Fourier and Schur Multipliers of Locally Compact Groups

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Declaration

The work in this thesis is my own except where otherwise stated.

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Statement of Originality

Chapter 1 is chiefly a compilation of material from [9], [29], and [10]. The exposition and arguments in Chapter 2 are in many cases my own, but there are no new results obtained. In Section 2.1, the definition of the extended trace comes from [34]. The approaches in Section 2.2 are in part informed by [9] and [10]. The overarching approach in Section 2.3 is inspired by [21], with important ideas coming from [36], [2] and [7]. In Chapter 3, the exposition on the left-regular representation largely follows [16], the remaining presentation of the group von Neumann algebra takes inspiration from [1], and the proofs of the trace properties are adapted from [9] (and [1], which is itself seemingly modelled off [9]). The ideas in Section 4.1 are globally based on [20]. My proof of Theorem 4.9 for the most part follows [20], and I have written these arguments out in much greater detail. Similarly, my proof of Theorem 4.32 closely follows and adds detail to the argument in [5]. In the remaining parts of Chapter 4, I have critically examined and modified the methods in [5] to produce simpler constructions and proofs in the case of second-countable, unimodular groups.

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Abstract

This thesis explores the relationship between Fourier and Schur multipliers on second-countable, unimodular groups. We examine Caspers and de la Salle's result [5], that for amenable groups there is an equality between the completely bounded norms of these multipliers on their respective noncommutative L^p spaces.

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$B(\mathcal{H})$	the $*$-algebra of bounded operators on $\mathcal{H}$, page 3
$B_h(\mathcal{H})$	the self-adjoint operators in $B(\mathcal{H})$, page 3
$B_r(\mathcal{H})$	the collection of bounded operators on $\mathcal{H}$ whose operator norm is at most r, page 3
$C_b(G)$	the space of bounded continuous complex-valued functions on G, page 65
E_x	the spectral measure associated with a self-adjoint operator x, page 13
G	a locally compact group, page 47
$L^p(\mathcal{M}, \tau)$	the L^p space associated to a von Neumann algebra $\mathcal{M}$ with faithful, normal, semifinite trace τ, page 22
M_ϕ	the Schur multiplier associated with the function ϕ, page 64
$P(\mathcal{M})$	the set of orthogonal projection in $\mathcal{M}$, page 6
T_ϕ	the Fourier multiplier associated with a function ϕ, page 75
$U(\mathcal{M})$	the set of unitary operators in $\mathcal{M}$, page 6
$\mathbf{1}_{\mathcal{H}}$,$\mathbf{1}_{B(\mathcal{H})}$,$\mathbf{1}$	the identity map on $\mathcal{H}$, page 3
$\mathcal{H}$	a separable Hilbert space, page 3
$\mathcal{M}$	a von Neumann algebra, page 6
$\mathcal{M}_+$	the set of positive operators in $\mathcal{M}$, page 8
$\mathbb{Q}_{>1}$	the set of rationals greater than 1, page 80
$\mathbf{T}$	the unit circle in $\mathbb{C}$, page 47

Introduction

Fourier multipliers abound in harmonic analysis and have been studied extensively in both classical and abstract settings. Most commonly, one considers Fourier multipliers as operators on $L^p(\mathbb{R}^n)$ or $L^p(\mathbf{T}^n)$, where $\mathbf{T}$ denotes the unit circle in $\mathbb{C}$ with normalised surface measure. Similarly, for any locally compact abelian group, one defines a Fourier transform, sending functions on the group to functions on the so-called Pontryagin dual, and thereby one defines Fourier multipliers on locally compact abelian groups. The suitable abstraction of such multipliers to nonabelian groups was introduced by Eymard [11], by means of a Banach algebra deemed the Fourier algebra of the group. Crucially, the Fourier algebra's dual space can be identified with the group von Neumann algebra, a $*$-algebra of bounded operators on a Hilbert space. Therefore, the Fourier multipliers of harmonic analysis dualise to maps on the group von Neumann algebra, and in the parlance of operator theory, these dual maps are termed Fourier multipliers. This operator-theoretic perspective has proven fruitful (see, for example, [8]) and gives rise to the notion of Fourier multipliers that are completely bounded. A major result in the study of such multipliers was achieved by Bożejko and Fendler [3], who characterised completely bounded Fourier multipliers in terms of the following simple operation on matrices.

Given a set X, let $\ell^2(X)$ denote the Hilbert space of square-summable complex-valued functions on X, and observe that any bounded linear operator on $\ell^2(X)$ can be written as a matrix $(A_{ij})_{i,j\in X}$. Classically, a Schur multiplier is a continuous mapping between such operators, acting via the entrywise matrix transformation $(A_{ij})_{i,j\in X} \mapsto (\psi(i,j)A_{ij})_{i,j\in X}$, for some function $\psi : X \times X \to \mathbb{C}$. Replacing X with a discrete group G, Herz [14] considered the class of Schur multipliers where $\psi(s,t) = \phi(st^{-1})$, for some function $\phi : G \to \mathbb{C}$, which we say defines a Herz-Schur multiplier. One can extend Herz's definition to nondiscrete locally compact groups by replacing matrix entries with integral kernels.

Bożejko and Fendler [3] proved that for any locally compact group, the space of functions defining Herz-Schur multipliers coincides with the space of functions defining completely bounded Fourier multipliers and that the completely bounded norms of these multipliers coincide. Later, Jolissaint [15] produced an alternative proof of this result, using what is essentially Grothendieck's inequality [26].

Herz-Schur and Fourier multipliers act on two different von Neumann algebras. Herz-Schur multipliers act on the von Neumann algebra of bounded operators on a particular Hilbert space, and Fourier multipliers act on the group von Neumann algebra. A central idea in the study of von Neumann algebras is that they are a noncommutative (that is, operator-theoretic) analogue to the space of essentially bounded functions on a measure space. Segal [30] continued this analogy by introducing noncommutative L^p spaces associated to a von Neumann algebra. These spaces provide a setting to study noncommutative abstractions of L^p-Fourier multipliers. This thesis will examine the extensions of Herz-Schur and Fourier multipliers to the noncommutative L^p spaces associated with their respective von Neumann algebras.

For discrete, amenable groups, the L^p-extensions of Herz-Schur and Fourier multipliers were shown to have the same completely bounded norms by Neuwirth and Ricard [23], partially extending Bożejko and Fendler's result to L^p spaces. More recently, Caspers and de la Salle [5] extended Neuwirth and Ricard's result to all amenable locally compact groups.

This thesis explores Caspers and de la Salle's result under the additional assumptions that the underlying group is second-countable and unimodular. There are four reasons to impose these assumptions: they are satisfied for large classes of interesting groups (in particular, any discrete, compact, or abelian locally compact group is unimodular); second-countable groups have convenient measure-theoretic properties; the von Neumann algebras associated to such groups admit faithful, normal, semifinite traces, providing an especially elegant construction of noncommutative L^p spaces; and, with these assumptions, we can meaningfully simplify Caspers and de la Salle's argument.

Chapter 1 contains a cursory overview of some background material, before we characterise noncommutative L^p spaces in Chapter 2. Chapter 3 introduces topological groups, as well as the group von Neumann algebra and its associated trace. Finally, in Chapter 4, we precisely define Herz-Schur and Fourier multipliers, and we prove the two key results in Caspers and de la Salle's paper.

Chapter 1

Preliminaries on Operators

In this chapter, we will present a terse outline of the general theories of von Neumann algebras and unbounded operators. The combination of these two matters leads us to the algebra of τ-measurable operators, which provides a suitable noncommutative setting to analogise the classical L^p spaces. Along the way, we will encounter several important notions for this thesis, including functional calculus and the partial order on self-adjoint operators. For more detailed expositions on this chapter's material, we direct the reader to [9], [29], and [10].

Throughout this thesis, let $\mathcal{H}$ be a separable Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and let $B(\mathcal{H})$ be the $*$-algebra of bounded (linear) operators on $\mathcal{H}$, with operator norm $\|\cdot\|_{L^\infty}$. For any $r \geq 0$, let $B_r(\mathcal{H})$ denote the closed ball of radius r centred at 0 in $B(\mathcal{H})$, and let $\mathbf{1}_{\mathcal{H}}$ (or $\mathbf{1}_{B(\mathcal{H})}$, or, when unambiguous, even simply $\mathbf{1}$) denote the identity map on $\mathcal{H}$. Also, let $B_h(\mathcal{H})$ denote the real vector space of self-adjoint operators on $\mathcal{H}$.

1.1 von Neumann Algebras

Von Neumann algebras play a central role in this thesis, by providing domains for Fourier and Herz-Schur multipliers to act on, as well as a foundation for noncommutative integration. Such algebras interweave algebraic and topological properties. To begin with, we will consider some operator topologies, which interact powerfully with the algebraic notion of commutativity. This interaction will lead us to several equivalent definitions of a von Neumann algebra. After examining some properties of these algebras, we define traces, which will lead to the notion of noncommutative integration in Chapter 2.

1.1.1 Topologies on $B(\mathcal{H})$

The operator topologies in this section will play a central role in the characterisation of von Neumann algebras, and they will appear repeatedly in this thesis. To define these topologies, we will simply characterise their convergent nets.

Definition 1.1. Let $\{x_\alpha\}$ be a net in $B(\mathcal{H})$ and let $x \in B(\mathcal{H})$. We say that $\{x_\alpha\}$ converges to x

(i) in the **strong operator topology** if for every $\xi \in \mathcal{H}$, $x_\alpha \xi \to x\xi$;

(ii) in the **weak operator topology** if for every $\xi, \eta \in \mathcal{H}$, $\langle x_\alpha \xi, \eta\rangle \to \langle x\xi, \eta\rangle$;

(iii) in the **σ-strong operator topology** if for every sequence $\{\xi_k\}_{k=1}^\infty$ in $\mathcal{H}$ with $\sum_{k=1}^\infty \|\xi_k\|_{\mathcal{H}}^2 < \infty$, we have $\sum_{k=1}^\infty \|x\xi_k - x_\alpha \xi_k\|_{\mathcal{H}}^2 \to 0$;

(iv) in the **σ-weak operator topology** if for any pair of sequences $\{\xi_k\}_{k=1}^\infty$, $\{\eta_k\}_{k=1}^\infty$ in $\mathcal{H}$ with $\sum_{k=1}^\infty \|\xi_k\|_{\mathcal{H}}^2 < \infty$ and $\sum_{k=1}^\infty \|\eta_k\|_{\mathcal{H}}^2 < \infty$, we have $\sum_{k=1}^\infty |\langle x\xi_k - x_\alpha \xi_k, \eta_k\rangle| \to 0$.

Each of these defines a Hausdorff topology on $B(\mathcal{H})$, by declaring a set closed if it contains its own limit points (for details, see Subsection 3.1 in Part I of [9]). In practice, however, we will analyse these topologies via their definitions of convergence. For instance, a comparison of the above conditions yields the following result.

Proposition 1.2. *The σ-strong topology is the finest of the above topologies, and the weak topology is the coarsest. Moreover, the operator norm topology is finer than all of them. This is captured in the following diagram, which can also be found in [4] and [9]. Here, $>$ means "is coarser than". Moreover, for any $r > 0$, the strong and σ-strong topologies coincide on $B_r(\mathcal{H})$, as do the weak and σ-weak topologies.*

$$\begin{array}{ccccc} \sigma\text{-weak} & > & \sigma\text{-strong} & > & \text{operator norm} \\ \wedge & & \wedge & & \\ \text{weak} & > & \text{strong} & & \end{array}$$

Importantly, each of these topologies respects the vector space structure of $B(\mathcal{H})$. That is, addition and scaling are continuous operations with respect to each of these topologies. Although multiplication is not bicontinuous in general (see Exercise 3.2 in Part I of [9]), the following facts are easily verified.

Proposition 1.3. *With respect to any of the above topologies, the multiplication map from $B(\mathcal{H}) \times B(\mathcal{H})$ to $B(\mathcal{H})$ is single-variable continuous in each variable. Moreover, for any $r > 0$, the multiplication map restricted to $B_r(\mathcal{H}) \times B(\mathcal{H})$ is jointly continuous with respect to the strong operator topology.*

Finally, since the norm topology is finer than any of the topologies in Definition 1.1, the closed graph theorem implies the following result.

Proposition 1.4. *Suppose $V \subset B(\mathcal{H})$ is a closed subspace with respect to the operator norm, and let $T : V \to B(\mathcal{H})$ be a linear map. If T is continuous with respect to any of the topologies in Definition 1.1, then T is norm-bounded.*

1.1.2 The Bicommutant Theorem

In this subsection, we provide both algebraic and topological characterisations of von Neumann algebras, and we introduce some of their elementary properties.

Definition 1.5. Let $\mathcal{M}$ be a subset of $B(\mathcal{H})$. The **commutant** of $\mathcal{M}$, denoted $\mathcal{M}'$, is the set of operators $x \in B(\mathcal{H})$ such that $xy = yx$ for every $y \in \mathcal{M}$.

When $\mathcal{M}$ is $*$-invariant, in that it contains its own adjoints, one easily sees that $\mathcal{M}'$ is a unital $*$-subalgebra of $B(\mathcal{H})$. Consider now the **bicommutant** of $\mathcal{M}$, $\mathcal{M}'' := (\mathcal{M}')'$. It is clear that $\mathcal{M} \subset \mathcal{M}''$, and we ask under what conditions the equality $\mathcal{M} = \mathcal{M}''$ holds. To answer this, we refine the class of subsets $\mathcal{M} \subset B(\mathcal{H})$ under consideration.

Definition 1.6. A $*$-subalgebra $\mathcal{M}$ in $B(\mathcal{H})$ is called **nondegenerate** if the set $\{x\xi \in \mathcal{H} : x \in \mathcal{M}, \xi \in \mathcal{H}\}$ is dense in $\mathcal{H}$.

Any unital $*$-subalgebra of $B(\mathcal{H})$ is nondegenerate, since $\mathbf{1}_{\mathcal{H}}$ is surjective. Hence, if $\mathcal{M}$ contains its own adjoints and $\mathcal{M} = \mathcal{M}''$, it must be the case that $\mathcal{M}$ is nondegenerate. Therefore, the following remarkable theorem answers the question we posed in the previous paragraph.

Theorem 1.7 (von Neumann's Bicommutant Theorem, Theorem 2.4.11 in [4])**.** *Let $\mathcal{M}$ be a nondegenerate $*$-subalgebra of $B(\mathcal{H})$. The following are equivalent:*

(i) $\mathcal{M} = \mathcal{M}''$;

(ii) $\mathcal{M}$ *is weakly closed;*

(iii) $\mathcal{M}$ *is σ-weakly closed;*

(iv) $\mathcal{M}$ *is strongly closed;*

(v) $\mathcal{M}$ *is* σ*-strongly closed.*

A unital $*$-subalgebra of $B(\mathcal{H})$ satisfying these conditions is called a **von Neumann algebra.** For any subset $\mathcal{M} \subset B(\mathcal{H})$ containing its own adjoints, we have already seen that $\mathcal{M}' \subset \mathcal{M}'''$, and since $\mathcal{M} \subset \mathcal{M}''$, any operator that commutes with $\mathcal{M}''$ must also commute with $\mathcal{M}$. That is, $\mathcal{M}''' \subset \mathcal{M}'$, so $\mathcal{M}'$ is a von Neumann algebra. In particular, $\mathcal{M}''$ is a von Neumann algebra containing $\mathcal{M}$. When $\mathcal{M}$ is a nondegenerate $*$-subalgebra of $B(\mathcal{H})$, the following proposition implies that $\mathcal{M}''$ is the smallest such von Neumann algebra.

Proposition 1.8 (Lemma 3.4.6 in Part I of [9])**.** *If* $\mathcal{M}$ *is a nondegenerate* $*$*-subalgebra of* $B(\mathcal{H})$*, then* $\mathcal{M}$ *is* σ*-strongly dense in* $\mathcal{M}''$*.*

Examples 1.9. A key motivating example in the study of von Neumann algebras, and the example that informs our definition of noncommutative L^p spaces, is the following. Let X be a σ-finite measure space and $L^\infty(X)$ the space of essentially bounded functions on X. To each $f \in L^\infty(X)$, associate a multiplication operator $m_f \in B(L^2(X))$, given by $m_f\xi(s) = f(s)\xi(s)$ for all $\xi \in L^2(X)$ and $s \in X$. The map $f \mapsto m_f$ is an isometric, unital $*$-homomorphism $L^\infty(X) \to B(\mathcal{H})$, and its image set is a von Neumann algebra. In fact, any commutative von Neumann algebra on a separable Hilbert space is $*$-isomorphic to $L^\infty(X)$ for some measure space X (see Theorem 4.4.4 in [22]). Therefore, when constructing noncommutative L^p spaces, von Neumann algebras will play the role of L^∞.

Importantly for this thesis, $B(\mathcal{H})$ is a von Neumann algebra. (Obviously, it is closed in any of the topologies in Definition 1.1.) We will construct another important von Neumann algebra in Chapter 3.

