PROBLEM SET 7
MATH 115
NUMBER THEORY
PROFESSOR PAUL VOJTA

NOAH RUDERMAN

Problems 7.1.1, 7.1.4, 7.1.5, 7.3.3abd, 7.3.5, 7.4.1, 7.4.3, 7.4.4, 7.4.7, 7.5.1, 7.5.3, 7.5.4, 7.6.4, and 7.6.5 from *An Introduction to The Theory of Numbers*, 5$^{\text{th}}$ edition, by Ivan Niven, Herbert S. Zuckerman, and Hugh L. Montgomery

Date: August 11, 2014.

Problem (7.1.1)

Expand the rational fractions 17/3, 3/17, and 8/1 into finite simple continued fractions.

Solution.

We start with $\frac{17}{3}$. Using the Euclidean algorithm, we see that

$$\begin{aligned} 17 &= 5 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

So we see that $a_0 = 5$, $a_1 = 1$ and $a_2 = 2$. Thus, we get

$$\begin{aligned} \frac{17}{3} &= \langle 5, 1, 2 \rangle \\ &= 5 + \frac{1}{1 + \frac{1}{2}}. \end{aligned}$$

Next we have $\frac{3}{17}$. Using the Euclidean algorithm, we see that

$$\begin{aligned} 3 &= 0 \cdot 17 + 3 \\ 17 &= 5 \cdot 3 + 2 \\ 3 &= 1 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

So we see that $a_0 = 0$, $a_1 = 5$, $a_2 = 1$ and $a_3 = 2$. Thus, we get

$$\begin{aligned} \frac{3}{17} &= \langle 0, 5, 1, 2 \rangle \\ &= \frac{1}{5 + \frac{1}{1 + \frac{1}{2}}}. \end{aligned}$$

Finally we have $\frac{8}{1}$. Using the Euclidean algorithm, we see that

$$\begin{aligned} 8 &= \langle 8 \rangle \\ &= 8 \cdot 1 + 0. \end{aligned}$$

We see that $a_0 = 8$. Thus, we get

$$8 = 8$$

$\square$

Problem (7.1.4)

Given positive integers b, c, d with $c > d$, prove that $\langle a, c\rangle < \langle a, d\rangle$ but $\langle a, b, c\rangle > \langle a, b, d\rangle$ for any integer a.

Solution.

We see that

$$\begin{aligned} c &> d \\ \frac{1}{d} &> \frac{1}{c} && d, c \in \mathbb{Z}^+ \\ a + \frac{1}{d} &> a + \frac{1}{c} \\ \langle a, d\rangle &> \langle a, c\rangle, \end{aligned}$$

proving the first part.

Next we see that

$$\begin{aligned} c &> d \\ \frac{1}{d} &> \frac{1}{c} && d, c \in \mathbb{Z}^+ \\ b + \frac{1}{d} &> b + \frac{1}{c} \\ \langle b, d\rangle &> \langle b, c\rangle \\ \frac{1}{\langle b, c\rangle} &> \frac{1}{\langle b, d\rangle} \\ a + \frac{1}{\langle b, c\rangle} &> a + \frac{1}{\langle b, d\rangle} \\ \langle a, b, c\rangle &> \langle a, b, d\rangle, \end{aligned}$$

proving the second part. $\square$

Problem (7.1.5)

Let a_1, a_2, $\cdots$, a_n and c be positive real numbers. Prove that

$$\langle a_0, a_1, \cdots, a_n\rangle > \langle a_0, a_1, \cdots, a_n + c\rangle$$

holds if n is odd, but is false if n is even.

Solution.

Let $m \in \mathbb{Z}^+$ and suppose $2 \leq m \leq n$. We use induction to show that if

$$\langle a_m, a_{m+1} \cdots, a_n\rangle > \langle a_m, a_{m+1} \cdots, a_n + c\rangle$$

then

$$\langle a_{m-2}, a_{m-1}, \cdots, a_n\rangle > \langle a_{m-2}, a_{m-1}, \cdots, a_n + c\rangle.$$

Suppose

$$\langle a_m, a_{m+1} \cdots, a_n\rangle > \langle a_m, a_{m+1} \cdots, a_n + c\rangle$$

We see that

$$\begin{aligned}
\langle a_m, a_{m+1} \cdots, a_n\rangle &> \langle a_m, a_{m+1} \cdots, a_n + c\rangle \\
\frac{1}{\langle a_m, a_{m+1} \cdots, a_n + c\rangle} &> \frac{1}{\langle a_m, a_{m+1} \cdots, a_n\rangle} \\
a_{m-1} + \frac{1}{\langle a_m, a_{m+1} \cdots, a_n + c\rangle} &> a_{m-1} + \frac{1}{\langle a_m, a_{m+1} \cdots, a_n\rangle} \\
\langle a_{m-1}, a_m \cdots, a_n + c\rangle &> \langle a_{m-1}, a_m \cdots, a_n\rangle \qquad (1)\\
\frac{1}{\langle a_{m-1}, a_m \cdots, a_n\rangle} &> \frac{1}{\langle a_{m-1}, a_m \cdots, a_n + c\rangle} \\
a_{m-2} + \frac{1}{\langle a_{m-1}, a_m \cdots, a_n\rangle} &> a_{m-2} + \frac{1}{\langle a_{m-1}, a_m \cdots, a_n + c\rangle} \\
\langle a_{m-2}, a_{m-1}, \cdots, a_n\rangle &> \langle a_{m-2}, a_{m-1}, \cdots, a_n + c\rangle, \qquad (2)
\end{aligned}$$

completing the induction step. It should be clear that this induction step also holds if we replace the $>$ sign with $<$.

