Final exam practice 4 (solutions)

UCLA: Math 3B, Winter 2019

Instructor: Noah White
Date:

- This exam has 7 questions, for a total of 80 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable, non graphing and non integration capable calculators are allowed.

Name: ______________________________

ID number: ______________________________

Discussion section (please circle):

Day/TA	Louis	Matthew
Tuesday	1A	1C
Thursday	1B	1D

Question	Points	Score
1	12	
2	12	
3	12	
4	10	
5	12	
6	12	
7	10	
Total:	80	

Question 1 and 2 are multiple choice. Indicate your answers in the table below. *The following three pages will not be graded, your answers must be indicated here.*

Part	A	B	C	D
1(a)				
1(b)				
1(c)				
1(d)				
1(e)				
1(f)				
2(a)				
2(b)				
2(c)				
2(d)				
2(e)				
2(f)				

1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) The function $f(x) = x^3 + x + 1$ is

A. Always increasing.
B. Always decreasing.
C. Always concave up.
D. Always concave down.

(b) (2 points) The function $f(x) = \ln(e^x + 1)$ is

A. Always increasing.
B. Always decreasing.
C. Always concave down.
D. None of the above.

(c) (2 points) The function $f(x) = \frac{2x^4}{x^3+1}$ has a

A. Horizontal asymptote at $y = -1$.
B. Vertical asymptote at $x = 1$.
C. Slanted asymptote with slope -1.
D. Slanted asymptote with slope 2.

(d) (2 points) The function $f(x) = \frac{\sin x}{x^2+1}$ has a

A. Horizontal asymptote at $y = 0$.

B. No horizontal asymptotes.

C. Horizontal asymptote at $y = 1$.

D. Horizontal asymptote at $y = -1$.

(e) (2 points) The function $f(x) = \ln(e^x + e^{-x})$ has a critical point at

A. $x = 0$

B. $x = e$

C. $x = 1$

D. It has no critical points

(f) (2 points) The function $f(x) = (x^2 - 4x + 5)^{-1}$ has a

A. minimum at $x = 2$.

B. maximum at $x = 2$.

C. minimum $x = 1$.

D. maximum $x = 1$.

2. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

   (a) (2 points) Suppose $F(x)$ is an antiderivative of $f(x) = xe^x$ such that $F(0) = 0$. What is $F(1)$?

   A. 3.
   B. 2.
   C. 1.
   D. 0.

   (b) (2 points) The indefinite integral $\int \frac{\cos x}{\sin x + 1}\, dx$ is

   A. $(\sin x + 1)^{-2} + C$.
   B. $\ln(\sin x + 1) + C$**.**
   C. $\ln(\cos x) + C$.
   D. $(\tan x + 1)^2 + C$.

   (c) (2 points) The solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = 2y$ when $y(0) = 1$ has

   A. $y(1) = 2e^2$
   B. $y(0.5) = e$**.**
   C. $y(1) = 2e^{-2}$.
   D. $y(0.5) = e^{-1}$.

(d) (2 points) The solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{e^{-y}}{(t+1)^2}$ when $y(0) = 0$ has

A. $y(-2) = \ln(3)$**.**
B. $y(-2) = \ln(2)$.
C. $y(1) = \ln(3)$.
D. $y(1) = \ln(2)$.

(e) (2 points) Given a function $f(x)$ what is the derivative of $\int_{-x}^{x} f(x)\,dx$

A. $f(x)$.
B. $2f(x)$.
C. $f(-x)$.
D. $f(x) + f(-x)$**.**

(f) (2 points) The differential equation $\frac{\mathrm{d}y}{\mathrm{d}t} = \ln(y^2+1) - \ln(5)$ has a

A. stable equilibrium at $y = 2$.
B. unstable equilibrium at $y = 1$.
C. stable equilibrium at $y = 1$.
D. unstable equilibrium at $y = 2$**.**

3. Let $f(x) = \dfrac{x^2-4}{x^2-9}$. Note that $f'(x) = -\dfrac{10x}{\left(x^2-9\right)^2}$ and $f''(x) = \dfrac{30\left(x^2+3\right)}{\left(x^2-9\right)^3}$.

   (a) (1 point) Does $f(x)$ cross the x and y axes? If so, where?

   Solution: Function can only be zero when $x^2 - 4 = 0$. Thus $x = \pm 2$ are the x-intercepts. $f(0) = 4/9$ is the y-intercept.

