

Quivers

A quiver $Q = (Q_0, Q_1, h, t)$ is defined as a four-tuple consisting of two sets and two functions. The set of vertices is Q_0 , the set of arrows is Q_1 , $h : Q_1 \rightarrow Q_0$ is the function mapping arrows to their terminal point, and $t : Q_1 \rightarrow Q_0$ is the function mapping arrows to their starting point. We only consider finite sets Q_0, Q_1 .

A quiver Q as below

$$1 \xrightarrow{\alpha} 2$$

has $Q_0 = \{1, 2\}$, $Q_1 = \{\alpha\}$, $h(\alpha) = 2$, and $t(\alpha) = 1$.

Quivers may also possess more complex structures such as double arrows and loops. Essentially any construction is allowed so long as each arrow has a starting and ending point.

$$\begin{array}{ccc} \curvearrowleft & & \curvearrowright \\ \alpha & 1 & \xrightarrow{\gamma} 2 \\ \curvearrowright & & \curvearrowleft \\ & \beta & \end{array}$$

Quiver Representations

A quiver representation is an object in the abelian \mathbb{C} -category $\text{Rep}(Q)$. An abelian category is a category that possesses some nice features such as a zero category, kernels and cokernels, short exact sequences, and direct sums. A quiver representation, $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is a collection of \mathbb{C} vector spaces, one for each vertex of Q , and linear maps, one for each arrow of Q .

We define a morphism between two quiver representations M, M' , f , as a collection of mappings $(f_i)_i \in Q_0$ where

$$\begin{array}{ccc} M_i & \xrightarrow{\varphi_\alpha} & M_j \\ f_i \downarrow & & \downarrow f_j \\ M'_i & \xrightarrow{\varphi'_\alpha} & M'_j \end{array}$$

commutes for each $i, j \in Q_0$. That is, $f_j \varphi_\alpha = \varphi'_\alpha f_i$.

Concretely, consider the following two quiver representations

$$M : \mathbb{C} \xrightarrow{1} \mathbb{C}$$

$$M' : \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 10 \\ 10 \\ 00 \end{pmatrix}} \mathbb{C}^3 .$$

Then we can construct a morphism $f = (f_1, f_2) : M \rightarrow M'$. We see that if we set $f_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, then the diagram commutes.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{1} & \mathbb{C} \\ f_1 \downarrow & & \downarrow f_2 \\ \mathbb{C}^2 & \xrightarrow{\begin{pmatrix} 10 \\ 10 \\ 00 \end{pmatrix}} & \mathbb{C}^3 \end{array} .$$

Consider again the representations $M = (M_i, \varphi_\alpha)_i \in Q_0, M' = (M'_i, \varphi'_\alpha)_i \in Q_0$. We define the direct sum between M and M' as

$$M \oplus M' := (M_i \oplus M'_i, \begin{pmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{pmatrix}).$$

Consider the example where Q is the quiver,

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

with representations

$$M : \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{0} 0$$

$$M' : \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 11 \\ 01 \end{pmatrix}} \mathbb{C}^2 \xleftarrow{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \mathbb{C}.$$

Then we can take their direct sum

$$M \oplus M' : \mathbb{C} \oplus \mathbb{C}^2 \xrightarrow{\begin{pmatrix} 100 \\ 011 \\ 001 \end{pmatrix}} \mathbb{C} \oplus \mathbb{C}^2 \xleftarrow{\begin{pmatrix} 00 \\ 01 \\ 01 \end{pmatrix}} 0 \oplus \mathbb{C}.$$

Note that this direct sum is isomorphic to

$$\mathbb{C}^3 \xrightarrow{\begin{pmatrix} 100 \\ 011 \\ 001 \end{pmatrix}} \mathbb{C}^3 \xleftarrow{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \mathbb{C}.$$

Path Algebra of a Quiver

Let Q be a quiver and $i, j \in Q_0$. We define a path c from i to j of length l as the sequence

$$c = (i|\alpha_1, \alpha_2, \dots, \alpha_l|j)$$

where each $\alpha_n \in Q_1$ for $n = 1, \dots, l$. For this to be a well-defined path, each arrow must pick up where its predecessor left off, so we have the following conditions

$$t(\alpha_1) = i$$

$$h(\alpha_n) = t(\alpha_{n+1})$$

$$h(\alpha_l) = j.$$

We also consider the path at vertex i of length zero, which is called the lazy or constant path, denoted e_i . This path never leaves the vertex at which it originates.

For example, we have the quiver

$$\begin{array}{ccccc} & \alpha & & & \\ & \curvearrowright & & & \\ & 1 & \xrightarrow{\beta} & 2 & \xleftarrow{\gamma} 3 \end{array}$$

where we have the paths $(1|\alpha|1)$, $(1|\alpha, \beta|2)$, $(1|\alpha, \alpha, \beta|2)$, but we may not have a path which originates at vertex 1 and ends at vertex 3.

