Sequences

A **sequence** is an ordered list of numbers

e.g. $2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$
that goes on forever.

We will write a_n for the **n^{th} term** of the sequence.

The sequence itself is denoted (a_n)

More examples:

* $1, -1, 1, -1, \ldots$ $\quad a_n = (-1)^n, \ n \geq 0$
* $1, \frac{1}{2}, \frac{1}{3}, \ldots$ $\quad a_n = \frac{1}{n-1}, \ n \geq 2$
* $2, 4, 6, \ldots$ $\quad a_n = 2^n, \ n \geq 1$

Sequences don't need to start with a_1.

Rmk * Some sequences don't have a formula for the n^{th} term, eg $a_n = n^{th}$ prime

* We can also define sequences recursively:

$$a_0 = 1, \ a_1 = 1, \ a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 2 \quad \text{so}$$

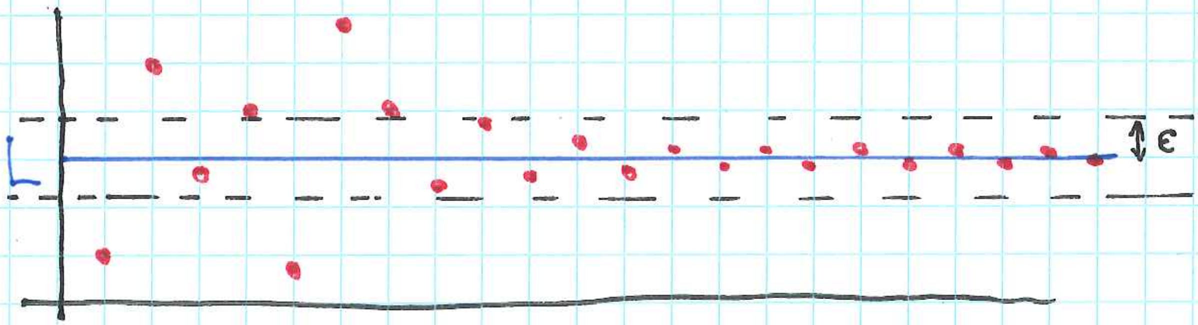
$$1, 1, 2, 3, 5, 8, 13, \ldots$$

Def

* A sequence (a_n) ~~approaches~~ converges to a limit L if it gets arbitrarily close to L

More precise

* A sequence (a_n) converges to a limit L if for any (small) $\varepsilon > 0$, the sequence is eventually within ε of L forever.



* A sequence (a_n) converges to a limit L if for any $\varepsilon > 0$, there exists an N such that
$$|a_n - L| < \varepsilon$$
for all $n > N$.

If (a_n) converges to L, we write
$$\lim_{n \to \infty} a_n = L$$

If (a_n) does not converge to any limit we say (a_n) diverges

If (a_n) grows without bound (i.e. for any large number $M > 0$, the sequence is eventually $a_n > M$) then (a_n) diverges to ∞

But how ~~&~~ can we calculate limits?

Thm If (a_n) is given by a function, i.e.

$$a_n = f(n)$$

then

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

Ex Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ then $a_n = \frac{1}{n} = f(n)$ where $f(x) = \frac{1}{x}$ so

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} \frac{1}{x} = 0.$$

Just like we have the usual ~~#~~ limit laws we have limit laws for sequences:

Thm Let (a_n) and (b_n) be convergent seq's, (c_n) a divergent seq, $f(x)$ a cts function and k a real number. Then:

1. $\lim\limits_{n\to\infty} k = k$

2. $\lim\limits_{n\to\infty} k \cdot a_n = k \cdot \left(\lim\limits_{n\to\infty} a_n\right)$

3. $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

4. $a_n + c_n$ is divergent.

5. $\lim\limits_{n\to\infty} (a_n b_n) = \left(\lim\limits_{n\to\infty} a_n\right) \cdot \left(\lim\limits_{n\to\infty} b_n\right)$

6. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}$, as long as $\lim\limits_{n\to\infty} b_n \neq 0$

7. $\lim\limits_{n\to\infty} f(a_n) = f\left(\lim\limits_{n\to\infty} a_n\right)$.

