RESEARCH STATEMENT

NOAH WHITE

Broad areas: Representation theory, combinatorics, algebraic geometry, quantum algebra.

Key words: Crystals, cactus groups, complex or unitary reflection groups, Cherednik algebras, Bethe algebras, Gaudin model, Schubert calculus, tableaux, dual equivalence, asymptotic Hecke algebra, reflection equation algebra, invariant theory, cluster combinatorics.

1. Introduction

My interests lie at the intersection of representation theory and geometry. In particular I am interested in representation theory and geometry that gives meaning to combinatorial phenomena. More precisely I study complex reflection groups; their associated geometry, combinatorics and representation theory; the combinatorics associated to (limits) of Bethe algebras; and the reflection equation algebra. A central aim of the work in my PhD thesis is to understand how one can produce the Kazhdan-Lusztig cells in type A using a relationship between Schubert calculus and the Calogero-Moser space, an object which can be constructed for any complex reflection group.

Calogero-Moser space can be thought of as the spectrum of certain commuting operators, whose eigenvectors are called *Bethe vectors*. Mukhin, Tarasov and Varchenko [MTV09] have developed a tight relationship between Schubert intersections and Bethe vectors. Both objects, Schubert intersections and Bethe vectors, vary naturally in families over a parameter space. I was interested in explaining how both objects vary, and describing the monodromy this produces. This problem is interesting as the monodromy realises the combinatorial pheneomena of *Jeu de taquin* and *Knuth moves* as well as the *Schutzenberger involution*. In particular one can use this monodromy action to recover the Kazhdan-Lusztig cells of the Symmetric group. This relates to a conjecture of Bonnafé and Rouquier [BR13] which aims to describe Kazhdan-Lusztig cells using Calogero-Moser space which is also related to the *rational Cherednik algebra*.

The monodromy action described above is the shadow of an action by the *cactus group*. This group can be considered as a crystal limit of the braid group and can be defined either combinatorially from the Dynkin diagram, or topologically, as a fundamental group (of the Deligne-Mumford compactification of the moduli space of genus 0, stable curves in type A). There exist cactus groups for every Coxeter group while it is not clear what a cactus group should be for a more general complex reflection group. It would also be interesting to relate the cactus group to Lusztig's asymptotic Hecke algebra.

Together with David Jordan, we have a project to investigate the reflection equation algebra. This algebra satisfies an interesting dual property: it is the locally finite part of the quantum group and the quantisation of the Semenov-Tian-Shansky Poisson bracket on the group GL_r. In particular we have conjectured explicit formulas for invariants of this algebra which give a description of the centre of $U_q(\mathfrak{gl}_r)$. This project comes with many extra questions such as understanding the quantisation of classical invariant theory objects in this situation, describing the higher Hochschild cohomology and questions about categorification of this algebra.

The Gaudin Hamiltonians are a certain set of commuting elements of the group algebra of the symmetric group depending on a set of complex parameters. By taking a suitable limit of these operators one obtains the *Jucys-Murphy elements*. In fact, one can take many kinds of limits and produce various commutative algebras analogous to the Jucys-Murphy elements. These algebras are not always maximally commutative but one can always (explicitly) produce new elements to make them so. It would be interesting to investigate the combinatorics of these algebras and their spectrum.

2. Bethe vectors and schubert calculus

For $\mathfrak{g}$ a simple lie algebra or $\mathfrak{gl}_r$, λ a dominant integral weight, let $L(\lambda)$ be the irreducible with highest weight λ. Understanding the multiplicity of $L(\mu)$ inside the tensor product

$$L(\lambda_\bullet) = L(\lambda^{(1)}) \otimes L(\lambda^{(2)}) \otimes \cdots \otimes L(\lambda^{(n)})$$

is a classical problem. In the case of $\mathfrak{g} = \mathfrak{gl}_r$ the multiplicity is given by the *Littlewood-Richardson coefficient* $c^{\mu}_{\lambda_\bullet}$. The Littlewood-Richardson coefficients (in the case of $\mathfrak{gl}_r$) count certain combinatorial objects, *dual equivalence cylindrical growth diagrams* (defined in [Spe14]). In the case $\lambda^{(s)} = (1)$ for all s, these are the same as *standard Young tableaux* of shape μ. It is an interesting problem to give meaning to this numerical coincidence by finding natural bases of the *multiplicity space* $L(\lambda_\bullet)^{\mathsf{sing}}_\mu$ which have a natural labelling by these combinatorial objects.