Now we define the path algebra of a quiver Q . We define the vector space $\mathbb{C}Q$ of Q as an algebra with basis the set of all paths in Q . Because this vector

space is extended to be an algebra, we define a multiplication between paths.

Say c, c' are two paths of Q , then

$$cc' = \begin{cases} c \cdot c' & \text{if } h(c) = t(c') \\ 0 & \text{otherwise} \end{cases}$$

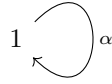
We call this multiplication concatenation of paths. Specifically, if $c = (i_1 | \alpha_1, \dots, \alpha_n | j_1)$ and $c' = (j_2 | \beta_1, \dots, \beta_m | j_2)$ with $i_1, i_2, j_1, j_2 \in Q_0$ and $\alpha_a, \beta_b \in Q_1$, then if $j_1 = i_2$, then

$$c \cdot c' = (i_1 | \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_m | j_2).$$

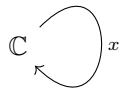
The unity element of this algebra is the sum of all constant paths

$$1_{\mathbb{C}Q} = \sum_{i \in Q_0} e_i.$$

Consider the quiver



with representation



We have the constant path, e in addition to x , but we also have x^2, x^3, \dots . Note that this looks very similar to a polynomial. In fact, this representation is isomorphic to $\mathbb{C}[x]$.

Some Category Theory Definitions

For this section, consider two categories, \mathcal{C}, \mathcal{D} .

A covariant functor $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ is a mapping such that for all $X \in \mathcal{C}$, $\mathcal{F}(X) \in \mathcal{D}$ and for all morphisms $f : X \rightarrow Y$ in \mathcal{C} , $\mathcal{F}(f) : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ in \mathcal{D} .

Consider two covariant functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \rightarrow \mathcal{D}$ and a natural transformation $\phi : \mathcal{F} \rightarrow \mathcal{G}$ that assigns each object $x \in \mathcal{C}$ a morphism $\phi_x : \mathcal{F}(X) \rightarrow \mathcal{G}(X)$ in \mathcal{D} such that

$$\begin{array}{ccc} \mathcal{F}(X) & \xrightarrow{\mathcal{F}(f)} & \mathcal{F}(Y) \\ \phi_X \downarrow & & \downarrow \phi_Y \\ \mathcal{G}(X) & \xrightarrow{\mathcal{G}(f)} & \mathcal{G}(Y) \end{array}$$

commutes. If ϕ_X is isomorphic for all $X \in \mathcal{C}$, then ϕ is a natural isomorphism and we say that \mathcal{F} is naturally isomorphic to \mathcal{G} .

We consider two categories \mathcal{C}, \mathcal{D} to be equivalent if there exist two functors \mathcal{F}, \mathcal{G} where $\mathcal{F} : \mathcal{C} \rightarrow \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{C}$ such that $\mathcal{F} \circ \mathcal{G}$ is naturally isomorphic to $1_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F}$ is naturally isomorphic to $1_{\mathcal{C}}$.

Theorem. *The categories $\text{Rep}(Q)$ and $\mathbb{C}Q\text{-mod}$ are equivalent.*

Proof. Set a functor $\mathcal{F} : \mathbb{C}Q\text{-mod} \rightarrow \text{Rep}(Q)$. If M is a $\mathbb{C}Q$ -module, then $\mathcal{F}(M)$ is a quiver representation. For each vertex $i \in Q_0$, we have

$$\mathcal{F}(M)(i) = e_i M$$

and for each arrow $\alpha \in Q_1$, we have

$$\mathcal{F}(M)(\alpha) : e_{t(\alpha)} M \rightarrow e_{h(\alpha)} M.$$

Now consider $\phi : M \rightarrow N$ a morphism of $\mathbb{C}Q$ -modules. Then, $\mathcal{F}(\phi)$ is defined as the map $\phi_i : M_i \rightarrow N_i$ sending me_i to $\phi(m)e_i$.

Now set a functor $\mathcal{G} : \text{Rep}(Q) \rightarrow \mathbb{C}Q\text{-mod}$. If V is a quiver representation, then $\mathcal{G}(V)$ is a $\mathbb{C}Q$ -module such that

$$\mathcal{G}(V) = \bigoplus_{i \in Q_0} V_i.$$

If p is a path in $\mathbb{C}Q\text{-mod}$ with $v \in V_i$, then $p \cdot v = V(p)v$ if $t(p) = i$ and 0

otherwise. This gives $\mathcal{G}(V)$ the structure of a $\mathbb{C}Q$ -module. Let $\psi : V \rightarrow W$ be a morphism of representations, then

$$\mathcal{G}(\psi)(m) = (\psi_i(m_i))_{i \in Q_0}.$$

It is easy to see that $\mathcal{F} \circ \mathcal{G}$ is naturally isomorphic to $1_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F}$ is naturally isomorphic to $1_{\mathcal{C}}$. □