

# Math 3B: Lecture 13

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# How to factorize polynomials

The normal method for factorizing a polynomial  $p(x)$  is to find a root  $\alpha$  and then writing

$$p(x) = q(x)(x - \alpha).$$

# How to factorize polynomials

The normal method for factorizing a polynomial  $p(x)$  is to find a root  $\alpha$  and then writing

$$p(x) = q(x)(x - \alpha).$$

What if we want to divide a polynomial  $p(x)$  by another polynomial  $q(x)$ ? We want to write

$$p(x) = q(x)d(x) + r(x)$$

for a polynomial  $d(x)$  (the **divisor**) and a **remainder**  $r(x)$ .

# Long division

We know how to do this with numbers! Long division.

$$\begin{array}{r} 176 \\ 34 \overline{) 6000} \\ \underline{3400} \\ 2600 \\ \underline{2380} \\ 220 \\ \underline{204} \\ 16 \end{array}$$

## Long division

We know how to do this with numbers! Long division.

$$\begin{array}{r} 176 \\ 34 \overline{) 6000} \\ \underline{3400} \\ 2600 \\ \underline{2380} \\ 220 \\ \underline{204} \\ 16 \end{array}$$

$$\text{So } 6000 = 34 \cdot 176 + 16$$

## Why?

Lets rewrite the equation  $p(x) = q(x)d(x) + r(x)$

$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

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$$\frac{p(x)}{q(x)} = d(x) + \frac{r(x)}{q(x)}.$$

E.g.

$$\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 1} = x + \frac{1}{(x + 1)^2}.$$

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Lets rewrite the equation  $p(x) = q(x)d(x) + r(x)$

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E.g.

$$\frac{x^3 + 2x^2 + x + 1}{x^2 + 2x + 1} = x + \frac{1}{(x + 1)^2}.$$

The left hand side is difficult to integrate. The right hand side is easy!



How?

$$x + 3 \overline{) x^2 + 5x + 4}$$

How?

$$\begin{array}{r} x \\ x+3 \overline{) x^2 + 5x + 4} \end{array}$$

How?

$$\begin{array}{r} \phantom{x+3}\overline{\phantom{x+3}x} \\ x+3 \overline{) x^2 + 5x + 4} \\ \underline{-x^2 - 3x} \phantom{+4} \end{array}$$

## How?

$$\begin{array}{r} x \\ x+3 \overline{) x^2 + 5x + 4} \\ \underline{-x^2 - 3x} \phantom{+ 4} \\ 2x + 4 \end{array}$$

## How?

$$\begin{array}{r} x+2 \\ x+3 \overline{) x^2+5x+4} \\ \underline{-x^2-3x} \phantom{+4} \\ 2x+4 \end{array}$$

How?

$$\begin{array}{r} \phantom{x+3)} \phantom{x^2+} x+2 \\ \hline x+3) \phantom{x^2+} x^2+5x+4 \\ \phantom{x+3)} \underline{-x^2-3x} \phantom{+4} \\ \phantom{x+3)} \phantom{x^2+} 2x+4 \\ \phantom{x+3)} \phantom{x^2+} \underline{-2x-6} \end{array}$$

## How?

$$\begin{array}{r} \phantom{x+3)} \phantom{x^2+} x+2 \\ x+3 \overline{) x^2+5x+4} \\ \underline{-x^2-3x} \phantom{4} \\ 2x+4 \\ \underline{-2x-6} \\ -2 \end{array}$$

## How?

$$\begin{array}{r} \phantom{x+3) } \phantom{x^2+} x+2 \\ x+3 \overline{) x^2+5x+4} \\ \underline{-x^2-3x} \phantom{+4} \\ \phantom{x+3) } 2x+4 \\ \underline{-2x-6} \\ \phantom{x+3) } \phantom{2x+} -2 \end{array}$$

So  $\frac{x^2 + 5x + 4}{x + 3} = x + 2 - \frac{2}{x + 3}$ .



