Midterm 2 practice

UCLA: Math 32B, Fall 2019

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Version: practice

- This exam has 4 questions, for a total of 35 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: ______________________________

ID number: ______________________________

Question	Points	Score
1	10	
2	8	
3	8	
4	9	
Total:	35	

1. (a) (5 points) Let $\mathcal{D}$ be the region in the xy-plane above the x-axis and below the curve $y = 1 - x^2$. Compute the integrals

$$I_1 = \frac{1}{A}\iint_{\mathcal{D}} x\;dA \text{ and } I_2 = \frac{1}{A}\iint_{\mathcal{D}} y\;dA$$

where A is the area of $\mathcal{D}$.

Solution: We describe $\mathcal{D}$ as a vertically simple region

$$\mathcal{D} = \{\ (x,y) \ \mid\ -1 \le x \le 1, 0 \le y \le 1 - x^2\ \}$$

We first compute the area

$$A = \iint_{\mathcal{D}} 1\;dA = \int_{-1}^{1}\int_{0}^{1-x^2} 1\;dy\;dx = \int_{-1}^{1} 1 - x^2\;dx = 2 - 2/3 = 4/3.$$

Now we compute

$$\iint_{\mathcal{D}} x\;dA = \int_{-1}^{1}\int_{0}^{1-x^2} x\;dy\;dx = \int_{-1}^{1}(1-x^2)x\;dx = 0.$$

Now we compute

$$\iint_{\mathcal{D}} y\;dA = \int_{-1}^{1}\int_{0}^{1-x^2} y\;dy\;dx = \int_{-1}^{1}\frac{1}{2}(1-x^2)^2\;dx = 16/30.$$

So

$$I_1 = 0 \text{ and } I_2 = 2/5.$$

(b) (5 points) Parametrize the paraboloid and find the normal vector for this parametrisation.

$$x^2 + y^2 = 2z, \quad 0 \le z \le 1.$$

Solution: We can express $z = \frac{1}{2}(x^2 + y^2)$ so we get an easy parametrisation in this case

$$G(x, y) = (x, y, \frac{1}{2}(x^2 + y^2))$$

But we need to understand the domain for (x, y). When $0 \le z \le 1$ we see that

$$0 \le \frac{1}{2}(x^2 + y^2) \le 1$$

so we can see that (x, y) should be contained in $\mathcal{D} \subset \mathbb{R}^2$ where $\mathcal{D}$ is the disk of radius $\sqrt{2}$ centred at the origin.

To find the normal vector, we first find two tangent vectors

$$T_x(x, y) = \langle 1, 0, x \rangle$$
$$T_y(x, y) = \langle 0, 1, y \rangle$$

Thus we get

$$N(x, y) = T_x \times T_y = \langle -x, -y, 1 \rangle.$$

2. (8 points) Consider the region $\mathcal{E}$ given by

$$0 \leq z \leq (y - x^2)^2, \quad x^2 \leq y \leq x.$$

Use the change of variables

$$x = u, y = v + u^2, z = wv^2,$$

to evaluate

$$\iiint_{\mathcal{E}} \frac{1}{y - x^2} \, \mathrm{d}V.$$

Solution: First we describe $\mathcal{E}$ in the form

$$\mathcal{E} = \{ \ (x, y, z) \ | \ (x, y) \in \mathcal{D} \text{ and } 0 \leq z \leq (y - x^2)^2 \ \}$$

where

$$\mathcal{D} = \{ \ (x, y) \ | \ 0 \leq x \leq 1 \text{ and } x^2 \leq y \leq x \ \}.$$

Our next job is to figure out which region in uvw-space is mapped to $\mathcal{E}$ when we apply $G(u, v, w) = (u, v + u^2, wv^2)$. We can use the inequalities given, in terms of u, v, w.

$$0 \leq wv^2 \leq v^2, u^2 \leq v + u^2 \leq u.$$

We can manipulate these to

$$0 \leq w \leq 1, \text{ and } 0 \leq v \leq u - u^2.$$

Thus if we take

$$\mathcal{E}' = \{ \ (u, v, w) \ | \ (u, v) \in \mathcal{D}' \text{ and } 0 \leq w \leq 1 \ \}$$

where

$$\mathcal{D}' = \{ \ (u, v) \ | \ 0 \leq u \leq 1, 0 \leq v \leq u - u^2 \ \}.$$

