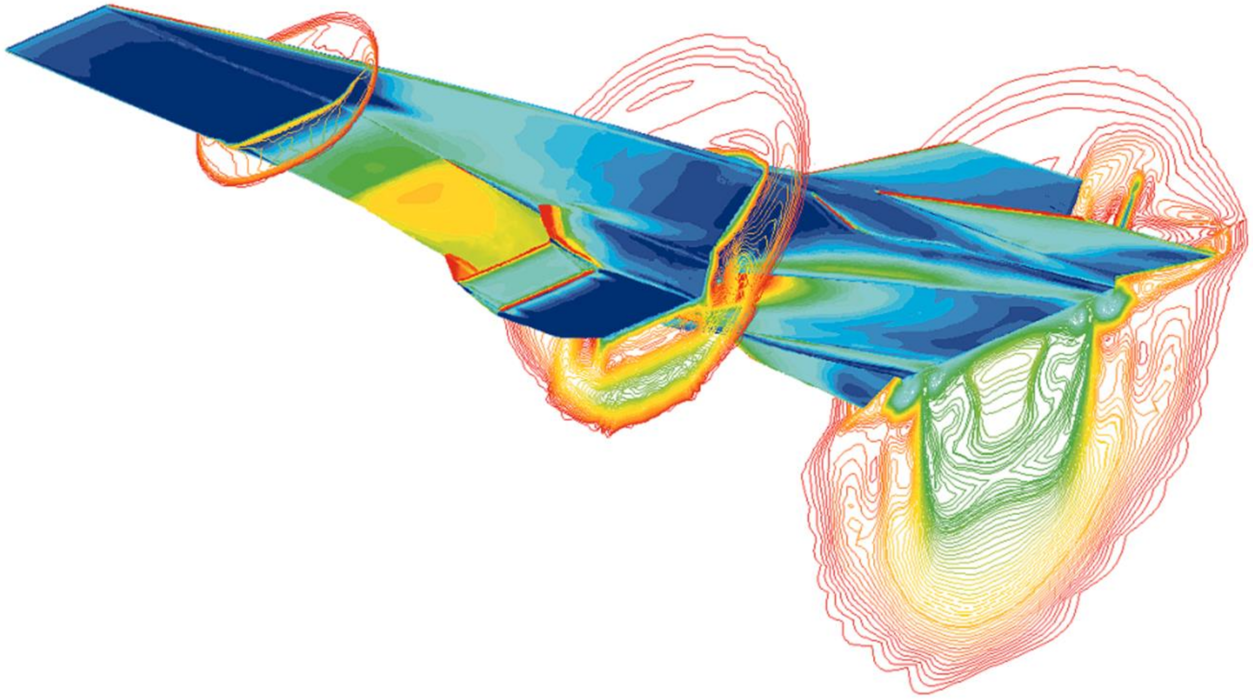
The Taylor Series Expansion

SEBASTIAN THOMAS



Taylor Series

- An approximating tool that *translates* between the worlds of the infinitesimal and the finite
- Developed by Brook Taylor in the early 18th century
- This ‘translator’ allows us to:

  - approximate functions with polynomials of arbitrary size when derivatives are known

