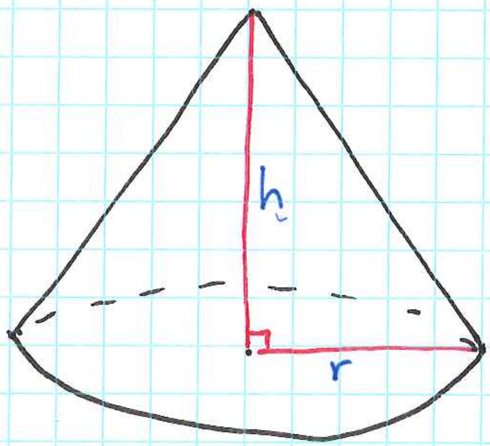
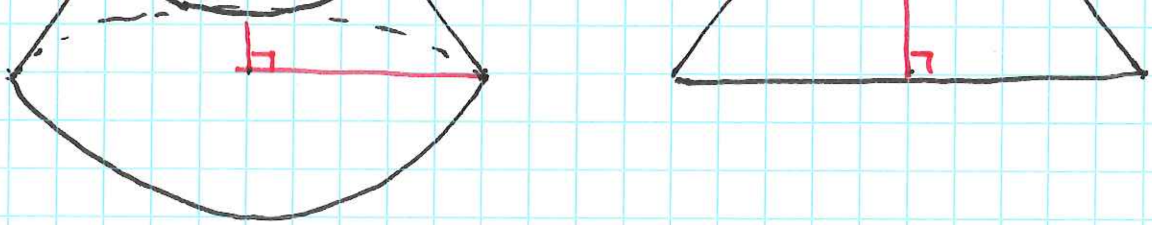
Lecture 12

- Another example similar to the previous lecture.

Ex What is the volume of a cone with height h and base radius r ?



- We start by trying to ~~est~~ estimate the volume by slicing the cone into n horizontal slices:



- If each slice is very thin (ie, n is big or $n \to \infty$) then we can assume that each slice is a cylinder.

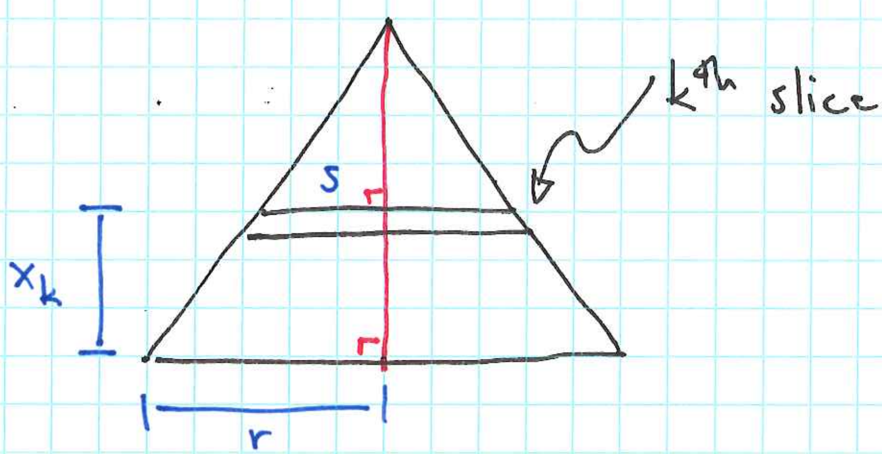
- Suppose $x = 0$ is the bottom of the cone and $x = h$ is the top.
- Each slice is $\Delta x = h/n$ thick.
- ~~Each~~ Let x_k be the height of the top of the k^{th} slice. Then

$$x_k = k\frac{h}{n} = k\,\Delta x$$

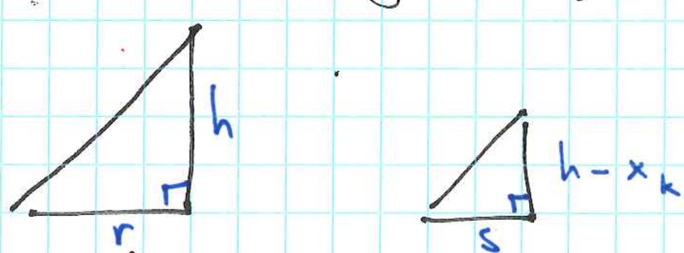
- If we want to know the volume of the k^{th} slice this is given by

$$Vol_k = \pi s^2 \Delta x$$

where s is the radius of the slice.



