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AN OPTIMAL ALGORITHM FOR THE MULTIPRODUCT CAPACITATED FACILITY LOCATION PROBLEM WITH A CHOICE OF FACILITY TYPE

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Scope and Purpose—The capacitated facility location problem is the problem of locating a number of facilities which have to serve a set of customers, at minimum cost, where each customer has an associated demand and there are constraints on the total demand that can be met from a facility which has the finite capacity. These facilities can represent industrial plants, warehouses, repair facilities, computing facilities in computer network systems and so forth. The problem is formulated as a mixed integer linear program which belongs to the class of *NP*-hard problems. Several algorithms for the solution of this problem have appeared in the literature. This paper extends the standard capacitated facility location problem to a generalization of a multiproduct-multitype capacitated facility location problem and presents an efficient algorithm based on Benders' decomposition, which ' generally recognized as one of the most efficient approaches to the mixed integer program.

Abstract—Consider a generalization of the multiproduct capacitated facility location problem in which each facility capacity has to be chosen from a given set of admissible levels. The objective is to select a set of facilities to open, along with their capacities, and to assign customer demands to them so as to minimize the total cost. In this model, in addition to a fixed cost for opening a facility, a fixed cost is incurred for each product that a facility is equipped to produce. Also, transportation costs are incurred for satisfying each customer's requirement for a particular product at each chosen facility. Benders' decomposition provides a natural partitioning of this model which defines a master problem for selecting a set of open facilities and equipment (or machines) and a subproblem in which the optimal assignment of demand to a specific facility is determined. A solution algorithm based on Benders' decomposition is developed, implemented and applied to a numerical example to discuss the behavior of the cutting planes generated in Benders' decomposition. Computational results of this algorithm are quite satisfactory and encouraging.

1. INTRODUCTION

A capacitated facility location problem has been defined as follows. The facilities of restricted size are placed among some possible locations with the objective of minimizing the total cost of satisfying fixed demands specified at demand points. Costs include a fixed charge for opening each facility and a constant amount for each unit of location's demand supplied from facility [1]. These facilities can represent industrial plants, warehouses, repair facilities, computer facilities in communication network systems and so forth.

The facility location problem is a well-known location problem of both theoretical and practical interest [1-4]. Recent surveys of work on this problem have been presented by Baker [5] and Sridharan [6]. Although many researchers have made significant contributions to optimization models [7-11], most of their models can be characterized as either a single-commodity problem or only a one-facility type problem. Their solution procedures for these types of problems are based on heuristic approaches, or branch-and-bound techniques with or without Lagrangian relaxation.

Here, consider a generalization of a capacitated facility location problem in which several different products are required by the customers with a choice of facility type. Each facility type offers a different capacity on a particular product with the different fixed set-up costs. This is called a multiproduct capacitated facility location problem with a choice of facility type. In this problem, in addition to the fixed cost for opening a facility, there is an added fixed cost incurred if an open facility is equipped to handle a particular product. Also, a transportation cost is incurred for

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satisfying a customer's demand requirement for a particular product at a given facility. In this paper, an attempt is made to develop the following:

(1) A more general model that integrates the multiproduct facility with a choice of facility types. The attempt is to study a multiproduct–multitype facility problem.
(2) A solution procedure which can find the optimal solution in a reasonable amount of time. To reduce computation time, Benders' decomposition technique is used which takes advantage of the special structure of the model. This algorithm has been implemented in Fortran.

Virtually all of the existing models are mixed integer linear programs. The result is a complex combinatorial model which belongs to the notoriously difficult class of NP-hard problems. This is due to the number of integer variables involved. Therefore the computation of an optimal solution is expected to be a challenging task. Most of the algorithms developed in the past have been based on branch-and-bound techniques with or without Lagrangian relaxation or heuristic approaches. Although these techniques and approaches are well-known, they have some inherent disadvantages. The main disadvantage of the branch-and-bound technique is that it takes much more time and effort to find the optimal solution to a realistic size problem. Also, heuristic approaches do not necessarily guarantee an exact optimal solution. These approaches generally produce only acceptable solutions.

In summary, the primary objective here is to develop a general model of a capacitated facility location problem which incorporates the multiproduct–multitype facility and to propose an optimal solution algorithm based on Benders' decomposition technique. A description of the model, the solution algorithm and an application of the model using data from a numerical example are presented.

2. LITERATURE REVIEW OF SOLUTION ALGORITHMS

The focus of this section is on potential solution algorithms for capacitated facility location problems which can be relevant to the problem of interest. The literature review provided by Baker [5] summarizes capacitated facility location problems.

For capacitated facility location problems, branch-and-bound [3, 4, 7, 12], relaxation [9, 13–15] and heuristic [16–19] methods have been suggested. These algorithms, however, are restricted to solving either small or simple (e.g. single-product or single-type) problems.

The multiproduct–multitype capacitated facility location problem is a large-scale integer programming problem and, for most practical applications, computationally complex using conventional branch-and-bound and implicit enumeration methodology. Benders' decomposition reduces the computational difficulty by exploiting the special structure of problem.

Benders' decomposition method was first applied to distribution/location problems by Balinski [20]. His method is based on separating a difficult problem into two simpler problems.

Geoffrion and Graves [14] were the first to solve the multicommodity capacitated location problem by Benders' decomposition. This model not only deals with facility location and commodity flows, but with customer assignment as well. These authors developed and implemented a solution technique based on Benders' decomposition and successfully applied it to a real problem for a major food firm. An optimal solution was found and proven with a surprisingly small number of Benders' cuts. Geoffrion and Graves concluded that this problem class appears to be amenable to the solution yielded by Benders' decomposition method. Geoffrion [21] later reported the results of a large-scale distribution warehouse location analysis using Benders' decomposition method for another company. Significant cost savings were estimated in his report. Geoffrion [22] also developed a generalized Benders' decomposition method to apply Benders' decomposition procedure to nonlinear problems.

Côtés and Laughton [23] and Hoang [24] applied a generalized Benders' decomposition method to an uncapacitated discrete transportation network design problem which was subject to a budget constraint. The experimental results reported in Hoang's peper [24] were very encouraging.

Franca and Luna [25] also applied Benders' decomposition method to a discrete stochastic transportation–location problem. With demand at each destination a random variable, the problem was to minimize the sum of expected holding and shortage costs, shipping costs and fixed construction costs. The resultant mixed integer nonlinear program was solved via stochastic transportation subproblems, whose optimal multipliers generated constraints (cuts) for a master integer location problem. This also generated a computational result which was also encouraging.