  - approximate derivatives of functions when function values are known

Taylor Series

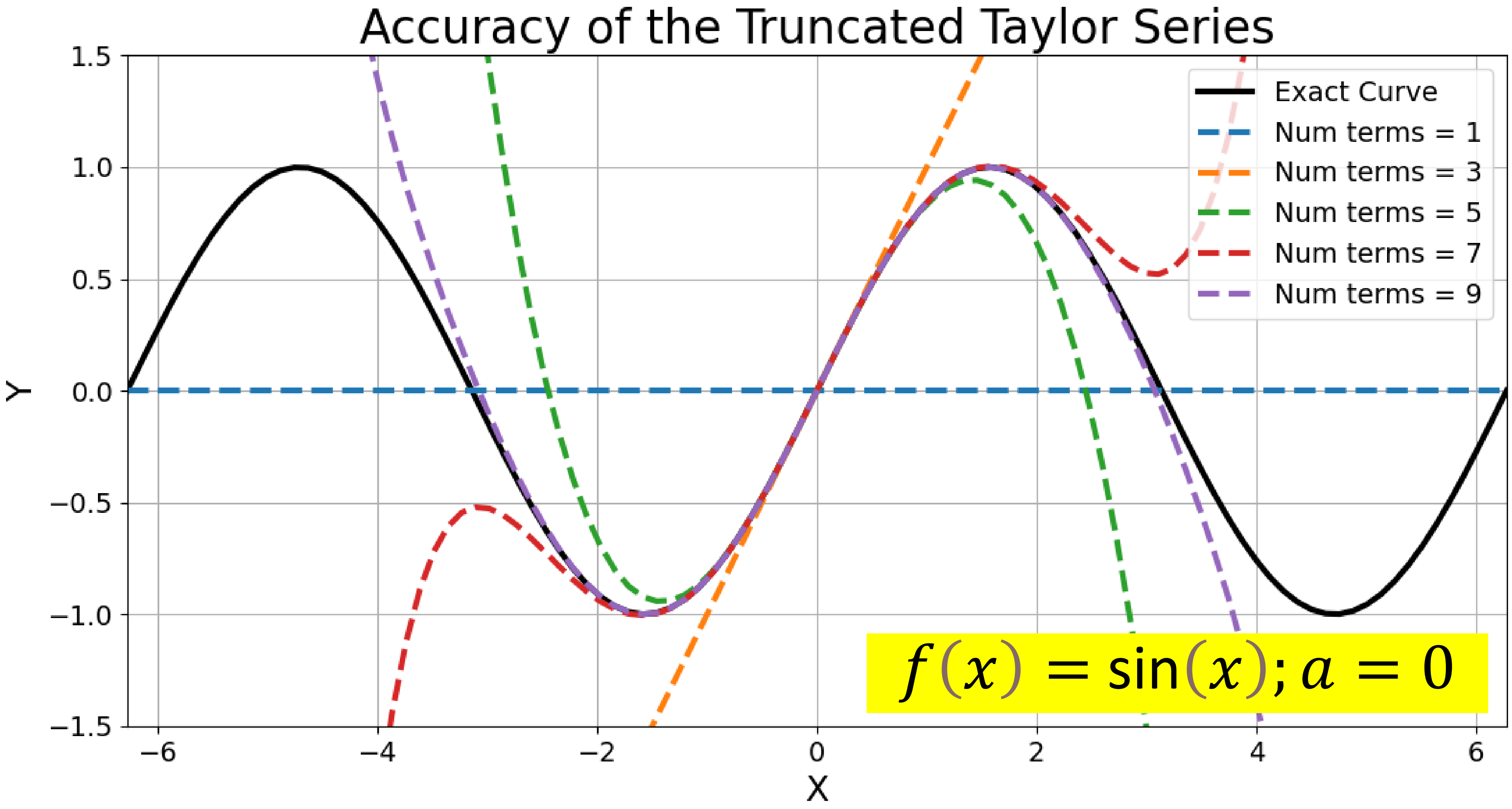
$$f(x) - f(a) = \int_a^x f'(t)\,dt$$

To derive, begin with the fundamental theorem of calculus and repeatedly apply the 'integration by parts' operation