   (b) (2 points) Does $f(x)$ have any horizontal asymptotes? If so what are they?

   Solution: We need to evaluate

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-4}{x^2-9} = 1.$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x^2-4}{x^2-9} = 1.$$

So we have horizontal asymptote in the positive and negative directions at $y = 1$.

   (c) (2 points) Does $f(x)$ have any vertical asymptotes? If so what are they?

   Solution: The denominator of $f(x)$ is zero when $x = \pm 3$, thus a vertical asymptote at $x = 3$ and $x = -3$.

   (d) (2 points) For what x is the first derivative $f'(x)$ positive?

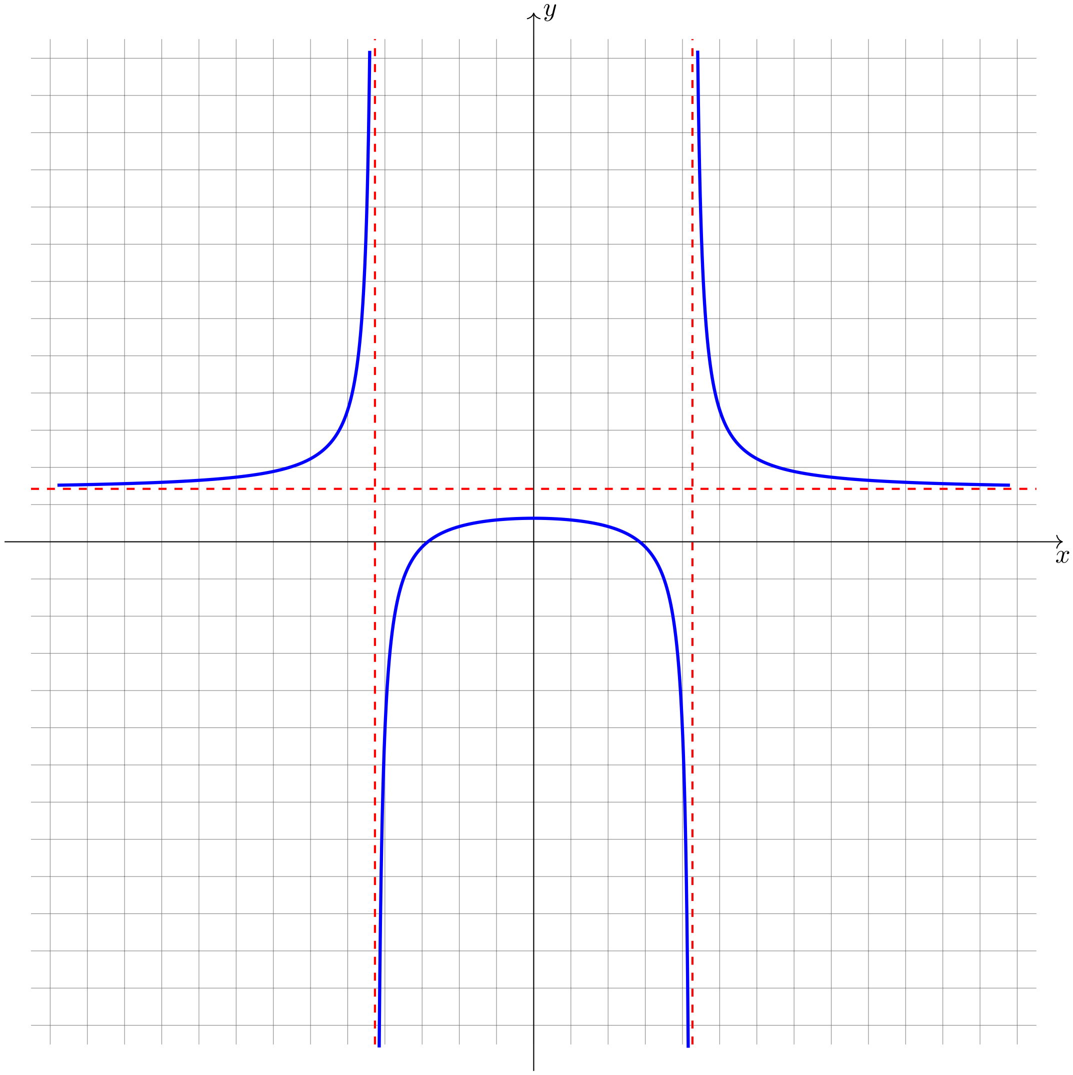
   Solution: The denominator is always positive. Thus $f'(x)$ is positive when $x < 0$.