We see that $\langle a_n\rangle = a_n < a_n + c = \langle a_n + c\rangle$. We have two cases

(1) n is even.
If $n = 0$, then $\langle a_0\rangle = a_0 < a_0 + c = \langle a_0 + c\rangle$, and we are done. If $n > 0$, then we can use the above result to get $\langle a_{n-2}, a_{n-1}, a_n\rangle < \langle a_{n-2}, a_{n-1}, a_n + c\rangle$. Clearly, $n - 2$ and n have the same parity. We can repeat this induction step until it terminates where we get

$$\langle a_0, a_1, \cdots, a_n\rangle < \langle a_0, a_1, \cdots, a_n + c\rangle$$

(2) n is odd.

If $n = 1$, then we use $\langle a_1 \rangle = a_1 < a_1 + c = \langle a_1 + c \rangle$ and equation 1 with $m = n = 1$ to get $\langle a_0, a_1 + c \rangle < \langle a_0, a_1 \rangle$ and we are done. If $n > 1$, then using equation 1 with $m = n$ and $\langle a_n \rangle < \langle a_n + c \rangle$ to get $\langle a_{n-1}, a_n + c \rangle < \langle a_{n-1}, a_n \rangle$. Now we can use the induction step to get $\langle a_{n-3}, \ldots, a_n + c \rangle < \langle a_{n-1}, \ldots, a_n \rangle$, and so on until it terminates. Clearly, $n - 1$ and $n - 3$ have the same parity. When the induction terminates, we get

$$\langle a_0, a_1, \cdots, a_n + c \rangle < \langle a_0, a_1, \cdots, a_n \rangle$$

Thus,

$$\langle a_0, a_1, \cdots, a_n \rangle > \langle a_0, a_1, \cdots, a_n + c \rangle$$

is true if n is odd, but false if n is even. $\square$

Problem (7.3.3)

Evalue the infinite continued fractions:

a. $\langle 2, 2, 2, 2, \cdots \rangle$
b. $\langle 1, 2, 1, 2, 1, 2 \cdots \rangle$
d. $\langle 1, 3, 1, 2, 1, 2, 1, 2 \cdots \rangle$

Solution.

a. Let $x = \langle 2, 2, 2, 2, \cdots \rangle$. We see that

$$\begin{aligned} x &= 2 + \frac{1}{x} \\ x^2 &= 2x + 1 \\ x^2 - 2x - 1 &= 0. \end{aligned}$$

From here we can use the general solution to quadratic equations to get

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4+4}}{2} \\ &= 1 \pm \frac{\sqrt{8}}{2} \\ &= 1 \pm \sqrt{2} \end{aligned}$$

But $x > 0$, so the only valid solution is $x = 1 + \sqrt{2}$.

b. Let $x = \langle 1, 2, 1, 2, \cdots \rangle$. We see that

$$\begin{aligned} x &= 1 + \frac{1}{2 + \frac{1}{x}} \\ x &= 1 + \frac{x}{2x+1} \\ (2x+1)x &= (2x+1) + x \\ 2x^2 + x &= 2x + 1 + x \\ 2x^2 - 2x - 1 &= 0. \end{aligned}$$

From here we can use the general solution to quadratic equations to get

$$\begin{aligned} x &= \frac{2 \pm \sqrt{4+8}}{4} \\ &= \frac{1}{2} \pm \frac{\sqrt{12}}{4} \\ &= \frac{1 \pm \sqrt{3}}{2} \end{aligned}$$

But $x > 0$, so the only valid solution is $x = \frac{1+\sqrt{3}}{2}$.

d. Let $x = \langle 1, 3, 1, 2, 1, 2, \cdots \rangle$. Let $y = \langle 1, 2, 1, 2, \cdots \rangle$. From our results in part b, we see that $y = \frac{1+\sqrt{3}}{2}$. We have

$$x = 1 + \cfrac{1}{3 + \frac{1}{y}}$$
$$x = 1 + \cfrac{1}{3 + \frac{1}{\frac{1}{2} + \frac{\sqrt{3}}{2}}}$$
$$x = 1 + \cfrac{1}{3 + \frac{2}{1+\sqrt{3}}}$$
$$x = 1 + \frac{1+\sqrt{3}}{3(1+\sqrt{3}) + 2}$$
$$x = 1 + \frac{1+\sqrt{3}}{5 + 3\sqrt{3}}$$
$$x = 1 + \frac{(1+\sqrt{3})(5-3\sqrt{3})}{25 - 27}$$
$$x = 1 + \frac{(1+\sqrt{3})(5-3\sqrt{3})}{-2}$$
$$x = 1 + \frac{5 - 3\sqrt{3} + 5\sqrt{3} - 9}{-2}$$
$$x = 1 + \frac{-4 + 2\sqrt{3}}{-2}$$
$$x = 1 + 2 - \sqrt{3}$$
$$x = 3 - \sqrt{3}.$$

