Noah Dcruz

Homework 2

MATH5131

1)

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1) a) Cigarette smoke $\rightarrow$ $\boxed{\text{bar}}$ $\xrightarrow{\text{ventilator}}$

$$\begin{bmatrix}\text{rate of change}\\ \text{of carbon monoxide}\end{bmatrix} = \begin{bmatrix}\text{rate of carbon}\\ \text{monoxide by}\\ \text{smoke}\end{bmatrix} - \begin{bmatrix}\text{rate carbon monoxide}\\ \text{removed by ventilators}\end{bmatrix}$$

$$C'(t) = \frac{R_{in}}{V} C_{in} - \frac{R_{out}}{V} C(t)$$

$$C'(t) + \frac{R_{out}}{V} C(t) = \frac{R_{in}}{V} C_{in}$$

$$e^{\int \frac{R_{out}}{V} dt} = e^{\left(\frac{R_{out}}{V}\right)t}$$

$$e^{\left(\frac{R_{out}}{V}\right)t} C(t) = \int e^{(R_{out}/V)t} \frac{R_{in}}{V} C_{in}\, dt$$

$$u = \left(\frac{R_{out}}{V}\right) t \rightarrow u' = \frac{R_{out}}{V}$$

(This constant use x instead of C)

$$dt = \frac{du}{u'} = \frac{du}{\frac{R_{out}}{V}} = \frac{V}{du\, R_{out}}$$

$$e^{\left(\frac{R_{out}}{V}\right)t} C(t) = e^{\left(\frac{R_{out}}{V}\right)t} \frac{R_{in}}{V} C_{in} \times \frac{V}{du\, R_{out}} + x$$

$$C(t) e^{(R_{out}/V)t} = C_{in} \frac{R_{in}}{R_{out}} e^{(R_{out}/V)t} + x$$

As $C(0) = 0$

$$0 e^{\left(\frac{R_{out}}{V}\right)0} = e^{\left(\frac{R_{out}}{V}\right)0} C_{in} \frac{R_{in}}{R_{out}} + x$$

$$0 = C_{in} \frac{R_{in}}{R_{out}} + x$$

$$x = -C_{in} \frac{R_{in}}{R_{out}}$$

$$C(t)e^{(R_{out}/V)t} = e^{\left(\frac{R_{out}}{V}\right)t} C_{in}\frac{R_{in}}{R_{out}} - C_{in}\frac{R_{in}}{R_{out}}$$

$$\frac{C(t)e^{\left(\frac{R_{out}}{V}\right)t}}{e^{(R_{out}/V)t}} = \frac{e^{\left(\frac{R_{out}}{V}\right)t}}{e^{(R_{out}/V)t}} C_{in}\frac{R_{in}}{R_{out}} - C_{in}\frac{R_{in}}{R_{out}} \times \frac{1}{e^{(R_{out}/V)t}}$$

$$C(t) = C_{in}\frac{R_{in}}{R_{out}} - C_{in}\frac{R_{in}}{R_{out}} e^{-\left(\frac{R_{out}}{V}\right)t}$$

$$C(t) = C_{in}\frac{R_{in}}{R_{out}}\left[1 - e^{-\left(\frac{R_{out}}{V}\right)t}\right]$$

b) $R_{in} = 0.006 m^3/min$, $R_{out} = 10R_{in} = 10 \times 0.006 m^3/min = 0.06 m^3/min$

$C_{in} = 0.04 = 4\%$, $C(t) = 0.012\%$, $V = 20m \times 15m \times 4m = 1200 m^3$

$$C(t) = C_{in}\frac{R_{in}}{R_{out}}\left[1 - e^{-\left(\frac{R_{out}}{V}\right)t}\right]$$

$$0.012\% = 4\% \frac{R_{in}}{10R_{in}}\left[1 - e^{-\left(\frac{0.06 m^3/min}{1200 m^3}\right)t}\right]$$

$$\frac{0.012}{4} = \frac{1}{10}\left[1 - e^{-5\times10^{-5}t}\right]$$

$$\frac{0.012\times10}{4} = 1 - e^{-5\times10^{-5}t}$$

$$0.03 = 1 - e^{-5\times10^{-5}t}$$

$$e^{-5\times10^{-5}t} = 1 - 0.03$$

$$e^{-5\times10^{-5}t} = 0.97$$

$$\ln e^{-5\times10^{-5}t} = \ln 0.97$$

$$-5\times10^{-5}t = \ln 0.97$$

$$t = \frac{\ln 0.97}{-5\times10^{-5}}$$

$$t = 609.1841497 \text{ min}$$

$$1 \text{ hr} = 60 \text{ min}$$

$$x = 609.1841497 \text{ min}$$

$$x = 10.15306916 \text{ hours}$$

A

The compartment diagram is the cigarette smoke being produced entering the bar and then being removed/filtered out by the ventilator.