## Example 1

$$x - 3 \overline{) x^3 - 12x^2 - 42}$$

## Example 1

$$\begin{array}{r} x^2 \\ x-3 \overline{) x^3 - 12x^2 \phantom{- 42} } \\ \phantom{x^3} - 12x^2 \phantom{- 42} \\ \phantom{x^3} \phantom{- 12x^2} + 42 \phantom{0} \\ \phantom{x^3} \phantom{- 12x^2} \phantom{+ 42} - 42 \\ \phantom{x^3} \phantom{- 12x^2} \phantom{+ 42} \phantom{- 42} 0 \end{array}$$

## Example 1

$$\begin{array}{r} x^2 \\ x-3 \overline{) x^3 - 12x^2 \phantom{+ 0x} - 42} \\ \underline{-x^3 \phantom{- 12x^2} + 3x^2} \phantom{- 42} \end{array}$$

## Example 1

$$\begin{array}{r} x^2 \\ x-3 \overline{) x^3 - 12x^2 \phantom{- 42} } \\ \underline{-x^3 + 3x^2} \phantom{- 42} \\ -9x^2 \phantom{- 42} \end{array}$$

## Example 1

$$\begin{array}{r} x^2 - 9x \\ x-3 \overline{) x^3 - 12x^2 \phantom{- 9x} - 42} \\ \underline{-x^3 + 3x^2} \phantom{- 42} \\ -9x^2 \phantom{- 42} \end{array}$$

## Example 1

$$\begin{array}{r} x^2 - 9x \\ x - 3 \overline{) x^3 - 12x^2 \phantom{00} - 42} \\ \underline{-x^3 + 3x^2} \phantom{00} \\ -9x^2 \phantom{00} - 42 \\ \underline{9x^2 - 27x} \phantom{00} \\ -42 \phantom{00} \end{array}$$

## Example 1

$$\begin{array}{r} x^2 - 9x \\ x - 3 \overline{) x^3 - 12x^2 \phantom{00} - 42} \\ \underline{-x^3 + 3x^2} \phantom{00} \\ -9x^2 \phantom{00} - 42 \\ \underline{9x^2 - 27x} \phantom{00} \\ -27x - 42 \end{array}$$

## Example 1

$$\begin{array}{r} x^2 - 9x - 27 \\ x - 3 \overline{) x^3 - 12x^2 \phantom{- 27x} - 42} \\ \underline{-x^3 + 3x^2} \phantom{- 42} \\ -9x^2 \phantom{- 27x} - 42 \\ \underline{9x^2 - 27x} \phantom{- 42} \\ -27x - 42 \end{array}$$



## Example 1

$$\begin{array}{r}
 x^2 - 9x - 27 \\
 x-3 \overline{) x^3 - 12x^2 \phantom{- 27x} - 42} \\
 \underline{-x^3 + 3x^2} \phantom{- 27x} \\
 -9x^2 \phantom{- 27x} \\
 \underline{9x^2 - 27x} \phantom{- 42} \\
 -27x - 42 \\
 \underline{27x} \phantom{- 42} \\
 -81
 \end{array}$$

## Example 1

$$\begin{array}{r} x^2 - 9x - 27 \\ x - 3 \overline{) x^3 - 12x^2 \phantom{- 27x} - 42} \\ \underline{-x^3 + 3x^2} \phantom{- 42} \\ -9x^2 \phantom{- 42} \\ \underline{9x^2 - 27x} \phantom{- 42} \\ -27x - 42 \\ \underline{27x - 81} \\ -123 \end{array}$$

## Example 1

$$\begin{array}{r} x^2 - 9x - 27 \\ x - 3 \overline{) x^3 - 12x^2 \phantom{- 27x} - 42} \\ \underline{-x^3 + 3x^2} \phantom{- 42} \\ -9x^2 \phantom{- 27x} - 42 \\ \underline{9x^2 - 27x} \phantom{- 42} \\ -27x - 42 \\ \underline{27x - 81} \\ -123 \end{array}$$

$$\text{So } \frac{x^3 - 12x^2 - 42}{x - 3} = x^2 - 9x - 27 - \frac{123}{x - 3}.$$

