# Even More Plural Theory in HOL

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September 13 & 20, 2018

# Game Plan

- At first, we work in HOL with basic types $\mathtt{e}$ and $\mathtt{t}$.
- For $A$ a type, an '$A$-set' means something of type $A \to \mathtt{t}$.
- We inroduce a Link-isch theory, using a unary type constructor $\mathsf{Agg}$ of **aggregates**.
- We make $\mathsf{Agg}$ into a monad.
- We then elaborate the theory to classify predicates and handle 'fancy' plurals.
- Eventually the theory will have to be framed in a richer type theory (at least with dependent sums indexed by the natural numbers) in order to handle predicates that can be predicated *only* of plurals.
- (Hyper-)intensionality will have to wait.

# Review of Useful Defined Terms for Set-ish Business

$$\{-\}_A := \lambda xy : A.x = y$$

$$\mathsf{nonempty}_A := \lambda S : A \to \mathsf{t}.\exists x : A.S\ x$$

$$\mathsf{singleton}_A := \lambda S : A \to \mathsf{t}.\exists x : A.S = \{x\}$$

$$\mathsf{pluralton}_A := \lambda S : A \to \mathsf{t}.\exists xy : A.(x \neq y) \wedge (S\ x) \wedge (S\ y)$$

$$\mathsf{injective}_{A,B} := \lambda f : A \to B.\forall xy : A.[(f\ x) = (f\ y)] \to x = y$$

$$\subseteq_A := \lambda ST : A \to \mathsf{t}.\forall x : A.(S\ x) \to (T\ x)$$

$$\bigcup_A := \lambda S : (A \to \mathsf{t}) \to \mathsf{t}.\lambda x : A.\exists T : A \to \mathsf{t}.(S\ T) \wedge (T\ x)$$

$$\cup_A := \lambda ST : A \to \mathsf{t}.\lambda x : A.(S\ x) \vee (T\ x)$$

# Link-isch Theory Basics (1/2)

- We introduce a unary type constructor $\mathsf{Agg}$ of *aggregates*.
- We introduce the type-indexed family of constants

$$\mathsf{atoms}_A : (\mathsf{Agg}\ A) \to A \to \mathsf{t}$$

axiomatized as bijections from the $A$-aggregates to the nonempty $A$-sets:

$$\vdash \mathsf{injective\ atoms}_A$$

$$\vdash \forall m : \mathsf{Agg}\ A.\mathsf{nonempty}\ (\mathsf{atoms}\ m)$$

$$\vdash \forall S : A \to \mathsf{t}.(\mathsf{nonempty}\ S) \to \exists m : \mathsf{Agg}\ A.S = (\mathsf{atoms}\ m)$$

# Link-isch Theory Basics (2/2)

- We define our counterpart of Link's part-of order as follows:

  $$\sqsubseteq_A := \lambda mn : \mathsf{Agg}\ A.(\mathsf{atoms}\ m) \subseteq (\mathsf{atoms}\ n)$$

  which makes the bijection from $A$-aggregates to nonempty $A$-sets into an order-isomorphism.

- We define singular and plural aggregates straightforwardly:

  $\mathsf{singular}_A := \lambda m : \mathsf{Agg}\ A.\mathsf{singleton}\ (\mathsf{atoms}\ m)$

  $\mathsf{plural}_A := \lambda m : \mathsf{Agg}\ A.\mathsf{pluralton}\ (\mathsf{atoms}\ m)$

# The Aggregate Monad (1/2)

The nonempty powerset functor has a well-known monad structure (aka the nondetermism monad), which we transfer to $\mathsf{Agg}$ via the $\mathsf{atoms}$ bijection:

Unit:

$$\eta_A : A \to (\mathsf{Agg}\ A)$$
$$\vdash \forall x : A.(\mathsf{atoms}\ (\eta_A\ x)) = \{x\}$$

