Quivers

A quiver $Q = (Q_0, Q_1, h, t)$ is defined as a four-tuple consisting of two sets and two functions. The set of vertices is Q_0, the set of arrows is Q_1, $h : Q_1 \to Q_0$ is the function mapping arrows to their terminal point, and $t : Q_1 \to_0$ is the function mapping arrows to their starting point. We only consider finite sets Q_0, Q_1.

A quiver Q as below

$$1 \xrightarrow{\alpha} 2$$

has $Q_0 = \{1, 2\}$, $Q_1 = \{\alpha\}$, $h(\alpha) = 2$, and $t(\alpha) = 1$.

Quivers may also possess more complex structures such as double arrows and loops. Essentially any construction is allowed so long as each arrow has a starting and ending point.

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\beta"'] & 2 \end{tikzcd}$$

Quiver Representations

A quiver representation is an object in the abelian $\mathbb{C}$-category $\mathrm{Rep}(Q)$. An abelian category is a category that possesses some nice features such as a zero category, kernels and cokernels, short exact sequences, and direct sums. A quiver representation, $M = (M_i, \varphi_\alpha)_{i \in Q_0, \alpha \in Q_1}$ is a collection of $\mathbb{C}$ vector spaces, one for each vertex of Q, and linear maps, one for each arrow of Q.

We define a morphism between two quiver representations M, M', f, as a collection of mappings $(f_i)_i \in Q_0$ where

$$\begin{tikzcd} M_i \arrow[r, "\varphi_\alpha"] \arrow[d, "f_i"'] & M_j \arrow[d, "f_j"] \\ M'_i \arrow[r, "\varphi'_\alpha"] & M'_j \end{tikzcd}$$

commutes for each $i, j \in Q_0$. That is, $f_j \varphi_\alpha = \varphi'_\alpha f_i$.

Concretely, consider the following two quiver representations

$$M: \ \mathbb{C} \xrightarrow{1} \mathbb{C}$$

$$M': \ \mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix}1&0\\1&0\\0&0\end{smallmatrix}\right)} \mathbb{C}^3 \ .$$

Then we can construct a morphism $f = (f_1, f_2) : M \to M'$. We see that if we set $f_1 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$ and $f_2 = \begin{pmatrix}1\\1\\0\end{pmatrix}$, then the diagram commutes.

$$\begin{tikzcd} \mathbb{C} \arrow[r, "1"] \arrow[d, "f_1"'] & \mathbb{C} \arrow[d, "f_2"] \\ \mathbb{C}^2 \arrow[r, "\left(\begin{smallmatrix}1&0\\1&0\\0&0\end{smallmatrix}\right)"] & \mathbb{C}^3 \end{tikzcd} .$$

Consider again the representations $M = (M_i, \varphi_\alpha)_i \in Q_0, M' = (M'_i, \varphi'_\alpha)_i \in Q_0$. We define the direct sum between M and M' as

$$M \oplus M' := (M_i \oplus M'_i, \begin{pmatrix} \varphi_\alpha & 0 \\ 0 & \varphi'_\alpha \end{pmatrix}).$$

Consider the example where Q is the quiver,

$$1 \xrightarrow{\alpha} 2 \xleftarrow{\beta} 3$$

with representations

$$M: \ \mathbb{C} \xrightarrow{1} \mathbb{C} \xleftarrow{0} 0$$

$$M': \ \mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)} \mathbb{C}^2 \xleftarrow{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)} \mathbb{C}.$$

Then we can take their direct sum

$$M \oplus M': \ \mathbb{C} \oplus \mathbb{C}^2 \xrightarrow{\left(\begin{smallmatrix}1&0&0\\0&1&1\\0&0&1\end{smallmatrix}\right)} \mathbb{C} \oplus \mathbb{C}^2 \xleftarrow{\left(\begin{smallmatrix}0&0\\0&1\\0&1\end{smallmatrix}\right)} 0 \oplus \mathbb{C}.$$

Note that this direct sum is isomorphic to

$$\mathbb{C}^3 \xrightarrow{\begin{pmatrix} 100 \\ 011 \\ 001 \end{pmatrix}} \mathbb{C}^3 \xleftarrow{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}} \mathbb{C}.$$

Path Algebra of a Quiver

Let Q be a quiver and $i, j \in Q_0$. We define a path c from i to j of length l as the sequence

$$c = (i|\alpha_1, \alpha_2, ..., \alpha_l|j)$$

where each $\alpha_n \in Q_1$ for $n = 1, ..., l$. For this to be a well-defined path, each arrow must pick up where its predecessor left off, so we have the following conditions

$$t(\alpha_1) = i$$
$$h(\alpha_n) = t(\alpha_{n+1})$$
$$h(\alpha_l) = j.$$

We also consider the path at vertex i of length zero, which is called the lazy or constant path, denoted e_i. This path never leaves the vertex at which it originates.

For example, we have the quiver

$$\begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[r, "\beta"] & 2 & 3 \arrow[l, "\gamma"'] \end{tikzcd}$$

where we have the paths $(1|\alpha|1), (1|\alpha, \beta|2), (1|\alpha, \alpha, \beta|2)$, but we may not have a path which originates at vertex 1 and ends at vertex 3.