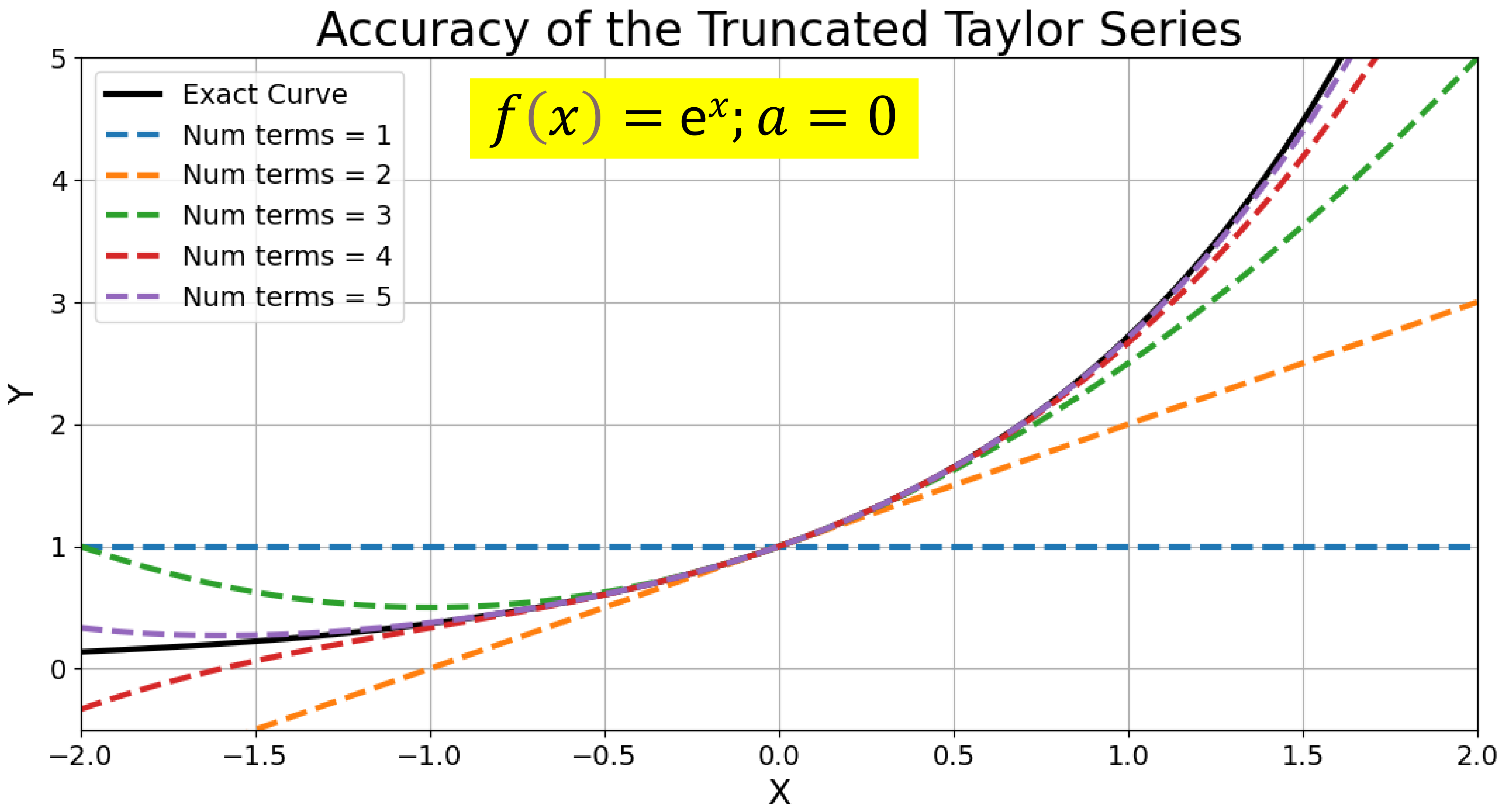
$$\Downarrow$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots$$

Error in lecture: replace term counts 3, 5, 7... with 2, 4, 6... (term indices in original Taylor series)