(e) (2 points) For what x is the second derivative $f''(x)$ positive?

Solution: The denominator is positive when $|x| > 3$. The numerator is always positive. Thus, $f''(x) > 0$ when both the numerator and denominator have the same sign, i.e. when

$$x < -3 \text{ or } x > 3.$$

(f) (3 points) On the graph provided, sketch $f(x)$



4. A 100 metre long chain is hanging off a bridge. The chain weighs 0.4 kilograms per metre. You may assume that the acceleration due to gravity is 10 m/s^2.

   (a) (6 points) Write a Riemann sum which represents the total work done pulling the chain up, onto the bridge.

   Solution: Let $\Delta y = 100/n$ and $y_k = k\Delta y$. If $y = 0$ is the top of the chain then y_k is the bottom of the k^{th} interval. The k^{th} interval of chain weighs $0.4\Delta y$ kilograms. It has to be pulled y_k metres up so the work done is $4y_k\Delta y$. Thus the total amount of work is

   $$W = \lim_{n\to\infty} \sum_{k=1}^{n} 4y_k\Delta y$$

   (b) (4 points) Use an integral to evaluate the Riemann sum above.

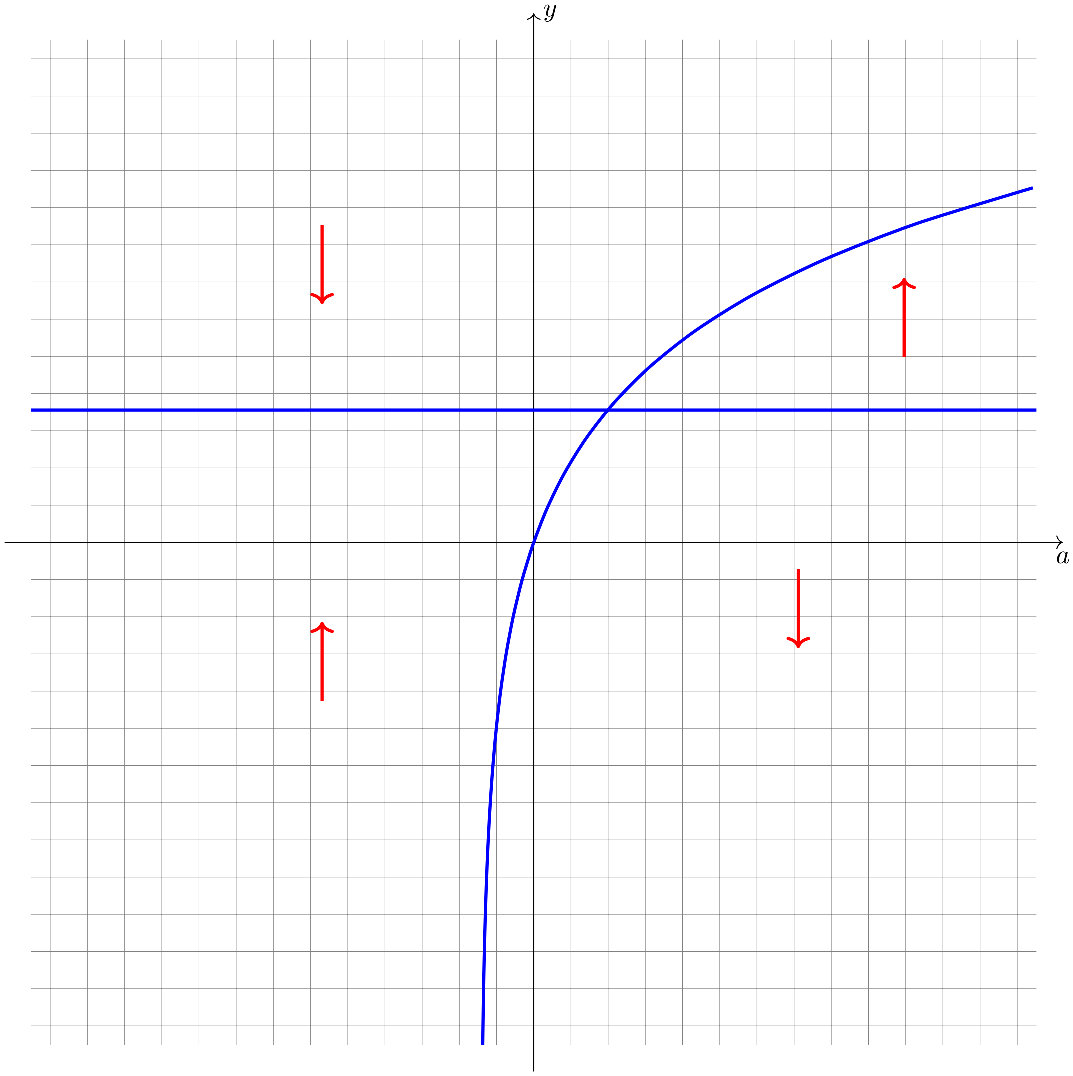
   Solution:

   $$\begin{aligned} W &= \int_0^{100} 4y\; dy \\ &= \left[2y^2\right]_0^{100} \\ &= 20000 \end{aligned}$$

5. In this question we will investigate the behaviour of the solutions of

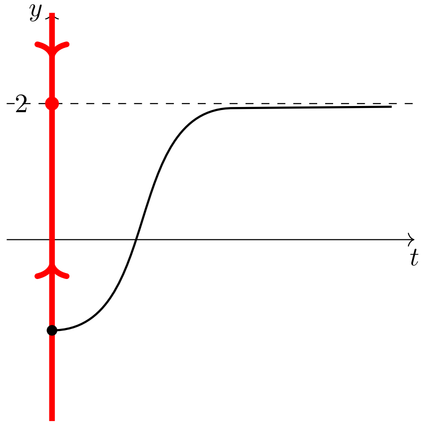
$$\frac{\mathrm{d}y}{\mathrm{d}t} = (y-2)(a - e^y + 1)$$

(a) (4 points) Draw a bifurcation diagram for this equation with parameter a. Make sure to label the regions of your diagram with up/down arrows according to the direction of the derivative.



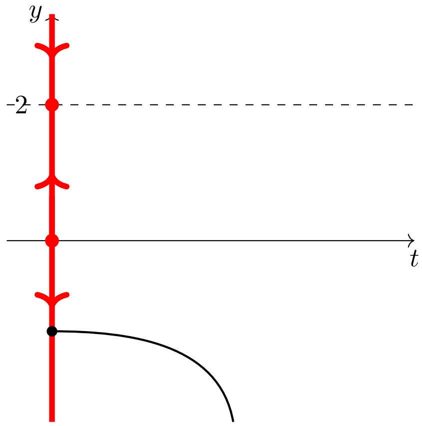
(b) (2 points) Draw a phase diagram when $a = -1$ and sketch the solution if $y(0) = -1$.

Solution:



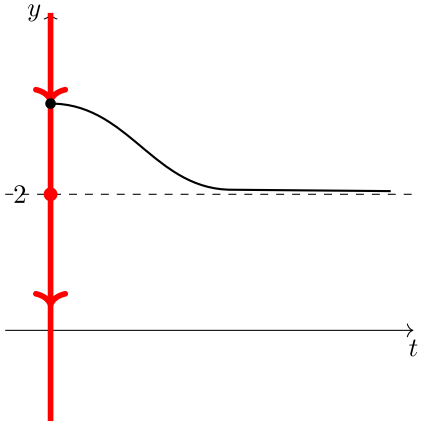
(c) (2 points) Draw a phase diagram when $a = 0$ and sketch the solution if $y(0) = -1$.

Solution:



(d) (2 points) Draw a phase diagram when $a = e^2 - 1$ and sketch the solution if $y(0) = 3$.

Solution:



(e) (2 points) The differential equation above has an a equilibrium solution of $y = 2$ for any value of a. For what a is this equilibrium stable?

Solution: It is stable before $a = e^y - 1$ intersects with $y = 2$, i.e. when $a = e^2 - 1$. That is, the equilibrium is stable when $a < e^2 - 1$ (by inspection it is semistable when $a = e^2 - 1$).

6. A solution of 4 mg/L of chlorine is being pumped into a tank at a rate of 3 L/hour. It is known that chlorine will *degas* from the water (it leaves the water) with a half-life of $4 \ln 2$ hours.

   (a) (4 points) Write a differential equation modelling the total amount of chlorine in the tank at time t.

