Logarithms

- For a number $b > 0$, the logarithm with base b of x,

$$\log_b x$$

answers the question:

"to what power do we raise b in order to obtain x?".

- i.e. if $a = \log_b x$ then $b^a = x$

or

if $b^a = x$ then $\log_b x = a$.

Ex $\log_2 4 = 2$ since $2^2 = 4$

$\log_4 2 = \frac{1}{2}$ since $4^{\frac{1}{2}} = 2$

$\log_b 1 = 0$ since $b^0 = 1$ for any b.

- By design $\log_b(x)$ is the inverse function to b^x.

$$b^{\log_b(x)} = x = \log_b(b^x)$$

- Note that b^x is always positive so,

$$\log_b(x)$$

does not make sense if $x \leq 0$

Def The function $\log_b : \mathbb{R}_{>0} \longrightarrow \mathbb{R}$ def, by

$$\log_b(x) = \log_b x$$

is the logarithm function.

We give a special name to the log of base e:

$$\log_e x = \ln x.$$

Rmk * Since $b^x b^y = b^{x+y}$

$$\log_b(xy) = \log_b(x) + \log_b(y)$$

* Since $b^{-x} = \frac{1}{b}$

$$\log_b\left(\frac{1}{x}\right) = -\log_b(x)$$

* $\log_b x = \frac{\log_a x}{\log_a b}$ in particular $\log_b x = \frac{\ln x}{\ln b}$

Thm

$$\frac{d}{dx} \ln x = \frac{1}{x}$$

proof: If $y = \ln x$ then $e^y = x$ diff'ing

$$\frac{d}{dx} e^y = \frac{d}{dx} x$$

$$e^y \frac{dy}{dx} = 1$$

$$x \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{x}$$

Now let us determine m_b from the prev lecture.

If

$$y = b^x$$

take natural log's see assside below.

$$\ln y = \ln(b^x) = x \ln(b)$$

differentiating

$$\frac{1}{y}\frac{dy}{dx} = \ln(b)$$

$$\frac{dy}{dx} = \ln(b)\, b^x$$

So $m_b = \ln(b)$.

Asside $\log_b(a^x) = x\log_b(a)$ since...

$$\log_b(a^x) = \frac{\log_a(a^x)}{\log_a(b)} \quad \text{(change of base)}$$

$$= \frac{x}{\log_a(b)} \quad \text{(def.)}$$

$$= x\left(\frac{\log_b(b)}{\log_b(a)}\right)^{-1} \quad \text{(change of base)}$$

$$= x\left(\frac{1}{\log_b(a)}\right)^{-1} = x\log_b(a).$$