- Calculate s by using similar triangles



Ratio of sides:

$$\frac{s}{r} = \frac{h - x_k}{h}$$

Thus

$$s = \frac{r}{h}(h - x_k)$$

- The volume of the k^{th} slice is

$$\pi \frac{r^2}{h^2}(h - x_k)^2 \cdot \Delta x$$

- Total volume is the sum of individual pieces (as $n \Rightarrow \infty$)

$$\sum_{k=1}^{n} \pi \frac{r^2}{h^2}(h - x_k)^2 \, \Delta x$$

Which we recognise as the integral

$$\int_0^h \pi \frac{r^2}{h^2}(h - x)^2 \, dx$$

$$= \frac{\pi r^2}{h^2}\left[-\frac{1}{3}(h - x)^3 \right]_0^h$$

$$= \frac{\pi r^2}{h^2}\left(0 + \frac{1}{3}h^3 \right)$$

$$= \frac{1}{3}\pi r^2 h .$$

What do these examples have in common?

- Add up quantity that is changing (eg. patients remaining / volume of disk)
- There is a "direction of change" (eg time / height)

Steps to solving these problems

- Identify "direction of change"
- divide into n subintervals
- assume all the change happens suddenly at the <u>end</u> of each subinterval
- Add together contributions from each subinterval and let $n \longrightarrow \infty$
- interpret as ~~a~~ Riemann sum
- Convert to integral
- solve!

Work

- Work is measured in Joules
- 1J = amount of energy expended moving a mass 1 metre using 1 Newton of force.
- From wikipedia: - 1J = energy required to lift a 100 g mass 1 meter above the Earth's surface.
  - Heat required to raise the temp of 1g of water by 0.24°C.

If F newtons of force are applied to move a mass d meters then the work done is

$$W = Fd \text{ J}.$$

Small example: Work done lifting 30 kg by 20 meters.

Solution Acceleration due to gravity = $-9.8\,m/s^2$

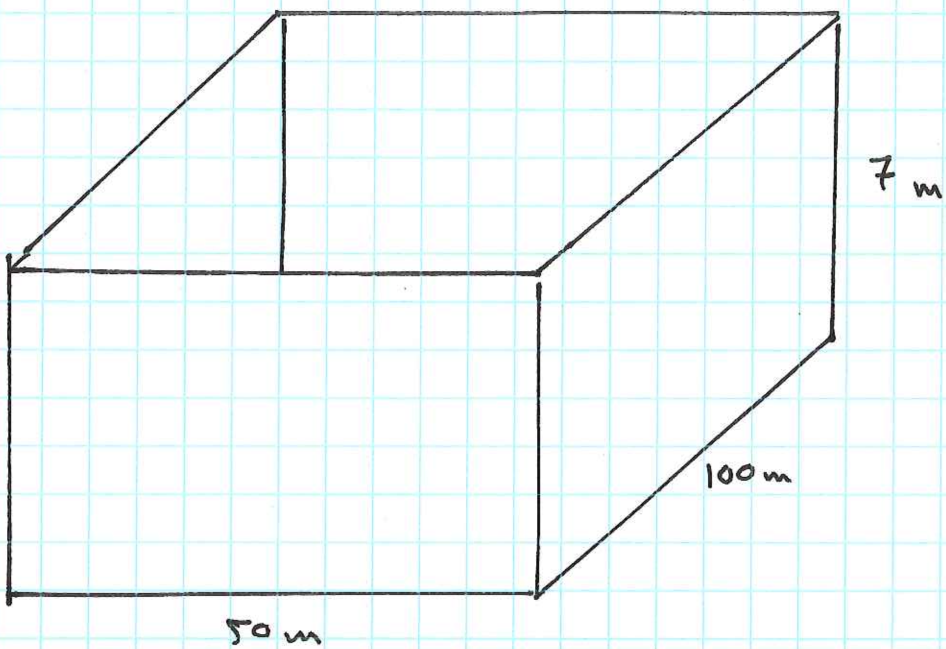
So force need to lift = $(9.8) \cdot 30$

$$W = (9.8) \cdot 30 \cdot 20$$
$$= 5886 \text{ J}.$$

Example 3

- A hole, 100 m × 50 m big, 7 m deep to be dug.
- Assume $1 m^3$ of dirt weights 1000 kg.

Q How ~~to~~ much work is being done by ~~dirt~~ digging the hole?



