Final exam practice 3

UCLA: Math 3B, Fall 2018

Instructor: Noah White
Date:

- This exam has 7 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ____________________

ID number: ____________________

Discussion section (please circle):

Day/TA	Ben	Ryan
Tuesday	1A	1C
Thursday	1B	1D

Question	Points	Score
1	12	
2	12	
3	12	
4	10	
5	12	
6	12	
7	10	
Total:	80	

Questions 1 and 2 are multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

Please note! The following four pages will not be graded. You must indicate your answers here for them to be graded!

Question 1.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

Question 2.

Part	A	B	C	D
(a)				
(b)				
(c)				
(d)				
(e)				
(f)				

Some tips for the exam:

- Have a go at every problem. You will get partial credit if you are on the correct track.
- Check your answers carefully, you have a lot of time to do the exam.
- Indicate clearly if your solution to a question is on a different page.

Good luck!

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) The function $f(x) = e^x + e^{-x}$ is

   A. Always increasing.

   B. Always decreasing.

   C. Always concave up.

   D. Always concave down.

   (b) (2 points) The function $f(x) = x(x^2 - 1)$ is

   A. Always increasing.

   B. Always decreasing.

   C. Always concave up.

   D. None of the above.

   (c) (2 points) The function $f(x) = \frac{2x^3+1}{x^2-1}$ has a

   A. Horizontal asymptote at $y = -1$.

   B. Vertical asymptote at $x = 1$.

   C. Slanted asymptote with slope -1.

   D. Slanted asymptote with slope 1.

(d) (2 points) The function $f(x) = \frac{\ln(x^2)}{\ln(x)-1}$ has a

A. Horizontal asymptote at $y = 1$.

B. No horizontal asymptotes.

C. Horizontal asymptote at $y = 2$.

D. Horizontal asymptote at $y = -3$.

(e) (2 points) The function $f(x) = x - \sin x$ has a critical point at

A. $x = 1$

B. $x = 2$

C. $x = \pi$

D. $x = 2\pi$

(f) (2 points) The function $f(x) = e^{x^3-3x^2}$ has a

A. minimum at $x = 2$.

B. maximum at $x = 2$.

C. minimum $x = 1$.

D. maximum $x = 1$.

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) The definite integral $\int_1^{e^3} x^{-1}\,\mathrm{d}x$ has a value of

A. 3.
B. 2.
C. 1.
D. 0.

(b) (2 points) The definite integral $\int_1^2 15x\sqrt{x-1}\,\mathrm{d}x$ has a value of

A. 0.
B. 4.
C. 16.
D. 20.

(c) (2 points) The solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = -3y$ when $y(0) = 2$ has

A. $y(1) = 2e^3$
B. $y(0.5) = 3e$.
C. $y(1) = 2e^{-3}$**.**
D. $y(0.5) = 2e^{-1}$.

(d) (2 points) The solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = 3t^2y^{-1}$ when $y(0) = 3$ has

A. $y(2) = 25$.

B. $y(2) = 5$**.**

C. $y(1) = 16$.

D. $y(1) = 4$.

(e) (2 points) Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y^2 + t$ with initial condition $y(0) = 1$. Using Euler's method with a step size of $h = 0.5$ to estimate $y(1)$ gives

A. $5/8 = 0.625$.

B. $7/4 = 1.75$.

C. 2.

D. $23/8 = 2.875$**.**

(f) (2 points) The differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = y - \sqrt{3y-2}$ has a

A. stable equilibrium at $y = 2$.

B. unstable equilibrium at $y = 1$.

C. stable equilibrium at $y = 1$**.**

D. unstable equilibrium at $y = -2$.

3. Let $f(x) = \dfrac{x}{x^2-1}$. Note that $f'(x) = -\dfrac{x^2+1}{\left(x^2-1\right)^2}$ and $f''(x) = \dfrac{2x\left(x^2+3\right)}{\left(x^2-1\right)^3}$.

(a) (1 point) Does $f(x)$ cross the x and y axes? If so, where?

Solution: Function can only be zero when $x = 0$. Thus
$x = 0$ and $y = 0$ are the x and y intercepts.

(b) (2 points) Does $f(x)$ have any horizontal asymptotes? If so what are they?

Solution: We need to evaluate

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x}{x^2-1} = 0.$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{x^2-1} = 0.$$

So we have horizontal asymptote in the positive and negative directions at $y = 0$.

(c) (2 points) Does $f(x)$ have any vertical asymptotes? If so what are they?

