Midterm 2 practice 1

UCLA: Math 32B, Spring 2018

Instructor: Noah White
Date: May, 2018
Version: *practice*

- This exam has 4 questions, for a total of 40 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: Solutions

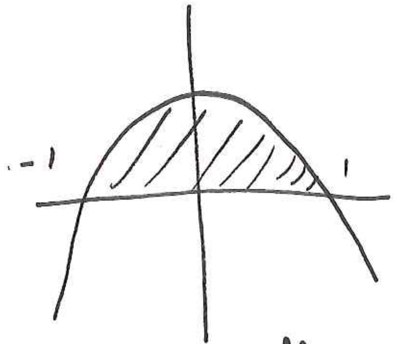
ID number: ______________________

Discussion section (please circle):

Day/TA	Ryan	Eli	Khang
Tuesday	1A	1C	1E
Thursday	1B	1D	1F

Question	Points	Score
1	10	
2	8	
3	8	
4	14	
Total:	40	

1. (a) (5 points) Compute the center of mass of the region in the xy-plane above the x-axis and below the curve $y = 1 - x^2$. Assume a constant mass density of 1.



$$(x_{cm}\ y_{cm}) = \frac{1}{\text{Area}\, D} \iint_D (x, y) \cdot \overset{\text{density}}{1}\, dA$$

$$D = \{(x,y) \mid -1 \leq x \leq 1,\ 0 \leq y \leq 1 - x^2\}$$

$$\text{Area}\, D = \iint_D 1\, dA = \int_{-1}^{1} \int_{0}^{1-x^2} dy\, dx$$

$$= \int_{-1}^{1} 1 - x^2\, dx = 4/3$$

$$\iint_D (x\ y)\, dA = \int_{-1}^{1} \int_{0}^{1-x^2} (x\ y)\, dy\, dx$$

$$= \int_{-1}^{1} \left(xy, \tfrac{1}{2}y^2\right)\Big|_{0}^{1-x^2} dx$$

$$= \int_{-1}^{1} \left(x - x^3, \tfrac{1}{2}(1-x^2)^2\right) dx$$

$$= \left(\tfrac{1}{2}x^2 - \tfrac{1}{4}x^4,\ \tfrac{1}{2}\left(x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\right)\right)\Big|_{-1}^{1}$$

$$= \left(0, \frac{8}{15}\right)$$

So

$$(x_{cm}\ y_{cm}) = \left(0, \frac{2}{5}\right)$$

(b) (5 points) Determine the surface area of the paraboloid

$$x^2 + y^2 = 2z, \quad 0 \leq z \leq 1.$$

We parametrise by

$$G(u\ v) = \left(u, v, \tfrac{1}{2}(u^2+v^2)\right)$$

If $z \in [0\ 1]$ then $0 \leq x^2 + y^2 \leq 2$ so

$(u,v) \in$ disk of radius $\sqrt{2}$

$T_{\underline{u}} = (1\ 0\ u)$ $\quad \underline{N}(u\ v) = (-u\ -v\ 1)$

$T_{\underline{v}} = (0\ 1\ v)$ $\quad \|\underline{N}(u\ v)\| = \sqrt{u^2+v^2+1}$

$$\iint_S 1\, dS = \iint_{\mathcal{D}} \sqrt{u^2+v^2+1}\, dA_{uv}$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} r\sqrt{r^2+1}\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{3}(r^2+1)^{3/2}\Big|_0^{\sqrt{2}}\, d\theta$$

$$= 2\pi\left(\sqrt{3} - \frac{1}{3}\right)$$

2. (8 points) Consider the region $\mathcal{E}$ given by

$$0 \le z \le (y - x^2)^2, \quad x^2 \le y \le x.$$

Use the change of variables

$$x = u, y = v + u^2, z = wv^2,$$

to evaluate

$$\int_{\mathcal{E}} \frac{1}{y - x^2}\, \mathrm{d}V.$$

The region is (in uvw-coordinates)

$0 \le wv^2 \le v^2 \rightsquigarrow 0 \le w \le 1$

$u^2 \le v + u^2 \le u \rightsquigarrow 0 \le v \le u - u^2$

from which we see (draw a pic!) $0 \le u \le 1$

$G(uvw) = (u, v + u^2, wv^2)$

Jacobian $J(G) = \det \begin{pmatrix} 1 & 0 & 0 \\ 2u & 1 & 0 \\ 0 & 2wv & v^2 \end{pmatrix} = v^2 > 0$

so

$$\iiint_{\mathcal{E}} \frac{1}{y - x^2}\, dV = \iiint_{\mathcal{E}_{\star}} \frac{1}{v} \cdot v^2 \, dV_{uvw}$$

$$= \int_0^1 \int_0^1 \int_0^{u-u^2} v \, dv\, du\, dw$$

$$= \int_0^1 \int_0^1 \frac{1}{2}(u - u^2)^2 \, du\, dw$$

$$= \int_0^1 \frac{1}{2}\left(\frac{1}{3}u^3 - \frac{1}{2}u^4 + \frac{1}{5}u^5\right)\Big|_0^1 dw$$

$$= \int_0^1 \frac{1}{60}\, dw = 1/60$$

3. Let $\mathbf{F}$ be a vector field given by

$$\mathbf{F}(x,y,z) = (y\cos z - yze^x, x\cos z - ze^x, -xy\sin z - ye^x).$$

(a) (4 points) Show that $\mathbf{F}$ is conservative.

