This weeks problem set focuses on the ideas of linear combinations, linear dependence and bases. A question marked with a † is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a * is especially important.

Homework: due Friday 11 October, uploaded to Gradescope before 11:50pm: questions 3 and 4 below.

1. From section 1.4, problems 1, 7, 8 ($\mathsf{P}_n(F)$ is the set of polynomials of degree less than or equal to n), 11, 12, 13*.

2. From section 1.5, problems 1, $2a, c, e$, 4^*, 5, 9^*, 15, 18^*.

3.* Let V be a vector space over a field $\mathbb{F}$ and W a subspace of V. For any $v \in V$, consider the set $\{v\} + W = \{v + w \mid w \in W\}$. We will denote it simply as $v + W$. Now consider the set

$$V/W = \{v + W \mid v \in V\}.$$

We can define addition and scalar multiplication on this set by

$$(v + W) + (w + W) = (v + w) + W \qquad \text{and} \qquad \lambda(v + W) = \lambda v + W.$$

Prove that V/W is a vector space. It is called the *quotient* of V by W.

Solution: We simply check each axiom one at a time.

VS1 Clearly $(v+W)+(u+W) = (v+u)+W = (u+v)+W = (u+W)+(v+W)$ since $v+u = u+v$ in V.

VS2 This holds in exactly the same way as above since $(v + u) + w = v + (u + w)$ in V.

VS3 The zero element is $0 + W = W$. Indeed

$$(0 + W) + (v + W) = (0 + v) + W = v + W$$

Note that 0 is the zero element of V and we denote the zero element of V/W by $0 + W$ or W for clarity.

VS4 The additive inverse of $v + W$ is $-v + W$. Indeed

$$(v + W) + (-v + W) = (v - v) + W = 0 + W$$

since $v - v = 0$ in V.

VS5 It is clear that $1 \cdot (v + W) = (1v) + W = v + W$ since $1v = v$ in V.

VS6-8 These all follow in exactly the same way as above. The relations are true in V so they are true in V/W.

4. Let $\mathbb{C}[x]$ be the vector space of polynomials and let $W = \operatorname{span}\{x^a \mid a > 2\}$.

   (a) Find a set of 3 linearly independent elements of $\mathbb{C}[x]/W$.

   Solution: Note that $x^a + W = W$ if $a > 2$. Thus we can choose $1 + W, x + W, x^2 + W$. These are linearly independent since if

   $$a(1 + W) + b(x + W) + c(x^2 + W) = W$$

then

$$(a + bx + cx^2) + W = W$$

and thus $a + bx + cx^2 \in W$, but this is impossible unless $a = b = c = 0$.

(b) Find 2 nonzero elements $p, q \in \mathbb{C}[x]$ that are linearly independant and such that $p + W$ and $q + W$ are linearly dependant and nonzero. *Note: you can only receive full points for this problem if your polynomials p and q and different from everyone elses! If you understand the problem then this will be easy to ensure.*

Solution: We want two elements p, q such that they are linearly independent. The easiest way to ensure this is to pick two polynomials with different degrees. We also want $p+W$ and $q+W$ to be non-zero. That is, we need to make sure $p, q \notin W$. The easiest way to ensure this is to make sure they have, for example, a constant term. Thirdly we want to make sure $p + W$ and $q + W$ are linearly dependent. The easiest way to ensure this is to make sure $p + W = q + W$, i.e. $(p - q) + W = W$, that is we want $p - q \in W$. This means their constant, linear and quadratic terms should agree. Lets summarise the conditions we want p, q to have.

- They should have different degrees.
- They should have a constant term
- Their constant, linear and quadratic terms should be the same.

So we could pick, for example, $p = 1 + x^3$ and $q = 1 + x^4$.

Note on problem 3: The astute reader might be worried that the addition and scalar multiplication might not be well defined. What do I mean by this? Well, it is entirely possible that $v + W = v' + W$ for two different elements $v, v' \in V$. This means we could calculate a sum in two different ways. As

$$(v + W) + (u + W) = (v + u) + W$$

or as

$$(v + W) + (u + W) = (v' + W) + (u + W) = (v' + u) + W$$

(since $v + W = v' + W$). So we need to check that $(v + u) + W = (v' + u) + W$. I will show you how to do this below. You might like to try to prove that the scalar multiplication is unambiguous for yourself.

Proof that $(v + u) + W = (v' + u) + W$: Note that $(v + u) + W = \{(v + u) + w \mid w \in W\}$ and $(v' + u) + W = \{(v' + u) + w \mid w \in W\}$. Also note that $v \in v + W$ since $v = v + 0$ and $0 \in W$.

Since $v + W = v' + W$ we see that $v \in v' + W$ and thus $v = v' + x$ for some $x \in W$. Now lets take an arbitrary elements $s \in (v + u) + W$, it will be of the form $s = v + u + w$. We know

$$s = v + u + w = v' + x + u + w = (v' + u) + (x + w).$$

Since $x + u \in W$ we see that $s = (v' + u) + (x + w) \in (v' + u) + W$. We have just shown that $(v + u) + W \subset (v' + u) + W$. To complete the proof we need to show the opposite containment.

We do this in almost the same way. Take an arbitrary element $t \in (v' + u) + W$. We have that $t = v' + u + w$ for some $w \in W$. Then

$$t = v' + u + w = v - x + u + w = (v + u) + (w - x) \in (v + u) + W.$$

Thus we have shown $(v' + u) + W \subset (v + u) + W$ and hence $(v + u) + W = (v' + u) + W$.