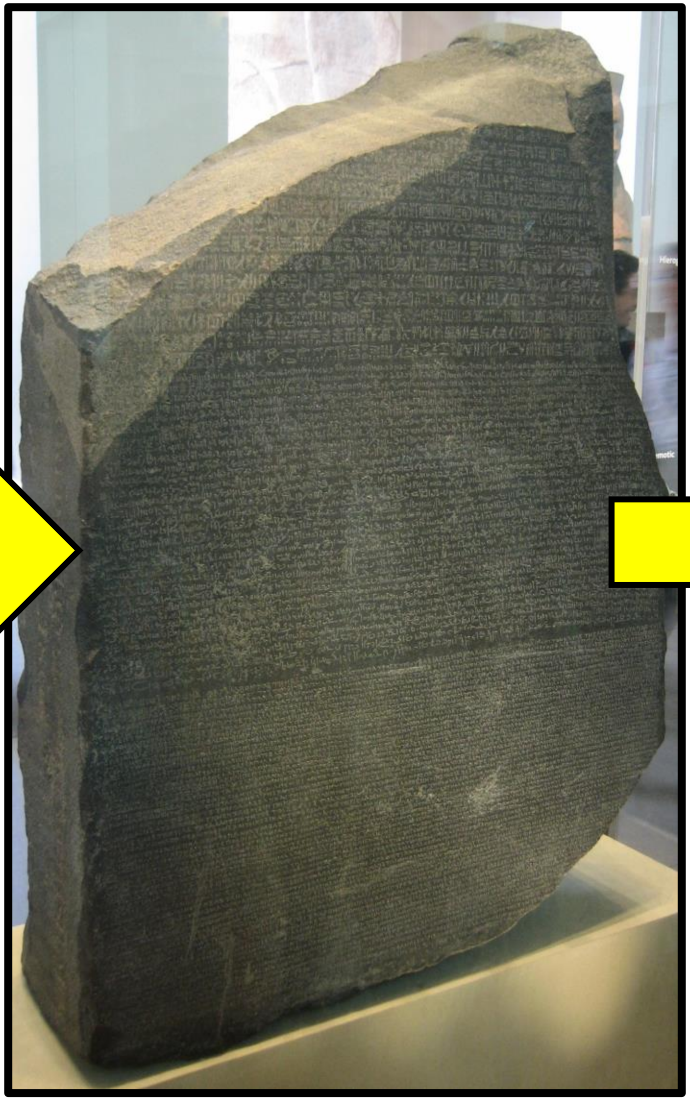


$f(x) = \sin(x); a = 0$


Accuracy of the Truncated Taylor Series
$f(x) = \mathrm{e}^x; a = 0$
Exact Curve
Num terms = 1
Num terms = 2
Num terms = 3
Num terms = 4
Num terms = 5
Y
X
5
4
3
2
1
0
−2.0
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5
2.0

Two ways in which the Taylor series ‘translator’ can work

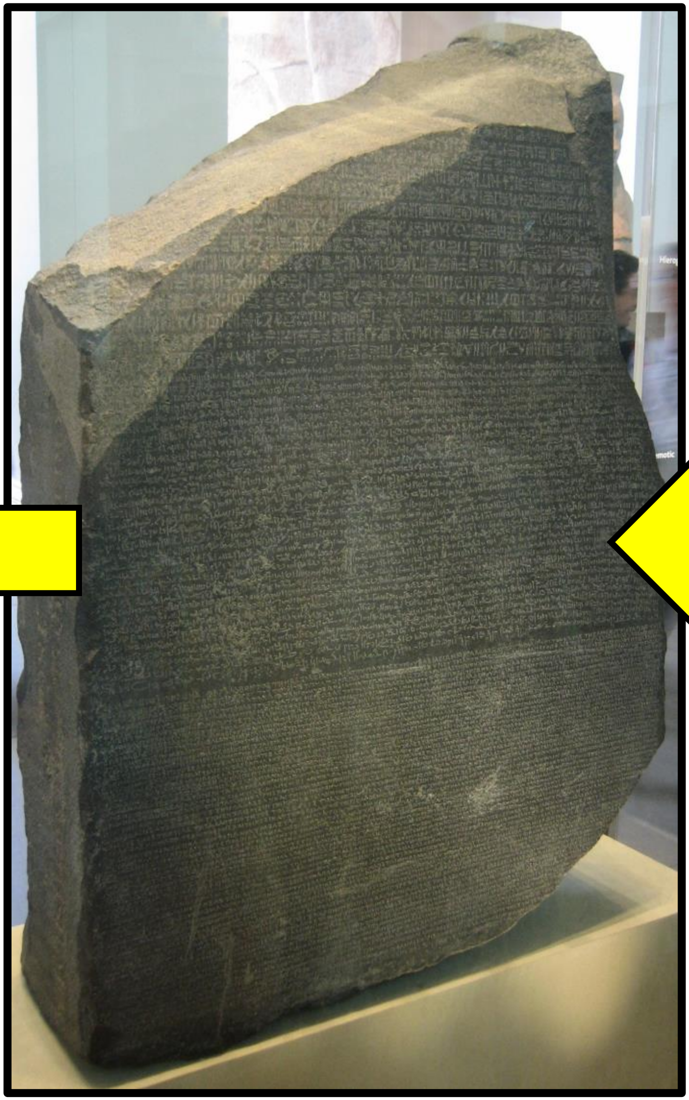
Function value and derivatives at point ‘a’



Taylor Series

Approximate function value at neighboring point ‘b’

Approximate derivatives at point ‘a’ ???