The rate of change of carbon monoxide is the rate of carbon monoxide produced by smoke minus the rate of carbon monoxide removed by the ventilators.

The rate of change of carbon monoxide is represented as C'(t). The rate of carbon monoxide produced by smoke is represented as $(R_{in}/ V) * C_{in.}$ The rate of carbon monoxide removed by the ventilators is $(R_{out} / V) * C(t)$.

The equation is then integrated by using an integrating factor then the constant X produced by integration is solved by using C(t) = 0 at time t = 0. Then for isolating the left side of the equation we divided the whole equation by $e^{-(R_{out}/V) *t}$. Then on the right side of the equation, I factored out $C_{in} * (R_{in}/R_{out})$ to get the following differential equation:

$$C(t) = C_{in} \frac{R_{in}}{R_{out}} \left[1 - e^{-\left(\frac{R_{out}}{V}\right)t} \right]$$

B

The given information needed to plug into the above equation.

R_{in}=0.006 m^3/min,

R_{out}= 10 * R_{in}= 10 * 0.006 m^3/min

C_{in}= 4%

C(t) = 0.012% (The lethal limit)

V = 20m*15m*4m = 1200m^3 (The volume of the room)

As seen in the calculations above, the time it takes to reach the lethal limit of 0.012% is 609.1841497 minutes (10.15 hours). Since, the bar opens at 6 pm it will reach the lethal limit at 4:09 am.

2)

A

2) a) $M'(t) = 0.01r\,M(t)$ Rule of 72 $\left[t = \frac{72}{r}\right]$

$$\frac{dM}{dt} = 0.01r\,M(t)$$

$$\frac{dM}{M(t)} = 0.01r\,dt$$

$$\int \frac{dM}{M(t)} = \int 0.01r\,dt$$

$$\ln|M(t)| = 0.01rt + C$$

$$e^{\ln|M(t)|} = e^{0.01rt}C$$

$$|M(t)| = Ce^{0.01rt}$$

$$M(t) = \pm Ce^{0.01rt}$$

investing and doubling original investment so only consider + (positive) no − (negative)

$$M(t) = Ce^{0.01rt}$$

Let C represent the initial amount of money invested. Since we want to find the time the original investment doubles. Let $M(t) = 2C$

$$2C = Ce^{0.01rt}$$

$$2 = e^{0.01rt}$$

$$\ln 2 = \ln e^{0.01rt}$$

$$\ln 2 = 0.01rt$$

$$\left[t = \frac{\ln 2}{0.01r}\right] \qquad \left[\text{Rule of 72: } t = \frac{72}{r}\right]$$

The equation derived above for an initial investment to double at r% interest rate

Rule of 72 equation for time for an investment to double at r% interest rate

Check for overestimation

$r = 2$

$$t = \frac{\ln 2}{0.01 \times 2} \qquad t = \frac{72}{2}$$

$$t = 34.66 < t = 36$$

$r = \cancel{5}\ 7$

$$t = \frac{\ln 2}{0.01 \times 7} \qquad t = \frac{72}{7}$$

$$t = 9.902 < t = 10.2857$$

$r = 10$

$$t = \frac{\ln 2}{0.01 \times 10} \qquad t = \frac{72}{10}$$

$$t = 6.93 < t = 7.2$$

rule of 72: $t = \frac{72}{r}$ always overestimates the time required for investment to double

I integrated the provided equation M'(t) = 0.01rM(t) as it is a separable differential equation.

I rewrote M'(t), the rate at which money compounds with respect to time, as dM/dt. Since the equation is separable, the left side of the equation leads to dM/M(t) and the right side of the equation is 0.01*r*dt.

Integrate both sides to get the left-hand side to get ln|M(t)| + constant and the right-hand side of the equation to get (0.01*r*t) + constant.

For simplification purposes, I wrote the constant on the right-hand side and skipped the previously written step of adding the constant to the left.

Then both sides are risen as the power of e. This results in M(t) = +-Ce$^{0.01*r*t}$.

Since, the investment is expected to double(grow) we drop the negative sign: M(t) = Ce$^{0.01*r*t}$.

