This week on the problem set you will get practice manipulating sets and applying them to probability models. Especially challenging questions, or questions that are not appropriate for an exam, are indicated with one or more asterisks.

Homework: The first homework will be due on Friday 13 October, at 12pm, the *start* of the lecture. It will consist of questions:

4 and 6.

1. From the textbook, chapter 1, problems 14, 17, 50, 55.
2. From the supplementary problems, chapter 1, problems 13, 31.
3. (*) From a well shuffled, standard deck of 52 cards (`https://en.wikipedia.org/wiki/Standard_52-card_deck`), calculate the probabilities of the standard poker hands (`https://en.wikipedia.org/wiki/List_of_poker_hands`) being drawn from the top,
   - one pair,
   - two pair,
   - three of a kind,
   - straight,
   - flush,
   - full house,
   - four of a kind, and
   - straight flush.

   Conclude that this list is ordered from most to least likely.
4. If a day is sunny the probability that the next day will be rainy is $\frac{1}{2}$. If exactly k consecutive days have been rainy probability that the following day will be sunny is $\frac{1}{k+1}$. If today is sunny what is the probability that the next n days will all be rainy?

   Solution: Let A_1 denote the event that today is a sunny day and tomorrow is rainy, let A_2 denote the event that in two days it will be rainy, A_3 the event that in 3 days it will be rainy, and so on. We want to find the probability $\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n)$. We use the multiplication rule:

   $$\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n) = \mathbf{P}(A_1)\mathbf{P}(A_2|A_1)\mathbf{P}(A_3|A_1 \cap A_2)\cdots\mathbf{P}(A_n|A_1 \cap A_2 \cap \cdots \cap A_{n-1}),$$

   We are given that $\mathbf{P}(A_1) = 1/2$, $\mathbf{P}(A_2|A_1)$ is the probability that that in 2 days it will be rainy given it rains tomorrow and today is sunny which is $1 - 1/2 = 1/2$. Similarly, $\mathbf{P}(A_3|A_1 \cap A_2)$ is the probability that in three days it will be rainy given that it was rainy exactly two preceding days which is $1-1/3 = 2/3$. In the same way one can show that $\mathbf{P}(A_k|A_1 \cap A_2 \cap \cdots \cap A_{k-1}) = 1 - 1/k = (k-1)/k$. Using the multiplication rule we get that

   $$\mathbf{P}(A_n|A_1 \cap A_2 \cap \cdots \cap A_n) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{2n}.$$
5. A burglar just stole your keychain which has n keys, exactly two of which open your apartment door. He will try to open your door using the keys one by one in a random order. Whenever a key does not open your door, he will not attempt to use this key again. What is the probability that the burglar opens your door on the kth attempt.

Solution: We can do this using the multiplication rule. The probability the burglar fails in the first attempt is $\frac{n-2}{n}$. Given the failure in the first try the probability of the failure in the second try is $\frac{n-3}{n-1}$. Given the failure in the first two tries the probability of the failure in the third try is $\frac{n-4}{n-2}$ and so on. Given the failure in the first $k-2$ tries the probability of the failure in the next try is $\frac{n-k}{n-k+2}$. Given the failure in the first $k-1$ tries the probability of the success in the next try is $\frac{2}{n-k+1}$. This gives the wanted probability

$$\frac{2(n-2)(n-3)\cdots(n-k)}{n(n-1)\cdots(n-k+1)} = \frac{2(n-k)}{n(n-1)}.$$

6. (*) You are running a day care center and today you are looking after n babies. At the end of the day, n parents arrive to collect their respective children. You are lazy and terrible at running a day care center so you give every parent a random baby.

   (a) What is the probability that no parent ends up with the correct baby?

Solution: Each assignment of a baby to a parent is equally likely. Let A_0 be the event that no baby is assigned to the correct parent. Label the babies and parents by $1, 2, \ldots, n$. Then

$$A_0^c = \bigcup_{i=1}^n B_i$$

where B_i is the event that baby i is assigned to the correct parent. Using the inclusion exclusion principle

$$\#A_0^c = \sum_{i=1}^n \#B_i - \sum_{i<j} \#B_i \cap B_j + \ldots (-1)^{n-1} \#B_1 \cap B_2 \cap \ldots B_n.$$

Note that $\#B_1 \cap B_2 = \#B_i \cap B_j$ for any $i \neq j$ and similarly for the intersection of more than two events. Thus

$$\begin{aligned}\#A_0^c &= \binom{n}{1}\#B_1 - \binom{n}{2}\#B_1 \cap B_2 + \binom{n}{3}\#B_1 \cap B_2 \cap B_3 \ldots + (-1)^{n-1}\binom{n}{n}\#B_1 \cap B_2 \cap \ldots B_n \\ &= \sum_{k=1}^n (-1)^{k-1}\binom{n}{k}\#B_1 \cap B_2 \cap \ldots B_k\end{aligned}$$

