PUSH-GR

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I. NOTES

A. Assorted basics

- We say that $\vec{v}$ is parallel-transported along a curve that has a tangent $\vec{u}$ if $u^\beta \nabla_\beta v^\alpha = 0$

B. 3+1 Formalism

- Split into a global time function t and spacelike hypersurfaces Σ
- Spacelike hypersurfaces
  - three-dimensional metric γ_{ij}, $i, j \in \{1, 2, 3\}$
  - **lapse function** α relates proper time measured between Σ, as measured by observes traveling along the normal to Σ $d\tau = \alpha(t, x^i)dt$
  - Instrinic curvature of Σ defined in terms of 3D Riemann tensors, defined with γ_{ij}
  - Extrinsic curvature defined in terms of how $\vec{n}$ changes as it is parallel transported along Σ; with tensor $K_{\alpha\beta}$
    * $K_{\mu\nu} = K_{\nu\mu}$
- global time
  - **shift vector** β^i is the relative velocity between Eulerian observes and lines of constant spatial coordinates $x^i_{t+dt} = x^i_t - \beta^i(t, x^j)dt$ Noah: How to express this more nicely? E.g. I don't recall '$t + dt$' showing up a lot in GR class..
- The big picture:
  - Foliate spacetime into spacelike hypersurfaces Σ and a global time t, choosing a gauage α and β^i (which, I guess, defines the coordinate system?)
  - Do a bunch of stuff to define the extrinsic curvature $K_{\mu\nu}$ of Σ within the 4D spacetime
  - Exploit properties of Σ to get an evolution equation for the spatial metric γ_{ij} (purely geometry, no actual GR dynamics introduced yet)
  - Construct the Hamiltonian/energy constraint ${}^{(3)}R + K^2 - K_{\mu\nu}K^{\mu\nu} = 16\pi\rho$ with $\rho = n^\mu n^\nu T_{\mu\nu}$ is the local energy density as measured by Eulerian observers
  - Construct a momentum constraint $D_\mu \left(K^{\alpha\mu} - \gamma^{\alpha\mu}K\right) = 8\pi j^\alpha$ with $j^\alpha = -P^{\alpha\mu}n^\nu T_{\mu\nu}$ (both constraints are independent of gauage (α, β^i))
  - ADM equations close the rest of the system (eqn. 2.5.5 in the book)
  - oh also a discretized form of the evolution equations for K_{ij}, γ_{ij} does not continually satisfy all the constraints b/c math (though exact forms do, as can be shown with the Bianchi identity). So, **we can take two approaches to try and evolve the equations**
    * Free Evolution: start with a valid solution (satisfies the constraints), and then evolve all 12 evolution equations simultaneously from there, assuming constraints will remain satisified. Also evaluate the constraint equations, to get an idea of our errors.
    * Constrainted Evolution: At each step, also evaluate the constraints, to see how wrong we are!
- A note about the *standard* ADM (from York) and version of ADM originally derived
  - the original version comes from a change of variables, so that they include the Hamiltonian/energy constraint (e.g. a bunch of stuff = 0); york version does not
  - they are *physically equivalent* as we expect the Hamiltonian constraint to vanish to 0 for physical systems (as it is the diff between one term and the energy density term... and should all equal 0)
  - *mathematically* they aren't equivalent; they are only equivalent on a *constraint hypersurface* in the space of possible solutions... but with numerical error... they might deviate!!
  - also the Hamiltonian has second derivatives of the spatial metric, whereas original ADM version does not

C. Horizons

1. Apparent Horizons

- Globally, we would look for the places where null (lightlike) lines stop going towards infinity, and instead bend inward towards a singularity
- But that requires full evolution of the whole spacetime!
- Locally, look for **apparent horizons**; the 'outermost marginally trapped surface on a given spatial hypersurface'... or... a 'closed 2D surface S such that the expansion of outgoing null geodesics orthogonal to S is 0 everywhere' (lightlike curves are like... not... 'going outward' or stuck in some sense...?)
- with lie derivatives and such, we define the expansion of these outgoing null geodesics, H (final form is eqn. 6.7.9, or in terms of some scalar function F (such that the surface is it's level sets Noah: why isn't this just the global time function t?) with equation 6.7.13)
- Example in spherical symmetry Noah: Work through this in complete!
  - outgoing null geodeiscs are unit radial vector!
  - $dl^2 = A(r,t)dr^2 + r^2 B(r,t)d\Omega^2$ $H = \frac{1}{\sqrt{A}}\left(\frac{2}{r} + \frac{\partial_r B}{B}\right) - 2K^\theta_\theta = 0$

