Chapter 7

Theorem 1 (7.1)**.** *If* $\langle a_0, a_1, \cdots, a_j\rangle = \langle b_0, b_1, \cdots, b_j\rangle$ *where these finite continued fractions are simple, and if* $a_j > 1$ *and* $b_n > 1$*, then* $j = n$ *and* $a_i = b_i$ *for* $i = 0, 1, \cdots, n$*.*

Theorem 2 (7.2)**.** *Any finite simple continued fraction represents a rational number. Conversely, any rational number can be expressed as a finite simple continued fraction, and in exactly two ways.*

Note. Suppose $u_0/u_1 \in \mathbb{Q}$. Then $\frac{u_0}{u_1} = \langle a_0, a_1, \cdots a_j\rangle = \langle a_0, a_1, \cdots a_j - 1, 1\rangle$. Those are its two sole representations.

Definition. Let $a_0, a_1, a_2, \ldots$ be an infinite sequence of integers, all positive except perhaps a_0. We define two sequences of integers $\{h_n\}$ and $\{k_n\}$ inductively as follows:

$$h_{-2} = 0, h_{-1} = 1, h_i = a_i h_{i-1} + h_{i-2}, \quad \text{for } i \geq 0$$
$$k_{-2} = 1, k_{-1} = 0, k_i = a_i k_{i-1} + k_{i-2}, \quad \text{for } i \geq 0$$

We note that $1 = k_0 < k_1 < k_2 < k_3 < \cdots < k_n < \cdots$.

Theorem 3. *For any positive real number* x*,*

$$\langle a_0, a_1, \cdots, a_{n-1}, x\rangle = \frac{xh_{n-1} + h_{n-2}}{xk_{n-1} + k_{n-2}}.$$

Theorem 4. *If we define* $r_n = \langle a_0, a_1, \cdots, a_n\rangle$ *for all integers* $n \geq 0$*, then* $r_n = \frac{h_n}{k_n}$*.*

Theorem 5. *The equations*

$$h_i k_{i-1} - h_{i-1} k_i = (-1)^{i-1} \quad \text{and} \quad r_i - r_{i-1} = \frac{(-1)^{i-1}}{k_i k_{i-1}}$$

hold for $i \geq 1$*. The identities*

$$h_i k_{i-2} - h_{i-2} k_i = (-1)^i a_i \quad \text{and} \quad r_i - r_{i-2} = \frac{(-1)^{i-1}}{k_i k_{i-2}}$$

hold for $i \geq 1$*. The fraction* h_i/k_i *is reduced, that is* $(h_i, k_i) = 1$*.*

Theorem 6. *The values* r_n *defined in Theorem 7.4 satisfy the infinite chan of inequalities* $r_0 < r_2 < r_4 < r_6 < \cdots < r_7 < r_5 < r_3 < r_1$*. Stated in words, the* r_n *with even subscripts form an increasing sequence, those with odd subscripts form a decreasing sequence, and every* r_{2n} *is less than every* r_{2j-1}*. Furthermore,* $\lim_{n\to\infty} r_n$ *exists, and for every* $j \geq 0$*,* $r_{2j} < \lim_{n\to\infty} r_n < r_{2j+1}$*.*

Definition. An infinite sequence $a_0, a_1, a_2, \cdots$ of integers, all positive except perhaps for a_0, determines an infinite simple continued fraction $\langle a_0, a_1, a_2, \cdots\rangle$. The value of $\langle a_0, a_1, a_2, \cdots\rangle$ is defined to be $\lim_{n\to\infty}\langle a_0, a_1, a_2, \cdots, a_n\rangle$.

