This week on the problem set you will get practice thinking about potential functions and calculating line integrals.

Homework: The second homework will be due on Monday 11 May and will consist of 3, 4, 5, and 6 below. *Numbers in parentheses indicate the question has been taken from the textbook:

J. Rogawski, C. Adams, *Calculus, Multivariable*, 3^{rd} Ed., W. H. Freeman & Company,

and refer to the section and question number in the textbook.

1. (Section 17.1) Questions 13 – 17, 22, 26, 28, 29, 38, 42, 44, 47, 52, 56*. (Use the following translations $4^{\text{th}} \mapsto 3^{\text{rd}}$ editions: $47 \mapsto 45$, $52 \mapsto 50$, $56 \mapsto 54$, otherwise the questions are the same).

2. (Section 17.2) 3, 10, 12, 13, 21, 24, 28, 43, 44, 46, 47, 54, 55, 57, 63, 64, 67. (Use the following translations $4^{\text{th}} \mapsto 3^{\text{rd}}$ editions: $43 \mapsto 41$, $44 \mapsto 42$, $46 \mapsto 44$, $47 \mapsto 45$, $54 \mapsto 52$, $55 \mapsto 53$, $57 \mapsto 55$, otherwise the questions are the same).

3. A parameterized curve $\mathbf{r} : [a, b] \to \mathbb{R}^2$ (the codomain could be $\mathbb{R}^3$ as well) is a *flow line* for the vector field $\mathbf{F}$ if for all $t \in (a, b)$ we have that $\mathbf{F}(\mathbf{r}(t)) = \mathbf{r}'(t)$. A flow line for a vector field is the path that a particle would follow if the vector field was a velocity vector field for a fluid.

   (a) Consider the vector field $\mathbf{F}(x, y) = \langle x, y\rangle$. Find a flow line for $\mathbf{F}$ (note that a vector field will have many different flow lines).

   Solution: Let's find a flow line through $(1, 0)$. At this point $\mathbf{F} = \langle 1, 0\rangle$ and in fact, on the x-axis $\mathbf{F}(x, 0) = \langle x, 0\rangle$. This means the flow line will simply traverse the positive x-axis. More formally we know if $\mathbf{r}(t) = \langle x(t), 0\rangle$ then

   $$\mathbf{r}'(t) = \langle x'(t), 0\rangle = \mathbf{F}(\mathbf{r}(t)) = \langle x(t), 0\rangle$$

   So we need $x'(t) = x(t)$, for example $x(t) = e^t$. The domain for t is given by $t \in (-\infty, \infty)$.

   (b) Find a collection of flow lines for $\mathbf{F}$ so that every point (x, y) is contained in exactly one of the flow lines in the collection.

   Solution: We generalise the example above. A flow line $\mathbf{r}(t) = \langle x(t), 0\rangle$ will need to satisfy

   $$\mathbf{r}'(t) = \langle x'(t), 0\rangle = \mathbf{F}(\mathbf{r}(t)) = \langle x(t), 0\rangle$$

   so in general we will have $x = ae^t$ and $y = be^t$ with $t \in (-\infty, \infty)$, for some a and b. This flow line passes through the point (a, b) but also through any positive multiple $(\lambda a, \lambda b)$.
   To avoid doubling up, we can simply ask that the flow line passes through $(\cos\theta, \sin\theta)$ for any particular $\theta \in [0, 2\pi)]$. The flow line will be

   $$\mathbf{r}(t) = \left(e^t \cos\theta, e^t \sin\theta\right) \quad \text{for } t \in (-\infty, \infty).$$

   But we are missing the flow line through $(0, 0)$ which is $\mathbf{r}(t) = (0, 0)$.

   (c) Consider the vector field $\mathbf{G}(x, y) = \left\langle \dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}} \right\rangle$. Find a collection of flow lines for $\mathbf{G}$ so that every point (x, y) is contained in exactly one of the flow lines in the collection.

   Solution: The vector field $\mathbf{G}$ has the same direction as $\mathbf{F}$ however it has been normalised so $\mathbf{G}$ is a unit vector field. We can apply the same thinking as last time, and notice that the vectors

along the ray connecting $(0,0)$ and (a,b) all point in the same direction, so the flow lines will be the same as last time, however the speed we travel along them will change!
We can see that along one of the above rays, the vector field $\mathbf{G}$ is constant. I.e. along the ray through $(\cos\theta, \sin\theta)$ we have that $\mathbf{G} = \langle\cos\theta, \sin\theta\rangle$. That means, $\mathbf{r}'(t) = \langle\cos\theta, \sin\theta\rangle$. Integrating this we see that $x(t) = t\cos\theta$ and $y(t) = t\sin\theta$. Thus, every point in the plane is contained in one of the flow lines

$$\mathbf{r}_\theta(t) = (t\cos\theta, t\sin\theta) \quad \text{for } t \in (0,\infty)$$

for some $\theta \in [0, 2\pi)$.

