Integration by parts

* Substitution reverses the chain rule
* Integration by parts "reverses" the product rule.

Notation

If $f(x)$ is a function let

$$df = \frac{df}{dx} \cdot dx$$
$$= f'(x)\, dx$$

Eg if $f(x) = x^2$, $\quad df = 2x\, dx$

$u = \ln x \quad du = \frac{dx}{x}$

If u and v are functions of x the product rules states

$$(uv)' = u'v + uv'$$

Integrating

$$uv = \int (uv)'dx = \int u'v\,dx + \int uv'dx$$
$$= \int v\,du + \int u\,dv$$

Thm

$$\int u\,dv = uv - \int v\,du$$

or for definite integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Ex $\int x e^x\,dx$ let $u = x$ $dv = e^x\,dx$

$du = dx$ $v = e^x$

$$= xe^x - \int e^x\,dx$$

$$= xe^x - e^x + C$$

$$= (x-1)e^x + C$$

Ex $\int \ln x\,dx$ let $u = \ln x$ $dv = dx$

$du = \dfrac{dx}{x}$ $v = x$

$$= x\ln x - \int x \cdot \frac{dx}{x}$$

$$= x\ln x - \int dx$$

$$= x\ln x - x + C$$

$$= x(\ln x - 1) + C$$

Ex

$$\int_0^2 x^2 e^x \, dx$$

let $u = x^2$ $\quad dv = e^x dx$

$du = 2x\,dx$ $\quad v = e^x$

$$= x^2 e^x \Big|_0^2 - \int_0^2 2x e^x \, dx$$

$\tilde{u} = 2x$ $\quad d\tilde{v} = e^x dx$

$d\tilde{u} = 2\,dx$ $\quad \tilde{v} = e^x$

$$= x^2 e^x \Big|_0^2 - \left(2x e^x \Big|_0^2 - \int_0^2 2e^x \, dx \right)$$

$$= x^2 e^x \Big|_0^2 - 2x e^x \Big|_0^2 + 2e^x \Big|_0^2$$

$$= (4e^2 - 0) - (4e^2 - 0) + (2e^2 - 2)$$

$$= 2(e^2 - 1).$$

L'Hopital's rule

L'Hopital's rule is used to evaluate limits of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

if, say, $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$. Eg:

$$\lim_{x \to \infty} \frac{x^2 - 3x + 2}{2 - 3x^2}$$

Thm (L'Hopitals rule)

Suppose $f(x)$ and $g(x)$ are diff'ble functions and

* either $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$

  or

  $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$, and

* $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists or is equal to $\pm\infty$

Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Ex $\displaystyle\lim_{x\to\infty} \frac{3x-2}{1-x} \overset{L'H}{=} \lim_{x\to\infty} \frac{3}{-1} = -3.$

∞ / $-\infty$

Ex $\displaystyle\lim_{x\to 0^+} \sqrt{x}\cdot \ln x = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}}$

$\sqrt{x} \to 0$, $\ln x \to -\infty$

$$\overset{L'H}{=} \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}}$$

$$= \lim_{x\to 0^+} -2\sqrt{x} = 0$$

Ex $\displaystyle\lim_{x\to\infty} (\ln x)^{1/x}$

Trick. Let $\displaystyle L = \lim_{x\to\infty} (\ln x)^{1/x}$ then

$$\ln(L) = \ln\left(\lim_{x\to\infty} (\ln x)^{1/x}\right)$$

$$= \lim_{x\to\infty} \ln\left((\ln x)^{1/x}\right) \quad \text{since } \ln \text{ is cts}$$

$$= \lim_{x\to\infty} \frac{1}{x}\ln(\ln x) \to \infty$$

$\to \infty$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x\ln x}}{x}$$

$$= \lim_{x\to\infty} \frac{1}{x^2 \ln x} = 0$$

So $\ln(L) = 0$ ie $L = 1$.

$$\lim_{x\to\infty} (\ln x)^{1/x} = 1$$

Asymptotic growth

We say $f(x)$ grows faster than $g(x)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$$

$\left(\text{or equivalently } \lim_{x\to\infty} \frac{g(x)}{f(x)} = 0 \right)$.

Ex e^x grows faster than x^n for any n:

We try to evaluate the limit

$$\lim_{x\to\infty} \frac{e^x \nearrow \infty}{x^n \searrow \infty} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x \to \infty}{n x^{n-1} \to \infty}$$

$$\overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{e^x}{n(n-1)x^{n-2}}$$

$$\vdots$$

$$= \lim_{x\to\infty} \frac{e^x}{n!} = \infty$$

L'H n times!