This weeks problem set focuses on the concept of a change of basis matrix. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework 3: due Friday 15 Feb: questions 2 and 5 below.

1. From section 2.5, problems 1, $2a, c$, $3a, c$, 5, 7, 10^*, 13^*.

2.* Let V be a finite dimensional vector space and W a subspace. Show that V and $W \times V/W$ are isomorphic by finding an explicit isomorphism (rather than simply computing the dimensions).

> **Solution:** Let $B = \{ v_1, \ldots, v_n \}$ be a basis of V so that $\{ v_1, \ldots, v_k \}$ is a basis for W. Now define a map $\phi : V \longrightarrow W \times V/W$ by
>
> $$\phi(v_i) = \begin{cases} (v_i, 0) & \text{if } 0 \leq i \leq k \\ (0, v_i + W) & \text{otherwise.} \end{cases}$$
>
> We claim this is an isomorphism. Indeed, for every element $(w, v + W) \in W \times V/W$ we can express this as
>
> $$(w, v + W) = \lambda_1(v_1, 0) + \ldots + \lambda_k(v_k, 0) + \mu_{k+1}(0, v_{k+1}) + \ldots + \mu_n(0, v_n + W)$$
>
> where $w = \lambda_1 v_1 + \ldots + \lambda_k v_k$ and $v = \mu_1 v_1 + \ldots + \mu_n v_n$. So ϕ is surjective.
>
> Now if
> $$\lambda_1(v_1, 0) + \ldots + \lambda_k(v_k, 0) + \lambda_{k+1}(0, v_{k+1}) + \ldots + \lambda_n(0, v_n + W) = 0$$
> then $\lambda_1 v_1 + \ldots + \lambda_k v_k = 0$ and $\lambda_{k+1} v_{k+1} + \ldots + \lambda_n v_n \in W$. Thus $\lambda_i = 0$ for all i and ϕ is injective and thus an isormorphism.

3.* Let V be a finite dimensional vector space and W a subspace. Show that $\dim(V/W) = \dim V - \dim W$. *Hint: consider a basis of W and extend it to V. Now find a basis for V/W. You can also prove it using the dimension theorem.*

> **Solution:** Let $B = \{ v_1, \ldots, v_n \}$ be a basis of V so that $\{ v_1, \ldots, v_k \}$ is a basis for W. Then $\{ v_{k+1} + W, \ldots, v_n + W \}$ is a basis for V/W. Hence
>
> $$\dim(V/W) = n - k = \dim V - \dim W.$$

4.* Let $T : V \longrightarrow W$ be a linear map.

(a) Show that $\operatorname{im} T$ and $V/\ker T$ are isomorphic.

> **Solution:** Define a map $\phi : V/\ker T \longrightarrow \operatorname{im} T$ by $\phi(v + \ker T) = T(v)$. We must first check that this is well defined. I.e. if $v + \ker T = v' + \ker T$ then we should check that $\phi(v + \ker T) = \phi(v' + \ker T)$. This translates to checking that $T(v) = T(v')$ if $v - v' \in \ker T$. In this situation, $T(v - v') = 0$ so $T(v) - T(v') = 0$ by linearity, so ϕ is well defined.
>
> To check this is an isomorphism, note first of all that it is surjective. Now suppose that $\phi(v + \ker T) = \phi(v' + \ker T)$. This means $T(v - v') = 0$ so $v - v' \in \ker T$, i.e $v + \ker T = v' + \ker T$ so ϕ is injective and thus an isomorphism.

(b) Use this (and the previous exercise) to give an alternative proof of the dimension theorem.

Solution: Note first that $\dim(V/\ker T) = \dim V - \dim\ker T$. Thus, using the previous part, we see that $\dim V - \dim\ker T = \dim\operatorname{im} T$ which is the dimension theorem.

5. A differential operator on $\mathbb{R}_n[x]$ is a linear combination of expressions of the form $x^a \frac{d^b}{dx^b}$ where $a - b \leq 0$ (otherwise the degree would potentially increase!). We can consider a differential operator as a linear map $\mathbb{R}_n[x] \longrightarrow \mathbb{R}_n[x]$.

(a) Let $D : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ be the differential operator given by $2 - 4\frac{d}{dx} + 2x^2\frac{d^2}{dx^2}$. Find the matrix of D relative to the basis $\{x^2, (x-1)^2, (x+1)^2\}$.

Solution:
$$\begin{pmatrix} 6 & 2 & 6 \\ 2 & 5 & 1 \\ -2 & -1 & -1 \end{pmatrix}$$

(b) Suppose $E : \mathbb{R}_2[x] \longrightarrow \mathbb{R}_2[x]$ is a differential operator and that the matrix of E, relative to the basis $\{1, x, x^2\}$ is
$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix}.$$

Find E.

Solution:
$$E = \frac{d}{dx} + x\frac{d^2}{dx^2}$$