$\square$

Problem (7.3.5)

Let u_0/u_1 be a rational number in lowest terms, and write $u_0/u_1 = \langle a_0, a_1, \cdots, a_n \rangle$. Show that if $0 \le i < n$, then $|r_i - u_0/u_1| \le 1/(k_i k_{i+1})$, with equality if and and only if $i = n - 1$.

Solution.

First we note that

$$\left| r_i - \frac{u_0}{u_1} \right| = |r_i - r_n|$$

Suppose $n = i + 1$. By theorem 7.5, we have

$$\begin{aligned} |r_i - r_n| &= |r_i - r_{i+1}| \\ &= |r_{i+1} - r_i| \\ &= \left| \frac{(-1)^i}{k_i k_{i+1}} \right| \\ &= \frac{1}{k_i k_{i+1}}, \end{aligned}$$

because $k_i \in \mathbb{Z}^+$ for all $i \in \mathbb{N}$. Now suppose $n - 1 > i$. We have four cases

(1) n is odd and i is even:
We see that

$$\begin{aligned} 0 < r_n - r_i &< r_{n-2} - r_i \\ &< r_{n-4} - r_i \\ &\vdots \\ &< r_{i+1} - r_i \\ &= \frac{1}{k_i k_{i+1}}. \end{aligned}$$

From this we can deduce that

$$\frac{1}{k_i k_{i+1}} < 0 < r_n - r_i < \frac{1}{k_i k_{i+1}},$$

so

$$|r_n - r_i| < \frac{1}{k_i k_{i+1}}.$$

(2) n is even and i is odd:
We see that

$$\begin{aligned} 0 < r_i - r_n &< r_i - r_{n-2} \\ &< r_i - r_{n-4} \\ &\vdots \\ &< r_i - r_{i+1} \\ &= -(r_{i+1} - r_i) \end{aligned}$$

$$= -\frac{(-1)^i}{k_{i+1}k_i}$$
$$= \frac{1}{k_i k_{i+1}}.$$

From this we can deduce that

$$\frac{1}{k_i k_{i+1}} < 0 < r_i - r_n < \frac{1}{k_i k_{i+1}},$$

so

$$|r_n - r_i| < \frac{1}{k_i k_{i+1}}.$$

(3) n is odd and i is odd:
We see that

$$0 < r_i - r_n < r_i - r_{n-1}.$$

Call $n - 1 = n'$. If $n' - 1 = i$, then

$$\begin{aligned} r_i - r_{n-1} &= r_i - r_{n'} \\ &= r_i - r_{i+1} \\ &= \frac{1}{k_i k_{i+1}}, \end{aligned}$$

in which case we see that

$$\frac{1}{k_i k_{i+1}} < 0 < r_i - r_n < \frac{1}{k_i k_{i+1}},$$

so

$$|r_n - r_i| < \frac{1}{k_i k_{i+1}}.$$

If $n' - 1 > i$, then we can appeal to case 2 to see that $r_i - r_{n'} < \frac{1}{k_i k_{i+1}}$ so again we have

$$\frac{1}{k_i k_{i+1}} < 0 < r_i - r_n < \frac{1}{k_i k_{i+1}},$$

and thus

$$|r_n - r_i| < \frac{1}{k_i k_{i+1}}.$$

(4) n is even and i is even:
We see that

$$0 < r_n - r_i < r_{n-1} - r_i.$$

Call $n - 1 = n'$. If $n' - 1 = i$, then

$$\begin{aligned} r_{n-1} - r_i &= r_{n'} - r_i \\ &= r_{i+1} - r_i \\ &< \frac{1}{k_i k_{i+1}}, \end{aligned}$$

in which case we see that

$$\frac{1}{k_i k_{i+1}} < 0 < r_n - r_i < \frac{1}{k_i k_{i+1}},$$

so

$$|r_n - r_i| < \frac{1}{k_i k_{i+1}}.$$

If $n' - 1 > i$, then we can appeal to case 1 to see that $r_{n'} - r_i < \frac{1}{k_i k_{i+1}}$ so again we have

$$\frac{1}{k_i k_{i+1}} < 0 < r_n - r_i < \frac{1}{k_i k_{i+1}},$$

and thus

$$|r_n - r_i| < \frac{1}{k_i k_{i+1}}.$$

$\square$

Problem (7.4.1)

Expand each of the following as infinite simple continued fractions: $\sqrt{2}$, $\sqrt{2}-1$, $\sqrt{2}/2$, $\sqrt{3}$, $1/\sqrt{3}$.

