

This weeks problem set provides practice with the Gram-Schmidt process and adjoint operators. A question marked with a \dagger is difficult and probably too hard for an exam (though still illustrates a useful point). A question marked with a $*$ is especially important.

Homework 6: due Friday 16 March: questions 22 from 6.2 and $4a, b$ below.

1. From section 6.2, problems 1, $2b, g, i, k, 5^*, 6, 7, 9, 13^*, 17^*, 22$.
2. From section 6.3, problems 1, $2a, 3a, c, 4, 6, 8^*$.
3. From section 6.4, problems 1, $2a, c, 4, 7, 11, 24$.
4. Let V be a finite dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} .
 - (a) Fix $y \in V$ and suppose $\langle x, y \rangle = 0$ for all $x \in V$. Show that $y = 0$.

Solution: Let $B = \{v_1, \dots, v_n\}$ be an orthonormal basis for V . Then

$$y = \sum_{i=1}^n \langle y, v_i \rangle v_i = 0$$

since $\langle y, v_i \rangle = 0$ for all i .

- (b) Let $T : V \longrightarrow V$ be a linear map such that $\langle T(x), T(y) \rangle = \langle x, y \rangle$ for all pairs $x, y \in V$ (we call such a map an *isometry*). Prove that T is an isomorphism.

Solution: First we show that T is injective. Suppose that $T(x) = T(y)$. On one hand we have

$$\|T(x)\| = \langle T(x), T(x) \rangle = \langle x, x \rangle.$$

On the other hand,

$$\|T(x)\| = \langle T(x), T(x) \rangle = \langle T(x), T(y) \rangle = \langle x, y \rangle.$$

Thus $\langle x, x \rangle = \langle x, y \rangle$, i.e. $\langle 0, x - y \rangle = 0$. Thus by the above, $x - y = 0$ or $x = y$.

Now, since T is an injective map from V to V , it must be surjective by the dimension theorem. Thus it is an isomorphism.

- (c) (not for homework) \dagger Find all isometries $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that have $\det T = 1$.

Solution: *Solution to 22 from 6.2.* Part a). We use the Gram-Schmidt process to define first an orthogonal basis $\{f_1, f_2\}$. Set $f_1 = t$. Then

$$f_2 = \sqrt{t} - \frac{\langle \sqrt{t}, t \rangle}{\|t\|^2} t.$$

To get an explicit expression we do some calculations.

$$\|t\|^2 = \int_0^1 t^2 dt = \frac{1}{3}.$$

$$\langle \sqrt{t}, t \rangle = \int_0^1 t^{3/2} dt = \frac{2}{5}.$$

$$\|\sqrt{t}\|^2 = \int_0^1 t dt = \frac{1}{2}.$$

Putting this together we get

$$f_2 = \sqrt{t} - \frac{6}{5}t.$$

To get an orthonormal basis, we need to normalise, so we need to calculate

$$\begin{aligned} \|f_2\|^2 &= \int_0^1 \left(\sqrt{t} - \frac{6}{5}t \right)^2 dt \\ &= \int_0^1 t - \frac{12}{5}t^{3/2} + \frac{36}{25}t^2 dt \\ &= \frac{1}{2} - \frac{24}{25} + \frac{36}{75} = \frac{1}{50} \end{aligned}$$

We already know that $\|f_1\| = \frac{1}{\sqrt{3}}$ and now we also know $\|f_2\| = \frac{1}{5\sqrt{2}}$ thus, an orthonormal basis is

$$\{g_1 = \sqrt{3}t, g_2 = \sqrt{2}(5\sqrt{t} - 6t)\}.$$

Part b). We want to project t^2 onto W . The result will be

$$\langle t^2, g_1 \rangle g_1 + \langle t^2, g_2 \rangle g_2.$$

We calculate the coefficients.

$$\begin{aligned} \langle t^2, g_1 \rangle &= \int_0^1 \sqrt{3}t^3 dt = \frac{\sqrt{3}}{4}. \\ \langle t^2, g_2 \rangle &= \int_0^1 \sqrt{2} \left(5t^{5/2} - 6t^{3/2} \right) dt \\ &= \sqrt{2} \left(\frac{10}{7} - \frac{12}{5} \right) \\ &= -\frac{34\sqrt{2}}{35}. \end{aligned}$$

Thus, the best approximation is

$$\frac{3}{4}t - \frac{68}{35} \left(5\sqrt{t} - 6t \right) = \frac{1737}{140}t - \frac{68}{7}\sqrt{t}.$$