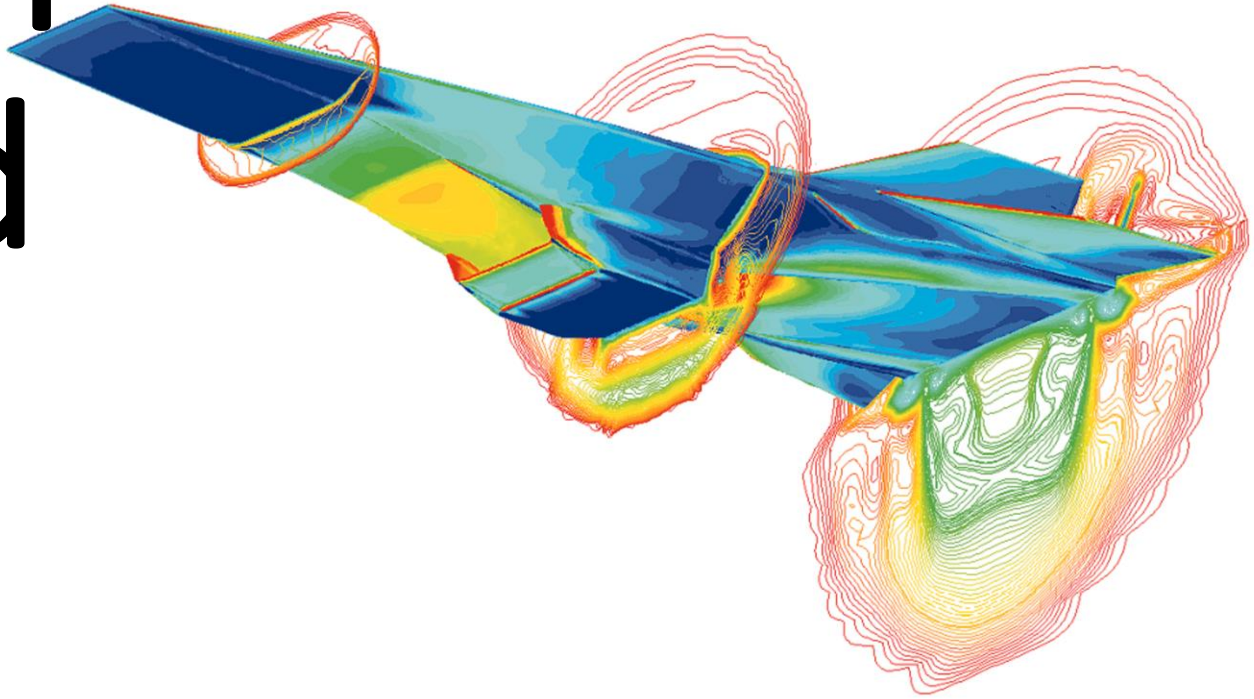
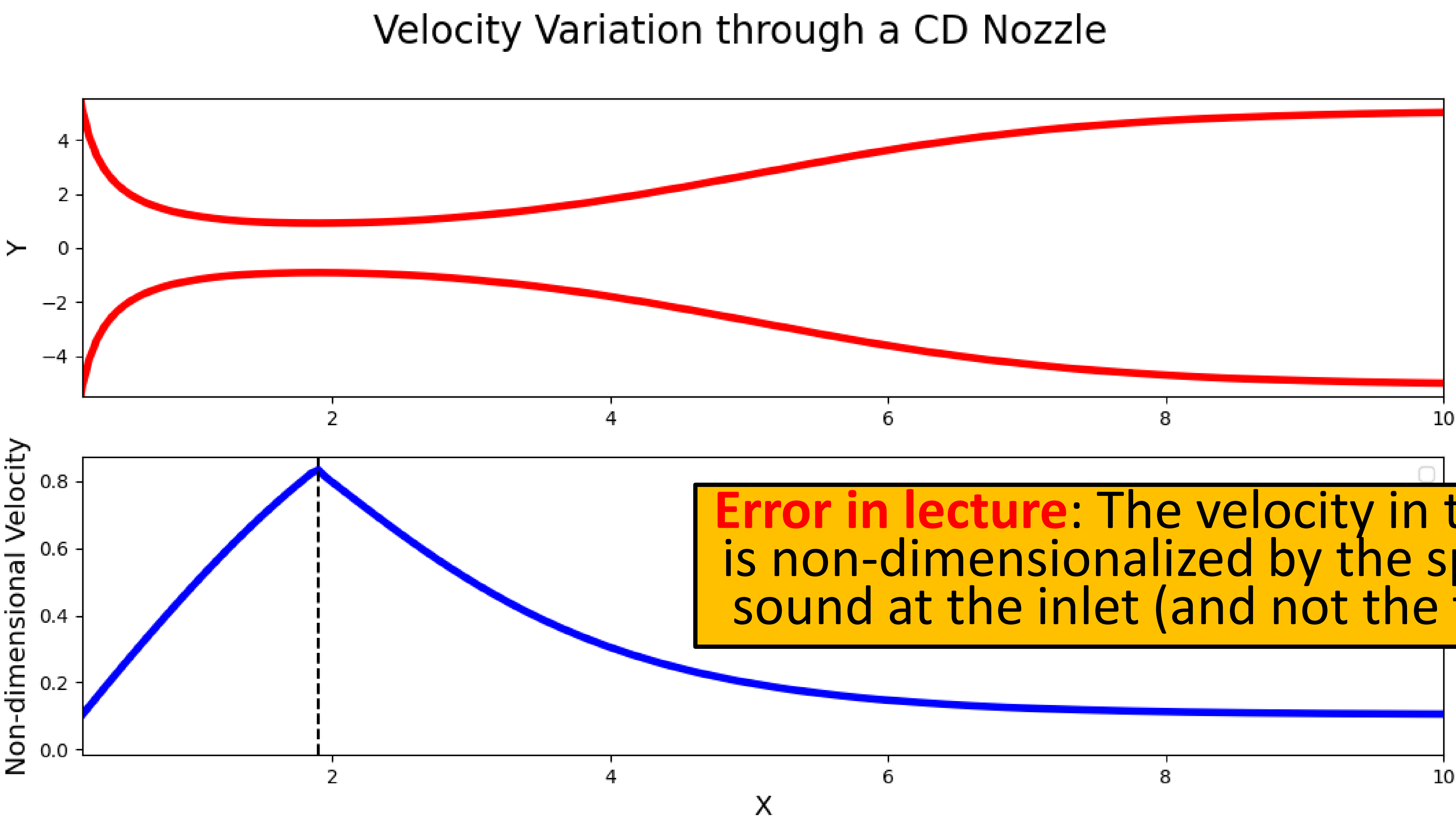
Conservation Laws of Fluid Dynamics



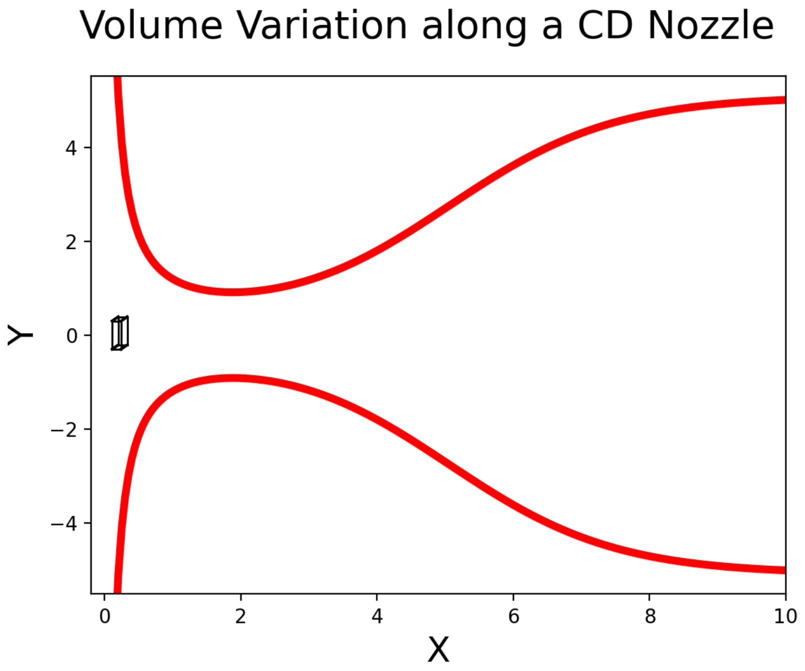
SEBASTIAN THOMAS

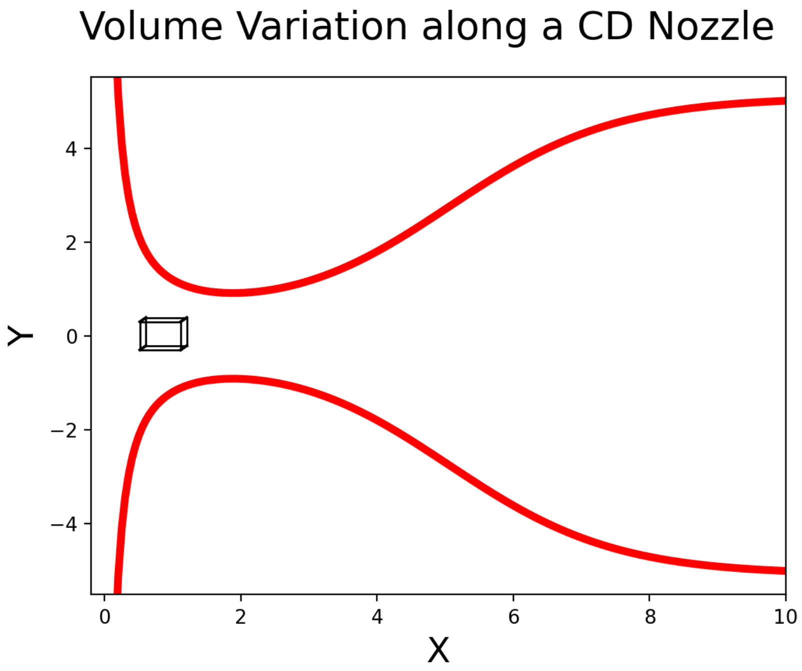
Two Vital Concepts

- Divergence
- Substantial Derivative



Error in lecture: The velocity in this plot is non-dimensionalized by the speed of sound at the inlet (and not the throat)


Volume Variation along a CD Nozzle
Y
X
4
2
0
−2
−4
0
2
4
6
8
10


Volume Variation along a CD Nozzle
Y
X
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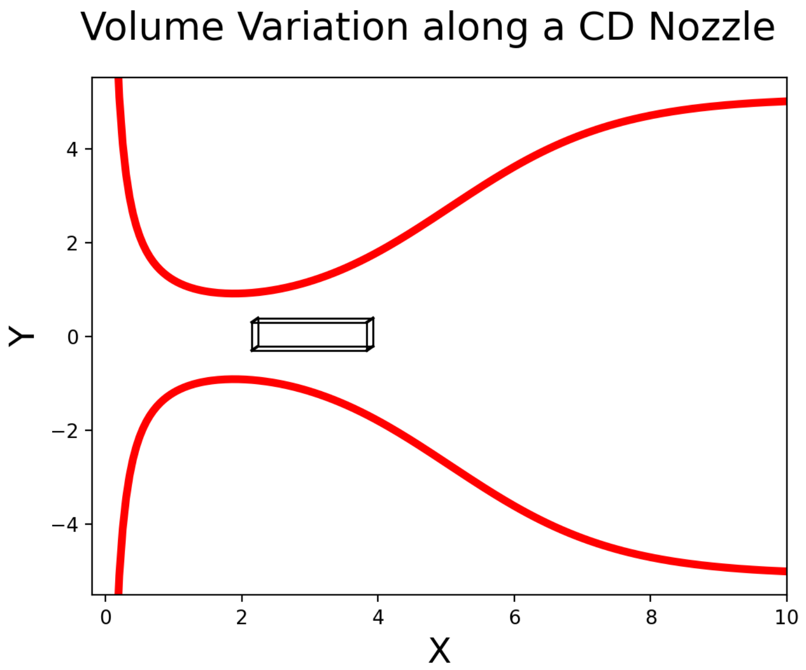
Volume Variation along a CD Nozzle
Y
X
4
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−2
−4
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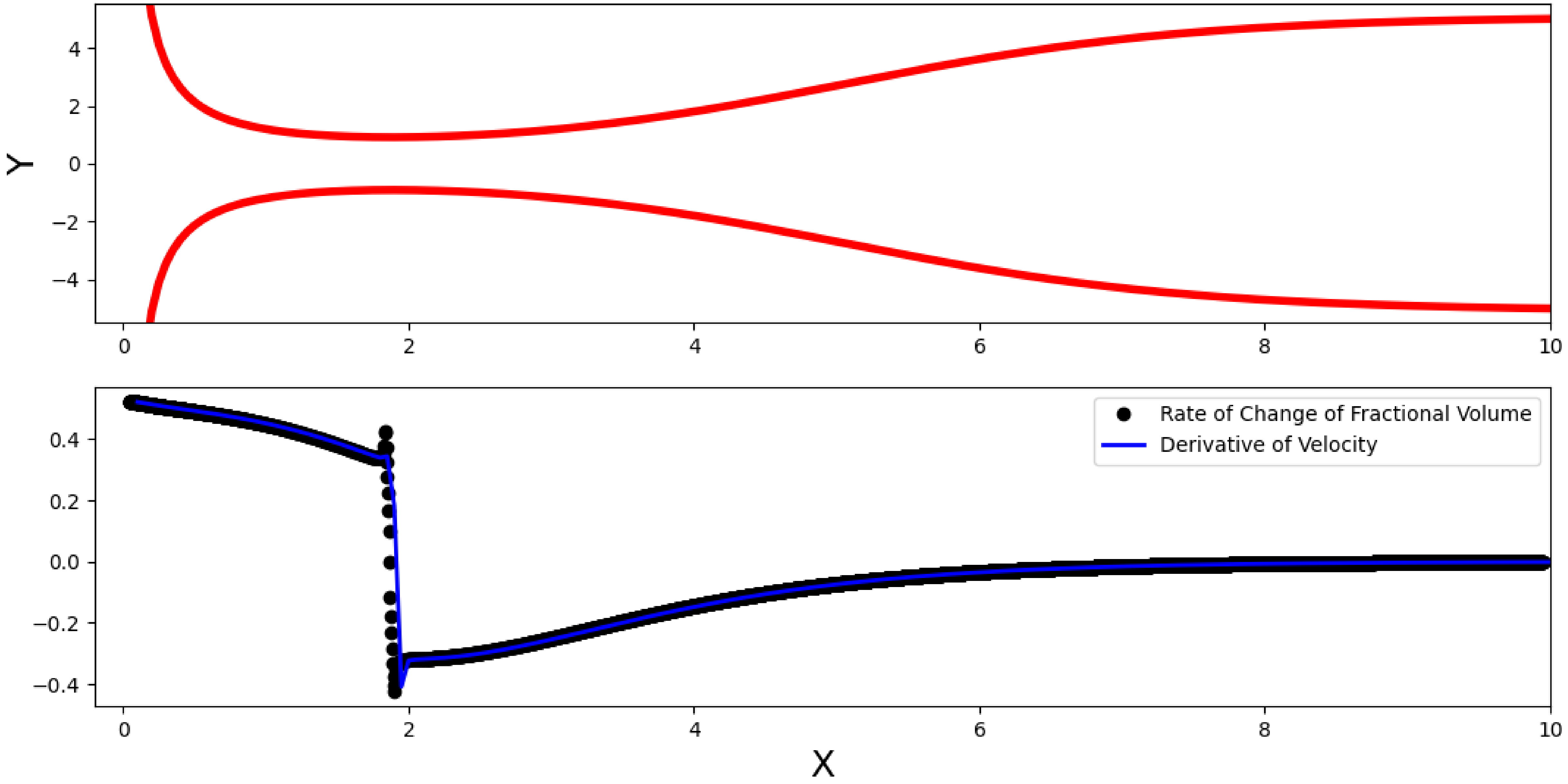
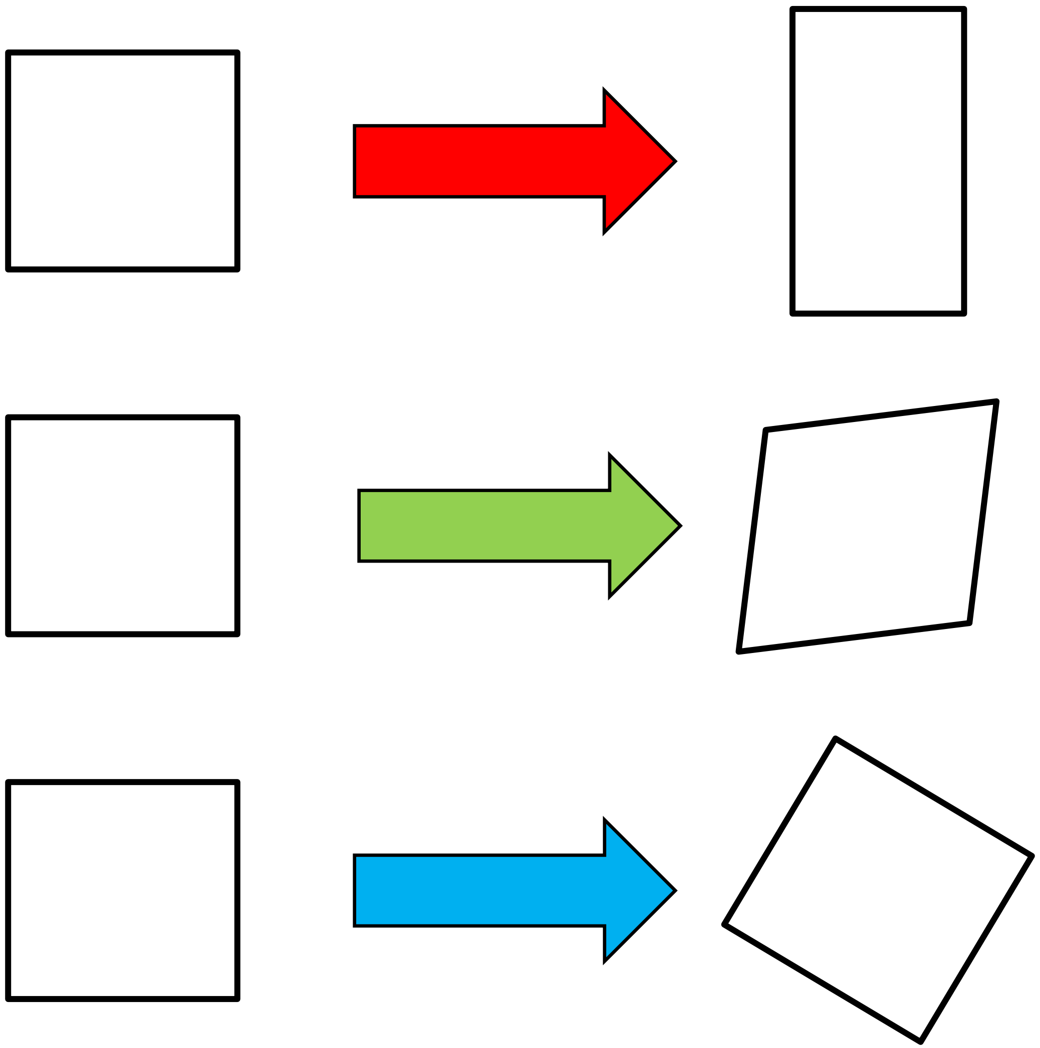
4
6
8
10