Herein, let $\mathcal{M}$ denote a von Neumann algebra in $B(\mathcal{H})$, let $P(\mathcal{M})$ be the set of projections in $\mathcal{M}$, and let $U(\mathcal{M})$ be the set of unitaries in $\mathcal{M}$. When we explore spectral theory in Subsection 1.2.1, we will construct many projections in $P(\mathcal{M})$. For now, we merely introduce some notation and make an observation.

Definition 1.10. For any projection p on $\mathcal{H}$, let $p^\perp := \mathbf{1}_\mathcal{H} - p$. That is, $p^\perp$ is the orthogonal projection onto $\operatorname{ran}(p)^\perp$. Also, for any set of projections $\{p_i\}_{i\in I}$ on $\mathcal{H}$, let $\bigwedge_{i\in I} p_i$ denote the orthogonal projection onto $\bigcap_{i\in I} \operatorname{ran}(p_i)$, and let $\bigvee_{i\in I} p_i$ denote the orthogonal projection onto the closed subspace of $\mathcal{H}$ generated by $\bigcup_{i\in I} \operatorname{ran}(p_i)$. That is, $\bigvee_{i\in I} p_i = \left(\bigwedge_{i\in I} p_i^\perp\right)^\perp$.

For any $x \in B(\mathcal{H})$ and any projection p on $\mathcal{H}$, the condition $xp = px$ says exactly that $x(\mathrm{ran}(p)) \subset \mathrm{ran}(p)$ and $x(\mathrm{ran}(p)^\perp) \subset \mathrm{ran}(p)^\perp$. Therefore, $p \in P(\mathcal{M})$ if and only if these inclusions hold for every $x \in \mathcal{M}'$. We deduce the following.

Proposition 1.11. *For any $p \in P(\mathcal{M})$, we have $p^\perp \in P(\mathcal{M})$. Also, for any set $\{p_i\}_{i\in I}$ in $P(\mathcal{M})$, we have $\bigwedge_{i\in I} p_i \in P(\mathcal{M})$ and $\bigvee_{i\in I} p_i \in P(\mathcal{M})$.*

The set $U(\mathcal{M})$ is important, in part because it generates $\mathcal{M}$.

Proposition 1.12 (Proposition 1.3.3 in Part I of [9])**.** *Every operator $x \in \mathcal{M}$ can be written as a linear combination of unitaries in $\mathcal{M}$.*

Finally, we record a convenient description of the predual of $\mathcal{M}$. As proven by Sakai [28], the existence of such a predual distinguishes von Neumann algebras amongst C^*-algebras.

Definition 1.13. Let $\mathcal{M}_*$ be the set of all σ-weakly continuous linear functionals on $\mathcal{M}$.

Since any σ-weakly continuous linear functional is also norm-continuous, $\mathcal{M}_*$ is a linear subspace of the Banach dual $\mathcal{M}^*$. In fact, the following holds.

Proposition 1.14. *The space $\mathcal{M}_*$ is a Banach subspace of $\mathcal{M}^*$. Moreover, the pairing $\langle \phi, y\rangle = \phi(y)$, with $\phi \in \mathcal{M}_*$ and $y \in \mathcal{M}$, determines an isometric isomorphism $\mathcal{M} \cong (\mathcal{M}_*)^*$. Under this isomorphism, the σ-weak topology on $\mathcal{M}$ coincides with the weak-* topology on $(\mathcal{M}_*)^*$.*

Proof. For the first two statements, see Theorem 2.6 in Chapter II of [34], and for the topological assertion, see Theorem 3.5 in Chapter III of [34]. □

For this reason, $\mathcal{M}_*$ is called the **predual** of $\mathcal{M}$.

1.1.3 Partial Ordering on Self-Adjoint Operators

To understand traces on von Neumann algebras, we must first examine the notion of positivity and the induced partial ordering on $B_h(\mathcal{H})$.

Definition 1.15. An operator $x \in B(\mathcal{H})$ is said to be **positive** if $\langle x\xi, \xi\rangle \geq 0$ for every $\xi \in \mathcal{H}$. In this case, we write $x \geq 0$. By polarisation, it follows that x is self-adjoint. We note that for any $x \in B(\mathcal{H})$, $x^*x \geq 0$.

Given two self-adjoint operators $x, y \in B_h(\mathcal{H})$, we write $x \leq y$ if $y - x \geq 0$. That is, $x \leq y$ precisely when $\langle x\xi, \xi\rangle \leq \langle y\xi, \xi\rangle$ for every $\xi \in \mathcal{H}$. This defines a partial ordering on $B_h(\mathcal{H})$, some basic properties of which are collected below.

Proposition 1.16. *Suppose $x, y \in B_h(\mathcal{H})$ satisfy $x \leq y$.*

(i) For any $z \in B(\mathcal{H})$, we have $zxz^* \leq zyz^*$.

(ii) For any $z \in B_h(\mathcal{H})$, we have $x + z \leq y + z$.

(iii) For any $\alpha \in \mathbb{R}_{\geq 0}$, $\alpha x \leq \alpha y$ and $-\alpha y \leq -\alpha x$.

(iv) $x \leq \|x\|_{L^\infty} \mathbf{1}_{\mathcal{H}}$.

(v) If x is positive and invertible, then y is invertible and $0 \leq y^{-1} \leq x^{-1}$.

Proof. The first four facts are trivial. For (v), see Proposition 4.2.8 in [17]. □

Example 1.17. For any orthogonal projections p and q, it is routine to check that $p \leq q$ if and only if $\mathrm{ran}(p) \subset \mathrm{ran}(q)$.

Proposition 1.18 (Appendix II in [9])**.** *With respect to the partial order defined above, let $\{x_\alpha\}$ be an increasing (resp. decreasing) net in $B_h(\mathcal{H})$. If there exists an operator $y \in B_h(\mathcal{H})$ such that $x_\alpha \leq y$ (resp. $y \leq x_\alpha$) for every α, then the net $\{x_\alpha\}$ has a supremum (resp. infimum) in $B_h(\mathcal{H})$. Moreover, this supremum (resp. infimum) is the strong limit of the net $\{x_\alpha\}$.*

We will write $x_\alpha \nearrow x$ if $\{x_\alpha\}$ is an increasing net in $B_h(\mathcal{H})$ with supremum x and $x_\alpha \searrow x$ if $\{x_\alpha\}$ is a decreasing net in $B_h(\mathcal{H})$ with infimum x. Since the von Neumann algebra $\mathcal{M}$ is strongly closed, it must contain its suprema and infima.

Henceforth, we write $\mathcal{M}_+$ for the set of positive elements in $\mathcal{M}$. This set has the following special properties.

Proposition 1.19. *If $x \in \mathcal{M}_+$, then there exists a unique element $x^{1/2} \in \mathcal{M}_+$ satisfying $(x^{1/2})^2 = x$.*

Proof. This is proven for C^*-algebras (of which von Neumann algebras are a subclass) in Theorem 2.2.1 in [22]. We will explain this further once we define functional calculus in Subsection 1.2.1. □

Proposition 1.20 (Lemma 2.1.6 in Part I of [9])**.** *Suppose $x, y \in \mathcal{M}_+$ with $x \leq y$. Then there exists a unique $a \in B(\mathcal{H})$ such that* $\ker(y) \subset \ker(a)$ *and $x^{1/2} = ay^{1/2}$. Moreover, we have $a \in \mathcal{M}$ and $\|a\|_{L^\infty} \leq 1$.*

Finally, we state an important result, relating the partial order above to the σ-weak topology. We will say that a linear functional $\phi \in \mathcal{M}^*$ is **positive** if $\phi(x) \geq 0$ for every $x \in \mathcal{M}_+$. A positive linear functional is **normal** if for every increasing net $\{x_\alpha\}$ in $\mathcal{M}_+$ with supremum x, we have $\phi(x) = \sup_\alpha \phi(x_\alpha)$.

Proposition 1.21 (Theorem 1.4.2 in Part I of [9]). *A positive linear functional in $\mathcal{M}^*$ is normal if and only if it is σ-weakly continuous.*

1.1.4 Traces

Finally, we define a trace on a von Neumann algebra, which we think of as the noncommutative analogue to integration on a measure space.

Definition 1.22. Suppose $\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra. A **trace** on $\mathcal{M}_+$ is a function $\tau : \mathcal{M}_+ \to [0, \infty]$ satisfying the following properties:

(i) for all $x, y \in \mathcal{M}_+$, $\tau(x + y) = \tau(x) + \tau(y)$;

(ii) for all $x \in \mathcal{M}_+$ and $\lambda \geq 0$, $\tau(\lambda x) = \lambda\tau(x)$;

(iii) for all $x \in \mathcal{M}$, $\tau(x^*x) = \tau(xx^*)$.

In addition, a trace is called

(iv) **faithful** if for any $x \in \mathcal{M}_+$, $\tau(x) = 0$ implies $x = 0$;

(v) **normal** if for any increasing net $\{x_\alpha\}$ in $\mathcal{M}_+$ with supremum x, we have $\tau(x) = \sup_\alpha \tau(x_\alpha)$;

(vi) **semifinite** if $\tau(x) = \sup\{\tau(y) : 0 \leq y \leq x, \tau(y) < \infty\}$ for every $x \in \mathcal{M}_+$.

By property (i), if $x \leq y$, then $\tau(y) = \tau(x) + \tau(y - x) \geq \tau(x)$. That is, τ is monotonic. Therefore, we only need to check property (vi) when $\tau(x) = \infty$.

Examples 1.23. Our motivation for supplying this definition was to generalise the notion of integration. Indeed, for any σ-finite measure space (X, ν), the von Neumann algebra $L^\infty(X)$ admits a faithful, normal, semifinite trace given by $f \mapsto \int_X f \, \mathrm{d}\nu$.

The von Neumann algebra $B(\mathcal{H})$, where $\mathcal{H}$ is a separable Hilbert space, admits a trace whose definition directly generalises the familiar trace on the algebra $M_n(\mathbb{C})$ of $n \times n$ complex matrices. Fixing an orthonormal basis $\{\xi_k\}_{k\in\mathbb{N}}$ for $\mathcal{H}$, we let $\mathrm{tr}(x) = \sum_{k=1}^\infty \langle x\xi_k, \xi_k\rangle$ for all positive operators $x \in B(\mathcal{H})$. In fact, $\mathrm{tr}(x)$ is independent of the chosen orthonormal basis and defines a faithful, normal, semifinite trace on $B(\mathcal{H})_+$ (see Theorem 6.6.5 in Part I of [9]).

In both of these examples, the trace can be extended to a certain class of not-necessarily-positive elements, namely the integrable functions and the trace-class operators. Such extensions will be considered in Chapter 2.

When a trace is already known to be normal, the condition for semifiniteness can be reformulated in the following ways.

Proposition 1.24. *Let $\mathcal{M}$ be a von Neumann algebra with a normal trace τ on $\mathcal{M}_+$. Then the following conditions are equivalent:*

(i) τ is semifinite;

(ii) for every non-zero $x \in \mathcal{M}_+$, there exists $y \in \mathcal{M}_+$ with $0 < y \leq x$ and $\tau(y) < \infty$;

(iii) for every non-zero $x \in \mathcal{M}_+$, there exists an increasing net $\{x_\alpha\}$ in $\mathcal{M}_+$ with $\tau(x_\alpha) < \infty$ and $x_\alpha \nearrow x$.

Proof. The implication (i) $\implies$ (ii) is immediate, and (iii) $\implies$ (i) follows from normality of the trace τ. The following proof of (ii) $\implies$ (iii) was adapted from a suggestion by my supervisor. Assume (ii) holds. If $\tau(x) < \infty$, then (iii) holds immediately, so assume $\tau(x) = \infty$.

Let $\mathcal{C}$ denote the collection of all increasing nets $\{y_\alpha\}$ in $\mathcal{M}_+$ with $y_\alpha \leq x$ and $\tau(y_\alpha) < \infty$. We partially order the nets in $\mathcal{C}$ by inclusion. By assumption, $\mathcal{C}$ is nonempty. Moreover, any increasing chain of nets in $\mathcal{C}$ has an upper bound, namely, their union. By Zorn's lemma, $\mathcal{C}$ contains a maximal net, which we denote by $\{x_\alpha\}_{\alpha\in I}$. Since $x_\alpha \leq x$ for every $\alpha \in I$, Proposition 1.18 implies $\{x_\alpha\}_{\alpha\in I}$ has a supremum $\tilde{x}$ in $\mathcal{M}_+$. If $\tilde{x} = x$, then we are done.

On the other hand, if $\tilde{x} < x$, our assumption in (ii) implies that there exists $y \in \mathcal{M}_+$ with $0 < y \leq x - \tilde{x}$ and $\tau(y) < \infty$. Then, for every α, $x_\alpha + y \leq x$ and $\tau(x_\alpha + y) = \tau(x_\alpha) + \tau(y) < \infty$, so $\{x_\alpha + ty\}_{(\alpha,t)\in I\times\{0,1\}}$ is a net in $\mathcal{C}$. But this net strictly contains the net $\{x_\alpha\}_{\alpha\in I}$, contradicting the maximality of $\{x_\alpha\}_{\alpha\in I}$. $\square$

1.2 Unbounded Linear Operators

Not all interesting linear operators are bounded. For example, suppose f is a measurable function on a σ-finite measure space X. For any $\xi \in L^2(X)$, let $m_f\xi(s) = f(s)\xi(s)$ for all $s \in X$. Unless $f \in L^\infty(X)$, the linear map m_f is not a bounded linear operator on $L^2(X)$, and we may not even have $m_f\xi \in L^2(X)$. We therefore regard m_f as a (partially defined) unbounded operator on $L^2(X)$. Hence, to capture noncommutative L^p spaces in their entirety, we will need to broaden our perspective to include such operators.

Definition 1.25. An (unbounded linear) **operator** on $\mathcal{H}$ is a linear map $x : \mathrm{dom}(x) \to \mathcal{H}$, where $\mathrm{dom}(x)$ is some linear subspace of $\mathcal{H}$. We call $\mathrm{dom}(x)$ the **domain** of the operator x, and we say that x is **densely defined** if $\mathrm{dom}(x)$ is a dense subspace of $\mathcal{H}$. Given linear operators x and y, we write $x \subseteq y$ if $\mathrm{dom}(x) \subset \mathrm{dom}(y)$ and $x\xi = y\xi$ for every $\xi \in \mathrm{dom}(x)$. Also, we write $\ker(x) = \{\xi \in \mathrm{dom}(x) : x\xi = 0\}$ and $\mathrm{ran}(x) = \{x\xi : \xi \in \mathrm{dom}(x)\}$.

If x and y are linear operators on $\mathcal{H}$, we denote by $x + y$ the linear operator with $\mathrm{dom}(x + y) = \mathrm{dom}(x) \cap \mathrm{dom}(y)$ and $(x + y)\xi = x\xi + y\xi$ for every $\xi \in \mathrm{dom}(x) \cap \mathrm{dom}(y)$. Similarly, define xy to be the linear operator with domain $\mathrm{dom}(xy) = \{\xi \in \mathrm{dom}(y) : y\xi \in \mathrm{dom}(x)\}$ and $(xy)\xi = x(y\xi)$ for all such ξ. Lastly, for any $\lambda \in \mathbb{C}$, we define λx to be the linear operator with $\mathrm{dom}(\lambda x) = \mathrm{dom}(x)$ and $(\lambda x)\xi = \lambda(x\xi)$ for all $\xi \in \mathrm{dom}(x)$.

Importantly, the linear operators on $\mathcal{H}$ do not form a vector space, much less an algebra, under these operations. Observe that the everywhere-defined zero operator, which we denote by 0, is an additive identity, and if x is a linear operator with $\mathrm{dom}(x) \subsetneq \mathcal{H}$, then $\mathrm{dom}(x+(-x)) = \mathrm{dom}(x) \subsetneq \mathcal{H}$, so $x+(-x) \neq 0$.

Definition 1.26. A linear operator x on $\mathcal{H}$ is said to be **closed** if its graph $\Gamma(x) = \{(\xi, \eta) \in \mathcal{H} \times \mathcal{H} : \xi \in \mathrm{dom}(x), \eta = x\xi\}$ is closed in $\mathcal{H} \times \mathcal{H}$. An operator x is **closable** if there exists a closed linear operator y with $x \subseteq y$. In fact, if x is closable, then there exists a linear operator, denoted $\overline{x}$, whose graph is the closure of $\Gamma(x)$ in $\mathcal{H} \times \mathcal{H}$ (see Propositon 1.5 in [29]). We call $\overline{x}$ the **closure** of x.

By the closed graph theorem, every operator in $B(\mathcal{H})$ is closed, and every closed operator with domain $\mathcal{H}$ must be bounded. In general, the sum or product of two closed linear operators may not be closed, nor even closable.

We devote the remainder of this subsection to extending familiar definitions about bounded operators to unbounded operators. We begin with adjoints.

