Math 3B: Lecture 9

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October 16, 2018

Differential equations (motivation)

A differential equation is an equation that involves derivatives of an unknown function.

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = y - 3y^2$$

or

$$x^2 y'' + xy' + x^2 y = 0$$

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The challenge is to find all the functions y = f(x) (or even just one) that satisfy a given equation.

Newton's second law (motivation)

The original differential equation!

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If $h(t)$ measures the height of an object (maybe an apple?) above the earth then

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The force due to gravity is roughly $-10m$ Newtons, so

$$-10m = mh''(t)$$

Population growth (motivation)

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If $P(t)$ is the population at time t:

$$\frac{\mathrm{d}P}{\mathrm{d}t} = rP(t)$$

Some more examples of differential equations

$$\frac{dy}{dx} = f(x)$$

$$\frac{dy}{dx} = y(1 - y)$$

$$y'' = \sqrt{a^2 - (y')^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k(A - y)^2$$

Antiderivatives

We have been solving differential equations of the form

$$\frac{dy}{dx} = f(x).$$

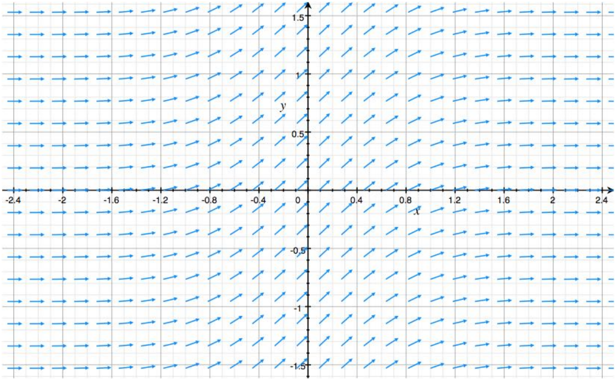
Any antiderivative $y = F(x)$ of $f(x)$ is a solutions to this differential equation!

Slope fields

In some cases it is impossible to find the antiderivative (without special functions). E.g.

$$f(x) = e^{-x^2}$$

But we can still (approximately) graph the antiderivative! First we draw the slope field



Slope fields (how to draw)

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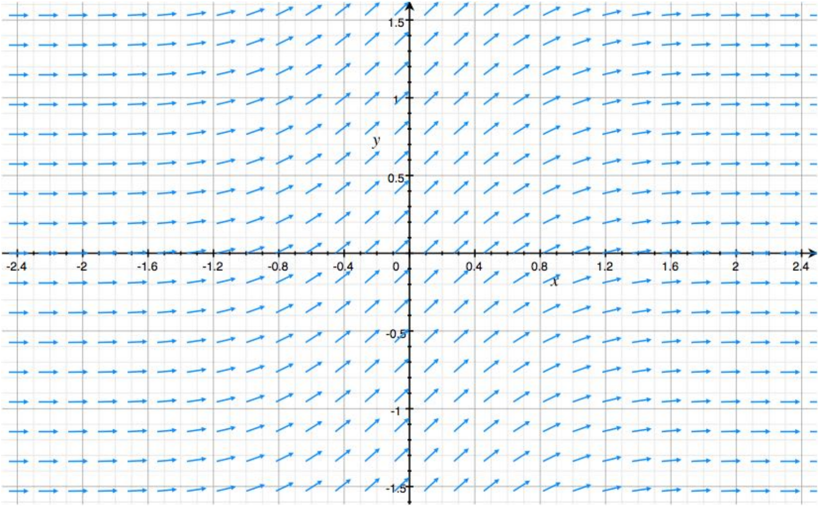
3. Answer given by differential equation above, slope is $f(x)$
4. Draw a small arrow with slope $f(x)$ and the point (x, y)
5. Do this for a grid of points on the xy-plane.