This is achieved by Mukhin, Tarasov and Varchenko in [MTV09] by using *Bethe algebras*. Bethe algebras are certain large commutative subalgebras

$$\mathcal{B}(\lambda_\bullet; z)_\mu \subseteq \mathrm{End}(L(\lambda_\bullet)^{\mathsf{sing}}_\mu),$$

depending on a tuple of distinct complex parameters $z = (z_1, z_2, \ldots, z_n)$. For generic z these algebras have simple spectrum, its eigenvectors are called *Bethe vectors*. In [MTV09] it is proven that $\mathcal{B}(\lambda_\bullet; z)_\mu$ is isomorphic to functions on

$$\Omega(\lambda^{(1)}; z_1) \cap \Omega(\lambda^{(2)}; z_2) \cap \ldots \cap \Omega(\lambda^{(n)}; z_n) \cap \Omega(\mu^{\mathsf{c}}; \infty) \tag{1}$$

for some special flags associated to the complex parameters z. In [Spe14], Speyer explains how one can label points in this intersection (for generic z) by certain combinatorial objects *dual equivalence cylindrical growth diagrams*, generalising standard tableaux. The following result (which was already studied and conjectured in [Ryb14]) describes how this labelling varies as we vary the parameters.

Theorem 2 ([Whi])**.** *The monodromy of the Bethe vectors for real z is described by the action of the cactus group J_n on $B(\lambda_\bullet)^{\mathsf{sing}}_\mu$.*

Here $B(\lambda_\bullet)$ is the crystal associated to $L(\lambda_\bullet)$. The category of crystals, unlike the category of modules for a simple Lie algebra, does not have a braiding. Instead it has a coboundary structure, first defined by Henriques and Kamnitzer in [HK06]. The cactus group J_n is the group which plays an analogous role to the braid group in this situation. Theorem 2 can be thought of as an analogue of the Kohno-Drinfeld theorem for the category of crystals.

In fact the Bethe algebra can be defined more generally. Let q be another n-tuple of complex parameters. The Bethe algebra $\mathcal{B}(\lambda_\bullet; q, z)_\mu$ is a large commutative algebra, this time acting as endomorphisms of $L(\lambda_\bullet)_\mu$. When $q = 0$ we are returned to the original case. In general the Bethe algebra commutes with the action of the Cartan $\mathfrak{h} \subseteq \mathfrak{g}$, and when $q = 0$ it commutes with the entirety of $\mathfrak{g}$ meaning we could restrict the action to singular vectors.

The spectrum of the Bethe algebras $\mathcal{B}(\lambda_\bullet; q, z)_\mu$ when $\lambda^{(s)} = (1)$ for all s (all the representations appearing are the vector representation $V \cong \mathbb{C}^r$) and $\mu = (1, 1, \ldots, 1)$ is isomorphic to *Calogero-Moser space*, a variety associated to the rational Cherednik algebra. This makes the connection to the work of Bonnafé and Rouquier [BR13] who use the geometry of Calogero-Moser space to define a partitioning of the Weyl group into *cells*. They conjecture these cells are in fact the Kazhdan-Lusztig cells. In the case of $\mathfrak{gl}_r$, the set $[B((1))^{\otimes n}]_{(1,1,\ldots,1)}$ can be identified with the symmetric group and the orbits of the action of the cactus group recover the Kazhdan-Lusztig cells.