Y
4
2
0
−2
−4
0
2
4
6
8
10
0.4
0.2
0.0
−0.2
−0.4
Rate of Change of Fractional Volume
Derivative of Velocity
X

What about higher dimensions?



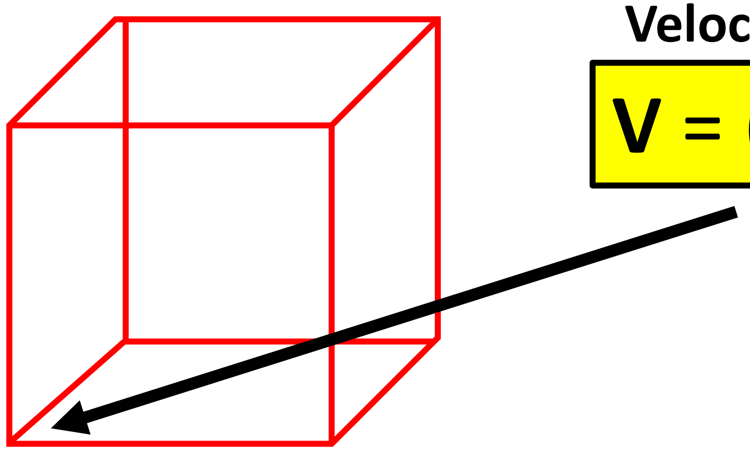
Expansion

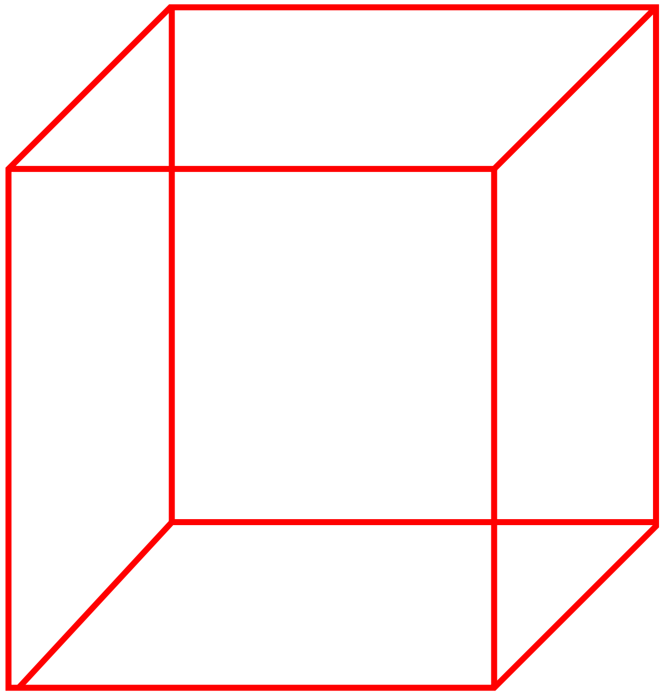
Shear

Rigid Rotation

Velocity vector

$\mathbf{V} = (u,v,w)$





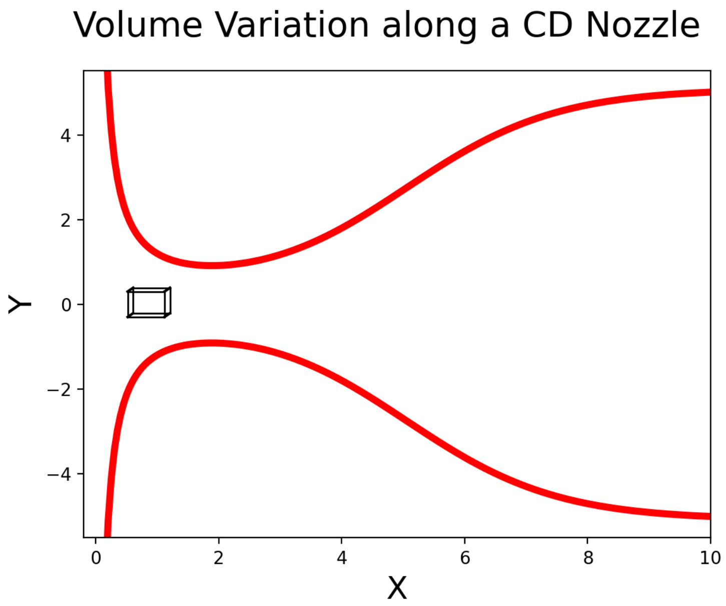
$$Vol_t = dx*dy*dz$$

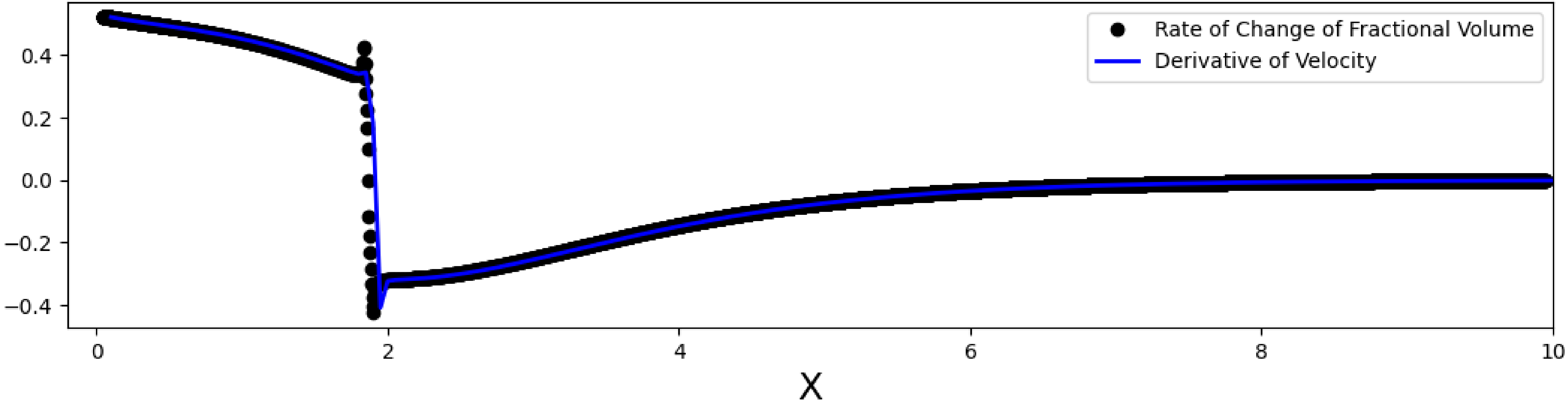
$$Vol_{t+dt} = (dx+u_x dxdt)*(dy+v_y dydt)*(dz+w_z dzdt)$$

$$\boxed{\frac{Vol_{t+dt} - Volt}{Vol_t * dt} = u_x + v_y + w_z}$$

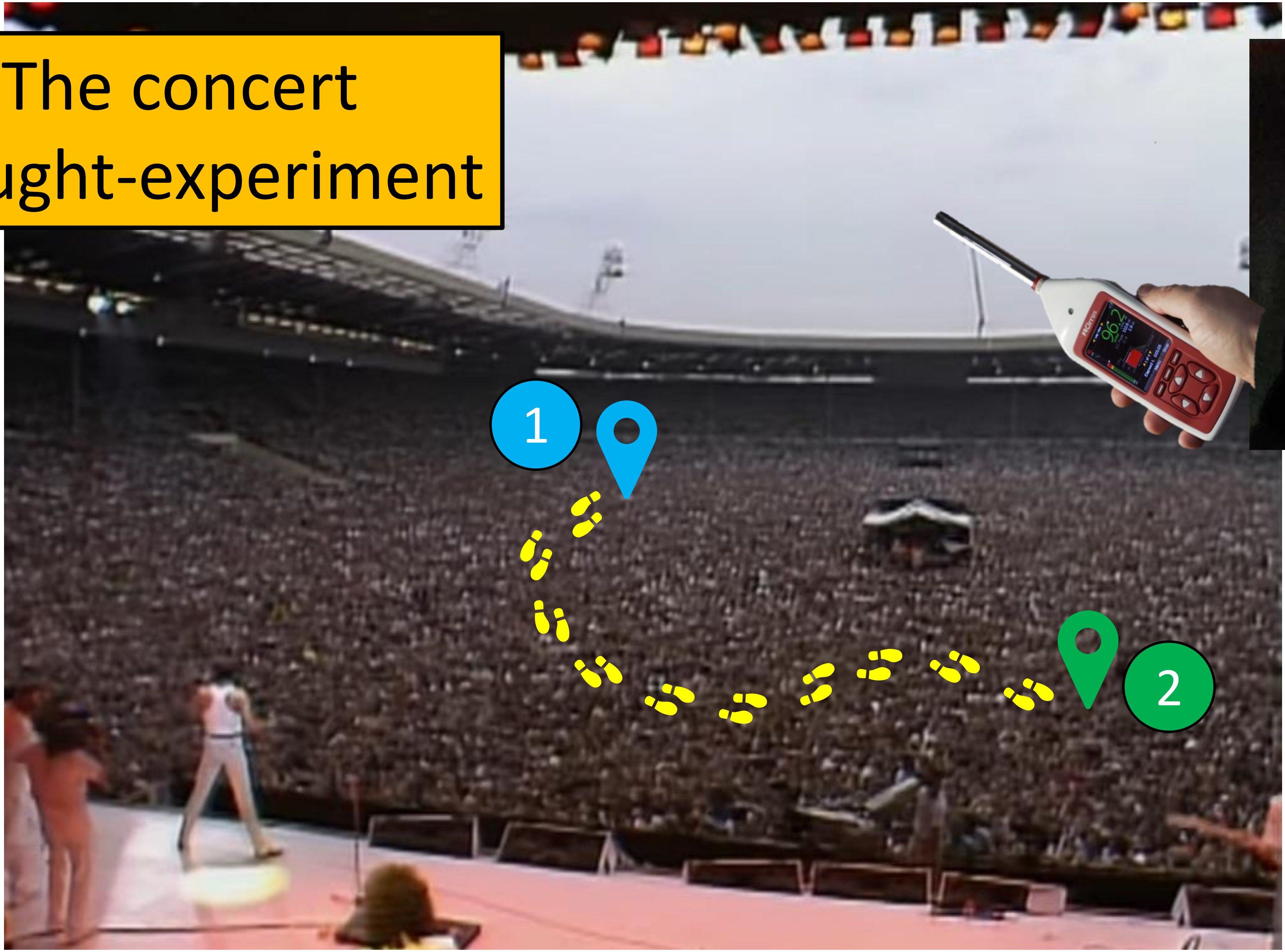
$$\nabla.\mathbf{V}$$

∇.V is simply the fractional rate of change of volume





The concert thought-experiment



Joseph-Louis Lagrange

Taylor Series

$$f = f(x)$$

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \frac{(b-a)^3}{3!}f'''(a) + \cdots$$

Multi-variate Taylor Series Expansion

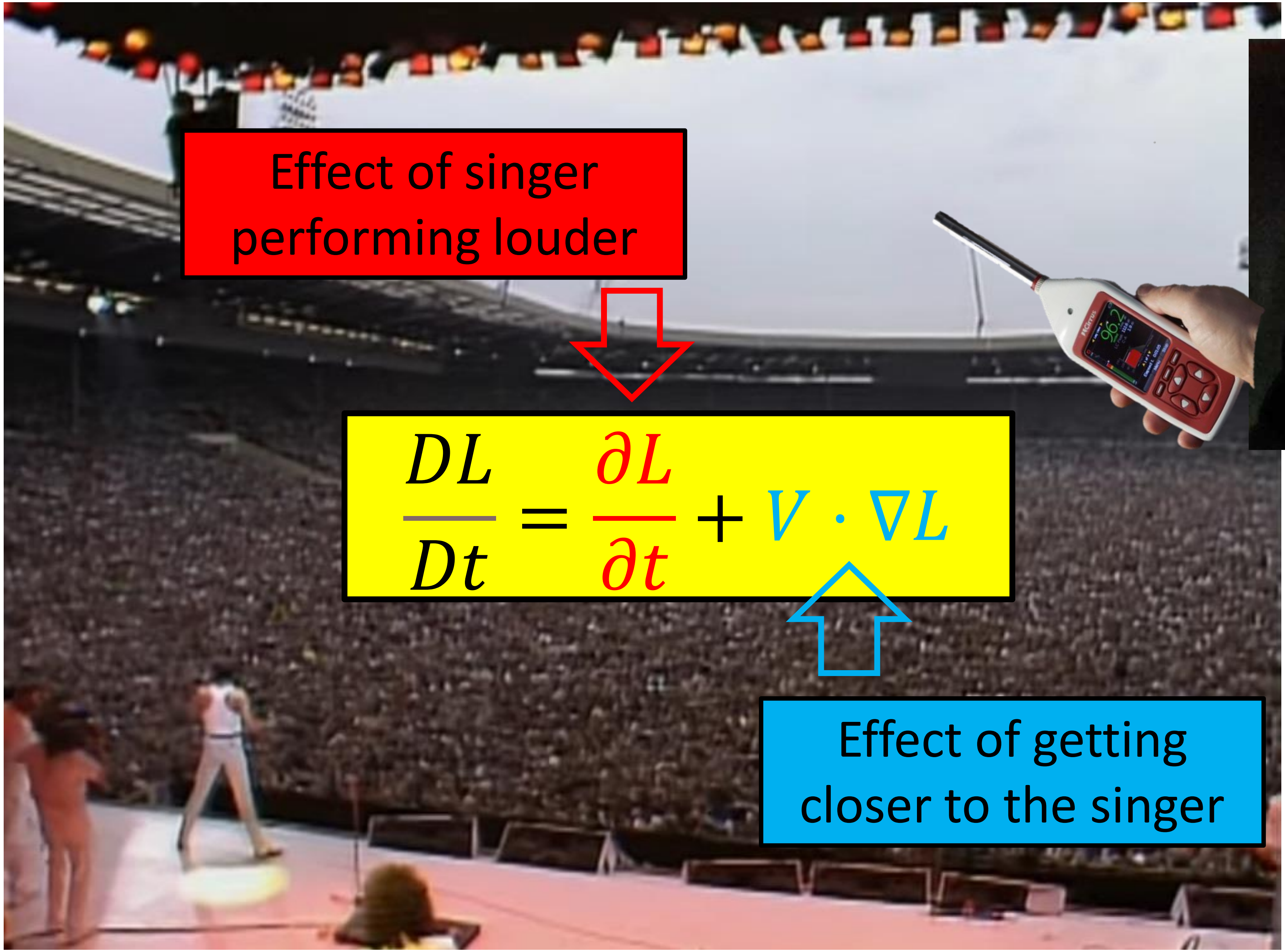
$$L = L(x, y, t)$$

$$L_2 = L_1 + \frac{\partial L}{\partial t}(t_2 - t_1) + \frac{\partial L}{\partial x}(x_2 - x_1) + \frac{\partial L}{\partial y}(y_2 - y_1) + \cdots$$