Solution.

We start with $\sqrt{2}$. We see that

$$\xi_0 = \sqrt{2}.$$

We see that $1 < \sqrt{2} < 2$ so $a_0 = 1$. Next we have

$$\begin{aligned}\xi_1 &= \frac{1}{\xi_0 - a_0} \\ &= \frac{1}{\sqrt{2}-1} \\ &= \sqrt{2}+1,\end{aligned}$$

so $a_1 = 2$. Next we have

$$\begin{aligned}\xi_2 &= \frac{1}{\xi_1 - a_1} \\ &= \frac{1}{\sqrt{2}+1-2} \\ &= \frac{1}{\sqrt{2}-1} \\ &= \sqrt{2}+1,\end{aligned}$$

so $a_2 = 2$. Let $\imath \in \mathbb{N}$. Evidently $\xi_i = \sqrt{2}+1$ implies $\xi_{i+1} = \xi_i$. $\xi_1 = \sqrt{2}+1$ so $\xi_i = \xi_1$ for $i \geq 1$. It should be clear that if $a_i = a_1$ for all $i \geq 1$. From this we get

$$\begin{aligned}\sqrt{2} &= \langle 1, a_1, a_1, a_1, \ldots \rangle \\ &= \langle 1, 2, 2, 2, \ldots \rangle.\end{aligned}$$

Now we continue with $\sqrt{2}-1$. We see that

$$\begin{aligned}\sqrt{2}-1 &= \langle 1, 2, 2, 2, \ldots \rangle - 1 \\ &= \left(1 + \frac{1}{\langle 2, 2, 2, \ldots \rangle}\right) - 1 \\ &= 0 + \frac{1}{\langle 2, 2, 2, \ldots \rangle} \\ &= \langle 0, 2, 2, 2, \ldots \rangle\end{aligned}$$

Next we have $\frac{\sqrt{2}}{2}$. We see that $\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$ and

$$\frac{1}{\sqrt{2}} = 0 + \frac{1}{\sqrt{2}}$$

$$
\begin{aligned}
&= 0 + \frac{1}{\langle 1, 2, 2, 2, \cdots \rangle} \\
&= \langle 0, 1, 2, 2, 2, \cdots \rangle
\end{aligned}
$$

We continue with $\sqrt{3}$. We start with

$$\xi_0 = \sqrt{3}.$$

We see that $1 < \sqrt{3} < 2$ so $a_0 = 1$. Next we have

$$
\begin{aligned}
\xi_1 &= \frac{1}{\xi_0 - a_0} \\
&= \frac{1}{\sqrt{3} - 1} \\
&= \frac{\sqrt{3} + 1}{2},
\end{aligned}
$$

so $a_1 = 1$. Next we have

$$
\begin{aligned}
\xi_2 &= \frac{1}{\xi_1 - a_1} \\
&= \frac{1}{\frac{\sqrt{3}+1}{2} - 1} \\
&= \frac{1}{\frac{\sqrt{3}-1}{2}} \\
&= \frac{2}{\sqrt{3} - 1} \\
&= \frac{2(\sqrt{3} + 1)}{2} \\
&= \sqrt{3} + 1
\end{aligned}
$$

so $a_2 = 2$. Next we have

$$
\begin{aligned}
\xi_3 &= \frac{1}{\xi_2 - a_2} \\
&= \frac{1}{\sqrt{3} + 1 - 2} \\
&= \frac{1}{\sqrt{3} - 1} \\
&= \frac{\sqrt{3} + 1}{2}
\end{aligned}
$$

Let $\imath \in \mathbb{N}$. Evidently $\xi_i = \frac{\sqrt{3}+1}{2}$ implies $\xi_{i+2} = \xi_i$. $\xi_1 = \frac{\sqrt{3}+1}{2}$ so $\xi_i = \xi_1$ for $i \equiv 1 \mod 2$. Furthermore, $\xi_i = \xi_2$ for $i \equiv 0 \mod 2$ and $i \geq 1$. Thus, $a_1 = a_3 = a_5 = \cdots$ and $a_2 = a_4 = a_6 \cdots$, so we have

$$\sqrt{3} = \langle a_0, a_1, a_2, a_1, a_2, \ldots \rangle$$

$$= \langle 1, 1, 2, 1, 2, \ldots \rangle.$$

We start with

$$\xi_0 = \frac{1}{\sqrt{3}}.$$

We finish with $\frac{1}{\sqrt{3}}$. We see that

$$\begin{aligned} \frac{1}{\sqrt{3}} &= 0 + \frac{1}{\sqrt{3}} \\ &= 0 + \frac{1}{\langle 1, 1, 2, 1, 2, \ldots \rangle} \\ &= \langle 0, 1, 1, 2, 1, 2, \ldots \rangle \end{aligned}$$

$\square$

Problem (7.4.3)

Let α, β, γ be irrational numbers satisfying $\alpha < \beta < \gamma$. If α and γ have identical convergents $h_0/k_0, h_1/k_1, \cdots$, up to h_n/k_n, prove that β also has these same convergents up to h_n/k_n.

