

This weeks problem set focuses on the ideas of bases and linear transformations. A question marked with a  $\dagger$  is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a  $*$  is especially important.

1. From section 2.1, problems 15, 17, 18, 19, 24, 26\*, 28, 31 $\dagger$ , 40\*.
2. From section 2.1, problems 1, 2, 5, 6, 9\*, 14, 14b.
- 3\* Let  $V$  be a finite dimensional vector space over  $\mathbb{F}$  and  $B\{v_1, \dots, v_n\}$  a basis. Let  $W$  be another vector space and  $w_1, \dots, w_n$  a collection of elements. Show that there is a unique linear map such that  $T(v_i) = w_i$ .

**Solution:** Since  $B$  is a basis, if we have any vector  $v \in V$  we can write

$$v = \lambda_1 v_1 + \dots + \lambda_n v_n.$$

Define  $T(v) = \lambda_1 T(v_1) + \dots + \lambda_n T(v_n) = \lambda_1 w_1 + \dots + \lambda_n w_n$ . We can check that it is linear by supposing that  $u, v \in V$  and that

$$\begin{aligned} u &= \mu_1 v_1 + \dots + \mu_n v_n \\ v &= \lambda_1 v_1 + \dots + \lambda_n v_n. \end{aligned}$$

Then we have  $u + v = v = (\mu_1 + \lambda_1)v_1 + \dots + (\mu_n + \lambda_n)v_n$ , so

$$\begin{aligned} T(u + v) &= (\mu_1 + \lambda_1)w_1 + \dots + (\mu_n + \lambda_n)w_n, \text{ but} \\ T(u) + T(v) &= \mu_1 w_1 + \dots + \mu_n w_n + \lambda_1 w_1 + \dots + \lambda_n w_n. \end{aligned}$$

Thus  $T(u + v) = T(u) + T(v)$ . Now lets suppose  $\mu \in \mathbb{F}$ , then  $\mu v = \mu \lambda_1 v_1 + \dots + \mu \lambda_n v_n$ , so

$$\begin{aligned} T(\mu v) &= \mu \lambda_1 w_1 + \dots + \mu \lambda_n w_n, \text{ but} \\ \mu T(v) &= \mu(\lambda_1 w_1 + \dots + \lambda_n w_n). \end{aligned}$$

Thus  $T(\mu v) = \mu T(v)$  and so  $T$  is linear.

To see that  $T$  is unique, suppose that  $SLV \rightarrow W$  is another linear map such that  $S(v_i) = w_i$ . Now let  $v \in V$  such that  $v = \lambda_1 v_1 + \dots + \lambda_n v_n$ . Then by linearity

$$\begin{aligned} S(v) &= S(\lambda_1 v_1 + \dots + \lambda_n v_n) \\ &= \lambda_1 S(v_1) + \dots + \lambda_n S(v_n) \\ &= \lambda_1 w_1 + \dots + \lambda_n w_n = T(v). \end{aligned}$$

Thus  $S = T$ .

- 4\* Let  $V$  and  $W$  be vector spaces over  $\mathbb{F}$ . Define  $\text{Hom}(V, W)$  to be the set of linear maps from  $V$  to  $W$ .
  - (a) Show that  $\text{Hom}(V, W)$  is itself a vector space.

**Solution:** We define the linear map  $T + S$  by  $(T + S)(v) = T(v) + S(v)$  and  $\lambda T$  by  $(\lambda T)(v) = \lambda T(v)$ . The zero element is the map  $O : V \rightarrow W$  defined by  $O(v) = 0$ . It is easy to check all of the axioms.

- (b) If  $V$  is finite dimensional and  $B$  is a basis for  $V$ , construct a basis for  $V^* = \text{Hom}(V, \mathbb{F})$ . The vector space  $V^*$  is called the *dual space* to  $V$ .