The Substantial Derivative

$$\boxed{\frac{DL}{Dt} = \frac{\partial L}{\partial t} + V \cdot \nabla L}$$

The concept of the 'substantial' (or 'material') derivative arises in fluid dynamics when a quantity is being measured inside a control volume that moves with the fluid



$$\frac{DL}{Dt} = \frac{\partial L}{\partial t} + V \cdot \nabla L$$

Joseph-Louis Lagrange

Conservation Laws

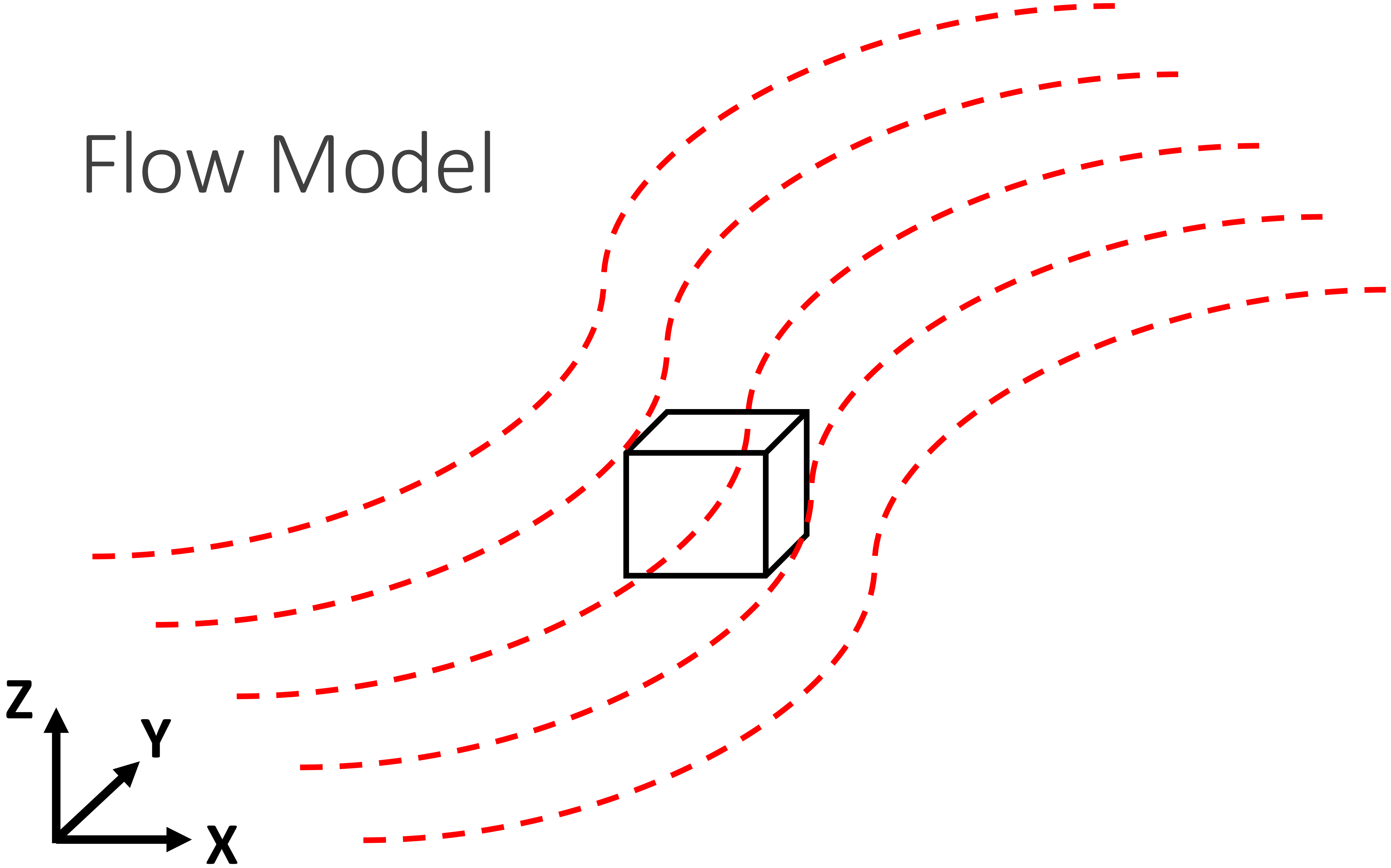
- For any fluid element, three laws are inviolable:

  - Mass is conserved

  - The rate of change of momentum is equal to the force applied

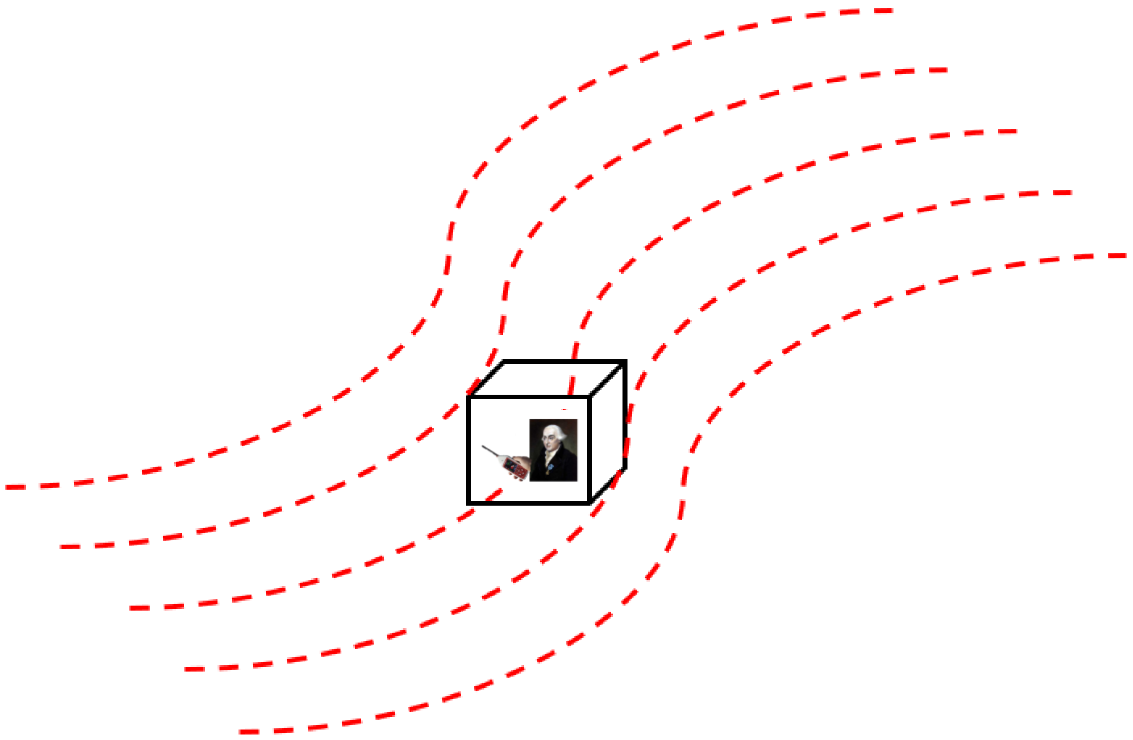
  - Energy is conserved

Flow Model



Conservation of Mass

- The mass of the fluid element – *as it moves through the fluid* - remains constant
- In other words, the substantial derivative of mass is zero



Conservation of Mass

density → volume → Product Rule

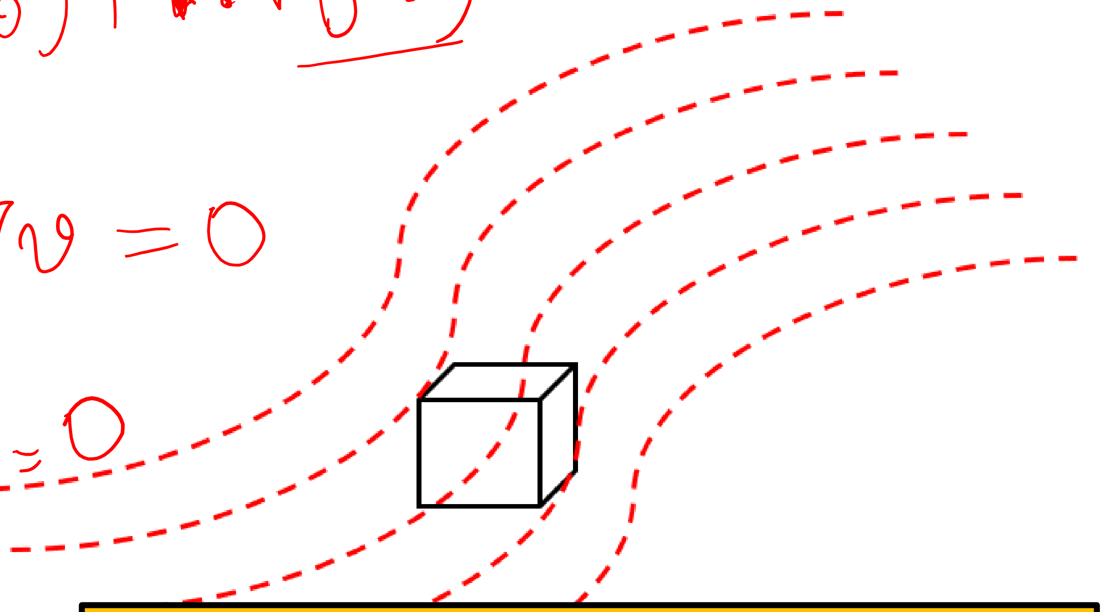
$$\frac{Dm}{Dt} = \frac{D(\rho v)}{Dt} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho v) + V\cdot\nabla(\rho v) = 0$$

$$\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t} + Vv\cdot\nabla\rho + V\rho\cdot\nabla v = 0$$

$$v\left[\frac{\partial \rho}{\partial t} + V\cdot\nabla\rho\right] + \rho\left[\frac{\partial v}{\partial t} + V\cdot\nabla v\right] = 0$$

$$\boxed{v\frac{D\rho}{Dt} + \rho\frac{Dv}{Dt} = 0}$$



Invoke the meaning of 'divergence' as fractional rate of change of volume

Conservation of Mass

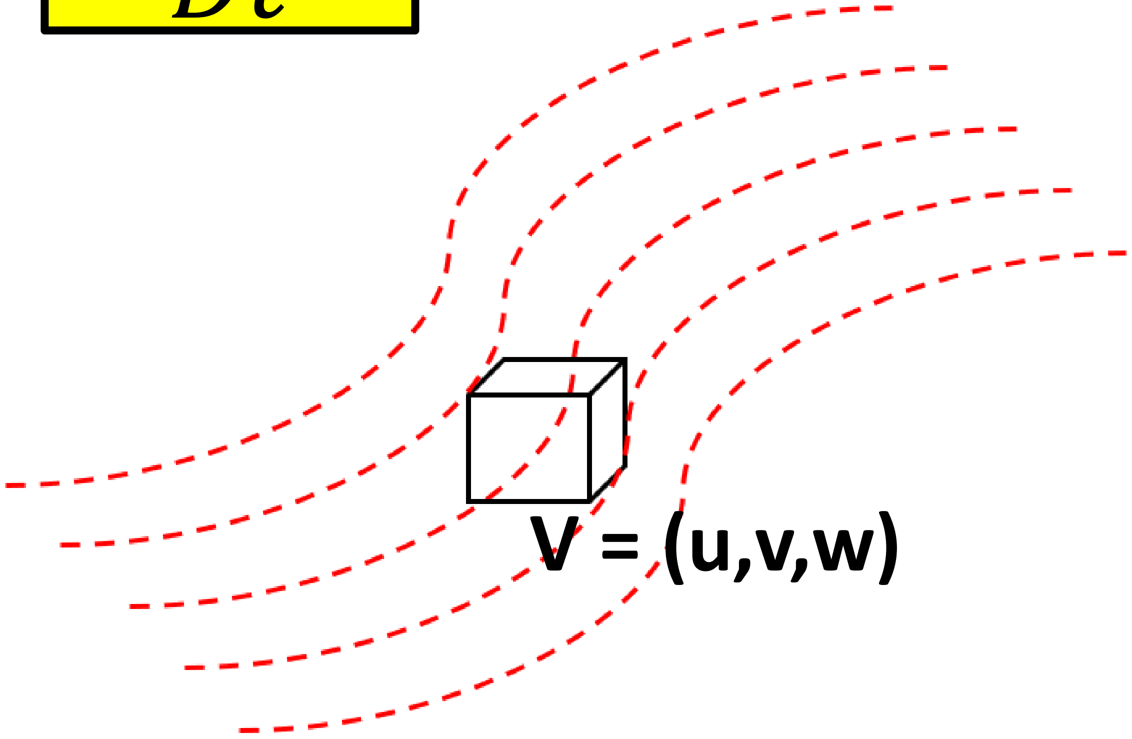
$$\boxed{\frac{D\rho}{Dt} + \rho\nabla.\mathbf{V} = 0}$$

Conservation of Momentum

- The rate at which the momentum of the fluid element changes **in the X direction** – *as it moves through the fluid* – is equal to the total force acting on it **in the X direction**

$$\boxed{\frac{D(mu)}{Dt}}$$

$$\boxed{\Sigma F_x}$$



Conservation of Momentum

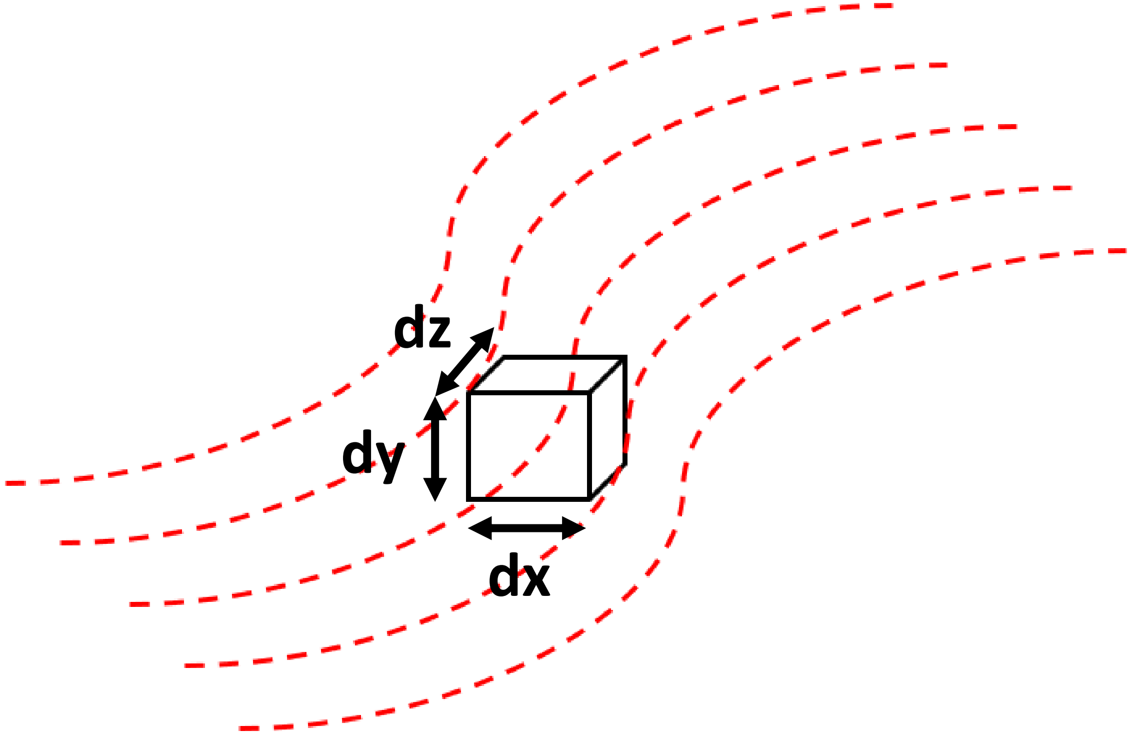
$$\boxed{\frac{D(mu)}{Dt} = \Sigma F_x}$$