## Example 2

$$x^2 + 1 \overline{) x^3 - x^2 + x - 1}$$

## Example 2

$$x^2 + 1 \overline{) \begin{array}{r} x^3 - x^2 + x - 1 \\ x \end{array}}$$

## Example 2

$$\begin{array}{r} \phantom{x^2 + 1) } \phantom{x^3 - x^2 + } x \\ \phantom{x^2 + 1) } \hline x^2 + 1) \phantom{x^3 - } x^3 - x^2 + x - 1 \\ \phantom{x^2 + 1) } \underline{- x^3} \phantom{+ x} - x \\ \phantom{x^2 + 1) } \phantom{x^3 - x^2 + } \phantom{x} - x \end{array}$$

## Example 2

$$\begin{array}{r} \phantom{x^2 + 1) } \phantom{x^3 - } x^2 + x - 1 \\ x^2 + 1 \overline{) \phantom{x^3 - } x^3 - x^2 + x - 1} \\ \underline{- x^3 \phantom{+ x - 1}} \phantom{- x^2 + } x - 1 \\ \phantom{- x^3 + } \underline{- x^2} \phantom{+ x - 1} - 1 \end{array}$$

## Example 2

$$\begin{array}{r} \phantom{x^2 + 1) } \phantom{x^3 - } x - 1 \\ \hline x^2 + 1) \phantom{x^3 - } x^3 - x^2 + x - 1 \\ \phantom{x^2 + 1) } \underline{- x^3} \phantom{+ x} - x \\ \phantom{x^2 + 1) } \phantom{- x^3} \phantom{+ x} \underline{- x^2} - 1 \end{array}$$



## Example 2

$$\begin{array}{r} \phantom{x^2 + 1) \phantom{x^3 - x^2 + x - 1}} \phantom{x^2 + 1) \phantom{x^3 - x^2 + x - 1}} x - 1 \\ \hline x^2 + 1) \phantom{x^2 + 1) \phantom{x^3 - x^2 + x - 1}} x^3 - x^2 + x - 1 \\ \phantom{x^2 + 1) \phantom{x^2 + 1) \phantom{x^3 - x^2 + x - 1}} - x^3 \phantom{- x} \\ \hline \phantom{x^2 + 1) \phantom{x^2 + 1) \phantom{x^3 - x^2 + x - 1}} - x^2 \phantom{- 1} \\ \phantom{x^2 + 1) \phantom{x^2 + 1) \phantom{x^3 - x^2 + x - 1}} x^2 \phantom{+ 1} \\ \hline \phantom{x^2 + 1) \phantom{x^2 + 1) \phantom{x^3 - x^2 + x - 1}} \phantom{- 1} + 1 \end{array}$$

## Example 2

$$\begin{array}{r} \phantom{x^2 + 1) } \phantom{x^3 - } x - 1 \\ \hline x^2 + 1) \phantom{x^3 - } x^3 - x^2 + x - 1 \\ \phantom{x^2 + 1) } \underline{- x^3} \phantom{+ x} - x \\ \phantom{x^2 + 1) } \phantom{- x^3} \phantom{+ x} - 1 \\ \phantom{x^2 + 1) } \phantom{- x^3} \phantom{+ x} \underline{x^2} \phantom{- 1} + 1 \\ \phantom{x^2 + 1) } \phantom{- x^3} \phantom{+ x} \phantom{- 1} \phantom{+ 1} \underline{\phantom{0}} \\ \phantom{x^2 + 1) } \phantom{- x^3} \phantom{+ x} \phantom{- 1} \phantom{+ 1} 0 \end{array}$$

