ASTR 541 Lecture Notes: Perturbations and Large Scale Structure, 2024

1 Plan for the next month

So far, we've studied homogeneous cosmology. Our universe has had the same density at all points in space and have merely evolved in time. This has allowed us to make statements about the thermal history and fate of the universe. However, if we want to study objects like galaxies, we have to consider departures from **homogeneity.**

Galaxies and clusters of galaxies are complex systems, but the aim of cosmologist is not to explain all their details - that's the goal of galaxy class, astrophysicist. Here we seek to explain the origin of large scale structure of the universe, and how does it develop in later universe. We define the density enhancement $\delta\rho$, and the density contrast $\Delta = \delta\rho/\rho$. When $\delta << 1$, these perturbations grow linearly (actually it is linearly with scale factor in standard models). However, when these amplitude approaches unity, their subsequent development becomes non-linear and they rapidly evolve into bound objects, such as galaxies and clusters of galaxies, when many non-linear astrophysical effects become important, such as star formation and feedback. Our second part of this course is to deal with mostly small, linear perturbations and the observations of large scale structure, and how to use them to probe our cosmological model. Here our objectives are twofold, (1) to understand how density perturbations evolve in an expanding universe, and (2) to derive and account for the initial conditions necessary for the formation of the structures we observe.

- **perturbation $\delta(t)$, and cosmological tests**
- **given cosmology and observations, initial condition**

In the first few lectures of this part of the course, we will mostly due with the first question. We will go over how density and velocity perturbations grow in the universe, and then introduce statistics of large scale structure and the means to observe them at low redshift. Then we will treat the fluctuations of the CMB as a special case, which is especially important in understanding some of the initial condition issues and its impact in our understanding of the early universe.

Before diving into the math, some back-of-the-envelop considerations of the scale that we are talking about. You can simply calculate the mass density of a galaxy ($\sim$ 10 kpc), a cluster of galaxy ($\sim$ 1 Mpc), and then a supercluster, or large scale structure you see in redshift surveys (~ 10 Mpc). Comparing that to the mass density of the universe, e.g, for $\Omega = 0.3$. Turns out that $\Delta = \delta\rho/\rho$ is 10^6, 10^3 and a few, respectively. Note that for a bound object, such as a galaxy, $\delta\rho$, its density will not change with the expansion of the universe: it is a bound object, which means that the local gravitational field has already overcome the expansion and it is viralized and doesn't grow or collapse anymore (to the zeroth order, it doesn't change subtly). This is an important point: when an object has already gone non-linear, our Friedmann universe model doesn't work anymore. A bound, collapsed object is decoupled from the expansion of the universe and dominated by the local gravitational force. We will show that the minimum density of a viralized object is of order 200 the cosmic mean, so it is no longer subject to Hubble expansion, and one can safely ignore the GR effect on cosmological scale when dealing with individual objects at cluster scale or smaller.

Since $\rho \sim (1+z)^3$, that means that for galaxies, we will reach $\Delta \sim 1$ at $z \sim 100$ or so if nothing else happens. This immediately gives you that we should begin to consider galaxy formation from redshift of a hundred or lower, and clusters at $z < 10$. Turns out that most galaxies started to form at even lower redshift but not by more than an order of magnitude. So this gives you the basic scale of the redshift range that we care about.

2 Growth of small perturbations in the expanding universe

The universe is obviously lumpy on small scales, and we have argued that it gets smoother on large scales. This is the justification for considering the inhomogeneity as a perturbation to the homogeneous solution.

We have already done this kind of analysis once in the case of the ISM. We found a Jeans instability, in which perturbations grew exponentially if they had a longer wavelength than the Jeans wavelength and were stabilized by pressure otherwise.

Now we'll repeat this analysis in the case of an expanding universe. Two key differences: we need to deal with an expanding universe, and we need to use comoving coordinate system to simplify expressions.

But let's start with the basic hydro equations

$$\frac{d_c\rho}{d_ct} + \rho\nabla_x \cdot \mathbf{v} = 0$$

$$\frac{d_c\mathbf{v}}{d_ct} = -\frac{1}{\rho}\nabla_x p - \nabla_x\Phi$$

$$\nabla_x^2\Phi = 4\pi G\rho$$

where the $d_c/d_c t$ notation indicates a Lagrangian derivative and the ∇_x are derivatives with respect to the physical coordinate $\mathbf{x}$. This is the notation in fluid dynamics with a coordinate system that motion of a particular fluid element is followed. The first equation is the continuity equation, the second is the equation of motion, or Euler equation, and the third describes the gravitational potential, the Poisson equation.

You might be more familiar with the other coordinate system in fluid dynamics, the Euler system, in which we apply partial derivatives to qualities as a function of a fixed grid of coordinates,

$$\frac{d_c}{d_c t} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla_x).$$

So in real calculations, such as when you carry out cosmological simulations, there are advantages and disadvantages of using either Euler or Lagrangian systems when setting up your calculations. We are dealing with cosmology, and with particles that are almost fixed in Hubble flow, so one more complication is that we want to express things in comoving coordinate. In particular, we define $\mathbf{x} = R\mathbf{r}$ where $R(t)$ is the usual expansion factor. This means $\nabla_r = R\nabla_x$.