Definition 1.27. Let x be a densely defined operator on $\mathcal{H}$. To define the adjoint x^*, we let $\mathrm{dom}(x^*)$ be the set of elements $\xi \in \mathcal{H}$ for which there exists an element $\psi \in \mathcal{H}$ satisfying $\langle x\eta, \xi\rangle = \langle \eta, \psi\rangle$ for all $\eta \in \mathrm{dom}(x)$. Since $\mathrm{dom}(x)$ is dense in $\mathcal{H}$, such a ψ must be unique, and so we set $x^*\xi = \psi$. One can check that $\mathrm{dom}(x)$ is a linear subspace of $\mathcal{H}$ and that x^* is linear.

Adjoints are related to closures and closability via the following result, which is contained in Proposition 1.6(i) and Theorem 1.8 in [29].

Proposition 1.28. *Let x be a densely defined operator on $\mathcal{H}$. Then x^* is closed, and x is closable if and only if x^* is densely defined. In this case, $\overline{x} = (x^*)^*$.*

Moreover, adjoints satisfy the usual relation between kernels and ranges.

Proposition 1.29 (Proposition 1.6(ii) in [29]). *If x is a densely defined operator on $\mathcal{H}$, then* $\ker(x^*) = \operatorname{ran}(x)^\perp$.

Definition 1.30. A densely defined operator x is **self-adjoint** if $x = x^*$. More weakly, an operator x is **symmetric** if $\langle x\xi, \eta\rangle = \langle \xi, x\eta\rangle$ for all $\xi, \eta \in \operatorname{dom}(x)$. (When x is densely defined, this says that $x \subseteq x^*$.) By polarisation (see Lemma 3.1 in [29]), x is symmetric if and only if $\langle x\xi, \xi\rangle \in \mathbb{R}$ for every $\xi \in \operatorname{dom}(x)$. Finally, an operator x is **positive** if $\langle x\xi, \xi\rangle \geq 0$ for every $\xi \in \operatorname{dom}(x)$. Of course, any positive operator is symmetric. We note also that any self-adjoint operator is closed, and any densely defined symmetric operator is closable.

1.2.1 Spectral Theory for Self-Adjoint Operators

Spectral theory is a rich and remarkable operator-valued integration theory. For our purposes, it is unfortunately only a tool, so we will present it just briefly.

Definition 1.31. Given a closed operator x, let $\sigma(x)$ be the set of $\lambda \in \mathbb{C}$ for which $x - \lambda\mathbf{1}$ does *not* have a bounded inverse, i.e., there does not exist $y \in B(\mathcal{H})$ with $(x - \lambda\mathbf{1})y = \mathbf{1}$ and $y(x - \lambda\mathbf{1}) = \mathbf{1}|_{\operatorname{dom}(x)}$. We call $\sigma(x)$ the **spectrum** of x.

Proposition 2.6 in [29] asserts that $\sigma(x)$ is a closed subset of $\mathbb{C}$. We will be particularly interested in the spectra of self-adjoint operators.

Proposition 1.32. *Let x be a self-adjoint operator on $\mathcal{H}$. Then $\sigma(x) \subset \mathbb{R}$, and x is positive if and only if $\sigma(x) \subset [0, \infty)$.*

Proof. Paired with Proposition 3.10(b) in [29], these statements follow from Proposition 3.2(i) and (ii) respectively. □

Recall that when $x \in B(\mathcal{H})$, $\sigma(x)$ is compact and every $\lambda \in \sigma(x)$ satisfies $|\lambda| \leq \|x\|_{L^\infty}$ (for example, see Theorem 3.6 in Chapter VII of [6]). In particular, if $x \in B(\mathcal{H})$ is positive, then $\sigma(x) \subset [0, \|x\|_{L^\infty}]$.

Definition 1.33. Let $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}$. A **spectral measure** on $\mathcal{B}$ is a projection-valued mapping $E : \mathcal{B} \to P(B(\mathcal{H}))$ such that

(i) $E(\mathbb{R}) = \mathbf{1}_{\mathcal{H}}$; and,

(ii) if $\{M_n\}_{n\in\mathbb{N}}$ is a countable, pairwise disjoint family in $\mathcal{B}$ and $M = \bigcup_{n=1}^\infty M_n$, then $E(M) = \sum_{n=1}^\infty E(M_n)$ in the strong operator topology.

We will call the projections $E(M)$ **spectral projections**. Importantly, for any $\xi, \eta \in \mathcal{H}$, E induces a finite, scalar-valued Borel measure $M \mapsto \langle E(M)\xi, \eta\rangle$.

From a spectral measure, one constructs a spectral integral. To this end, let $\mathcal{B}(\mathbb{C})$ denote the space of complex-valued Borel functions on $\mathbb{R}$. With pointwise operations and ordering, this is a partially-ordered $*$-algebra. For any function $f \in \mathcal{B}(\mathbb{C})$, one associates a closed, densely defined operator on $\mathcal{H}$, which we denote by $\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E(\lambda)$, with domain $\{\xi \in \mathcal{H} : \int_{\mathbb{R}} |f(\lambda)|^2\,\mathrm{d}\langle E(\lambda)\xi, \xi\rangle < \infty\}$. This operator-valued integral is defined as expected for simple functions, before being extended via approximation to arbitrary $f \in \mathcal{B}(\mathbb{C})$. The details of this construction are laid out in Section 4.3 of [29].

We now state the remarkable spectral theorem for self-adjoint operators.

Theorem 1.34 (Theorem 5.7 in [29])**.** *For any self-adjoint operator x on $\mathcal{H}$, there exists a unique spectral measure E_x on $\mathcal{B}$ with $\int_{\mathbb{R}} \lambda\,\mathrm{d}E_x(\lambda) = x$.*

For any self-adjoint operator x and $f \in \mathcal{B}(\mathbb{C})$, we write $f(x)$ for the linear operator $\int_{\mathbb{R}} f(\lambda)\,\mathrm{d}E_x(\lambda)$, and we call the mapping $f \mapsto f(x)$ the functional calculus of x. Functional calculus appears frequently in this thesis, largely because it provides a means of translating notions from real analysis to operator theory. We present a few of its properties in the following proposition and will explore some of its richer features when we discuss τ-measurable operators.

Proposition 1.35. *Let x be a self-adjoint operator on $\mathcal{H}$ and $f, g \in \mathcal{B}(\mathbb{C})$.*

(i) For all $\xi, \eta \in \mathrm{dom}(f(x))$, $\langle f(x)\xi, \eta\rangle = \int_{\mathbb{R}} f(\lambda)\,\mathrm{d}\langle E_x(\lambda)\xi, \eta\rangle$.

(ii) If $f(\lambda) = g(\lambda)$ for all $\lambda \in \sigma(x)$, then $f(x) = g(x)$.

(iii) We have $\sigma(f(x)) \subset \overline{f(\sigma(x))}$.

(iv) If f is bounded, then $f(x) \in B(\mathcal{H})$ with $\|f(x)\|_{L^\infty} \leq \sup_{\lambda\in\sigma(x)} |f(\lambda)|$.

Proof. Statement (i) is Theorem 5.9(1) in [29]. Statement (ii) follows from Propositions 4.17(i) and 5.10(i) in [29]. Statement (iii) follows from Proposition 4.20 in [29]. Statement (iv) follows from Proposition 4.1.2(iv) in [29]. □

The second property implies that, to define the operator $f(x)$, we only need to specify f on $\sigma(x)$. Functional calculus is related to commutation via the following result, which follows from Propositions 4.23 and 5.15 in [29].

Proposition 1.36. *Let x be a self-adjoint operator on $\mathcal{H}$. For any $y \in B(\mathcal{H})$, we have $yx \subseteq xy$ if and only if $yf(x) \subseteq f(x)y$ for all $f \in \mathcal{B}(\mathbb{C})$.*

Remark 1.37. If $x \in \mathcal{M}$ is self-adjoint, this proposition implies that $f(x) \in \mathcal{M}$ whenever f is bounded. In particular, for any Borel set $M \in \mathcal{B}$, $\mathcal{M}$ contains the spectral projection $E_x(M)$. This justifies our claim that $P(\mathcal{M})$ is plentiful.

1.3 The Algebra of τ-Measurable Operators

It was observed in Section 1.2 that the unbounded operators on $\mathcal{H}$ do not form a vector space. Given a von Neumann algebra $\mathcal{M}$ with a faithful, normal, semifinite trace τ, we will construct a certain class of unbounded operators that does form a vector space and in fact a $*$-algebra. This will be an unbounded extension of $\mathcal{M}$ and will provide an ambient space for noncommutative L^p spaces. Throughout, let τ be a faithful, normal, semifinite trace on $\mathcal{M}_+$.

Definition 1.38. A closed linear operator x on $\mathcal{H}$ is **τ-measurable** if

(i) $yx \subset xy$ for all $y \in \mathcal{M}'$; and,

(ii) $\mathrm{dom}(x)$ is **τ-dense**, i.e., there is an increasing sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(\mathcal{M})$ with $\mathrm{ran}(p_n) \subset \mathrm{dom}(x)$, $\tau(p_n^{\perp}) < \infty$ and $p_n \nearrow \mathbf{1}_{\mathcal{H}}$ as $n \to \infty$.

We denote by $S(\mathcal{M}, \tau)$ the set of τ-measurable operators.

Notice that condition (i) extends the bicommutant definition of $\mathcal{M}$. In particular, if $x \in B(\mathcal{H})$ then the condition $yx \subseteq xy$ actually says that $yx = xy$, so $x \in S(\mathcal{M}, \tau)$ if and only if $x \in \mathcal{M}$. In this way, $S(\mathcal{M}, \tau)$ may be thought of as an unbounded extension of $\mathcal{M}$.

Next, if $x \in S(\mathcal{M}, \tau)$, then letting $\{p_n\}_{n=1}^{\infty}$ be a sequence as in condition (ii), we have $\langle p_n\xi, \xi\rangle \nearrow \langle \xi, \xi\rangle$ for every $\xi \in \mathcal{H}$. Since $\mathrm{ran}(p_n) \subset \mathrm{dom}(x)$, this implies that $\mathrm{dom}(x)$ is dense in $\mathcal{H}$. Therefore, all τ-measurable operators admit adjoints.

Remarkably, for any $x, y \in S(\mathcal{M}, \tau)$, $x+y$ and xy are closable (see Proposition 2.2.15 in [10]). Hence, one defines the **strong sum** $\overline{x+y}$ and **strong product** $\overline{xy}$. In fact, $\overline{x+y}$ and $\overline{xy}$ are τ-measurable, and we have the following result.

Theorem 1.39 (Theorem 2.38 in [10])**.** *When equipped with the operations of strong addition and strong multiplication, $S(\mathcal{M}, \tau)$ is a $*$-algebra.*

Henceforth, unless otherwise stated, we will simply write $x + y$ and xy for the strong sum and strong product of τ-measurable operators x and y.

An important fact is that if $x, y \in S(\mathcal{M}, \tau)$ satisfy $x \subseteq y$, then in fact $x = y$ (this is proven in Proposition 2.2.14 of [10] for measurable operators, of which τ-measurable operators form a subclass). Therefore, for τ-measurable operators, there is no distinction between symmetricity and self-adjointness. In particular, all positive operators are self-adjoint. Moving forward, we will write $S_h(\mathcal{M}, \tau)$ for the real vector space of self-adjoint τ-measurable operators and $S(\mathcal{M}, \tau)_+$ for the set of positive τ-measurable operators. We will also need the following results about domains of τ-measurable operators.

Proposition 1.40 (Propositions 2.3.3 and 2.3.4 in [10])**.** *A subspace $D \subset \mathcal{H}$ is τ-dense if and only if the following condition holds: for all $\varepsilon > 0$, there exists a projection $p \in P(\mathcal{M})$ with $\operatorname{ran}(p) \subset D$ and $\tau(p^\perp) < \varepsilon$. Moreover, any countable intersection of τ-dense domains is τ-dense.*

Example 1.41. Consider the von Neumann algebra $B(\mathcal{H})$ with faithful, normal, semifinite trace tr, as defined in Subsection 1.1.4. Suppose D is a tr-dense subspace of $\mathcal{H}$. By Proposition 1.40, there exists a projection p with $\operatorname{ran}(p) \subset D$ and $\operatorname{tr}(p^\perp) < 1$. But $\operatorname{tr}(p^\perp)$ is just the rank of $p^\perp$, so we must have $p^\perp = 0$, and hence $p = \mathbf{1}_{\mathcal{H}}$. Therefore, the only tr-dense subspace of $\mathcal{H}$ is $\mathcal{H}$ itself. This implies that every operator in $S(B(\mathcal{H}), \operatorname{tr})$ is everywhere-defined, so by the closed graph theorem, $S(B(\mathcal{H}), \operatorname{tr}) = B(\mathcal{H})$.

Let us equip the self-adjoint elements of $S(\mathcal{M}, \tau)$ with a partial ordering, which extends the partial ordering on the self-adjoint elements of $\mathcal{M}$. Given $x, y \in S_h(\mathcal{M}, \tau)$, we write $x \leq y$ if $y - x$ (i.e., the strong difference) is positive. In fact, it is easy to see that a closable operator is positive if and only if its closure is positive, so $x \leq y$ precisely when $\langle x\xi, \xi \rangle \leq \langle y\xi, \xi \rangle$ for all $\xi \in \operatorname{dom}(x) \cap \operatorname{dom}(y)$. Properties (i)-(iii) and (v) in Proposition 1.16 continue to hold in $S_h(\mathcal{M}, \tau)$. (This is proven for measurable operators in Proposition 2.2.24 of [10].)

1.3.1 Spectral Theory on $S(\mathcal{M}, \tau)$

For spectral theory on $S(\mathcal{M}, \tau)$, to ensure that our operators are τ-densely defined, we consider only the $*$-algebra $\mathcal{B}_{bc}(\mathbb{C})$, consisting of Borel functions that are bounded on compact subsets of $\mathbb{R}$. Suppose $x \in S_h(\mathcal{M}, \tau)$ and $f \in \mathcal{B}_{bc}(\mathbb{C})$. Once it is verified that $f(x)$ has τ-dense domain (see Proposition 2.3.14 in [10]), it follows from Proposition 1.36 that $f(x) \in S(\mathcal{M}, \tau)$. In fact, when dealing with the $*$-algebra of τ-measurable operators, several properties of the functional calculus (see Theorem 5.9 of [29]) coalesce into the following elegant result.

Proposition 1.42. *For any self-adjoint $x \in S(\mathcal{M}, \tau)$, the map $f \mapsto f(x)$ is an order-preserving $*$-homomorphism $\mathcal{B}_{bc}(\mathbb{C}) \to S(\mathcal{M}, \tau)$.*

Two familiar convergence theorems for integrals are analogised as follows.

Proposition 1.43. *Let $x \in S_h(\mathcal{M}, \tau)$. Suppose $\{f_n\}_{n=1}^{\infty}$ is a uniformly bounded sequence of functions in $\mathcal{B}_{bc}(\mathbb{C})$ and $f \in \mathcal{B}_{bc}(\mathbb{C})$ satisfies $f_n(\lambda) \to f(\lambda)$ for every $\lambda \in \mathbb{R}$. Then $f_n \to f$ in the weak operator topology.*

Proof. This follows from Proposition 1.35(i) and the dominated convergence theorem applied to the scalar measure $\langle E_x(\cdot)\xi, \eta\rangle$ for any $\xi, \eta \in \mathcal{H}$. $\square$

Proposition 1.44. *Let $x \in S_h(\mathcal{M}, \tau)$. If $\{f_\alpha\}$ is an increasing net of positive-valued functions in $\mathcal{B}_{bc}(\mathbb{C})$ with supremum $f \in \mathcal{B}_{bc}(\mathbb{C})$, then $f_\alpha(x) \nearrow f(x)$.*

Proof. From the definition of $\operatorname{dom}(f(x))$, we have $\operatorname{dom}(f(x)) \subset \bigcap_\alpha \operatorname{dom}(f_\alpha(x))$. In view of Proposition 1.35(i), the monotone convergence theorem implies that $\langle f_\alpha(x)\xi, \xi\rangle \nearrow \langle f(x)\xi, \xi\rangle$ for all $\xi \in \operatorname{dom}(f(x))$. That is, $f_\alpha(x) \nearrow f(x)$. $\square$

1.3.2 Polar Decomposition

An important application of functional calculus is polar decomposition. For our purposes, it is convenient to present this only for τ-measurable operators.

As a first observation, we can explain the existence of positive square roots. If $x \in S(\mathcal{M}, \tau)_+$, Proposition 1.32 implies that $\sigma(x) \subset [0, \infty)$. Therefore, we define a positive operator $x^{1/2} = \int_{[0,\infty)} \lambda^{1/2}\, \mathrm{d}E_x(\lambda)$, and by the homomorphism property in Proposition 1.42, we have $(x^{1/2})^2 = x$. In fact, Proposition 5.13 in [29] proves that $x^{1/2}$ is the unique positive self-adjoint square root of x.