In joint work with Adrien Brochier and Iain Gordon [BGW], we strengthen this result, showing how to connect Theorem 2 to a continuous version of the RSK correspondence, thereby relating Calogero-Moser space to tableaux combinatorics, as conjectured by Bonnafe and Rouquier in [BR13] (in different but equivalent language).

There are several possible directions in which this research project could progress. Firstly it is important to note Theorem 2 only describes the monodromy over real points. It would be desirable to understand what happens over the full parameter space. Over the real numbers Mukhin, Tarasov and Varchenko showed that the Bethe algebras have simple spectrum, however this is only true for *generic* values of the complex parameters. Describing the locus on which the Bethe algebras fail to have simple spectrum is a very difficult problem, however there is still hope we may be able to say something about the monodromy.

Question 1. *Do the Bethe vectors have extra monodromy not described by the action of the cactus group? It is expected the answer should be no.*

A related question would be to describe the monodromy of the spectrum of the Bethe algebras $\mathcal{B}(\lambda_\bullet; q, z)_\mu$ when $q \neq 0$. It is possible this would give some insight into the Bonnafe-Rouquier conjecture in types other than A.

Question 2. *Is it possible to extend the cactus group action to the spectrum of Bethe algebras when $q \neq 0$? Does this describe all the monodromy?*

There is an important collection of elements of the Bethe algebra called the *Gaudin Hamiltonians*. These are important operators coming from the study of an integrable system, the *Gaudin model*. In fact much of our work above was inspired by a result of Aguirre, Felder and Veselov [AFV11] about the moduli space of the algebras these Hamiltonians generate. In the case $\lambda^{(s)} = (1)$ for $s = 1, 2, \ldots, n$ these Hamiltonians in fact generate the entire Bethe algebra. This is not true in general.

Question 3. *In what cases do the Gaudin Hamiltonians generate the Bethe algebra (for generic parameter values)? What about the case $q \neq 0$?*

A key fact which makes calculating the monodromy of the Bethe vectors over real parameters possible is that Speyer [Spe14] has defined a compactification $\mathcal{S}(\lambda_\bullet, \mu^c)$ of the family of Schubert intersections (the variety obtained by letting z vary in (1)). In a similar way, Rybnikov in [Ryb14] defined a compactification $\mathcal{A}(\lambda_\bullet)_\mu$ of the family of Bethe algebras.

Question 4. *Is the spectrum of $\mathcal{A}(\lambda_\bullet)_\mu$ isomorphic to the variety $\mathcal{S}(\lambda_\bullet, \mu^c)$?*

3. Cactus groups and the asymptotic Hecke algebra

Let W be a complex reflection group with reflection representation $\mathfrak{h}$. Let $\mathfrak{h}^{\mathrm{reg}} \subseteq \mathfrak{h}$ be the part on which W acts freely (i.e. the complement of the reflecting hyperplanes). Associated to W we can define the *pure braid group* PB_W as the fundamental group $\pi_1(\mathfrak{h}^{\mathrm{reg}})$ and the *braid group* B_W, as the fundamental group $\pi_1(\mathfrak{h}^{\mathrm{reg}}/W)$.

Now restrict to the case W is a finite Coxeter group. The cactus group of W can be thought of as a kind of crystal limit of the braid group and can be defined in two ways. First combinatorially. If C is the Coxeter diagram associated to W, every connected subdiagram $E \subseteq C$ determines a subgroup $W_E \subseteq W$. We denote the longest element of W_E by w_E. The cactus group J_W is the group generated by elements s_E for $E \subseteq C$ and relations

$$\begin{aligned} s_E^2 &= 1, && \\ s_D s_E &= s_E s_D && \text{if } D \cap E = \emptyset, \text{ and} \\ s_D s_E &= s_F s_D && \text{if } E \subseteq D, \text{ where } F = w_D \cdot E. \end{aligned}$$