Therefore, Benders' decomposition method seems to be the most promising tool for solving large-scale facility location problems, especially when some effective techniques for quick convergence, e.g. generating sharp cuts and preprocessing of integer variables [26], are incorporated.

3. MODEL DESCRIPTION

Consider a set of demand points indexed by $I = \{1, 2, \ldots, N\}$, each with a unit demand, a set of possible facility sites indexed by $J = \{1, 2, \ldots, N\}$, where the capacity of facility type m can be chosen from among $\{S_m \mid m \in M = \{1, 2, \ldots, Q\}\}$, and product $f \in F = \{1, 2, \ldots, P\}$ which are required by a number of customers.

The demand of product f by each customer is fulfilled and the total demand assigned to each facility does not exceed its capacity. The problem is, then, first to choose some subset of the possible facility type m at j to be open; for each open facility, it must also be decided which products are produced by that facility.

For each product, customers are assigned to the facilities that handle that product. It is assumed that there are explicit constriants on the capacity of a facility type m to handle the products produced at the given facility location j. The objective is to minimize total costs which consist of: fixed set-up costs F_{jm} for opening facility type m at j; fixed set-up costs E_{jf} for equipping facility at j to handle product f; and transportation costs C_{ijf} for serving the requirements of demand point i for product f from the facility at j.

Let Z_{jm} be a binary variable that indicates whether facility type m at j is open ($Z_{jm} = 1$) or not ($Z_{jm} = 0$); let Y_{jf} indicate whether equipment (or machines) for product f is installed at j ($Y_{jf} = 1$) or not ($Y_{jf} = 0$). Let X_{ijf} ($0 \leqslant X_{ijf} \leqslant 1$) denote the fraction of demand for product f at demand point i that is provided by facility j. Note that any optimal solution to the model here will consist of values of 0 or 1 for all the variables, the continuous as well as the integer variables. The problem (P) can be formulated as follows:

$$\min \quad \sum_i \sum_j \sum_f C_{ijf} X_{ijf} + \sum_j \sum_f E_{jf} Y_{jf} + \sum_m \sum_j F_{jm} Z_{jm} \tag{1}$$

$$\text{s.t.} \quad \sum_j X_{ijf} = 1 \qquad \forall i, f \tag{2}$$

$$\sum_i X_{ijf} \leqslant N Y_{jf} \qquad \forall j, f \tag{3}$$

$$\sum_f Y_{jf} \leqslant \sum_m Z_{jm} P \qquad \forall j \tag{4}$$

$$\sum_f Y_{jf} S_f \leqslant \sum_m Z_{jm} S_m \qquad \forall j \tag{5}$$

$$\text{linear configuration constraints on } Ys \text{ and } Zs \tag{6}$$

$$Y_{jf} = 0, 1 \qquad \forall j, f \tag{7}$$

$$Z_{jm} = 0, 1 \qquad \forall j, m \tag{8}$$

$$X_{ijf} \geqslant 0 \qquad \forall i, j, f \tag{9}$$

Constraint set (2) assures that every customer's demand for each product is assigned to some facility. Constraint set (3) assures that a customer is assigned for a given product only to a facility that has the equipment (or machines) to handle that product. Constraint set (4) assures that only open facilities are equipped to handle products. Constraint set (5) assures that the capacity of a facility type is large enough to handle all the products produced at the given facility location.

Linear configuration constraint set (6) can be used to impose a variety of requirements. For example, at least one out of three specified facilities must be equipped for a particular product, no more than a predetermined number of facilities or equipment can be placed in the system, certain locations are excluded to have equipment for a particular product for environmental reasons and a certain location must have a facility (maybe because one is already in place) etc. Constraint set (5) always implies constraint set (4); therefore the redundant constraint set, constraint set (4), can be dropped from the model.

In summary, the model is a mixed integer linear program with the objective being to minimize total costs. The model has N^2P continuous variables, $N(P+Q)$ integer variables and $N(2P+1)$ functional constraints in addition to the linear configuration, integer and non-negativity constraints. The decision of whether or not to open a facility at a potential location with the resulting substantial set-up cost is explicitly handled in the model. Since equipment can only be placed where a facility is open, the equipment cannot be installed independently of one another. It is apparent that once a set of open facilities and equipment are chosen, the problem is reduced to an easy problem that may be solved by inspection. A natural partitioning of this model, therefore, defines a master problem for selecting a set of open facilities and equipment to be evaluated, and a subproblem in which the optimal assignment of demand to a specific facility is determined. Benders' decomposition provides a convenient mechanism for schieving this decomposition.

4. ALGORITHM

Most real-life applications of this model are too large to be solved economically by existing general mixed integer linear programming codes [27]. The model does, however, have a special structure that enables it to be decomposed in such a way that the multiproduct aspect becomes much less burdensome: since there are no coupling constraints, when the binary variables (Ys and Zs) are temporarily held fixed so as to satisfy constraint sets (4)–(6), the remaining optimization in X separates into as many independent transportation problems as there are products. The transportation problem for the product f is of the form:

$$\min \quad \sum_i \sum_j \sum_f C_{ijf} X_{ijf} \tag{10}$$

$$\text{s.t.} \quad \sum_j X_{ijf} = 1 \qquad \forall i \tag{11}$$

$$\sum_i X_{ijf} \leqslant N Y_{jf} \qquad \forall j \tag{12}$$

$$X_{ijf} \geqslant 0 \qquad \forall i, j. \tag{13}$$

This can be seen either from the physical interpretation of the problem or directly from formulation (1)–(9). The simplicity of the problem for fixed Y_{jf} and Z_{jm} suggests the application of Benders' decomposition [28, 29]. The derivation of the master problem is given in Section 4.1, and Section 4.2 explains how to solve the subproblem. A conventional specialization of this approach is made in Section 4.3, and Section 4.4 describes the embedded Benders' decomposition which is a variant of Benders' decomposition.