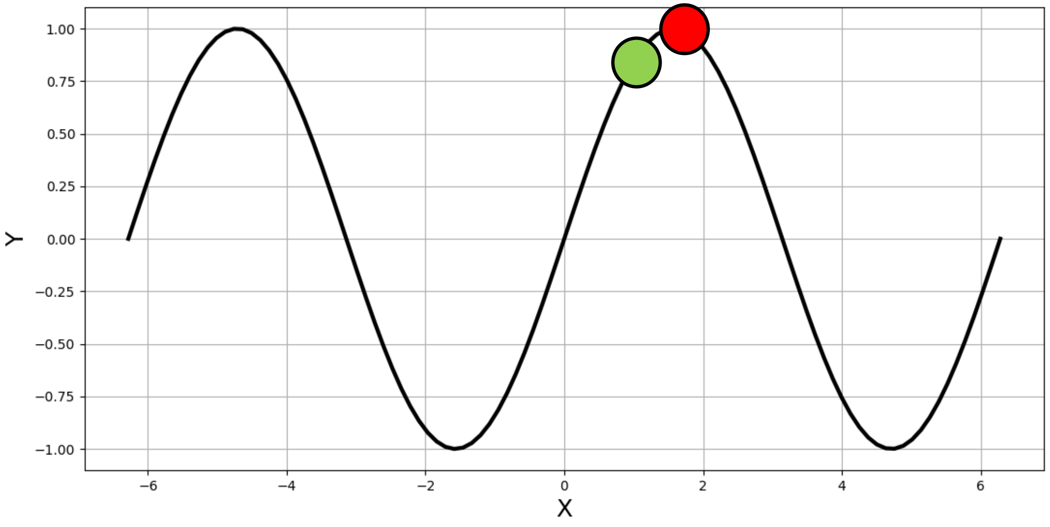
Taylor Series

Function values at ‘a’ and neighboring points

Example of a Taylor Series Approximation

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \frac{(b-a)^3}{3!}f'''(a) + \cdots$$

Let $f(x) = \sin(x); a = \frac{\pi}{2} - 0.35, b = \frac{\pi}{2} + 0.15$



Example of a Taylor Series Approximation

$$f'(a) = \frac{f(b) - f(a)}{b - a} + O\{(b - a)\}$$

$$f'(a) \approx \frac{f(b) - f(a)}{b - a}$$

Let $f(x) = \sin(x); a = \frac{\pi}{2} - 0.35, b = \frac{\pi}{2} + 0.15$

Exact Value of $f'(a)$	0.3429
Approx Value of $f'(a)$	0.0987

Can we do better?