Multiplication:

$$\mu_A : (\mathsf{Agg}^2 A) \to (\mathsf{Agg}\ A)$$
$$\vdash \forall m : \mathsf{Agg}^2 A.\mathsf{atoms}\ (\mu_A\ m) =$$
$$\bigcup(\lambda S : A \to \mathsf{t}.\exists n : \mathsf{Agg} A.(\mathsf{atoms}\ m\ n) \wedge (S = (\mathsf{atoms}\ n)))$$

Functor at level of terms:

$$\mathsf{agg}_{A,B} : (A \to B) \to (\mathsf{Agg}\ A) \to (\mathsf{Agg}\ B)$$
$$\vdash \forall f : A \to B.\forall m : \mathsf{Agg} A.\mathsf{atoms}\ (\mathsf{agg}_{A,B}\ f\ m) =$$
$$\lambda y : B.\exists x : A.(\mathsf{atoms}\ m\ x) \wedge y = (f\ x)$$

# The Aggregate Monad (2/2)

More standard monadic gadgetry (not used here):

Kleisli star:

$$\star_{A,B} : (\mathsf{Agg}\ A) \rightarrow (\mathsf{A} \rightarrow (\mathsf{Agg}\ B)) \rightarrow (\mathsf{Agg}\ B)$$
$$\vdash \forall m : \mathsf{AggA}.\forall f : A \rightarrow (\mathsf{Agg}\ B).(\mathsf{atoms}\ m \star f) =$$
$$\lambda y : B.\exists x : A.(\mathsf{atoms}\ m\ x) \wedge (\mathsf{atoms}\ (f\ x)\ y)$$

Monadic application:

$$\mathsf{mapp}_{A,B} : (\mathsf{Agg}(A \rightarrow B)) \rightarrow (\mathsf{Agg}\ A) \rightarrow (Agg\ B)$$
$$\vdash \forall h : \mathsf{Agg}\ (A \rightarrow B).\forall m : \mathsf{Agg}\ A.(\mathsf{atoms}\ (\mathsf{mapp}\ h\ m)) =$$
$$\lambda y : B.\exists f : A \rightarrow B.\exists x : A.(\mathsf{atoms}\ h\ f) \wedge (\mathsf{atoms}\ m\ x) \wedge$$
$$(y = (f\ x))$$

# Aggregate Sum

- We introduce a family of constants corresponding to Link's (binary) sum:

$$\vdash \sqcup_A : (\mathsf{Agg}\ A) \to (\mathsf{Agg}\ A) \to (\mathsf{Agg}\ A)$$

$$\vdash \forall mn : \mathsf{Agg}\ A.(\mathsf{atoms}\ (m \sqcup n)) = (\mathsf{atoms}\ m) \cup (\mathsf{atoms}\ n)$$

- The new axiom schema makes the order isomorphisms from aggregates to nonempty sets of atoms into join-semilattice isomorphisms.
- We lack a counterpart to Link's infinitary sum (so the join semilattices of aggregates are not complete).
- We didn't really need infinitary sums anyway.

# Nonquantificational NPs

- As in traditional accounts, we translate names of entities with constants of type $\mathsf{e}$, e.g. $\mathsf{j} : \mathsf{e}$ (*John*), $\mathsf{m} : \mathsf{e}$ (*Mary*).
- *And* is treated as ambiguous between its familiar boolean meaning (for conjoining truth values or functions with final result type $\mathsf{t}$) and the new meaning $\sqcup$.
- Entities can't be summed, but the corresponding singular aggregates can, e.g. $(\eta\ \mathsf{j}) \sqcup (\eta\ \mathsf{m}) : \mathsf{Agg}\ \mathsf{e}$ (*John and Mary*).

# Indifferent Predicates

- Predicates which can predicate of both singlars and plurals, such as *performed*, are treated as sets of aggregates, i.e. (for entities) $(\mathsf{Agg}\ \mathsf{e}) \to t$:

  $\mathsf{perform}\ ((\eta\ \mathsf{m}) \sqcup (\eta\ \mathsf{j}))$ (*Mary and John performed.* [as a unit])
  $\mathsf{perform}\ (\eta\ \mathsf{m})$ (*Mary performed.*)

- But *Mary and John performed* also has a distributive) reading, usually expressed using boolean conjunction. We'll come back to that.
- And *Mary performed* is standardly analyzed as having an entity predicate (type $\mathsf{e} \to \mathsf{t}$). We'll come back to that too.