Conservation of Momentum

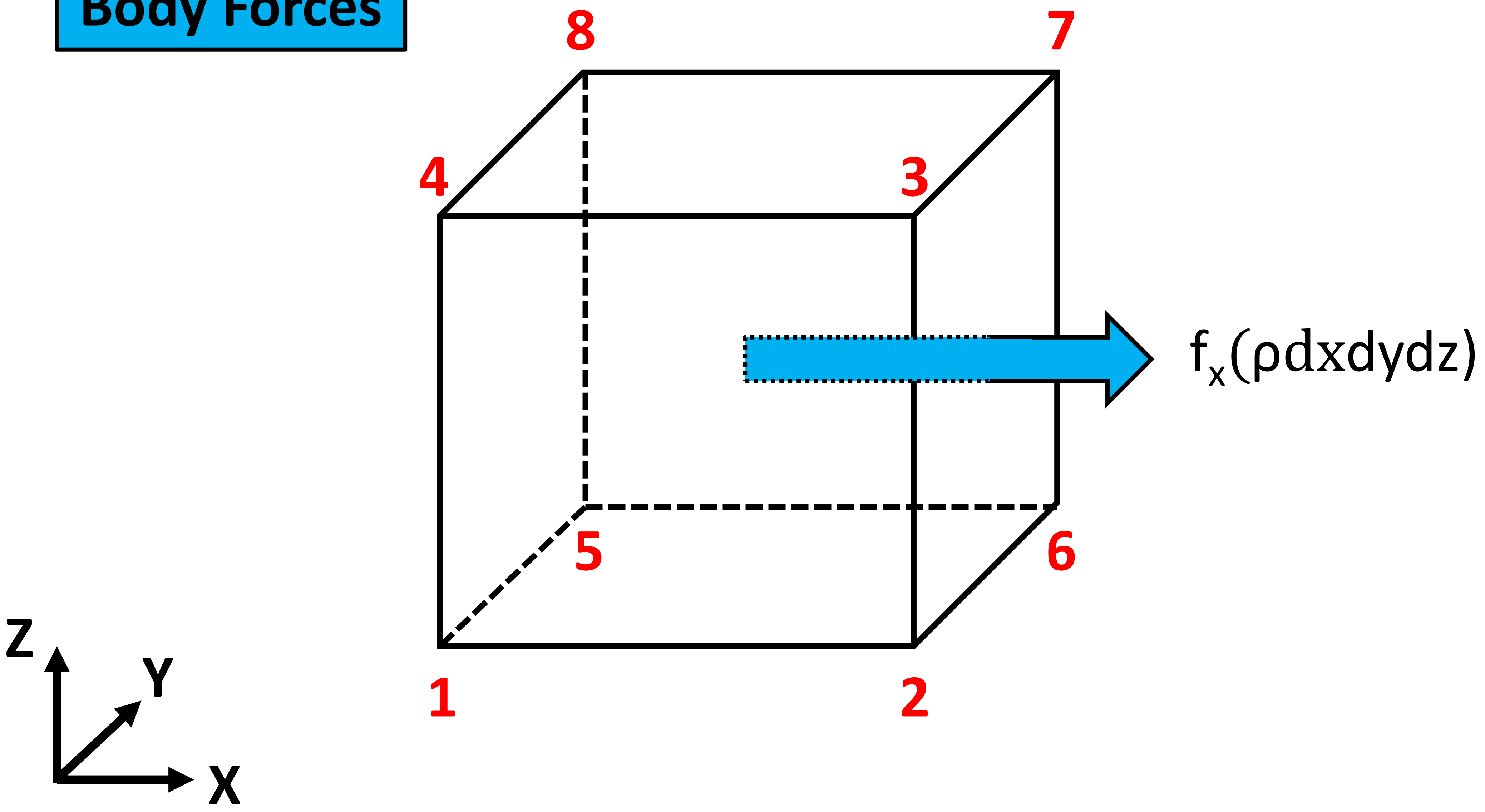
$$\frac{(\rho dxdydz)D(u)}{Dt} = \Sigma F_x$$

Body Forces	
1	Gravity
2	Magnetic

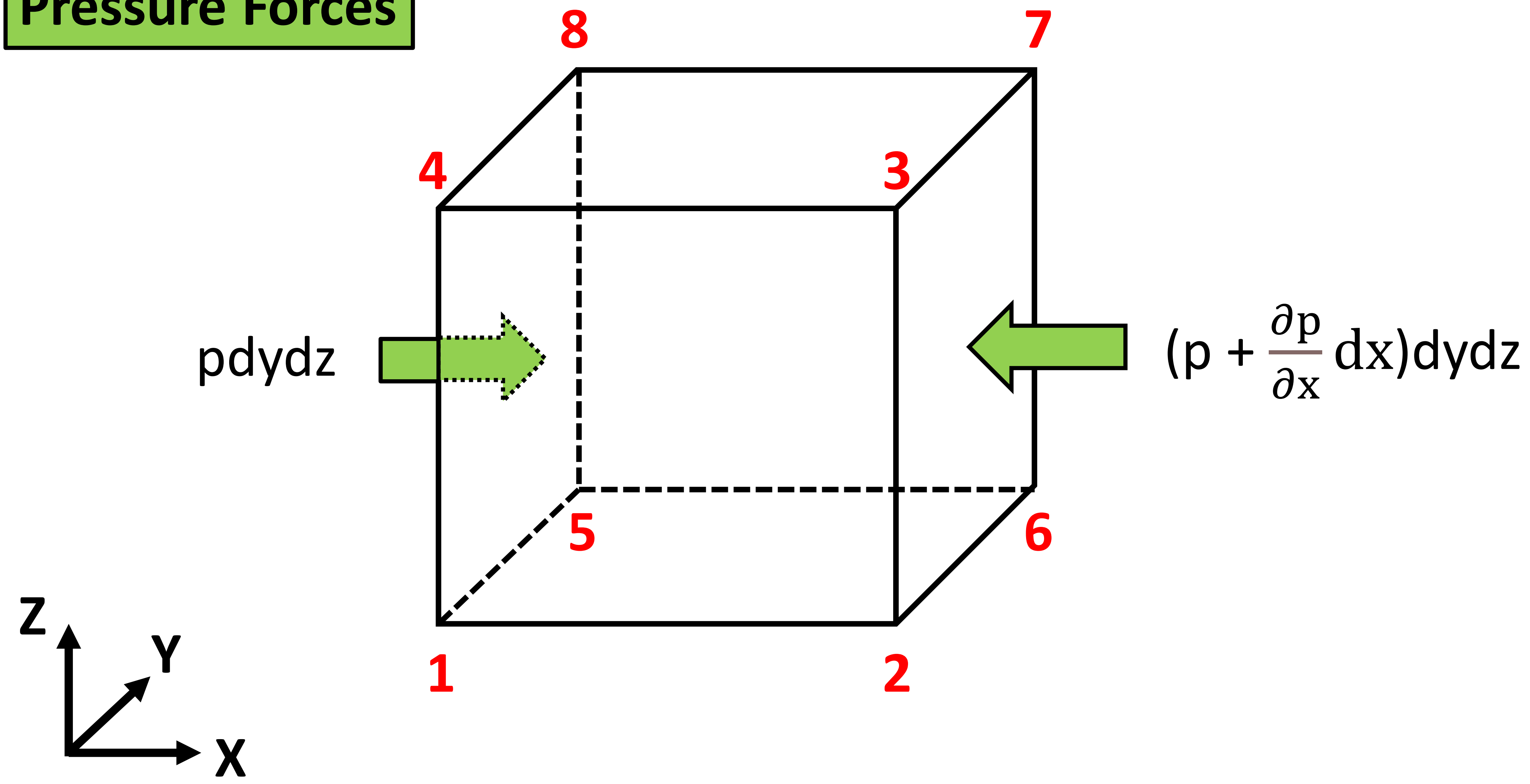
Surface Forces	
1	Pressure
2	Viscous Forces



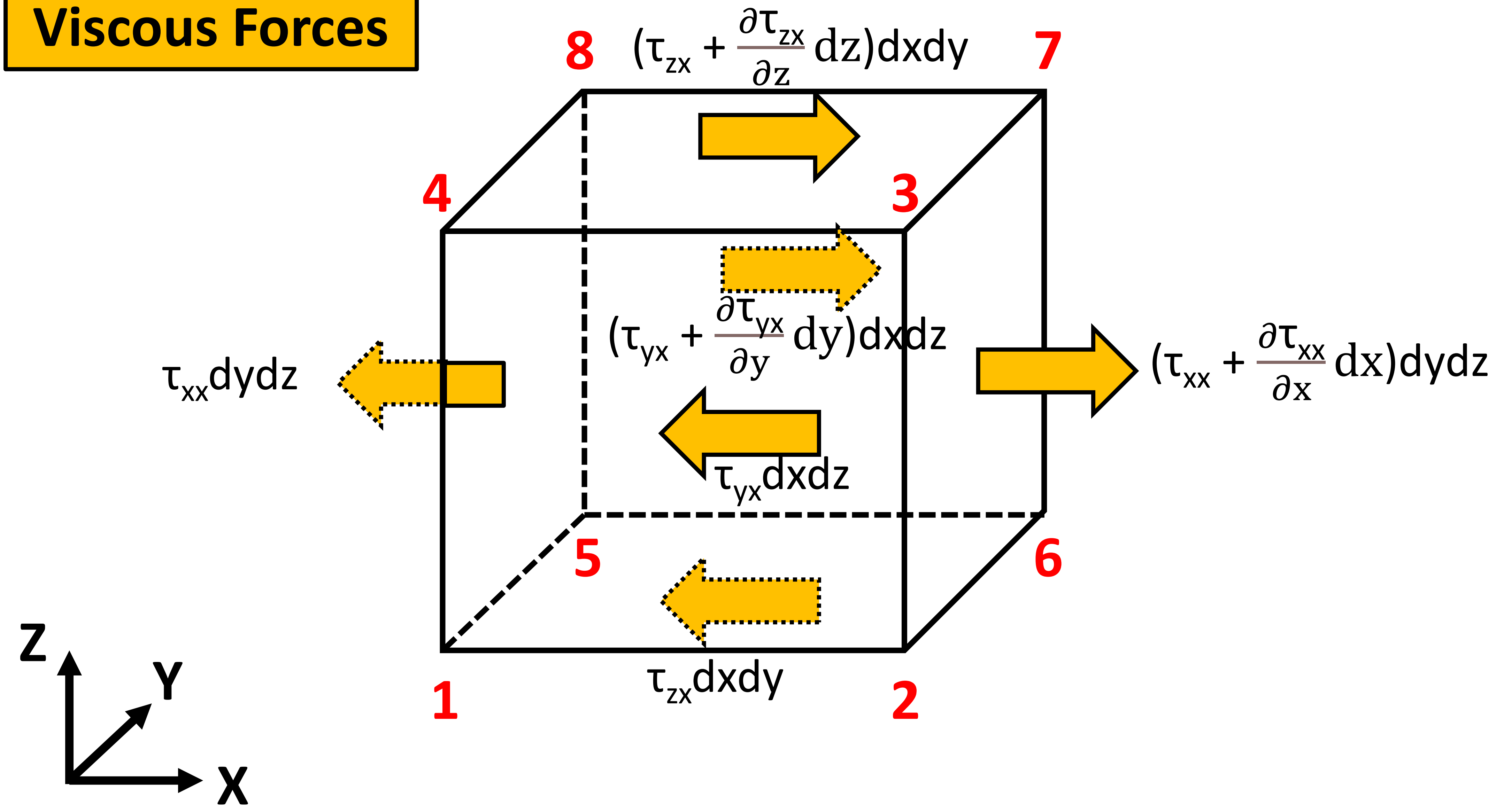
Body Forces



Pressure Forces



Viscous Forces



Conservation of Momentum

$$\frac{(\rho\mathrm{dxdydz})D(u)}{Dt} = \frac{\partial \mathrm{p}}{\partial \mathrm{x}}\,\mathrm{dxdydz} + \frac{\partial \tau_{xx}}{\partial \mathrm{x}}\,\mathrm{dxdydz} + \frac{\partial \tau_{yx}}{\partial \mathrm{y}}\,\mathrm{dxdydz} + \frac{\partial \tau_{zx}}{\partial \mathrm{z}}\,\mathrm{dxdydz} + f_x(\rho\mathrm{dxdydz})$$

$$\frac{\rho D(u)}{Dt} = -\frac{\partial \mathrm{p}}{\partial \mathrm{x}} + \frac{\partial \tau_{xx}}{\partial \mathrm{x}} + \frac{\partial \tau_{yx}}{\partial \mathrm{y}} + \frac{\partial \tau_{zx}}{\partial \mathrm{z}} + f_x\rho$$

Conservation of Momentum

$$\frac{\rho D(u)}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x\rho$$

$$\frac{\rho D(v)}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y\rho$$

$$\frac{\rho D(w)}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z\rho$$

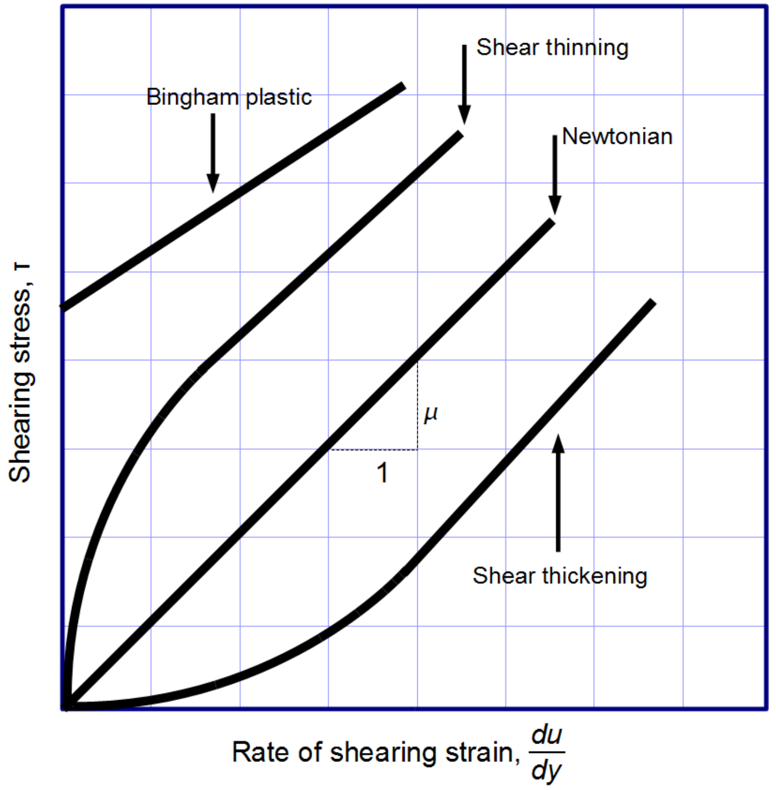
Newtonian Fluid

Newtonian fluids exhibit a linear relationship between viscous stress and velocity gradients

$$\frac{\rho D(u)}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x\rho$$

$$\frac{\rho D(v)}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y\rho$$

$$\frac{\rho D(w)}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z\rho$$



$$
\begin{bmatrix} \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \end{bmatrix}
=
\begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\
a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\
a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\
a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} & a_{49} \\
a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} & a_{58} & a_{59} \\
a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} & a_{68} & a_{69} \\
a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} & a_{79} \\
a_{81} & a_{82} & a_{83} & a_{84} & a_{85} & a_{86} & a_{87} & a_{88} & a_{89} \\
a_{91} & a_{92} & a_{93} & a_{94} & a_{95} & a_{96} & a_{97} & a_{98} & a_{99}
\end{bmatrix}
\begin{bmatrix} \frac{\partial \mathbf{u}}{\partial x} \\ \frac{\partial \mathbf{v}}{\partial x} \\ \frac{\partial \mathbf{w}}{\partial x} \\ \frac{\partial \mathbf{u}}{\partial y} \\ \frac{\partial \mathbf{v}}{\partial y} \\ \frac{\partial \mathbf{w}}{\partial y} \\ \frac{\partial \mathbf{u}}{\partial z} \\ \frac{\partial \mathbf{v}}{\partial z} \\ \frac{\partial \mathbf{w}}{\partial z} \end{bmatrix}
$$

