

This week you will get practice drawing and understanding bifurcation diagrams.

\*Numbers in parentheses indicate the question has been taken from the textbook:

S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,

and refer to the section and question number in the textbook.

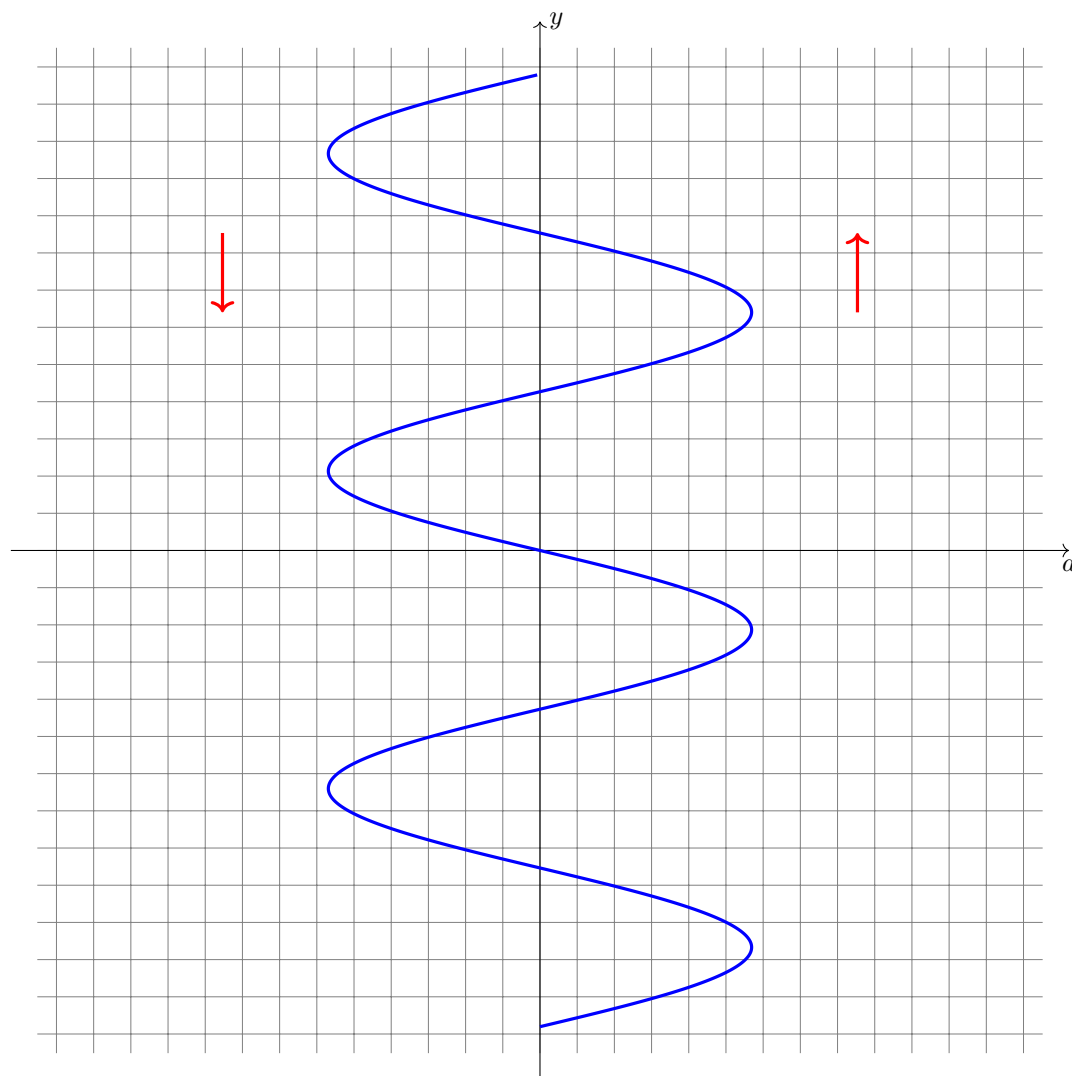
1. (6.6) Sketch the bifurcation diagrams for the equations in the following.