4.1. *Derivation of the Master Problem*

Benders' decomposition is an algorithm that exploits the special structure of the multiproduct facility location problem. This is accomplished by fixing Y_{jf} and Z_{jm} and rewriting problem (P) as follows:

$$\min_{Y,Z=0,1} \left[\sum_j \sum_f E_{jf} Y_{jf} + \sum_m \sum_j F_{jm} Z_{jm} + \min_{X \geqslant 0} \left\{ \sum_i \sum_j \sum_f C_{ijf} X_{ijf} \,\middle|\, \sum_j X_{ijf} = 1, \forall i, f; \sum_i X_{ijf} \leqslant N Y_{jf}, \forall j, f \right\} \right] \tag{14}$$

$$\text{s.t.} \quad \sum_f Y_{jf} S_f - \sum_m Z_{jm} S_m \leqslant 0 \qquad \forall j. \tag{15}$$

Of course, only values of Y_{jf} and Z_{jm} for which there exist $X_{ijf} \geqslant 0$, satisfying the resulting linear constraint sets (11) and (12), may be considered. The existence of $X_{ijf} \geqslant 0$ satisfying constraint sets (11) and (12) is guaranteed iff

$$\sum_j Y_{jf} \geqslant 1 \qquad \forall f. \tag{16}$$

This master problem provides sites for the location of facilities and a specific assignment of products given an optimal transportation routing. Replacing the inner minimization problem by its linear programming dual:

$$\min_{Y,Z=0,1} \Bigg[\sum_j \sum_f E_{jf} Y_{jf} + \sum_m \sum_j F_{jm} Z_{jm} + \max_{U \geqslant 0} \left\{ \sum_i \sum_f V_{if} - \sum_j \sum_f (NY_{jf}) U_{jf} \mid V_{if} - U_{jf} \leqslant C_{ijf}, \forall i, j, f \right\} \Bigg] \tag{17}$$

$$\text{s.t.} \qquad \sum_f Y_{jf} S_f - \sum_m Z_{jm} S_m \leqslant 0 \qquad \forall j. \tag{18}$$

Since the optimal dual solution occurs at an extreme point of the feasible space, the resulting problem (MP_A) can be written with an explicit set of constraint (16) as follows:

$$\min_{Y,Z=0,1} \quad W \tag{19}$$

$$\text{s.t.} \qquad W \geqslant \sum_j \sum_f E_{jf} Y_{jf} + \sum_m \sum_j F_{jm} Z_{jm} + \sum_i \sum_f V_{if}^{\mathrm{p}} - \sum_j \sum_f (NY_{jf}) U_{jf}^{\mathrm{p}} \qquad (V^{\mathrm{p}}, U^{\mathrm{p}}) \in T \tag{20}$$

$$\sum_j Y_{jf} \geqslant 1 \qquad \forall f \tag{21}$$

$$\sum_f Y_{jf} S_f = \sum_m Z_{jm} S_m \leqslant 0 \qquad \forall j. \tag{22}$$

linear configuration constraints on Ys and Zs, (23)

where $(V^{\mathrm{p}}, U^{\mathrm{p}})$ is the set of extreme points of the polyhedron,

$$P = \{(V, U) \mid V_{if} - U_{jf} \leqslant C_{ijf}, \forall i, j, f, U_{jf} \geqslant 0\}. \tag{24}$$

These constraints (20) are called "Benders' cuts". The number of Benders' cuts is potentially enormous. Benders proposed to solve a realxation, (MP), of problem (MP_A) by taking only a subset of Benders' cuts, and to generate cuts only when necessary becauseonly a small fraction of these cuts will be ginding at the optimal solution. Therefore, a realxed version (MP) of problem (MP_A) is constructed:

$$W^k = \min_{Y,Z=0,1;W} \quad W \tag{25}$$

$$\text{s.t.} \qquad W - \sum_m \sum_j F_{jm} Z_{jm} + \sum_j \sum_f (-E_{jf} + NU_{jf}^{\mathrm{p}}) Y_{jf} \geqslant \sum_i \sum_f V_{if}^{\mathrm{p}} \qquad (V^{\mathrm{p}}, U^{\mathrm{p}}) \in T^k \tag{26}$$

$$\sum_j Y_{jf} \geqslant 1 \qquad \forall f \tag{27}$$

$$\sum_f Y_{jf} S_f - \sum_m Z_{jm} S_m \qquad \forall j \tag{28}$$

linear configuration constraints in Ys and Zs, (29)

where T^k is the kth subset of T.

Problem (MP) is the realxed version of problem (MP_A) since only a subset of the constraints of problem (MP_A) are included. An optimal solution to problem (MP) is an optimal solution to problem (MP_A) iff the solution satisfies all the constraints of problem (MP_A).

Motice that, while the master problem (MP) is still an integer linear program, the subproblem has a special structure and can be solved by well-known methods. Thus, most of the required solution time involves the master problem.

4.2. *Solution of the Subproblem*

In deriving the master problem in Section 4.1, replace the subproblem

Primal

$$\min_{X\geq 0} \quad \sum_i \sum_j \sum_f C_{ijf} X_{ijf} \tag{30}$$

$$\text{s.t.} \quad \sum_j X_{ijf} = 1 \qquad \forall i, f \tag{31}$$

$$\sum_i X_{ijf} \leq N Y_{jf} \qquad \forall j, f \tag{32}$$

by its linear programming dual

Dual

$$\max_{U\geq 0} \quad \sum_i \sum_f V_{if} - \sum_j \sum_f (N Y_{jf}) U_{jf} \tag{33}$$

$$\text{s.t.} \quad V_{if} - U_{jf} \leq C_{ijf} \qquad \forall i, j, f. \tag{34}$$

The solution to the primal subproblem is simple. Once an assignment of equipment (or machines) for all products is determined, the problem is reduced to an easy problem that can be separated into as many subproblems as there are products and solved by inspection since there are no coupling constraints.

For each subproblem, the variable corresponding to the minimum cost (C_{ijf}) among all the nodes j where the equipment (or machines) is placed set equal to 1. All other variables are set equal to 0:

let

$$K_f = \{ j \mid Y_{jf} = 1 \} \qquad \forall f$$

let

$$J_{if} = \left\{ k \mid C_{ikf} = \min_{j\epsilon k_f} C_{ijf} \right\} \qquad \forall i, f$$

then,

$$X_{ijf} = 1 \qquad \text{if } j = J_{if} \qquad \forall i, j, f \tag{35}$$

and

$$X_{ijf} = 0 \qquad \text{if } j \neq J_{if} \qquad \forall i, j, f. \tag{36}$$

The solution to the dual problem can be obtained from complementary slackness conditions as follows:

$$V_{if} = U_{J_{if}f} + C_{iJ_{if}f} \qquad \forall i, f \tag{37}$$

and

$$U_{jf} = 0 \qquad \forall j \in K_f. \tag{38}$$

Since $J_{if} \in K_f$ $\forall i, f$, we have, from conditions (27) and (38),

$$V_{if} = C_{iJ_{if}f} \qquad \forall i, f \tag{39}$$

The remaining dual constraints are:

$$U_{jf} \geq 0 \qquad \forall j \notin K_f, \forall i, f \tag{40}$$

and

$$[U_{jf} \geqslant V_{if} - C_{ijf} \qquad \forall j \notin K_f, \forall i, f.] \tag{41}$$