Solution.

Let

$$
\begin{aligned}
r_i^\alpha &= \langle a_0, a_1, \cdots, a_i \rangle \\
r_i^\beta &= \langle b_0, b_1, \cdots, b_i \rangle \\
r_i^\gamma &= \langle c_0, c_1, \cdots, c_i \rangle
\end{aligned}
$$

We are given $r_i^\alpha = r_i^\gamma$ for $0 \leq i \leq n$. By theorem 7.1, $a_i = c_i$ for $0 \leq i \leq n$. We use induction to show that $a_i = b_i = c_i$ for all $0 \leq i \leq n$.

Consider the usual algorithm for calculating the n^{th} convergent to an irrational number, ξ. For each term x_i in the $\langle x_0, x_1, x_2, \cdots, x_n \rangle$, we have $x_i = [\xi_i]$, $\xi_{i+1} = \frac{1}{\xi_i - x_i}$ and $\xi_0 = \xi$. Suppose $\xi_i^\alpha < \xi_i^\beta < \xi_i^\gamma$ and that $a_i = b_i = c_i$. We see that

$$
\begin{aligned}
\xi_i^\beta &< \xi_i^\gamma \\
\xi_i^\beta - b_i &< \xi_i^\gamma - c_i \\
\frac{1}{\xi_i^\gamma - c_i} &< \frac{1}{\xi_i^\beta - b_i} \\
\xi_{i+1}^\gamma &< \xi_{i+1}^\beta.
\end{aligned}
$$

Likewise, we see that

$$
\begin{aligned}
\xi_i^\alpha &< \xi_i^\beta \\
\xi_i^\alpha - a_i &< \xi_i^\beta - b_i \\
\frac{1}{\xi_i^\beta - b_i} &< \frac{1}{\xi_i^\alpha - a_i} \\
\xi_{i+1}^\beta &< \xi_{i+1}^\alpha.
\end{aligned}
$$

Thus we have

$$\xi_{i+1}^\gamma < \xi_{i+1}^\beta < \xi_{i+1}^\alpha.$$

Furthermore, we see that

$$
\begin{aligned}
\xi_{i+1}^\gamma &< \xi_{i+1}^\beta < \xi_{i+1}^\alpha \\
\left[\xi_{i+1}^\gamma\right] &\leq \left[\xi_{i+1}^\beta\right] \leq \left[\xi_{i+1}^\alpha\right] \\
c_{i+1} &\leq b_{i+1} \leq a_{i+1}.
\end{aligned}
$$

But $a_{i+1} = c_{i+1}$ for $0 \leq i+1 \leq n$, so

$$a_{i+1} = b_{i+1} = c_{i+1}.$$

We have proved $\xi_i^\alpha < \xi_i^\beta < \xi_i^\gamma$ and $a_i = b_i = c_i$ implies $\xi_{i+1}^\gamma < \xi_{i+1}^\beta < \xi_{i+1}^\alpha$ and $a_{i+1} = b_{i+1} = c_{i+1}$. A similar argument shows $\xi_i^\gamma < \xi_i^\beta < \xi_i^\alpha$ and $a_i = b_i = c_i$ implies $\xi_{i+1}^\alpha < \xi_{i+1}^\beta < \xi_{i+1}^\gamma$ and $a_{i+1} = b_{i+1} = c_{i+1}$ for $0 \le i < n$.

As our base case, we see that

$$\alpha < \beta < \gamma$$
$$\xi_0^\alpha < \xi_0^\beta < \xi_0^\gamma$$
$$[\xi_0^\alpha] \le [\xi_0^\beta] \le [\xi_0^\gamma]$$
$$a_0 \le b_0 \le c_0.$$

Since $a_0 = c_0$, we see that $a_0 = b_0 = c_0$.

By the induction hypothesis, we see that $a_i = b_i = c_i$ for all $0 \le i \le n$. From this we see that

$$\langle a_0, a_1, \cdots, a_i \rangle = \langle b_0, b_1, \cdots, b_i \rangle = \langle c_0, c_1, \cdots, c_i \rangle,$$

so

$$r_i^\alpha = r_i^\beta = r_i^\gamma,$$

and the convergents are equal for $0 \le i \le n$. $\square$

Problem (7.4.4)

Let ξ be an irrational number with continued fraction expansion $\langle a_0, a_1, a_2, a_3, \cdots \rangle$. Let $b_1, b_2, b_3, \cdots$ be any finite or infinite sequence of positive integers. Prove that

$$\lim_{n\to\infty} \langle a_0, a_1, a_2, \cdots, a_n, b_1, b_2, b_3, \cdots \rangle = \xi.$$

Solution.