Since the time needs to be found for the original investment to double, I let C represent the initial amount of money invested and let M(t) = 2C as this represents the doubling of the original investment. The formula is then 2C = Ce$^{0.01*r*t}$, cancel the C's to get 2=e$^{0.01*r*t}$.

Then I took the log of both sides to get ln 2 = ln e$^{0.01*r*t}$

The next step is ln 2 = 0.01*r*t.

Isolating t to find the time it takes for an original investment to double at r% interest rate is:
t = ln 2/(0.01*r).

Then taking the equation I derived above and the Rule of 72, I plugged in different interest rates to both equations to find the amount of time it takes for an investment to double.

I plugged in 2% and got 34.66 years from the equation I derived above and 36 years from the Rule of 72. Since, 36 years is higher than 34.66 years; the Rule of 72 overestimates the time it takes for an investment to double.

I then checked at different rates 7% and 10% and further proved that the Rule of 72 overestimates the time it takes for an investment to double.

B

b) $M(t) = Ce^{0.01rt}$ let $C = M(0)$

C is the initial investment, r is the interest rate, t is time

$M(t)$ is the amount expected from compounding interest on the investment, causing the initial amount to grow.

$M(65)$: Amount after 65 years $t = 65$ years

$M(0)$: initial investment of \$7000

r: 7%

$M(t) = M(0)e^{0.01rt}$

$M(65) = M(0)e^{0.01rt}$

$M(65) = \$7000e^{0.01(7)(65)}$

$M(65) = \$7000 \times 94.63240831$

$M(65) = \$662,426.8582$

$M(65)_{r=7} < \$1,000,000$

$\$662,426.8582 < \$1,000,000$

r: 8%

$M(65) = M(0)e^{0.01(8)(65)}$

$M(65) = 7000\,e^{0.01(8)(65)}$

$M(65) = 7000 \times 181.2722419$

$M(65) = 1,268,905.693$

$M(65)_{r=8} > \$1,000,000$

$\$1,268,905.693 > \$1,000,000$

The two rates were picked from smartasset.com roth ira return which provides a range of 7% - 10%. Comparing 7% to 8%, the influencer would be right at 8%.

Check for exact r to hit 1,000,000 USD 65 years later with initial investment of 7000 USD.

$M(t) = M(0)e^{0.01rt}$

$\$1,000,000 = \$7000e^{0.01r(65)}$

$\frac{\$1,000,000}{\$7000} = e^{0.65r}$

$\frac{1000}{7} = e^{0.65r}$

$\ln\left(\frac{1000}{7}\right) = \ln e^{0.65r}$

$\ln\left(\frac{1000}{7}\right) = 0.65r$

$r = \frac{\ln(1000/7)}{0.65}$

$r = 7.6336$

The exact rate for an initial investment of \$7000 to hit \$1,000,000 after 65 years is 7.63%

I took the equation from part A: $M(t) = Ce^{0.01*r*t}$.

Then I described the variables in the equation.

C is the initial investment. I changed the variable C to M(0), as I found it easier to work with.

R is the interest rate.

T is time.

M(t) is the amount expected from compounding interest on the investment, causing the initial investment to grow.

Then reading the influencer's statement, the initial investment M(0): $7000.00, t:65 years and M(65) is the amount that the $7,000 will grow to after compounding interest after 65 years: $1,000,000.00

I picked two rates: 7% and 8%, from smartasset.com Roth Ira return ***(1, Ashley Kilroy)*** which provides a range of 7% to 10%.

So, I plugged in the given numbers to find M(65) at 7% and 8%. At r=7%, M(65) = $662,426.8582 and at r=8%, M(65) = $1,268,905.693.

When r=7%, the investment grows to $662,426.8582 which is less than $1,000,000 as mentioned in the influencer's statement. While when r=8%, the investment grows to $1,268,905.693 which is more than $1,000,000.

So, the rate at which an original investment of $7,000.00 grows to $1,000,000.00 after 65 years is between 7% and 8%.

I then wanted to find the exact rate so I go back to the equation $M(t) = M(0)e^{0.01*r*t}$.