(a) (6.6-7)  $\frac{dy}{dt} = ay - y^2$

(b) (6.6-10)  $\frac{dy}{dt} = 1 - ay^2$

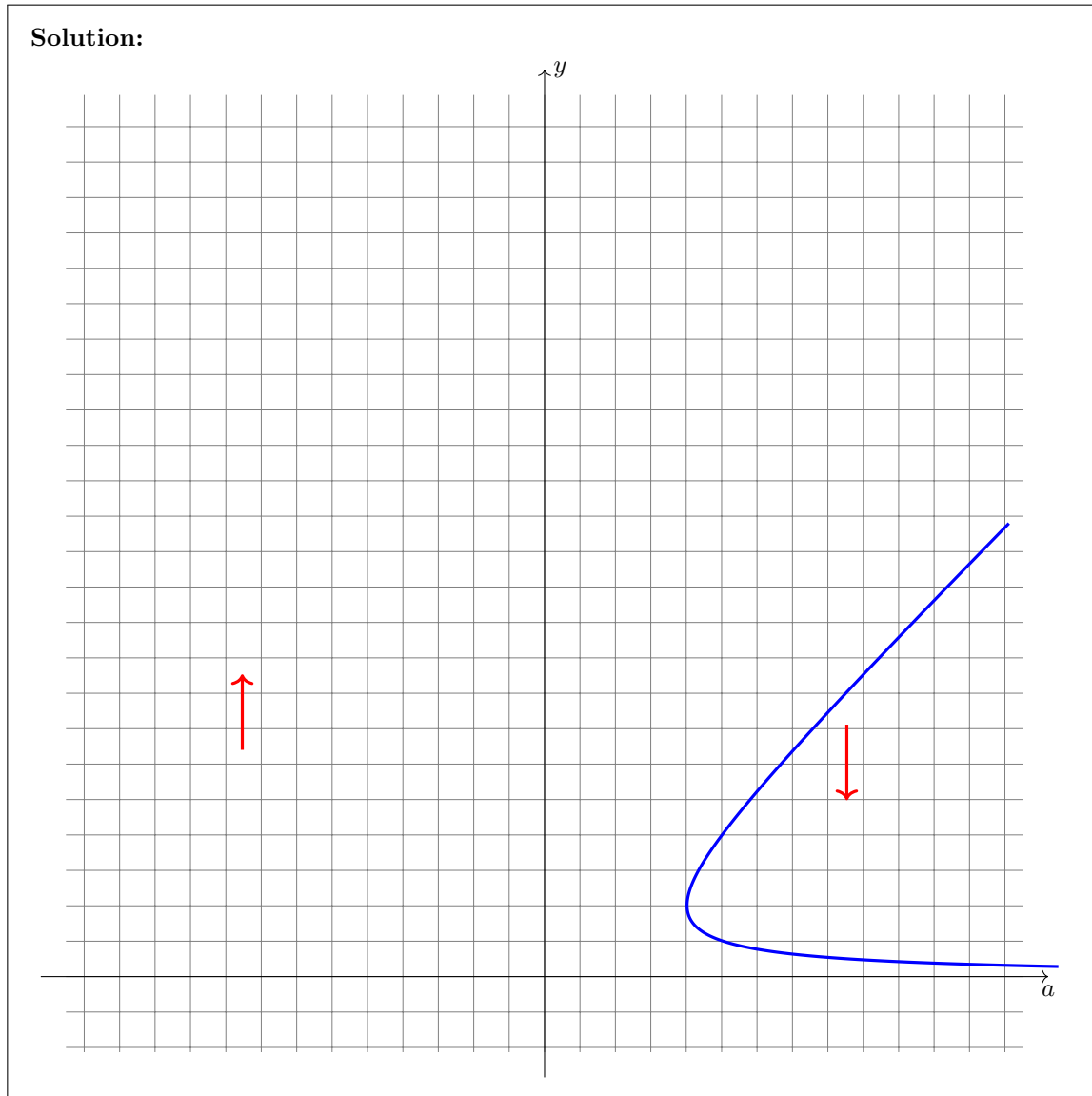
(c) (6.6-11)  $\frac{dy}{dt} = \sin y + a$

**Solution:**



(d) (6.6-12)  $\frac{dy}{dt} = y^2 - ay + 2$  for  $a > 0$ .

Solution:



2. (6.6) Consider the model

$$\frac{dy}{dt} = \frac{ay^2}{k^2 + y^2} - cy$$

of an autocatalytic gene from question 5. In each of the following cases, two of the parameters are specified. Sketch a bifurcation diagram with respect to the third parameter.

- (a) (6.6-13)  $k = 1$ ,  $c = 2$  with  $a$  as the bifurcation parameter.
  - (b) (6.6-14)  $k = 2$ ,  $c = 1$  with  $a$  as the bifurcation parameter.
  - (c) (6.6-15)  $a = 10$ ,  $k = 1$  with  $c$  as the bifurcation parameter.
  - (d) (6.6-16)  $a = 10$ ,  $c = 1$  with  $k^2$  as the bifurcation parameter.
3. (6.6-40) Suppose the growth rate of a whale population at density  $N$  (individuals per million square kilometers of ocean), harvested at a rate  $h$ , is given by

$$\frac{dN}{dt} = 0.07N \left( \frac{N}{10} - 1 \right) \left( 1 - \frac{N}{80} \right) - h$$

where the units of  $t$  are years.

- (a) Sketch a bifurcation diagram with respect to the parameter  $h$  as it varies over the interval  $[0, 8]$ .