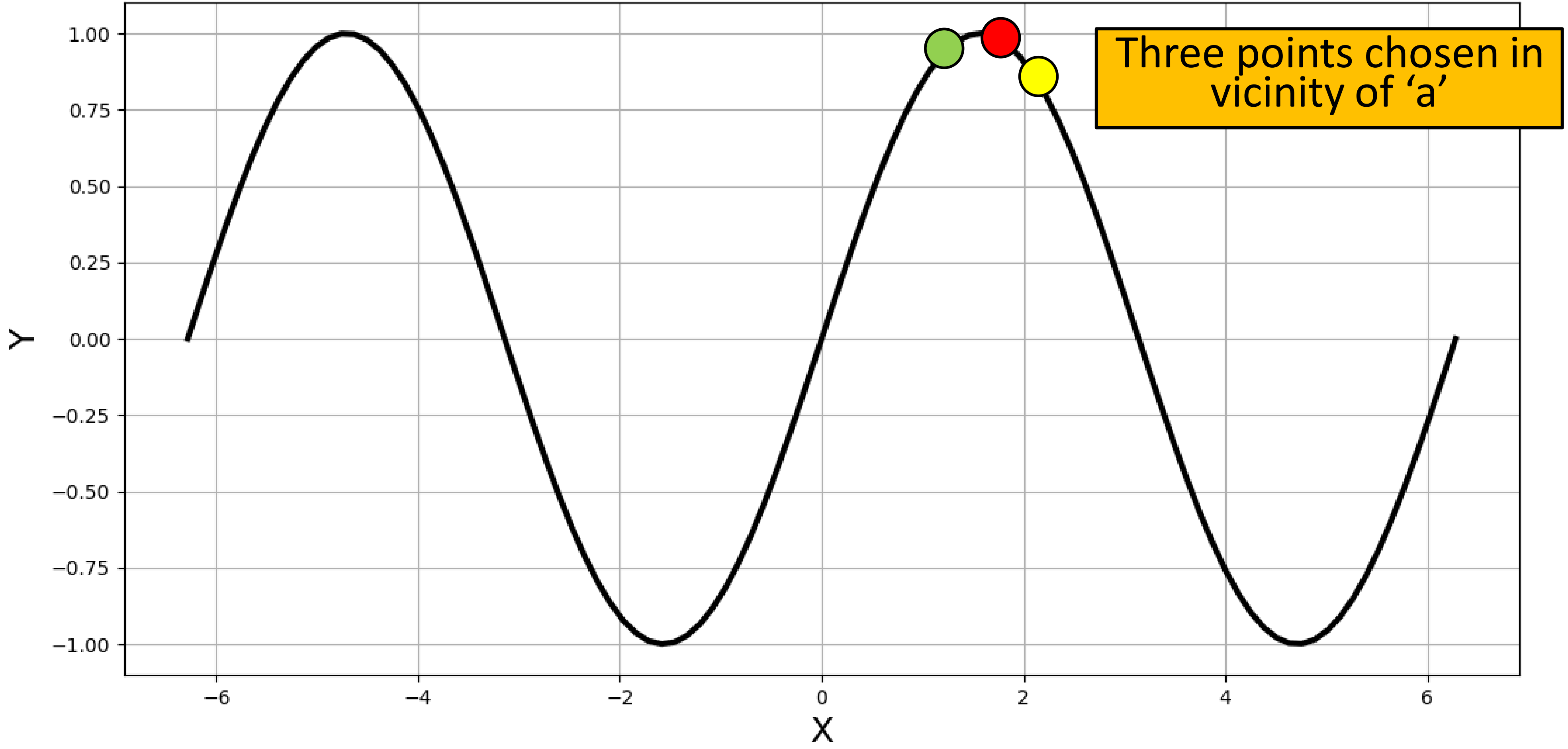
A Quick Variable Change

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \frac{(b-a)^3}{3!}f'''(a) + \cdots$$

Let $b = a + \Delta x$

$$f(a+\Delta x) = f(a) + (\Delta x)f'(a) + \frac{(\Delta x)^2}{2!}f''(a) + \frac{(\Delta x)^3}{3!}f'''(a) + \cdots$$

Derivative Approximation Revisited



Derivative Approximation Revisited

$$f(a) = f(a)$$

$$f(a+\Delta x) = f(a) + (\Delta x)f'(a) + \frac{(\Delta x)^2}{2!}f''(a) + \frac{(\Delta x)^3}{3!}f'''(a) + \cdots$$

$$f(a+2\Delta x) = f(a) + (2\Delta x)f'(a) + \frac{(2\Delta x)^2}{2!}f''(a) + \frac{(2\Delta x)^3}{3!}f'''(a) + \cdots$$

Find k_1, k_2 and k_3 such that:

$$f'(a) = \frac{k_1 f(a) + \mathrm{k}_2 f(a+\Delta x) + \mathrm{k}_3 f(a+2\Delta x)}{\Delta x} + Error$$

$$f(a) = f(a)$$

$$f(a+\Delta x) = f(a) + (\Delta x) f'(a) + \frac{(\Delta x)^2}{2!} f''(a) + \frac{(\Delta x)^3}{3!} f'''(a) + \cdots$$

$$f(a+2\Delta x) = f(a) + (2\Delta x) f'(a) + \frac{(2\Delta x)^2}{2!} f''(a) + \frac{(2\Delta x)^3}{3!} f'''(a) + \cdots$$