# Distributivity (1/2)

- We define an aggregate predicate to be *distributive* provided it holds of an aggregate iff it holds of all the aggregate's singular subparts:

$$\mathsf{distrib}_A := \lambda T : (\mathsf{Agg}\ A) \to \mathsf{t}.\forall m : \mathsf{Agg}\ A.(T\ m) \leftrightarrow (\forall n : \mathsf{Agg} A.((\mathsf{singular}\ n) \wedge (n \sqsubseteq m)) \to (T\ n))$$

- We analyze distributive predicates (e.g. *die*) as aggregate predicates which are *axiomatically* distributive:

$$\vdash \mathsf{die} : (\mathsf{Agg}\ \mathsf{e}) \to \mathsf{t}$$

$$\vdash (\mathsf{distrib}\ \mathsf{die})$$

# Distributivity (2/2)

- For each type $A$ we can define two functions that set up a bijection between the $A$-predicates and the distributive $(\mathsf{Agg}\ A)$-predicates, called **individualization** and **distributivization**:

$$\mathsf{indiv}_A := \lambda T : (\mathsf{Agg}\ A) \to \mathsf{t}.\lambda x : A.T(\eta\ x)$$

$$\mathsf{dist}_A := \lambda S : A \to \mathsf{t}.\lambda m : \mathsf{Agg}\ A.\forall x : A.(\mathsf{atoms}\ m\ x) \to (S\ x)$$

# Indifferent Predicates Revisited

- Any aggregate predicate $T$ can be mapped to a distributive predicate, namely $\mathsf{dist}\ (\mathsf{indiv}\ T)$.
- For example, the distributive reading of *Mary and John performed* can be expressed (*without* boolean conjunction):

$$\mathsf{dist}\ (\mathsf{indiv}\ \mathsf{perform})\ ((\eta\ \mathsf{m}) \sqcup (\eta\ \mathsf{j}))$$

- Also, *Mary performed* can be expressed with an entity predicate:

$$\mathsf{indiv}\ \mathsf{perform}\ \mathsf{m}$$

# Singular and Plural Nouns (1/4)

N.B.: Here, by ‘noun’, we really mean ‘count noun’.

- On a first pass, we’ll treat (entity-)plural noun denotations as distributive aggregate predicates $((\mathsf{Agg}\ \mathsf{e}) \to \mathsf{t})$ and singular nouns as their individualizations (*a fortiori*, entity predicates $(\mathsf{e}\ \to \mathsf{t})$:

$$\vdash \mathsf{bees} : (\mathsf{Agg}\ \mathsf{e}) \to \mathsf{t}$$

$$\vdash \mathsf{distrib}\ \mathsf{bees}$$

$$\mathsf{bee} := (\mathsf{indiv}\ \mathsf{bees}) : \mathsf{e} \to \mathsf{t}$$

- $\mathsf{bees}\ ((\eta\ \mathsf{e}) \sqcup (\eta\ \mathsf{d}))$ (*Eric and Derek are bees.*)
  $\mathsf{bee}\ \mathsf{s}$ (*Sam is a bee.*)

# Singular and Plural Nouns (2/4)

- For some common nouns such as *swarm*, the *singular* form already denotes a predicate of aggregates, which moreover holds only of plurals. We analyze the corresponding plural nouns as denoting aggregates of aggregates:

$$\vdash \mathsf{swarms} : (\mathsf{Agg}^2\ \mathsf{e}) \to \mathsf{t}$$

$$\vdash \mathsf{distrib\ swarms}$$

$$\mathsf{swarm} := (\mathsf{indiv\ swarms}) : (\mathsf{Agg\ e}) \to \mathsf{t}$$

$$\vdash \forall m : \mathsf{Agg\ e}.(\mathsf{swarm}\ m) \to (\mathsf{plural}\ m)$$

- $\mathsf{swarm}\ ((\eta\ \mathsf{e}) \sqcup (\eta\ \mathsf{d}) \sqcup (\eta\ \mathsf{b}) \sqcup (\eta\ \mathsf{s}))$ (*Eric, Derek, Buzz, and Sam are a swarm.*)

  $\mathsf{swarm}\ (\eta\ \mathsf{e})$ (*Eric is a swarm.*)
  (merely false; cf. * *Eric is bees*)

# Singular and Plural Nouns (3/4)

- This treatment of plural nouns isn't quite right, because entity-plural noun denotations can't hold of entities, or even of singular aggregates:

  a. Eric/the bee is/are/wants to be bees.