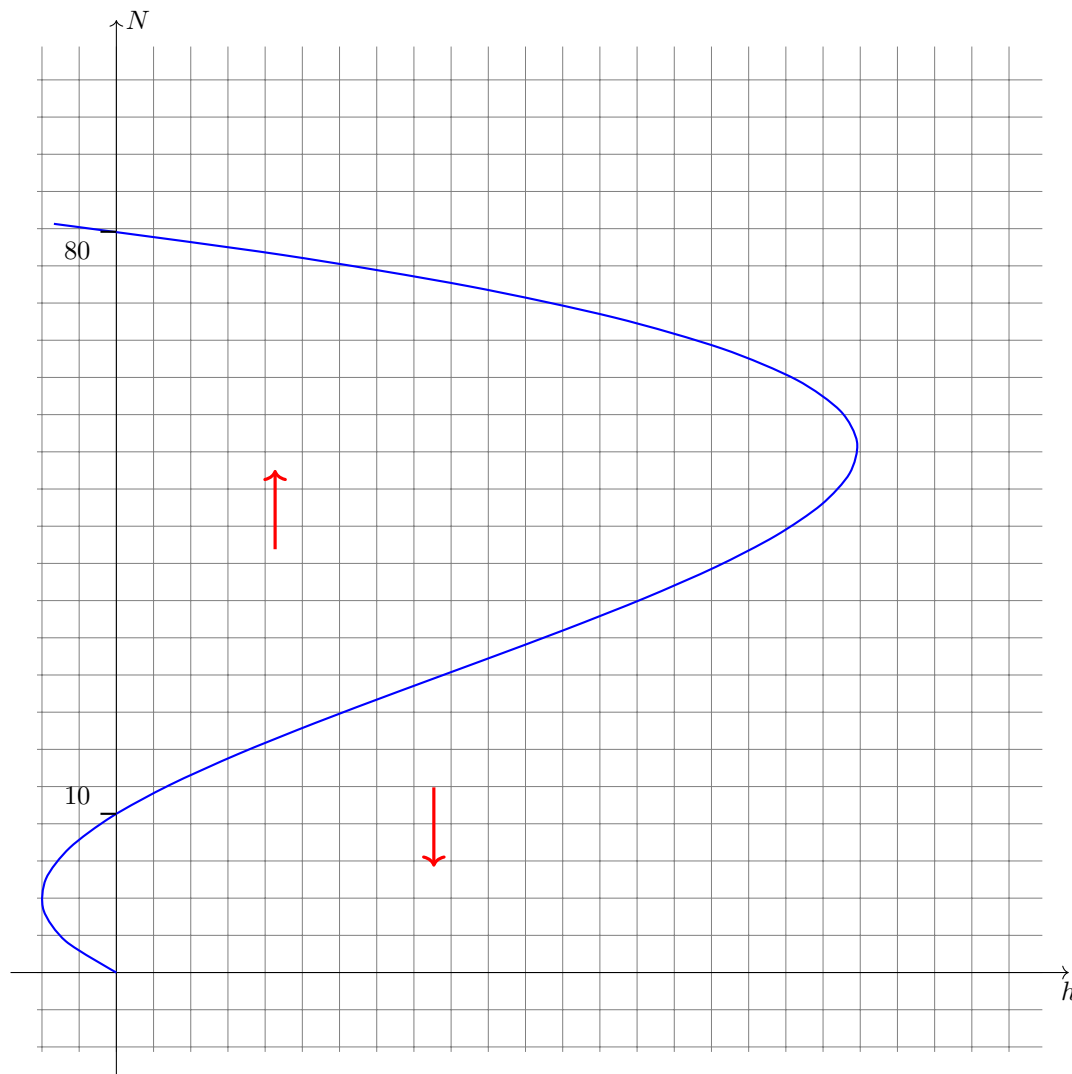
**Solution:** We want to sketch the points that satisfy

$$0.07N \left( \frac{N}{10} - 1 \right) \left( 1 - \frac{N}{80} \right) - h = 0$$

i.e. where

$$h = 0.07N \left( \frac{N}{10} - 1 \right) \left( 1 - \frac{N}{80} \right)$$

To do this we can sketch it in the  $hN$ -axis and then flip to the  $Nh$ -axis:



The direction of the arrows is found by subbing in points from each region (e.g.  $(0, 40)$  and  $(1, 0)$ ) into

$$0.07N \left( \frac{N}{10} - 1 \right) \left( 1 - \frac{N}{80} \right) - h$$

and determining whether it is positive or negative.

- (b) If  $h = vN$ , then sketch a bifurcation diagram with respect to the parameter  $v$  as it varies over the interval  $[0, 0.12]$ .