In particular, if $x \in \mathcal{M}$, then $\sigma(x) \subset [0, \|x\|_{L^\infty}]$, so $x^{1/2} = \int_{[0,\|x\|_{L^\infty}]} \lambda^{1/2}\, \mathrm{d}E_x(\lambda)$ is a bounded operator by Proposition 1.35(iv). Now, Proposition 1.36 says that $yx^{1/2} = x^{1/2}y$ for all $y \in \mathcal{M}'$, and therefore $x^{1/2} \in \mathcal{M}$.

For any $x \in S(\mathcal{M}, \tau)$, the operator $x^*x \in S(\mathcal{M}, \tau)$ is positive, so we can define $|x| = (x^*x)^{1/2}$. Note that if $x \in B(\mathcal{H})$, then $x^*x \in B(\mathcal{H})$, and so $|x| \in B(\mathcal{H})$. Also, if x is self-adjoint, then $|x| = \int_{\mathbb{R}} |\lambda|\, \mathrm{d}E_x(\lambda)$, since this is the unique positive square root of $x^*x = x^2 = \int_{\mathbb{R}} \lambda^2\, \mathrm{d}E_x(\lambda)$. The operator $|x|$, which we call the absolute value of x, bears close resemblance to x.

Proposition 1.45 (Lemma 7.1 in [29])**.** *For any $x \in S(\mathcal{M}, \tau)$, we have $\operatorname{dom}(x) = \operatorname{dom}(|x|)$ and $\ker(x) = \ker(|x|)$.*

Given closed subspaces $D_1, D_2 \subset \mathcal{H}$, we say that an operator $u \in B(\mathcal{H})$ is a **partial isometry** from D_1 to D_2 if u maps D_1 onto D_2 isometrically, and $\ker(u) = D_1^{\perp}$. In this case, u^*u is the orthogonal projection onto D_1 and uu^* is the orthogonal projection onto D_2. This notion gives rise to polar decomposition, which emulates the polar decomposition of a complex number. The next result follows from Theorem 7.2 in [29] and Proposition 2.1.4 in [10].

Theorem 1.46. *Any $x \in S(\mathcal{M}, \tau)$ has a unique decomposition $x = uy$, where y is a positive self-adjoint operator, and u is a partial isometry from $\overline{\operatorname{ran}(y)}$ to $\overline{\operatorname{ran}(x)}$. In particular, $y = |x|$ and $u \in \mathcal{M}$.*

We call this decomposition the **polar decomposition** of x. In the following proposition, we collect some useful, technical results.

Proposition 1.47. *Let $x \in S(\mathcal{M}, \tau)$, with polar decomposition $x = u|x|$. Also, let $p > 0$ and $f \in \mathcal{B}_{bc}(\mathbb{C})$.*

(i) $|xf(|x|)| = |x||f|(|x|)$;

(ii) $u^*u = E_{|x|}(0, \infty)$;

(iii) $|x^*|^p = u|x|^p u^*$, *and in particular* $xx^* = ux^*xu^*$;

(iv) $\big|u|x|^p f(|x|)\big| = |x|^p |f|(|x|)$.

Proof. For (i), define $g(\lambda) = \lambda|f(\lambda)|$ for $\lambda \in [0, \infty)$. We have

$$(xf(|x|))^*(xf(|x|)) = f(|x|)^* x^* x f(|x|) = f(|x|)^* |x|^2 f(|x|) = g(|x|)^2.$$

Since $g(\lambda) \geq 0$ for all $\lambda \in [0, \infty)$ and the functional calculus is order-preserving, we have that $g(|x|)$ is positive, and the uniqueness of positive square roots implies that $|xf(|x|)| = g(|x|) = |x|f(|x|)$.

Next, we prove (ii). By Proposition 1.42, $E_{|x|}(0, \infty)|x| = |x|$. Since u^*u is projection onto $\overline{\operatorname{ran}(|x|)}$, it follows that $E_{|x|}(0, \infty) \geq u^*u$. So we just need to prove that $\operatorname{ran}(E_{|x|}(0, \infty)) \subset \overline{\operatorname{ran}(|x|)}$. To do this, we will take orthogonal complements and show that $\ker(|x|) \subset \operatorname{ran}(E_{|x|}(0, \infty))^{\perp}$. Suppose $\xi \in \ker(|x|)$. By Proposition 1.44, we have $\langle E_{|x|}(\varepsilon, \infty)\xi, \xi\rangle \nearrow \langle E_{|x|}(0, \infty)\xi, \xi\rangle$ as $\varepsilon \searrow 0$. However, for every $\varepsilon > 0$, since the functional calculus is order-preserving, we also have $\varepsilon E_{|x|}(\varepsilon, \infty) \leq E_{|x|}(\varepsilon, \infty)|x|$. Thus, $\langle E_{|x|}(\varepsilon, \infty)\xi, \xi\rangle \leq \varepsilon^{-1}\langle E_{|x|}(\varepsilon, \infty)|x|\xi, \xi\rangle = 0$. It follows that $\langle E_{|x|}(0, \infty)\xi, \xi\rangle = 0$, so we must have $\xi \in \operatorname{ran}(E_{|x|}(0, \infty))^{\perp}$.

For (iii), recall that $|x^*| = (xx^*)^{1/2}$. Since u^*u is the projection onto $\overline{\operatorname{ran}(|x|)}$, we have $|x| = u^*u|x|$, and so $xx^* = u|x|^2u^* = u|x|u^*u|x|u^*$. Since $u|x|u^*$ is positive, the uniqueness of positive square roots implies $|x^*| = u|x|u^*$. Now, for every Borel set $M \in \mathcal{B}$, define $E(M) = uE_{|x|}(M)u^*$. In view of (ii), it is easily shown that E is a spectral measure on $\mathcal{B}$. Moreover, for any $g \in \mathcal{B}_{bc}(\mathbb{C})$, we have $\int_{\mathbb{R}} g(\lambda)\,\mathrm{d}E(\lambda) = ug(|x|)u^*$. In particular, $\int_{\mathbb{R}} \lambda\,\mathrm{d}E(\lambda) = u|x|u^* = |x^*|$. By the uniqueness assertion in Theorem 1.34, we have $E = E_{|x^*|}$, and so $|x^*|^p = \int_{[0,\infty)} \lambda^p\,\mathrm{d}E(\lambda) = u|x|^p u^*$.

For (iv), we have by (ii) that

$$\begin{aligned}(u|x|^p f(|x|))^*(u|x|^p f(|x|)) &= f(|x|)^*|x|^p E_{|x|}(0,\infty)|x|^p f(|x|)\\ &= (|x|^p f(|x|))^*(|x|^p f(|x|)),\end{aligned}$$

and now (iv) follows in exactly the same way (i) did. □

Finally, these considerations imply a useful fact about projections in $\mathcal{M}$.

Proposition 1.48. *For any nonzero $p \in P(\mathcal{M})$ and any $0 < b < \tau(p)$, there exists a projection $q \in P(\mathcal{M})$ satisfying $q \leq p$ and $b < \tau(q) < \infty$.*

Proof. Since τ is semifinite, choose a nonzero $y \in \mathcal{M}_+$ so that $y \leq p$ and $b < \tau(y) < \infty$. Then $\|y\|_{L^\infty} \leq 1$, so $\sigma(y) \subset [0,1]$. We will set $q = E_y([s,1])$, for a suitable choice of $s \in (0,1)$. By Proposition 1.42, we have $E_y([s,1]) \in \mathcal{M}$, and since $E_y([s,1]) \leq s^{-1}y$, we have $\tau(E_y([s,1])) \leq s^{-1}\tau(y) < \infty$. Also, since we have $sE_y([s,1]) \leq y \leq p$, and $E_y([s,1])$ and p are projections, it follows that $E_y([s,1]) \leq p$. Now, we are only left to ensure that $b < \tau(E_y([s,1]))$. Notice that as $s \to 0$, Proposition 1.44 implies $E_y([s,1]) \nearrow E_y((0,1]) \geq y$. Therefore, $\tau(E_y((0,1])) \geq \tau(y) > b$, and the normality of τ implies that for sufficiently small $s > 0$, we have $\tau(E_y([s,1])) > b$, as required. □

Chapter 2

Noncommutative L^p Spaces

In this chapter, we construct noncommutative L^p spaces associated to a von Neumann algebra with a faithful, normal, semifinite trace. In the classical setting of measure theory, the integral defines an extended-valued trace, not only for positive $f \in L^\infty$, but *all* positive measurable functions f. We then let L^p denote the space of such functions f satisfying $\int |f|^p < \infty$. We mirror this construction in this chapter, first by extending the trace on $\mathcal{M}_+$ to a trace on the set of positive τ-measurable operators. The resulting L^p spaces enjoy many of the familiar properties of classical L^p spaces. To prove these, we will use the powerful machinery of interpolation theory.

Throughout, let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, semifinite trace τ, and let $S(\mathcal{M},\tau)$ be the $*$-algebra of τ-measurable operators constructed in Section 1.3.

2.1 Extension of the Trace to $S(\mathcal{M},\tau)$

Recall that properties (i)-(iii) and (v) in Proposition 1.16 continue to hold for self-adjoint elements of $S(\mathcal{M},\tau)$. Given $x \in S(\mathcal{M},\tau)_+$ and $\varepsilon > 0$, we have $\mathbf{1}+\varepsilon x \geq \mathbf{1}$, so $\mathbf{1}+\varepsilon x$ is invertible. Thus, we define $x_\varepsilon = x(\mathbf{1}+\varepsilon x)^{-1} = \int_{[0,\infty)} \frac{\lambda}{1+\varepsilon\lambda}\,\mathrm{d}E_x(\lambda)$. Since this integrand is bounded and positive, one sees that $x_\varepsilon \in \mathcal{M}_+$. Also, since $\frac{\lambda}{1+\varepsilon\lambda} = \frac{1}{\varepsilon} - \frac{1}{\varepsilon+\varepsilon^2\lambda}$, we have $x_\varepsilon = \varepsilon^{-1}\mathbf{1} - (\varepsilon\mathbf{1}+\varepsilon^2 x)^{-1})$. Finally, as $\varepsilon \to 0$, we have $\frac{\lambda}{1+\varepsilon\lambda} \nearrow \lambda$ for all $\lambda \in [0,\infty)$. Therefore, by Proposition 1.44, $x_\varepsilon \nearrow x$.

The normality of the trace τ implies that for any $x \in \mathcal{M}_+$, $\tau(x_\varepsilon) \nearrow \tau(x)$ as $\varepsilon \to 0$. Therefore, abusing notation, we will extend the trace τ to $S(\mathcal{M},\tau)_+$ as a lower integral.

Definition 2.1. For any $x \in S(\mathcal{M}, \tau)_+$, set $\tau(x) = \lim_{\varepsilon \to 0} \tau(x_\varepsilon)$. (We understand this limit to be infinite if $\tau(x_\varepsilon)$ grows without bound.)

To see that τ satisfies the properties of a trace, we begin by collecting some basic facts about the mapping $x \mapsto x_\varepsilon$.

Lemma 2.2. *Let $x, y \in S(\mathcal{M}, \tau)_+$, $\varepsilon > 0$, and $\lambda \in \mathbb{R}_{\geq 0}$.*

(i) if $x \leq y$, then $x_\varepsilon \leq y_\varepsilon$

(ii) $(\lambda x)_\varepsilon = \lambda x_{\lambda\varepsilon}$

(iii) $\tau((x+y)_\varepsilon) \leq \tau(x_\varepsilon) + \tau(y_\varepsilon)$

Proof. For (i), if $x \leq y$, then $\varepsilon \mathbf{1} + \varepsilon^2 x \leq \varepsilon \mathbf{1} + \varepsilon^2 y$, and so

$$x_\varepsilon = \varepsilon^{-1}\mathbf{1} - (\varepsilon\mathbf{1} + \varepsilon^2 x)^{-1} \leq \varepsilon^{-1}\mathbf{1} - (\varepsilon\mathbf{1} + \varepsilon^2 y)^{-1} = y_\varepsilon.$$

For (ii), we have $(\lambda x)_\varepsilon = \lambda x(\mathbf{1} + \varepsilon\lambda x)^{-1} = \lambda x_{\lambda\varepsilon}$.

Finally, for (iii), the homomorphism properties in Proposition 1.42 imply that $(x+y)_\varepsilon = (\mathbf{1}+\varepsilon(x+y))^{-1/2}(x+y)(\mathbf{1}+\varepsilon(x+y))^{-1/2}$. Since $\mathbf{1}+\varepsilon x \leq \mathbf{1}+\varepsilon(x+y)$, we have $(\mathbf{1}+\varepsilon(x+y))^{-1} \leq (\mathbf{1}+\varepsilon x)^{-1}$. Applying property (iii) in Definition 1.22 and then the monotonicity of τ, we have

$$\begin{aligned} \tau\left((\mathbf{1}+\varepsilon(x+y))^{-1/2}x(\mathbf{1}+\varepsilon(x+y))^{-1/2}\right) &= \tau\left(x^{1/2}(\mathbf{1}+\varepsilon(x+y))^{-1}x^{1/2}\right) \\ &\leq \tau\left(x^{1/2}(\mathbf{1}+\varepsilon x)^{-1}x^{1/2}\right) \\ &= \tau(x_\varepsilon). \end{aligned}$$

Similarly, $\tau\left((\mathbf{1}+\varepsilon(x+y))^{-1/2}y(\mathbf{1}+\varepsilon(x+y)\right)^{-1/2}) \leq \tau(y_\varepsilon)$. Therefore, we have $\tau((x+y)_\varepsilon) \leq \tau(x_\varepsilon) + \tau(y_\varepsilon)$. $\square$

Lemma 2.3. *For all $x \in S(\mathcal{M}, \tau)$, with polar decomposition $x = u|x|$, we have $(xx^*)_\varepsilon = u(x^*x)_\varepsilon u^*$*

Proof. Since $xx^* = u|x|^2u^*$, we have $(xx^*)_\varepsilon = u|x|^2u^*(\mathbf{1}+\varepsilon u|x|^2u^*)^{-1}$. Thus, it will suffice to prove that $u^*(\mathbf{1}+\varepsilon u|x|^2u^*)^{-1} = (\mathbf{1}+\varepsilon|x|^2)^{-1}u^*$, or equivalently $(\mathbf{1}+\varepsilon|x|^2)u^* = u^*(\mathbf{1}+\varepsilon u|x|^2u^*)$. But since u^*u is projection onto $\overline{\operatorname{ran}(|x|)}$, we have $u^*u|x| = |x|$, so the equality holds. $\square$

With these results, we can show that the extension of τ to $S(\mathcal{M}, \tau)_+$ is a faithful, normal, semifinite trace.

Proposition 2.4. *The function $\tau : S(\mathcal{M},\tau)_+ \to [0,\infty]$ satisfies the properties in Definition 1.22. That is,*

(i) $\tau(x+y) = \tau(x) + \tau(y)$*, for all* $x, y \in S(\mathcal{M},\tau)_+$*;*

(ii) $\tau(\lambda x) = \lambda\tau(x)$*, for all* $x \in S(\mathcal{M},\tau)_+$*,* $\lambda \in \mathbb{R}_{\geq 0}$*;*

(iii) $\tau(x^*x) = \tau(xx^*)$ *for all* $x \in S(\mathcal{M},\tau)$*;*

(iv) if $x \in S(\mathcal{M},\tau)_+$ *and* $\tau(x) = 0$*, then* $x = 0$*;*

(v) if $\{x_\alpha\}$ *is an increasing net in* $S(\mathcal{M},\tau)_+$ *with supremum* $x \in S(\mathcal{M},\tau)_+$*, then* $\tau(x_\alpha) \nearrow \tau(x)$*;*

(vi) $\tau(x) = \sup\{\tau(y) : y \in S(\mathcal{M},\tau)_+, y \leq x, \tau(y) < \infty\}$ *for all* $x \in S(\mathcal{M},\tau)_+$*.*

Proof. First, by Lemma 2.2(i), τ is monotonic. Now, to prove (i), the inequality $\tau(x+y) \leq \tau(x) + \tau(y)$ follows from Lemma 2.2(iii). For the reverse inequality, note that $x_\varepsilon + y_\varepsilon \leq x + y$ for all $\varepsilon > 0$. The monotonicity of the extended trace τ now implies $\tau(x) + \tau(y) = \lim_{\varepsilon\to 0} \tau(x_\varepsilon + y_\varepsilon) \leq \tau(x+y)$.

For (ii), Lemma 2.2(ii) implies that $\tau(\lambda x) = \lim_{\varepsilon\to 0} \lambda\tau(x_{\varepsilon\lambda}) = \lambda\tau(x)$.

