

# Math 3B: Lecture 19

Noah White

November 15, 2017

# Linear models

## Definition

A first order ODE is **linear** if it is of the form

$$\frac{dy}{dt} = a + by$$

for constants  $a$  and  $b$ .

# Linear models

## Definition

A first order ODE is **linear** if it is of the form

$$\frac{dy}{dt} = a + by$$

for constants  $a$  and  $b$ .

# Linear models

## Definition

A first order ODE is **linear** if it is of the form

$$\frac{dy}{dt} = a + by$$

for constants  $a$  and  $b$ . In other words, the right hand side is a linear function of  $y$ .

# Linear models

## Definition

A first order ODE is **linear** if it is of the form

$$\frac{dy}{dt} = a + by$$

for constants  $a$  and  $b$ . In other words, the right hand side is a linear function of  $y$ .

## Examples

$$\frac{dy}{dt} = ay, \quad \frac{dy}{dt} = -\lambda y.$$

# Mixing models

A mixing model describes the concentration of something over time, if

- rate in is constant

so

# Mixing models

A mixing model describes the concentration of something over time, if

- rate in is constant
- rate out is proportional to concentration

so

# Mixing models

A mixing model describes the concentration of something over time, if

- rate in is constant
- rate out is proportional to concentration

so



# Mixing models

A mixing model describes the concentration of something over time, if

- rate in is constant
- rate out is proportional to concentration

so

$$\begin{aligned}\frac{dy}{dt} &= \text{rate in} - \text{rate out} \\ &= a - by\end{aligned}$$

# Mixing models

A mixing model describes the concentration of something over time, if

- rate in is constant
- rate out is proportional to concentration

so

$$\begin{aligned}\frac{dy}{dt} &= \text{rate in} - \text{rate out} \\ &= a - by\end{aligned}$$

Note

Something could mean (for example)

# Mixing models

A mixing model describes the concentration of **something** over time, if

- rate in is constant
- rate out is proportional to concentration

so

$$\begin{aligned}\frac{dy}{dt} &= \text{rate in} - \text{rate out} \\ &= a - by\end{aligned}$$

## Note

**Something** could mean (for example)

- concentration of a drug in bloodstream

# Mixing models

A mixing model describes the concentration of **something** over time, if

- rate in is constant
- rate out is proportional to concentration

so

$$\begin{aligned}\frac{dy}{dt} &= \text{rate in} - \text{rate out} \\ &= a - by\end{aligned}$$

## Note

**Something** could mean (for example)

- concentration of a drug in bloodstream
- pollutant in water supply

## General solution

Using separation of variables, we can show that the general solution to

$$\frac{dy}{dt} = a - by$$

is

$$y(t) = \frac{a}{b} - Ce^{-bt}$$

where  $C$  is an arbitrary constant.

## Example 1

A drug with a half-life of 2 hours is injected into the bloodstream with an infusion rate of 10 mg/h. Determine the concentration  $y(t)$  at time  $t$ .

Solution

## Example 1

A drug with a half-life of 2 hours is injected into the bloodstream with an infusion rate of 10 mg/h. Determine the concentration  $y(t)$  at time  $t$ .

### Solution

- Ignoring infusion, every 2 hours the amount of drug halves.

## Example 1

A drug with a half-life of 2 hours is injected into the bloodstream with an **infusion rate** of 10 mg/h. Determine the concentration  $y(t)$  at time  $t$ .

### Solution

- Ignoring infusion, every 2 hours the amount of drug halves.
- Starting with  $M$  mg, after  $t$  hours there will be

$$M \left( \frac{1}{2} \right)^{t/2} = M e^{-0.5t \ln(2)} \text{ mg left}$$



## Example 1

A drug with a half-life of 2 hours is injected into the bloodstream with an **infusion rate** of 10 mg/h. Determine the concentration  $y(t)$  at time  $t$ .

### Solution

- Ignoring infusion, every 2 hours the amount of drug halves.
- Starting with  $M$  mg, after  $t$  hours there will be

$$M \left( \frac{1}{2} \right)^{t/2} = Me^{-0.5t \ln(2)} \text{ mg left}$$

- Thus the rate at which the drug is leaving (at time  $t$ ) is given by

$$0.5 \ln(2) Me^{-0.5t \ln(2)} = 0.5 \ln(2)(\text{current concentration}) \text{ mg/h.}$$

## Example 1

- If we infuse the drug at a rate of 10 mg/h we have

$$\frac{dy}{dt} = 10 - 0.5 \ln(2)y$$

## Example 1

- If we infuse the drug at a rate of 10 mg/h we have

$$\frac{dy}{dt} = 10 - 0.5 \ln(2)y$$

- The general solution to this is

$$y(t) = \frac{10}{0.5 \ln(2)} - Ce^{-0.5 \ln(2)t}.$$

## Example 1

- If we infuse the drug at a rate of 10 mg/h we have

$$\frac{dy}{dt} = 10 - 0.5 \ln(2)y$$

- The general solution to this is

$$y(t) = \frac{10}{0.5 \ln(2)} - Ce^{-0.5 \ln(2)t}.$$

- Since there was initially no drug in the bloodstream,  $y(0) = 0$ ,

$$0 = \frac{20}{\ln(2)} - C \approx 28.9 - C$$

## Example 1

- If we infuse the drug at a rate of 10 mg/h we have

$$\frac{dy}{dt} = 10 - 0.5 \ln(2)y$$

- The general solution to this is

$$y(t) = \frac{10}{0.5 \ln(2)} - Ce^{-0.5 \ln(2)t}.$$

- Since there was initially no drug in the bloodstream,  $y(0) = 0$ ,

$$0 = \frac{20}{\ln(2)} - C \approx 28.9 - C$$

- Thus at time  $t$  the concentration is

$$y(t) = 28.9 - 28.9e^{-0.3t} = 28.9(1 - e^{-0.3t})$$

# Newton's Law of Cooling

Isaac Newton stated that

## Newton's Law of Cooling

The temperature  $T$  of a body changes at a rate proportional to the difference between the ambient temperature  $A$  and  $T$ .