Example 1

$$f(x) = e^{-x^2} \text{ with } F(0) = -0.25$$



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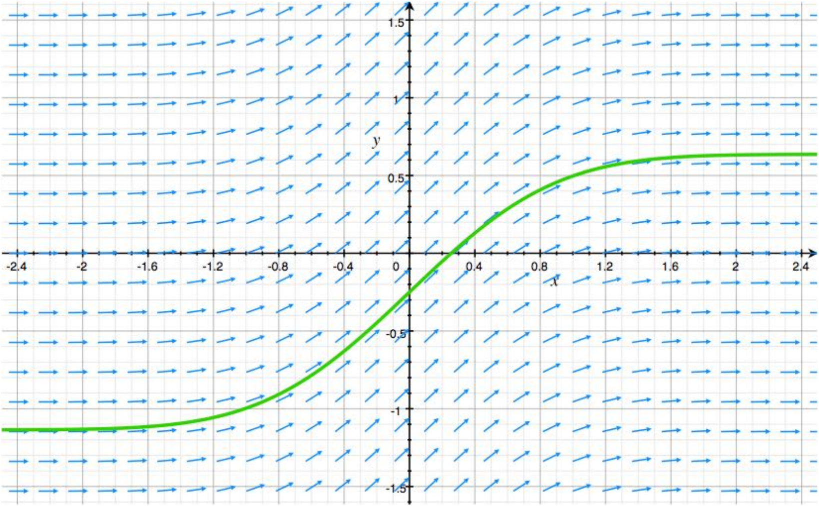
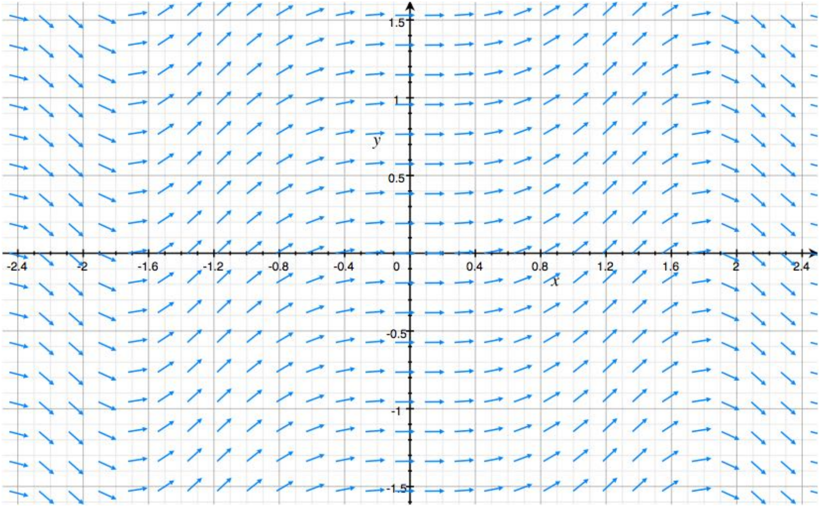


Example 2

$$f(x) = \sin(x^2) \text{ with } F(0) = 0.1$$



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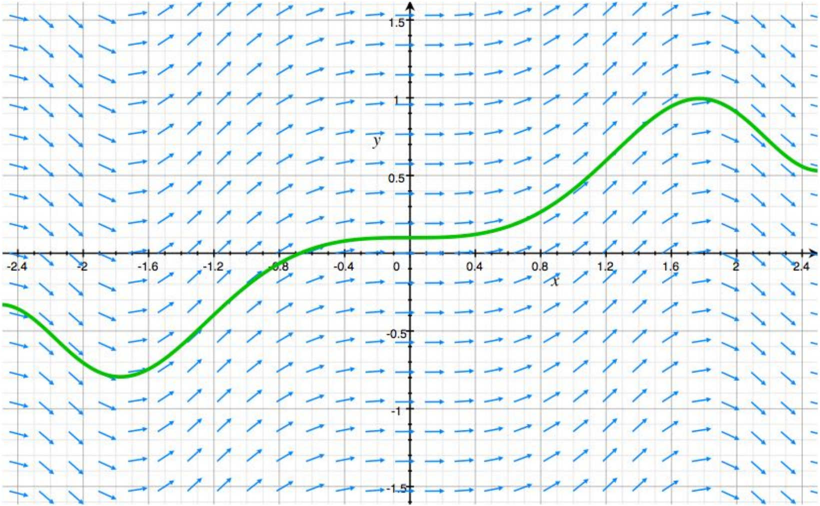

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These problems involve finding the area under some curve.

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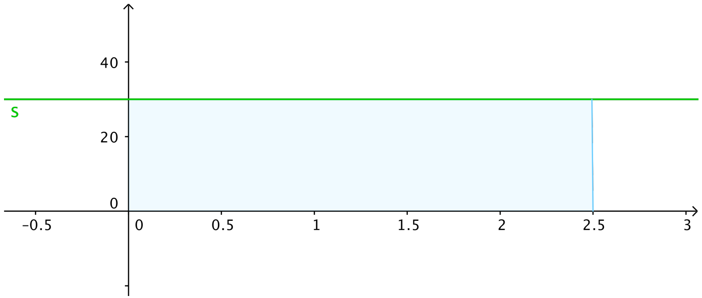
Solution

We model the car's speed using the function $s(t) = 30$. So we can see that the area under this curve



is the distance travelled (75 miles)

Example 2

If a car accellerates for 20 seconds at a rate of $2m/s^2$ and then decelerates for 30 seconds at a rate of $1m/s^2$, how far has it travelled?