To maximize the remaining terms of the objective function of the dual problem the remaining U_{jf} should be set as small as possible while still satisfying the constraints. Therefore, the optimal dual solution is:

$$V_{if} = C_{iJ_{if}f} \qquad \forall i, f, \tag{42}$$

$$U_{jf} = 0 \qquad \forall j \in K_f, \forall i, f \tag{43}$$

and

$$U_{jf} = \max\{0, \{V_{if} - C_{ijf}\}\} \qquad \forall j \notin K_f, \forall i, f. \tag{44}$$

4.3. Specialization of Benders' Decomposition

Specialization of Benders' decomposition to the model (1)–(9) in the standard way leads to the following algorithm:

Step 1. [Initialize problem (MP)]
Select a convergence tolerance parameter $\varepsilon \geqslant 0$.
Let $K = 0$; $\mathrm{UB} = \infty$; $\mathrm{LB} = -\infty$; $T^k = \varnothing$.

Step 2. [Solve the current integer master problem (MP)]

$$W^k = \min_{Y,Z=0,1} \quad W \tag{45}$$

$$\text{s.t.} \quad W - \sum_m \sum_j F_{jm} Z_{jm} + \sum_j \sum_f (-E_{jf} + NU^p_{jf}) Y_{jf} \geqslant \sum_i \sum_f V^p_{if} \qquad (V^p, U^p) \in T^k \tag{46}$$

$$\sum_j Y_{jf} \geqslant 1 \qquad \forall f \tag{47}$$

$$\sum_f Y_{jf} S_f - \sum_m Z_{jm} S_m \leqslant 0 \qquad \forall j \tag{48}$$

linear configuration constraints on Ys and Zs. (49)

(1) If the current master problem does not have a feasible solution, then neither does problem (P).
(2) If the current master problem has an optimal solution (W^k, Y^k, Z^k) then, since problem (MP) is a relaxation problem of $(\mathrm{MP_A})$, W^k is a lower bound on the optimal value of the original mixed integer programming problem. Let $\mathrm{LB} = W^k$. If $\mathrm{UB} - \mathrm{LB} > \varepsilon$ then go to Step 3 or else terminate.

Step 3. (Solve the primal LP subproblem)

(a) $$W^*(Y^*, Z^*) = \sum_m \sum_j F_{jm} Z^k_{jm} + \sum_j \sum_f Y^k_{jf} + \min_{x \geqslant 0} \sum_i \sum_j \sum_f C_{ijf} X_{ijf} \tag{50}$$

$$\text{s.t.} \quad \sum_j X_{ijf} = 1 \qquad \forall i, f \tag{51}$$

$$\sum_i X_{ijf} \leqslant N Y^k_{jf} \qquad \forall j, f. \tag{52}$$

Denote the optimal solution by X^k. The vector (X^k, Y^k, Z^k) represents a feasible solution to problem (P). If $W^*(Y^k, Z^k) < \mathrm{UB}$ then let $\mathrm{UB} = W^*(Y^k, Z^k)$ and store (X^k, Y^k, Z^k) as the incumbent solution. If $\mathrm{UB} - \mathrm{LB} \leqslant \varepsilon$ then terminate; otherwise go to Step 3(b).

(b) Find an optimal dual solution (V^k, U^k) for the primal subproblem in Step 3(a).
(c) Let $T^{k+1} = T^k \cup \{(V^k, U^k)\}$. Let $k = k + 1$ and go to Step 2.

A few remarks on this algorithm are in order. First, note that an ε-optimal terminal criterion has been used. The available UB and LB on the optimal value of the model coincide to within ε upon termination, at which time the incumbent has been demonstrated to be ε-optimal in the model. Prior to termination it is known only that the incumbent is within UB − LB of the optimal value. By using UB − ε instead of UB, the constraint set (46) is made tighter. No solution with an optimal value less than the current UB − ε will be overlooked. Even though solutions with values greater than UB − ε may still be generated, because problem (MP) is a relaxation of problem ($\mathrm{MP_A}$) it has proven to be helpful in reducing CPU time to find the optimal solution. In other words, one possible way of reducing computation time at the possible expense of not finding the optimal solution is to reduce the UB by ε. Finite convergence is assured for any $\varepsilon \geqslant 0$. Second, problem (P) may be more easily solved if the constraint set (3) is replaced by the set of more explicit constraints:

$$X_{ijf} - Y_{jf} \leqslant 0 \qquad \forall i, j, f. \tag{53}$$

Since the disaggregated constraint set (53) more tightly constrains the feasible space than the equivalent aggregate constraint set (3), the solution time can be reduced.

4.4. Embedded Benders' Decomposition Algorithm

There are numerous variants of the pure Benders' decomposition algorithm described in Section 4.3. One variant of particular interest is not to solve the current master problem at Step 2 to optimality, but rather to stop as soon as a feasible solution to it is produced which has a value below UB − ε. In order to improve the algorithm in terms of computation time, the decomposition concept will be embedded in a 0–1 implicit enumeration algorithm. Specifically, rather than solving the master problem to optimality at each iteration, the improved algorithm, called the embedded Benders' algorithm, looks for a feasible solution to the master problem possessing an objective value which is better than the previous best feasible solution. Then, the improved feasible solution and new constraint are added to the master problem at the next iteration. The motivation is that, initially only a few extreme points of problem (P) are known and, consequently, solving the master problem to optimality with this limited information may lead to a great deal of wasted computational effort. Any portion of the tree enumerated during one master problem remains enumerated during subsequent master problems. Each master problem enumerates part of the tree which is not previously enumerated. Therefore, the enumeration may be restarted at the node reached during the previous enumeration. Consequently, the enumeration tree is completed only once at the end of this algorithm.