We then split the velocity into a Hubble expansion term and a peculiar velocity term, $\mathbf{v} = H\mathbf{x} + \delta\mathbf{v} = H\mathbf{x} + R\mathbf{u}$. $\mathbf{u}$ is the comoving peculiar velocity, i.e. how far a particle moves in comoving coordinates per unit time.

$\delta\mathbf{v}$ is called a peculiar velocity. It's the velocity difference relative to the expectation of the Hubble law.

We will then switch from a Lagrangian derivative in physical units to a derivative in the comoving frame. This means that

$$\frac{d_c}{d_c t} = \frac{\partial}{\partial t} + (\mathbf{v} \cdot \nabla_x) = \frac{\partial}{\partial t} + (H\mathbf{x} \cdot \nabla_x) + (\delta\mathbf{v} \cdot \nabla_x) = \frac{d}{dt} + (\mathbf{u} \cdot \nabla_r),$$

where d/dt is differential over the comoving grids.

Finally, we will find it convenient to measure densities in fractional units relative to the background density $\rho_h(t)$. So $\rho = (1+\delta)\rho_h$.

We now can insert these substitutions and remove the homogeneous part of the solution.

Let's see how our hydro equations become in the next lecture.

For the continuity equation, we get

$$\frac{d\rho}{dt} + (\mathbf{u} \cdot \nabla_r)\rho + \rho\nabla_x \cdot (H\mathbf{x} + R\mathbf{u}) = 0$$

$$\rho_h \frac{d\delta}{dt} + (1+\delta)\frac{d\rho_h}{dt} + \rho_h(\mathbf{u} \cdot \nabla_r)\delta + (1+\delta)\rho_h(3H + \nabla_r \cdot \mathbf{u}) = 0$$

$$\frac{d\rho_h}{dt} = \frac{d(\rho(0)R^{-3})}{dt} = -3\rho_0 R^{-4} dR/dr = -3H\rho_h$$

Note that $\nabla_x \cdot \mathbf{x} = 3$. This means that the homogeneous terms cancel, leaving us with

$$\frac{d\delta}{dt} + (\mathbf{u} \cdot \nabla_r)\delta + (1+\delta)\nabla_r \cdot \mathbf{u} = 0$$

This looks very much like a continuity equation, but now it's on perturbed quantities.

We can play similar games with the Euler and Poisson equation. We get

$$\frac{d\mathbf{v}}{dt} + (\mathbf{u} \cdot \nabla_r)\mathbf{v} = -\frac{1}{\rho}\nabla_x p - \nabla_x \Phi$$

The first term becomes $d\mathbf{v}/dt = d(H\mathbf{x})/dt + (dR/dt)\mathbf{u} + R(d\mathbf{u}/dt)$, and the first term of this cancels the homogeneous part of the potential. The second term here generates two terms, the first of which is $(\mathbf{u} \cdot \nabla_r)H\mathbf{x} = R\mathbf{u}H$. This will have important consequences.

$$\frac{d\mathbf{u}}{dt} + 2H\mathbf{u} + (\mathbf{u} \cdot \nabla_r)\mathbf{u} = -\frac{c_s^2}{R^2}\frac{\nabla_r \delta}{1+\delta} - \frac{1}{R^2}\nabla_r \phi$$

Here we consider the universe is expanding adiabatically, and the perturbation in energy and density are related to the adiabatic sound speed:

$$\partial p/\partial \rho = c_s^2$$

, and Poisson equation becomes:

$$\nabla_r^2 \phi = 4\pi G R^2 \rho_h \delta$$

Here $\Phi = \phi_0 + \phi$

These are the full equations for gravitational perturbations in comoving coordinates. They are non-linear.

Now let's consider small perturbations. We keep only linear terms.

$$\frac{d\delta}{dt} = -\nabla_r \cdot \mathbf{u}$$

$$\frac{d\mathbf{u}}{dt} + 2H\mathbf{u} = -\frac{c_s^2}{R^2}\nabla_r \delta - \frac{1}{R^2}\nabla_r \phi$$

$$\nabla_r^2 \phi = 4\pi G R^2 \rho_h \delta$$

We take the negative divergence of the Euler equation

$$-\frac{d\nabla_r \cdot \mathbf{u}}{dt} - 2H\nabla_r \cdot \mathbf{u} = \frac{c_s^2}{R^2}\nabla_r^2 \delta + \frac{1}{R^2}\nabla_r^2 \phi$$

$$\frac{d^2\delta}{dt^2} + 2H\frac{d\delta}{dt} = \frac{c_s^2}{R^2}\nabla_r^2\delta + 4\pi G\rho_h\delta$$

Because this is a linear equation, we expand in spatial Fourier modes: $\delta = D(t)\exp(-i\mathbf{k}\cdot\mathbf{r})$ (obviously, we're assuming that only the real part matters). This gives

$$\ddot{D} + 2H\dot{D} = \left(4\pi G\rho_h - \frac{c_s^2k^2}{R^2}\right)D$$

$D(t)$ is called the growth function. Note that only ratios of growth functions matter!