Now we need to find the Jacobian:

$$J(G) = \det \begin{pmatrix} 1 & 0 & 0 \\ 2u & 1 & 0 \\ 0 & 2vw & v^2 \end{pmatrix} = v^2$$

This is always positive! Thus

$$\begin{aligned}
\iiint_{\mathcal{E}} \frac{1}{y - x^2} \, dV &= \iiint_{\mathcal{E}'} \frac{1}{v} \| J(G) \| \, dV_{uvw} \\
&= \iiint_{\mathcal{E}'} v \, dV_{uvw} \\
&= \iint_{\mathcal{D}'} \int_0^1 v \, dw \, dA_{uv} \\
&= \int_0^1 \int_0^{u - u^2} \int_0^1 v \, dw \, dv \, du \\
&= \int_0^1 \int_0^{u - u^2} v \, dv \, du \\
&= \int_0^1 \frac{1}{2} (u - u^2) \, du = 1/60
\end{aligned}$$

3. Let $\mathbf{F}$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{F}(x,y,z) = (y\cos z - yze^x, x\cos z - ze^x, -xy\sin z - ye^x).$$

(a) (4 points) Show that $\mathbf{F}$ is conservative.

Solution: Our vector field is defined on a simply connected domain. This means being conservative is equivalent to having curl zero. So we simply check this:

$$\begin{aligned}\nabla \times \mathbf{F} &= \langle \partial_x, \partial_y, \partial_z \rangle \times \langle y\cos z - yze^x, x\cos z - ze^x, -xy\sin z - ye^x \rangle \\ &= \langle -x\sin z - e^x - (-x\sin z - e^x), -y\sin z - ye^x - (-y\sin z - ye^x), \cos z - ze^x - (\cos z - ze^x) \rangle = 0.\end{aligned}$$

(b) (4 points) Find a potential function for $\mathbf{F}$.

Solution: We need a function f such that

$$\begin{aligned}\partial_x f &= y\cos z - yze^x \\ \partial_y f &= x\cos z - ze^x \\ \partial_z f &= -xy\sin z - ye^x\end{aligned}$$

This means we get three conditions

$$\begin{aligned}f &= xy\cos z - yze^x + \alpha(y,z) \\ f &= xy\cos z - yze^x + \beta(x,z) \\ f &= xy\cos z = yze^x + \gamma(x,y)\end{aligned}$$

We can simply let $\alpha = \beta = \gamma = 0$ and take

$$f = xy\cos z - yze^x.$$

4. In this question we will calculate the surface area of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + z^2 = 1$.

   (a) (4 points) Find a parameterisation of the ellipsoid given above.

Solution: Here we can take our idea from spherical coordinates:

$$G'(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi).$$

This obviously doesn't work unless $a = 1$, so we should adjust for this. We notice that if we multiply the first two coordinates by a we do get something that works:

$$G(\theta, \phi) = (a\cos\theta \sin\phi, a\sin\theta \sin\phi, \cos\phi).$$

Now also thinking about spherical coordinates gives us the fact that we should let $(\theta, \phi) \in [0, 2\pi] \times [0, \pi]$.

   (b) (5 points) Find the normal vector to this parameterisation and its length.

Solution: We find the two tangent vectors

$$T_\theta(\theta, \phi) = \langle -a\sin\theta \sin\phi, a\cos\theta \sin\phi, 0\rangle$$
$$T_\phi(\theta, \phi) = \langle a\cos\theta \cos\phi, a\sin\theta \cos\phi, -\sin\phi\rangle$$

Thus we get

$$N(\theta, \phi) = T_\theta \times T_\phi = \langle -a\cos\theta \sin^2\phi, -a\sin\theta \sin^2\phi, -a^2 \sin\phi \cos\phi\rangle$$

and

$$\|N\|^2 = a^2 \sin^4\phi + a^4 \sin^2\phi \cos^2\phi$$

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