M(t): M(65): $1,000,000.00

M(0): $7,000.00

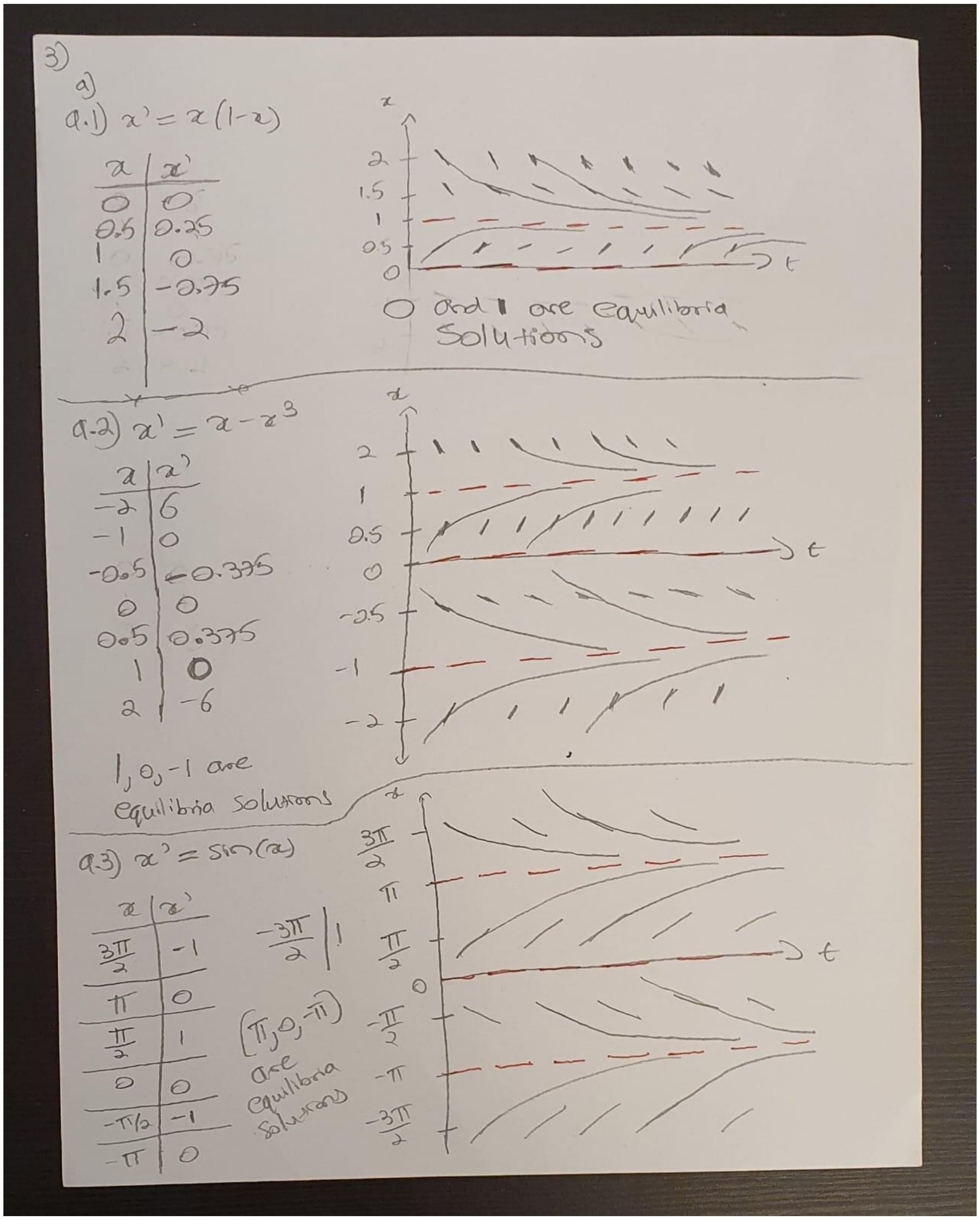
T : 65

Solving for r, we take 7000.00 to the left side to get 1000/7. Taking the log of both sides of the equation to get $\ln(1000/7) = \ln(e^{0.65r})$. Then, $\ln(1000/7) = (0.65r)$. Then $r = (\ln(1000/7))/0.65$

The result is r=7.63% for an initial investment of $7,000.00 to reach $1,000,000.00 after 65 years.

3)

a.1, a.2, a.3



<u>B</u>

The system which can relate to sin(x) is the pendulum. As when the pendulum is at its highest position the velocity is zero as it has to swing back down and when the pendulum is at its lowest position, the velocity is the highest as it has to swing back up.

