

CHAPTER 3

Lemma 3.24. *Let $f(x, y) = ax^2 + bxy + cy^2$ be a reduced positive definite form. If for some pair of integers x and y we have $\gcd(x, y) = 1$ and $f(x, y) \leq c$, then $f(x, y) = a$ or c , and the point (x, y) is one of the six points $\pm(1, 0), \pm(0, 1), \pm(1, -1)$. Moreover, the number of proper representations of a by f is*

$$\begin{cases} 2 & \text{if } a < c \\ 4 & \text{if } 0 \leq b < a = c, \text{ and} \\ 6 & \text{if } a = b = c. \end{cases}$$

Theorem 3.25. *Let $f(x, y) = ax^2 + bxy + cy^2$ and $g(x, y) = Ax^2 + Bxy + Cy^2$ be reduced positive definite quadratic forms. If $f \sim g$ then $f = g$.*

Definition. Let f be a positive definite binary quadratic form. A matrix $M \in \Gamma$ is called an automorph of f if M takes f to itself, that is, $f(m_{11}x + m_{12}y, m_{21}x + m_{22}y) = f(x, y)$. The number of automorphs of f is denoted by $w(f)$.

Theorem 3.26. *Let f and g be equivalent positive definite binary quadratic forms. Then $w(f) = w(g)$, there are exactly $w(f)$ matrices $M \in \Gamma$ that take f to g , and there are exactly $w(f)$ matrices $M \in \Gamma$ that take g to f . Moreover, the only values of $w(f)$ are 2, 4, and 6. If f is reduced then*

$$\begin{aligned} w(f) &= 6 \text{ if } a = b = c, \text{ and} \\ w(f) &= 4 \text{ if } a = c \text{ and } b = 0, \\ w(f) &= 2 \text{ otherwise.} \end{aligned}$$