## Example 2

$$\begin{array}{r} \phantom{x^2 + 1) } \phantom{x^3 - x^2 + x - 1} x - 1 \\ \hline x^2 + 1) \phantom{x^3 - } x^3 - x^2 + x - 1 \\ \phantom{x^2 + 1) } \underline{- x^3} \phantom{+ x - 1} - x \\ \phantom{x^2 + 1) } \phantom{- x^3} \phantom{- x} - x^2 \phantom{- 1} \\ \phantom{x^2 + 1) } \phantom{- x^3} \phantom{- x} \phantom{- x^2} \underline{+ 1} \\ \phantom{x^2 + 1) } \phantom{- x^3} \phantom{- x} \phantom{- x^2} \phantom{+ 1} 0 \end{array}$$

So  $\frac{x^3 - x^2 + x - 1}{x^2 + 1} = x - 1.$

### Example 3

$$x^2 + x + 1 \overline{) x^3 \phantom{00} - 1}$$

### Example 3

$$\begin{array}{r} x^2 + x + 1 \overline{) x^3 \phantom{00} - 1} \end{array}$$

## Example 3

$$\begin{array}{r} x^2 + x + 1 \overline{) \begin{array}{r} x^3 \phantom{- x^2 - x} - 1 \\ - x^3 - x^2 - x \\ \hline \end{array}} \end{array}$$

### Example 3

$$\begin{array}{r} x^2 + x + 1 \overline{) \begin{array}{r} x^3 \phantom{- x^2 - x - 1} \\ - x^3 - x^2 - x \phantom{- 1} \\ \hline - x^2 - x - 1 \end{array}} \end{array}$$

### Example 3

$$\begin{array}{r} x^2 + x + 1 \overline{) \begin{array}{r} x^3 \phantom{- x^2 - x - 1} \\ - x^3 - x^2 - x \phantom{- 1} \\ \hline - x^2 - x - 1 \end{array}} \end{array}$$



### Example 3

$$\begin{array}{r} x^2 + x + 1 \overline{) \begin{array}{r} x^3 \phantom{- x^2 - x - 1} \\ - x^3 - x^2 - x \phantom{- 1} \\ \hline - x^2 - x - 1 \\ x^2 + x + 1 \\ \hline \end{array}} \end{array}$$

### Example 3

$$\begin{array}{r} x^2 + x + 1 \overline{) \begin{array}{r} x^3 \phantom{- x^2 - x - 1} \\ - x^3 - x^2 - x \phantom{- 1} \\ \hline - x^2 - x - 1 \\ x^2 + x + 1 \\ \hline 0 \end{array}} \end{array}$$

### Example 3

$$\begin{array}{r} x^2 + x + 1 \overline{) \begin{array}{r} x^3 \phantom{- x^2 - x - 1} \\ - x^3 - x^2 - x \phantom{- 1} \\ \hline - x^2 - x - 1 \\ x^2 + x + 1 \\ \hline 0 \end{array}} \end{array}$$

$$\text{So } \frac{x^3 - 1}{x^2 + x + 1} = x - 1.$$

## Example 4

$$3x - 1 \overline{) 2x^3 - 4x^2 + 1}$$

## Example 4

$$\begin{array}{r} \phantom{3x-1)} \phantom{2x^3} \phantom{-4x^2} \phantom{+1} \phantom{+1} \\ \phantom{3x-1)} \phantom{2x^3} \phantom{-4x^2} \phantom{+1} \phantom{+1} \\ \hline 3x-1) \phantom{2x^3} \phantom{-4x^2} \phantom{+1} \phantom{+1} \phantom{+1} \\ \phantom{3x-1)} \phantom{2x^3} \phantom{-4x^2} \phantom{+1} \phantom{+1} \phantom{+1} \\ \phantom{3x-1)} \phantom{2x^3} \phantom{-4x^2} \phantom{+1} \phantom{+1} \phantom{+1} \end{array}$$

## Example 4

$$\begin{array}{r} \phantom{3x-1)} \phantom{2x^3} - \frac{2}{3}x^2 \\ \underline{3x-1) \phantom{2x^3} - 4x^2} \phantom{+1} \\ \phantom{3x-1)} -2x^3 + \frac{2}{3}x^2 \end{array}$$