For (iii), letting $x = u|x|$ be the polar decomposition of x, Lemma 2.3 implies $\tau(xx^*) = \lim_{\varepsilon\to 0} \tau(u(x^*x)_\varepsilon u^*)$. Since $((x^*x)_\varepsilon)^{1/2} = |x|(\mathbf{1} + \varepsilon x^*x)^{-1/2}$, and u^*u is orthogonal projection onto $\overline{\mathrm{ran}(|x|)}$, we have $u^*u\,((x^*x)_\varepsilon)^{1/2} = ((x^*x)_\varepsilon)^{1/2}$. Therefore, applying property (iii) in Definition 1.22, we see that

$$\tau(u(x^*x)_\varepsilon u^*) = \tau\left(((x^*x)_\varepsilon)^{1/2}\, u^*u\, ((x^*x)_\varepsilon)^{1/2}\right) = \tau((x^*x)_\varepsilon).$$

We conclude that $\tau(xx^*) = \tau(x^*x)$.

To see that τ is faithful, suppose $\tau(x) = 0$. Since $x_\varepsilon \leq x$ for every $\varepsilon > 0$, the monotonicity and faithfulness of the trace τ on $\mathcal{M}_+$ imply $x_\varepsilon = 0$ for every $\varepsilon > 0$. Since $x_\varepsilon \nearrow x$, it follows that $x = 0$.

To see that τ is normal, suppose that $\{x_\alpha\}$ is an increasing net in $S(\mathcal{M},\tau)_+$ with supremum x. Since τ is monotonic, we have $\lim_\alpha \tau(x_\alpha) \leq \tau(x)$. To get the reverse inequality, we notice that for any $\varepsilon > 0$, $\varepsilon\mathbf{1} + \varepsilon^2 x_\alpha \nearrow \varepsilon\mathbf{1} + \varepsilon^2 x$, and hence we have $(\varepsilon\mathbf{1} + \varepsilon^2 x_\alpha)^{-1} \searrow (\varepsilon\mathbf{1} + \varepsilon^2 x)^{-1}$. From this, it follows that $(x_\alpha)_\varepsilon = \varepsilon^{-1}\mathbf{1} - (\varepsilon\mathbf{1} + \varepsilon^2 x_\alpha)^{-1} \nearrow \varepsilon^{-1}\mathbf{1} - (\varepsilon\mathbf{1} + \varepsilon^2 x) = x_\varepsilon$. The normality of the extended trace τ now follows from the normality of the trace on $\mathcal{M}_+$.

Finally, we prove that τ is semifinite. As in Proposition 1.24, Zorn's lemma implies that it suffices to prove that every $x \in S(\mathcal{M},\tau)_+$ majorises some $y \in S(\mathcal{M},\tau)_+$ with $\tau(y) < \infty$. Fixing $\varepsilon > 0$, choose $y \in \mathcal{M}_+$ with $y \leq x_\varepsilon$ and $\tau(y) < \infty$. Then $y \leq x$ also, and we are done. □

We immediately deduce the following important corollary.

Corollary 2.5. *For any $x \in S(\mathcal{M},\tau)_+$ and $u \in U(\mathcal{M})$, $\tau(x) = \tau(uxu^*)$.*

Proof. Since $uxu^* = (ux^{1/2})(ux^{1/2})^*$ and $x = x^{1/2}u^*ux^{1/2} = (ux^{1/2})^*(ux^{1/2})$, this follows from Proposition 2.4(iii). □

2.2 The Spaces $L^1(\mathcal{M},\tau)$ and $L^2(\mathcal{M},\tau)$

In analogy with the classical L^p spaces, we can now define the spaces $L^p(\mathcal{M},\tau)$. In this section, we will chiefly examine $L^1(\mathcal{M},\tau)$ and $L^2(\mathcal{M},\tau)$. Our approach is largely influenced by the methods in [9] and [10]. Arbitrary L^p spaces will require more complex methods, which we leave for the next section. First, recall that for any $x \in S(\mathcal{M},\tau)$ and $p > 0$, we define $|x|^p = \int_{[0,\infty)} \lambda^p \, \mathrm{d}E_{|x|}(\lambda) \in S(\mathcal{M},\tau)_+$.

Definition 2.6. For $1 \leq p < \infty$, let $L^p(\mathcal{M},\tau) = \{x \in S(\mathcal{M},\tau) : \tau(|x|^p) < \infty\}$, and for any $x \in L^p(\mathcal{M},\tau)$, write $\|x\|_{L^p} = \tau(|x|^p)^{1/p}$. Also, set $L^\infty(\mathcal{M},\tau) = \mathcal{M}$. We write $L^p(\mathcal{M},\tau)_+$ for the set of positive elements in $L^p(\mathcal{M},\tau)$, and we will sometimes write L^p instead of $L^p(\mathcal{M},\tau)$.

It is not immediately clear that $L^p(\mathcal{M},\tau)$ is a vector space, nor that $\|\cdot\|_{L^p}$ is a norm. Proving this for $p=1$ and $p=2$ will be the primary goal of this section. That $L^2(\mathcal{M},\tau)$ is a vector space is a consequence of the following inequality.

Lemma 2.7. *For any $x, y \in S(\mathcal{M},\tau)$, $|x+y|^2 \leq 2(|x|^2 + |y|^2)$.*

Proof. Since $|x+y|^2 = (x+y)^*(x+y)$, we have that for any $\xi \in \mathcal{H}$,

$$\langle |x+y|^2\xi, \xi\rangle = \|(x+y)\xi\|^2 \leq (\|x\xi\| + \|y\xi\|)^2.$$

By the AM-QM inequality, we have

$$(\|x\xi\| + \|y\xi\|)^2 \leq 2\left(\|x\xi\|^2 + \|y\xi\|^2\right) = 2\left(\langle |x|^2\xi,\xi\rangle + \langle |y|^2\xi,\xi\rangle\right).$$

Combining these two estimates, we conclude that $|x+y|^2 \leq 2(|x|^2 + |y|^2)$. □

Proposition 2.8. *The set $L^2(\mathcal{M},\tau)$ is a $*$-invariant vector space. In fact, it is an $\mathcal{M}$-bimodule, i.e., for any $x \in L^2(\mathcal{M},\tau)$ and $y \in \mathcal{M}$, one has $yx, xy \in L^2(\mathcal{M},\tau)$.*

Proof. If $x, y \in L^2(\mathcal{M},\tau)$, the inequality $|x+y|^2 \leq 2(|x|^2+|y|^2)$ and the monotonicity of τ imply $x+y \in L^2(\mathcal{M},\tau)$. Since $|\lambda x|^2 = |\lambda|^2|x|^2$, it follows that $L^2(\mathcal{M},\tau)$ is a vector space. Moreover, Proposition 2.4(iii) implies that $L^2(\mathcal{M},\tau)$

is $*$-invariant. We now show that it is an $\mathcal{M}$-bimodule. Suppose $x \in L^2(\mathcal{M},\tau)$. If $u \in U(\mathcal{M})$, then $|ux|^2 = x^*u^*ux = x^*x = |x|^2$, so $ux \in L^2(\mathcal{M},\tau)$. For any $y \in \mathcal{M}$, Proposition 1.12 says that y is a linear combination of unitaries in $U(\mathcal{M})$, and thus $yx \in L^2(\mathcal{M},\tau)$. Since $L^2(\mathcal{M},\tau)$ is $*$-invariant, it follows that $xy = (y^*x^*)^* \in L^2(\mathcal{M},\tau)$ too. □

Next, we describe the set $L^1(\mathcal{M},\tau)$ and show that it is an $\mathcal{M}$-bimodule too.

Proposition 2.9. *We have that $L^1(\mathcal{M},\tau) = \{xy : x, y \in L^2(\mathcal{M},\tau)\}$, and $L^1(\mathcal{M},\tau)$ is an $\mathcal{M}$-bimodule, in the sense described in Proposition 2.8.*

Proof. To see the left-to-right inclusion, let $x \in L^1(\mathcal{M},\tau)$ with polar decomposition $x = u|x|$. Then $|x|^{1/2} \in L^2(\mathcal{M},\tau)$, and since $L^2(\mathcal{M},\tau)$ is an $\mathcal{M}$-bimodule, $u|x|^{1/2} \in L^2(\mathcal{M},\tau)$, proving the desired inclusion. To prove the remaining statements, let $Y = \operatorname{span}\{xy : x, y \in L^2(\mathcal{M},\tau)\}$, so that we have $L^1(\mathcal{M},\tau) \subset \{xy : x, y \in L^2(\mathcal{M},\tau)\} \subset Y$. Since $L^2(\mathcal{M},\tau)$ is an $\mathcal{M}$-bimodule, it is clear that Y is too. Therefore, we just need to show that $Y \subset L^1(\mathcal{M},\tau)$.

For any $x \in S(\mathcal{M},\tau)$, with polar decomposition $x = u|x|$, recall that we have $|x| = u^*u|x| = u^*x$. Since Y is an $\mathcal{M}$-bimodule, it follows that $x \in Y$ if and only if $|x| \in Y$. By definition, we also have $x \in L^1$ if and only if $|x| \in L^1$, so it will suffice to show that the positive elements of Y are contained in L^1.

Suppose $x \in Y$ is positive, and write $x = \sum_{i=1}^k y_i^* z_i$, with $y_i, z_i \in L^2(\mathcal{M},\tau)$. (The adjoint here is purely for convenience. We may as well write $x = \sum y_i z_i$.) Since x is self-adjoint, we also have $x = \sum_{i=1}^k z_i^* y_i$, and therefore

$$\begin{aligned} x &= \frac{1}{2}\sum_{i=1}^k (y_i^* z_i + z_i^* y_i) \\ &= \frac{1}{4}\sum_{i=1}^k \big((y_i^* + z_i^*)(y_i + z_i) - (y_i^* - z_i^*)(y_i - z_i)\big) \\ &= \frac{1}{4}\sum_{i=1}^k (y_i + z_i)^*(y_i + z_i) - \frac{1}{4}\sum_{i=1}^k (y_i - z_i)^*(y_i - z_i) \end{aligned}$$

Thus, $0 \le x \le \frac{1}{4}\sum_{i=1}^k (y_i + z_i)^*(y_i + z_i)$. Since $y_i + z_i \in L^2(\mathcal{M},\tau)$ for every $i = 1, \dots, k$, the monotonicity of τ implies that $\tau(x) < \infty$, so $x \in L^1(\mathcal{M},\tau)$. □

In light of this description of the space $L^1(\mathcal{M},\tau)$, we can now see that $L^1(\mathcal{M},\tau)$ is generated by its positive elements.

Corollary 2.10. *Every element in $L^1(\mathcal{M},\tau)$ is a linear combination of at most three positive elements of $L^1(\mathcal{M},\tau)$.*

Proof. We need to show that for any $x, y \in L^2(\mathcal{M}, \tau)$, x^*y is a linear combination of three positive elements of $L^1(\mathcal{M}, \tau)$. Indeed,

$$x^*y = \frac{1}{2}\Big((x+y)^*(x+y) + i(x-iy)^*(x-iy) - (1+i)(x^*x + y^*y)\Big),$$

as one can verify by expanding the right-hand side. □

Since $L^1(\mathcal{M}, \tau)$ is the span of its positive elements, the properties of τ in Proposition 2.4 imply that τ extends uniquely to a linear functional on $L^1(\mathcal{M}, \tau)$. Abusively, we denote this linear functional by τ as well. In the analogy to measure theory, this is just the integral on the space of integrable functions. The following proposition captures two important properties of this functional.

Proposition 2.11. *For any $x \in L^1(\mathcal{M}, \tau)$, $y \in \mathcal{M}$, we have $\tau(xy) = \tau(yx)$, and $\tau(x^*) = \overline{\tau(x)}$.*

Proof. Fix $x \in L^1(\mathcal{M}, \tau)$ and write $x = \sum_{i=1}^3 a_i x_i$, where $x_i \in L^1(\mathcal{M}, \tau)_+$ and $a_i \in \mathbb{C}$. For any $u \in U(\mathcal{M})$, Corollary 2.5 implies that

$$\tau(uxu^*) = \sum_{i=1}^{3} a_i \tau(u x_i u^*) = \sum_{i=1}^{3} a_i \tau(x_i) = \tau(x).$$

In particular, since $ux \in L^1(\mathcal{M}, \tau)$, we have $\tau(ux) = \tau(u(xu)u^*) = \tau(xu)$. Since every $y \in \mathcal{M}$ is a linear combination of unitaries in $U(\mathcal{M})$ (Proposition 1.12), it follows that $\tau(yx) = \tau(xy)$ for every $y \in \mathcal{M}$, $x \in L^1(\mathcal{M}, \tau)$.

For the second assertion, we note that $x^* = \sum_{i=1}^3 \overline{a_i} x_i$. Since $\tau(x_i) \in \mathbb{R}$ for every $i = 1, \ldots, 3$, we have $\tau(x^*) = \sum_{i=1}^3 \overline{a_i}\tau(x_i) = \sum_{i=1}^3 \overline{a_i \tau(x_i)} = \overline{\tau(x)}$. □

We call the first of these properties the cyclicity property. Next, we establish an important identity, which shows that every set $L^p(\mathcal{M}, \tau)$ is $*$-invariant.

Corollary 2.12. *For all $x \in S(\mathcal{M}, \tau)$ and $1 \leq p < \infty$, $\tau(|x|^p) = \tau(|x^*|^p)$. Thus, $x \in L^p(\mathcal{M}, \tau)$ if and only if $x^* \in L^p(\mathcal{M}, \tau)$, and in this case $\|x\|_{L^p} = \|x^*\|_{L^p}$.*

Proof. Without loss of generality, assume $\tau(|x|^p) < \infty$, so $|x|^p \in L^1(\mathcal{M}, \tau)$. Write $x = u|x|$ for the polar decomposition of x. Applying Proposition 1.47(iii), the cyclity property in Proposition 2.11, and then Proposition 1.47(ii), we have $\tau(|x^*|^p) = \tau(u|x|^p u^*) = \tau(u^*u|x|^p) = \tau(|x|^p)$. □

To supplement our algebraic understanding of $L^2(\mathcal{M}, \tau)$, we now equip the space $L^2(\mathcal{M}, \tau)$ with an inner product. This structure shows that $L^2(\mathcal{M}, \tau)$ is a normed vector space, and eventually the same will follow for $L^1(\mathcal{M}, \tau)$.

Proposition 2.13. *The map $(x,y) \mapsto \tau(y^*x)$ for $x,y \in L^2(\mathcal{M},\tau)$ is an inner product on $L^2(\mathcal{M},\tau)$, and for any $x,y \in L^2(\mathcal{M},\tau)$, $|\tau(xy)| \leq \|x\|_{L^2}\|y\|_{L^2}$.*

Proof. It is clear that this map is sesquilinear. Its conjugate symmetry (i.e., the identity $\tau(x^*y) = \overline{\tau(y^*x)}$) follows from the second statement in Proposition 2.11. To see positive-definiteness, recall that for every $x \in L^2(\mathcal{M},\tau)$, $x^*x \geq 0$. Since the extended trace τ is faithful, if $\tau(x^*x) = 0$, it follows that $x^*x = 0$. Hence, $|x| = 0$, so $x = 0$. Thus, the map is an inner product. The second statement now follows from the Cauchy-Schwarz inequality and Corollary 2.12. □

Since $x^*x = |x|^2$, the norm induced by this inner product agrees with $\|\cdot\|_{L^2}$, so we have proven that $(L^2(\mathcal{M},\tau), \|\cdot\|_{L^2})$ is a normed vector space. In fact, we have an even richer structure on $L^2(\mathcal{M},\tau)$, as we show now.

Proposition 2.14. *The space $L^2(\mathcal{M},\tau)$ is a normed $\mathcal{M}$-bimodule, in that for any $x \in L^2(\mathcal{M},\tau)$ and $y \in \mathcal{M}$, $\|yx\|_{L^2} \leq \|y\|_{L^\infty}\|x\|_{L^2}$ and $\|xy\|_{L^2} \leq \|y\|_{L^\infty}\|x\|_{L^2}$.*

Proof. We will only prove the first inequality. Together with Corollary 2.12, this implies the second inequality. For any $x \in L^2(\mathcal{M},\tau)$ and $y \in \mathcal{M}$, notice that $y^*y \leq \|y\|_{L^\infty}^2 \mathbf{1}_{\mathcal{H}}$ and hence $x^*y^*yx \leq \|y\|_{L^\infty}^2|x|^2$ by Proposition 1.16. Therefore, $\tau(|yx|^2) = \tau(x^*y^*yx) \leq \|y\|_{L^\infty}^2\tau(|x|^2)$, so $\|yx\|_{L^2} \leq \|y\|_{L^\infty}\|x\|_{L^2}$. □

From this, we can now deduce the elementary properties of $L^1(\mathcal{M},\tau)$. Throughout, note that for any $x \in L^1(\mathcal{M},\tau)$, $\||x|^{1/2}\|_{L^2} = \tau(|x|)^{1/2}$. First, we establish the triangle inequality.