2. Event Horizons

- Global definition
- 'Shoot off null light rays and see where they go'

3. Isolated and Dynamical Horizons

- Let's us look at black hole properties like mass, angular momentum, more specifically
- neat! really advanced / modern stuff?
- more precise definitions, closer to astrophysical understanding of black holes

II. TO-DOS AND SUCH

Aim each of these to be quarters of the semester... leave a buffer... phase I and II by the end of the holiday break Aim to have the python done by November 29thPhase I: reading, pen-and-paper; what an apparent horizon is, how it works, how to fit it into a 1D GR code –¿ Done by November

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  - 3+1 numerical relativity, ADM formalism (chapter 2)
  - physics of apparent horizons, physics of geodesic convergent + singularity theorems
  - quasi-normal measures; how to compute the mass of a BH from the horizon
    - no-hair theorem doesn't apply if black holes are evolving!
    - LIGO BHs ... if no-hair was always true, then at every point in the merger, the BHs would always be 'perfect' from just mass and spin... but they're not!
    - so similar thing applies during gravitational collapse; no-hair only satisfied when stuff isn't changing quickly
    - quasi-normal measures are the math for computing BH mass generically
    - integrating the stuff inside of it will give baryonic mass, but won't contain the gravitational mass (a small difference, but important)... but use gauss' law approach, integrating over the surface of the apparent horizon

- Phase II: implement it in a simpler context than AGILE –¿ take, e.g., some output from PUSH... implement a horizon finder in python... post-process the data, find the horizon location and the BH mass and stuff –¿ Done by November 29th

- Phase III: convert into FORTRAN, put it into PUSH in the right place make sure we don't degrade the performance too much

- Phase IV: run some test on simulations, write up the work... some connection to a broader project Dr. Frohlich is doing...

read 2.7 as well 2.8 goes into 3D numerical relativity is done; not relevant here

geoemtric structure of 3 + 1 split and for edification, the elegance of the split between ... as evolution + constraint + gauge freedom

apparent horizons, section 6.7; *6.7.1*

what is an apparent horizon vs. what is an event horizon

Homework

- In general, it's easier to do things with the lie derivative version than covariant derivative, because it doesn't involve Christoffel symbols.

- Mathematica is not cheating!

- Type up a paragraph for each homework assignment!

- Assignment 1

  – Identify A and B terms in the metric that AGILE uses... look at mattias liebendorfer's thesis (finding this is Dr. Frohlich's homework)

  – Identify in the 3+1 split the lapse and the shift from Liebendorfer's thesis.... because ADM is standard now, but not when it was written

  – Figure out what formula to use to compute the extrinsic curvature, and how we might do that. How might we modify this formula to raise one of the indices. Hint: it's in section 2.3; DO THE COMPUTATION Could multiply the inverse of the spatial metric... but can't just do it on both sides of the equation, because there is a non-commutativity of operators issue.

- Assignment 2

  – Find the equation for the mass of all material integrated inside a surface. (quasi-normal measures stuff)

- Assignment 3

  – Do the geometric optics that appeal to me in the MTW chapters. 'Follow your heart'

- Read: section 22.5 in MTW on "geometric optics in curved spacetime"; of particular interest is the "focusing theorem"... which is an exercise, number 22.13 (the area of a bundle of rays).

- Read: section 34.6 in MTW on singularity theorems. Doesn't look like they prove them, but they describe them. Endpoint of gravitational collapse; the point is that ... we can prove singularities are inevitable in GR, a key piece of that proof is the focusing theorem.

- Read: Chapter 19 and 20 of MTW (whole chapters); there's a conservation law/gauss' law of mass and angular momentum... they don't call it this, but outside of MTW these are the 'ADM mass'. Also definitely in Carroll, maybe Schutz

  If an apparent horizon forms, there is an event horizon somewhere behind it.