Number of parameters can be reduced from 81 to 2 through logical arguments

$$
\begin{bmatrix} \tau_{xx} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yx} \\ \tau_{yy} \\ \tau_{yz} \\ \tau_{zx} \\ \tau_{zy} \\ \tau_{zz} \end{bmatrix}
=
\begin{bmatrix}
\lambda+2\mu & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda \\
0 & \mu & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mu & 0 & 0 & 0 & \mu & 0 & 0 \\
0 & \mu & 0 & \mu & 0 & 0 & 0 & 0 & 0 \\
\lambda & 0 & 0 & 0 & \lambda+2\mu & 0 & 0 & 0 & \lambda \\
0 & 0 & 0 & 0 & 0 & \mu & 0 & \mu & 0 \\
0 & 0 & \mu & 0 & 0 & 0 & \mu & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \mu & 0 & \mu & 0 \\
\lambda & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda+2\mu
\end{bmatrix}
\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial x} \\ \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial y} \\ \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial z} \end{bmatrix}
$$

Stokes' Hypothesis

$$\boxed{\lambda = -\frac{2}{3}\mu}$$

Sir George Stokes

Viscous Stresses

$$\tau_{xx} = \frac{2\mu}{3}\left[2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right]$$

$$\tau_{yy} = \frac{2\mu}{3}\left[-\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right]$$

$$\tau_{zz} = \frac{2\mu}{3}\left[-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + 2\frac{\partial w}{\partial z}\right]$$

$$\tau_{xy}, \tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\tau_{xz}, \tau_{zx} = \mu\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]$$

$$\tau_{yz}, \tau_{zy} = \mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

Conservation of Momentum

$$\tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\tau_{zx} = \mu\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]$$

$$\frac{\rho D(u)}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x\rho$$

$$\tau_{xx} = \frac{2\mu}{3}\left[2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right]$$

Conservation of Momentum

$$\rho \frac{\partial u}{\partial t} + V.\nabla u = -\frac{\partial p}{\partial x} + \frac{2\mu}{3}\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right] + \mu \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] + \mu \frac{\partial}{\partial z}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] + f_x\rho$$

$$\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] = -\frac{\partial p}{\partial x} + \frac{2\mu}{3}\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right] + \mu \frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right] + \mu \frac{\partial}{\partial z}\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right] + f_x\rho$$

$$\rho\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right]=-\frac{\partial p}{\partial x}+\frac{2\mu}{3}\frac{\partial}{\partial x}\left[2\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}\right]+\mu\frac{\partial}{\partial y}\left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right]+\mu\frac{\partial}{\partial z}\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right]+f_x\rho$$

$$\rho\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right]=-\frac{\partial p}{\partial y}+\mu\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right]+\frac{2\mu}{3}\frac{\partial}{\partial y}\left[-\frac{\partial u}{\partial x}+2\frac{\partial v}{\partial y}-\frac{\partial w}{\partial z}\right]+\mu\frac{\partial}{\partial z}\left[\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right]+f_y\rho$$

$$\rho\left[\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right]=-\frac{\partial p}{\partial z}+\mu\frac{\partial}{\partial x}\left[\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right]+\mu\frac{\partial}{\partial y}\left[\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right]+\frac{2\mu}{3}\frac{\partial}{\partial z}\left[-\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+2\frac{\partial w}{\partial z}\right]+f_z\rho$$

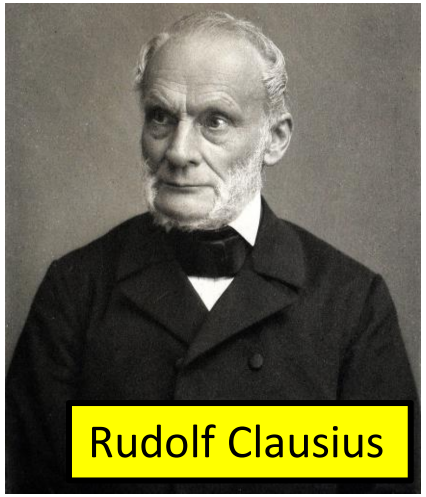
A Much More Convenient Form

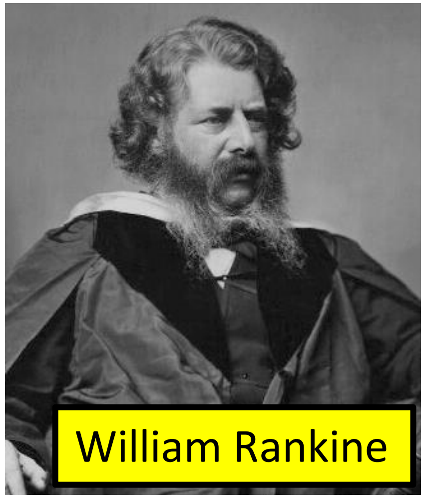
$$\boxed{\frac{\rho D(u)}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x \rho}$$

$$\boxed{\frac{\rho D(v)}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y \rho}$$

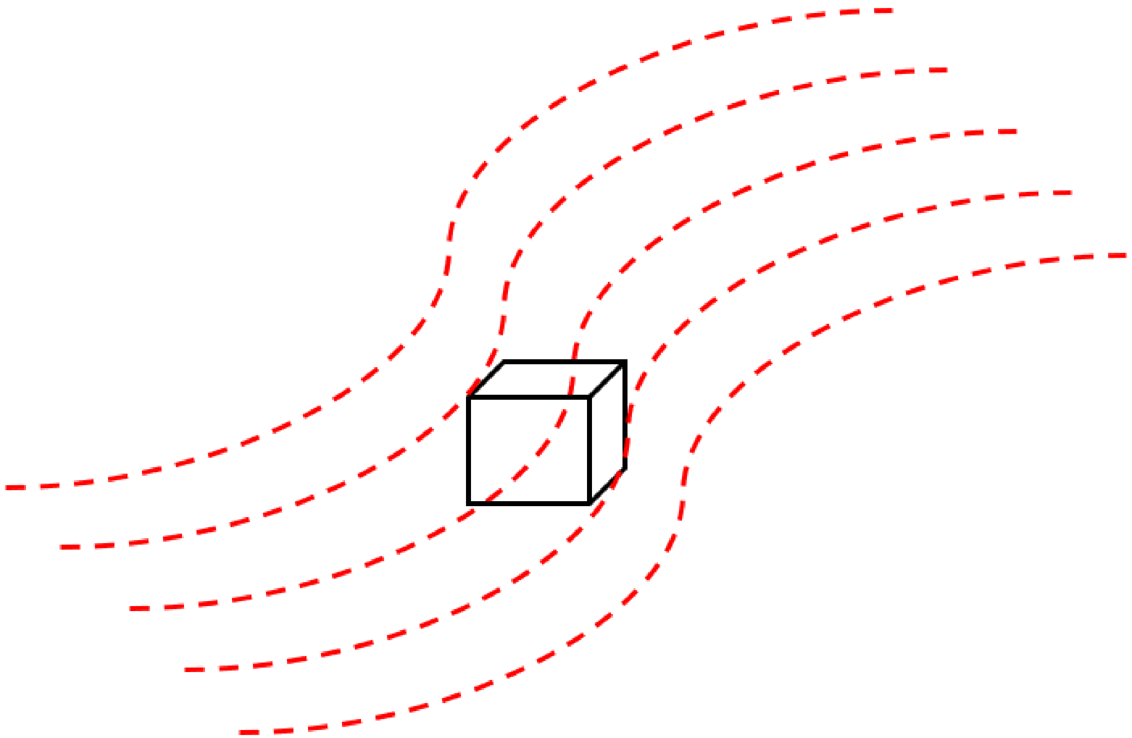
$$\boxed{\frac{\rho D(w)}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z \rho}$$

Conservation of Energy





- The rate at which the energy of the fluid element changes – *as it moves through the fluid* – is equal to the rate of work done on it and the rate at which energy is transferred into it



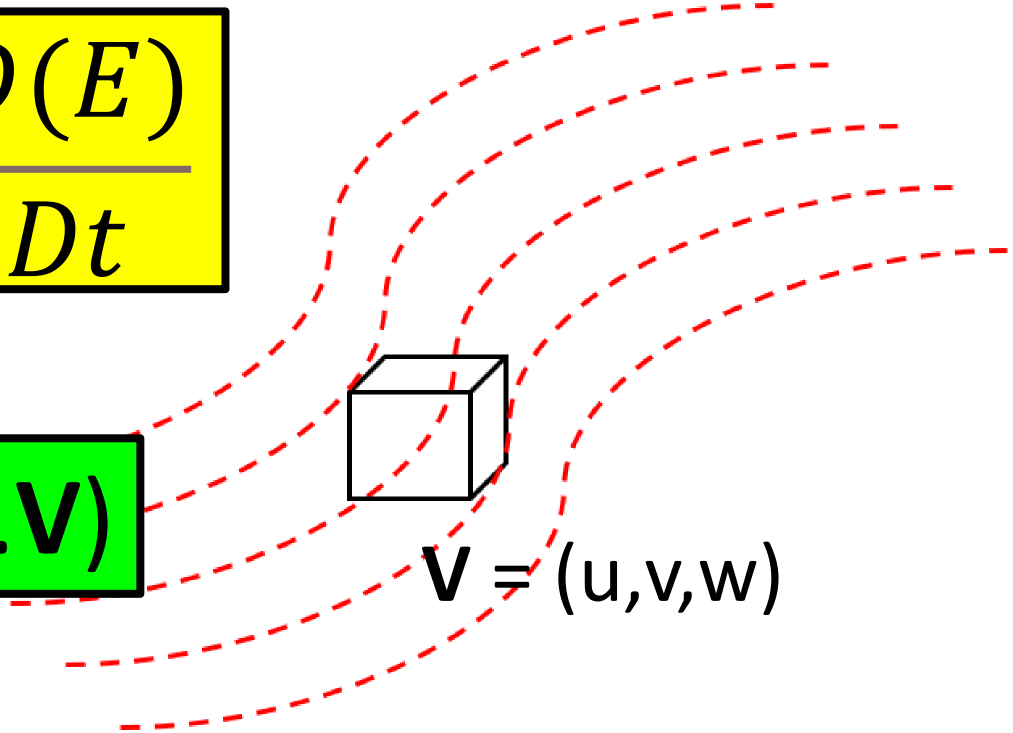
Conservation of Energy

- The rate at which the energy of the fluid element changes – *as it moves through the fluid* – is equal to the rate of work done on it and the rate at which energy is transferred into it

$$\boxed{\frac{D(E)}{Dt}}$$

$$\boxed{\Sigma(\mathbf{F}.\mathbf{V})}$$

$$\boxed{\Sigma(\dot{Q})}$$



Conservation of Energy

$$\boxed{\frac{D(E)}{Dt} = \Sigma(\mathbf{F}.\mathbf{V}) + \Sigma(\dot{Q})}$$

Rate of Change of *What Exactly*?

$$\frac{D(E)}{Dt}$$

The 'bees-in-a-car' thought-experiment

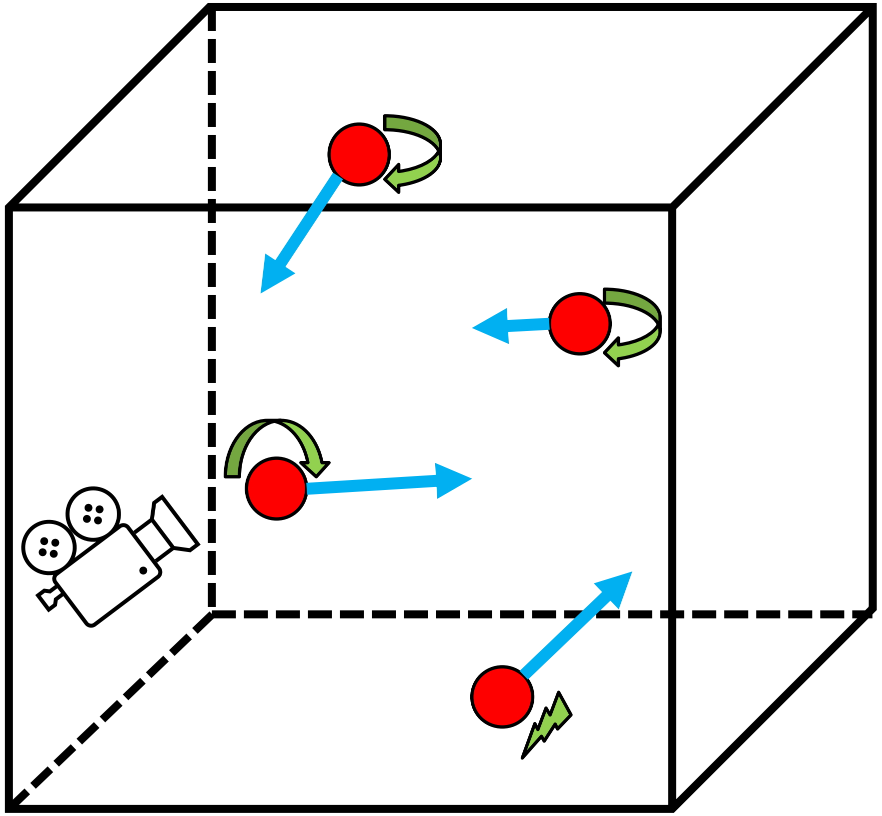
Source: Josh Dutton, "*'That's crazy': Baffling photo of woman driving a car full of bees,*" Yahoo News Australia, 22 March 2021

Energy of the bees

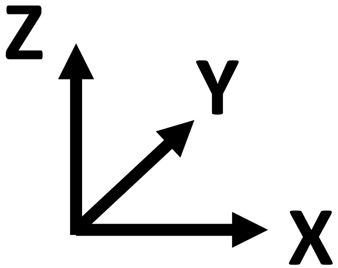
$$\underbrace{\text{Energy due to the bees flapping their wings} + \text{Energy due to the bees flying around within the car}}_{\text{“INTERNAL” ENERGY}} + \underbrace{\text{Energy due to the car’s motion}}_{\text{KINETIC ENERGY}}$$

Rate of Change of *What Exactly*?

Atoms/Molecules have translational velocities as measured in the reference frame attached to the moving volume

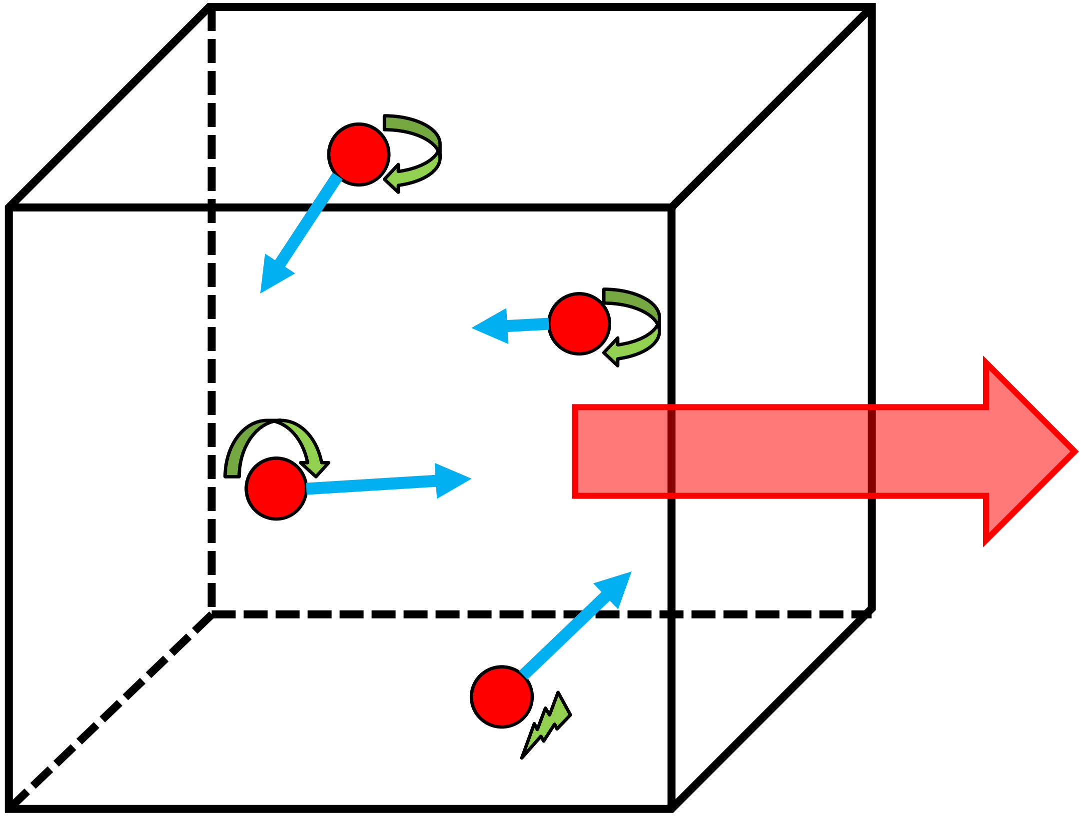


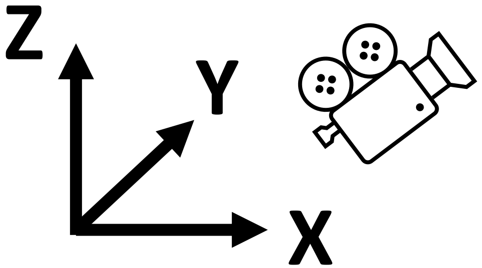
Molecules can also have rotational and vibrational energy



Rate of Change of *What Exactly*?