$$k_1(\text{Eq } 1) + k_2(\text{Eq } 2) + k_3(\text{Eq } 3)$$

$$k_1 f(a) + \mathrm{k}_2 f(a+\Delta x) + \mathrm{k}_3 f(a+2\Delta x)$$

	Coefficient of $f(a)$	Coefficient of $(\Delta x) f'(a)$	Coefficient of $\frac{(\Delta x)^2}{2!} f''(a)$	Coefficient of $\frac{(\Delta x)^3}{3!} f'''(a)$
	k_1	0	0	0
	k_2	k_2	k_2	k_2
	k_3	$2k_3$	$4k_3$	$8k_3$
Sum →	$\mathbf{k_1 + k2 + k_3}$	$\mathbf{k_2 + 2k_3}$	$\mathbf{k_2 + 4k_3}$	$\mathbf{k_2 + 8k_3}$

	Coefficient of $f(a)$	Coefficient of $(\Delta x)f'(a)$	Coefficient of $\frac{(\Delta x)^2}{2!}f''(a)$	Coefficient of $\frac{(\Delta x)^3}{3!}f'''(a)$
	k_1	0	0	0
	k_2	k_2	k_2	k_2
	k_3	$2k_3$	$4k_3$	$8k_3$
Sum →	$\boldsymbol{k_1+k_2+k_3}$	$\boldsymbol{k_2+2k_3}$	$\boldsymbol{k_2+4k_3}$	$\boldsymbol{k_2+8k_3}$

$$\begin{aligned} \boldsymbol{k_1+k_2+k_3} &= 0 \\ \boldsymbol{k_2+2k_3} &= 1 \\ \boldsymbol{k_2+4k_3} &= 0 \end{aligned} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \boldsymbol{k_1} \\ \boldsymbol{k_2} \\ \boldsymbol{k_3} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$\boldsymbol{k_1}$	-1.5
$\boldsymbol{k_2}$	2
$\boldsymbol{k_3}$	-0.5

$$f(a) \quad = \quad f(a)$$

$$f(a+\Delta x) \quad = \quad f(a) + (\Delta x)f'(a) + \frac{(\Delta x)^2}{2!}f''(a) + \frac{(\Delta x)^3}{3!}f'''(a) + \cdots$$

$$f(a+2\Delta x) = f(a) + (2\Delta x)f'(a) + \frac{(2\Delta x)^2}{2!}f''(a) + \frac{(2\Delta x)^3}{3!}f'''(a) + \cdots$$

$$\boxed{k_1(\text{Eq }1) + k_2(\text{Eq }2) + k_3(\text{Eq }3)}$$

$$\boxed{-1.5f(a) + 2f(a+\Delta x) - 0.5f(a+2\Delta x)}$$

	Coefficient of $f(a)$	Coefficient of $(\Delta x)f'(a)$	Coefficient of $\frac{(\Delta x)^2}{2!}f''(a)$	Coefficient of $\frac{(\Delta x)^3}{3!}f'''(a)$
	-1.5	0	0	0
	2	2	2	2
	-0.5	-1	-2	-4
Sum →	**0**	**1**	**0**	**−2**

$$-1.5f(a) + 2f(a+\Delta x) - 0.5f(a+2\Delta x) = (\Delta x)f'(a) + \frac{(\Delta x)^3}{3!}f'''(a) + \cdots$$

$$f'(a) = \frac{-0.5f(a+2\Delta x) + 2f(a+\Delta x) - 1.5f(a)}{\Delta x} + O\{(\Delta x)^2\}$$

$$f'(a) \approx \frac{-0.5f(a+2\Delta x) + 2f(a+\Delta x) - 1.5f(a)}{\Delta x}$$

$$f'(a) \approx \frac{-f(a+2\Delta x) + 4f(a+\Delta x) - 3f(a)}{2\Delta x}$$

An example of a ‘**central**’ scheme. Could also have chosen points on one side of f(a) in which case the scheme would have been ‘**forward**’ or ‘**backward**’

$$f'(a) \approx \frac{-f(a+2\Delta x) + 4f(a+\Delta x) - 3f(a)}{2\Delta x}$$

		Error %
Exact Value of $f'(a)$	0.3429	N/A
1st-order Approx Value of $f'(a)$	0.0987	71.2%
2nd-order Approx Value of $f'(a)$	0.3409	0.6%