III. HOMEWORK

A. Assignment 1

1. ADM identification in AGILE

- $ds^2 = e^{-2\Phi(a,t)}dt^2 + e^{2\Lambda(a,t)}da^2 + r^2(a,t)d\Omega^2$
- $ds^2 = -\alpha^2 dt^2 + \left(\frac{1}{\Gamma}\frac{\partial r}{\partial a}\right)^2 da^2 + r^2(a,t)d\Omega^2$
- $ds^2 = (-\alpha^2 + \beta_i\beta^i)dt^2 + 2\beta_i dt dx^i + A(r,t)dr^2 + r^2 B(r,t)d\Omega^2$
- So I think $B(r,t) = 1$, $A(r,t) = 0(?)$, and that I need a relation between r and a to confirm $A(r,t) = 0$ and to get α (And I'm not sure how to get β)
  - Maybe use the relation between enclosed mass, a, and r? But it's an integral...
  - Appendix gives relation between $\frac{\partial\Phi(a,t)}{\partial a}$, r, and Christoffel Γ^t_{at}; then might try using xact, get generic Christoffel symbols for the generic spherical symmetry metric, and see what hpapens? And then try to dig up another relation so I have a system of DE's...
- **Solution:** Instead of trying to start with dr total derivative, get da (with a dr part) and sub. into AGILE metric, like so:
  - $da = c_1 dt + c_2 dr$, where we can identify c_1, c_2 from the Jacobian for da... but for that, we'd need $a(r,t)$.
  - However, we can get Jacobian for dr, and invert that to get Jacobian for da (**sidenote:** the reason the Jacobian matters is, because as we recall, our real analysis Taylor expansion is $f(x) = f(a) + J(a)(x-a) + H(a)(x-a) + ...$ and from this we get that the total derivative is the best linear approximation to $f(x)$, i.e. the Jacobian term, and so that's the matrix which gives us our coefficients in the total derivative
  - So we have $J(a,t) = \left(\frac{\partial r}{\partial a}, \frac{\partial r}{\partial t}, 1, 1\right)$ and then invert this for a $J(r,t)$, which really just flips the two derivatives (b/c this is a very simple matrix).
  - So, $da = \left(\frac{\partial r}{\partial t}\right)^{-1} dt + \left(\frac{\partial r}{\partial a}\right)^{-1} dr$
  - Also, from Liebendorfer,

$$\frac{\partial r}{\partial t} = \alpha u = (c_1)^{-1}$$
$$\frac{\partial r}{\partial a} = \frac{\Gamma}{4\pi r^2 \rho} = (c_2)^{-1}$$

  - We substitute this into the AGILE metric and get

$$ds^2 = \left[-\alpha^2 + \left(\frac{1}{\Gamma}\frac{\partial r}{\partial a}c_1\right)^2\right]dt^2 + \left(\frac{1}{\Gamma}\frac{\partial r}{\partial a}c_2\right)^2 dr^2 + 2\left(\frac{1}{\Gamma}\frac{\partial r}{\partial a}\right)^2 c_1 c_2 dr dt + r^2 d\Omega^2$$
$$ds^2 = \left[-\alpha^2 + \left(\frac{1}{\Gamma}\frac{\partial r}{\partial a}\frac{1}{\alpha u}\right)^2\right]dt^2 + \frac{1}{\Gamma^2}dr^2 + 2\left(\frac{1}{\Gamma^2}\right)\left(\frac{\partial r}{\partial a}\right)\frac{1}{\alpha u}dr dt + r^2 d\Omega^2$$
$$ds^2 = \left[-\alpha^2 + \left(4\pi r^2 \rho\alpha u\right)^{-2}\right]dt^2 + \frac{1}{\Gamma^2}dr^2 + 2\left(4\pi r^2\rho\Gamma\alpha u\right)^{-1} dr dt + r^2 d\Omega^2$$

  So, we have

$$2\beta_r = 2\left(\frac{1}{\Gamma^2}\right)\left(\frac{\partial r}{\partial a}\right)\frac{1}{\alpha u}$$
$$\therefore \beta_r = \frac{1}{\alpha u}\frac{1}{\Gamma^2}\frac{\Gamma}{4\pi r^2\rho}$$
$$= \left(4\pi r^2 \rho\Gamma\alpha u\right)^{-1},$$

and all other $\beta_\theta = \beta\phi = 0$

$$\therefore \boxed{\beta_\mu = (0, \beta_r, 0, 0)}$$

And we can also identify

$$\boxed{B(r,t) = 1}$$
$$\boxed{A(r,t) = \frac{1}{\Gamma^2}}$$

– (cross-check with the lapse function)