- Rather, they (and nondistributive plural predicates, such as *be alike* and *hate each other*) can only hold of *plural* aggregates (*John and Mary, the children*).
- In fact, it seems we really should say something stronger: that they can only be *predicated* of plural aggregates.
- But as yet we can't formalize that idea, because there are no *types* of plural aggregates.

# Singular and Plural Nouns (4/4)

- The following examples aren't merely false:
  a. The honeybee/Eric is/are/wants to be bumblebees.
  b. The farmer/Pedro is/are alike.
  c. The mathematician/Fermat hated each other.
- Negating them does not improve them.
- We'll ignore this issue for now; we'll eventually resolve it by adding a separate type constructor for plurals .
- But not today.

# Definites (1/2)

- We assume *the* is ambiguous:

  $\mathsf{the}^{\mathsf{sng}}_{A} : (A \to \mathsf{t}) \to A$, which presupposes a contextually salient member of the argument predicate and returns it.

  $\mathsf{the}^{\mathsf{plu}}_{A} : ((\mathsf{Agg}\ A) \to \mathsf{t})) \to (\mathsf{Agg}\ A)$, which presupposes a contextually salient *plural* member of the predicate and returns it.

- Note that $\mathsf{the}^{\mathsf{sng}}_{(\mathsf{Agg}\ A)}$ and $\mathsf{the}^{\mathsf{plu}}_{A}$ have the same type but different presuppositions.
- [Eric, Derek, Buzz, and Sam]$_1$ were bees. They$_1$ gave a four-hour joint presentation on waggle dance semantics. Then [the exhausted swarm]$_1$ returned to it$_1$s colony.

# Definites (2/2)

- $(\mathsf{the}^{\mathsf{sng}}_{\mathsf{e}}\ \mathsf{bee}) : \mathsf{e}$
  $(\mathsf{the}^{\mathsf{plu}}_{\mathsf{e}}\ \mathsf{bees}) : \mathsf{Agg}\ \mathsf{e}$
  $(\mathsf{the}^{\mathsf{sng}}_{(\mathsf{Agg}\ \mathsf{e})}\ \mathsf{swarm}) : \mathsf{Agg}\ \mathsf{e}$
  $(\mathsf{the}^{\mathsf{plu}}_{(\mathsf{Agg}\ \mathsf{e})}\ \mathsf{swarms}) : \mathsf{Agg}^2\ \mathsf{e}$
- $(\mathsf{the}^{\mathsf{plu}}_{\mathsf{e}}\ \mathsf{bees}) \sqcup_{\mathsf{e}} (\mathsf{the}^{\mathsf{plu}}_{\mathsf{e}}\ \mathsf{wasps}) : \mathsf{Agg}\ \mathsf{e}$ (an aggregate each of whose atoms is either one of the bees or one of the wasps)
  $(\eta_{(\mathsf{Agg}\ \mathsf{e})}\ (\mathsf{the}^{\mathsf{plu}}_{\mathsf{e}}\ \mathsf{bees})) \sqcup_{(\mathsf{Agg}\ \mathsf{e})} (\eta_{(\mathsf{Agg}\ \mathsf{e})}\ (\mathsf{the}^{\mathsf{plu}}_{\mathsf{e}}\ \mathsf{wasps})) : \mathsf{Agg}^2\ \mathsf{e}$ (an aggregate with two atoms: the bees and the wasps)