3 Jeans' Instability

Before we proceed with cosmology, let's step back and look at the original problem that started these discussion which has many more astrophysical applications. The Jeans' instability posted by Sir James Jeans in 1912, which has been the foundation of astrophysical collapse in many environment and you might have seen a much simpler version of our derivation when studying star formation and galaxy formation before.

The only difference is that now we are assuming a static universe, so H=0, or $dR/dt = 0$. Let's consider a wave in the form of

$$\delta(t) = \delta_0\exp[-i(\mathbf{k}\cdot\mathbf{r} - \omega t]$$

Solving the second order differential equation, you will have the dispersion relation,

$$\omega^2 = c_s^2k^2 - 4\pi G\rho.$$

This dispersion relation describes oscillations (stable) or instabilities depending on the sign of its right hand side. Introducing Jeans wavelength

$$\lambda_J = 2\pi/k_J = c_s(\pi/G\rho)^{1/2}$$

- If $c_s^2k^2 > 4\pi G\rho$, the RHS is positive and the perturbation is oscillatory. They are sound waves in which the pressure gradient is sufficient to provide support. So for perturbation with wavelength shorter than λ_J, they will not grow.
- If $c_s^2k^2 < 4\pi G\rho$, the RHS is negative, this is the unstable mode, in which the perturbation will either grow or fall exponentially

$$\delta \propto \exp(\Gamma t),$$

Where $\Gamma = \pm[4\pi\rho G(1-\lambda_J^2/\lambda^2)]^{-1/2}$. The positive sign corresponds to growing mode. For very long wavelength, the growth rate $\Gamma \sim (G\rho)^{1/2}$, or the growth time $\tau \sim (G\rho)^{-1/2}$. This is your dynamical timescale for a spherical collapse.

The physics for Jeans instability is simple. We have hydrostatic pressure support:

$$dp/dr = -GM(< r)/r^2,$$

The region becomes unstable when gravity overwhelms pressure support. So $dp/dr \sim p/r \sim c_s^2\rho/r$ and $M \sim \rho r^3$, then we will have for $r > r_J \sim c_s/(G\rho)^{1/2}$, it becomes Jeans unstable. The other way to put it is sound crossing time r/c_s is less than dynamical time $1/(G/\rho)^{1/2}$, then it becomes unstable. We will see Jeans instability again when discussing galaxy formation.

4 Jeans' instability in an expanding universe

Before, we had an equation with time-independent coefficients, so we could use time exponentials. This led to a dispersion relation that indicated that $k < k_J$ grew exponentially.

Now, H and ρ_h both depend on time, so exponentials aren't the correct solution. Since we are worrying out large scale structure, we will only consider long wavelength $\lambda >> \lambda_J$, in which case we neglect the pressure term.

$$\ddot{D} + 2H\dot{D} = 4\pi G\rho_h D$$

Let's consider a matter-dominated universe ($\Omega = 1$) in which $c_s = 0$. This means that $H = \frac{2}{3t}$ and $8\pi G\rho_h = 3H^2 = \frac{4}{3t^2}$. Hence, we have

$$\ddot{D} + \frac{4}{3t}\dot{D} - \frac{2}{3t^2}D = 0$$

This has a form that can be solved by power-laws, so let's try $D = t^n$. This gives

$$n(n-1) + \frac{4n}{3} - \frac{2}{3} = 0$$

which has solutions $2/3$ and -1. This means that we have one mode that grows in time, and another that decays.

$$D = At^{2/3} + Bt^{-1}$$

Next, we remember that $R = t^{2/3}$. So our growing mode has its amplitude growing as R, while the decaying mode decreases as $R^{-3/2}$.

We no longer have exponential growth! The expansion of the universe slows the growth to a power-law! In particular, since the perturbation growth with the exact same dependent on t as R, this means perturbation growth self-similarly as the expansion of the universe, $\delta \propto (1+z)^{-1}$, so in a critical universe, new structure is always forming because the universe is self-similar.

We can also work out the case in which the universe is empty, $\dot{R} = constant$, and $\rho = 0$, i.e., the Milne model, in this case,

$$\ddot{D} + 2H\dot{D} = 0,$$

and $H = 1/t$. Seeking power law solution, we find $n = 0$ or $n = -1$, so there is a constant mode and a decay mode. Perturbation won't grow in an empty universe.

$$D = At^0 + Bt^{-1}$$

In general, one can write down the solution for the growing mode as

$$\Delta(R) = 5/2\Omega_{m0}H \int_0^R dR'/(dR'/dt)^3$$

where R is the scale factor. And for a low density universe, it goes through two stages, at $z >> 1$, the density parameter $\Omega_m \sim 1$, so the perturbation grows linearly. At low redshift, $\Omega_m \sim 0$, so the grow freezes out, with the transition happens at $\Omega_0 z \sim 1$. This gives us yet another way to probe cosmological parameters: by looking at the density of objects in the universe, or the abundance of structures in the universe, if the universe is of critical density, then the number density will decline rapidly towards high-z; if it is low density, then the number density will be roughly constant to $\Omega_0 z \sim 1$ and then decline towards high redshift. The dN/dz test is a very powerful test since the growth factor depends both on Ω_m and Λ through the evolution of scale factor. It is another important way to probe dark energy. It is a particularly powerful probe to use number count of collapsed objects, such as cluster of galaxies, for cosmological test, which we will see later.