Now we define the path algebra of a quiver Q. We define the vector space $\mathbb{C}Q$ of Q as an algebra with basis the set of all paths in Q. Because this vector space is extended to be an algebra, we define a multiplication between paths. Say c, c' are two paths of Q, then

$$cc' = \begin{cases} c \cdot c' \text{ if h(c) = t(c')} \\ 0 \text{ otherwise} \end{cases}$$

We call this multiplication concatenation of paths. Specifically, if $c = (i_1|\alpha_1, ..., \alpha_n|j_1)$ and $c' = (j_2|\beta_1, ..., \beta_m|j_2)$ with $i_1, i_2, j_1, j_2 \in Q_0$ and $\alpha_a, \beta_b \in Q_1$, then if $j_1 = i_2$, then

$$c \cdot c' = (i_1|\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_m|j_2).$$

The unity element of this algebra is the sum of all constant paths

$$1_{\mathbb{C}Q} = \sum_{i \in Q_0} e_i.$$

Consider the quiver

$$\begin{tikzcd} 1 \arrow[loop right, "\alpha"] \end{tikzcd}$$

with representation

$$\begin{tikzcd} \mathbb{C} \arrow[loop right, "x"] \end{tikzcd}$$

We have the constant path, e in addition to x, but we also have $x^2, x^3, ...$ Note that this looks very similar to a polynomial. In fact, this representation is isomorphic to $\mathbb{C}[x]$.

Some Category Theory Definitions

For this section, consider two categories, $\mathcal{C}, \mathcal{D}$.

A covariant functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a mapping such that for all $X \in \mathcal{C}$, $\mathcal{F}(X) \in \mathcal{D}$ and for all morphisms $f : X \to Y$ in $\mathcal{C}$, $\mathcal{F}(f) : \mathcal{F}(X) \to \mathcal{F}(Y)$ in $\mathcal{D}$.

Consider two covariant functors $\mathcal{F}, \mathcal{G} : \mathcal{C} \to \mathcal{D}$ and a natural transformation $\phi : \mathcal{F} \to \mathcal{G}$ that assigns each object $x \in \mathcal{C}$ a morphism $\phi_X : \mathcal{F}(X) \to \mathcal{G}(X)$ in $\mathcal{D}$ such that

$$\begin{tikzcd} \mathcal{F}(X) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\phi_X"'] & \mathcal{F}(Y) \arrow[d, "\phi_Y"] \\ \mathcal{G}(X) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(Y) \end{tikzcd}$$

commutes. If ϕ_X is isomorphic for all $X \in \mathcal{C}$, then ϕ is a natural isomorphism and we say that $\mathcal{F}$ is naturally isomorphic to $\mathcal{G}$.

We consider two categories $\mathcal{C}, \mathcal{D}$ to be equivalent if there exist two functors $\mathcal{F}, \mathcal{G}$ where $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ and $\mathcal{G} : \mathcal{D} \to \mathcal{C}$ such that $\mathcal{F} \circ \mathcal{G}$ is naturally isomorphic to $1_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F}$ is naturally isomorphic to $1_{\mathcal{C}}$.

Theorem. *The categories Rep*(Q) *and* $\mathbb{C}Q$*-mod are equivalent.*

Proof. Set a functor $\mathcal{F} : \mathbb{C}Q - \text{mod} \to \text{Rep}(Q)$. If M is a $\mathbb{C}Q$-module, then $\mathcal{F}(M)$ is a quiver representation. For each vertex $i \in Q_0$, we have

$$\mathcal{F}(M)(i) = e_i M$$

and for each arrow $\alpha \in Q_1$, we have

$$\mathcal{F}(M)(a) : e_{t(\alpha)}M \to e_{h(\alpha)}M.$$

Now consider $\phi : M \to N$ a morphism of $\mathbb{C}Q$-modules. Then, $\mathcal{F}(\phi)$ is defined as the map $\phi_i : M_i \to M_i'$ sending me_i to $\phi(m)e_i$.

Now set a functor $\mathcal{G} : \text{Rep}(Q) \to \mathbb{C}Q - \text{mod}$. If V is a quiver representation, then $\mathcal{G}(V)$ is a $\mathbb{C}Q$ − module such that

$$\mathcal{F}(V) = \oplus_{i \in Q_0} V_i.$$

If p is a path in $\mathbb{C}Q$ − mod with $v \in V_i$, then $p \cdot v = V(p)v$ if $t(p) = i$ and 0 otherwise. This gives $\mathcal{G}(V)$ the structure of a $\mathbb{C}Q$-module. Let $\psi : V \to W$ be a morphism of representations, then

$$\mathcal{G}(\psi)(m) = (\psi_i(m_i))_{i \in Q_0}.$$

It is easy to see that $\mathcal{F} \circ \mathcal{G}$ is naturally isomorphic to $1_{\mathcal{D}}$ and $\mathcal{G} \circ \mathcal{F}$ is naturally isomorphic to $1_{\mathcal{C}}$. $\square$