift  $m$  kg of material,  $d$  meters up is given by

$$W = 10md$$

where we have approximated the acceleration due to gravity as  $10 \text{ m/s}^2$ . How much work needs to be done in order to lift the  $k^{\text{th}}$  slice out of the hole if we assume  $1 \text{ m}^3$  of dirt weighs 1000 kg?

**Solution:**

$$10000\pi(1 - h_k^2)h_k\Delta h$$

- (g) (2 points) Write a Riemann sum which represents the total amount of work needed to dig the dirt out of the hole.

**Solution:**

$$W = \lim_{n \rightarrow \infty} \Delta h \sum_{k=0}^{n-1} 10000\pi(1 - h_k^2)h_k$$

- (h) (2 points) Use an integral to evaluate the Riemann sum above.

**Solution:**

$$\begin{aligned} W &= \int_0^1 10000\pi(1 - h^2)h \, dh \\ &= \left[ 10000\pi \left( h - \frac{1}{3}h^3 \right) \right]_0^1 \\ &= 10000\pi \left( 1 - \frac{1}{3} \right) - 10000\pi(0) \\ &= \frac{20000}{3}\pi \end{aligned}$$

2. Solve the following differential equations. If no initial condition is given, find the general solution.

(a) (4 points)  $\frac{dy}{dx} = \frac{y}{2x+4}$ .

**Solution:**

$$y = C\sqrt{2x-4}$$

(b) (4 points)  $\frac{dy}{dt} = \frac{e^{y^2}}{y}$  where  $y(1) = 0$ .

**Solution:**

$$y = \sqrt{\ln\left(\frac{1}{3-2t}\right)}$$

(c) (4 points)  $\frac{dy}{dx} = e^{x+y}$  where  $y(0) = 0$ .

**Solution:**

$$y(x) = -\ln(2 - e^e)$$

3. A river flows into a small lake and another river flows out of the lake such that the lake has a constant volume of  $2000 \text{ m}^3$  (the rate of water flowing in equals the rate of water flowing out). The river flowing into the lake contains a pollutant present at  $0.5 \text{ g/m}^3$ . In this question you will model the total amount of pollutant,  $y(t)$ , present at time  $t$  (Note that  $y(t)$  is the total amount of pollutant in the lake and not a concentration).

- (a) (1 point) Assume that the river flowing in, flows at a constant rate of  $20 \text{ m}^3/\text{h}$ . At what rate is the pollutant flowing into the lake (in  $\text{mg/h}$ )?

**Solution:** Every hour there is  $0.5$  grams of pollutant entering the lake *per meter cubed* of water. Since there are  $20 \text{ m}^3$  of water entering the lake every hour, there is  $10 \text{ g/h}$  of pollutant entering the lake.

- (b) (4 points) Under the above assumption, write a differential equation describing the change in the level of pollution in the lake.

**Solution:** The differential equation will take the form

$$\frac{dy}{dt} = \text{rate in} - \text{rate out}.$$

Thus we need to find the rate out. There are  $20 \text{ m}^3$  flowing out every hour. At time  $t$  the concentration of pollutant in the lake is

$$\frac{y(t)}{2000} \text{ g/m}^3.$$

Thus at time  $t$  there is

$$\frac{20y(t)}{2000} = \frac{y(t)}{100} \text{ g/h}$$

of pollutant leaving the lake. Thus our differential equation is

$$\frac{dy}{dt} = 10 - \frac{y}{100}$$

- (c) (3 points) Assuming that initially there is no pollutant in the lake, solve this differential equation.

**Solution:** We can use the general solution of the linear model to get

$$y(t) = 1000 - Ce^{-0.01t}.$$

We assume that  $y(0) = 0$  to get that

$$0 = 1000 - C$$

so the final solution is

$$y(t) = 1000 (1 - e^{-0.01t})$$

- (d) (5 points) Now assume that there is some seasonal variability and that the river flowing in (and thus also the river flowing out), flow at a rate of  $40 \sin^2 t \text{ m}^3/\text{h}$ . Write and solve a differential equation to model this situation, assuming there is initially no pollution in the lake.

**Solution:** Here we repeat the analysis above with the changed assumption. At time  $t$ , there is  $40 \sin^2 t \text{ m}^3$  of water entering the lake every hour. Thus there is  $20 \sin^2 t \text{ g/h}$  of pollutant entering the lake at time  $t$ .

Now, at time  $t$ , there is  $y(t)$  grams of pollutant in the lake and thus the concentration of pollutant is

$$\frac{y(t)}{2000} \text{ g/m}^3.$$

Thus there is

$$\frac{40y(t) \sin^2 t}{2000} = \frac{y(t) \sin^2 t}{50} \text{ g/h}$$

flowing out of the lake. The differential equation is

$$\frac{dy}{dt} = 20 \sin^2 t - \frac{y(t) \sin^2 t}{50} = \left(20 - \frac{y}{50}\right) \sin^2 t.$$

To solve this we separate variables and integrate

$$\int \frac{50}{1000 - y} dy = \int \sin^2 t dt.$$

We use the hint to obtain

$$-50 \ln(1000 - y) = \frac{1}{2}(t - \sin(t) \cos(t)) + C.$$

Rearranging we get that the solution is

$$y(t) = 1000 - C \exp(-0.01(t - \sin(t) \cos(t)))$$

We can use the fact that  $y(0) = 0$  to get

$$C = 1000$$

so the final solution is

$$y(t) = 1000 \left(1 - e^{-0.01(t - \sin(t) \cos(t))}\right).$$

- (e) (1 point) Compare the long term behaviour of the two solutions.

**Solution:** In the long term, both solution approach 1000 as the  $\sin(t) \cos(t)$  term becomes insignificant.



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