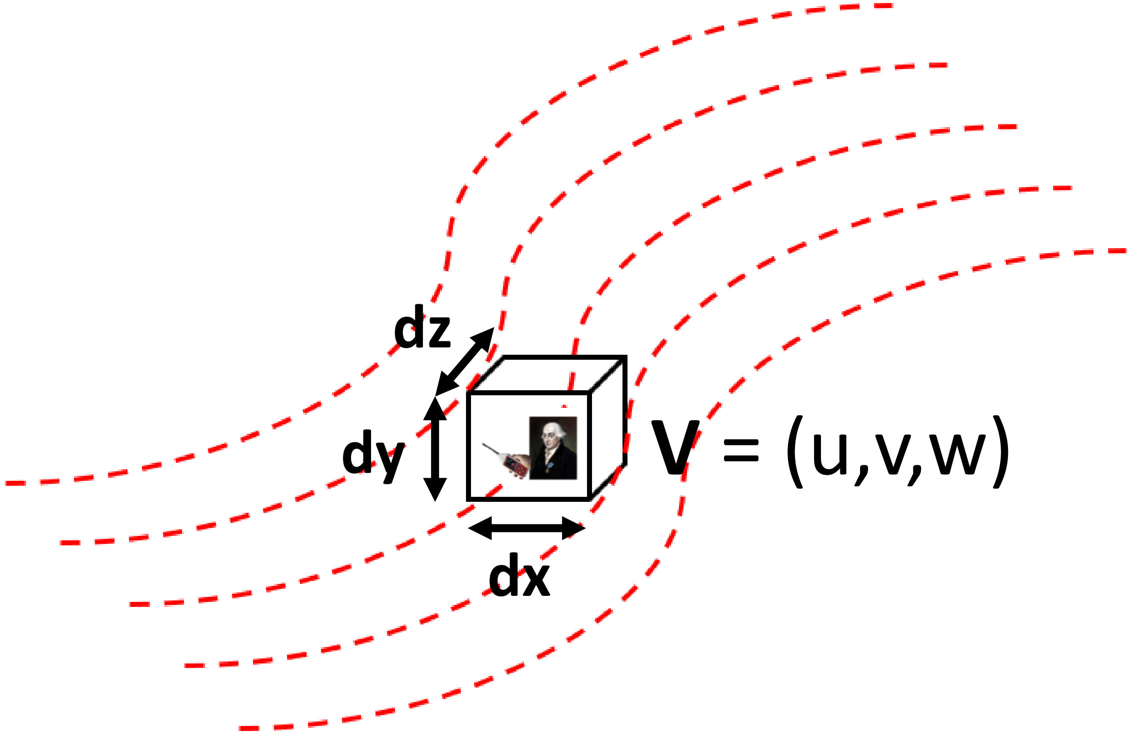
Atoms/Molecules can also have kinetic energy due to moving *en masse*





Rate of Change of *What Exactly*?

$$\frac{D(E)}{Dt} = \frac{D\left(me + \frac{1}{2}m|\mathbf{V}|^2\right)}{Dt}$$



Internal Energy

$$\boxed{e = function(T, p)}$$

↓ 'Calorically perfect gas'

$$\boxed{e = c_v T}$$

↓ 'Ideal gas'

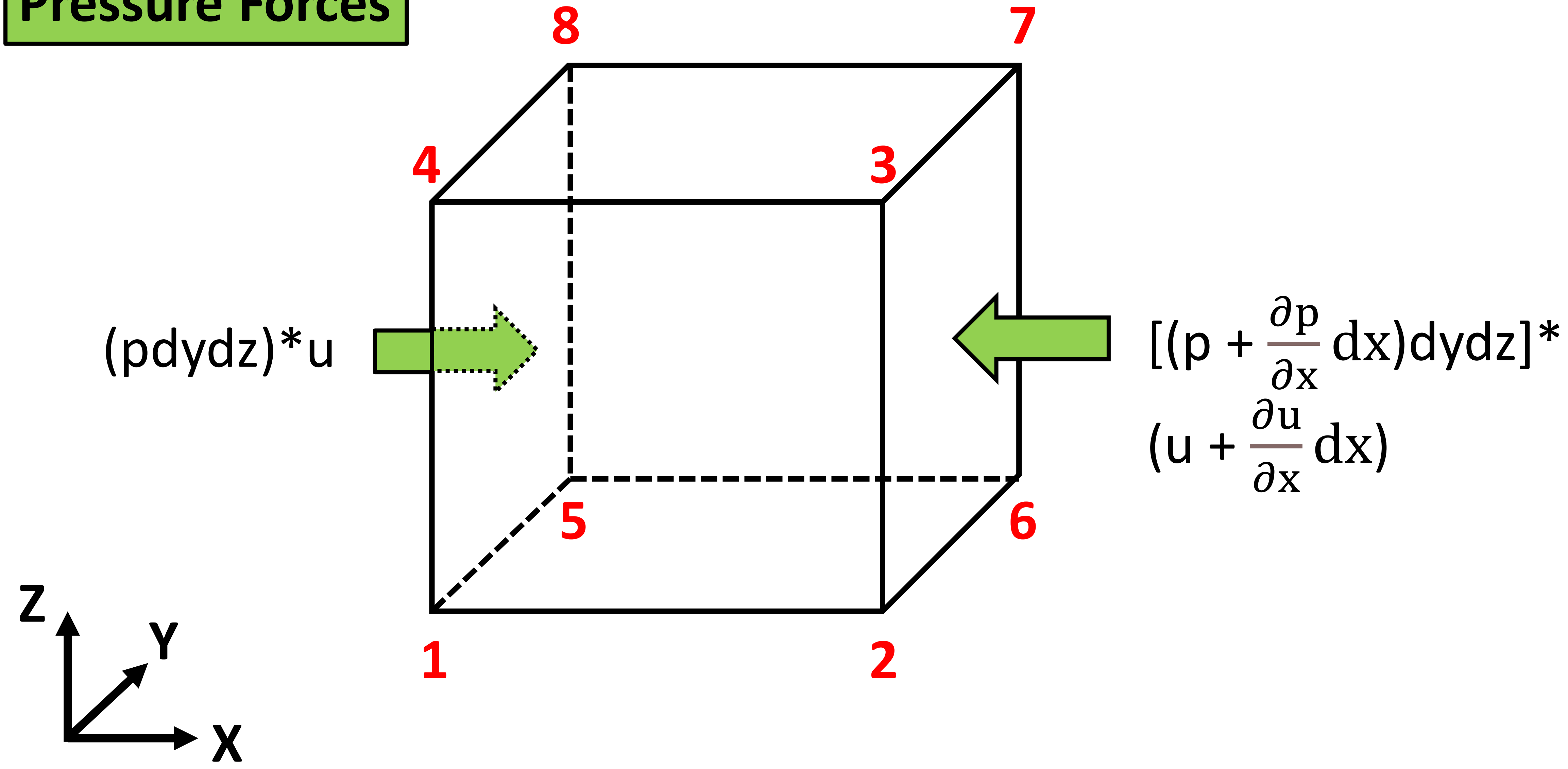
$$\boxed{e = \frac{p}{(\gamma - 1)\rho}}$$

Conservation of Energy

$$(\rho dxdydz)\frac{D\left(e+\frac{1}{2}|\mathbf{V}|^2\right)}{Dt}$$

$$\frac{D(E)}{Dt} = \Sigma(\mathbf{F.V}) + \Sigma(\dot{Q})$$

Pressure Forces

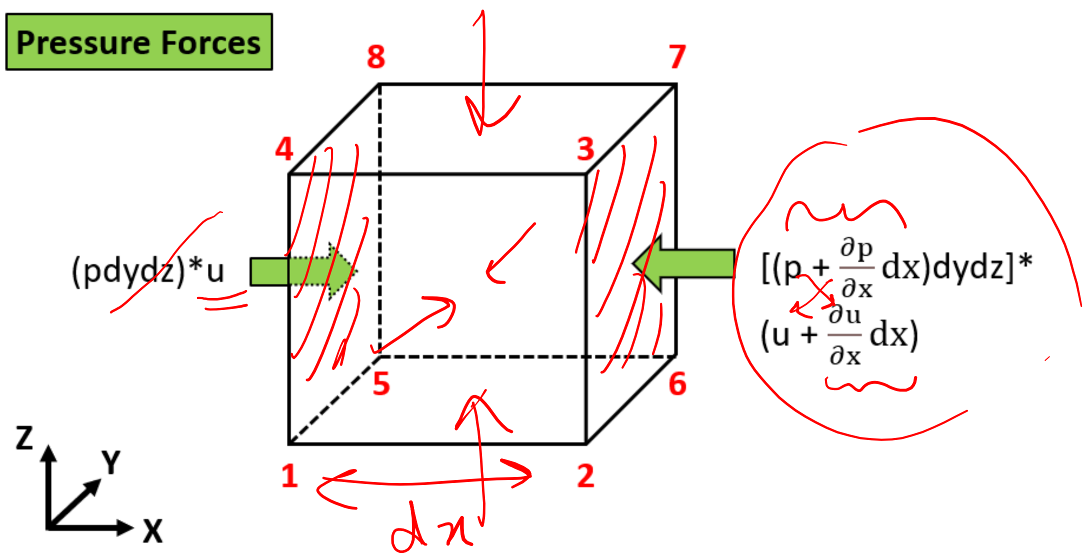


$$-\left[u\frac{\partial p}{\partial x}\,dx\,dy\,dz + p\frac{\partial u}{\partial x}\,dx\,dy\,dz\right]$$

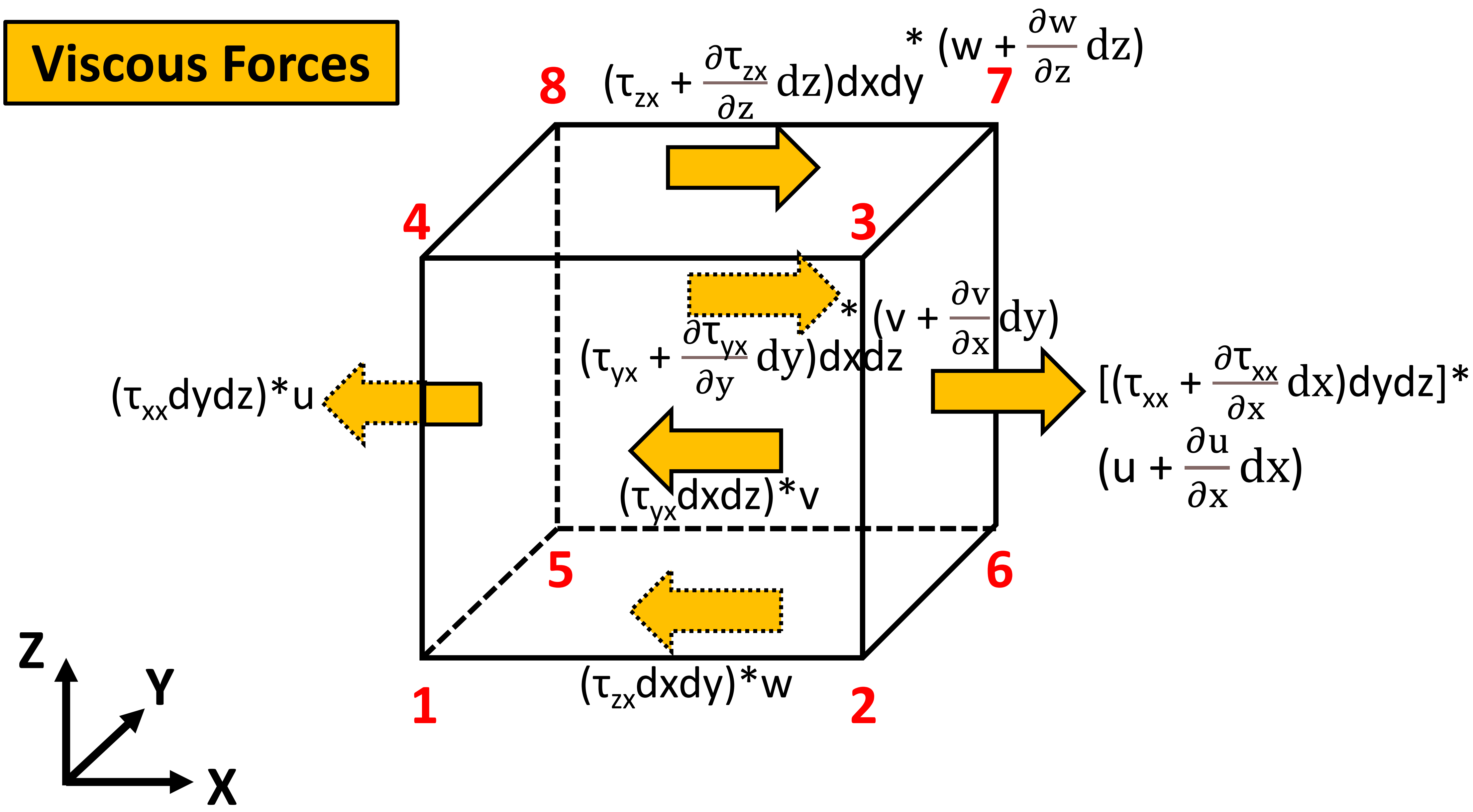
$$= -(dx\,dy\,dz)\left[u\frac{\partial p}{\partial x} + p\frac{\partial u}{\partial x}\right]$$

$$= -(\underbrace{dx\,dy\,dz})\left[\frac{\partial}{\partial x}(up)\right]$$

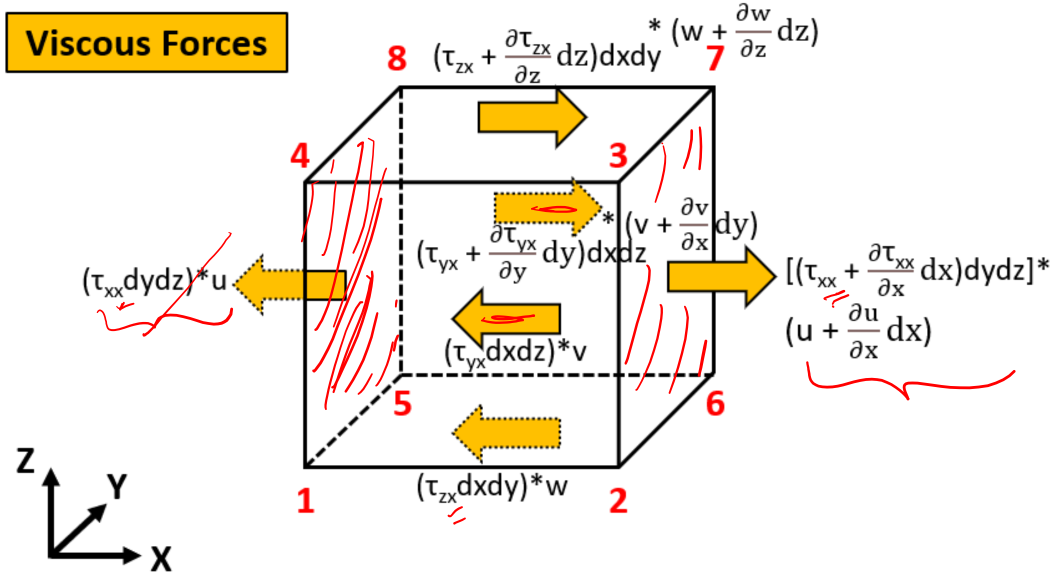
$$= -(dx\,dy\,dz)\underbrace{\left[\frac{\partial}{\partial x}(pu) + \frac{\partial}{\partial y}(pv) + \frac{\partial}{\partial z}(pw)\right]}$$