5 Peculiar Velocity

The development of velocity perturbations in the expanding universe can be derived from the Euler equation we derived before:

$$\frac{d\mathbf{u}}{dt} + 2H\mathbf{u} = -\frac{c_s^2}{R^2}\nabla_r\delta - \frac{1}{R^2}\nabla_r\phi$$

and ignore pressure term.

$$\frac{d\mathbf{u}}{dt} + 2H\mathbf{u} = -\frac{1}{R^2}\nabla_r\phi$$

Note that this is the comoving peculiar velocity. Let's split the velocity vector into components parallel and perpendicular to the gravitational potential gradient.

$$\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}.$$

The parallel term is referred as potential motion, and the perpendicular part the vortex or rotational motions.

For rotational motions,

$$\frac{d\mathbf{u}_{\perp}}{dt} + 2H\mathbf{u}_{\perp} = 0$$

For $\Omega = 1$, such modes scale in time as $\mathbf{u}_{\perp} \propto R^{-2}$. Hence, they decay. This is important because it means that primordial vorticity is erased. We expect the velocity field today to be curl-free. This poses issue with galaxy formation model involving primordial turbulence.

Neglect perpendicular part, the velocities that match the density modes are proportional to the gradient of the potential. Let's try

$$\mathbf{u} = -F(t)\frac{\nabla_r \phi}{R} = F(t)\mathbf{g},$$

where $\mathbf{g} = -\nabla_r \phi / R$ is the gravitational force. The continuity equation becomes

$$\frac{d\delta}{dt} = -\nabla_r \cdot \mathbf{u} = F(t)\nabla_r^2 \phi / R = F(t) 4\pi G R \rho_h \delta$$

We put in $8\pi G \rho = 3H^2\Omega$, and use $\delta \propto D(t)$. This means that

$$F(t) = \frac{\dot{D}}{D}\frac{1}{4\pi G \rho_h R} = \frac{\dot{D}}{D}\frac{2}{3H^2\Omega R} = \frac{2}{3H\Omega}\frac{dD/dR}{D}$$

$$\delta v = RF\mathbf{g} = \frac{2f}{3H\Omega}\mathbf{g}$$

where $f = (R/D)(dD/dR) \approx \Omega^{0.6}$ for the growing mode.

The growing mode has a velocity that is proportional to the gravitational acceleration. For $\Omega = 1$, $\delta\mathbf{v} = t\mathbf{g}$, it's simply the present-day gravitational acceleration times the age of the universe!

The growing mode of velocity field will map the growing mode of density field. Consider a plane wave density perturbation. $\delta = D(t)\delta_0 \exp(-i\mathbf{k}\cdot\mathbf{r})$. The potential must have the same exponential dependence: $\phi = \phi_0 \exp(-i\mathbf{k}\cdot\mathbf{r})$.

Then we have $-k^2\phi_0 = 4\pi G R^2 \rho_h \delta_0 D(t)$ from the Poisson equation, and

$$\mathbf{g} = -\nabla\phi/R = -i\mathbf{k}(\phi_0/R)\exp(-i\mathbf{k}\cdot\mathbf{r}) = \frac{i\mathbf{k}}{k^2} 4\pi G R \rho_h \delta_0 D(t) \exp(-i\mathbf{k}\cdot\mathbf{r}) = \frac{i\mathbf{k}}{k^2}\frac{3H^2\Omega R}{2}\delta$$

$$\delta\mathbf{v} = \frac{i\mathbf{k}}{k^2} HRf\delta$$

What this means is that the mapping of density and velocity field depends on this factor f, which is related to density parameter mainly. So by mapping the large scale density and velocity field, this gives us yet another way of measuring Ω. Measuring peculiar velocity is not easy. But this is fairly popular in the 1990s, and for a while had produced the only observational evidence that $\Omega \sim 1$. Later, with better peculiar velocity measurements, this seems to have gone away.

6 Non-linear perturbations

So far we've described the evolution of small perturbations. However, this is certainly not the whole story.

When overdensities become non-linear ($\delta \approx 1$), the linear approximation does not break gracefully. Gravity is an attractive force, and the overdensities quickly run away to very large density!

Higher-order perturbation theory is generally a very poor approximation in cosmology. We quickly reach a regime of $\delta \gg 1$.

Numerical simulations show that the matter accumulates in dense regions. We call these regions halos, because of our expectation that they correspond to the dark matter halos of galaxies and clusters.

What stops the density from going to infinity? Random motions of the particles. As the particles enter the dense region from various directions, they interpenetrate (the gas shocks). This produces an effective pressure term that opposes further collapse.

Simulations and analytic models suggest that collapse is halted at an average overdensity of about 200. The interior of the halos have higher densities, but it's not because of cosmological infall.

An important aspect of gravitational collapse is that non-linear collapse on small scales does not spoil the linear evolution of large-scale perturbations. These perturbations don't care whether the matter on small scales is lumpy or not. Credit Gauss's law!

Hence, in the non-linear regime, halos become our standard description.

Halos have different sizes and masses. They can merge together. There are models and simulation results (that agree) to describe the mass functions and the merger rates.