Proposition 2.15. *For any $x \in L^1(\mathcal{M},\tau)$, $|\tau(x)| \leq \tau(|x|)$.*

Proof. Write $x = u|x|$ for the polar decomposition of x. Since $|x|^{1/2} \in L^2(\mathcal{M},\tau)$, Proposition 2.14 implies $\|u|x|^{1/2}\|_{L^2} \leq \||x|^{1/2}\|_{L^2}$. By Proposition 2.13,

$$|\tau(x)| = |\tau(u|x|^{1/2}|x|^{1/2})| \leq \|u|x|^{1/2}\|_{L^2}\||x|^{1/2}\|_{L^2} \leq \tau(|x|),$$

as required. □

Proposition 2.16. *For any $x \in L^1(\mathcal{M},\tau)$ and $y \in \mathcal{M}$, we have*

$$|\tau(xy)| \leq \tau(|xy|) \leq \|y\|_{L^\infty}\tau(|x|).$$

Proof. Since L^1 is an $\mathcal{M}$-bimodule, the first inequality is just Proposition 2.15. For the second inequality, write $x = u|x|$ and $xy = v|xy|$ for the polar decompositions of x and xy respectively. Then $|xy| = (v^*u|x|^{1/2})(|x|^{1/2}y)$. By Propositions 2.13 and 2.14, we therefore have

$$\tau(|xy|) \leq \|v^*u|x|^{1/2}\|_{L^2}\||x|^{1/2}y\|_{L^2} \leq \|y\|_{L^\infty}\||x|^{1/2}\|_{L^2}^2 = \|y\|_{L^\infty}\tau(|x|),$$

as required. □

Corollary 2.17. *The pair* $(L^1(\mathcal{M},\tau), \|\cdot\|_{L^1})$ *is a normed* $\mathcal{M}$*-bimodule.*

Proof. Let $x, y \in L^1(\mathcal{M},\tau)$ and write $x+y = u|x+y|$ for the polar decomposition of $x+y$, so that $\tau(|x+y|) = \tau(u^*(x+y))$. By Proposition 2.9, we have that $u^*x, u^*y \in L^1(\mathcal{M},\tau)$, so $\tau(u^*(x+y)) = \tau(u^*x) + \tau(u^*y)$. Now, applying the triangle inequality and Proposition 2.16, we have

$$|\tau(u^*(x+y))| \le |\tau(u^*x)| + |\tau(u^*y)| \le \tau(|x|) + \tau(|y|),$$

which proves the triangle inequality for $\|\cdot\|_{L^1}$. Homogeneity is clear, and positive-definiteness follows from the faithfulness of τ, so $\|\cdot\|_{L^1}$ is a norm. The bimodule property now follows from Proposition 2.16 and Corollary 2.12. □

Before, proceeding further, we establish three more useful results. The first is a strengthening of the Cauchy-Schwarz inequality.

Proposition 2.18. *For any* $x, y \in L^2(\mathcal{M},\tau)$*,* $\|xy\|_{L^1} \le \|x\|_{L^2}\|y\|_{L^2}$*.*

Proof. Let $xy = u|xy|$ be the polar decomposition of xy, so that $\|xy\|_{L^1} = \tau(u^*xy)$. By Propositions 2.13 and 2.14, we have $|\tau(u^*xy)| \le \|x\|_{L^2}\|y\|_{L^2}$. □

We now obtain a cyclicity property similar to the one in Proposition 2.11.

Proposition 2.19. *For any* $x, y \in L^2(\mathcal{M},\tau)$*,* $\tau(xy) = \tau(yx)$*.*

Proof. Arguing just as in Proposition 2.13, we define an inner product on $L^2(\mathcal{M},\tau)$ given by $(x, y) \mapsto \tau(xy^*)$. Since $\tau(xx^*) = \tau(x^*x)$ for all $x \in L^2(\mathcal{M},\tau)$, these two inner products on $L^2(\mathcal{M},\tau)$ induce the same norm. By polarisation, the inner products therefore coincide. That is, $\tau(y^*x) = \tau(xy^*)$ for all $x, y \in L^2(\mathcal{M},\tau)$. □

The following result will be useful in the following subsection.

Proposition 2.20. *Let* $x \in S(\mathcal{M},\tau)$*, and assume that for every* $y \in L^1 \cap \mathcal{M}$*, we have* $yx \in L^1(\mathcal{M},\tau)$ *with* $\tau(yx) = 0$*. Then* $x = 0$*.*

Proof. For any $y \in L^1 \cap \mathcal{M}$, let $xy = u|xy|$ be the polar decomposition of xy. Since $L^1(\mathcal{M},\tau)$ is an $\mathcal{M}$-bimodule, we have $u^*y \in L^1 \cap \mathcal{M}$, so $\tau(|yx|) = \tau(u^*yx) = 0$. Therefore, $yx = 0$ for every $y \in L^1 \cap \mathcal{M}$. Let $x = w|x|$ be the polar decomposition of x, and recall that ww^* is the orthogonal projection onto $\overline{\mathrm{ran}(x)}$. If $x \ne 0$, Proposition 1.48 implies that there exists a nonzero projection $p \in L^1 \cap \mathcal{M}$ with $\mathrm{ran}(p) \subset \overline{\mathrm{ran}(x)}$. But then $px \ne 0$, a contradiction. □

2.2.1 Completeness and Preduality of $L^1(\mathcal{M},\tau)$

In the commutative setting, L^1 is a Banach space whose Banach dual is L^∞. This fact generalises to the noncommutative setting and we prove it here. The following observation will be useful to us. For any $z \in L^1(\mathcal{M},\tau)$ with polar decomposition $z = u|z|$, consider the decomposition of z in Corollary 2.10 with $x = (u|z|^{1/2})^*$ and $y = |z|^{1/2}$. It is easy to check that the L^1-norm of each term in this decomposition is bounded above by $4\|z\|_{L^1}$.

Theorem 2.21. *$L^1(\mathcal{M},\tau)$ is a Banach space.*

Proof. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in $L^1(\mathcal{M},\tau)$ with $\sum_{n=1}^\infty \|x_n\|_{L^1} < \infty$. We are required to show that the infinite series $\sum_{n=1}^\infty x_n$ converges in $L^1(\mathcal{M},\tau)$. Via the decomposition in Corollary 2.10, we may assume that each $x_n \geq 0$. By forgetting the first terms in the series and grouping terms together if necessary, we may assume that $\|x_n\|_{L^1} < 2^{-2n}$ for every $n \in \mathbb{N}$. Now, define spectral projections $p_n = E_{x_n}[0, 2^{-n}]$, so that $\|x_n p_n\|_{L^\infty} \leq 2^{-n}$. We have that $p_n^\perp = \int_{(2^{-n},\infty)} \mathrm{d}E_{x_n}(\lambda) \leq 2^n x_n$, so $\tau(p_n^\perp) \leq 2^n \tau(x_n) < 2^{-n}$.

We now define a candidate limit for the series $\sum_{n=1}^\infty x_n$, following a construction given to me by my supervisor. For each $n \in \mathbb{N}$, let $q_n = \bigwedge_{m\geq n} p_m \in \mathcal{M}$, as in Proposition 1.11. Set $D = \bigcap_{n=1}^\infty \operatorname{dom}(x_n) \cap \bigcup_{n=1}^\infty \operatorname{ran}(q_n)$. Since the sequence of projections q_n is increasing in n, D is a vector subspace of $\mathcal{H}$. Fix $\xi \in \bigcap_{n=1}^\infty \operatorname{dom}(x_n) \cap \operatorname{ran}(q_n)$. For any $m \geq n$, we have $\xi \in \operatorname{ran}(p_m)$, so $\|x_m \xi\|_{\mathcal{H}} = \|x_m p_m \xi\|_{\mathcal{H}} \leq 2^{-m}\|\xi\|_{\mathcal{H}}$, since $\|x_m p_m\|_{L^\infty} \leq 2^{-m}$. Therefore, the series $\sum_{m=1}^\infty x_m \xi$ converges in $\mathcal{H}$, so we define an operator x with $\operatorname{dom}(x) = D$ and $x\xi = \sum_{m=1}^\infty x_m \xi$, for all $\xi \in D$. We claim that x is closable and $\overline{x}$ is τ-measurable. Later, we will show that $\overline{x}$ is the desired limit in $L^1(\mathcal{M},\tau)$.

First, we show that D is τ-dense. Since a countable intersection of τ-dense domains is τ-dense (Proposition 1.40), we only need to establish the τ-density of $\bigcup_{n=1}^\infty \operatorname{ran}(q_n)$. For all $n, k \in \mathbb{N}$, let $q_{n,k} = \bigwedge_{m=n}^{n+k} p_m$. From Definition 1.10, $q_{n,k}^\perp = \bigvee_{m=n}^{n+k} p_m^\perp \leq \sum_{m=n}^{n+k} p_m^\perp$, so

$$\tau(q_{n,k}^\perp) \leq \sum_{m=n}^{n+k} \tau(p_m^\perp) \leq \sum_{m=n}^{n+k} 2^{-m} \leq 2^{-n+1}.$$

It is easily seen that $q_{n,k} \searrow q_n$ and so $q_{n,k}^\perp \nearrow q_n^\perp$. By the normality of τ, it follows that $\tau(q_n^\perp) \leq 2^{-n+1}$, and by Proposition 1.40 this implies D is τ-dense.

In particular, x is densely defined, so to show that x is closable, it suffices to show that x is symmetric. Fix $\xi, \eta \in D$. Since each x_n is self-adjoint, we have

$$\langle x\xi, \eta\rangle = \sum_{n=1}^{\infty}\langle x_n\xi, \eta\rangle = \sum_{n=1}^{\infty}\langle \xi, x_n\eta\rangle = \langle \xi, x\eta\rangle.$$

From this, it follows that $\overline{x}$ is τ-densely defined too. To see that $\overline{x}$ is τ-measurable, we must show that $y\overline{x} \subseteq \overline{x}y$ for all $y \in \mathcal{M}'$. As a first step, we show that $yx \subseteq xy$. Fix $y \in \mathcal{M}'$ and $\xi \in \bigcap_{m=1}^{\infty} \mathrm{dom}(x_m) \cap \mathrm{ran}(q_n)$. Since $q_n \in \mathcal{M}$, we have $y\xi \in \mathrm{ran}(q_n)$. Moreover, for every $m \in \mathbb{N}$, since x_m is τ-measurable, we have $y\xi \in \mathrm{dom}(x_m)$, hence $y\xi \in D$, and $yx_m\xi = x_m y\xi$. Since y is continuous, we have

$$yx\xi = \sum_{m=1}^{\infty} yx_m\xi = \sum_{m=1}^{\infty} x_m y\xi = xy\xi.$$

This shows that $yx \subseteq xy$. Now, fix $\xi \in \mathrm{dom}(\overline{x})$. By definition, there exists a sequence $\{\xi_k\}$ in $\mathrm{dom}(x)$ with $\xi_k \to \xi$ and $x\xi_k \to \overline{x}\xi$. Since y is bounded, it follows that $y\xi_k \to y\xi$ and $yx\xi_k \to y\overline{x}\xi$. We have already shown that for every $k \in \mathbb{N}$, $y\xi_k \in \mathrm{dom}(x)$ and $xy\xi_k = yx\xi_k$. Since $\overline{x}$ is closed, it follows that $y\xi \in \mathrm{dom}(\overline{x})$ and $\overline{x}y\xi = y\overline{x}\xi$. That is, $y\overline{x} \subseteq \overline{x}y$, so $\overline{x}$ is τ-measurable.

We claim now that $\overline{x} = \sum_{n=1}^{\infty} x_n$ in $L^1(\mathcal{M}, \tau)$. For any $\xi \in D$, we have $\langle x\xi, \xi\rangle = \sum_{n=1}^{\infty}\langle x_n\xi, \xi\rangle$, and it follows that this same equality holds for any $\xi \in \mathrm{dom}(\overline{x}) \cap \bigcap_{n\in\mathbb{N}} \mathrm{dom}(x_n)$. That is, $\sum_{n=1}^{k} x_n \nearrow \overline{x}$. By the normality of the extended trace τ, it follows that $\tau(\overline{x}) = \lim_{k\to\infty} \tau(\sum_{n=1}^{k} x_n) < \infty$. Thus, $\tau(\overline{x} - \sum_{n=1}^{k} x_k) \searrow 0$ as $k \to \infty$, so $\overline{x} = \sum_{n=1}^{\infty} x_n$ in $L^1(\mathcal{M}, \tau)$. □

The following proof is adapted Lemmas 3.4.22 and 3.4.23 in [10].

Theorem 2.22. *Recall that $\mathcal{M}_*$ is the Banach space of σ-weakly continuous linear functionals on $\mathcal{M}$. To each $x \in L^1(\mathcal{M}, \tau)$, associate a linear functional $\phi_x \in \mathcal{M}^*$, given by $\phi_x(y) = \tau(xy)$. The map $x \mapsto \phi_x$ is an isometric isomorphism $L^1(\mathcal{M}, \tau) \cong \mathcal{M}_*$.*

Proof. First, by Proposition 2.16, the linear functional ϕ_x is bounded on $\mathcal{M}$, with $\|\phi_x\|_{\mathcal{M}^*} \le \|x\|_{L^1}$. To see that $\|\phi_x\|_{\mathcal{M}^*} = \|x\|_{L^1}$, let $x = u|x|$ be the polar decomposition of x. Since $|x| = u^*x$ and $\|u^*\|_{L^\infty} = 1$, it follows from the cyclicity property in Proposition 2.11 that $\|x\|_{L^1} = \tau(u^*x) = \tau(xu^*) \le \|\phi_x\|_{\mathcal{M}^*}$.

Next, we show that ϕ_x is σ-weakly continuous. Assume at first that $x \ge 0$. For any $y \in \mathcal{M}_+$, we have $x^{1/2}yx^{1/2} \ge 0$, and by Proposition 2.19, we have $\tau(xy) = \tau(x^{1/2}yx^{1/2}) \ge 0$, so ϕ_x is a positive linear functional. Moreover, if $\{y_\alpha\}$ is an increasing net in $\mathcal{M}_+$ with $y_\alpha \nearrow y$, then $x^{1/2}y_\alpha x^{1/2} \nearrow x^{1/2}yx^{1/2}$. Since the extended trace τ is normal, it follows that $\tau(xy_\alpha) \nearrow \tau(xy)$, proving that ϕ_x is normal. By Proposition 1.21, we conclude that ϕ_x is σ-weakly continuous when x is positive. By Corollary 2.10, any $x \in L^1(\mathcal{M}, \tau)$ is a linear combination of positive elements in $L^1(\mathcal{M}, \tau)$, so ϕ_x is σ-weakly continuous for all $x \in L^1(\mathcal{M}, \tau)$.

Thus, the map $x \mapsto \phi_x$ is an isometry from $L^1(\mathcal{M}, \tau)$ to $\mathcal{M}_*$. Therefore, its image is closed in $\mathcal{M}_*$. For the sake of contradiction, assume that this map is not surjective. By the Hahn-Banach theorem, there exists $0 \neq y \in (\mathcal{M}_*)^*$ with $y(\phi_x) = 0$ for all $x \in L^1(\mathcal{M}, \tau)$. By identifying $(\mathcal{M}_*)^*$ with $\mathcal{M}$ as in Proposition 1.14, this says exactly that there exists a nonzero $y \in \mathcal{M}$ such that $\tau(xy) = 0$ for all $x \in L^1(\mathcal{M}, \tau)$. In particular, $\tau(xy) = 0$ for all $x \in L^1 \cap \mathcal{M}$, and so Proposition 2.20 implies that $y = 0$, a contradiction. $\square$

The desired duality result now follows from Proposition 1.14.

Corollary 2.23. *We have an isometric isomorphism $\mathcal{M} \cong L^1(\mathcal{M}, \tau)^*$, via the pairing $\langle x, y\rangle = \tau(xy)$ with $x \in L^1(\mathcal{M}, \tau)$ and $y \in \mathcal{M}$. Under this isomorphism, the σ-weak topology on $\mathcal{M}$ coincides with the weak-* topology on $L^1(\mathcal{M}, \tau)^*$.*

2.3 Interpolation of Noncommutative L^p Spaces

To study noncommutative L^p spaces for $1 < p < \infty$, it will be convenient to use the language and machinery of interpolation theory. This theory is powerful. It will help us establish Hölder-like inequalities for $L^p(\mathcal{M}, \tau)$, and it will show that each $L^p(\mathcal{M}, \tau)$ is Banach. Additionally, we will establish the usual results about the Banach duality of L^p spaces, as well as a noncommutative analogue to the Riesz-Thorin theorem. We begin by quickly introducing general interpolation theory as laid out in [2], before returning to the study of noncommutative L^p spaces. The overarching approach taken in this section is inspired by [21].