(d) What do the flow lines look like in $\mathbb{R}^2$ for the vector fields $\mathbf{F}$ and $\mathbf{G}$? Relate how the flow lines are similar and different to how the vector fields $\mathbf{F}$ and $\mathbf{G}$ are similar and different.

Solution: The first difference is that $\mathbf{F}$ is defined at $(0,0)$ whereas $\mathbf{G}$ is not. This means we get a constant flow line at the origin for $\mathbf{F}$ but not for $\mathbf{G}$. The second difference is that $\mathbf{G}$ is a unit vector field so the flow lines for $\mathbf{G}$ all move at unit speed.

(e) A particle is dropped into the plane at the point $(-1,1)$ at time $t=0$. If the particle is located at (x,y) in the plane its velocity vector is $(1, 2x)$. What is the position of the particle at time $t=3$?

Solution: What the question is asking us to do is to find a flow line $\mathbf{r}(t)$, through the point $(-1,1)$ for the vector field $\mathbf{H} = \langle 1, 2x\rangle$ such that $\mathbf{r}(0) = (-1,1)$.
We know that we want $x'(t) = 1$ and $y'(t) = 2x(t)$. Thus $x(t) = t + C$ and $y'(t) = t^2 + 2Ct + D$. We also know that $x(0) = -1$, so $C = -1$ and $y(0) = 1$, so $D = 1$. Summarising, the flow line is

$$\mathbf{r}(t) = (t-1, t^2 - 2t + 1) \quad \text{for } t \in (-\infty,\infty).$$

Now we want to know what $\mathbf{r}(3)$ is. $\mathbf{r}(3) = (2,4)$.

4. Consider the vector field $\mathbf{F} = \left\langle \dfrac{1-y}{x^2+(y-1)^2} + \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+(y-1)^2} + \dfrac{x}{x^2+y^2} \right\rangle$

(a) Show that the curl of $\mathbf{F}$ is zero.

Solution: It will be useful to split $\mathbf{F}$ as the sum of two vector fields. $\mathbf{F} = \mathbf{G} + \mathbf{H}$ where

$$\mathbf{G} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \quad \text{and} \quad \mathbf{H} = \left\langle \frac{1-y}{x^2+(y-1)^2}, \frac{x}{x^2+(y-1)^2} \right\rangle.$$

Then $\nabla\times\mathbf{F} = \nabla\times\mathbf{G} + \nabla\times\mathbf{H}$. A quick calculation shows that $\nabla\times\mathbf{G} = 0$ and $\nabla\times\mathbf{H} = 0$

(b) Show that $\mathbf{F}$ is not conservative on the largest domain on which it is defined.

Solution: The only points where $\mathbf{F}$ is not defined are $(0,0)$ and $(0,1)$. So the largest possible domain is D, the set of all points apart from these two.
Since $\nabla\times\mathbf{F} = 0$, we cannot use this to rule out $\mathbf{F}$ not being conservative. We will need to find a loop around which to integrate. A useful choice is $\mathcal{C}$, the circle of radius $1/2$, center $(0,0)$, oriented counter clockwise. Then

$$\oint_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{G}\cdot d\mathbf{r} + \oint_{\mathcal{C}} \mathbf{H}\cdot d\mathbf{r}$$

Here we notice a big simplification. The curve $\mathcal{C}$ is contained in a simply connected domain for $\mathbf{G}$ (e.g. the points where $y < 1$). Since $\nabla \times \mathbf{H} = 0$, it is conservative on this domain and so $\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{r} = 0$. Using the parametrisation $\mathbf{r}(t) = (0.5\cos t, 0.5\sin t)$, $t \in [0, 2\pi]$ we get

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{G} \cdot d\mathbf{r} \tag{1}$$

$$= \int_0^{2\pi} 4(sin^2 t + \cos^2 t)\, dt = 8\pi \neq 0 \tag{2}$$

Thus $\mathbf{F}$ is not conservative.

(c) Show that $\mathbf{F}$ *is* conservative on the right half plane and find a potential function.