Halos are viewed as the sites of galaxy and cluster formation. Hence, the formation of galaxy-mass-sized halo is the precursor to the formation of luminous galaxies, while the merging of those halos is the precursor to galaxy merging and cluster formation. We'll talk more about this later on.

One cannot solve the gravitational collapse problem from arbitrary initial conditions in the non-linear regime. There are some analytic models (some of which have been remarkably useful), but the primary workhorse here are numerical simulations.

In the simplest form, one represents the cosmological density field by millions of particles and then follows the motions of these particles according to their self-gravity. At the initial time, the particles are nearly on a grid, but have been either displaced in position or in mass to install some small perturbations, randomly generated to mimic the linear regime power spectrum of the chosen cosmological model.

The particles generically clump into halos and one can study the inner properties and the external correlations of these halos.

However, comparing to the real world requires building some statistics, so this is what we'll talk about next!

- $\delta\rho/\rho = \Delta \ll 1$: **linear perturbation.**
- $\delta\rho/\rho = \Delta \gg 1$: **non-linear, simulations.**
- $\delta\rho/\rho = \Delta \sim 200$: **collapsed, viralized objects, "halo".**
- **halo formation $\rightarrow$ galaxy formation**
- **halo merger $\rightarrow$ galaxy merger and cluster formation**

7 Correlation Function

From these redshift surveys we know that galaxies are clustered. So how do we describe the clustering mathematically. Here I am going to introduce two very important statistics. Correlation function, and power spectrum.

One way to describe the tendency that galaxies like to stay together is the two point correlation function $\xi(r)$. Correlation functions $\xi(r)$ is the excess probability that a galaxy is found at a distance r from a known one. Hence,

$$\xi(r) = \left\langle \frac{\rho(\mathbf{x}+\mathbf{r})\rho(\mathbf{x})}{\rho_0^2} \right\rangle - 1 = \langle \delta(\mathbf{x}+\mathbf{r})\delta(\mathbf{x}) \rangle$$

Here $\delta(r)$ is the density fluctuation field of galaxies in the universe

$$\delta(\mathbf{x}) = \frac{\rho(\mathbf{x}) - \rho_0}{\rho_0}.$$

It is the density contrast to the random density. You might have thought the argument should be a vector, but the claim is that the universe is statistically isotropic.

This idea of a correlation function can be extended to triplets and so forth. These are called higher-order correlation functions. Note that these depend on the geometry of the locations, so the vector directions of the separations now matter.

The other way to put it: let us have two small volumes ΔV_1 and ΔV_2, separated by distance of r. The average spatial density of galaxies is ρ. Then the number of galaxies in ΔV_1 is $\rho \Delta V_1$. Since ρ and ΔV are small numbers, it means that the chance of finding a galaxy in ΔV_1 is $\Delta P_1 = \rho \Delta V_1$. Now if galaxies are truly randomly distributed, then the chance of find a galaxy in ΔV_2 is $\Delta P_2 = \rho \Delta V_2$. So the chance of finding a galaxy in both ΔV_1 and ΔV_2 is

$$\Delta P = \rho^2 \Delta V_1 \Delta V_2.$$

In this case, galaxy distribution is uncorrelated. If galaxies are clustered, or correlated, then the chance of finding a galaxy in ΔV_1 and at the same time in ΔV_2 is bigger than random:

$$\Delta P = \rho^2 \Delta V_1 \Delta V_2 (1 + \xi(r)).$$

So $\xi(r)$ describes how much the two volumes, separated by r, are correlated, or how strong the galaxies are clustered. If $\xi(r) > 0$, they are clustered. If $\xi(r) < 0$, they are anti-clustered, in other words, they tend to avoid each other. Clearly, in order to compute $\xi(r)$, we need to measure redshift in order to derive the real space distance. Also clear is that ξ is a function of distance separation. At very large distance separation, the galaxy distribution will have nothing to do with each other, so $\xi(r) \to 0$ when $r \to \infty$. On the other hand, if galaxies are clustered, then the closer the separation is, the higher the possibility of finding a galaxy next to a known galaxy is, comparing to the random field. When $r \to 0$, this probability is going to go to 1, or $\xi(r) \to \infty$. From observations, people find that it is a very good power law:

$$\xi(r) = (r/r_0)^{-\gamma}.$$

Here r_0 is called the correlation length, the scale at which the correlation function is 1, or the probability of finding a galaxy from a known galaxy at a distance r is twice as high as finding a galaxy in the random field. It is a very important parameter to measure in galaxy surveys. At a few times r_0, the galaxy distribution becomes to be very close to random.

Most galaxy redshift survey shows that $r_0 \sim 5/hMpc$ and $\gamma \sim 1.8$.

Deviation from this power law on both small and large scales.

- small scale: non-linear growth of halo will matter. Two halo term. ($< 1Mpc$).
- large scale: can't ignore that the universe is not pure dark matter. Baryon in the early universe causes acoustic waves. BAO. $> 100Mpc$.

8 Power Spectrum

A common reformulation of these ideas is to study the density field in Fourier space. First, we Fourier transform

$$\hat{\delta}_{\mathbf{k}} = \int d^3x \; \delta(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}}$$

and then we consider the statistical properties of the Fourier coefficients (which are complex numbers).