# Newton's Law of Cooling

Isaac Newton stated that

## Newton's Law of Cooling

The temperature  $T$  of a body changes at a rate proportional to the difference between the ambient temperature  $A$  and  $T$ .

# Newton's Law of Cooling

Isaac Newton stated that

## Newton's Law of Cooling

The temperature  $T$  of a body changes at a rate proportional to the difference between the ambient temperature  $A$  and  $T$ .

$$\frac{dT}{dt} = k(A - T)$$

## General solution

$$T(t) = A - Ce^{-kt}.$$



## Example 2

An object takes 20 minutes to cool from  $90^{\circ}$  to  $86^{\circ}$  in a room which is  $70^{\circ}$ . At what time will it be  $75^{\circ}$ ?

Solution

## Example 2

An object takes 20 minutes to cool from  $90^\circ$  to  $86^\circ$  in a room which is  $70^\circ$ . At what time will it be  $75^\circ$ ?

### Solution

- The temp is described by the equation

$$\frac{dT}{dt} = k(70 - T).$$

## Example 2

An object takes 20 minutes to cool from  $90^\circ$  to  $86^\circ$  in a room which is  $70^\circ$ . At what time will it be  $75^\circ$ ?

### Solution

- The temp is described by the equation

$$\frac{dT}{dt} = k(70 - T).$$

- The solution is given by

$$T(t) = 70 - Ce^{-kt}$$

## Example 2

An object takes 20 minutes to cool from  $90^\circ$  to  $86^\circ$  in a room which is  $70^\circ$ . At what time will it be  $75^\circ$ ?

### Solution

- The temp is described by the equation

$$\frac{dT}{dt} = k(70 - T).$$

- The solution is given by

$$T(t) = 70 - Ce^{-kt}$$

- We know  $T(0) = 90$  and  $T(20) = 86$ .

## Example 2

An object takes 20 minutes to cool from  $90^\circ$  to  $86^\circ$  in a room which is  $70^\circ$ . At what time will it be  $75^\circ$ ?

### Solution

- The temp is described by the equation

$$\frac{dT}{dt} = k(70 - T).$$

- The solution is given by

$$T(t) = 70 - Ce^{-kt}$$

- We know  $T(0) = 90$  and  $T(20) = 86$ .
- Thus

$$90 = 70 - C \quad \text{so} \quad C = -20.$$

## Example 2

- $T(t) = 70 + 20e^{-kt}$

## Example 2

- $T(t) = 70 + 20e^{-kt}$
- To find  $k$  we use  $T(20) = 86$

$$86 = 70 + 20e^{-20k}$$

## Example 2

- $T(t) = 70 + 20e^{-kt}$
- To find  $k$  we use  $T(20) = 86$

$$86 = 70 + 20e^{-20k}$$

- Thus

$$e^{-20k} = \frac{86 - 70}{20} = \frac{4}{5} \quad \text{so} \quad k = -\frac{1}{20} \ln \left( \frac{4}{5} \right) \approx -0.01.$$



## Example 2

- $T(t) = 70 + 20e^{-kt}$
- To find  $k$  we use  $T(20) = 86$

$$86 = 70 + 20e^{-20k}$$

- Thus

$$e^{-20k} = \frac{86 - 70}{20} = \frac{4}{5} \quad \text{so} \quad k = -\frac{1}{20} \ln \left( \frac{4}{5} \right) \approx -0.01.$$

- The model is thus  $T(t) = 70 + 20e^{-0.01t}$ . We want to solve

$$75 = 70 + 20e^{-0.01t}.$$

## Example 2

- $T(t) = 70 + 20e^{-kt}$
- To find  $k$  we use  $T(20) = 86$

$$86 = 70 + 20e^{-20k}$$

- Thus

$$e^{-20k} = \frac{86 - 70}{20} = \frac{4}{5} \quad \text{so} \quad k = -\frac{1}{20} \ln \left( \frac{4}{5} \right) \approx -0.01.$$

- The model is thus  $T(t) = 70 + 20e^{-0.01t}$ . We want to solve

$$75 = 70 + 20e^{-0.01t}.$$

- Rearranging we get  $20e^{-0.01t} = 5$  i.e.

## Example 2

- $20e^{-0.01t} = 5$  becomes

$$e^{-0.01t} = \frac{1}{4}$$

## Example 2

- $20e^{-0.01t} = 5$  becomes

$$e^{-0.01t} = \frac{1}{4}$$

- Applying a logarithm

$$-0.01t = \ln\left(\frac{1}{4}\right)$$

## Example 2

- $20e^{-0.01t} = 5$  becomes

$$e^{-0.01t} = \frac{1}{4}$$

- Applying a logarithm

$$-0.01t = \ln\left(\frac{1}{4}\right)$$

- So we get

$$t = -100 \ln\left(\frac{1}{4}\right) \approx 138 = 2 \text{ hours } 18 \text{ minutes.}$$