2. *Calculation of Extrinsic Curvature*

- From eqn. 2.3.6 in Alcubierre, $K_{\mu\nu} = -\frac{1}{2}\pounds_{\vec{n}}\gamma_{\mu\nu}$ but this isn't immediately super useful, because I don't want to try and figure out $\vec{n}$ at each step...
  - Maybe it wouldn't be too bad, actually, with α and β– may need to look more into the earlier relation Alcubierre gives for $\vec{n}$
- Alcubierre reduces it to a nicer form (eqn. 2.3.10), and with $\pounds_{\vec{t}} = \partial_t$ in our 3+1 split (this intuitively makes sense, but I can't actually explain it, hmm...)

$$\left(\pounds_{\vec{t}} - \pounds_{\vec{\beta}}\right)\gamma_{\mu\nu} = -2\alpha K_{\mu\nu}$$
$$\left(\partial_t - \pounds_{\vec{\beta}}\right)\gamma_{\mu\nu} = -2\alpha K_{\mu\nu}$$

- It's reduced to an even nicer form, at least for the spatial components, in eqn. 2.3.12

$$\partial_t\gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i$$

- From MTW Exercise 21.5 (b)

$$\pounds_{\vec{n}}\gamma_{\mu\nu} = (\gamma_{\mu\nu,\alpha}n^\alpha + \gamma_{\alpha\nu}n^\alpha{}_{,\mu} + \gamma_{\mu\alpha}n^\alpha{}_{,\nu})\mathbf{d}x^\mu \otimes \mathbf{d}x^\nu$$

- **Attempt 1 at raising one index**:
  - $\gamma^{\mu\sigma}\partial_t\gamma_{\sigma\nu} = -\gamma_{\sigma\nu}\partial_t\gamma^{\mu\sigma}$
  - We confirm a similar relationship for the Lie derivative. First, we explicitly show that $\pounds_{\vec{\beta}}\gamma^\mu{}_\nu = 0$ (under the assumption that $\gamma^\mu{}_\nu$ behaves like $g^\mu{}_\nu$)

$$\begin{aligned}\pounds_{\vec{\beta}}\gamma^\mu{}_\nu &= \beta^\alpha\partial_\alpha\gamma^\mu{}_\nu - \gamma^\alpha{}_\nu\partial_\alpha\beta^\mu + \gamma^\mu{}_\alpha\partial_\nu\beta^\alpha \\ &= -\delta^\alpha{}_\nu\partial_\alpha\beta^\mu + \delta^\mu{}_\alpha\partial_\nu\beta^\alpha \\ &= -\partial_\nu\beta^\mu + \partial_\nu\beta^\mu \\ &= 0\end{aligned}$$

  Then, we have that, by the product rule,

$$\begin{aligned}\pounds_{\vec{\beta}}\gamma^\mu{}_\nu &= \pounds_{\vec{\beta}}\left(\gamma^{\mu\sigma}\gamma_{\sigma\nu}\right) \\ \therefore 0 &= \gamma^{\mu\sigma}\pounds_{\vec{\beta}}\gamma_{\sigma\nu} + \gamma_{\sigma\nu}\pounds_{\vec{\beta}}\gamma^{\mu\sigma} \\ \therefore \gamma^{\mu\sigma}\pounds_{\vec{\beta}}\gamma_{\sigma\nu} &= -\gamma_{\sigma\nu}\pounds_{\vec{\beta}}\gamma^{\mu\sigma}\end{aligned}$$

  - From these relations, we can determine that

$$\begin{aligned}\gamma^{\mu\sigma}\left(\partial_t - \pounds_{\vec{\beta}}\right)\gamma_{\sigma\nu} &= \gamma^{\mu\sigma}\partial_t\gamma_{\sigma\nu} - \gamma^{\mu\sigma}\pounds_{\vec{\beta}}\gamma_{\sigma\nu} \\ &= -\gamma_{\sigma\nu}\partial_t\gamma^{\mu\sigma} + \gamma_{\sigma\nu}\pounds_{\vec{\beta}}\gamma^{\mu\sigma} \\ &= -\gamma_{\sigma\nu}\left(\partial_t - \pounds_{\vec{\beta}}\right)\gamma^{\mu\sigma}\end{aligned}$$