**Solution:** Suppose  $B = \{v_1, \dots, v_n\}$ . For any  $1 \leq i \leq n$  define the linear map  $\varepsilon_i : V \rightarrow \mathbb{F}$  by  $\varepsilon_i(v_j) = \delta_{ij}$ . Here  $\delta_{ij} = 0$  if  $i \neq j$  and  $\delta_{ii} = 1$ .

We claim that  $B^* = \{\varepsilon_1, \dots, \varepsilon_n\}$  is a basis for  $V^*$ . Here is a sketch of the proof. To see that  $B^*$  spans  $V^*$ , take an arbitrary linear map  $\chi : V \rightarrow \mathbb{F}$ . One can see that this map is completely determined by the values  $\chi(v_i)$ . Then

$$\chi = \chi(v_1)\varepsilon_1 + \dots + \chi(v_n)\varepsilon_n.$$

To see that it is a linearly independent subset, take an arbitrary linear combination

$$\lambda_1\varepsilon_1 + \lambda_n\varepsilon_n = 0.$$

This means that by evaluating this map at  $v_i$ , we get

$$\lambda_i = \lambda_1\varepsilon_1(v_i) + \dots + \lambda_n\varepsilon_n(v_i) = 0$$

5\* Let  $T : V \rightarrow W$  be an injective linear map. Show that, if we consider  $T$ , instead, as a linear map  $V \rightarrow \text{im } T$  (just restrict what we consider to be the codomain), then it defines an isomorphism and shows that  $V \cong \text{im } T$ .

6\* Let  $V$  and  $W$  be vector spaces over  $\mathbb{F}$ . Define the set

$$V \times W = \{(v, w) \mid v \in V \text{ and } w \in W\}.$$

This is called the *product* of the vector spaces.

(a) Show that  $V \times W$  is a vector space.

**Solution:** We define addition and scalar multiplication componentwise. So  $(v, w) + (v', w') = (v + v', w + w')$  and  $\lambda(v, w) = (\lambda v, \lambda w)$ . The axioms are now not hard to check.

(b) Define a map  $\iota_V : V \rightarrow V \times W$  by  $\iota_V(v) = (v, 0)$ . Show that  $\iota_V$  is an injective linear map. Note that we can define a similar map  $\iota_W$ .

**Solution:** We have

$$\iota_V(v + w) = (v + w, 0) = (v, 0) + (w, 0) = \iota_V(v) + \iota_V(w)$$

and

$$\iota_V(\lambda v) = (\lambda v, 0) = \lambda(v, 0) = \lambda \iota_V(v).$$

(c) If  $U \subset V$  is a subspace, show that  $U \times W$  is a subspace of  $V \times W$ .

**Solution:** Let  $x, y \in U$ , thus  $x = (u, w)$  and  $y = (u', w')$  for some  $u, u' \in U$  and  $w, w' \in W$ . Then  $x + y = (u + u', w + w') \in U \times W$ . If  $\lambda \in \mathbb{F}$ , then  $\lambda x = (\lambda u, \lambda w) \in U \times W$ . Hence  $U \times W$  is closed under scalar multiplication and addition and is thus a subspace.

(d) Show that  $V \times W = (V \times \{0\}) \oplus (\{0\} \times W)$ .

**Solution:** First of all, it is clear that  $(V \times \{0\}) \cap (\{0\} \times W) = \{0\}$ . Now observe that  $(v, w) = (v, 0) + (0, w)$  so  $V \times W = (V \times \{0\}) + (\{0\} \times W)$ .

Note that we can consider  $V \times \{0\}$  as a copy of  $V$  in  $V \times W$ . For this reason, often mathematicians write  $V \oplus W$  instead of  $V \times W$  and call it the external direct product. Though this is a little confusing so we won't talk about it in this way in this class.

- 7\* (2.1.18) Give an example of a linear transformation  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  such that  $\ker T = \operatorname{im} T$ .
- 8\* (2.1.19) Give an example of distinct linear transformations  $T$  and  $U$  such that  $\ker T = \ker U$  and  $\operatorname{im} T = \operatorname{im} U$ .