The second definition is topological. De Concini and Procesi [DP95] define a compactification of $\mathbb{P}\mathfrak{h}^{\mathrm{reg}}$, denoted $\overline{\mathbb{P}\mathfrak{h}^{\mathrm{reg}}}$. We define the pure cactus group PJ_W to be the fundamental group of the real locus $\mathbb{P}\mathfrak{h}^{\mathrm{reg}}_{\mathbb{R}}$. Note W preserves the real locus of $\mathbb{P}\mathfrak{h}^{\mathrm{reg}}$ since it is a Coxeter group. We define the cactus group J_W to be the equivariant fundamental group of the stack $[\mathbb{P}\mathfrak{h}^{\mathrm{reg}}/W]$. Just as in the braid group case we have a short exact sequence

$$1 \longrightarrow PJ_W \longrightarrow J_W \longrightarrow W \longrightarrow 1.$$

The equivalence of the two definitions is demonstrated in [DJS03]. As noted above, the cactus group of type A acts on the symmetric group and the orbits of this action are exactly the Kazhdan-Lusztig cells. This is almost true for other Coxeter groups as demonstrated by Losev [Los15] and Bonnafé [Bon15]. The orbits are not quite the full cells in general, there is an obstruction which disappears in type A.

It would be interesting to find a group which plays a role analogous to the cactus group for a general complex reflection group. In particular it would be interesting to make the link with *Calogero-Moser cells* [BR13] which are the conjectural analogue of Kazhdan-Lusztig cells. One may think the initial problem in constructing such a group is that there is no analogue of the real points $\mathbb{P}\mathfrak{h}^{\mathrm{reg}}_{\mathbb{R}}$. However in [MM10] it is shown that the collection of reflecting hyperplanes for any complex reflection group W have a $\mathbb{R}$-form (in fact a $\mathbb{Q}$-form). One could then use this $\mathbb{R}$-form to define a compactification and thus a candidate for a pure cactus group. Whether this group has the correct properties remains to be investigated. Of course, W does not fix this $\mathbb{R}$-form in general so it is unclear how this definition would extend to a full cactus group.

Another approach would be to try and define these groups combinatorially. There exist Coxeter-like diagrams which display a set of generators and relations for a complex refection group (see for example [Bro10]).

Question 5. *Can one define a (pure) cactus group for an arbitrary complex reflection group? Does this cactus group have an action on the complex reflection group which preserves the Calogero-Moser cells?*

An important point is that in type A, one does not need the action of the full cactus group to produce the Kazhdan-Lusztig cells. The orbits of the pure cactus group are enough.

The group algebra of the braid group has an interesting quotient, the *Hecke algebra* of W. For Coxeter groups, Lusztig defined the *asymptotic Hecke algebra* $\mathcal{J}$. This algebra is defined by taking asymptotic versions of the structure constants in the Hecke algebra and can be thought of as a crystal limit. In fact $\mathcal{J}$ is isomorphic over $\mathbb{C}$ to the group algebra $\mathbb{C}W$. In some ways it has a much simpler structure than the Hecke algebra, for example it contains ideals which are isomorphic to the left cell modules.

Question 6. *Is there a relationship between the asymptotic Hecke algebra and the cactus group? Very optimistically one could hope that $\mathcal{J}$ is a quotient of the group algebra of J_W.*

4. Quantisation in the reflection equation algebra

The *reflection equation algebra* $\mathcal{A}_q$ is the quantisation of the *Semenov-Tian-Shansky Poisson bracket* on $\mathcal{O}(\mathrm{GL}_r)$, functions on GL_r. Abstractly, it is the algebra freely generated by a^i_j for $1 \le i, j \le r$, (collected into a matrix A) subject to relations

$$R_{21}A_1R_{12}A_2 = A_2R_{21}A_1R_{12}.$$

Here R is the R-matrix for the vector representation of the Drinfeld-Jimbo quantum group $U_q(\mathfrak{gl}_r)$. Importantly $\mathcal{A}_q$ can be embedded in $U_q(\mathfrak{gl}_r)$ as the the locally finite part. By restricting the adjoint action of $U_q(\mathfrak{gl}_r)$ on itself, we obtain an adjoint action on $\mathcal{A}_q$. By construction $\mathcal{A}_q$ contains the centre of $U_q(\mathfrak{gl}_r)$.