4)

4) $f^{(3)}(x) \approx \dfrac{f(x+2h) - 2f(x+h) + 2f(x-h) - f(x-2h)}{2h^3}$

Actual - Approximations = Error

$f(x+2h)$

$f(x) + f'(x)(2h) + \dfrac{f''(x)(2h)^2}{2} + \dfrac{f'''(x)(2h)^3}{6} + h.o.t$

$f(x) + 2f'(x)h + 2f''(x)h^2 + \frac{4}{3}f'''(x)h^3 + h.o.t$

$-2f(x+h) = -2[f(x+h)]$

$-2\left[f(x) + f'(x)h + \dfrac{f''(x)h^2}{2} + \dfrac{f'''(x)h^3}{6} + h.o.t\right]$

$-2f(x) - 2f'(x)h - f''(x)h^2 - \dfrac{f'''(x)h^3}{3} + h.o.t$

$+2f(x-h) = 2[f(x-h)]$

$2\left[f(x) + f'(x)(-h) + \dfrac{f''(x)(-h)^2}{2} + \dfrac{f'''(x)(-h)^3}{6} + h.o.t\right]$

$2\left[f(x) - f'(x)h + \dfrac{f''(x)h^2}{2} - \dfrac{f'''(x)h^3}{6} + h.o.t\right]$

$2f(x) - 2f'(x)h + f''(x)h^2 - \dfrac{f'''(x)h^3}{3} + h.o.t$

$$-[f(x-2h)]$$

$$-\left[f(x) + f'(x)(-2h) + \frac{f''(x)(-2h)^2}{2} + \frac{f'''(x)(-2h)^3}{6} + h.o.t\right]$$

$$-\left[f(x) - 2f'(x)h + 2f''(x)h^2 - \frac{4}{3}f'''(x)h^3 + h.o.t\right]$$

$$-f(x) + 2f'(x)h - 2f''(x)h^2 + \frac{4}{3}f'''(x)h^3 + h.o.t$$

Put it all together

$$\cancel{f(x)} + 2\cancel{f'(x)}h + 2\cancel{f''(x)}h^2 + \frac{4}{3}f'''(x)h^3$$

$$-\cancel{2f(x)} - 2\cancel{f'(x)}h - \cancel{f''(x)}h^2 - \frac{f'''(x)h^3}{3}$$

$$+\cancel{2f(x)} - 2\cancel{f'(x)}h + \cancel{f''(x)}h^2 - \frac{f'''(x)h^3}{3}$$

$$-\cancel{f(x)} + 2\cancel{f'(x)}h - \cancel{2f''(x)}h^2 + \frac{4}{3}f'''(x)h^3 + h.o.t$$

$$\frac{8}{3}f'''(x)h^3 - \frac{2}{3}f'''(x)h^3 + h.o.t$$

$$\frac{6}{3}f'''(x)h^3 + h.o.t$$

$$\left[2f'''(x)h^3 + h.o.t\right]\frac{1}{2h^3}$$

$$f'''(x) + h.o.t$$

$$f'''(x) - f'''(x) = 0$$

Redo but Look to terms greater than 3 [4 and 5]

$$f(x+2h) = \frac{f^{(4)}(x)(2h)^4}{4!} + \frac{f^{(5)}(x)(2h)^5}{5!} = \frac{16 f^{(4)}(x) h^4}{4!} + \frac{f^{(5)}(x)(2h)^5}{5!}$$

$$-2[f(x+h)] = -2\left[\frac{f^4(x)h^4}{4!} + \frac{f^5(x)h^5}{5!}\right]$$

$$-\frac{2f^4(x)h^4}{4!} - \frac{2f^5(x)h^5}{5!}$$

$$2[f(x-h)] = 2\left[\frac{f^4(x)(-h)^4}{4!} + \frac{f^5(x)(-h)^5}{5!}\right]$$

$$\frac{2f^4(x)h^4}{4!} - \frac{2f^5(x)h^5}{5!}$$

$$-[f(x-2h)] = -\left[\frac{f^4(x)(-2h)^4}{4!} + \frac{f^5(x)(-2h)^5}{5!}\right]$$

$$-\frac{16 f^4(x)h^4}{4!} + \frac{32 f^5(x)h^5}{5!}$$

Add all up [first 4's]

$$\cancel{\frac{16 f^4(x)h^4}{4!}} \cancel{- \frac{16 f^4(x)h^4}{4!}} \cancel{- \frac{2f^4(x)h^4}{4!}} \cancel{+ \frac{2f^4(x)h^4}{4!}}$$