   Solution: Let $y(t)$ be the amount of chlorine at time t. There is 12 mg/hour entering the tank. Due to half-life there is $y/4$ mg/hour leaving the tank. Thus

   $$\frac{\mathrm{d}y}{\mathrm{d}t} = 12 - \frac{1}{4}y.$$

   (b) (3 points) If, initially, the tank contains only water with no chlorine solve the differential equation.

   Solution:

   $$y(t) = 48 - Ce^{-\frac{1}{4}t}$$

   and since $y(0) = 0$ we have $C = 12$. So

   $$y(t) = 48\left(1 - e^{-t/4}\right).$$

(c) (1 point) How many grams of the chlorine is in the tank after 24 hours? You may leave your answer in terms of e.

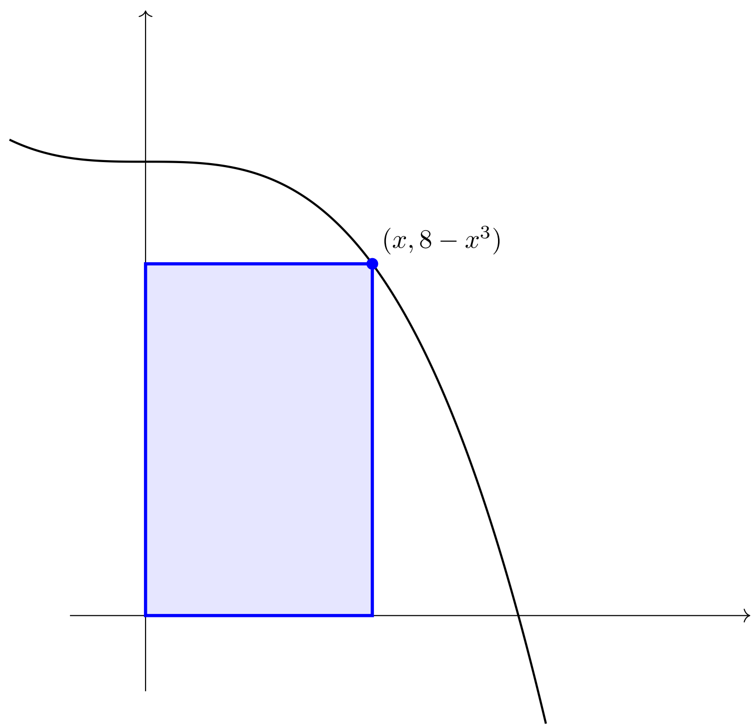
> **Solution:** When $t = 24$ there are
> $$y(12) = 48\left(1 - e^{-6}\right) \text{ milligrams.}$$

(d) (4 points) At the 24 hour mark, a tap is turned and the tank starts draining at a rate of 3 L/hour (the incoming solution continues as well). If we assume that the tank contains exactly 100 litres of water at the 24 hour mark, what is the amount of chlorine in the tank in the long term?

> **Solution:** Let $z(t)$ be the amount of chlorine in the tank t hours after the tap in turned on. We still have 12 mg/hour coming into the tank and $z/4$ mg/hour leaving the tank due to the half life. In addition to this we also have 3 L/hour leaving the tank. Each litre contains $z/100$ mg and thus we have $3z/100$ mg/hour leaving the tank through the tap. Thus we have
> $$\frac{dz}{dt} = 12 - \left(\frac{1}{4} + \frac{3}{100}\right) y = 12 - \frac{7}{25} y.$$
> This has a single stable equilibrium so in the long term the solution will approach this value which is
> $$\frac{300}{7} \text{ grams.}$$

7. (10 points) What is the area of the largest rectangle that fits in the region bounded by the x-axis, the y-axis and the curve $y = 8 - x^3$?

Solution: The situation we are interested in is illustrated in the following picture.



The width of the rectangle is x and the height is $8 - x^3$. Thus the area is $A(x) = 8x - x^4$. We want to find the critical points, the derivative is $A'(x) = 8 - 4x^3$ which is zero when $x^3 = 2$, i.e. when $x = \sqrt[3]{2}$. The function $8 - x^3$ crosses the x axis when $x = 2$. Thus the domain of A is $x \in [0, 2]$. This is a closed interval. Since $A(0) = A(2) = 0$ we have that $A(\sqrt[3]{2}) = 6\sqrt[3]{2}$ must be the maximum area.

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