The aim of this joint project with David Jordan, is to calculate the a full set of polynomial generators of the centre of $U_q(\mathfrak{gl}_r)$. By the above discussion this is the same as the invariants of the adjoint action on $\mathcal{A}_q$.

Definition 3 ([JW])**.** We define the following elements of $\mathcal{A}_q \subseteq U_q(\mathfrak{gl}_r)$.

$$c_k = \sum_{\substack{I \subset [r] \\ \#I = k}} q^{-2|I|} \sum_{\sigma \in S_I} (-1)^\sigma q^{l(\sigma)+e(\sigma)} a^{i_1}_{\sigma(i_1)} a^{i_2}_{\sigma(i_2)} \cdots a^{i_k}_{\sigma(i_k)},$$

where l and e denote the length and *excedence statistics* and $S_I \subseteq S_r$ is the subgroup of permutations fixing $[r] \setminus I$.

In [JW] we conjecture that the c_k are a full set of algebraically independent invariants. We also conjecture a Cayley-Hamilton type identity.

Conjecture 4. *The c_k are invariant under the adjoint action. The matrix A satisfies the Cayley-Hamilton identity:*

$$A^N - q^2 c_1 A^{N-1} + q^4 c_2 A^{N-2} - \ldots + (-1)^N q^{2N} c_N I = 0.$$

The problem of understanding the centre of $U_q(\mathfrak{gl}_r)$ has a significant literature (see [Ros90], [JL94], [Dri89], [Res89], [RTF89], and [Bau98]). The algebra $\mathcal{A}_q$ has received attention recently, in particular because it computes the Hochschild homology of the braided monoidal category $\mathrm{Rep}_q(\mathrm{GL}_r)$ and has thus appeared in work [CK15] and [BBJ15]. It would be desirable to have an explicit description of the centre in terms of this algebra.

In [JW], we succeed in proving this conjecture for $k = 1, 2$ and r. In particular for $k = 2$ we produce a set of elements $c(ij|kl)$ for any $i, j, k, l \in \{1, 2, \ldots, r\}$, which quantise the 2×2 minors. The element c_2 is then a weighted sum of the diagonal minors and we are able to use this to prove invariance. For the other c_k we would like formulas for the $k \times k$ minors.

Question 7. *What are formulas for the $k \times k$ minors in $\mathcal{A}_q$? What is the action of $U_q(\mathfrak{gl}_r)$ on these minors?*

This question is interesting as it would provide a quantisation of the determinantal variety. This has been well studied in the case of the RTT-algebra (see [DL03]). The conjectures above come from extensive computational experimentation using the MAGMA computer algebra package [BCP97]. During this project we have developed a large set of code for experimenting with $\mathcal{A}_q$ and the adjoint action of $U_q(\mathfrak{gl}_r)$. Future research includes the following questions.

Question 8. *What is the combinatorial meaning of the appearance of the excedance? Can we categorify $\mathcal{A}_q$? What can one say about Hochschild cohomology of $\mathcal{A}_q$ in higher degree?*

5. Combinatorics of Gaudin Hamiltonians

The Gaudin Hamiltonians for the symmetric group are operators

$$H_a(z) = \sum_{b \neq a} \frac{(a,b)}{z_a - z_b} \in \mathbb{C}S_n$$

depending on n distinct complex parameters $z = (z_1, z_2, \ldots, z_n)$. They generate a large commutative subalgebra $G(z) \subseteq \mathbb{C}S_n$. In fact, Aguirre, Felder and Veselov [AFV11] showed that these algebras can be defined for any point in $\overline{M}_{0,n+1}(\mathbb{C})$, the moduli space of stable rational curves with $n+1$ marked points. In this project we are interested in the limits of these algebras. That is, the algebras $G(z)$ for points $z \in \overline{M}_{0,n+1}$ corresponding to curves with the maximal number of irreducible components.