The translational invariance of the statistical distribution gives useful properties. In particular, $\left\langle \hat{\delta}_{\mathbf{k}} \right\rangle = 0$.

The most common statistic in Fourier space is the power spectrum. This is the two-point correlations of the Fourier coefficients. In particular, we have

$$\left\langle \hat{\delta}_{\mathbf{k}} \hat{\delta}^*_{\mathbf{k}'} \right\rangle = (2\pi)^3 \delta^3_D(\mathbf{k} - \mathbf{k}') P(k)$$

$\delta^3_D(\mathbf{k} - \mathbf{k}')$is the Dirac delta function.

The power spectrum and correlation function are really just two representations of the same information. In fact, the two functions are 3-d Fourier transforms of one another.

$$\delta(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \delta_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$\xi(\mathbf{r}) = \langle \delta(\mathbf{r})\delta(0) \rangle = \langle \delta(\mathbf{r})\delta(0)^* \rangle = \left\langle \int \frac{d^3k}{(2\pi)^3} \delta_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \int \frac{d^3k'}{(2\pi)^3} \delta^*_{\mathbf{k}'} \right\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} \langle \delta_{\mathbf{k}} \delta^*_{\mathbf{k}'} \rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d^3k'}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} (2\pi)^3 \delta^3_D(\mathbf{k}-\mathbf{k}') P(k) = \int \frac{d^3k}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{x}} P(k)$$

While power spectrum and correlation function are basically the same thing, they are measured in quite different ways, one by counting galaxy pairs, the other by deriving the density field and its Fourier transform. They have different error properties and can have pros and cons each in real life measurements.

What is a power spectrum, really? It's the mean square amplitude of perturbations of a given wavenumber. Since we found that each Fourier mode evolves independently in linear theory, power spectra are particularly useful on linear scales.

A useful quantity to compute is the rms density fluctuation in a sphere of radius R. To do this, first write that the density fluctuation in the sphere is

$$\Delta_R = \frac{3}{4\pi R^3}\int_{sphere} d^3r\ \delta(\mathbf{r})$$

This has

$$\langle \Delta_R \rangle = \frac{3}{4\pi R^3}\int_{sphere} d^3r\ \ \langle \delta(\mathbf{r}) \rangle = 0$$

The rms is then $\sigma(R) = \langle \Delta_R^2 \rangle^{1/2}$.

For notational simplicity, let $W(\mathbf{r}) = 1$ if the point is inside the sphere and $W(\mathbf{r}) = 0$ otherwise. W is called the window function. This special case is called a top-hat window. Then

$$\sigma^2(R) = \left\langle \Delta_R^2 \right\rangle = \left\langle \frac{1}{V}\int d^3r\ \delta(\mathbf{r})W(\mathbf{r})\frac{1}{V}\int d^3r'\ \delta(\mathbf{r}')^* W(\mathbf{r}') \right\rangle = \frac{1}{V^2}\int d^3r\ d^3r'\ W(\mathbf{r})W(\mathbf{r}')\xi(|\mathbf{r}-\mathbf{r}'|))$$

Without going through a few lines of simple but long math that deal with changing integrals in Fourier transformation, we will get a simple one dimensional integral

$$\sigma^2(R) = \int \frac{d^3k}{(2\pi)^3} P(k) \frac{|W_{\mathbf{k}}|^2}{V^2}$$

where $W_{\mathbf{k}}$ is the Fourier transform of our window function.

Today, the rms fluctuation in spheres of $8h^{-1}$ Mpc radius is roughly 1. This corresponds to masses of about $2 \times 10^{14} h^{-1}$ $M_\odot$, which not coincidentally is the typical cluster mass. This is yet another fundamental cosmological parameter: σ_8.

Note that transforming between power spectrum, which tells you the amplitude of fluctuations in different Fourier modes, and measured density fluctuation, involves details of your window function. In observations, this window function is the geometry of your survey, including both angular shape and vertical completeness as a function of redshift. Deriving power spectrum, which is what we really care as a physical quantity to be compared with theoretical predications from the real observations requires one to understand the survey window function and model it carefully. And ideally, you want to design your survey to have as simple window function as possible. But it is almost never possible to have a volume limited survey with a top-hat window function.

It is important to note that the contribution to σ goes as $\int \frac{dk}{k} k^3 P(k)$. This means that $P(k)$ is *not* the appropriate quantity to base one's intuition on where fluctuation power is coming from. Rather, it is useful to think in terms of logarithmic intervals, in which case $k^3 P(k)$ is the relevant quantity. Basically, P is the fluctuation per point in k space, but the number of points in k that contribute to the fluctuations scales as k^3.

Indeed, it is increasingly popular to define $\Delta^2(k) = k^3 P(k)/2\pi^2$ as the power spectrum (I call it the dimensionless power spectrum). We then have $\sigma^2 = \int \frac{dk}{k} \Delta^2(k) |W_{\mathbf{k}}|^2$.

Window functions are defined to have $W_{k=0} = 1$, and they then usually cut off at some particular k. For most power spectra of interest, Δ^2 is increasing with k, and so it is a rough approximation that $\sigma^2 \approx \Delta^2$ where the latter is evaluated at the the cutoff scale of W, or $\sigma^2 \approx \Delta^2(1/R)$.