Viscous Forces



Viscous Forces



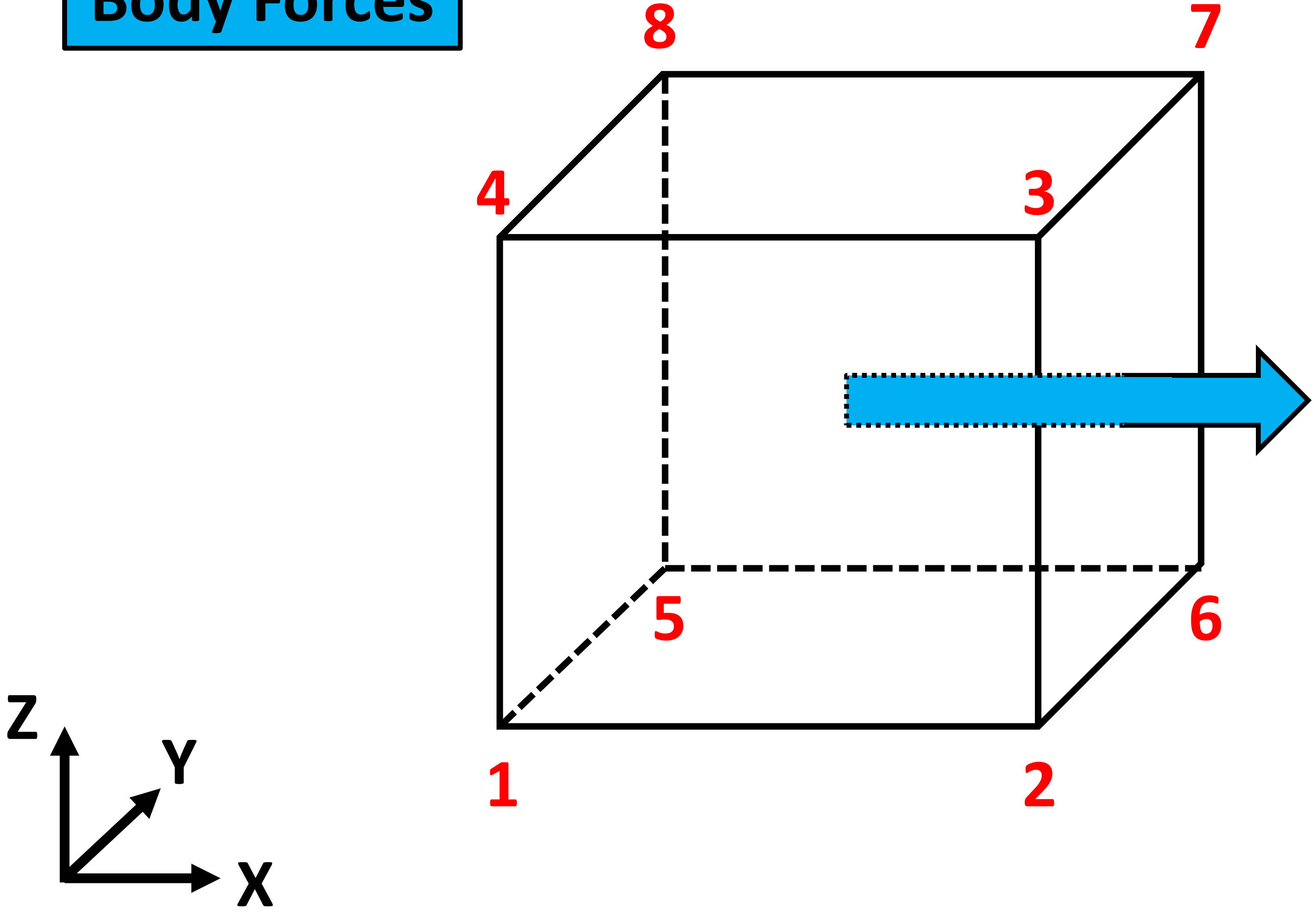
$$dx\,dy\,dz\underbrace{\left[u\frac{\partial \tau_{xx}}{\partial x}+\tau_{xx}\frac{\partial u}{\partial x}\right]}$$

$$dx\,dy\,dz\left[\frac{\partial}{\partial x}(u\tau_{xx})\right]$$

$$+dx\,dy\,dz\left[\frac{\partial}{\partial y}(v\tau_{yx})\right]$$

$$+dx\,dy\,dz\left[\frac{\partial}{\partial z}(w\tau_{zx})\right]$$

Body Forces



$$\overset{g}{f_x}\underbrace{(\rho dxdydz)}_{m}{}^{*}u$$

$$+\ f_y(\quad)\,v$$

$$+\ f_z(\quad)\,w$$

Conservation of Energy

$$(\rho dxdydz)\frac{D(e+\frac{1}{2}|\mathbf{V}|^2)}{Dt}$$

$$\frac{D(E)}{Dt} = \Sigma(\mathbf{F.V}) + \Sigma(\dot{Q})$$

$$(\rho dxdydz)[-\frac{\partial(pu)}{\partial x} - \frac{\partial(pv)}{\partial y} - \frac{\partial(pw)}{\partial z} + \frac{\partial(\tau_{xx}u)}{\partial x} + \frac{\partial(\tau_{yx}u)}{\partial y} + \frac{\partial(\tau_{zx}u)}{\partial z} + \frac{\partial(\tau_{xy}v)}{\partial x}$$

$$+ \frac{\partial(\tau_{yy}v)}{\partial y} + \frac{\partial(\tau_{zy}v)}{\partial z} + \frac{\partial(\tau_{xz}w)}{\partial x} + \frac{\partial(\tau_{yz}w)}{\partial y} + \frac{\partial(\tau_{zz}w)}{\partial z} + f_x u + f_y v + f_z w]$$

Conservation of Energy

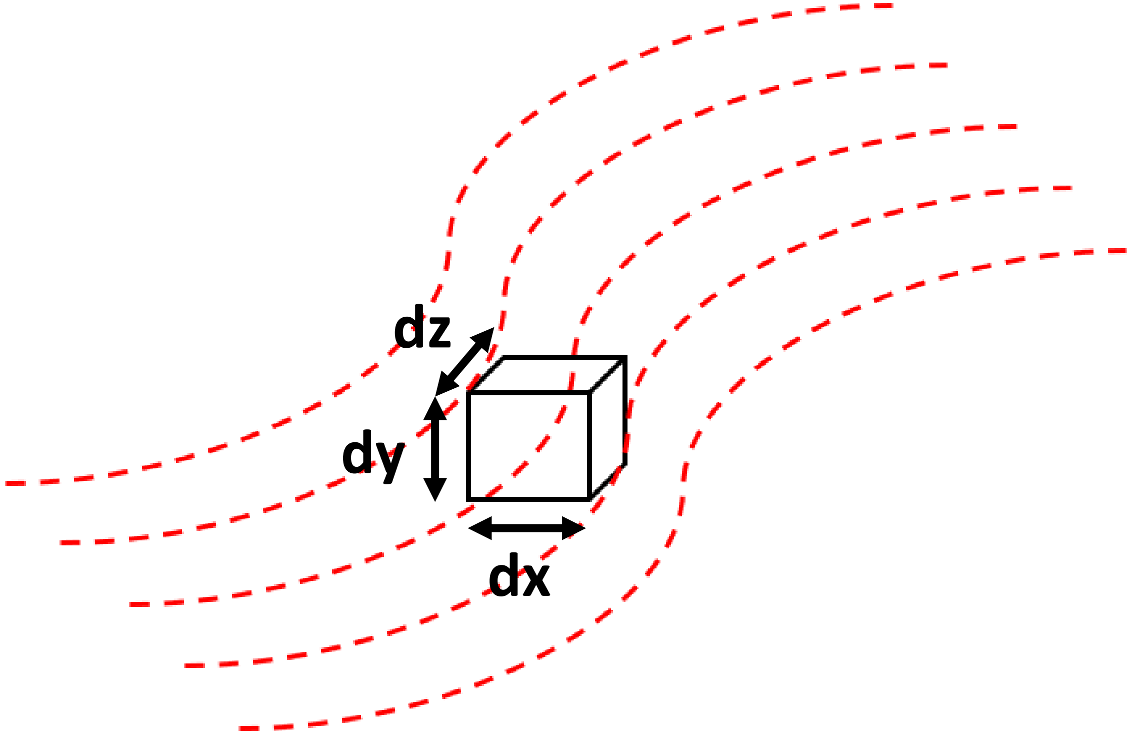
$$\frac{D(E)}{Dt} = \Sigma(F.V) + \Sigma(\dot{Q})$$

Volumetric Heating

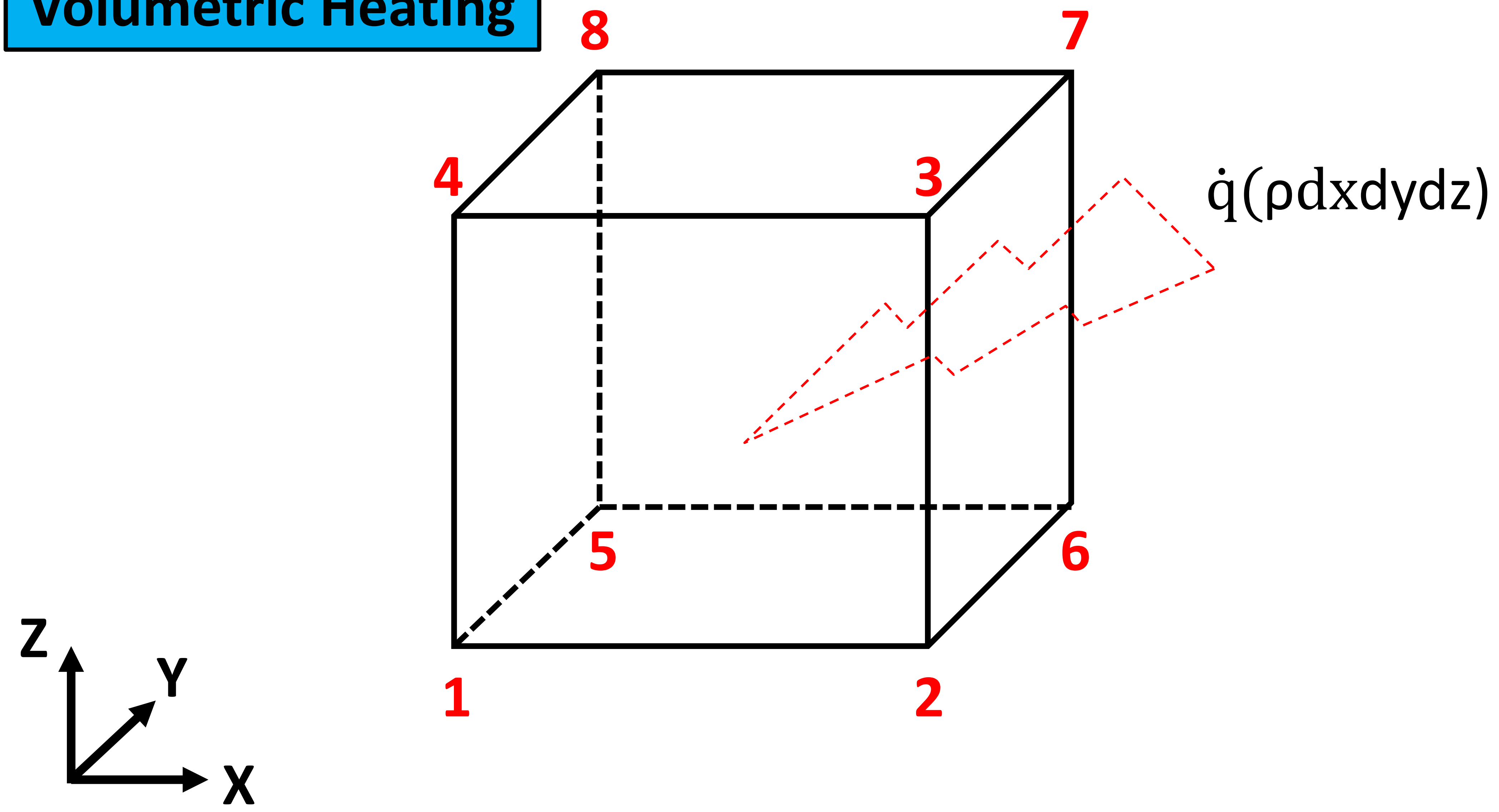
1	**Radiation**

Surface Heating

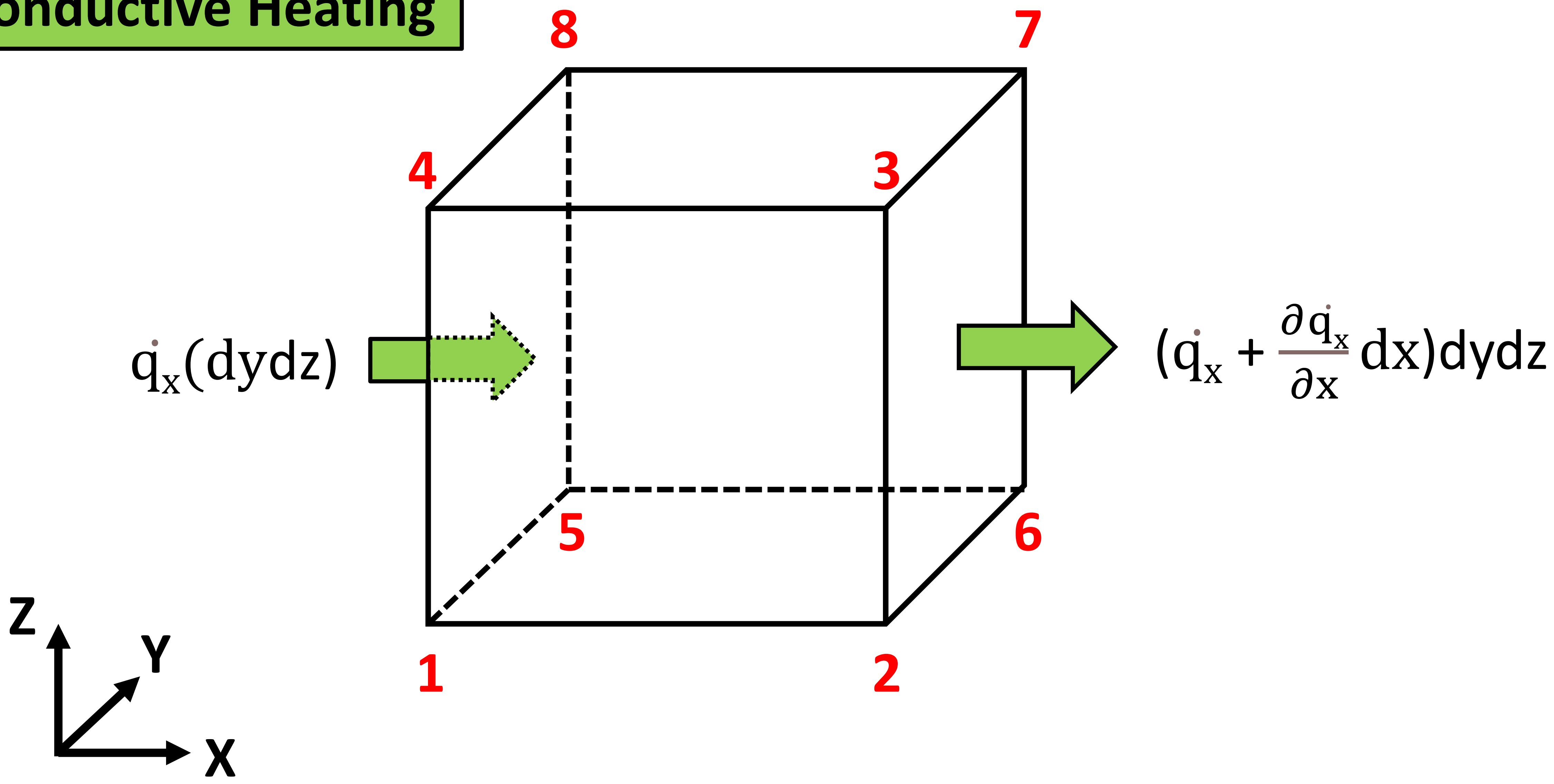
1	**Conduction**



Volumetric Heating



Conductive Heating



Conservation of Energy

$$(\rho dxdydz)\frac{D\left(e + \frac{1}{2}|\mathbf{V}|^2\right)}{Dt}$$

$$(dxdydz)\left[-\frac{\partial \dot{q}_x}{\partial x} - \frac{\partial \dot{q}_y}{\partial y} - \frac{\partial \dot{q}_z}{\partial z} + \rho\dot{q}\right]$$

$$\frac{D(E)}{Dt} = \Sigma(\mathbf{F.V}) + \Sigma(\dot{Q})$$

$$(\rho dxdydz)\left[-\frac{\partial(pu)}{\partial x} - \frac{\partial(pv)}{\partial y} - \frac{\partial(pw)}{\partial z} + \frac{\partial(\tau_{xx}u)}{\partial x} + \frac{\partial(\tau_{yx}u)}{\partial y} + \frac{\partial(\tau_{zx}u)}{\partial z} + \frac{\partial(\tau_{xy}v)}{\partial x} + \frac{\partial(\tau_{yy}v)}{\partial y} + \frac{\partial(\tau_{zy}v)}{\partial z} + \frac{\partial(\tau_{xz}w)}{\partial x} + \frac{\partial(\tau_{yz}w)}{\partial y} + \frac{\partial(\tau_{zz}w)}{\partial z} + f_x u + f_y v + f_z w\right]$$