One very special limit, when we take (for real z) $z_1 << z_2 << \ldots << z_n \to \infty$, produces the well known Jucys-Murphy elements in $\mathbb{C}S_n$. The spectrum of the Jucys-Murphy elements on irreducible representations is described using the combinatorics of tableaux and their contents and corresponds to a sequence of inductions over the chain of subgroups

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_n$$

where we embed S_i as the subgroup of S_{i+1} fixing $i+1$. Similarly other limit points in $\overline{M}_{0,n+1}$ produce algebras whose eigenvalues are described by the combinatorics corresponding to sequences of inductions over other chains of subgroups and their branching graphs.

For example let $n = 6$ and $z \in \overline{M}_{0,7}$ be the point obtained by colliding marked points as described by the bracketing $((12)3)((45)6)$. This corresponds to the sequence

$$S_1 \subseteq S_2 \subseteq S_3 \subseteq S_3 \times S_2 \subseteq S_6.$$

The restrictions in this chain are not all multiplicity free and accordingly the algebra $G(z)$ does not have simple spectrum on all irreducible representations of S_6. We can still describe the eigenvalues using the combinatorics of the branching graph. Amazingly we can fix the problem of non simplicity of the spectrum by adding extra operators, using work in [MTV13] we can describe these *higher Gaudin Hamiltonians* explicitly as limits of operators

$$X_a^{k+1}(z) = \sum_{\substack{b_1 < \ldots < b_k \\ b_i \neq a}} \frac{B_{a,b_1,\ldots,b_k}}{\prod_{s=1}^{k}(z_a - z_{b_s})}.$$

Here $B_{a,b_1,\ldots,b_k}$ is a very explicit alternating sum of elements in $\mathbb{C}S_n$.

Question 9. *What are the limiting eigenvalues of the operators $X_a^{k+1}(z)$? Can we describe them combinatorially?*

In [LY15], Lenagan and Yakimov use chains of subalgebras to produce cluster combinatorics for quantum Schubert cells and quantum Richardson varieties. It is possible we may be able to produce some cluster-type combinatorics in our analogous situation. This is, however, much more speculative.

References

[AFV11] Leonardo Aguirre, Giovanni Felder, and Alexander P Veselov. "Gaudin subalgebras and stable rational curves". In: *Compositio Mathematica* 147.5 (2011), pp. 1463–1478.

[Bau98] Pierre Baumann. "On the center of quantized enveloping algebras". In: *Journal of Algebra* 203.1 (1998), pp. 244–260.

[BBJ15] David Ben-Zvi, Adrien Brochier, and David Jordan. "Integrating quantum groups over surfaces: quantum character varieties and topological field theory". In: *arXiv.org* (Jan. 2015). arXiv: 1501.04652v1 [math.QA].

[BCP97] Wieb Bosma, John Cannon, and Catherine Playoust. "The Magma algebra system. I. The user language". In: *Journal of Symbolic Computation* 24.3-4 (1997), pp. 235–265.

[BGW] Adrien Brochier, Iain Gordon, and Noah White. *Calogero-Moser cells in type A and the RSK correspondence*. In preparation.

[Bon15] Cédric Bonnafé. "Cells and cacti". In: *arXiv.org* (June 2015). arXiv: 1506.07737v3 [math.RT].

[BR13] Cédric Bonnafé and Raphaël Rouquier. "Cellules de Calogero-Moser". In: (Feb. 2013). arXiv: 1302.2720.

[Bro10] Michel Broué. *Introduction to complex reflection groups and their braid groups*. Vol. 1988. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer-Verlag, Berlin, 2010.

