Lecture 21

- We are now at a point in the course where we have defined all the pieces and we can try and ~~st~~ say something about their relationships

- This course has basically been about various types of derivative

1D	2D	3D
functions	functions	functions
$\downarrow \frac{d}{dx}$	$\downarrow \nabla$	$\downarrow \nabla$
functions	vector fields	vector fields
	$\downarrow$ curl	$\downarrow$ curl
	functions	vector fields
		$\downarrow$ div.
		functions

- One aim is to say something about the antiderivatives of these operations.

Eg. * When can we find an φ st $\nabla\varphi = \underline{F}$ for a given vector field
* When can we find an $\underline{A}$ st $\mathrm{curl}(\underline{A}) = \underline{F}$ for a given $\underline{F}$?

- We have given some hint, but a full explanation is beyond this course (this is the realm of "topology" and something called "cohomology".)

- Another aim is to generalise the following theorems:

<u>Thm</u> (FTC) If $\frac{d}{dx}f(x) = g(x)$ then
$$\int_a^b g(x)\,dx = f(b) - f(a)$$

<u>Thm</u> (FTC for conservative vf's) If $\nabla\varphi = \underline{F}$ then
$$\int_{\mathcal{C}} \underline{F}\cdot d\underline{r} = \varphi(Q) - \varphi(P).$$

- We want statements ~~It~~ like: If $\text{curl}(\underline{A}) = \underline{F}$ then

$$\iint_S \underline{F} \cdot d\underline{S} = \cdots \underline{A} \cdots ?$$

- We will answer this question.

Green's theorem

- For now we stick to 2D world.
- If $D \subseteq \mathbb{R}^2$ is a region of 2D space denote by ∂D it boundary
- We define an orientation on the closed curve ∂D by
    * If we walk around the boundary in the direction of orientation the region is on our left
- This is called the boundary orientation

~~- We say $\underline{F}$ have a vector potential $\underline{A}$ if $\text{curl}(\underline{A})$~~

- Recall $\text{curl}(\underline{F}) = \partial_x F_2 - \partial_y F_1$.

Thm If $\underline{F}$ is defined on D, then

$$\oint_{\partial D} \underline{F} \cdot d\underline{r} = \iint_D \operatorname{curl}(\underline{F})\, dA$$

where ∂D is equipped w/ the boundary orientation.

Area

- The area of D is
$$\iint_D 1 \, dA$$
- If we find $\underline{F}$ s.t. $\operatorname{curl}(\underline{F}) = 1$ then by Green's theorem
$$\oint_{\partial D} \underline{F} \cdot d\underline{r} = \operatorname{Area}(D).$$
- Eg. $\langle 0, x \rangle$, $\langle -y, 0 \rangle$, $\frac{1}{2}\langle -y, x \rangle$.

Flux and Green's theorem

- Let $\underline{F}^\perp = \langle -F_2, +F_1 \rangle$ be the vector field orthogonal to $\underline{F}$.

Prop $\operatorname{curl}(\underline{F}^\perp) = \operatorname{div}(\underline{F})$

proof $\operatorname{curl}(\underline{F}^\perp) = \partial_x F_1 - \partial_y(-F_2) = \operatorname{div}(\underline{F})$

Thm (GT for flux)

$$\oint_{\partial D} (\underline{F} \cdot \underline{n})\, ds \; \cancel{\oint_{\partial D} \underline{F} \cdot d\underline{r}} = \iint_D \operatorname{div}(\underline{F})\, dA$$

proof:
$$\oint_{\partial D} (\underline{F} \cdot \underline{n})\, ds = \cancel{\iint_{D}} \oint_{\partial D} \underline{F}^{\perp} \cdot d\underline{r}$$
$$= \iint_D \operatorname{curl}(\underline{F}^{\perp})\, dA$$
$$= \iint_D \operatorname{div}(\underline{F})\, dA \qquad \square.$$