## Example 4

$$\begin{array}{r} \phantom{3x-1)} \phantom{2x^3} \phantom{-} \frac{2}{3}x^2 \\ \hline 3x-1) \phantom{2x^3} -4x^2 \phantom{+} \phantom{\frac{2}{3}x^2} +1 \\ \phantom{3x-1)} \underline{-2x^3 + \frac{2}{3}x^2} \phantom{+1} \\ \phantom{3x-1)} \phantom{-2x^3} -\frac{10}{3}x^2 \phantom{+1} \end{array}$$

## Example 4

$$\begin{array}{r} \phantom{3x-1)} \phantom{2x^3} - \frac{2}{3}x^2 - \frac{10}{9}x \\ \underline{\phantom{3x-1)} 2x^3 - 4x^2} \phantom{+ 1} \\ \phantom{3x-1)} -2x^3 + \frac{2}{3}x^2 \\ \underline{\phantom{3x-1)} -2x^3 + \frac{2}{3}x^2} \\ \phantom{3x-1)} \phantom{-2x^3} - \frac{10}{3}x^2 \end{array}$$



## Example 4

$$\begin{array}{r} \phantom{3x-1)} \phantom{2x^3-4x^2} \frac{\frac{2}{3}x^2 - \frac{10}{9}x}{\phantom{2x^3-4x^2}} \\ 3x-1) \phantom{2x^3-4x^2} \phantom{+1} \\ \underline{-2x^3 + \frac{2}{3}x^2} \phantom{+1} \\ \phantom{3x-1)} \phantom{2x^3-4x^2} -\frac{10}{3}x^2 \\ \phantom{3x-1)} \phantom{2x^3-4x^2} \underline{\frac{10}{3}x^2 - \frac{10}{9}x} \end{array}$$

## Example 4

$$\begin{array}{r}
 \frac{2}{3}x^2 - \frac{10}{9}x \\
 \hline
 3x - 1 \bigg) \quad 2x^3 - 4x^2 \qquad \qquad + 1 \\
 \underline{- 2x^3 + \frac{2}{3}x^2} \qquad \qquad \qquad \\
 \qquad \qquad - \frac{10}{3}x^2 \\
 \qquad \qquad \underline{\frac{10}{3}x^2 - \frac{10}{9}x} \\
 \qquad \qquad \qquad - \frac{10}{9}x + 1
 \end{array}$$

## Example 4

$$\begin{array}{r}
 \frac{2}{3}x^2 - \frac{10}{9}x - \frac{10}{27} \\
 \hline
 3x - 1 \big) \quad 2x^3 - 4x^2 \qquad \qquad + 1 \\
 \underline{- 2x^3 + \frac{2}{3}x^2} \qquad \qquad \qquad \\
 \qquad \qquad - \frac{10}{3}x^2 \\
 \qquad \qquad \underline{\frac{10}{3}x^2 - \frac{10}{9}x} \\
 \qquad \qquad \qquad \qquad - \frac{10}{9}x + 1
 \end{array}$$

## Example 4

$$\begin{array}{r} \phantom{3x-1)} \phantom{2x^3-4x^2} \frac{2}{3}x^2 - \frac{10}{9}x - \frac{10}{27} \\ 3x-1) \phantom{2x^3-4x^2} \phantom{2x^3-4x^2} + 1 \\ \underline{-2x^3 + \frac{2}{3}x^2} \phantom{+ 1} \\ \phantom{3x-1)} \phantom{2x^3-4x^2} -\frac{10}{3}x^2 \\ \phantom{3x-1)} \phantom{2x^3-4x^2} \underline{\frac{10}{3}x^2 - \frac{10}{9}x} \\ \phantom{3x-1)} \phantom{2x^3-4x^2} \phantom{\frac{10}{3}x^2 - \frac{10}{9}x} -\frac{10}{9}x + 1 \\ \phantom{3x-1)} \phantom{2x^3-4x^2} \phantom{\frac{10}{3}x^2 - \frac{10}{9}x} \underline{\frac{10}{9}x - \frac{10}{27}} \end{array}$$