[CK15] Sabin Cautis and Joel Kamnitzer. "Quantum K-theoretic geometric Satake". In: *arXiv.org* (Sept. 2015). arXiv: 1509.00112v1 [math.RT].

[DJS03] M Davis, T Januszkiewicz, and R Scott. “Fundamental groups of blow-ups”. In: *Advances in Mathematics* 177.1 (2003), pp. 115–179.

[DL03] M Domokos and T H Lenagan. “Conjugation coinvariants of quantum matrices”. In: *The Bulletin of the London Mathematical Society* 35.1 (2003), pp. 117–127.

[DP95] C De Concini and C Procesi. “Wonderful models of subspace arrangements”. In: *Selecta Mathematica. New Series* 1.3 (1995), pp. 459–494.

[Dri89] V G Drinfel d. “Almost cocommutative Hopf algebras”. In: *Rossi skaya Akademiya Nauk. Algebra i Analiz* 1.2 (1989), pp. 30–46.

[HK06] André Henriques and Joel Kamnitzer. “Crystals and coboundary categories”. In: *Duke Mathematical Journal* 132.2 (2006), pp. 191–216.

[JL94] Anthony Joseph and Gail Letzter. “Separation of variables for quantized enveloping algebras”. In: *American Journal of Mathematics* 116.1 (1994), pp. 127–177.

[JW] David Jordan and Noah White. *Quantum Cayley-Hamilton identities for the reflection equation algebra*. In preparation.

[Los15] Ivan Losev. “Cacti and cells”. In: *arXiv.org* (June 2015). arXiv: `1506.04400v1 [math.RT]`.

[LY15] T H Lenagan and M T Yakimov. “Prime factors of quantum Schubert cell algebras and clusters for quantum Richardson varieties”. In: *arXiv.org* (Mar. 2015). arXiv: `1503.06297v1 [math.QA]`.

[MM10] I Marin and J Michel. “Automorphisms of complex reflection groups”. In: *Representation Theory. An Electronic Journal of the American Mathematical Society* 14.21 (2010), pp. 747–788.

[MTV09] E Mukhin, V Tarasov, and A N Varchenko. “Schubert calculus and representations of the general linear group”. In: *Journal of the American Mathematical Society* 22.4 (2009), pp. 909–940.

[MTV13] E Mukhin, V Tarasov, and A N Varchenko. “Bethe subalgebras of the group algebra of the symmetric group”. In: *Transformation Groups* 18.3 (2013), pp. 767–801.

[Res89] N Yu Reshetikhin. “Quasitriangular Hopf algebras and invariants of links”. In: *Rossi skaya Akademiya Nauk. Algebra i Analiz* 1.2 (1989), pp. 169–188.

[Ros90] Marc Rosso. “Analogues de la forme de Killing et du théorème d’Harish-Chandra pour les groupes quantiques”. In: *Annales Scientifiques de l’École Normale Supérieure. Quatrième Série* 23.3 (1990), pp. 445–467.

[RTF89] N Yu Reshetikhin, L A Takhtadzhyan, and L D Faddeev. “Quantization of Lie groups and Lie algebras”. In: *Rossi skaya Akademiya Nauk. Algebra i Analiz* 1.1 (1989), pp. 178–206.

[Ryb14] Leonid Rybnikov. “Cactus group and monodromy of Bethe vectors”. In: *arXiv.org* (Aug. 2014). arXiv: `1409.0131v2 [math.QA]`.

[Spe14] David E Speyer. “Schubert problems with respect to osculating flags of stable rational curves”. In: *Algebraic Geometry* 1.1 (2014), pp. 14–45.

[Whi] Noah White. *Monodromy of real Bethe vectors*. In preparation.

Noah White, School of Mathematics, James Clerk Maxwell Building, The King’s Buildings, Peter Guthrie Tait Road, Edinburgh, EH9 3FD, UK.

E-mail address: noah.white@ed.ac.uk