The embedded Benders' decomposition solves the relaxed problem based on 0–1 implicit enumeration [30]. More precisely, let (Y^k, Z^k, W^k) be an optimal solution of problem (MP) at iteration k. Then the next cut to be included in problem (MP) is generated, using the dual solution of the primal subproblem, and added to problem (MP). The relaxed version (MP) yields a new LB on the optimal objective value of the problem (P). Also, after solving a subproblem a feasible solution of problem (P) is obtained, so the best-known UB on the objective value of problem (P) may or may not be updated, depending on the value of the incumbent solution. The procedure is:

Step 1. (Initialization)

Let V_0 be the root vertex and V_k the vertex of iteration k.

Let P_k, called a partial solution which represents an assignment of binary values to a subset of the variables, be a unique path from V_0 to V_k in the enumeration tree.

Let $k = 0$; UB = &; $T^k = \varnothing$. Begin with a vertex V_k for enumeration and the enumeration tree path P_k. Initialize S_k^+, S_k^-, F_k to represent the fixed and free variables at vertex V_k.

Step 2. [Fathoming by checking the feasibility of problem (MP)]

(a) Let (Y^k, Z^k) be such that

$$Z_{jm}^k = \begin{cases} 1 & \text{if } m_1(j, m) \in S_k^+ \\ 0 & \text{if } m_1(j, m) \in S_k^- \cup F_k \end{cases}$$

and

$$Y_{jf}^k = \begin{cases} 1 & \text{if } m_2(j, f) \in S_k^+ \\ 0 & \text{if } m_2(j, f) \in S_k^- \cup F_k, \end{cases}$$

where m_1 is a one-to-one mapping of the set $\{j, m\}$ onto the set $\{1, 2, \ldots, mq\}$ and m_2 is a one-to-one mapping of the set $\{j, f\}$ onto the set $\{mq+1, \ldots, m(q+p)\}$. S_k^+, S_k^- and F_k are subsets of $\{1, 2, \ldots, m(q+p)\}$. If (Y^k, Z^k) does not satisfy

$$\left(\mathrm{UB} - \sum_i \sum_f V_{if}\right) - \sum_m \sum_j F_{jm} Z_{jm} + \sum_j \sum_f (-E_{jf} + NU_{jf}^{p}) Y_{jf} \geq 0 \qquad (V^p, U^p) \in T^k \quad (54)$$

$$-1 + \sum_j Y_{jf} \geq 0 \qquad \forall f \quad (55)$$

$$\sum_m Z_{jm} S_m - \sum_f Y_{jf} S_f \qquad \forall j \quad (56)$$

linear configuration constraints on Ys and Zs, (57)

then go to Step 2(b) else go to Step 3.

(b) Problem (MP) has no feasible solution and then the vertex V_k is fathomed. If it can not be possible to have feasible completion of the partial solution defined by S_k^+ and S_k^- then go to Step 4(a) or else go to Step 4(b).

Step 3. (Solving the primal linear programming subproblem)

(a) Solve the primal linear programming subproblem

$$W^*(Y^*, Z^*) = \sum_m \sum_j F_{jm} Z_{jm}^k + \sum_j \sum_f E_{jf} Y_{jf}^k + \min_{x \geq 0} \sum_i \sum_j \sum_f C_{ijf} X_{ijf} \quad (58)$$

$$\text{s.t.} \quad \sum_j X_{ijf} = 1 \qquad \forall i, f \quad (59)$$

$$\sum_i X_{ijf} \leq N Y_{jf}^k \qquad \forall j, f. \quad (60)$$

Denote the optimal solution by X^k. The vector (X^k, Y^k, Z^k) is a feasible solution to problem (P). If $W^*(Y^k, Z^k) < \mathrm{UB}$ then update $\mathrm{UB} = W^*(Y^k, Z^k)$ and store (X^k, Y^k, Z^k) as the incumbent solution and update the constraint set (54) to reflect the new UB.

(b) Find an optimal dual solution (V^k, U^k) for the primal LP subproblem in Step 3(a). Let $T^{k+1} = T^k \cup \{V^k, U^k\}$. Let $k = k+1$. Go to Step 4(b).

Step 4. (Backtracking)

(a) (Backtracking)
If P_k has no non-underlined elements, then go to Step 5 or else let P_j be P_k after underlining and changing the sign of its right-most non-underlined entry and erasing all elements to the right of it. The new path P_j defines a vertex V_j to which to backtrack. Go to Step 4(c).

(b) (Separation and branching)
Choose a variable for separation and select a branch to a new vertex V_j.

(c) Determine the path P_j, and the variable sets S_j^+, S_j^-, F_j.
Let $T^j = T^k$, and let $k = j$. Go to Step 2.

Step 5. (Termination)
The incumbent solution is the optimal solution to problem (P). If none, problem (P) is infeasible.

Also, by using $\mathrm{UB} - \varepsilon$ instead of UB to the constraint set (54) is made tighter. The improved computation time can be obtained.

5. A NUMERICAL EXAMPLE

A simple example can be used to illustrate the characteristics of the algorithm. The parameters and solution of such an example are presented here, both to clarify the structure of the general model and to provide a basis for later developments.

5.1. Summarized Data

Consider the following problem of a three possible facility location, two product, two facility-type example. In the following data, all costs are expressed in US$ and the unit of time is 1 month.

Using the formulation of problem (P) in Section 3, the following values are given:

(1) The fixed monthly set-up costs for opening facility type m at location j, F_{jm}, are \$4000 with production capacity 4×10^3 units/month for the first type of facility (i.e. $m = 1$) and \$2000 with production capacity 2×10^3 units/month for the second type of facility (i.e. $m = 2$) regardless of location j.

(2) The fixed monthly set-up cost for equipping a facility for product f at location j, E_{jf}, are given as follows:

E_{jf}

	f	
j	1	2
1	1815	2255
2	1975	2015
3	2215	2575

(3) The unit transportation costs of product f from a facility at j to a customer at i, C_{ijf}, are given as follows:

C_{ijf}, for $f = 1$

	j		
1	1920	2880	3840
2	960	640	1280
3	1280	1280	640

C_{ijf}, for $f = 2$

	j		
1	640	960	1280
2	2840	2560	5120
3	1280	1280	640

The formulation is obtained by using the model in Section 3 without linear configuration constraints as follows:

$$
\begin{aligned}
\min \quad & 1920X_{111} + 640X_{112} + 2880X_{121} + 960X_{122} + 3840X_{131} \\
& + 1280X_{132} + 960X_{211} + 3840X_{212} + 640X_{221} + 2560X_{222} \\
& + 1280X_{231} + 5120X_{232} + 1280X_{311} + 1280X_{312} + 1280X_{321} \\
& + 1280X_{322} + 640X_{331} + 640X_{332} + 1815Y_{11} + 2255Y_{12} \\
& + 1975Y_{21} + 2015Y_{22} + 2215Y_{31} + 2575Y_{32} + 4000Z_{11} \\
& + 2000Z_{12} + 4000Z_{21} + 2000Z_{22} + 4000Z_{31} + 2000Z_{32}
\end{aligned}
$$

$$
\begin{aligned}
\text{s.t.} \quad & X_{111} + X_{121} + X_{131} = 1 \\
& X_{112} + X_{122} + X_{132} = 1 \\
& X_{211} + X_{221} + X_{231} = 1 \\
& X_{212} + X_{222} + X_{232} = 1 \\
& X_{311} + X_{321} + X_{331} = 1 \\
& X_{312} + X_{322} + X_{332} = 1 \\
& X_{111} + X_{211} + X_{311} - 3Y_{11} \leqslant 0 \\
& X_{112} + X_{212} + X_{212} - 3Y_{12} \leqslant 0 \\
& X_{121} + X_{221} + X_{321} - 3Y_{21} \leqslant 0 \\
& X_{122} + X_{222} + X_{322} - 3Y_{22} \leqslant 0 \\
& X_{131} + X_{231} + X_{331} - 3Y_{31} \leqslant 0 \\
& X_{132} + X_{332} + X_{332} - 3Y_{32} \leqslant 0 \\
& 6Y_{11} + 6Y_{12} - 40Z_{11} - 20Z_{12} \leqslant 0
\end{aligned}
$$

$$6Y_{21}+6Y_{22}-40Z_{21}-20Z_{22}\leqslant 0$$
$$6Y_{31}+6Y_{32}-40Z_{31}-20Z_{32}\leqslant 0$$
$$X_{ijf}\geqslant 0 \qquad \forall i,j,f$$
$$Y_{jf}=0,1 \qquad \forall j,f$$
$$Z_{jm}=0,1 \qquad \forall j,m.$$

The optimal solution to this simple example is $Y_{21}=Y_{22}=1$, $Z_{22}=1$ and $X_{121}=X_{122}=X_{221}=X_{222}=X_{321}=X_{322}=1$, with objective function value 15,590 (all other decision variables equal to 0). This optimal solution implies that a total cost of \$15,590/month is required to meet all the demands when the second type of facility produces both products 1 and 2 at node 2.

5.2. *Application of Benders' Decomposition*

The example provides a convenient setting for studying the cutting planes generated in Benders' decomposition. Table 1 summarizes this cutting plane activity with the master problem and subproblem solutions at each iteration of the pure Benders' decomposition algorithm.

At iteration 1, equipment for each product is installed at node 2 which has facility type 2. The subproblem solution is 15,590, which includes the costs of facility and equipment. The dual prices from the subproblem yield the master problem constraint:

$$\begin{aligned}&W+1065Y_{11}-1295Y_{12}-1975Y_{21}-2015Y_{22}-295Y_{31}-655Y_{32}\\&-4000Z_{11}-2000Z_{12}-400Z_{21}-2000Z_{22}-4000Z_{31}-2000Z_{32}\\&\geqslant 9600,\end{aligned}\tag{61}$$

where Y_{jf} is the assignment variable for product f at node j and Z_{jm} is the assignment variable for facility type m at node j. The r.h.s. of this constraint, 9600, represents the total transportation cost of all customers' demands, excluding the costs for equipment and facility, for the assignment of equipment and facility at iteration 1. If this same assignment is substituted in this constraint, the objective value $W=15{,}590$ as expected. However, the positive technology coefficients in this constraint indicate the decremental cost to reversing the assignment of equipment and facility from iteration 1, i.e. setting $Y_{11}=1$. Since equipment 1 resides at node 1 and the objective is simply to minimize W one facility is needed at node 1 with minimum cost. Equipment 2 also resides at node 1 because at least one set of equipment for each product should exist. In other words, we have a new solution to problem (MP), namely $Y_{11}=Y_{12}=Z_{12}=1$ (all other variables equal 0).

Table 1. Cutting plane activity for a sample example

			MCHN ASSGNT/CNST COEFF*		FCLT ASSGNT/CNST COEFF†		
Iter.	LB	UB	MCHN1	MCHN2	FCLT1	FCLT2	r.h.s.
1	5990	15,590‡	0/1065	0/−1295	0/−4000	0/−2000	9600
			1/−1975	1/−2015	0/−4000	1/−2000	
			0/−295	0/−655	0/−4000	0/−2000	
2	11,830	15,990	1/−1815	1/−2255	0/−4000	1/−2000	9920
			0/−1015	0/1825	0/−4000	0/−2000	
			0/−295	0/−655	0/−4000	0/−2000	
3	12,870	19,590	0/3945	0/3505	0/−4000	0/−2000	12,800
			0/905	0/5665	0/−4000	0/−2000	
			1/−2215	1/−2575	0/−4000	1/−2000	
4	14,550	16,800	1/−1815	0/−1295	0/−4000	1/−2000	8960
			0/−1015	1/−2015	0/−4000	1/−2000	
			0/295	0/−655	0/−4000	0/−2000	
5	15,590	15,590‡	0/1065	0/−1295	0/−4000	0/−2000	9600
			1/−1975	1/−2015	0/−4000	1/−2000	
			0/−295	0/−655	0/−4000	0/−2000	

* A/B denotes machine allocation A and resulting constraint coefficient B: the first two such entries apply to node 1, the second two to node 2 and the third two to node 3.

† C/D denotes facility location C and resulting constraint coefficient D: the first two such entries apply to node 1, the second two to node 2 and the third two to node 3.

‡ Indicates a new incumbent solution.

Substituting this solution in constraint (61) yields:

$$W + 1065 - 1295 - 2000 \geqslant 9600 \rightarrow W = 11{,}830.$$

Therefore W would be 11,830.

At iteration 2, the equipment assignment in the dual subproblem is the one just derived, resulting in a subproblem cost of 15,990 and constaint (62):

$$\begin{aligned} W &- 1815Y_{11} - 2255Y_{12} - 1015Y_{21} + 1825Y_{22} - 4000Z_{31} - 655Y_{32} \\ &- 4000Z_{11} - 2000Z_{12} - 4000Z_{21} - 2000Z_{22} - 4000Z_{31} - 2000Z_{32} \\ &\geqslant 9920. \end{aligned} \tag{62}$$

The optimal solution to the master problem with this additional constraint sets $Y_{31} = Y_{32} = Z_{32} = 1$ and the remaining variables to 0, yielding:

$$W - 295 - 655 - 2000 \geqslant 9600 \rightarrow W \geqslant 12{,}550 \tag{61}$$

$$W - 295 - 655 - 2000 \geqslant 9920 \rightarrow W \geqslant 12{,}870. \tag{62}$$

Therefore, the optimal master problem solution at the end of iteration 2 is 12,870. This procedure applies in the remaining iterations, although it is more difficult to follow manually due to the number of constraints in the master problem.