Now $\#B_1 \cap B_2 \cap \ldots B_k = (n-k)!$ and $\binom{n}{k} = n!/k!(n-k)!$ so

$$\#A_0^c = n! \sum_{k=1}^n \frac{(-1)^{k-1}}{k!}.$$

The total number of possible ways of assigning the babies is $n!$ thus

$$\mathbb{P}(A_0) = 1 - \mathbb{P}(A_0^c) = 1 - \frac{n! \sum_{k=1}^n \frac{(-1)^{k-1}}{k!}}{n!} = 1 + \sum_{k=1}^n \frac{(-1)^k}{k!} = \sum_{k=0}^n \frac{(-1)^k}{k!}.$$

   (b) What is the probability that exactly k parents end up with the correct baby?

Solution: Let A_k be the event that exactly k parents get the correct baby. Let us count the number of ways to give the babies to parents, so that exactly k are given the correct baby. There are $\binom{n}{k}$ ways to choose which parents get the correct baby, and then from the previous part, there are $(n-k)!\sum_{i=0}^{n-k}(-1)^i/i!$ ways of giving the remaining babies to the remaining parents in a way that no-one gets the correct baby. Thus the probability is

$$\mathbb{P}(A_k) = \frac{\binom{n}{k}(n-k)!\sum_{i=0}^{n-k}(-1)^i/i!}{n!} = \frac{1}{k!}\sum_{i=0}^{n-k}(-1)^i/i!.$$

(c) What do these probabilities tend to as n becomes very large?

Solution: Note that $e^{-1} = \sum_{i\geq 0}(-1)^i/i!$ so as $n \to \infty$ we have $\mathbb{P}(A_0) \to e^{-1}$ and $\mathbb{P}(A_k) \to e^{-1}/k!$.

7. A friend of yours plays a game with his opponent. The game goes as follows: your friend draws two cards from a well shuffled deck of twenty cards marked with numbers 1 to 20 and his opponent then draws two cards from the rest of the deck. The winner is the person who has the highest card. After dealing the cards, both players have the option to withdraw from the game. Your friend asks you to tell him if it is more likely to loose (in which case he will want to withdraw) or to win (in which case he will not withdraw). After receiving his cards he tells you that the higher of his two cards is k. For which values of k would you advise him to withdraw and for which values of k would you advise him to stay in the game.

Solution: Let's find the conditional probability that your friend wins if the higher card he got is k. The number of all possible outcomes in which his higher card is k is $(k-1)\times\binom{18}{2}$ (choose his other card in $k-1$ ways and the other player can have any 2 cards from the other 18 possibilities). The number of possibilities for your friend to have k as the highest card and to win is $(k-1)\times\binom{k-2}{2}$ (again $k-1$ choices for his other card and now the other player can have any two cards between 1 and k, except the ones taken by your friend. So the conditional probability is

$$\frac{\binom{k-2}{2}}{\binom{18}{2}} = \frac{(k-2)(k-3)}{306}.$$

This larger than 1/2 when $(k-2)(k-3) > 153 \Leftrightarrow k^2-5k-147 > 0$. The polynomial on the left hand side has zeros at

$$\frac{5\pm\sqrt{25+4\times 147}}{2} \approx 14.88,\ -9.88$$

So the polynomial $k^2-5k-147$ is positive for $k = 15, 16, \ldots, 20$ and for this values you would advise the friend to stay in the game. For $k = 2, 3, \ldots 14$ it's negative so you would advise the friend to withdraw.

8. You have d indistinguishable balls that need to be placed into n labelled buckets. How many ways are there of doing this?

9. Suppose Ω is a finite set. How many elements does the power set $\mathcal{P}(\Omega)$ have?

10. You are walking on the plane $\mathbb{R}^2$ starting at the origin $(0,0)$. You can only take steps of size 1 and only in the positive x or y directions. So your first step will either be to $(0,1)$ or $(1,0)$.

(a) How many paths are there to (n, m)?

(b) How many paths are there to (n, m) that pass through (i, j) where $i \leq n$ and $j \leq m$?

(c) If (i, j) is north-east of $(p.q)$, who many paths are there that pass though (i, j) *and* (p, q)?

(d) What about paths that pass through (i, j) *or* (p, q)?

(e) Repeat the previous two parts with the assumption that (i, j) is to the north-west of (p, q).

It is fine to leave your answers in terms of binomial coefficients.