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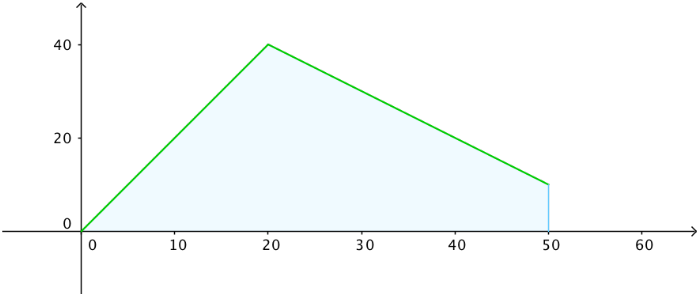
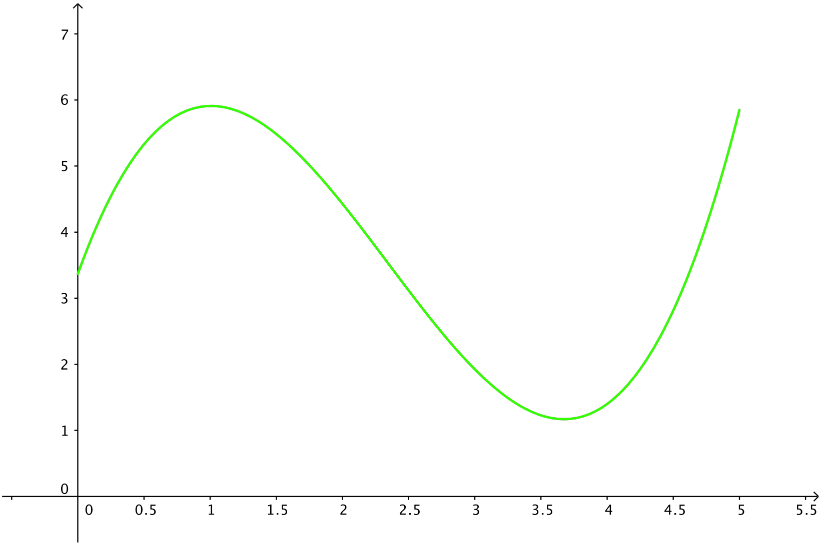
Solution

The car's speed is given by $s(t) = 2t$ when $0 \leq t \leq 20$ and $s(t) = 60 - t$ when $20 \leq t \leq 50$. So the graph looks like



More complicated rates of change

Suppose we have a car whose speed is descibed by the following curve. How far has it travelled in this time?



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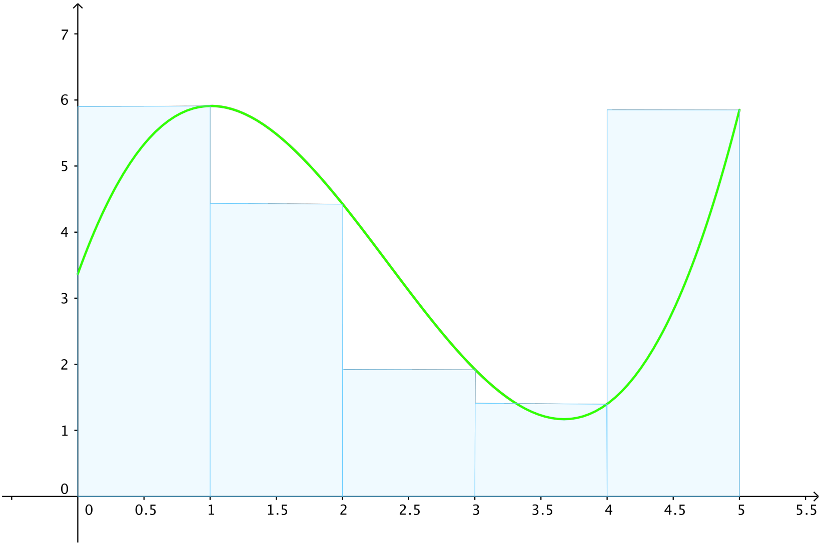

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- Answer: area under $f(t)$ between a and b.