– With this result we can determine that:

$$\begin{aligned}
\gamma^{\mu\sigma}K_{\sigma\nu} &= \gamma^{\mu\sigma}\left[-\frac{1}{2\alpha}\left(\partial_t - \pounds_{\vec{\beta}}\right)\gamma_{\sigma\nu}\right] \\
&= -\frac{1}{2\alpha}\gamma^{\mu\sigma}\left(\partial_t - \pounds_{\vec{\beta}}\right)\gamma_{\sigma\nu} \\
&= \frac{1}{2\alpha}\gamma_{\sigma\nu}\left(\partial_t - \pounds_{\vec{\beta}}\right)\gamma^{\mu\sigma} \\
&= -\gamma_{\sigma\nu}K^{\mu\sigma}
\end{aligned}$$

– So, we have

$$\begin{aligned}
\gamma^{\mu\sigma}K_{\sigma\nu} + \gamma_{\sigma\nu}K^{\mu\sigma} &= 0 \\
\therefore \text{ if these terms are both representations of } &K^{\mu}{}_{\nu}, \\
2K^{\mu}{}_{\nu} &= 0 \\
\therefore K^{\mu}{}_{\nu} &= 0?
\end{aligned}$$

– And also (but I haven't written down the derivation yet) $K^{\mu\nu} = -\gamma^{\mu\sigma}\gamma^{\nu\lambda}K_{\sigma\lambda}$

- **Attempt 2 (cheating?)**
  – If the above doesn't work, we can try to cheat with the definition of the extrinsic curvature:

$$\begin{aligned}
K_{\mu\nu} &= -\left(\nabla_\mu n_\nu + n_\mu n^\alpha \nabla_\alpha n_\nu\right) \\
\text{Permuting } \mu,\nu \text{ for aesthetics, } K_{\nu\mu} &= -\left(\nabla_\nu n_\mu + n_\nu n^\alpha \nabla_\alpha n_\mu\right)
\end{aligned}$$

In a PY509 (GR) homework, I derived

$$g^{\mu\sigma}\nabla_\alpha n_\sigma = \nabla_\alpha n^\mu$$

So,

$$\begin{aligned}
g^{\mu\sigma}K_{\nu\sigma} &= -g^{\mu\sigma}\left(\nabla_\nu n_\sigma + n_\nu n^\alpha \nabla_\alpha n_\sigma\right) \\
&= -\left(\nabla_\nu n^\mu + n_\nu n^\alpha \nabla_\alpha n^\mu\right) \\
&= K_\nu{}^\mu
\end{aligned}$$

And, using Mathematica, I confirmed that this yields the correct $K_\theta{}^\theta = 0$ for the Schwarszchild metric. **THOUGH I GUESS I'M ASSUMING THAT** $K_\theta{}^\theta = K^\theta{}_\theta$. *This is safe*

B. Assignment 2

1. *Rederivation of expansion from eqn. 6.7.3 (without explicitly using extrinsic curvature)*

From equation 6.7.3 of Alcubierre, the expansion is

$$H = -\frac{1}{2}h^{\mu\nu}\left(\pounds_{\vec{s}}h_{\mu\nu} + \pounds_{\vec{n}}h_{\mu\nu}\right),$$

where $\vec{s}$ is a free parameter, which is a unit normal vector with respect to the a two-dimensional surface embedded in the three-dimensional spacelike hypersurface Σ, $\vec{n}$ is the timelike unit normal vector with respect to Σ, and $h_{\mu\nu}$ is defined as (in equation 6.7.2 of Alcubierre),

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu - s_\mu s_\nu.$$

In AGILE's coordinate system, we are given the lapse α. The shift vector $\beta^i = 0$ as AGILE's coordinate system is Lagrangian; β^i is the relative velocity between an Eulerian observer and lines of constant spatial coordinates, but in a Lagrangian system where the coordinates follow each parcel of material, our observers are always at constant spatial coordinates. [TODO: I know this doesn't make a lot of sense right now... I feel like $\beta_i = 0$ b/c we're in a Lagrangian system but I don't have a good explanation.]