(b) (4 points) Find a potential function for $\mathbf{F}$.

a) • $\underline{F}$ is well def. on $\mathbb{R}^2$ which is simply connected.

• $\text{curl}(\underline{F}) = \langle (-x\sin z - e^x) - (-x\sin z - e^x),$
$-(-y\sin z - ye^x) + (-y\sin z - ye^x),$
$(\cos z - ze^x) - (\cos z - ze^x) \rangle$

$= 0$

Thus $\underline{F}$ is ~~simply~~ conservative

b) Suppose $\nabla\varphi = \underline{F}$, then

$\partial_x \varphi = y\cos z - yze^x \Rightarrow \varphi = xy\cos z - yze^x + \alpha(y\ z)$

$\partial_y \varphi = x\cos z - ze^x \Rightarrow \varphi = xy\cos z - yze^x + \beta(x\ z)$

$\partial_z \varphi = -xy\sin z - ye^x \Rightarrow \varphi = xy\cos z - yze^x + \gamma(x\ y)$

so choose $\alpha = \beta = \gamma =$ constant thus

$\varphi = xy\cos z - yze^x + C$

4. In this question we will calculate the surface area of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{a^2} + z^2 = 1$.
   (a) (4 points) Find a parameterisation of the ellipsoid given above.
   (b) (5 points) Express the surface area as a double iterated integral.
   (c) (5 points) Evaluate the surface area when $a = 2$. You may use the fact that

$$\int \sqrt{1+x^2}\,dx = \frac{1}{2}\sqrt{1+x^2} + \frac{1}{2}\ln\left(\sqrt{1+x^2}+x\right) + C.$$

a) Note that a sphere is parametrised by

$$G' = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$$

so stretching the x, y directions gives us

$$G(\theta\,\phi) = (a\cos\theta\sin\phi,\ a\sin\theta\sin\phi,\ \cos\phi)$$

$$(\theta\,\phi) \in [0\ 2\pi]\times[0\ \pi] = \mathcal{D}$$

b) $T_\theta = (-a\sin\theta\sin\phi,\ a\cos\theta\sin\phi,\ 0)$

$$T_\phi = (a\cos\theta\cos\phi,\ a\sin\theta\cos\phi,\ -\sin\phi)$$

so

$$\underline{N}(\theta\,\phi) = (-a\cos\theta\sin^2\phi,\ -a\sin\theta\sin^2\phi,\ -a^2\sin\phi\cos\phi)$$

and

$$\|\underline{N}(\theta\,\phi)\|^2 = a^2\sin^2\phi\,(\sin^2\phi + a^2\cos^2\phi)$$

$$= a^2\sin^2\phi\,(1 + (a^2-1)\cos^2\phi)$$

$$\iint_S 1\,dS = \iint_{\mathcal{D}} a\sin\phi\sqrt{1+(a^2-1)\cos^2\phi}\,d\theta\,d\phi = \int_0^{2\pi}\int_0^{\pi} \text{—''—}\,d\theta\,d\phi$$

c) Let $x = \cos\phi$, then $dx = -\sin\phi\,d\phi$ so

$$= \int_0^{2\pi} -\int_1^{-1} a\sqrt{1+(a^2-1)x^2}\,dx = 2\pi a\int_{-1}^{1}\sqrt{1+(a^2-1)x^2}\,dx$$

use $u = \sqrt{a^2-1}\,x$ and $du = \sqrt{a^2-1}\,dx$ so

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$$S.A. = \frac{2\pi a}{\sqrt{a^2-1}} \int_{x=-1}^{x=1} \sqrt{1+u^2}\, du$$

$$= \frac{2\pi a}{\sqrt{a^2-1}} \left(\frac{1}{2}\sqrt{1+u^2} + \frac{1}{2}\ln\left(\sqrt{1+u^2}+u\right)\right)\Big|_{x=-1}^{x=1}$$

$$= \frac{2\pi a}{\sqrt{a^2-1}} \left(\frac{1}{2}\sqrt{1+(a^2-1)x^2} + \frac{1}{2}\ln\left(\sqrt{1+(a^2-1)x^2} + \sqrt{a^2-1}\,x\right)\right)\Big|_{-1}^{1}$$

$$= \frac{\pi a}{\sqrt{a^2-1}} \ln\left(\frac{a+\sqrt{a^2-1}}{a-\sqrt{a^2-1}}\right)$$

when $a = 2$

$$= \frac{2\pi}{\sqrt{3}} \ln\left(\frac{2+\sqrt{3}}{2-\sqrt{3}}\right)$$

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