Solution: Again we can use the fact that $\mathbf{F} = \mathbf{G} + \mathbf{H}$. We can find potential functions for both of these fields. A useful fact is $\partial_x \tan^{-1} t = \dfrac{1}{1+t^2}$. Let $\nabla \phi_{\mathrm{G}} = \mathbf{G}$ and $\nabla \phi_{\mathrm{H}} = \mathbf{H}$. Thus

$$\partial_x \phi_{\mathbf{G}} = \frac{-y}{x^2+y^2} \quad \text{so} \quad \phi_{\mathbf{G}}(x,y) = \tan^{-1}\frac{y}{x} + \alpha(y) \tag{3}$$

$$\partial_y \phi_{\mathbf{G}} = \frac{x}{x^2+y^2} \quad \text{so} \quad \phi_{\mathbf{G}}(x,y) = \tan^{-1}\frac{y}{x} + \beta(x) \tag{4}$$

$$\partial_x \phi_{\mathbf{H}} = \frac{1-y}{x^2+(y-1)^2} \quad \text{so} \quad \phi_{\mathbf{H}}(x,y) = \tan^{-1}\frac{y-1}{x} + \gamma(y) \tag{5}$$

$$\partial_x \phi_{\mathbf{H}} = \frac{x}{x^2+(y-1)^2} \quad \text{so} \quad \phi_{\mathbf{H}}(x,y) = \tan^{-1}\frac{y-1}{x} + \delta(x) \tag{6}$$

$$\tag{7}$$

using the formula above and the substitution $t = y/x$ or $t = (y-1)/x$. Thus we get

$$\phi_{\mathbf{G}}(x,y) = \tan^{-1}\frac{y}{x} \tag{8}$$

$$\phi_{\mathbf{H}}(x,y) = \tan^{-1}\frac{y-1}{x} \tag{9}$$

And so $\phi = \phi_{\mathbf{G}} + \phi_{\mathbf{H}}$ is a potential function for $\mathbf{F}$.

Hint: For all of these, it will be useful to think of $\mathbf{F}$ as the sum of two more familiar vector fields.

5. A vector field $\mathbf{F} = \langle F_1, F_2 \rangle$ is called *holomorphic* if it satisfies the *Cauchy-Riemann* equations:

$$\frac{\partial F_1}{\partial x} = \frac{\partial F_2}{\partial y} \text{ and } \frac{\partial F_2}{\partial x} = -\frac{\partial F_1}{\partial y}.$$

For example, the vector field $\mathbf{G} = \langle x+p, y+q \rangle$ is holomorphic for any fixed $p, q \in \mathbb{R}$, but $\langle x^2, F_2 \rangle$ is not holomorphic, no matter what F_2 is! The purpose of this question is to convince you that holomorphic vector fields satisfy some spooky properties. (If you want to know more about this, take Math 132).

(a) Give an example of a holomophic vector field $\mathbf{H} = \langle H_1, H_2 \rangle$ (other than simple multiples or additions of the above examples). This is in general quite difficult, so to get to started, look for a vector field where

$$H_1 = x^2 + ay^2 + y$$

You will need to work out an appropriate value of a. Include verification that your example is in fact holomorphic.

Solution: If we let $H_1 = x^2 + ay^2 + y$ then

$$\partial_y H_2 = \partial_x H_1 = 2x \quad \text{so} \quad H_2 = 2xy + f(x)$$

We also have that

$$\partial_x H_2 = -\partial_y H_1 = -2ay - 1 \quad \text{so} \quad H_2 = -2axy - x + g(y)$$

So we should choose $a = -1$, $f(x) = -x$ and $g(y) = 0$. Thus

$$\mathbf{H} = \left\langle x^2 - y^2 + y, 2xy - x \right\rangle$$

is holomorphic.

(b) Let C be the unit circle centred at $(a, b) \in \mathbb{R}^2$ oriented counter clockwise. An amazing fact is that if $\mathbf{F}$ is holomorphic then

$$F_1(a,b) = \frac{1}{2\pi} \oint_C F_1 \, ds \quad \text{and} \quad F_2(a,b) = \frac{1}{2\pi} \oint_C F_2 \, ds$$

Notice the right hand side only involves values of $\mathbf{F}$ on C, and yet, somehow this knows about the value of $\mathbf{F}$ inside C! Verify that the formulas hold for the example $\mathbf{G}$ and your example $\mathbf{H}$ from part (a).