Solution: The denominator of $f(x)$ is zero when $x = \pm 1$, thus a vertical asymptote at $x = 1$ and $x = -1$.

(d) (2 points) For what x is the first derivative $f'(x)$ positive?

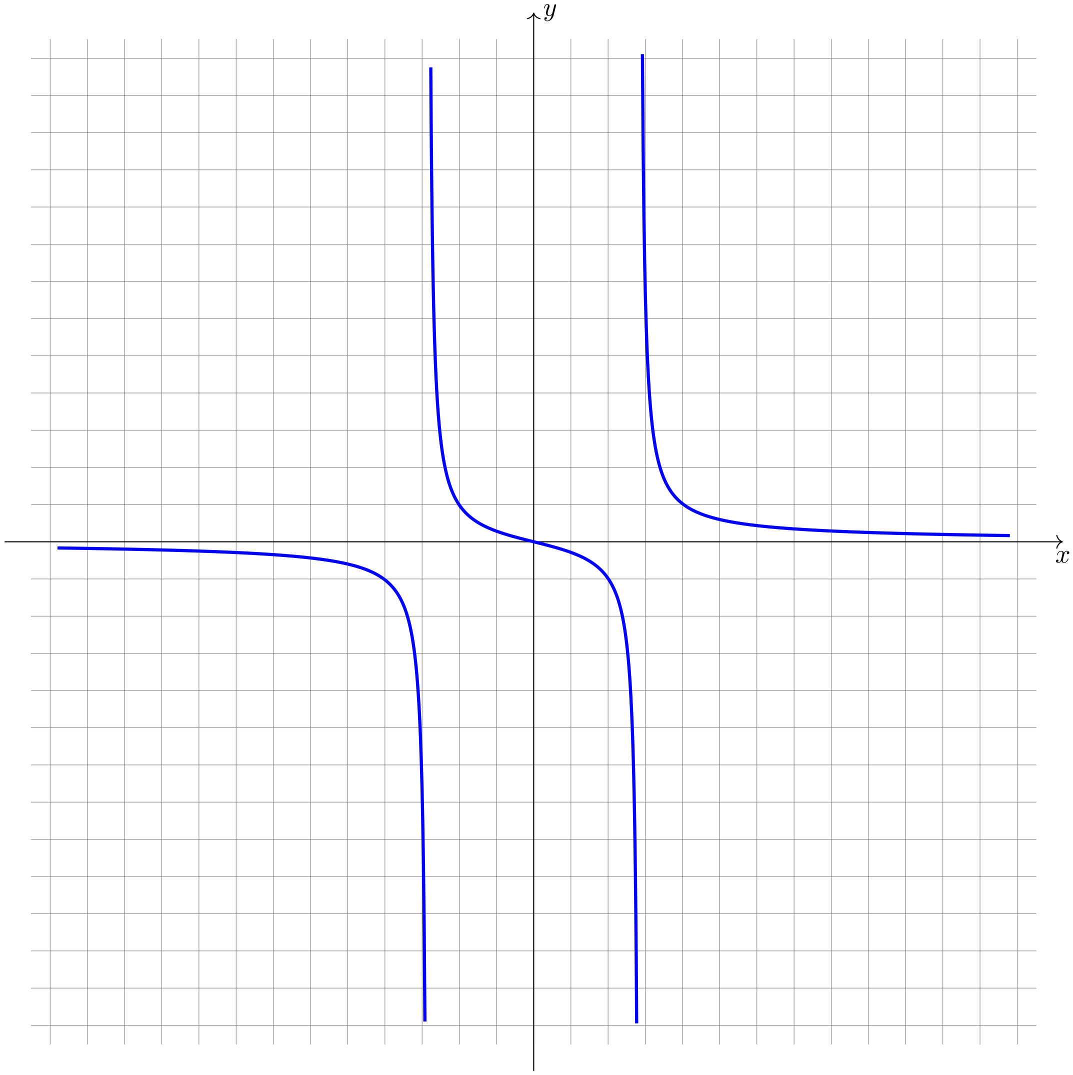
Solution: The denominator and the numerator are always positive. Since we have a negative out the front, $f'(x)$ is negative for all x.

(e) (2 points) For what x is the second derivative $f''(x)$ positive?