## Example 4

$$\begin{array}{r} \phantom{3x-1)} \phantom{2x^3-4x^2} \frac{2}{3}x^2 - \frac{10}{9}x - \frac{10}{27} \\ 3x-1) \phantom{2x^3-4x^2} \phantom{+1} \\ \underline{-2x^3 + \frac{2}{3}x^2} \phantom{+1} \\ \phantom{3x-1)} \phantom{2x^3-4x^2} -\frac{10}{3}x^2 \\ \phantom{3x-1)} \phantom{2x^3-4x^2} \underline{\frac{10}{3}x^2 - \frac{10}{9}x} \\ \phantom{3x-1)} \phantom{2x^3-4x^2} \phantom{+1} -\frac{10}{9}x + 1 \\ \phantom{3x-1)} \phantom{2x^3-4x^2} \phantom{+1} \underline{\frac{10}{9}x - \frac{10}{27}} \\ \phantom{3x-1)} \phantom{2x^3-4x^2} \phantom{+1} \phantom{+1} \frac{17}{27} \end{array}$$

## Example 4

$$\begin{array}{r}
 \frac{2}{3}x^2 - \frac{10}{9}x - \frac{10}{27} \\
 \hline
 3x - 1) \quad 2x^3 - 4x^2 + 1 \\
 \underline{- 2x^3 + \frac{2}{3}x^2} \phantom{+ 1} \\
 \phantom{3x - 1)} \quad -\frac{10}{3}x^2 \\
 \phantom{3x - 1)} \quad \underline{\frac{10}{3}x^2 - \frac{10}{9}x} \\
 \phantom{3x - 1)} \phantom{\frac{10}{3}x^2 -} -\frac{10}{9}x + 1 \\
 \phantom{3x - 1)} \phantom{\frac{10}{3}x^2 -} \quad \underline{\frac{10}{9}x - \frac{10}{27}} \\
 \phantom{3x - 1)} \phantom{\frac{10}{3}x^2 -} \phantom{-\frac{10}{9}x +} \frac{17}{27}
 \end{array}$$

$$\text{So } \frac{2x^3 - 4x^2 + 1}{3x - 1} = \frac{2}{3} \left( x^2 - \frac{5}{3}x - \frac{5}{9} \right) - \frac{17}{27(3x - 1)}.$$

## Example 5

$$\begin{array}{r} x^2 - 2x + 5 \overline{) x^4 \phantom{- 2x^3} - x^2 \phantom{+ x} - 4} \end{array}$$

## Example 5

$$\begin{array}{r} x^2 \\ x^2 - 2x + 5 \overline{) x^4 \phantom{- x^3} - x^2 + x - 4} \end{array}$$



## Example 5

$$\begin{array}{r} \phantom{x^2 - 2x + 5) } \phantom{x^4} \phantom{-} x^2 \phantom{+ x} \phantom{- 4} \\ x^2 - 2x + 5 \overline{) \phantom{x^4} \phantom{-} x^2 \phantom{+ x} \phantom{- 4}} \\ \underline{-x^4 + 2x^3 - 5x^2} \phantom{+ x} \phantom{- 4} \end{array}$$

## Example 5

$$\begin{array}{r} x^2 \\ x^2 - 2x + 5 \overline{) \quad x^4 \quad \quad - x^2 \quad + x \quad - 4} \\ \underline{-x^4 + 2x^3 - 5x^2} \phantom{+ x - 4} \\ 2x^3 - 6x^2 \phantom{+ x} \end{array}$$

## Example 5

$$\begin{array}{r} x^2 - 2x + 5 \overline{) \begin{array}{r} x^4 \phantom{+ 2x^3} - x^2 + x - 4 \\ - x^4 + 2x^3 - 5x^2 \\ \hline 2x^3 - 6x^2 + x \end{array}} \end{array}$$