But one important issue in cosmology is that galaxies may or may not trace mass because of the presence of dark matter. And in cosmology we mostly care about matter. So the real parameter we concern about, and the parameter that theory is going to tell us, the the mass fluctuation $\sigma_8(mass)$. It may or may not be one. And the ratio of the fluctuation in the counts of galaxies to the fluctuation of underlying mass distribution is called the bias factor:

$$b = \frac{\sigma_8(\text{galaxies})}{\sigma_8(\text{mass})}.$$

Since we know $\sigma_8(gal) = 1$, to determine bias factor is the same as to determine $\sigma_8(\text{mass})$, or to determine the normalization of the mass power spectrum. A lot of effort in galaxy survey business is to determine both the shape and the normalization of galaxy power spectrum, and compare them with simulations, to constrain both cosmological parameters, as well as different flavors of dark matter models, in particular the thermal properties of dark matter particles.

9 Types of dark matter

We know that most matter in the universe is dark matter. Let us first recall the reasons for talking about dark matter, in particular non-baryonic dark matter, seriously in the context of galaxy formation.

There are mostly three lines of evidence:

(1) the mass to light ratio in the MW, in galaxies, and in cluster of galaxies is much to high for normal stellar population, in other words, the dynamical mass of a stellar system bigger

than globular cluster is always much bigger than stellar mass. So there must be missing, or dark, matter.

(2) the determination of cosmological parameters should that $\Omega_m \sim 0.3$, and from BBN, $\Omega_b < 0.05$, so there must be non-baryonic dark matter.

(3) a purely baryonic model of galaxy formation could not fit CMB data, nor could it generate the structure that we see today.

So most people believe that the mass of the universe is dominated by non-baryonic dark matter, i.e. mass in forms other than quarks, which makes protons and neutrons.

There are two flavors of DM people are still considering:

1. Hot dark matter. These are particles that decouple from the rest of the universe when it was relativistic, and which have a number density roughly equal to that of photons. In this case, there rest energy is a few eV, particular, few ev-mass neutrinos. Why we call them "hot"? For neutrinos with T = 1.95K today (neutrino background), their velocity is going to be: $v = 158(m/eV)^{-1}\text{km s}^{-1}$. So the thermal velocity is very high, and it is even higher at high redshift, this leads major effects on the development of self-gravitating structures.

2. Cold dark matter. It the particles decouple while they are nonrelativistic, there thermal velocity today is effectively zero, and thus "cold". The typical candidates are WIMPS, or weak-interacting massive particles, with possible mass of a few GeV.

One of the key considerations for theories of structure formation in which non-baryonic dark matter dominant, is the damping of density perturbation by free-streaming. So long as the DM particles are strongly coupled, they behave no different from ordinary particles. At later epochs, however, DM particles no longer interact with other particles. If the particle were relativistic at the froze-out tome, they would continue to travel in straight lines at the speed of light. Thus, if the particles belonged to some density perturbation, as they continue to travel freely, they are going to damped out, or reduce the density fluctutation. The masses which are damped out depend upon how far the free-streaming particles can traevel at a given epoch. The comoving distance which a free-streaming particle can travel by epoch t is just:

$$r_{FS} = \int_0^t v(t)/R(t)dt.$$

Note that before the decoupling, the particle has the speed of light; however, after the decoupling the particle will start to cool down as its speed will slow down with $1/(1+z)$. Taking this into account, one find that the mass scale of free-streaming damping is:

$$M_{FS} = 4 \times 10^{15}(m/30eV)^{-2}M_\odot.$$

Therefore, if the hot dark matter neutrino mass is 30 eV, all density perturbations on mass scale less than this will be damped out as soon as they come through the horizon. Since these masses are of the order of massive clusters, in this picture, then, only structures at larger scales can survive. There will be no or little power in the mass power spectrum at smaller scales. So in hot dark matter cosmology, you have to make very large structure, the so-called "zeldovich pancake" first, and then the pancake will break down to make small structure. This scenario is called the "top-down" structure formation.

On the other hand, in the cold dark matter scenario, the CDM particles decoupled early in the universe, after they had become non-relativistic. Free-streaming is unimportant. So in CDM model, all the initial perturbations are preserved. The CDM model is the popular, and working model of structure formation.

The CDM model can be studied using computer simulations in great detail, once the initial power spectrum of fluctuation is given. As I will show you in a moment, these spectra are such that there is most power on the small scale, so the lowest mass objects form first. These then undergo hierarchical clustering under the influence of perturbations on large scales and so the larger scale, which has smaller initial perturbation, and therefore takes longer time to grow, are built-up later. This bottom-up, or hierarchical structure formation is the base of all the galaxy formation theory today.

10 Primordial power spectrum

The density fluctuation power spectrum of the universe consists of two parts: the primordial power spectrum, which is power spectrum the universe began with, and the effect that happens on top of that after the scale in question has entered the horizon as the universe is expanding. The function that describes the changes to the input initial spectrum of the perturbation is called the transfer function.