Solution: The denominator is positive when $|x| > 1$. The numerator is positive when $x > 0$. Thus, $f''(x) > 0$ when both the numerator and denominator have the same sign, i.e. when

$$-1 < x < 0 \text{ or } x > 1.$$

(f) (3 points) On the graph provided, sketch $f(x)$



4. The queue at Ricky's Fish Tacos can get pretty long. Today he is running late and customers are turning up looking for his food truck. Customers arrive at a rate of 2 per minute starting at 11 am. The longer a customer waits, the more likely they are to leave out of frustration. The proportion of customers that leave after t minutes of waiting is

$$0.2e^{-0.1t}.$$

(a) (6 points) Write a Riemann sum which represents the total number of customers waiting for Ricky at $11:15$. Be sure to define any symbols that you use (e.g. t_k, Δt, etc).

Solution: Let $\Delta t = 15/n$ and $t_k = k\Delta t$.

$$T = \lim_{n\to\infty} \sum_{k=1}^{n} 0.4e^{-0.1(15-t_k)}\Delta t$$

(b) (4 points) Use an integral to evaluate the Riemann sum above. You may leave your answer in terms of e.

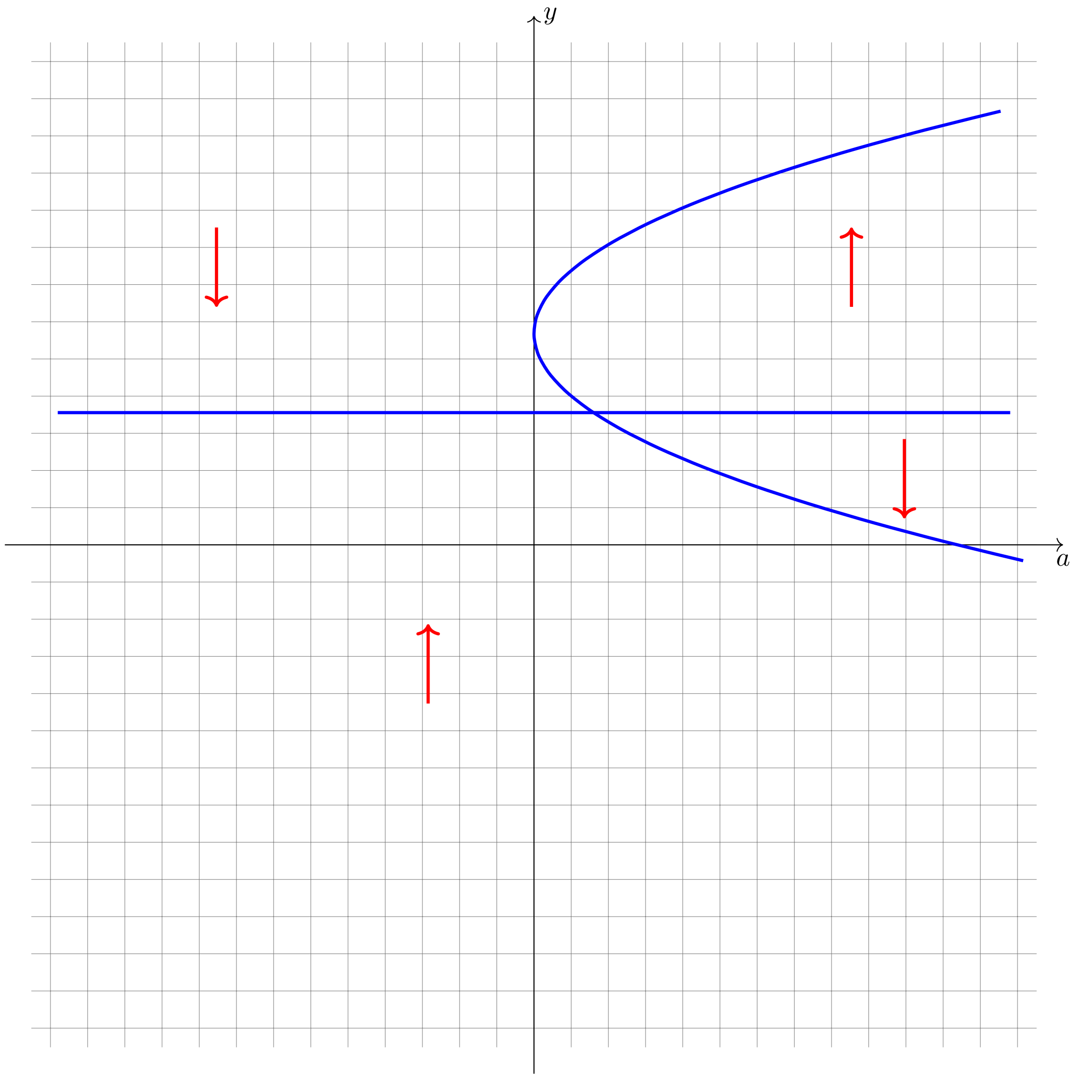
Solution:

$$\begin{aligned} W &= \int_0^{15} 0.4e^{-0.1(15-t)}\,\mathrm{d}t \\ &= \left[4e^{-0.1(15-t)}\right]_0^{15} \\ &= 4 - \left(4e^{-1.5}\right) \\ &\approx 3.1 \end{aligned}$$