## Example 5

$$\begin{array}{r} x^2 - 2x + 5 \overline{) x^4 - x^2 + x - 4} \\ \underline{-x^4 + 2x^3 - 5x^2} \phantom{+ x - 4} \\ 2x^3 - 6x^2 + x \phantom{- 4} \\ \underline{-2x^3 + 4x^2 - 10x} \phantom{- 4} \\ 10x^2 - 9x - 4 \end{array}$$

## Example 5

$$\begin{array}{r} \phantom{x^2 - 2x + 5) } \phantom{x^4} \phantom{- x^2} \phantom{+ x} \phantom{- 4} x^2 + 2x \\ x^2 - 2x + 5 \overline{) \phantom{x^4} x^4 \phantom{- x^2} \phantom{+ x} \phantom{- 4}} \\ \underline{- x^4 + 2x^3 - 5x^2} \phantom{+ x} \phantom{- 4} \\ \phantom{x^2 - 2x + 5) } 2x^3 - 6x^2 \phantom{+ x} \phantom{- 4} \\ \underline{- 2x^3 + 4x^2 - 10x} \phantom{- 4} \\ \phantom{x^2 - 2x + 5) } \phantom{2x^3} - 2x^2 - 9x - 4 \end{array}$$

## Example 5

$$\begin{array}{r} \phantom{x^2 - 2x + 5) } \phantom{x^4} \phantom{- x^2} \phantom{+ x} \phantom{- 4} x^2 + 2x - 2 \\ x^2 - 2x + 5) \phantom{x^4} \phantom{- x^2} \phantom{+ x} \phantom{- 4} \\ \underline{- x^4 + 2x^3 - 5x^2} \phantom{+ x} \phantom{- 4} \\ \phantom{x^2 - 2x + 5) } 2x^3 - 6x^2 \phantom{+ x} \phantom{- 4} \\ \underline{- 2x^3 + 4x^2 - 10x} \phantom{- 4} \\ \phantom{x^2 - 2x + 5) } \phantom{2x^3 - 6x^2} - 2x^2 - 9x - 4 \end{array}$$

## Example 5

$$\begin{array}{r} \phantom{x^2 - 2x + 5) } \phantom{x^4} \phantom{- x^2} \phantom{+ x} \phantom{- 4} x^2 + 2x - 2 \\ x^2 - 2x + 5) \phantom{x^4} \phantom{- x^2} \phantom{+ x} \phantom{- 4} \\ \underline{- x^4 + 2x^3 - 5x^2} \phantom{+ x} \phantom{- 4} \\ \phantom{x^2 - 2x + 5) } 2x^3 - 6x^2 \phantom{+ x} \phantom{- 4} \\ \underline{- 2x^3 + 4x^2 - 10x} \phantom{- 4} \\ \phantom{x^2 - 2x + 5) } \phantom{2x^3 - 6x^2} - 2x^2 - 9x - 4 \\ \phantom{x^2 - 2x + 5) } \phantom{2x^3 - 6x^2} \phantom{- 2x^2 - 9x} 2x^2 - 4x + 10 \\ \hline \end{array}$$

## Example 5

$$\begin{array}{r} \phantom{x^2 - 2x + 5) } \phantom{x^4} \phantom{- x^4 + 2x^3 - 5x^2} x^2 + 2x - 2 \\ \hline x^2 - 2x + 5) \phantom{x^4} x^4 \phantom{- x^4 + 2x^3 - 5x^2} - x^2 + x - 4 \\ \phantom{x^2 - 2x + 5) } \underline{- x^4 + 2x^3 - 5x^2} \phantom{- 2x^3 + 4x^2 - 10x} \\ \phantom{x^2 - 2x + 5) } \phantom{x^4} 2x^3 - 6x^2 + x \\ \phantom{x^2 - 2x + 5) } \phantom{x^4} \underline{- 2x^3 + 4x^2 - 10x} \\ \phantom{x^2 - 2x + 5) } \phantom{x^4} \phantom{2x^3 - 6x^2 + x} - 2x^2 - 9x - 4 \\ \phantom{x^2 - 2x + 5) } \phantom{x^4} \phantom{2x^3 - 6x^2 + x} \phantom{- 2x^2 - 9x - 4} \underline{2x^2 - 4x + 10} \\ \phantom{x^2 - 2x + 5) } \phantom{x^4} \phantom{2x^3 - 6x^2 + x} \phantom{- 2x^2 - 9x - 4} \phantom{2x^2 - 4x + 10} - 13x + 6 \end{array}$$