The initial power spectrum is typically described as having a power-law form:

$$P(k) \sim k^n.$$

Let's call the density fluctuation at mass scale M to be $\sigma(M)$, one can show (find proof in Longair) that: $\sigma(M) \sim M^{-(n+3)/6}$, so as long as $n > -3$, the mass fluctuation decreases to large mass scales. This is good. We want the universe to be uniform on large scales.

The $n = 1$ case is of special interests. This spectrum has the property that the density fluctuation $\sigma(M)$ had the same amplitude on all scales when the perturbations came through the horizon.

It is useful to look at the density perturbation of a certain size/mass scale at the time it enters the particle horizon $r = r_H = ct$ where t is the age of the universe at a certain time. We mentioned before that in radiation dominated era, the perturbation grows as $\delta \sim R^2$. So the fluctuation grows and changes with horizon mass scale as:

$$\sigma(M_H) \sim R^2 M_H^{-(n+3)/6}.$$

And the horizon mass scale is $M_H \sim \rho_D t^3$. In radiation dominated era: $R \sim t^{1/2}$, $\rho \sim R^{-3}$, so $M_H \sim R^3$. Plug all these things in, you will find:

$$\sigma(M_H) \sim M^{2/3} M^{-(n+3)/6.}$$

And if $n = 1$, the amplitudes of the density perturbation were all the same when the came through their particle horizons during the radiation dominated era.

This particular form of primordial power spectrum is called the Harrison-Zeldovich spectrum, first suggested by Harrison and Zeldovich, independently, in early 1970s. It has a number of very appealing features. In particular, if $n = 1$, then the universe is fractal, or self-similar, in the sense that every perturbation came through the horizon with the same amplitude, and as the universe expands, we always find perturbations of the same amplitude appearing on the horizon; the universe looks the same when viewed on the scale of the horizon. It is simple, and it occurs very naturally in inflationary models. It is sometimes also called the scale-invariant spectrum. Deviations of the power-law exponent from this special value are known as 'tilts'. Modern measurements find that n is rather close to 1, with a measurable tile, e.g., Planck finds that $n = 0.97 \pm 0.01$.

11 Transfer Function

Transfer function, or the effect on the perturbation power spectrum after it has entered horizon. The transfer function T(k) describes how the shape of the initial power spectrum in the dark matter is modified by different physical processes:

$$\Delta_k(z = 0) = T(z) D(z) \Delta_k(z)$$

Here D(z) is the growth function we discussed before $D(t) \propto 1/(1 + z)$ for critical universe. There are two effects that I want to mention:

(1) the free-streaming effect that we just talked about. This will reduce the small scale power in hot DM models because the gravitationally important neutrinos are free-streaming around and damped the fluctuation.

(2) effect related to pressure in radiation dominated era. In the matter-dominated era, all scales grow equally. However, in the radiation-dominated era, things behave differently. Here, pressure is important; indeed, the sound speed is $c/\sqrt{3}$! Remember that in the previous derivation of growth function, we ignored the sound speed and pressure term by assuming CDM. But it does matter in early times due to coupling. On scales smaller than the horizon, the growth is stalled by the presence of the radiation pressure. Basically, the universe expands too quickly for the dark matter to collapse. So on small scales the radiation remains smooth, but on large scales it has to cluster. So, scales that enter the horizon (i.e. suffer from smooth radiation) during the radiation-dominated epoch grow less than those that enter the horizon during the matter-dominated epoch. As soon as the perturbations came through the horizon, they cease to grow until the epoch of equality, after which they grow as described in the growth function.

One thing we didn't do is to work out the growth of perturbation in radiation dominated era. Going through the same process as we did before, one can show that in radiation dominated era,

$$\Delta \propto t \propto (1+z)^{-2}.$$

Thus, for the small scale (large k) which enters horizon early, Δ will not grow and suppressed by a factor of $(R_H/R_{eq})^2$, or by a factor of k^2. Therefore, the power spectrum itself will be suppressed by a factor of k^4 on small scale or large k. The transition happens at the horizon size of matter/radiation equality, which is about 100 Mpc co-moving.

This effect results in that the density fluctuation $\sigma(M) \sim M^{-(n+3)/6}$ is roughly flat on small scales. So the processed power spectrum will have $P(k) \sim k^{-3}$ on small scales, $P \propto k$ on large scales, the scale of the horizon at matter-radiation equality, which is about 100 Mpc for LCDM model in comoving units.

12 Power spectra from redshift survey

SDSS Figure.

The measurement of power spectra from galaxies and other cosmic structures have provided a powerful probe to cosmology.

(1) the overall shape of power spectrum provide information on the initial power spectrum, whether it is Harrison-Zeldovich, or tilted.

(2) The small scale shape shows that the perturbation is mostly from CDM. HDM has very

different shapes.

(3) The peak of power spectrum corresponds to the horizon scale of equality of mater and radiation. This wave number is related to both the redshift of equality, and the size of the universe at that time which is related to the expansion history. Turns out $t_{eq} = 7.3\times10^{-2}\Omega_0 h^2$ Mpc^{-1}. Therefore, measurement of power spectrum directly measures $\Omega_0 h$ – the other factor of h is absorbed in the distance dependence of Hubble constant.

(4) There are detailed difference between open and Λ dominated models, but they are subtle. The power to determine Λ from LSS comes from combining with CMB which shows that the universe is flat.