5. In this question we will investigate the behaviour of the solutions of

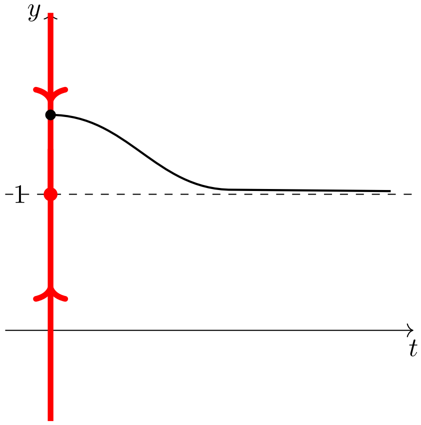
$$\frac{\mathrm{d}y}{\mathrm{d}t} = (a - y^2 + 4y - 4)(y - 1)$$

   (a) (4 points) Draw a bifurcation diagram for this equation with parameter a. Make sure to label the regions of your diagram with up/down arrows according to the direction of the derivative.



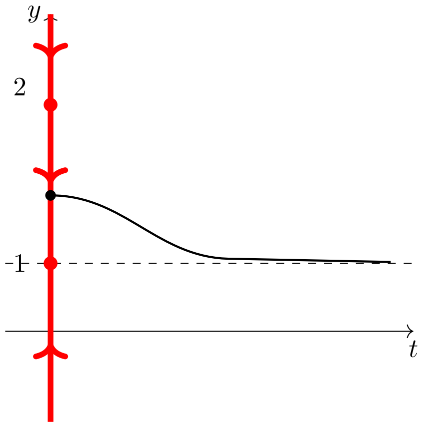
(b) (2 points) Draw a phase diagram when $a = -1$ and sketch the solution if $y(0) = 2$.

Solution:



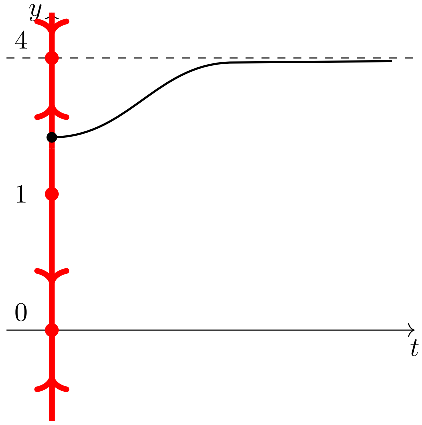
(c) (2 points) Draw a phase diagram when $a = 0$ and sketch the solution if $y(0) = 1.5$.

Solution:



(d) (2 points) Draw a phase diagram when $a = 4$ and sketch the solution if $y(0) = 2$.

Solution:



(e) (2 points) The differential equation above has an a equilibrium solution of $y = 1$ for any value of a. For what a is this equilibrium stable?

Solution: It is stable before $a = y^2 - 4y + 4$ intersects with $y = 1$, i.e. when $a = 1$. That is, the equilibrium is stable when $a < 1$.

6. A patient is administered a drug through a drip at a rate of 3 milligrams per hour. This particular drug has a half life in the body of $4\ln 2$ hours.

   (a) (4 points) Write a differential equation modelling the total amount of drug in the patients body at time t.

   Solution:
   $$\frac{\mathrm{d}y}{\mathrm{d}t} = 3 - \frac{\ln 2}{4\ln 4}y = 3 - \frac{1}{4}y$$

   (b) (3 points) If, initially, the patients body does not contain any drug, solve the differential equation.

   Solution:
   $$y(t) = 12 - Ce^{-\frac{1}{4}t}$$
   and since $y(0) = 0$ we have $C = 12$. So
   $$y(t) = 12\left(1 - e^{-t/4}\right).$$

(c) (1 point) How many milligrams of the drug is in the patients body after 12 hours? You may leave your answer in terms of e.