2.3.1 Complex Interpolation

Let A_0 and A_1 be Banach spaces. We say that the pair (A_0, A_1) is a **compatible couple** if there exists a normed vector space $\mathcal{A}$ that admits continuous linear inclusions $A_0, A_1 \hookrightarrow \mathcal{A}$. Consider the subspaces $A_0 \cap A_1$ and $A_0 + A_1$ inside $\mathcal{A}$. These admit natural norms, namely,

$$\|a\|_{A_0 \cap A_1} := \max\{\|a\|_{A_0}, \|a\|_{A_1}\} \quad \text{and} \quad \|a\|_{A_0+A_1} := \inf_{\substack{a_0+a_1=a \\ a_0 \in A_0, a_1 \in A_1}} \{\|a_0\|_{A_0} + \|a_1\|_{A_1}\},$$

and these make $A_0 \cap A_1$ and $A_0 + A_1$ into Banach spaces (see Lemma 2.3.1 in [2]). We will say that a Banach space A is an **intermediate space** with respect to (A_0, A_1) if we have continuous inclusions $A_0 \cap A_1 \hookrightarrow A \hookrightarrow A_0 + A_1$. For example, A_0 and A_1 are themselves intermediate spaces with respect to (A_0, A_1).

To construct complex interpolation spaces, we must study Banach-valued complex-analytic functions. These are functions $f : U \to X$, where $U \subset \mathbb{C}$ is open and X is a Banach space, for which the limit

$$f'(z) \coloneqq \lim_{w \to z} \frac{f(w) - f(z)}{w - z} \tag{2.1}$$

exists in X for every $z \in U$. For basic results about such functions, see Appendix A. Classical complex analysis is also important for us in this section. In particular, we will frequently use the following result.

Theorem 2.24 (Three-line theorem, Theorem 12.8 in [27])**.** *Let Ω denote the open strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$, with closure $\overline{\Omega}$. Assume a function $f : \overline{\Omega} \to \mathbb{C}$ is bounded and continuous on $\overline{\Omega}$ and holomorphic on Ω. Let $M_0 = \sup_{t\in\mathbb{R}} |f(it)|$ and $M_1 = \sup_{t\in\mathbb{R}} |f(1 + it)|$. Then for any $z \in \Omega$, one has $|f(z)| \leq M_0^{1-\theta} M_1^{\theta}$, where $\theta = \mathrm{Re}(z)$.*

Let (A_0, A_1) be a compatible couple of Banach spaces, and recall that $A_0 + A_1$ is Banach too. Let Ω again denote the open strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$, let $\mathcal{F}(A_0, A_1)$ denote the space of functions $f : \overline{\Omega} \to A_0 + A_1$ that are bounded and continuous on $\overline{\Omega}$ and analytic on Ω, and such that

(i) the map $t \mapsto f(it)$ is a continuous map from $\mathbb{R}$ to A_0, and $f(it) \to 0$ in A_0 as $|t| \to \infty$; and,

(ii) the map $t \mapsto f(1+it)$ is a continuous map from $\mathbb{R}$ to A_1, and $f(1+it) \to 0$ in A_1 as $|t| \to \infty$.

Lemma 4.1.1 in [2] shows that $\mathcal{F}(A_0, A_1)$ is a Banach space with norm

$$\|f\|_{\mathcal{F}} = \max\{\sup_{t\in\mathbb{R}} \|f(it)\|_{A_0}, \sup_{t\in\mathbb{R}} \|f(1 + it)\|_{A_1}\}, \tag{2.2}$$

for all $f \in \mathcal{F}(A_0, A_1)$. With this, we define complex interpolation spaces.

Definition 2.25. For any $0 < \theta < 1$ and $a \in A_0 + A_1$, we put $a \in (A_0, A_1)_\theta$ (or more concisely, $a \in A_\theta$) if there exists $f \in \mathcal{F}(A_0, A_1)$ with $f(\theta) = a$. One easily sees that the set A_θ is a vector subspace of $A_0 + A_1$. For each $a \in A_\theta$, we define

$$\|a\|_\theta = \inf\{\|f\|_{\mathcal{F}} : f \in \mathcal{F}(A_0, A_1),\ f(\theta) = a\}. \tag{2.3}$$

The following result captures the Riesz-Thorin property that we wish to extend to noncommutative L^p spaces.

Proposition 2.26 (Theorem 4.1.2 in [2])**.** *The pair $(A_\theta, \|\cdot\|_\theta)$ is a Banach space and is intermediate with respect to (A_0, A_1). Moreover, for any compatible couples (A_0, A_1) and (B_0, B_1), if $T : A_0 + A_1 \to B_0 + B_1$ is a linear map that restricts to bounded linear maps $T : A_0 \to B_0$ and $T : A_1 \to B_1$, then T restricts to a bounded map $T : A_\theta \to B_\theta$ with*

$$\|T\|_{A_\theta \to B_\theta} \leq \|T\|_{A_0 \to B_0}^{1-\theta} \|T\|_{A_1 \to B_1}^{\theta}. \tag{2.4}$$

Finally, we state two important results about complex interpolation spaces. Throughout, let (A_0, A_1) be a compatible couple.

Proposition 2.27 (Corollary 4.5.2 in [2])**.** *If $A_0 \cap A_1$ is dense in both A_0 and A_1, and at least one of A_0 or A_1 is reflexive, then A_θ is reflexive and $A_\theta^* = (A_0^*, A_1^*)_\theta$ for all $0 < \theta < 1$.*

Proposition 2.28 (Theorem 4.6.1 in [2])**.** *Let (A_0, A_1) be a compatible couple, and fix $0 < \theta_1, \theta_2 < 1$. We have that $(A_{\theta_1}, A_{\theta_2})$ is a compatible couple, and for any $0 < \eta < 1$, we have $(A_{\theta_1}, A_{\theta_2})_\eta = (A_0, A_1)_\theta$, where $\theta = (1-\eta)\theta_1 + \eta\theta_2$.*

2.3.2 Hölder-like Inequalities on L^p Spaces

The pair $(L^1, \mathcal{M})$ is a compatible couple of Banach spaces. Indeed, we can construct the spaces $L^1 \cap \mathcal{M}$ and $L^1 + \mathcal{M}$ explicitly as subspaces of $S(\mathcal{M}, \tau)$ and equip them with the norms in the previous section. We wish to identify the spaces $L^p(\mathcal{M}, \tau)$ introduced in Definition 2.6 with the interpolation space $(L^1, \mathcal{M})_\theta$, for some suitable θ. As a first step, we will use the language of interpolation theory to establish Hölder-like inequalities. We begin by proving the following inclusion of sets, which suggests that L^p is an intermediate space with respect to the compatible couple $(L^1, \mathcal{M})$.

Proposition 2.29. *For any $1 \leq p \leq \infty$, we have*

$$L^1(\mathcal{M}, \tau) \cap \mathcal{M} \subset L^p(\mathcal{M}, \tau) \subset L^1(\mathcal{M}, \tau) + \mathcal{M}.$$

Proof. For $p = 1$ or $p = \infty$, these inclusions are obvious, so assume $1 < p < \infty$.

First, suppose $x \in L^1 \cap \mathcal{M}$. By Proposition 2.16 and Proposition 1.35(iv),

$$\tau(|x|^p) \leq \left\| |x|^{p-1} \right\|_{L^\infty} \|x\|_{L^1} \leq \|x\|_{L^\infty}^{p-1} \|x\|_{L^1}.$$

So $x \in L^p(\mathcal{M}, \tau)$. In particular,

$$\tau(|x|^p)^{1/p} \leq \|x\|_{L^\infty}^{1-1/p} \|x\|_{L^1}^{1/p} \leq \max\{\|x\|_{L^\infty}, \|x\|_{L^1}\} = \|x\|_{L^1 \cap \mathcal{M}}. \tag{2.5}$$

Now, suppose $x \in L^p(\mathcal{M}, \tau)$. Let $r = \|x\|_{L^p}$, and write $x_0 = xE_{|x|}([0, r])$ and $x_1 = xE_{|x|}(r, \infty)$, so that $x = x_0 + x_1$. By Proposition 1.47(i), $|x_0| = |x|E_{|x|}([0, r])$, so $x_0 \in \mathcal{M}$ with $\|x_0\|_{L^\infty} \leq r$. Similarly, $|x_1| = |x|E_{|x|}(r, \infty) = \int_{(r,\infty)} \lambda \, \mathrm{d}E_{|x|}(\lambda)$. Since $\lambda \leq r^{1-p}\lambda^p$ for all $\lambda \in (r, \infty)$, we have $|x_1| \leq r^{1-p}|x|^p$, so $x_1 \in L^1(\mathcal{M}, \tau)$, with $\|x_1\|_{L^1} \leq r^{1-p}\|x\|_{L^p}^p = \|x\|_{L^p}$. In particular,

$$\|x\|_{L^1 + \mathcal{M}} \leq \|x_0\|_{L^\infty} + \|x_1\|_{L^1} \leq 2\|x\|_{L^p}, \tag{2.6}$$

so $L^p(\mathcal{M}, \tau) \subset L^1 + \mathcal{M}$. $\square$

Once we show that $L^p(\mathcal{M}, \tau)$ is a normed vector space, the estimates (2.5) and (2.6) will imply that these inclusions are in fact continuous. Next, we consider an important pairing between $L^1 \cap \mathcal{M}$ and $L^1 + \mathcal{M}$, which will provide a basis for our Hölder-like inequalities. To do this, we make the following observation.

Lemma 2.30. *If $x \in L^1 + \mathcal{M}$ and $y \in L^1 \cap \mathcal{M}$, then $xy, yx \in L^1$.*

Proof. Write $x = x_0 + x_1$, with $x_0 \in \mathcal{M}$ and $x_1 \in L^1$, so that $xy = x_0y + x_1y$. Since L^1 is an $\mathcal{M}$-bimodule, we have $x_0y \in L^1$ and $x_1y \in L^1$, so $xy \in L^1$. The proof that $yx \in L^1$ is similar. $\square$

Proposition 2.31. *The map $(L^1 + \mathcal{M}) \times (L^1 \cap \mathcal{M}) \to \mathbb{C}$ given by $(x, y) \mapsto \tau(xy)$ is a continuous bilinear form, with $|\tau(xy)| \leq \|x\|_{L^1+\mathcal{M}}\|y\|_{L^1\cap\mathcal{M}}$. This determines a continuous inclusion $L^1 + \mathcal{M} \hookrightarrow (L^1 \cap \mathcal{M})^*$ given by $x \mapsto \phi_x$, where $\phi_x(y) = \tau(xy)$.*

Proof. Bilinearity is clear, and continuity will follow from the desired inequality. Writing $x = x_0 + x_1$, with $x_1 \in L^1$ and $x_0 \in \mathcal{M}$, Proposition 2.16 and the cyclicity property in Proposition 2.11 imply that

$$\begin{aligned}
|\tau(xy)| &\leq |\tau(x_1y)| + |\tau(x_0y)| \\
&\leq \|x_1\|_{L^1}\|y\|_{L^\infty} + \|x_0\|_{L^\infty}\|y\|_{L^1} \\
&\leq (\|x_1\|_{L^1} + \|x_0\|_{L^\infty}) \max\{\|y\|_{L^\infty}, \|y\|_{L^1}\}.
\end{aligned}$$

Since we can make $\|x_1\|_{L^1} + \|x_0\|_{L^\infty}$ arbitrarily close to $\|x\|_{L^1+\mathcal{M}}$, this proves the desired inequality. For any $x \in L^1 + \mathcal{M}$, the linear functional ϕ_x is therefore bounded with $\|\phi_x\| \leq \|x\|_{L^1+\mathcal{M}}$. Moreover, if $\phi_x = 0$, then Proposition 2.20 and the cyclicity property imply $x = 0$, so the mapping $x \mapsto \phi_x$ is injective. $\square$

When $x \in L^1 + \mathcal{M}$ and $y \in L^1 \cap \mathcal{M}$, we will write $\langle x, y\rangle := \tau(xy)$, and in view of the cyclicity property in Proposition 2.11, we notice that $\langle x, y\rangle = \tau(yx)$ as well. The following definition, and the construction that follows, will be convenient for our Hölder-like inequalities.

Definition 2.32. Let $\mathcal{D} = \{x \in L^1 \cap \mathcal{M} : \sigma(|x|) \subset \{0\} \cup [\frac{1}{n}, n], n \in \mathbb{N}\}$. Also, given $x \in S(\mathcal{M}, \tau)$, define $x_n = xE_{|x|}([\frac{1}{n}, n])$ for every $n \in \mathbb{N}$.

The set $\mathcal{D}$ will be a useful class of operators on which to apply complex analysis. We claim that, for a large class of $x \in S(\mathcal{M}, \tau)$, $x_n \in \mathcal{D}$.

Lemma 2.33. *Let $x \in S(\mathcal{M}, \tau)$ and assume $\tau(E_{|x|}[s, \infty)) < \infty$ for all $s > 0$. Then, for any $n \in \mathbb{N}$, $x_n \in \mathcal{D}$. In particular, for any $1 \le p < \infty$, this holds for all $x \in L^p(\mathcal{M}, \tau)$.*

Proof. By Proposition 1.47(i), $|x_n| = |x|E_{|x|}([\frac{1}{n}, n]) = \int_{[1/n,n]} \lambda \, \mathrm{d}E_{|x|}(\lambda)$. By Proposition 1.35(iv), $x_n \in \mathcal{M}$, and since $|x_n| \le nE_{|x|}([\frac{1}{n}, n]) \le nE_{|x|}([\frac{1}{n}, \infty))$, our assumption implies $\tau(|x_n|) < \infty$. Finally, it follows from Proposition 1.35(iii) that $\sigma(|x_n|) \subset \{0\} \cup [\frac{1}{n}, n]$, so $x_n \in \mathcal{D}$.

Now, assuming $x \in L^p(\mathcal{M}, \tau)$ for $1 \le p < \infty$, and fixing $s > 0$, we need to show that $\tau(E_{|x|}[s, \infty)) < \infty$. For all $\lambda \in [s, \infty)$, we have $1 \le s^{-p}\lambda^p$, so $E_{|x|}[s, \infty) \le s^{-p}|x|^p$, and hence $\tau(E_{|x|}[s, \infty)) \le s^{-p}\tau(|x|^p) < \infty$. □

The next proposition collects several density results about $\mathcal{D}$.

Proposition 2.34. *If $x \in L^p(\mathcal{M}, \tau)$, $1 \le p < \infty$, then as $n \to \infty$ we have $\tau(|x - x_n|^p) = \tau(|x|^p) - \tau(|x_n|^p) \to 0$ and $\sup_{m \ge n} \tau(|x_m - x_n|^p) \to 0$. Moreover, $x_n \to x$ in $L^1 + \mathcal{M}$, and if $x \in L^1 \cap \mathcal{M}$, then $x_n \to x$ in $L^1 \cap \mathcal{M}$.*

Proof. By Proposition 1.47(i), $|x - x_n|^p = |x|^p E_{|x|}([0, \frac{1}{n}) \cup (n, \infty)) = |x|^p - |x_n|^p$. It is clear from Proposition 1.44 that $|x_n|^p \nearrow |x|^p$, and since $\tau(|x|^p) < \infty$, the normality of τ implies that $\tau(|x - x_n|^p) = \tau(|x|^p) - \tau(|x_n|^p) \searrow 0$. From (2.6), it follows that $\|x - x_n\|_{L^1+\mathcal{M}} \to 0$. The convergence $\sup_{m\ge n} \tau(|x_m - x_n|^p) \to 0$ as $n \to \infty$ is proven in almost identical fashion.

Now, assume $x \in L^1 \cap \mathcal{M}$, so that $\sigma(|x|) \subset [0, \|x\|_{L^\infty}]$. We have already seen that $\|x - x_n\|_{L^1} \to 0$. Convergence in the L^∞-norm follows from Proposition 1.35(iv), since $|x - x_n| = |x|E_{|x|}([0, 1/n))$ when $n \ge \|x\|_{L^\infty}$. □

Once we show that $L^p(\mathcal{M}, \tau)$ is a normed vector space, this proposition will say that $\mathcal{D}$ is dense in $L^p(\mathcal{M}, \tau)$, for every $1 \le p < \infty$. The same is not true for $p = \infty$, but we do have the following density result.

Proposition 2.35. *$L^1 \cap \mathcal{M}$ is a σ-weakly dense subset of $\mathcal{M}$.*

Proof. Let $x \in \mathcal{M}$ with polar decomposition $x = u|x|$. By Proposition 1.24 we choose a net $\{y_\alpha\}$ in $L^1 \cap \mathcal{M}_+$ with $y_\alpha \nearrow |x|$. By Proposition 1.18, $y_\alpha \to |x|$ weakly, and since $\|y_\alpha\|_{L^\infty} \leq \|x\|_{L^\infty}$ for all α, the last assertion in Proposition 1.2 implies $y_\alpha \to |x|$ σ-weakly. By Proposition 1.3, it follows that $uy_\alpha \to x$ σ-weakly. Since $L^1(\mathcal{M}, \tau)$ and $\mathcal{M}$ are both $\mathcal{M}$-bimodules, we have $uy_\alpha \in L^1 \cap \mathcal{M}$, so $L^1 \cap \mathcal{M}$ is σ-weakly dense in $\mathcal{M}$. $\square$

As a final step before establishing the desired Hölder inequalities, we prove a technical lemma that will be pivotal in the remainder of this section.