All cancel

Add all 5's

$$\frac{32f^5(x)h^5}{5!} - \frac{2f^5(x)h^5}{5!} - \frac{2f^5(x)h^5}{5!} + \frac{32f^5(x)h^5}{5!}$$

$$\frac{60f^5(x)h^5}{5!}$$

$$\frac{60f^5(x)h^5}{5\times \cancel{4\times 3}\times 2\times 1} = \frac{f^5(x)h^5}{2}\times\frac{1}{2h^3}$$

$$\frac{f^5(x)\,h^2}{4} \qquad O(h^x) = O(h^{2+1}) = O(h^3)$$

Error linked to power of h+1

Plug back into original formula

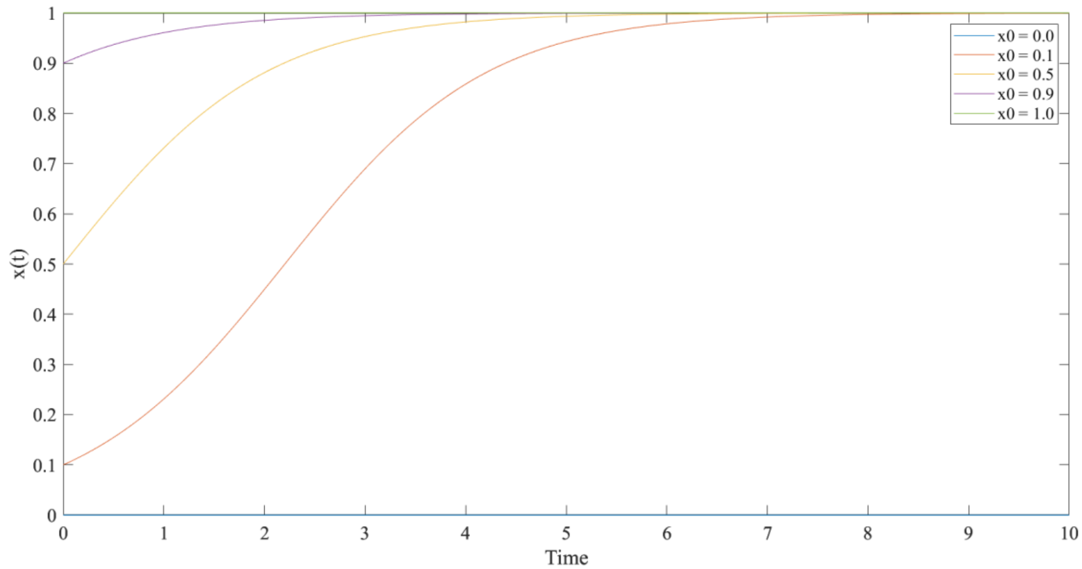
$$f'''(x) - \left[\frac{f(x+2h)-2f(x+h)+2f(x-h)-f(x-2h)}{2h^3}\right]$$

$$f'''(x) - \left[\frac{1}{4}f^5(x)h^2 + O(h^3) + f'''(x)\right]$$

$$-\frac{1}{4}f^5(x)h^2 + O(h^3)$$

5)

```
clear;
close all;
clc;

% time interval and step size
t0 = 0;
tf = 10;
h = 0.01;

% number of steps
N = (tf - t0)/h;

% Define the time array
t = linspace(t0, tf, N + 1);

% initial conditions
x0_vals = [0, 0.1, 0.5, 0.9, 1];

% Store the solutions
x = zeros(N + 1, length(x0_vals));

% Solve the ODE for each initial condition
for i = 1:length(x0_vals)
    x0 = x0_vals(i);
    x(1, i) = x0;

    for j = 1:N
        x(j + 1, i) = x(j, i) + h * x(j, i) * (1 - x(j, i));
    end

    % Plot solution for each initial condition
    plot(t, x(:, i), '-', 'DisplayName', sprintf('x0 = %.1f', x0));
    hold on;
end

% legend
legend('show');

% Make the plot
set(gcf, 'color', 'w');
set(gca, 'fontname', 'Times New Roman', 'fontsize', 14);
box on;
xlabel('Time');
ylabel('x(t)');
```



x(t)
Time
x0 = 0.0
x0 = 0.1
x0 = 0.5
x0 = 0.9
x0 = 1.0

CODE:

```
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box on;
xlabel('Time');
ylabel('x(t)');
```

Citations

1) Kilroy, Ashley. “What Is an Average Roth IRA Return?” *Yahoo!*, Yahoo!, 2023, https://www.yahoo.com/now/average-roth-ira-return-200250996.html.