Theorem 7. *The value of any infinite simple continued fraction* $\langle a_0, a_1, a_2, \cdots\rangle$ *is irrational.*

Theorem 8. *Two distinct infinite simple continued fractions converge to different values.*

Lemma 1. *Let* $\theta = \langle a_0, a_1, a_2, \cdots\rangle$ *be a simple continued fraction. Then* $a_0 = [\theta]$*. Furthermore, if* θ_1 *denotes* $\langle a_1, a_2, a_3, \cdots\rangle$*, then* $\theta = a_0 + 1/\theta_1$*.*

Note. To construct the continued fraction of an irrational number:

$$\xi_0 = \xi$$
$$a_i = [\xi_i]$$
$$\xi_{i+1} = \frac{1}{\xi_i - a_i}.$$

Note that this implies $\xi_i = a_i + \frac{1}{\xi_{i+1}}$. Now we get

$$\begin{aligned}
\xi = \xi_0 = a_0 + \frac{1}{\xi_1} &= \langle a_0, \xi_1 \rangle \\
&= \left\langle a_0, a_1 + \frac{1}{\xi_2} \right\rangle = \langle a_0, a_1, \xi_2 \rangle \\
&= \qquad \vdots \\
&= \left\langle a_0, a_1, a_2, \cdots, a_{m-2}, a_{m-1} + \frac{1}{\xi_m} \right\rangle = \langle a_0, a_1, \cdots, a_{m-1}, \xi_m \rangle,
\end{aligned}$$

so

$$\xi = \langle a_0, a_1, \cdots, a_{n-1}, \xi_n \rangle = \frac{\xi_n h_{n-1} + h_{n-2}}{\xi_n k_{n-1} + k_{n-2}}.$$

Theorem 9. *Any irrational number ξ is uniquely expressible, by the procedure gave in the previous note, as an infinite simple continued fraction $\langle a_0, a_1, a_2, \cdots \rangle$. Conversely, any such continued fraction determined by integers a_i that are positive for all $i > 0$ represents an irrational number, ξ. The finite simple continued fraction $\langle a_0, a_1, \cdots, a_n \rangle$ has the rational value $\frac{h_n}{k_n} = r_n$, and is called the nth convergent to ξ. Even convergents form a monotonically increasing sequence with ξ as the limit. Similarly, odd convergents form a monotonically decreasing sequence tending to ξ. The denominators k_n of the convergents are an increasing sequence of positive integers for $n > 0$. Finally, with ξ_i defined as in our previous note, we have $\langle a_0, a_1, \cdots \rangle = \langle a_0, a_1, \cdots, a_{n-1}, \xi_n \rangle$ and $\xi_n = \langle a_n, a_{n+1}, a_{n+2}, \cdots \rangle$.*

Theorem 10. *We have for any $n \geq 0$,*

$$\left| \xi - \frac{h_n}{k_n} \right| < \frac{1}{k_n k_{n+1}} \quad \text{and} \quad |\xi k_n - h_n| < \frac{1}{k_{n+1}}$$

Theorem 11. *The convergents h_n/k_n are successively closer to ξ, that is*

$$\left| \xi - \frac{h_n}{k_n} \right| < \left| \xi - \frac{h_{n-1}}{k_{n-1}} \right|,$$

in fact the stronger inequality

$$|\xi k_n - h_n| < |\xi k_{n-1} - h_{n-1}|$$

holds.

Theorem 12. *If a/b is a rational number with positive denominator such that $|\xi - a/b| < |\xi - h_n/k_n|$ for some $n \geq 1$, then $b > k_n$. In fact if $|\xi b - a| < |\xi k_n - h_n|$ for some $n \geq 0$, then $b \geq k_{n+1}$.*

Theorem 13. *Let ξ denote any irrational number. If there is a rational number a/b with $b \geq 1$ such that*

$$\left| \xi - \frac{a}{b} \right| < \frac{1}{2b^2},$$

then a/b equals one of the convergents of the simple continued fraction expansion of ξ.

Theorem 14. *The nth convergent of $1/x$ is the reciprocal of the $(n-1)$st convergetn of x if x is any real number > 1.*

Theorem 15. *Hurwitz. Given any irrational number ξ, there exist infinitely many rational numbers h/k such that*

$$\left| \xi - \frac{h}{k} \right| < \frac{1}{\sqrt{5} k^2}.$$

Theorem 16. *The constant $\sqrt{5}$ in the preceding theorem is the best possible. In other words, the theorem does not hold if $\sqrt{5}$ is replaced by a larger number.*