From the master problem formulation in the previous section, it is apparent that each master problem's additional constraint may be simplified to

$$W + \sum_j \sum_f (-E_{jf} + NU^{p}_{jf})Y_{jf} - \sum_m \sum_j F_{jm}Z_{jm} \geqslant \sum_i \sum_f V^{p}_{if} \qquad (V^{p}, U^{p}) \in T^{k}. \tag{63}$$

If $Y_{jf} = Z_{jm} = 1$ at a particular iteration of the dual subproblem solution procedure, the coefficient of Y_{jf} and Z_{jm} in the resulting constraint (63) must be negative. A negative coefficient at the next iteration indicates that the preferred choice for this variable, considering this constraint alone, is $Y_{jf} = Z_{jm} = 0$, i.e. the opposite of its current value. If $Y_{jf} = Z_{jm} = 0$ in the current iteration, then the resulting constraint coefficient may be positive. If the resulting coefficient is not positive, the decreased cost due to deleting this equipment or facility does not offset the cost for installing that equipment or the cost for opening that facility, and therefore it should not be assigned.

At each iteration, the dual subproblem generates a maximally violated constraint to be added to the master problem. This constraint or cutting plane eliminates infeasible values of W from the solution space. However, no reduction in the space of feasible assignments of equipment and facility occurs. That is, no suboptimal assignments of equipment are identified and eliminated. The recurrence of a constraint already generated at an earlier iteration must therefore signal that an optimal solution has been found, since no further reduction in the space of feasible objective function values will occur. This situation arises in Table 1 at iteration 5, hence the incumbent solution is optimal. In this problem, therefore, only 5 of the 2^{12} possible cutting planes were generated.

5.3. *Computational Results*

The example has $N(P+Q) = 12$ integer variables, $N^2P = 18$ continuous variables and $N(2P+1) = 15$ functional constraints in the problem. It has an optimal solution with a value of \$15,590/month. The optimal configuration for this facility location problem consists of a facility type 2 at node 2 (the most active node) and one set of equipment for each product at node 2 (see problem No. 97). In other words, the centralized configuration is the most economical one for this problem. Given this solution, the corresponding dual variables determine the quantities NU^{p}_{jf} which are the values at the margin of adding a set of equipment for product f at node 2. These values are again equal to the reduction in the transportation cost to all demand points from that node 2 which can be realized if a set of equipment for product f is placed at node 2. If the transportation costs C_{ijf} are fairly equal for a given product f the quantities U_{jf} will be relatively small. The coefficient of Y_{jf}, $-E_{jf} + NU^{p}_{jf}$, is the net benefit of adding a set of equipment for product f after subtracting the fixed set-up costs for equipping facility to produce product f. Clearly, a sufficient condition for a centralized optimal solution is that the coefficients of the Y_{jf}s are all negative for every dual constraint. This is, obviously, the case of this problem, No. 97. The

CPU time to solve this problem is 1.01 sec on the IBM 4381 Model Group 14 System at the University of Iowa.

In this study, some attempt is made to see when a distributed system might be more economical. As it turns out, substantial changes in one or more of the problem parameters are required to change the optimal solution for this problem, No. 97 in Table 2, from a purely centralized one to a distributed one. Problem No. 98 in Table 2 is obtained by reducing the fixed set-up costs for opening facilities by one-tenth. This problem needs a distributed configuration to minimize the total cost. Therefore, node 1 has the type 2 facility with equipment for product 1, while node 2 has the type 2 facility with equipment for product 2. Notice that, in this configuration, there is still only one single set of equipment for each product. If we have products whose equipment is 6 times cheaper and 10 times more demand, transportation costs which are 5 times higher and a fixed set-up cost for opening the facility which is one-tenth of original charge, this problem, No. 99, will have an optimal solution involving duplicated sets of equipment for each product. The summarized results of the optimal solutions of the above three problems are tabulated in Table 2.

Generally, the factors that favor distributed solutions are those that make the transportation cost coefficients C_{ijf} relatively large and that make the fixed set-up cost coefficients E_{jf} and F_{jm} relatively small. Thus, high transportation costs, high demand rates, low fixed set-up cost for installing equipment and low fixed charges for facilities all favor distributed configurations. Clearly, these observations coincide with the interpretation of physical parameters and configurations in the facility location problem.

6. COMPUTATIONAL EXPERIENCE

The embedded Benders' decomposition algorithm was coded in Fortran and implemented on a computer to obtain some computational experience. The code consists of approx. 950 lines, and its listing will be provided upon request. The algorithm is tested on problems varying in size from $N = 3$ to 10, $F = 2$ to 6 and $M = 1$ to 3, with satisfying results.

In order to evaluate the performance of the algorithm, we have actually solved the same randomly generated problems using different algorithms rather than comparing the implemented algorithm with other algorithms on the execution times for incomparable problems, since a standard set of test problems is not available in the literature.

Table 3 displays summarized information for a set of problems which are generated randomly. The column headings stand for number of integer variables ($N1$), number of continuous variables ($N2$), number of constraints in problem (P) (NCON) and CPU time, in sec, needed to solve the problem by the embedded Benders' decomposition algorithm (BEN), by a branch-and-bound algorithm (B&B) [31] and by an implicit enumeration algorithm (ENU) [32], on the IBM 4381 Model Group 14 System at the University of Iowa. Unfortunately, no other published computational results for problem (P) are available for comparison. However, it can be concluded that the performance of the embedded Benders' algorithm is quite satisfactory and encouraging considering the CPU times in Table 3. The standard branch-and-bound algorithm and implicit enumeration algorithm were used just for reference purposes.

The CPU times needed to solve the problems within this size are < 500 sec. Thus, the algorithm solves problems of practical sizes in a reasonable amount of time. From Table 3, it is apparent that the CPU times for the algorithm presented here are remarkably less than those for both the branch-and-bound and implicit enumeration algorithms. As the problem size becomes larger, Benders' decomposition algorithm outperforms other algorithms in both the time and space dimensions by a considerable margin. Also, the memory requirements for the algorithm are very small compared to those for other algorithms.