Let $x_n = \langle a_0, a_1, a_2, \cdots, a_n, b_1, b_2, b_3, \cdots \rangle$. Let r_n be the n^{th} convergent of x_n. We know that even convergents form a monotonically increasing sequence whose limit is x_n. Likewise, the odd convergents form a monotonically decreasing sequence whose limit is x_n. Thus, if n is odd, then

$$r_{n-1} < x_n < r_n$$

and if n is even, then

$$r_n < x_n < r_{n-1}.$$

But $r_n = \langle a_0, a_1, a_2, \cdots, a_n \rangle$, which is also the n^{th} convergent of ξ. Thus, we see that

$$\begin{aligned} \lim_{\substack{n\to\infty \\ n \text{ even}}} r_n \leq \lim_{n\to\infty} x_n &\leq \lim_{\substack{n\to\infty \\ n \text{ odd}}} r_n \\ \xi \leq \lim_{n\to\infty} x_n &\leq \xi \\ \lim_{n\to\infty} x_n &= \xi, \end{aligned}$$

completing the proof. $\square$

Problem (7.4.7)

Prove that
$$k_n|k_{n-1}\xi - h_{n-1}| + k_{n-1}|k_n\xi - h_n| = 1.$$

Solution.

Suppose the above equality is true, then
$$\begin{aligned} k_n|k_{n-1}\xi - h_{n-1}| + k_{n-1}|k_n\xi - h_n| &= 1 \\ \left|\xi - \frac{h_{n-1}}{k_{n-1}}\right| + \left|\xi - \frac{h_n}{k_n}\right| &= \frac{1}{k_nk_{n-1}} \\ |\xi - r_{n-1}| + |\xi - r_n| &= \frac{1}{k_nk_{n-1}}. \end{aligned}$$
Thus, it is sufficient to prove that
$$|\xi - r_{n-1}| + |\xi - r_n| = \frac{1}{k_nk_{n-1}}.$$

We know that odd convergents are larger than their limit and that even convergents are smaller than their limit. We have two cases:

(1) n is even:
Thus, $r_n < \xi$ and $r_{n-1} > \xi$. We have
$$\begin{aligned} |\xi - r_{n-1}| + |\xi - r_n| &= -(\xi - r_{n-1}) + (\xi - r_n) \\ &= r_{n-1} - r_n \\ &= r_n - r_{n-1} \\ &= -\frac{(-1)^{n-1}}{k_nk_{n-1}} && \text{by theorem } 7.5, n \geq 1 \\ &= \frac{(-1)^n}{k_nk_{n-1}} \\ &= \frac{1}{k_nk_{n-1}}. \end{aligned}$$

(2) n is odd:
Thus, $r_n > \xi$ and $r_{n-1} < \xi$. We have
$$\begin{aligned} |\xi - r_{n-1}| + |\xi - r_n| &= (\xi - r_{n-1}) - (\xi - r_n) \\ &= r_n - r_{n-1} \\ &= \frac{(-1)^{n-1}}{k_nk_{n-1}} && \text{by theorem } 7.5, n \geq 1 \\ &= \frac{1}{k_nk_{n-1}}. \end{aligned}$$

In each case, we have proved a necessary condition to imply the inequality.

However, we have used theorem 7.5 which only holds for $n \geq 1$. It remains to show that the equality holds for $n = 0, -1$. We note that $h_{-2} = k_{-1} = 0$ and $h_{-1} = k_{-2} = k_0 = 1$. Suppose $n = 0$. Then we have

$$\begin{aligned} k_n|k_{n-1}\xi - h_{n-1}| + k_{n-1}|k_n\xi - h_n| &= k_0|k_{-1}\xi - h_{-1}| + k_{-1}|k_0\xi - h_0| \\ &= 1 \cdot |0 \cdot \xi - 1| + 0 \cdot |k_0\xi - h_0| \\ &= 1. \end{aligned}$$

Suppose $n = -1$. Then we have

$$\begin{aligned} k_n|k_{n-1}\xi - h_{n-1}| + k_{n-1}|k_n\xi - h_n| &= k_{-1}|k_{-2}\xi - h_{-2}| + k_{-2}|k_{-1}\xi - h_{-1}| \\ &= 0 \cdot |k_{-2}\xi - h_{-2}| + 1 \cdot |0 \cdot \xi - 1| \\ &= 1. \end{aligned}$$

Now we have showed the equality holds for all $n \geq -1$. $\square$

Problem (7.5.1)

Prove that the first assertio nin theorem 7.13 holds in case $n = 0$ if $k_1 > 1$.

Solution.

We proceed in the same way that is outlined in NZM. Suppose the first part of theorem 7.13 if false. Then

$$\left|\xi - \frac{a}{b}\right| < \left|\xi - \frac{h_n}{k_n}\right| \qquad b \leq k_n$$
$$|\xi b - a| < |\xi k_n - h_n|.$$

Using the second part of theorem 7.13, we see that this implies $b \geq k_{n+1}$. If $n = 0$, then $b \geq k_1$ and $b \leq k_0$. But $k_0 = 1$. So if $k_1 > 1$, then $b > 1$ and $b \leq 1$, which is a contradiction. Thus, the assumption must have been false. Thus, $b > k_n$ for $n = 0$ when $k_1 > 1$. $\square$

Problem (7.5.3)

... Prove that every convergent to ξ is a good approximation.