Areas under general curves

We would like to calculate the area between a function $f(x)$ and the x-axis, between $x = a$ and $x = b$.

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(Too hard to draw, lets look at an animation)

The definite integral

Defintion

The definite integral of a function $f(x)$ is defined to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \Delta x \sum_{k=1}^{n} f(a + k\Delta x)$$

where $\Delta x = \frac{b-a}{n}$.

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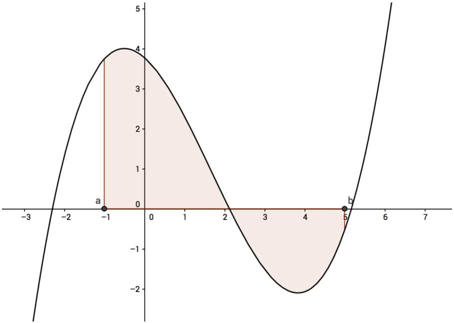
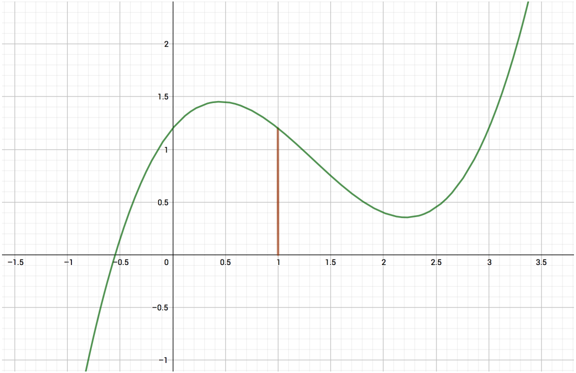


Properties of definite integrals

Zero area

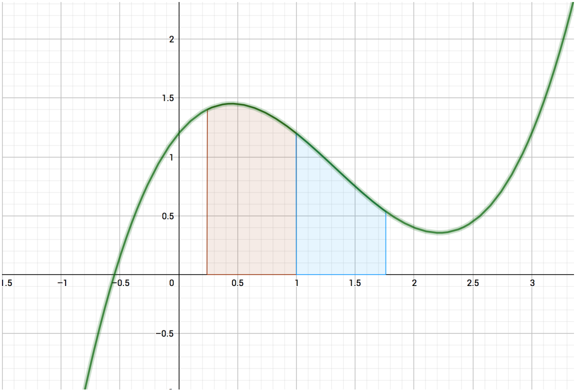
$$\int_a^a f(x)\,dx = 0$$



Properties of definite integrals

Adding areas

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$



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Reversing the area

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Lineararity (scalars factor out)

$$\int_a^b \alpha f(x)\ dx = \alpha \int_a^b f(x)\ dx$$

The fundamental theorem of calculus

Theorem

For any a,

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That is, $F(x) = \int_a^x f(t)\,dt$ is an antiderivative of $f(x)$!

Note

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Note

- $F(x) = \int_a^x f(t)\,dt$ is a function of x.
- every input x produces a number as an output.

A consequence (corrollary)

Corollary

For any antiderivative $F(x)$ of $f(x)$

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Why?

Well $F(x) = \int_a^x f(t)\,dt + C$ for some a and C. So

$$\begin{aligned} F(b) - F(a) &= \int_a^b f(t)\,dt + C - \int_a^a f(t)\,dt - C \\ &= \int_a^b f(t)\,dt \end{aligned}$$

Example 1

Question

Evaluate the definite integral

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Solution

An antiderivative of $x^2 - 4$ is $\frac{1}{3}x^3 - 4x$ so

$$\begin{aligned} \int_0^1 x^2 - 4 \, dx &= \frac{1}{3} \cdot 1^3 - 4 - \frac{1}{3} \cdot 0^3 + 4 \cdot 0 \\ &= \frac{1}{3} - 4 = -\frac{11}{3} \end{aligned}$$

Example 2

Question

Evaluate the definite integral

$$\int_0^{\pi} \sin x \; dx$$

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Solution

An antiderivative of $\sin x$ is $-\cos x$ so

$$\begin{aligned} \int_0^{\pi} \sin x \; dx &= -\cos \pi + \cos 0 \\ &= -(-1) + 1 = 2 \end{aligned}$$

Why is the FTC true?