Solution: First we parametrise C by $\mathbf{r}(t) = (\cos t + a, \sin t + b)$ which has speed 1, so

$$\begin{aligned}
\frac{1}{2\pi} \oint_C G_1 \, ds &= \frac{1}{2\pi} \int_0^{2\pi} \cos t + a + p \, dt = a + p = G_1(a,b) \\
\frac{1}{2\pi} \oint_C G_2 \, ds &= \frac{1}{2\pi} \int_0^{2\pi} \sin t + b + q \, dt = b + q = G_2(a,b) \\
\frac{1}{2\pi} \oint_C H_1 \, ds &= \frac{1}{2\pi} \int_0^{2\pi} \cos^2 t - \sin^2 t + 2a \cos t - 2b \sin t + a^2 - b^2 + \sin t + b \, dt \\
&= a^2 - b^2 + b = H_1(a,b) \\
\frac{1}{2\pi} \oint_C H_2 \, ds &= \frac{1}{2\pi} \int_0^{2\pi} 2 \cos t \sin t + 2b \cos t + 2b \sin t + 2ab - \cos t - a \, dt \\
&= \frac{1}{2\pi} \left[\sin^2 t + 2abt - at \right]_0^{2\pi} = 2ab - a = H_2(a,b)
\end{aligned}$$

Here we have used the fact that $\int_0^{2\pi} \cos^2 \, dt = \int_0^{2\pi} \sin^2 t \, dt$ by symmetry and $\int_0^{2\pi} \cos \, dt = \int_0^{2\pi} \sin \, dt = 0$.

(c) Maybe you aren't yet convinced that there is something strange going on with holomorphic vector fields. Well it turns out the values of $\mathbf{F}$ on a circle know even a little more, namely

$$\frac{\partial F_1}{\partial x}(a,b) = \frac{1}{2\pi} \oint_C (x-a)F_1 + (y-b)F_2 \, ds \quad \text{and} \quad \frac{\partial F_2}{\partial x}(a,b) = \frac{1}{2\pi} \oint_C (x-a)F_2 - (y-b)F_1 \, ds$$

Verify these formulas as well. Note the above aren't as symmetrical as the previous formulas. You will most likely need to use some formulas for the antiderivative of things like $\sin^n t$. You can look these up.

Solution: We use the same parameterisation.

$$\frac{1}{2\pi}\oint_C (x-a)G_1 + (y-b)G_2\, ds = \frac{1}{2\pi}\int_0^{2\pi} \cos^2 t + a\cos t + p\cos t + \sin^2 t + b\sin t + q\sin t\, dt$$
$$= 1 = \partial_x G_1(a,b)$$
$$\frac{1}{2\pi}\oint_C (x-a)G_2 - (y-b)G_1\, ds = \frac{1}{2\pi}\int_0^{2\pi} b\cos t + q\cos t - a\sin t - p\sin t\, dt = 0 = \partial_x G_2(a,b)$$

Where we have again used the fact that the integral of sine or cosine over $[0,2\pi]$ is zero. The next two integrals are a little more difficult.

$$\frac{1}{2\pi}\oint_C (x-a)H_1 + (y-b)H_2\, ds = \frac{1}{2\pi}\int_0^{2\pi} \cos t(\cos^2 t - \sin^2 t + 2a\cos t - 2b\sin t + a^2 - b^2 + \sin t + b)$$
$$+ \sin t(2\cos t\sin t + 2b\cos t + 2a\sin t + 2ab - \cos t - a)\, dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \cos^3 t - \cos t\sin^2 t + 2a + (a^2 - b^2 + b)\cos t - a\sin t\, dt$$
$$= \frac{1}{2\pi}\left[-\frac{1}{3}\sin^3 t\right]_0^{2\pi} + 2a = 2a = \partial_x H_1(a,b)$$

Here we have used that $\int_0^{2\pi}\sin^3 t\, dt = \int_0^{2\pi}\cos^3 t\, dt = 0$ as well as the identities mentioned above. We use the same identities to evaluate the second integral:

$$\frac{1}{2\pi}\oint_C (x-a)H_2 - (y-b)H_1\, ds = \frac{1}{2\pi}\int_0^{2\pi} \cos t(2\cos t\sin t + 2b\cos t + 2a\sin t + 2ab - \cos t - a)$$
$$- \sin t(\cos^2 t - \sin^2 t + 2a\cos t - 2b\sin t + a^2 - b^2 + \sin t + b)\, dt$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \cos^2 \sin t + 2b + (2ab - a)\cos t - 1 + \sin^3 t + (a^2 - b^2 + b)\sin t\, dt$$
$$= \frac{1}{2\pi}\left[-\frac{1}{3}\cos^3 t\right]_0^{2\pi} + 2b - 1 = 2b - 1 = \partial_x H_2(a,b).$$

(d) What are the analogous formulas for the y-derivatives of F_1 and F_2. In general, we will have formulas for all x and y derivatives but they become very complicated, the language of complex analysis allows us to understand this phenomenon is a clearer way.