Solution.

We use theorem 7.13 to see that

$$|\xi b - a| < |\xi k_n - h_n|$$

For $n \geq 1$ implies $b \geq k_{n+1} > k_n$, so $b > k_n$. Thus,

$$|\xi k_n - h_n| = \min_{\substack{\text{all } x \\ 0<y\leq k_n}} |\xi y - x|,$$

so $\frac{h_n}{k_n} = r_n$ is a "good approximation" to ξ. We proved in problem 7.5.1 that theorem 7.13 holds for $n = 0$ when $k_1 > 1$. From the recursive definition of k_i, it is easy to see that $k_1 = a_1$. But $a_1 \in \mathbb{Z}^+$, so $a_1 \geq 1$. If $a_1 > 1$, then $k_1 > 1$ and we are done so we assume that $k_1 = a_1 = 1$. Thus,

$$\begin{aligned}\min_{\substack{\text{all } x \\ 0<y\leq k_1}} |\xi y - x| &= \min_{\substack{\text{all } x \\ 0<y\leq 1}} |\xi y - x| \\ &= \min_{\text{all } x} |\xi - x|.\end{aligned}$$

Furthermore, $h_0 = a_0 = [\xi]$, so $|\xi k_0 - h_0| = |\xi - [\xi]|$.

It actually turns out that the 0^{th} convergent is not necessarily a "good approximation". By theorem 4.1(1), we know that $0 \leq \xi - [\xi] < 1$ so $|\xi - [\xi]| = \xi - [\xi]$, and $|\xi - [\xi] - 1| = -\xi + [\xi] + 1$. We see that if $\xi - [\xi] > \frac{1}{2}$, then

$$\begin{aligned}\xi - [\xi] &> \frac{1}{2} \\ 2\xi - 2[\xi] &> 1 \\ \xi - [\xi] &> -\xi + [\xi] + 1 \\ |\xi - [\xi]| &> |\xi - ([\xi] + 1)| \\ |\xi k_0 - h_0| &> \min_{\substack{\text{all } x \\ 0<y\leq k_0}} |\xi y - x|.\end{aligned}$$

Thus, every convergent r_n is a "good approximation" to ξ except for r_0 when $k_1 = 1$. $\square$

Problem (7.5.4)

Prove that every "good approximation" to ξ is convergent.

Solution.

Let $\frac{a}{b} \in \mathbb{Q}$ with $\gcd(a, b) = 1$. Suppose $\frac{a}{b}$ is a "good approximation" to ξ but isn't a convergent. First we will show by contradiction that $b = k_i$ for some $i \in \mathbb{N}$ and then that $a = h_i$.

If $b = k_i$, we are done, so we suppose that $k_j < b < k_{j+1}$ for $j \geq 1$ or $j \geq 0$ if $k_1 > 1$. Thus,

$$\begin{aligned} |\xi b - a| &< \min_{\substack{\text{all } x \\ 0<y\leq b}} |\xi y - x| \\ |\xi b - a| &< |\xi k_j - h_j|, \end{aligned}$$

so $b \geq k_{j+1}$ by theorem 7.13. This contradicts our assumption that $b < k_{j+1}$. So if $\frac{a}{b}$ is a "good approximation" to ξ, then $b = k_i$ for $i \in \mathbb{N}$.

Let $\frac{a}{b} = \frac{a}{k_i}$ be our good approximation to ξ. Using our work in problem 7.5.3, we see that the i^{th} convergent is a good approximation to ξ, so

$$\begin{aligned} \min_{\substack{\text{all } x \\ 0<y\leq b}} |\xi y - x| &= \min_{\substack{\text{all } x \\ 0<y\leq k_i}} |\xi y - x| \\ &= |\xi k_i - h_i|. \end{aligned}$$

Thus, if $\frac{a}{k_i}$ is also a good approximation to ξ, then

$$|\xi k_i - a| = |\xi k_i - h_i|.$$

We have two cases:

(1) $\xi k_i - a = \xi k_i - h_i$:
We see that

$$\begin{aligned} |\xi k_i - a| &= |\xi k_i - h_i| \\ \pm(\xi k_i - a) &= \pm(\xi k_i - h_i) \\ \pm\xi k_i \mp a &= \pm\xi k_i \mp h_i \\ \mp a &= \mp h_i \\ a &= h_i \end{aligned}$$

(2) $(\xi k_i - a) = -(\xi k_i - h_i)$:
We see that

$$\begin{aligned} |\xi k_i - a| &= |\xi k_i - h_i| \\ \mp(\xi k_i - a) &= \pm(\xi k_i - h_i) \\ \mp\xi k_i \pm a &= \pm\xi k_i \mp h_i \\ \pm a &= \pm 2\xi k_i \mp h_i \\ a &= 2\xi k_i - h_i, \end{aligned}$$

which is impossible since $a \in \mathbb{Z}$.

