Chapter 4

Theorem 1. *Let x and y be real numbers. Then*

(1) $[x] \le x \le [x]+1], x-1 < [x] \le x, 0 \le x-[x] < 1$.
(2) $[x] = \sum_{1\le i\le x} 1 \quad$ *if* $x \le 0$.
(3) $[x+m] = [x]+m$ *if m is an integer.*
(4) $[x]+[y] \le [x+y] \le [x]+[y]+1$.
(5) $[x]+[-x] = \begin{cases} 0 & \text{if } x \text{ is an integer} \\ -1 & \text{otherwise} \end{cases}$
(6) $\left[\frac{[x]}{m}\right] = \left[\frac{x}{m}\right]$ *if m is a positive integer.*
(7) $\cdots$
$\cdots$

Theorem 2. *de Policgnac's formula. Let p denote a prime. Then the largest exponent e such that $p^e \parallel n!$ is*

$$e = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right].$$

Definition. A function f is arithmetic if its domain is the positive integers and whose range is a subset of the complex numbers. In other words, $f(n)$ is defined for all positive integers n. *Arithmetic functions* are also called *number theoretic functions*, or *numerical functions.*

Note. An arithmetic function does not need to be defined for 0. Also, ϕ, or Euler's function, is an arithmetic function.

Definition. For positive integers n we make the following definitions

$d(n)$ is the number of positive divisors of n.
$\sigma(n)$ is the sum of the positive divisors of n.
$\sigma_k(n)$ is the sum of the kth powers of the positive divisors of n.
$\omega(n)$ is the number of distinct primes dividing n.
$\Omega(n)$ is the number of primes dividing n, counting multiplicity.

Note. Prime numbers are positive by definition.

Definition. If $f(n)$ is an arithmetic function <u>not identically zero</u> such that $f(mn) = f(m)f(n)$ for every pair of <u>positive</u> integers m, n satisfying $\gcd(m,n) = 1$, then $f(n)$ is said to be multiplicitive. If $f(mn) = f(m)f(n)$ whether m and n are relatively prime or not, then $f(n)$ is said to be totally multiplicitive or completely multiplicitive.

Note. If f is a multiplicitive function, then $f(n) = f(n)f(1)$. Since there is an n such that $f(n) \ne 0$, we can divide by $f(n)$ to reveal that $f(1) = 1$. Thus, an easy way to exclude a function as multiplicitive is to find $f(1) \ne 1$.

Note. For a multiplicitive function f, we see that for $n = \prod p^\alpha$, we have $f(n) = f\left(\prod p^\alpha\right) = \prod f(p^\alpha)$.

Theorem 3. *Let $f(n)$ be a multiplicitive function and let $F(n) = \sum_{d\mid n} f(d)$. Then $F(n)$ is multiplicitive.*

Theorem 4. *For every positive integer n,*

$$\sigma(n) = \prod_{p^\alpha \parallel n} \left(\frac{p^{\alpha+1}-1}{p-1}\right).$$

Definition. For positive integers n put $\mu(n) = (-1)^{\omega(n)}$ if n is square-free, and set $\mu(n) = 0$ otherwise. Then $\mu(n)$ is the Möbius mu function.

Theorem 5. *The function $\mu(n)$ is multiplicitive and*

$$\sum_{d\mid n} \mu(d) = \begin{cases} 1 & \text{if} \quad n = 1 \\ 0 & \text{if} \quad n > 1. \end{cases}$$

Theorem 6. *Möbius inversion formula. If* $F(n) = \sum_{d|n} f(d)$ *for every positive integer* n*, then*

$$f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).$$

Theorem 7. *If* $f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right)$ *for every positive integer* n*, then* $F(n) = \sum_{d|n} f(d)$*.*