Fourier’s Law

$$\boxed{\dot{q}_x = -k\,\frac{\partial T}{\partial x}}$$

Conservation of Energy

$$(\rho dxdydz)\frac{D(e+\frac{1}{2}|\mathbf{V}|^2)}{Dt}$$

$$(dxdydz)[\frac{\partial}{\partial x}(k\frac{\partial T}{\partial x})+\frac{\partial}{\partial y}(k\frac{\partial T}{\partial y})+\frac{\partial}{\partial z}(k\frac{\partial T}{\partial z})+\rho\dot{q}]$$

$$\frac{D(E)}{Dt} = \Sigma(\mathbf{F.V}) + \Sigma(\dot{Q})$$

$$(dxdydz)[\frac{\partial(pu)}{\partial x}+\frac{\partial(pv)}{\partial y}+\frac{\partial(pw)}{\partial z}+\frac{\partial(\tau_{xx}u)}{\partial x}+\frac{\partial(\tau_{xx}u)}{\partial x}+\frac{\partial(\tau_{xx}u)}{\partial x}+\frac{\partial(\tau_{xx}u)}{\partial x}$$
$$+\frac{\partial(\tau_{xx}u)}{\partial x}+\frac{\partial(\tau_{xx}u)}{\partial x}+\frac{\partial(\tau_{xx}u)}{\partial x}+\frac{\partial(\tau_{xx}u)}{\partial x}+\frac{\partial(\tau_{xx}u)}{\partial x}+\rho\,(f_xu+f_yv+f_zw)]$$

Conservation of Energy

$$(\rho dxdydz)\frac{D\left(e+\frac{1}{2}|\mathbf{V}|^2\right)}{Dt} = (dxdydz)\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\rho\dot{q}\right] + (dxdydz)\left[-\frac{\partial(pu)}{\partial x}-\frac{\partial(pv)}{\partial y}-\frac{\partial(pw)}{\partial z}+\frac{\partial(\tau_{xx}u)}{\partial x}+\frac{\partial(\tau_{yx}u)}{\partial y}+\frac{\partial(\tau_{zx}u)}{\partial z}+\frac{\partial(\tau_{xy}v)}{\partial x}+\frac{\partial(\tau_{yy}v)}{\partial y}+\frac{\partial(\tau_{zy}v)}{\partial z}+\frac{\partial(\tau_{xz}w)}{\partial x}+\frac{\partial(\tau_{yz}w)}{\partial y}+\frac{\partial(\tau_{zz}w)}{\partial z}+\rho\,(f_x u+f_y v+f_z w)\right]$$

Conservation of Energy

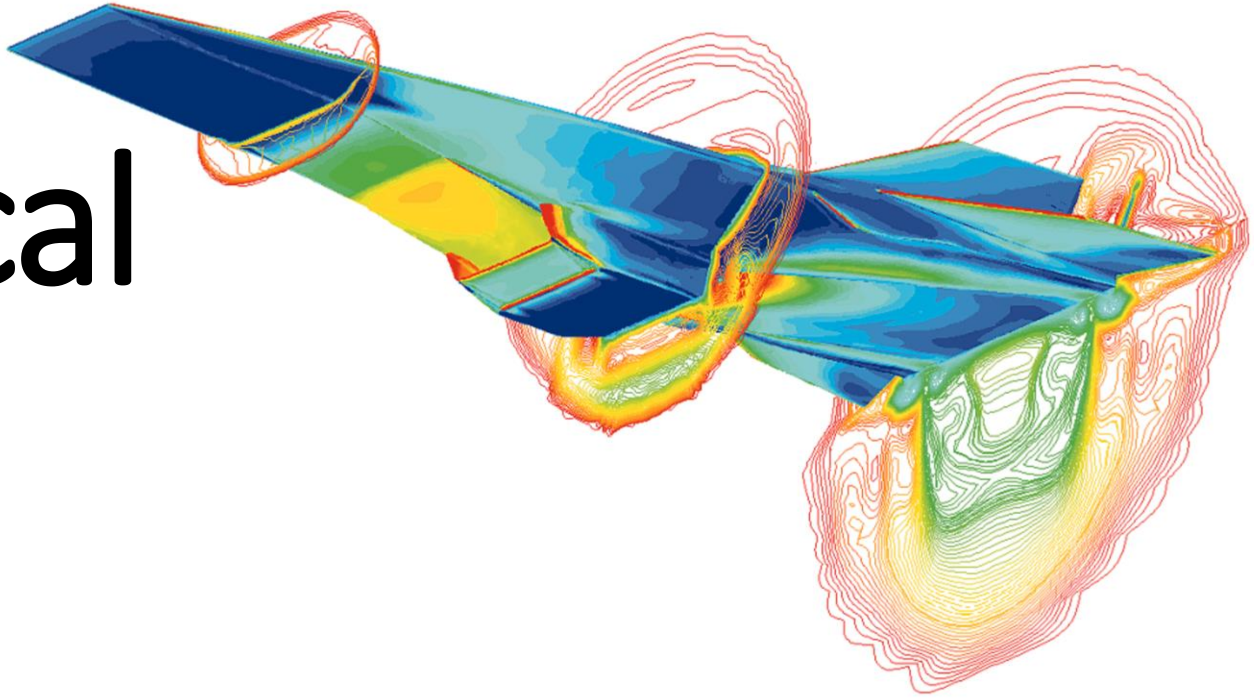
$$\frac{\rho D\left(e+\frac{1}{2}|\mathbf{V}|^2\right)}{Dt} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) +$$

$$\rho\dot{q} - \frac{\partial(pu)}{\partial x} - \frac{\partial(pv)}{\partial y} - \frac{\partial(pw)}{\partial z} + \frac{\partial(\tau_{xx}u)}{\partial x} + \frac{\partial(\tau_{yx}u)}{\partial y} +$$

$$\frac{\partial(\tau_{zx}u)}{\partial z} + \frac{\partial(\tau_{xy}v)}{\partial x} + \frac{\partial(\tau_{yy}v)}{\partial y} + \frac{\partial(\tau_{zy}v)}{\partial z} + \frac{\partial(\tau_{xz}w)}{\partial x} +$$

$$\frac{\partial(\tau_{yz}w)}{\partial y} + \frac{\partial(\tau_{zz}w)}{\partial z} + \rho\,(fxu + fyv + fzw)$$

Hierarchy of Fluid Dynamical Models



SEBASTIAN THOMAS

$$\frac{D\rho}{Dt} + \rho\nabla.\mathbf{V} = 0$$

$$\frac{\rho D(u)}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x\rho$$

$$\frac{\rho D(v)}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y\rho$$

$$\frac{\rho D(w)}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + f_z\rho$$

$$\frac{\rho D\left(e + \frac{1}{2}|\mathbf{V}|^2\right)}{Dt} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \rho\dot{q} - \frac{\partial(pu)}{\partial x} - \frac{\partial(pv)}{\partial y} - \frac{\partial(pw)}{\partial z} + \frac{\partial(\tau_{xx}u)}{\partial x} + \frac{\partial(\tau_{yx}u)}{\partial y} + \frac{\partial(\tau_{zx}u)}{\partial z} + \frac{\partial(\tau_{xy}v)}{\partial x} + \frac{\partial(\tau_{yy}v)}{\partial y} + \frac{\partial(\tau_{zy}v)}{\partial z} + \frac{\partial(\tau_{xz}w)}{\partial x} + \frac{\partial(\tau_{yz}w)}{\partial y} + \frac{\partial(\tau_{zz}w)}{\partial z} + \rho\,(fxu + fyv + fzw)$$

$$e = C_v T$$

$$p = \rho R T$$

$$\tau_{xx} = \frac{2\mu}{3}\left[2\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right]$$

$$\tau_{xy}, \tau_{yx} = \mu\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

$$\tau_{yy} = \frac{2\mu}{3}\left[-\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right]$$

$$\tau_{xz}, \tau_{zx} = \mu\left[\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right]$$

$$\tau_{zz} = \frac{2\mu}{3}\left[-\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + 2\frac{\partial w}{\partial z}\right]$$

$$\tau_{yz}, \tau_{zy} = \mu\left[\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right]$$

$$\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + \boldsymbol{V} \cdot \nabla(\,)$$

$$\mu = \mu_{ref}\left(\frac{T}{T_{ref}}\right)^{3/2}\left[\frac{T_{ref}+S}{T+S}\right]$$

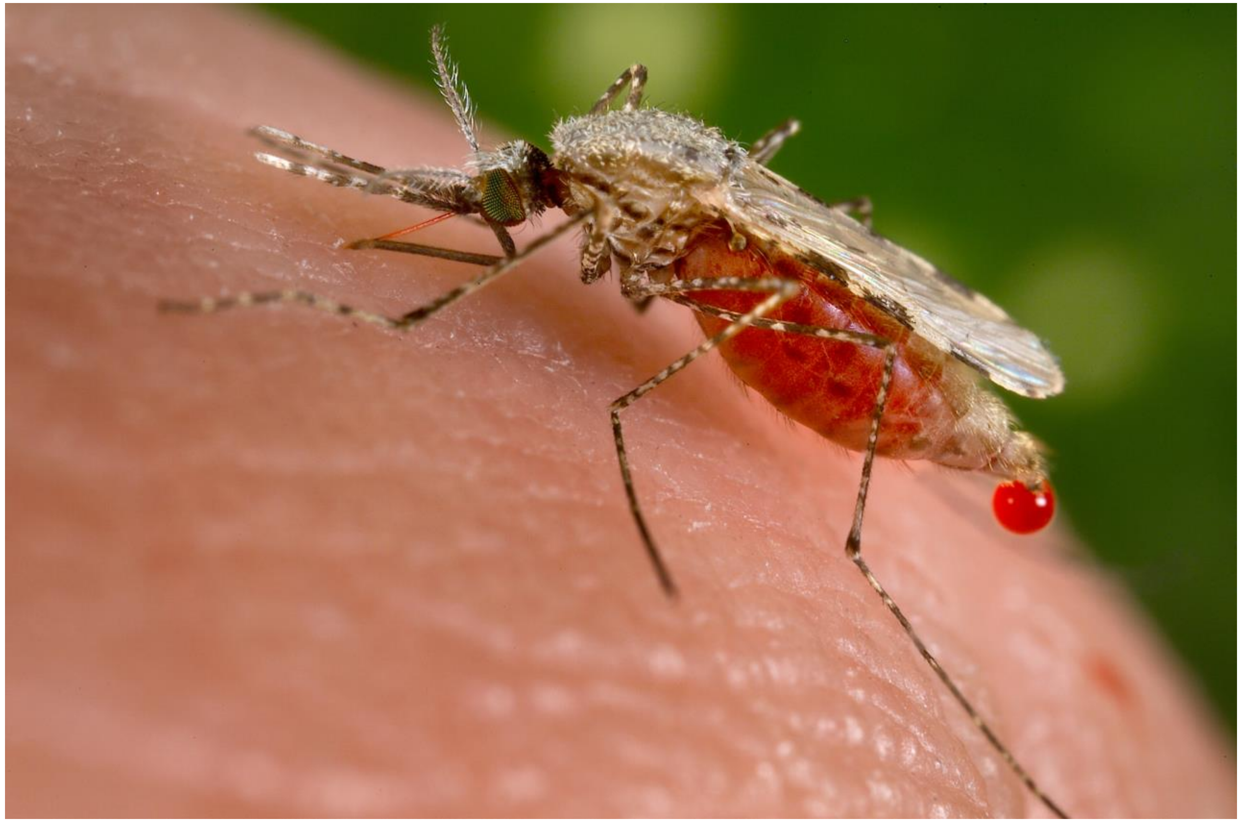
List of unknowns

Unknowns	Appearance(s)
ρ	Mass, Momentum, Energy, Ideal Gas
p	Momentum, Ideal Gas
u, v, w	Mass, Momentum, Energy
T	Energy, Ideal Gas
e	Energy, Linear function of temperature
μ	Function of temperature ('Sutherland's Law')

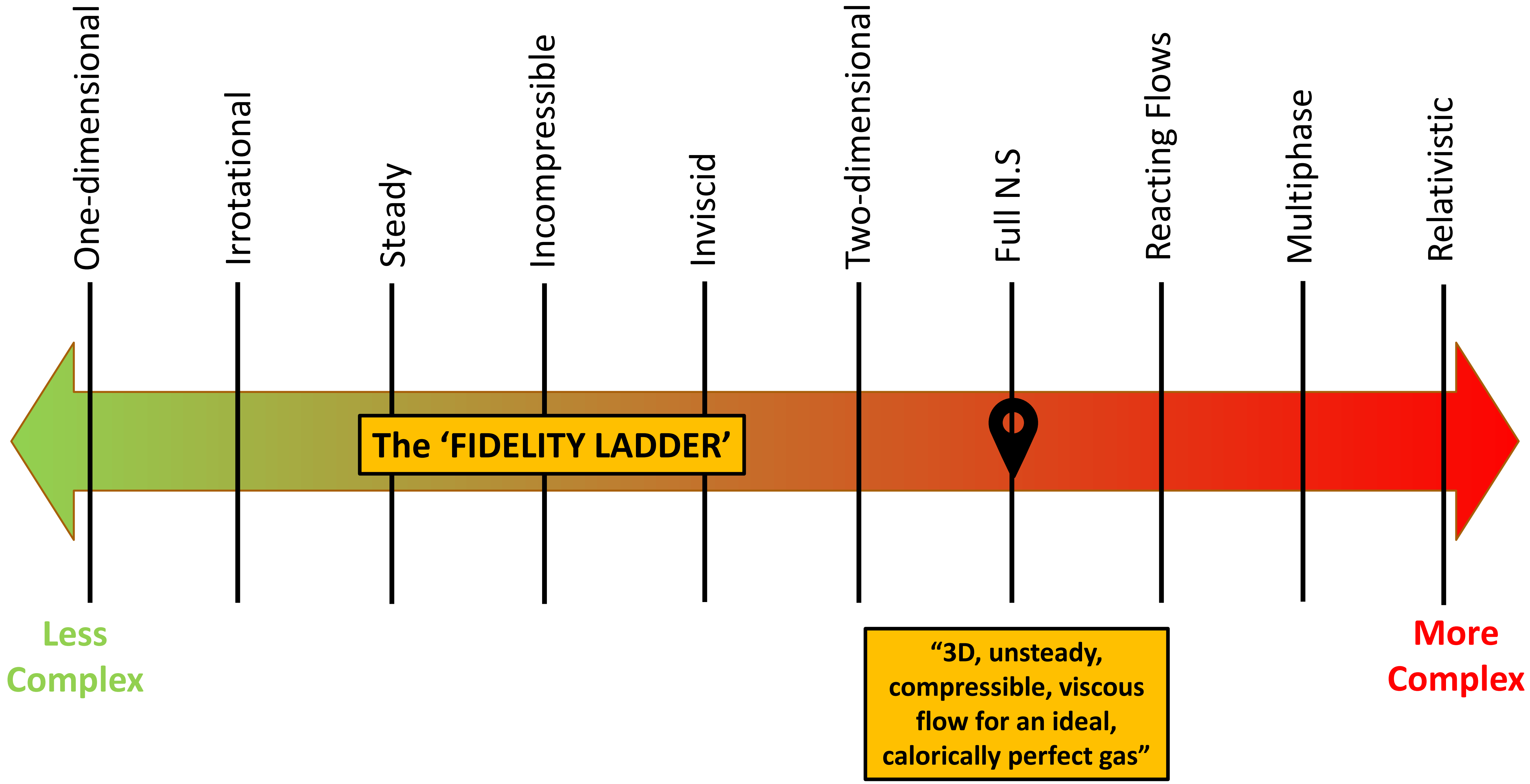
The Case for Simplification

Simplify the Navier-Stokes equations whenever justifiable!

The Navier-Stokes System of Equations



The problem you're simulating



One-dimensional
Irrotational
Steady
Incompressible
Inviscid
Two-dimensional
Full N.S
Reacting Flows
Multiphase
Relativistic
The 'FIDELITY LADDER'
"3D, unsteady, compressible, viscous flow for an ideal, calorically perfect gas"
Less Complex
More Complex