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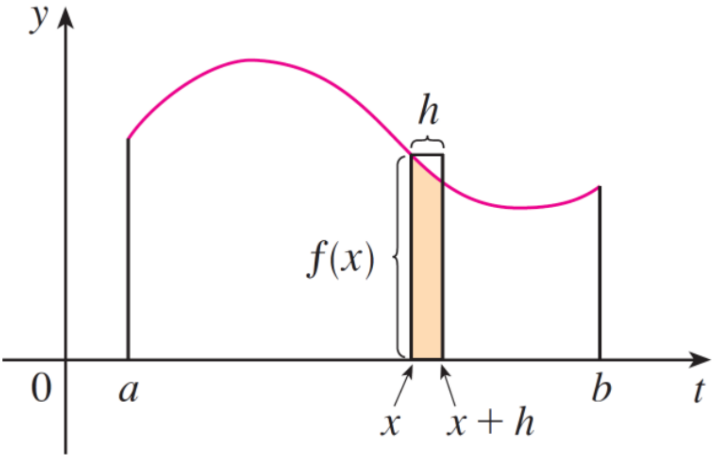
When h is very small $\quad \displaystyle\int_x^{x+h} f(t) \approx$



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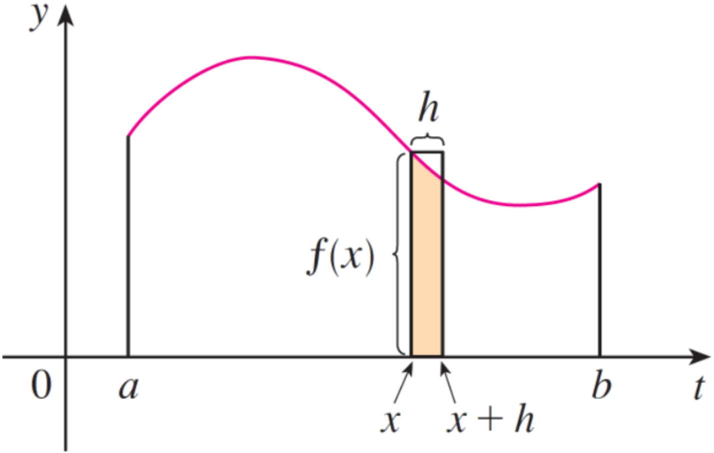

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Example

$$\int \sin(x) - x\,dx = -\cos(x) - \frac{1}{2}x^2 + C$$

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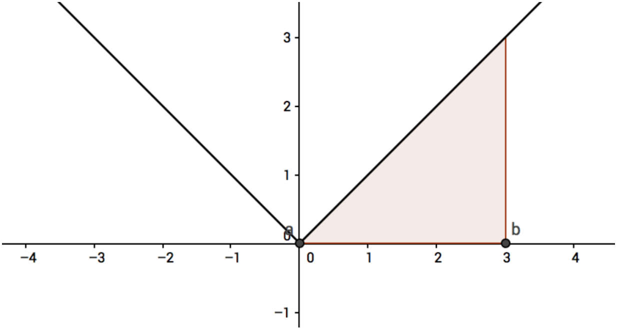
- Lets use $a = 0$.
- How should we calculate $F(x)$?

Example

Use the defintition!

$$F(x) = \int_0^x |t| \; dt$$

is the area under $|t|$!



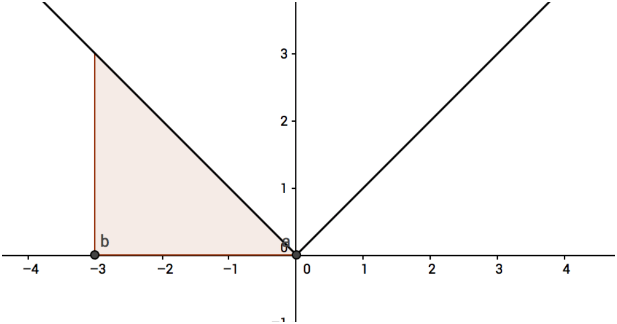
$$F(x) = \frac{1}{2}x^2 \quad \text{if } x \geq 0$$

Example

If $x \leq 0$ then

$$F(x) = \int_0^x |t| \; dt = -\int_x^0 |t| \; dt$$

is the negative of the area under $|t|$!



$$F(x) = -\frac{1}{2}x^2 \quad \text{if } x \leq 0$$

Example

In summary

$$F(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0 \\ -\frac{1}{2}x^2 & \text{if } x \leq 0 \end{cases}$$

or

$$F(x) = \frac{1}{2}x\,|x|$$