# Nondistributable Plural Predicates (1/3)

- *Nondistributable* plural predicates differ from plural common nouns in having no individual counterparts:
  a. The bees/Sam and Buzz are alike/converged/buzzed each other.
  b. *The bee/Sam is alike/converged/buzzed each other.
- Some nondistributable plural predicates aren't fussy about what their arguments are plurals *of*:
  c. Eric and Derek/juggling and miming/donkeys and burros/the Riemann Hypothesis and the Goldbach Conjecture/17 and 37/conjunction and sum are alike.
- We can analyze such predicates as families of type-indexed (ordinary) predicates, e.g.

$$\mathsf{alike}_A, \mathsf{converge}_A : (\mathsf{Agg}\ A) \to \mathsf{t}$$

# Nondistributable Plural Predicates (2/3)

- $\mathsf{converge_e} : (\mathsf{Agg\ e}) \to \mathsf{t}$
  $\mathsf{converge_{(Agg\ e)}} : (\mathsf{Agg}^2\ \mathsf{e}) \to \mathsf{t}$
  $(\mathsf{dist\ converge_e}) : (\mathsf{Agg}^2\ \mathsf{e}) \to \mathsf{t}$
- $(\mathsf{converge_e}\ ((\eta\ \mathsf{s}) \sqcup (\eta\ \mathsf{b}))$ (*Sam and Buzz converged.*)
  $(\mathsf{converge_e}\ (\mathsf{the}^{\mathsf{sng}}_{(\mathsf{Agg\ e})}\ \mathsf{swarm})$ (*The swarm converged.*)
- $(\mathsf{converge_{Agg\ e}}\ (\mathsf{the}^{\mathsf{plu}}_{(\mathsf{Agg\ e})}\ \mathsf{swarms}))$ (*The swarms converged.*)
  [They all headed to the same location.]
  $(\mathsf{dist\ converge_e}\ (\mathsf{the}^{\mathsf{plu}}_{(\mathsf{Agg\ e})}\ \mathsf{swarms}))$ (*The swarms converged.*)
  [Each of them converged.]
  $(\mathsf{converge_e}\ (\mu_{\mathsf{e}}\ (\mathsf{the}^{\mathsf{plu}}_{(\mathsf{Agg\ e})}\ \mathsf{swarms})))$ (*The swarms converged.*)
  [The bees in the swarms all headed to the same location.]

# Nondistributable Plural Predicates (3/3)

- $\mathsf{alike_e} : (\mathsf{Agg\ e}) \to \mathsf{t}$
  $\mathsf{alike_{(Agg\ e)}} : (\mathsf{Agg}^2\ \mathsf{e}) \to \mathsf{t}$
  $(\mathsf{dist\ alike_e}) : (\mathsf{Agg}^2\ \mathsf{e}) \to \mathsf{t}$
- $(\mathsf{alike_e}\ ((\eta\ \mathsf{s}) \sqcup (\eta\ \mathsf{b}))$ (*Sam and Buzz are alike.*)
  $(\mathsf{alike_e}\ (\mathsf{the}_e^{\mathsf{plu}}\ \mathsf{bees}))$ (*The bees are alike.*)
- Abbreviations:
  $\mathsf{bw} := (\mathsf{the_e^{plu}}\ \mathsf{bees}) \sqcup_{\mathsf{e}} (\mathsf{the_e^{plu}}\ \mathsf{wasps})$
  $\mathsf{BW} := (\eta_{(\mathsf{Agg\ e})}\ (\mathsf{the_e^{plu}}\ \mathsf{bees})) \sqcup_{(\mathsf{Agg\ e})} (\eta_{(\mathsf{Agg\ e})}\ (\mathsf{the_e^{plu}}\ \mathsf{wasps}))$
- $(\mathsf{alike_{(Agg\ e)}}\ \mathsf{BW})$ (*The bees and the wasps are alike.*) [They are similar aggregates.]

  $(\mathsf{dist\ alike_e}\ \mathsf{BW})$ (*The bees and the wasps are alike.*) [The bees are alike, and so are the wasps.]

  $(\mathsf{alike_e}\ \mathsf{bw})$ (*The bees and the wasps are alike.*) [The insects, which comprise the bees and the wasps, are alike.]