**Solution:** We want to sketch the points that satisfy

$$0.07N \left( \frac{N}{10} - 1 \right) \left( 1 - \frac{N}{80} \right) - vN = 0$$

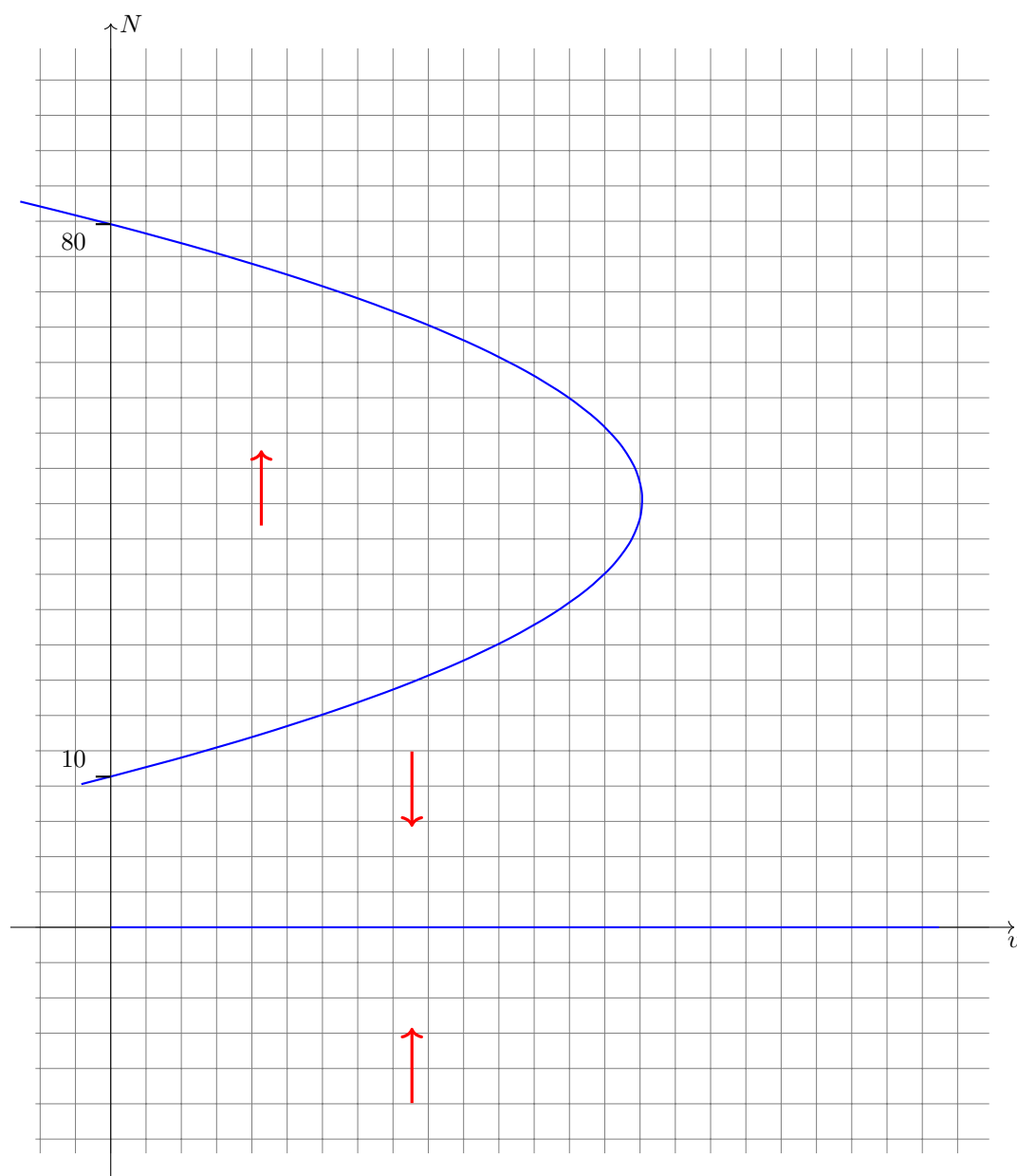
i.e. where

$$\left[ 0.07 \left( \frac{N}{10} - 1 \right) \left( 1 - \frac{N}{80} \right) - v \right] N = 0$$

Thus we want to sketch the points where either  $N = 0$  or

$$v = 0.07 \left( \frac{N}{10} - 1 \right) \left( 1 - \frac{N}{80} \right).$$

To do this we can first sketch the line  $N = 0$  and then the sideways parabola:



The direction of the arrows is found by subbing in points from each region (e.g.  $(0, 40)$ ,  $(0, 5)$  and  $(0, -1)$ ) into

$$0.07N \left( \frac{N}{10} - 1 \right) \left( 1 - \frac{N}{80} \right) - vN$$

and determining whether it is positive or negative.