Thus, $a = h_i$. In conclusion, if $\frac{a}{b}$ is a good approximation to ξ, then $a = h_i$ and $b = k_i$ for some $i \in \mathbb{N}$ $\square$

Problem (7.6.4)

Given any constant c, prove that there exists an irrational number ξ and infinitely many rational numbers h/k such that

$$\left|\xi - \frac{h}{k}\right| < \frac{1}{k^c}. \tag{3}$$

Solution.

By theorem 7.11, we see that

$$\left|\xi - \frac{h_n}{k_n}\right| < \frac{1}{k_n k_{n+1}}.$$

Let c be an arbitrary real number. We see that proving $k_{n+1} \geq k_n^{c-1}$ is a sufficient condition to complete the proof because

$$\begin{aligned}\frac{1}{k_{n+1}} &\leq \frac{1}{k_n^{c-1}}\\ \frac{1}{k_n k_{n+1}} &\leq \frac{1}{k_n^c}.\end{aligned}$$

By definition, $k_{n+1} = a_{n+1}k_n + k_{n-1}$. Thus, $a_{n+1} \geq k_n^{c-2}$ is also a sufficient condition because

$$\begin{aligned}k_{n+1} &= a_{n+1}k_n + k_{n-1}\\ &\geq k_n^{c-1} + k_{n-1}\\ &\geq k_n^{c-1}.\end{aligned}$$

Appealing to the recursive definition of k_n, we see that k_n is a function of $a_0, a_1, \cdots, a_n$. Since a_{n+1} is a function of k_n, we see that it is also a function of $a_0, a_1, \cdots, a_n$. Thus, we can construct an infinite continued fraction $\langle a_0, a_1, a_2, \cdots\rangle$ such that $a_n \geq k_n^{c-2}$ for all $n \in \mathbb{N}$. By theorem 7.7, the value of any infinite simple continued fraction is irrational, so such a number $\xi = \langle a_0, a_1, a_2, \cdots\rangle$ is guaranteed to exist. The rational numbers which satisfy equation 3 are the convergents of ξ, of which there are infinitely many. $\square$

Problem (7.6.5)

Prove that of every two consecutive convergents h_n/k_n to ξ with $n \geq 0$, at least one satisfies

$$\left|\xi - \frac{h}{k}\right| < \frac{1}{2k^2}. \tag{4}$$

Solution.

We proceed by contradiction. Suppose there are two consecutive convergents $\frac{h_n}{k_n}$ and $\frac{h_{n+1}}{k_{n+1}}$ for which

$$\left|\xi - \frac{h}{k}\right| > \frac{1}{2k^2}.$$

Then we see that

$$\left|\xi - \frac{h_n}{k_n}\right| + \left|\xi - \frac{h_{n+1}}{k_{n+1}}\right| > \frac{1}{2k_n^2} + \frac{1}{2k_{n+1}^2}. \tag{5}$$

But $r_n < \xi < r_{n+1}$ or $r_{n+1} < \xi < r_n$ because odd convergents are greater than their limit and even convergents are less than their limit. We suppose that $r_{n+1} > r_n$, but note that the same results follow when $r_{n+1} < r_n$. We see that

$$\begin{aligned}
|r_{n+1} - r_n| &= |r_{n+1} - \xi + \xi - r_n| \\
&= |r_{n+1} - \xi| + |\xi - r_n| \\
&= \left|\xi - \frac{h_{n+1}}{k_{n+1}}\right| + \left|\xi - \frac{h_n}{k_n}\right|.
\end{aligned}$$

Furthermore, by theorem 7.5 we see that

$$|r_{n+1} - r_n| = \frac{1}{k_{n+1}k_n}.$$

Now we can plug this into equation 5 to get

$$\begin{aligned}
\frac{1}{k_{n+1}k_n} &> \frac{1}{2k_n^2} + \frac{1}{2k_{n+1}^2} \\
2 &> \frac{k_{n+1}}{k_n} + \frac{k_n}{k_{n+1}}.
\end{aligned}$$

Call $x = \frac{k_{n+1}}{k_n}$. Clearly, $x > 0$. We wish to find solutions to the equation

$$2 > x + x^{-1}.$$

We see that

$$\begin{aligned}
2 &> x + x^{-1} \\
2x &> x^2 + 1 && \text{no sign change as } x > 0 \\
0 &> x^2 - 2x + 1 \\
0 &> (x-1)^2,
\end{aligned}$$

but $(x-1)^2 > 0$ for all $x \in \mathbb{Q}^+$, so we have arrived at a contradiction. Thus, one of the convergents $\frac{h_n}{k_n}$ and $\frac{h_{n+1}}{k_{n+1}}$ must satisfy equation 4.

$\square$