## Example 5

$$\begin{array}{r} \phantom{x^2 - 2x + 5) } \phantom{x^4} \phantom{- x^2} \phantom{+ x} \phantom{- 4} \phantom{- 2} \phantom{+ 2x} \phantom{+ x^2} \\ x^2 - 2x + 5 \overline{) \phantom{x^4} \phantom{- x^2} \phantom{+ x} \phantom{- 4} \phantom{- 2} \phantom{+ 2x} \phantom{+ x^2}} \\ \underline{- x^4 + 2x^3 - 5x^2} \phantom{+ x} \phantom{- 4} \phantom{- 2} \phantom{+ 2x} \phantom{+ x^2} \\ \phantom{x^2 - 2x + 5) } 2x^3 - 6x^2 \phantom{+ x} \phantom{- 4} \phantom{- 2} \phantom{+ 2x} \phantom{+ x^2} \\ \underline{- 2x^3 + 4x^2 - 10x} \phantom{- 4} \phantom{- 2} \phantom{+ 2x} \phantom{+ x^2} \\ \phantom{x^2 - 2x + 5) } \phantom{2x^3 - 6x^2} - 2x^2 - 9x - 4 \phantom{+ 2x} \phantom{+ x^2} \\ \phantom{x^2 - 2x + 5) } \phantom{2x^3 - 6x^2} \phantom{- 2x^2 - 9x} 2x^2 - 4x + 10 \phantom{+ 2x} \phantom{+ x^2} \\ \phantom{x^2 - 2x + 5) } \phantom{2x^3 - 6x^2} \phantom{- 2x^2 - 9x} \phantom{2x^2 - 4x} \underline{- 13x + 6} \phantom{+ x^2} \end{array}$$

$$\text{So } \frac{x^4 - x^2 + x - 4}{x^2 - 2x + 5} = x^2 + 2x - 2 + \frac{-13x + 6}{x^2 - 2x + 5}.$$

# How to deal with rational functions?

How can we integrate something like

$$\int \frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} dx$$

or

$$\int \frac{x + 2}{x^3 - x} dx?$$

## Long division of polynomials

For the first example we can rewrite it in the form

$$\frac{3x^3 - 7x^2 + 17x - 3}{x^2 - 2x - 8} = 3x - 1 + \frac{39x - 11}{x^2 - 2x - 8}$$

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This is still not something we can integrate so we need to go further.

# Partial fractions

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How do we reverse this process?

Answer: partial fractions

When the denominator is  $(ax + b)(cx + d) \cdots$

We want to rewrite  $\frac{P(x)}{Q(x)}$  as a sum. Let

$$Q(x) = (a_1x + b_1)(a_2x + b_2) \cdots (a_nx + b_n)$$



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we can always find constants  $A_1, A_2, \dots, A_n$  so that

$$\frac{P(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_n}{a_nx + b_n}$$

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Multiplying both sides by  $(x-1)(x+1)$

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x+1} + \frac{B(x-1)(x+1)}{x-1} \\ &= A(x-1) + B(x+1) \\ &= (A+B)x + (B-A) \end{aligned}$$

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So

$$-2A = 1 \quad \text{hence} \quad A = -\frac{1}{2} \quad \text{and} \quad B = \frac{1}{2}.$$