Ex Lets use these rules to carefully evaluate

$$\lim_{n\to\infty} \frac{\sqrt{4n^2+2n+1}}{\sqrt{9n^2+3n+227}}$$

Dividing top + bottom by n:

$$\frac{\sqrt{4n^2+2n+1}}{\sqrt{9n^2+3n+227}} = \frac{\sqrt{4 + 2/n + 1/n^2}}{\sqrt{9 + 3/n + 227/n^2}}$$

$$\lim_{n\to\infty} \left(9 + \frac{3}{n} + \frac{227}{n^2}\right) \overset{3.}{=} \lim_{n\to\infty} 9 + \lim_{n\to\infty} \frac{3}{n} + \lim_{n\to\infty} \frac{1}{n^2} \quad 3.$$

$$\overset{2.+5.}{=} \lim_{n\to\infty} 9 + 3 \lim_{n\to\infty} \frac{1}{n} + 227 \left(\lim_{n\to\infty} \frac{1}{n}\right)\left(\lim_{n\to\infty} \frac{1}{n}\right)$$

$$\overset{1.}{=} 9 + 3\cdot 0 + 227\cdot 0\cdot 0 = 9$$

So $\displaystyle\lim_{n\to\infty}\sqrt{9+\frac{3}{n}+\frac{227}{n^2}} \overset{7.}{=} \sqrt{\lim_{n\to\infty}\left(9+\frac{3}{n}+\frac{227}{n^2}\right)}$

$$= \sqrt{9} = 3.$$

Similarly

$$\lim_{n\to\infty}\sqrt{4+\frac{2}{n}+\frac{1}{n^2}} = \sqrt{4} = 2.$$

Thus

$$\lim_{n\to\infty}\frac{\sqrt{4+\frac{2}{n}+\frac{1}{n^2}}}{\sqrt{9+3/n+227/n^2}} \overset{6.}{=} \frac{2}{3}$$

Rmk (Important) We only care about what is happening "at infinity" when calculating limits.

If we have two sequences (a_n) and (b_n) so that for n large enough (ie all $n > M$ for some number M, $a_n = b_n$) then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

eg: $1 \quad 2 \quad 3 \quad 20 \quad 2 \quad 5 \quad 100 \quad 100 \quad \frac{1}{9} \quad \frac{1}{10} \quad \frac{1}{11} \quad \frac{1}{12} \ldots$

$\frac{1}{1} \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \quad \frac{1}{6} \quad \frac{1}{7} \quad \frac{1}{8} \quad \frac{1}{9} \quad \frac{1}{10} \quad \frac{1}{11} \quad \frac{1}{12} \ldots$

Examples

* Geometric sequences. For numbers r, c

(a_n) where $a_n = cr^n$ $n \geq 0$

is called a geometric sequence.

$$\lim_{n\to\infty} cr^n = c \cdot \lim_{n\to\infty} r^n = \begin{cases} 0 & \text{if } -1<r<1 \\ c & \text{if } r=1 \\ \infty & \text{if } r>1 \end{cases}$$

What about $r=-1$? then

(a_n): $c, -c, c, -c, c, \ldots$

This does not converge unless $c=0$.

Thm (Squeeze thm)

Suppose we have three sequences $(a_n), (b_n), (c_n)$ so that eventually (for all n larger than some fixed number)

$$b_n \leq a_n \leq c_n \quad \text{and} \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = L$$

Then $\lim_{n\to\infty} a_n = L$.

Rmk We use the theorem in the following ~~way~~ way:

* Suppose we want to find $\lim_{n\to\infty} a_n$
* Finds seq's b_n, c_n as above,
* use these to "squeeze" a_n.

Ex Lets try to find $\lim_{n\to\infty} \frac{\sin n}{n}$.

We notice that $-1 \leq \sin n \leq 1$ so

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$$

But we know that $\lim_{n\to\infty} \frac{1}{n} = \lim_{n\to\infty} -\frac{1}{n} = 0$

so by the squeeze theorem $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

Ex $\lim_{n\to\infty} \frac{2^n}{n!}$.

This one is a little trickier. Here we use a trick we have seen before:

$$\frac{2^n}{n!} = \underbrace{\frac{2}{1}\cdot\frac{2}{2}}_{=2}\cdot\underbrace{\frac{2}{3}\cdot\frac{2}{4}\cdot \text{—} \cdot\frac{2}{n}}_{n-2 \text{ factors}}$$

$$\leq 2\cdot\left(\frac{2}{3}\right)^{n-2} \quad \text{since } \frac{2}{3} \geq \frac{2}{k} \text{ when } k \geq 3.$$

This is a good upper bound. $c_n = 2\cdot\left(\frac{2}{3}\right)^{n-2}$.
What about a lower bound? Well

$$\lim_{n\to\infty} 2\cdot\left(\frac{2}{3}\right)^{n-2} = 0$$

Do we know any other sequences ~~tt~~ that have limits of 0? How about $b_n = 0$?

Certainly $$0 \leqslant \frac{2^n}{n!} \leqslant 2\left(\frac{2}{3}\right)^{n-2}$$

and since $$\lim_{n\to\infty} 0 = \lim_{n\to\infty} 2\left(\frac{2}{3}\right)^{n-2} = 0$$

the squeeze thm says
$$\lim_{n\to\infty} \frac{2^n}{n!} = 0.$$