7. CONCLUSIONS

The embedded Benders' decomposition algorithm is an effective method for solving the multiproduct capacitated facility location problem with a choice of facility type. It outperforms, in both the execution time and space dimension, the standard branch-and-bound algorithm for mixed integer linear programs and the implicit enumeration algorithm by very large margins. The results occur because the decomposition approach maintains the special structure in both the linear

Table 2. Summarized results of the optimal solutions for a sample example

ID	N	F	M	$FMAX$	EQUP ALLOCAT*		FCLT LOCAT†		NF	NC	OPT
					EQUP1	EQUP2	TYPE1	TYPE2			
97	3	2	2	3	0	0	0	0	2	1	15,590
					1	1	0	1			
					0	0	0	0			
98	3	2	2	3	1	0	0	1	2	2	13,190
					0	1	0	1			
					0	0	0	0			
99	3	2	2	3	1	1	0	1	6	3	289,294
					1	1	0	1			
					1	1	0	1			

ID = problem number. $N/F/M$ = number of nodes/number of distinct products/number of facility types. $FMAX$ = maximum number of sets of each equipment allowed.
*Denotes optimal equipment allocation; the first two entries apply to node 1, the second two to node 2 and the third two to node 3.
†Denotes optimal facility location; the first two entries apply to node 1, the second two to node 2 and the third two to node 3.
NF = total number of equipment sets in the optimal solution (each equipment for a product is counted separately). NC = number of facilities in the optimal solution. OPT = optimal objective function value.

Table 3. Summarized information for a set of random problems

ID	$N/F/M$	FMAX	$N1/N2/NCON$	BEN	B&B	ENU
1	3/2/1	3	9/ 18/15	0.36	1.36	4.87
2	3/2/1	3	9/ 18/15	0.33	2.57	2.87
3	3/2/2	3	12/ 18/15	2.21	7.26	9.70
4	3/2/1	3	12/ 18/15	1.85	6.89	3.21
5	3/3/1	3	12/ 27/21	1.56	7.18	419.22
6	3/3/1	3	12/ 27/21	0.60	2.50	591.85
7	3/3/2	3	15/ 27/21	3.36	49.99	468.14
8	3/3/2	3	15/ 27/21	3.98	39.97	821.29
9	3/4/1	3	15/ 36/27	4.20	33.15	3618.88
10	3/4/1	3	15/ 36/27	4.90	27.58	≪
11	3/4/2	3	18/ 36/27	25.55	150.31	≪
12	3/4/2	3	18/ 36/27	16.62	110.04	≪
13	3/4/3	3	21/ 36/27	64.30	294.92	≪
14	3/4/3	3	21/ 36/27	117.15	1624.65	≪
15	3/5/1	3	18/ 45/33	5.82	71.31	≪
16	3/5/1	3	18/ 45/33	5.74	59.81	≪
17	3/5/2	3	21/ 45/33	64.41	346.25	≪
18	3/5/2	3	21/ 45/33	62.60	483.49	≪
19	3/5/3	3	24/ 45/33	325.36	≪	≪
20	3/6/1	3	21/ 54/39	191.63	≪	≪
21	3/6/2	3	27/ 54/39	267.93	≪	≪
22	4/2/1	4	12/ 32/20	0.85	13.32	354.23
23	4/2/2	4	16/ 32/20	4.25	98.41	250.76
24	4/3/1	4	16/ 48/28	8.41	33.53	≪
25	4/3/2	4	20/ 48/28	13.84	486.53	≪
26	4/4/1	4	20/ 64/36	34.46	≪	≪
27	4/4/2	4	24/ 64/36	475.70	≪	≪
28	4/4/3	4	28/ 64/36	803.33	≪	≪
29	4/5/1	4	24/ 80/44	372.44	≪	≪
30	5/2/1	5	15/ 50/25	1.51	52.63	≪
31	5/2/2	5	20/ 50/25	46.70	≪	≪
32	5/3/1	5	20/ 75/35	26.23	≪	≪
33	5/3/2	5	25/ 75/35	29.11	≪	≪
34	5/3/3	5	30/ 75/35	232.44	≪	≪
35	6/2/1	6	18/ 72/30	3.55	453.45	≪
36	6/2/2	6	24/ 72/30	13.30	≪	≪
37	6/3/1	6	24/108/42	29.27	≪	≪
38	7/2/1	3	21/ 98/35	12.38	≪	≪
39	7/2/2	3	28/ 98/35	100.26	≪	≪
40	7/3/1	3	28/147/49	70.21	≪	≪
41	8/2/1	4	24/128/40	13.99	≪	≪
42	9/2/1	4	27/162/45	61.97	≪	≪
43	10/2/1	5	30/200/50	143.09	≪	≪

≪ Indicates that the CPU time exceeds 5000 sec.

and integer subproblems and exploits this special structure. The algorithm solves problems of practical sizes in very acceptable times. The implementation of the algorithm provides flexible means to model various special configurations and to limit the search to any portion of the enumeration tree. Finally, this algorithm, like other enumeration algorithms, is a primal algorithm. Once a feasible solution has been obtained, it provides successively better ones and the process can be interrupted at any time. Very often, optimal solutions are generated in the early stages of its execution. In such cases, the rest of the time is spent making small improvements and/or verifying optimality.

Although the model proposed here is comprehensive, it is by no means complete. Several issues such as capacity of route, are not considered in the model. Also, since no proper solution algorithm has been reported previously for the problem of interest, hopefully the proposed solution algorithm becomes a basis upon which an improved algorithm can be developed or other future solution methods can be evaluated. One possibility is the cross decomposition method. The algorithm may be improved by using cross decomposition, developed by Van Roy [33, 34]. Van Roy has noted the possibility of using "cross decomposition" to couple a primal decomposition method, such as Benders' decomposition, to a dual one, such as a Lagrangian relaxation. The method is designed to exploit both the primal and the dual structure simultaneously, which is accomplished by using two different restrictions of the problems, yielding, respectively, the primal and the dual subproblem. It unifies Benders' decomposition and Lagrangian relaxation into a single framework. The early generation of good feasible solutions using a primal decomposition might considerably trim the branch-and-bound tree for a Lagrangian relaxation. Switching to the dual procedure when the convergence rate of the primal formulation dropped to a predetermined level might then facilitate convergence and reduce the overall solution time.

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