We also note that, from Appendix 7 of Liebendorfer, the metric $g_{\mu\nu}$ is

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 & 0 & 0 & 0 \\ 0 & \left(\frac{r'}{\Gamma}\right)^2 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}.$$

With $\beta^i = 0$, our vector $\vec{n}$ is

$$n_\mu = (-\alpha, 0, 0, 0), \quad n^\mu = (1/\alpha, 0, 0, 0).$$

Additionally, since $\vec{s}$ is a free parameter, we give it the form $(0, s, 0, 0)$, so that it will be perpendicular to the $(\theta, \varphi) = (0, 0)$ surface pointing in the $+a$ direction. We need $\vec{s}$ to have unit length, i.e.

$$\begin{aligned} s^\mu g_{\mu\nu} s^\nu &= 1, \\ \therefore s^a g_{aa} s^a &= 1, \text{as } g_{\mu\nu} \text{ is diagonal and only } s^a \neq 0, \\ \therefore s \left(\frac{r'}{\Gamma}\right)^2 s &= 1, \end{aligned}$$

$$\therefore \boxed{s = \frac{\Gamma}{r'}}, \text{taking } + \text{ in the square root to } \vec{s} \text{ points in } +a.$$

Having specified s^μ, we can easily determine s_μ,

$$\begin{aligned} s_\mu &= g_{\mu\nu} s^\nu \\ \therefore s_t &= g_{t\nu} s^\nu = 0 \\ s_a &= g_{a\nu} s^\nu = g_{aa} s^a = \frac{1}{s^2} s = \frac{1}{s} \\ s_\theta &= g_{\theta\nu} s^\nu = g_{\theta\theta} s^\theta = 0 \\ s_\varphi &= g_{\varphi\nu} s^\nu = g_\varphi s^\varphi = 0. \end{aligned}$$

So, with $g_{\mu\nu}$, n_μ, and s_μ in hand,

$$h_{\mu\nu} = \begin{pmatrix} -\alpha^2 & 0 & 0 & 0 \\ 0 & s^{-2} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} + \begin{pmatrix} \alpha^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & s^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\therefore \boxed{h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix}}$$

Using Mathematica, we raise both indices to also find,

$$\boxed{h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r^{-2} & 0 \\ 0 & 0 & 0 & r^{-2} \csc^2\theta \end{pmatrix}}$$

With $h_{\mu\nu}, h^{\mu\nu}, n^\mu, s^\mu$, we can now calculate the expansion. First, we calculate the two Lie derivatives; from the definition of the Lie derivative in terms of partial derivatives from MTW Exercise 21.5 (b), we have,

$$\begin{aligned} \pounds_{\vec{s}} h_{\mu\nu} &= s^\sigma \partial_\sigma h_{\mu\nu} + h_{\sigma\nu} \partial_\mu s^\sigma + h_{\mu\sigma} \partial_\nu s^\sigma \\ &= s^a \partial_a h_{\mu\nu} + h_{a\nu} \partial_\mu s^a + h_{\mu a} \partial_\nu s^a \\ &= s \partial_a h_{\mu\nu} + h_{a\nu} \partial_\mu s + h_{\mu a} \partial_\nu s, \end{aligned}$$

where, since only the $\theta\theta$ and $\varphi\varphi$ components of $h_{\mu\nu}$ are non-zero, those are the only terms of $\pounds_{\vec{s}} h_{\mu\nu}$ which may be non-zero. Recalling that $s = \Gamma/r' \equiv \Gamma/\partial_a r$, those terms are

$$\begin{aligned}
\pounds_{\vec{s}} h_{\theta\theta} &= s\partial_a h_{\theta\theta} + \cancelto{0}{h_{a\theta}}\partial_\theta s + \cancelto{0}{h_{\theta a}}\partial_\theta s \\
&= s\partial_a r^2 \\
&= 2sr\partial_a r \\
&= 2r\frac{\Gamma}{r'}\partial_a r \\
&= 2r\Gamma \\
\pounds_{\vec{s}} h_{\varphi\varphi} &= s\partial_a h_{\varphi\varphi} + \cancelto{0}{h_{a\varphi}}\partial_\varphi s + \cancelto{0}{h_{\varphi a}}\partial_\varphi s \\
&= s\partial_a r^2 \sin^2\theta \\
&= 2sr\sin^2\theta\partial_a r \\
&= 2r\sin^2\theta\frac{\Gamma}{r'}\partial_a r \\
&= 2r\Gamma\sin^2\theta.
\end{aligned}$$