Silly solution: There is $5000 \times 7 \ m^3$ of dirt

$= 35\,000\,000$ kg of dirt

- Needs to be moved 7 m up
- Force needed $= (9.8) \cdot 35\,000\,000$

$= 343\,000\,000 \ N$

- Work $W = 343\,000\,000 \cdot 7$

$= 2\,401\,000\,000 \ J.$

But we shouldn't have to lift all the dirt the entire 7m up! E.g. the top layer only needs to be lifted ~ 0m

<u>Important</u>: How far below the surface the dirt is, determines how far it needs to be lifted!

- Direction of change = distance below surface = depth! = d.

- Subdivide into layers (n layers) each one $\Delta d = \frac{7}{n}$ m thick.
- The k^{th} layer (starting at $k = \cancel{0}1$) is

$$d_k = k \cdot \Delta d = \frac{7k}{n} \text{ m deep}$$

- The k^{th} layer contains

$$100 \cdot 50 \cdot \Delta d \text{ m}^3 \text{ of dirt}$$

So weighs $5000\,000\ \Delta d$ kg, thus the work is

$$W_k = \underbrace{\underbrace{(9.8)}_{\text{acceleration}}\underbrace{5000\,000\ \Delta d}_{\text{mass}}}_{\text{Force}} \cdot \underbrace{d_k}_{\text{distance}}$$

- Adding together and letting $n \to \infty$

$$W = \lim_{n \to \infty} \sum_{k=\cancel{0}1}^{n} 49000\,000\ \Delta d\ d_k$$

- Interpret as integral:

$$W = \int_0^7 49\,000\,000\, d \, dd$$

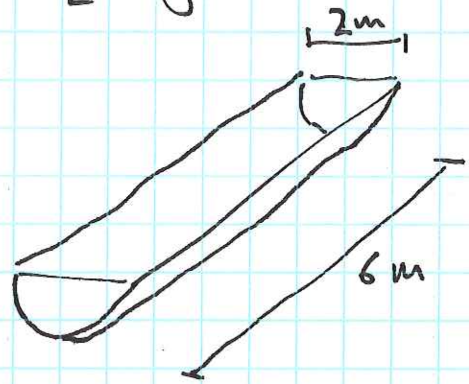
↑ oops! terrible variable choice!

$$= \left[\frac{1}{2} 49\,000\,000\, d^2 \right]_0^7$$

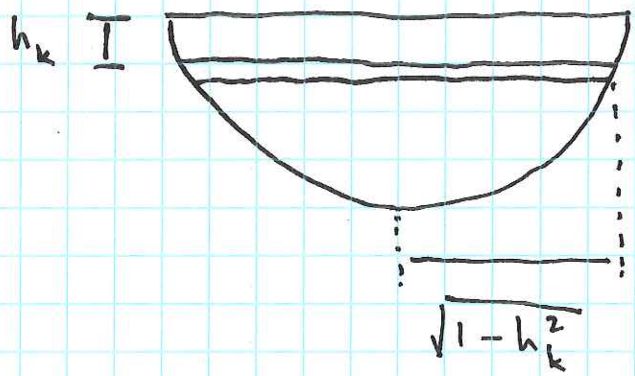
$$= 1\,200\,500\,000 \text{ J}.$$

Example 4

What about a $\frac{1}{2}$ cylindrical trench?



- Now the size of each layer is ~~so~~ changing w/ the depth as well.
- Let h = depth below ground.
- divide into n layers, $\Delta h = \frac{1}{n}$ m thick
- The k^{th} layer
  - $h_k = k\frac{1}{n}$ m below ground.



so k^{th} layer is

$$6\cdot 2\sqrt{1-h_k^2}\ \Delta h\ m^3 = 12000\sqrt{1-h_k^2}\,\Delta h\, kg$$

of dirt.

- Work needed to lift k^{th} slice:

$$(9.8)\cdot 12000\sqrt{1-h_k^2}\ \Delta h\cdot h_k$$

- Adding

$$W = \lim_{n\to\infty}\sum_{k=1}^{n} 117600\, h_k\sqrt{1-h_k^2}\ \Delta h$$

- As an integral

$$= \int_0^1 117600\, h\sqrt{1-h^2}\ dh$$

$$= \int_1^0 -58800\sqrt{u}\ du$$

$$= \left[58800\cdot 2\cdot\frac{1}{3}u^{3/2}\right]_0^1 = 19\ 200$$