Solution: From the definition of a holomorphic vector field

$$\partial_y F_1(a,b) = -\partial_x F_2(a,b) = \frac{1}{2\pi}\oint_C (y-b)F_1 - (x-a)F_2\, ds$$

and

$$\partial_y F_2(a,b) = \partial_x F_1(a,b) = \frac{1}{2\pi}\oint_C (x-a)F_1 + (y-b)F_2\, ds$$

6. Let $\mathcal{C}$ be the portion of the curve defined by the intersection of the surfaces $y = x^2$ and $x = y + z$ where

$z \geq 0$. Take the orientation to be away from the origin. Let

$$\mathbf{F} = \left\langle yz + \frac{1}{\sqrt{2-x^3}}, xz - \frac{1}{\sqrt{2-y^3}}, xy \right\rangle$$

.

(a) What is curl($\mathbf{F}$)?

> **Solution:** Let $f(t) = (2-t^3)^{-1/2}$. Then $\mathbf{F} = \langle yz, xz, xy\rangle + \langle f(x), -f(y), 0\rangle$. Note that $\langle yz, xz, xy\rangle = \nabla(xyz)$ so
>
> $$\begin{aligned} \nabla \times \mathbf{F} &= \nabla \times \nabla(xyz) + \nabla \times \langle f(x), -f(y), 0\rangle \\ &= 0 + \langle \partial_z f(y), \partial_z f(x), -\partial_x f(y) - \partial_y f(x)\rangle = 0. \end{aligned}$$

(b) Parameterise the curve and write out the integral of $\mathbf{F}$ as a single integral. There is no need to evaluate this integral.

> **Solution:** Let $x = t$, then $y = t^2$ and $z = x - y = t - t^2$. So we can take as a parametrisation $\mathbf{r}(t) = (t, t^2, t - t^2)$. We are looking at the portion of the curve where $z \geq 0$. This happens when $t \geq t^2$, i.e. when $t \in [0,1]$. The velocity is $\mathbf{r}'(t) = \langle 1, 2t, 1-2t\rangle$ and
>
> $$\mathbf{F}(\mathbf{r}(t)) = \left\langle t^2(t-t^2) + \frac{1}{\sqrt{2-t^3}}, t(t-t^2) - \frac{1}{\sqrt{2-t^6}}, t^3 \right\rangle$$
>
> so we can write the integral as
>
> $$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 t^3 - t^4 + 2t^3 - 2t^4 + t^3 - 2t^4 + \frac{1}{\sqrt{2-t^3}} - \frac{2t}{\sqrt{2-t^6}}\, dt \\ &= \int_0^1 4t^3 - 5t^4 + \frac{1}{\sqrt{2-t^3}} - \frac{2t}{\sqrt{2-t^6}}\, dt \end{aligned}$$

(c) Calculate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. *Hint: the integral in the previous question is very difficult so you should find another way to compute this.*

> **Solution:** The first thing we note, is that the vector field is defined everywhere in the region $x, y < \sqrt[3]{2}$. This is a simply connected region and since the curl is zero, the vector field is conservative on this domain. The curve $\mathcal{C}$ has endpoints $(0,0,0)$ and $(1,1,0)$ so we can replace $\mathcal{C}$ with any path with the same endpoints. The simplest is a straight line $\mathcal{L}$ parametrised by $\tilde{\mathbf{r}}(t) = (t,t,0)$ where $t \in [0,1]$. Then $\tilde{\mathbf{r}}'(t) = \langle 1,1,0\rangle$ and
>
> $$\mathbf{F}(\mathbf{r}(t)) = \left\langle \frac{1}{\sqrt{2-t^3}}, -\frac{1}{\sqrt{2-t^3}}, t^2 \right\rangle$$
>
> so
>
> $$\int_{\mathcal{C}} \mathbf{F} \cdot dt = \int_{\mathcal{L}} \mathbf{F} \cdot dt = \int_0^1 0\, dt = 0.$$

*The questions marked with an asterisk are more difficult or are of a form that would not appear on an exam. Nonetheless they are worth thinking about as they often test understanding at a deeper conceptual level.