Similarly, with n^μ,

$$\begin{aligned}
\pounds_{\vec{n}} h_{\mu\nu} &= n^\sigma\partial_\sigma h_{\mu\nu} + h_{\sigma\nu}\partial_\mu n^\sigma + h_{\mu\sigma}\partial_\nu n^\sigma \\
&= n^t\partial_t h_{\mu\nu} + h_{t\nu}\partial_\mu n^t + h_{\mu t}\partial_\nu n^t
\end{aligned}$$

with, as for $\pounds_{\vec{s}} h_{\mu\nu}$, since $h_{\mu\nu}$ is diagonal with $h_{aa} = 0$, the only non-zero terms are

$$\begin{aligned}
\pounds_{\vec{n}} h_{\theta\theta} &= n^t\partial_t h_{\theta\theta} + \cancelto{0}{h_{t\theta}}\partial_\theta n^t + \cancelto{0}{h_{\theta t}}\partial_\theta n^t \\
&= \frac{1}{\alpha}\partial_t r^2 \\
&= \frac{2r}{\alpha}\partial_t r \\
\pounds_{\vec{n}} h_{\varphi\varphi} &= n^t\partial_t h_{\varphi\varphi} + \cancelto{0}{h_{t\varphi}}\partial_\varphi n^t + \cancelto{0}{h_{\varphi t}}\partial_\varphi n^t \\
&= \frac{1}{\alpha}\partial_t r^2\sin^2\theta \\
&= \frac{2r\sin^2\theta}{\alpha}\partial_t r
\end{aligned}$$

With $h^{\mu\nu}$, $\pounds_{\vec{s}} h_{\mu\nu}$, and $\pounds_{\vec{n}} h_{\mu\nu}$, the expansion in AGILE coordinates becomes,

$$\begin{aligned}
H &= -\frac{1}{2}h^{\mu\nu}\left(\pounds_{\vec{s}} h_{\mu\nu} + \pounds_{\vec{n}} h_{\mu\nu}\right) \\
&= -\frac{1}{2}\left[h^{\theta\theta}\left(\pounds_{\vec{s}} h_{\theta\theta} + \pounds_{\vec{n}} h_{\theta\theta}\right) + h^{\varphi\varphi}\left(\pounds_{\vec{s}} h_{\varphi\varphi} + \pounds_{\vec{n}} h_{\varphi\varphi}\right)\right] \\
&= -\frac{1}{2}\left[\frac{1}{r^2}\left(2r\Gamma + \frac{2r\partial_t r}{\alpha}\right) + \frac{1}{r^2\sin^2\theta}\left(2r\Gamma\sin^2\theta + \frac{2r\sin^2\theta\partial_t r}{\alpha}\right)\right] \\
&= -\left(\frac{2\Gamma}{r} + \frac{2\partial_t r}{r\alpha}\right) \\
&= -\frac{2}{r}\left(\Gamma + \frac{\partial_t r}{\alpha}\right)
\end{aligned}$$

Our condition for the apperance of an apparent horizon is that $H = 0$; in AGILE, this then reduces to the condition that, recalling in Liebendorfer's thesis, $u \equiv \dot{r}/\alpha$,

$$\begin{aligned}
\Gamma + \frac{\partial_t r}{\alpha} &= 0 \\
\therefore \Gamma + u &= 0
\end{aligned}$$

or, equivalently,

$$\alpha\Gamma + \partial_t r = 0.$$

IV. AGILE NOTES

A. Notation

- In Libendorfer's Thesis
  - ρ: rest mass density (Appendix 7; page 70)
  - p: isotropic pressure (Appendix 7; page 70)
  - e: specific energy (Appendix 7; page 70)
  - $u \equiv \dot{r}/\alpha$: a "velocity" that describes the change of areal radius (r) with proper time of the comoving observer (Appendix 7; page 70)
  - m gravitational mass
- In code
  - the printed density is the rest mass density (see line 265 of 'state_vector_module.f'; then, later, we print in the stp files all of the primitive variables in the first columns, in order, and this is the fifth... which corresponds with the stp files...