Solution: When $t = 12$ there are

$$y(12) = 12\left(1 - e^{-3}\right) \text{ milligrams.}$$

(d) (4 points) At the 12 hour mark, a nurse checks on the patient and realises the dosage was way too high. They immediately remove the drip. If the patient has 5 liters of blood, how long after the nurse has checked on the patient, does it take for the concentration of the drug to fall below 2 milligrams per liter? You can leave your answer in terms of e.

Solution: Let $z(t)$ be the amount of drug in the system t hours after the drip is pulled out. Now that the drip has been removed, no more drug is flowing in, so the ODE is

$$\frac{\mathrm{d}z}{\mathrm{d}t} = -\frac{1}{4}y$$

The solution is $z(t) = Ce^{-t/4}$. We use the initial condition $z(0) = y(12) = 12\left(1 - e^{-3}\right)$ which gives

$$12\left(1 - e^{-3}\right) = C$$

So our solution is

$$z(t) = 12\left(1 - e^{-3}\right)e^{-t/4}.$$

Now we would like to work out when this falls below $2 \cdot 5 = 10$ milligrams. To do this we solve $z(t) = 10$. We get

$$10 = 12\left(1 - e^{-3}\right)e^{-t/4}$$

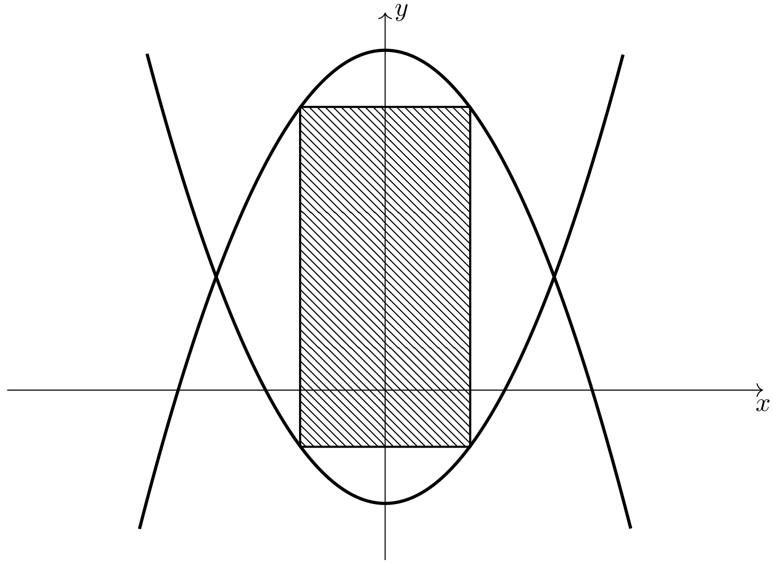
Dividing by $12\left(1 - e^{-3}\right)$ and rearranging gives

$$e^{-t/4} = \frac{5}{6\left(1 - e^{-3}\right)}$$

Now we take a log and multiply by -4 to get

$$t = 4\ln\left[\frac{6}{5}\left(1 - e^{-3}\right)\right] \approx 0.525 = 31.5 \text{ minutes}$$

7. (10 points) Consider the parabolas $y = 9 - x^2$ and $y = x^2 - 3$. They bound a region in the plane in which a rectangle can be inscribed (see the picture below). What is the maximum area this rectangle can have?



Solution: Let $(a, 9 - a^2)$ be the top right point. Then the vertices in clockwise order are

$$(a, 9 - a^2),\ \ (a, a^2 - 3),\ \ (-a, a^2 - 3),\ \ (-a, 9 - a^2).$$

Thus the width of the rectangle is $2a$ and the height is $12 - 2a^2$. Hence the area is

$$A(a) = 2a(12 - 2a^2) = 24a - 4a^3.$$

The coordinate a can be between 0 and the intersection point, this is where $9 - a^2 = a^2 - 3$, i.e $a = \sqrt{6}$. So the domain of A is $[0, \sqrt{6}]$.

To maximise we first find the critical points. We have that

$$A'(a) = 24 - 12a^2.$$

So if $A'(a) = 0$ we have $a = \sqrt{2}$. This is our only critical point.

Either we can observe that it is our only critical point and the second derivative is negative, so it must be a global maximum, or we can use the closed interval method to calculate the area there and at the endpoints,

$$A(0) = 0,\ \ A(\sqrt{2}) = 16\sqrt{2},\ \ A(\sqrt{6}) = 0$$

and thus conclude that $a = \sqrt{2}$ is a global max.

The maximum area is thus $16\sqrt{2}$.

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