Lemma 2.36. *Suppose $x \in \mathcal{D}$. For any $p > 0$, the function $f : \overline{\Omega} \to L^1 \cap \mathcal{M}$ defined by $f(z) = |x|^{pz}$ is bounded and continuous on $\overline{\Omega}$, and analytic on Ω.*

Proof. Choose $n \in \mathbb{N}$ so that $\sigma(|x|) \subset \{0\} \cup [\frac{1}{n}, n]$. First, notice that

$$\left||x|^{pz}\right| = \int_{[1/n,n]} |\lambda^{pz}| \,\mathrm{d}E_{|x|}(\lambda) = \int_{[1/n,n]} \lambda^{p\mathrm{Re}(z)} \,\mathrm{d}E_{|x|}(\lambda).$$

For all $\lambda \in [\frac{1}{n}, n]$, since $0 \leq \mathrm{Re}(z) \leq 1$, we have $\lambda^{p\mathrm{Re}(z)} \leq n^p \leq n^{p+1}\lambda$. By Proposition 1.35(iv), it follows that $\|f(z)\|_{L^\infty} \leq n^p$ for all $z \in \overline{\Omega}$. It also follows that $\|f(z)\|_{L^1} \leq n^{p+1}\tau(|x|)$, for all $z \in \overline{\Omega}$, proving that f is bounded as a map into $L^1 \cap \mathcal{M}$. Next, we demonstrate the continuity of f. For any $w, z \in \overline{\Omega}$,

$$|x|^{pw} - |x|^{pz} = \int_{[1/n,n]} \lambda^{pw} - \lambda^{pz} \,\mathrm{d}E_{|x|}(\lambda). \tag{2.7}$$

Since the map $(\lambda, w) \mapsto \lambda^{pw}$ is continuous on $[\frac{1}{n}, n] \times \overline{\Omega}$, and the set $[\frac{1}{n}, n]$ is compact, the above integrand converges to 0 uniformly in λ as $w \to z$. That is, $\|f(w) - f(z)\|_{L^\infty} \to 0$ as $w \to z$, so f is continuous into $\mathcal{M}$.

Next, we show that f is continuous into $L^1(\mathcal{M}, \tau)$. For any $\varepsilon > 0$ and $z \in \overline{\Omega}$, the uniform convergence asserted above implies that we can choose a neighbourhood U of z such that $|\lambda^{pw} - \lambda^{pz}| < \frac{\varepsilon}{n}$ for all $\lambda \in [\frac{1}{n}, n]$ and $w \in U$. For all $w \in U$, we then have

$$|f(w) - f(z)| \leq \int_{[1/n,n]} \frac{\varepsilon}{n} \,\mathrm{d}E_{|x|}(\lambda) \leq \int_{[1/n,n]} \varepsilon\lambda \,\mathrm{d}E_{|x|}(\lambda) = \varepsilon|x|.$$

Therefore, $\|f(w) - f(z)\|_{L^1} \leq \varepsilon\|x\|_{L^1}$. Hence, $f(w) \to f(z)$ in $L^1(\mathcal{M}, \tau)$ as $w \to z$.

Finally, we show that f is analytic on Ω, as a map into $L^1(\mathcal{M},\tau)\cap\mathcal{M}$. Fix $z\in\Omega$. We claim that, with respect to both the L^∞-norm and the L^1-norm,

$$\lim_{w\to z}\frac{|x|^{pw}-|x|^{pz}}{w-z}=p\ln(|x|)|x|^{pz},$$

where $\ln(|x|)=\int_{[1/n,n]}\ln(\lambda)\,\mathrm{d}E_{|x|}(\lambda)$. For all $w\neq z$, we have

$$\frac{|x|^{pw}-|x|^{pz}}{w-z}=\int_{[1/n,n]}\frac{\lambda^{pw}-\lambda^{pz}}{w-z}\,\mathrm{d}E_x(\lambda). \tag{2.8}$$

Next, notice that the map $g:[1/n,n]\times\Omega\to\mathbb{C}$ given by

$$g(\lambda,w)=\begin{cases}\frac{\lambda^{pw}-\lambda^{pz}}{w-z}, & \text{if } w\neq z\\ p\ln(\lambda)\lambda^{pz}, & \text{if } w=z,\end{cases}$$

is continuous. Since $[\frac{1}{n},n]$ is compact, it follows that the integrand in (2.8) converges uniformly to $p\ln(\lambda)\lambda^{pz}$ as $w\to z$. As in our above proof of continuity, this lifts to convergence in $L^1\cap\mathcal{M}$, as required. $\square$

By an entirely symmetrical argument, one observes the same properties for the map $z\mapsto|x|^{p(1-z)}$. Now, we arrive at our special case of Hölder's inequality, the proof of which is amended from [7].

Proposition 2.37. *Fix $1\leq p\leq\infty$, and let $\frac{1}{p}+\frac{1}{p'}=1$. For any $x\in L^p$ and $y\in L^1\cap\mathcal{M}$, $\|yx\|_{L^1}\leq\|x\|_{L^p}\|y\|_{L^{p'}}$.*

Proof. For $p=\infty$, this is the second inequality in Proposition 2.16, and the $p=1$ result follows by taking adjoints and applying Corollary 2.12. Therefore, we assume $1<p,p'<\infty$.

Given $x\in L^p$ and $y\in L^1\cap\mathcal{M}$, let $yx=w|yx|$ be the polar decomposition of yx. For every $n\in\mathbb{N}$, define x_n and y_n as in Definition 2.32, and let $x_n=u_n|x_n|$ and $y_n=v_n|y_n|$ be the polar decompositions of x_n and y_n respectively. By Lemma 2.33, we have $x_n,y_n\in\mathcal{D}$, and from Lemma 2.36, we see that the functions $g_n,h_n:\overline{\Omega}\to L^1\cap\mathcal{M}$ given by $g_n(z)=|x_n|^{pz}$ and $h_n(z)=|y_n|^{p'(1-z)}$ are continuous and bounded on $\overline{\Omega}$ and analytic on Ω. Since left-multiplication by a fixed element of $\mathcal{M}$ is continuous and linear on $L^1\cap\mathcal{M}$, it follows that the functions $z\mapsto u_n|x_n|^{pz}$ and $z\mapsto w^*v_n|x_n|^{p'(1-z)}$ have these same properties.

Now, define $f_n:\overline{\Omega}\to\mathbb{C}$ by $f_n(z)=\tau(w^*v_n|y_n|^{p'(1-z)}u_n|x_n|^{pz})$. That is, $f_n(z)=\langle w^*v_nh_n(z),u_ng_n(z)\rangle$. Since the bilinear pairing on $(L^1+\mathcal{M})\times(L^1\cap\mathcal{M})$ is continuous and we have a continuous inclusion $L^1\cap\mathcal{M}\hookrightarrow L^1+\mathcal{M}$, we see that

f_n is bounded and continuous on $\overline{\Omega}$ and holomorphic on Ω. (For a more detailed treatment of the complex analysis here, see Appendix A.)

We observe that $f_n(\frac{1}{p}) = \tau(w^* v_n|y_n|u_n|x_n|) = \tau(w^* y_n x_n)$. From Proposition 2.34, we know that $y_n \to y$ in $L^1 \cap \mathcal{M}$ and $x_n \to x$ in $L^1 + \mathcal{M}$ as $n \to \infty$. Since left-multiplication by w^* is continuous on $L^1 \cap \mathcal{M}$, it follows that $w^* y_n \to w^* y$ in $L^1 \cap \mathcal{M}$ as $n \to \infty$, and by Proposition 2.31, it follows that $f_n(\frac{1}{p}) \to \tau(w^* yx) = \|yx\|_{L^1}$ as $n \to \infty$. Therefore, we just need to show that $|f_n(\frac{1}{p})| \leq \|x\|_{L^p}\|y\|_{L^{p'}}$ for every $n \in \mathbb{N}$.

Letting $M_0 = \sup_{t\in\mathbb{R}} |f_n(it)|$ and $M_1 = \sup_{t\in\mathbb{R}} |f_n(1+it)|$, the three-line theorem (Theorem 2.24), specialised to the case $X = \mathbb{C}$, implies $|f_n(\frac{1}{p})| \leq M_0^{1/p'} M_1^{1/p}$, so we are left to estimate M_0 and M_1. For any $t \in \mathbb{R}$, Proposition 2.16 implies

$$\begin{aligned} |f_n(it)| &= |\tau(w^* v_n |y_n|^{p'-ip't} u_n |x_n|^{ipt})| \\ &\leq \|w^* v_n |y_n|^{-ip't}\|_{L^\infty} \|u_n |x_n|^{ipt}\|_{L^\infty} \tau(|y_n|^{p'}) \\ &\leq \tau(|y_n|^{p'}), \end{aligned}$$

so $M_0 \leq \tau(|y|^{p'})$. Similarly, $M_1 \leq \tau(|x_n|^p)$, so $|f_n(\frac{1}{p})| \leq \tau(|y_n|^{p'})^{1/p'}\tau(|x_n|^p)^{1/p}$. By Proposition 1.47(i), we have $|x_n|^p = |x|^p E_{|x|}([\frac{1}{n}, n]) \leq |x|^p$, and similarly $|y_n|^{p'} \leq |y|^{p'}$. Thus, $|f_n(\frac{1}{p})| \leq \|x\|_{L^p}\|y\|_{L^{p'}}$, as desired. $\square$

In fact, Hölder's inequality holds for all $x \in L^p(\mathcal{M}, \tau)$, $y \in L^{p'}(\mathcal{M}, \tau)$ (see, for example, Theorem 3.4 in [36]). Guided by the approach in Lemma 3.5 of [36], we now prove a partial converse to Hölder's inequality.

Proposition 2.38. *Let $x \in L^1 + \mathcal{M}$, $C \geq 0$, and $1 \leq p, p' \leq \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$. If $|\tau(yx)| \leq C\|y\|_{L^{p'}}$ for all $y \in \mathcal{D}$, then $x \in L^p$ and $\|x\|_{L^p} \leq C$.*

Proof. Let $x = u|x|$ be the polar decomposition of x. For $p = \infty$, the result follows from Corollary 2.23 and the density of $\mathcal{D}$ in $L^1(\mathcal{M}, \tau)$ established in Proposition 2.34. Therefore, assume $p < \infty$. The density results in Proposition 2.34, paired with Proposition 2.31, imply that the inequality $|\tau(yx)| \leq C\|y\|_{L^{p'}}\|x\|_{L^{p'}}$ holds for all $y \in L^1 \cap \mathcal{M}$.

Next, we claim that $\tau(E_{|x|}([\varepsilon, \infty))) < \infty$ for all $\varepsilon > 0$. For the sake of contradiction, assume there exists $\varepsilon > 0$ with $\tau(E_{|x|}([\varepsilon, \infty))) = \infty$. By Proposition 1.48, there exists $q \in P(\mathcal{M})$ with $q \leq E_{|x|}([\varepsilon, \infty))$ and $\frac{C^p}{\varepsilon^p} < \tau(q) < \infty$. Since $|x|E_{|x|}([\varepsilon, \infty)) \geq \varepsilon E_{|x|}([\varepsilon, \infty))$, it follows from Proposition 1.16(i) that $q|x|E_{|x|}([\varepsilon, \infty))q \geq \varepsilon q E_{|x|}([\varepsilon, \infty))q$. But since $q \leq E_{|x|}([\varepsilon, \infty))$, we have that $E_{|x|}([\varepsilon, \infty))q = q$ and so $q|x|q \geq \varepsilon q$. By the cyclicity property in Proposition 2.11, we have that $\tau(qu^* x) = \tau(q^2|x|) = \tau(q|x|q) \geq \varepsilon\tau(q)$. Also, since $q \in L^1 \cap \mathcal{M}$ and

$L^1(\mathcal{M},\tau)$ is an $\mathcal{M}$-bimodule, we have $qu^* \in L^1 \cap \mathcal{M}$. Since $q \le E_{|x|}(0,\infty) = u^*u$ by Proposition 1.47(ii), $|qu^*|^2 = uqu^*$ is an orthogonal projection. Therefore, $|qu^*| = uqu^*$, so $|qu^*|^{p'} = uqu^*$ and $\|qu^*\|_{L^{p'}} = \tau(uqu^*)^{1/p'} = \tau(q)^{1/p'}$ by the cyclicity property in Proposition 2.11. Therefore, by our assumption, we have

$$C \ge \frac{|\tau(qu^*x)|}{\|qu^*\|_{L^{p'}}} \ge \frac{\varepsilon\tau(q)}{\tau(q)^{1/p'}} = \varepsilon\tau(q)^{1/p} > C,$$

a contradiction. Therefore, $E_{|x|}([\frac{1}{n},n]) \in L^1 \cap \mathcal{M}$ for every $n \in \mathbb{N}$. We define a sequence $y_n = |x|^{p-1}E_{|x|}([\frac{1}{n},n])u^*$, so that $y_n \in L^1 \cap \mathcal{M}$ and $y_n x = |x|^p E_{|x|}([\frac{1}{n},n])$. First, assume $p > 1$. By Proposition 1.47(iv), $|y_n^*| = |x|^{p-1}E_{|x|}([\frac{1}{n},n])$. Thus, applying Corollary 2.12 and then the equality $(p-1)p' = p$, we have

$$\|y_n\|_{L^{p'}} = \|y_n^*\|_{L^{p'}} = \tau\left(|x|^p E_{|x|}([1/n,n])\right)^{1/p'}.$$

Now, by our assumption,

$$C \ge \frac{\tau(y_n x)}{\|y_n\|_{L^{p'}}} = \frac{\tau(|x|^p E_{|x|}([\frac{1}{n},n]))}{\tau(|x|^p E_{|x|}([\frac{1}{n},n]))^{1/p'}} = \tau(|x|^p E_{|x|}([1/n,n]))^{1/p}. \tag{2.9}$$

By Proposition 1.44, $|x|^p E_{|x|}([\frac{1}{n},n]) \nearrow |x|^p$ as $n \to \infty$, and the normality of τ implies $\tau(|x|^p E_{|x|}([\frac{1}{n},n])) \nearrow \tau(|x|^p)$. Combining this with (2.9), we see that $\tau(|x|^p)^{1/p} \le C$, as desired.

Now, for $p = 1$, we have $y_n = E_{|x|}([\frac{1}{n},n])u^*$ and $\|y_n\|_{L^\infty} \le 1$, so $|\tau(y_n x)| \le C$. Since $y_n x = E_{|x|}([\frac{1}{n},n])|x| \nearrow |x|$, the normality of τ implies $\tau(|x|) \le C$. □

2.3.3 Complex Interpolation for L^p Spaces

We are now well-equipped to characterise the interpolation of L^p spaces, with the help of the following lemma.

Lemma 2.39. *Given $0 < \theta < 1$ and $1 < p < \infty$, assume $\|x\|_\theta \le \|x\|_{L^p}$ for all $x \in \mathcal{D}$. Then $L^p(\mathcal{M},\tau) \subset (L^1,\mathcal{M})_\theta$ and $\|x\|_\theta \le \|x\|_{L^p}$ for every $x \in L^p(\mathcal{M},\tau)$.*

Proof. For any $x \in L^p(\mathcal{M},\tau)$, consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ from Definition 2.32. Proposition 2.34 shows that $\sup_{m\ge n}\tau(|x_m - x_n|^p) \to 0$ as $n \to \infty$, and since $\mathcal{D}$ is a vector space, our assumption implies $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in $(L^1,\mathcal{M})_\theta$. Since $(L^1,\mathcal{M})_\theta$ is a Banach space, let $y \in (L^1,\mathcal{M})_\theta$ be the limit of this sequence. Since $(L^1,\mathcal{M})_\theta$ is continuously included in $L^1 + \mathcal{M}$, it follows that $x_n \to y$ in $L^1 + \mathcal{M}$, but Proposition 2.34 also showed that $x_n \to x$ in $L^1 + \mathcal{M}$, so $x = y \in (L^1,\mathcal{M})_\theta$ and $\|x\|_\theta = \lim_{n\to\infty}\|x_n\|_\theta \le \lim_{n\to\infty}\|x_n\|_{L^p} = \|x\|_{L^p}$ by Proposition 2.34. □

Theorem 2.40. *For any $0 < \theta < 1$, choose $p = \frac{1}{1-\theta}$. Then $(L^1, \mathcal{M})_\theta = L^p$, with equality of norms.*

Proof. First, we will show that $L^p \subset (L^1, \mathcal{M})_\theta$ and $\|x\|_\theta \leq \|x\|_{L^p}$ for all $x \in L^p$. By the previous lemma, it suffices to assume that $x \in \mathcal{D}$. Therefore, choose $n \in \mathbb{N}$ so that $\sigma(|x|) \subset \{0\} \cup [\frac{1}{n}, n]$. Fixing $\delta > 0$ and $\rho \in \mathbb{R}$, we define for each $z \in \overline{\Omega}$,

$$f(z) = e^{\delta z^2 - \delta\theta^2 + \rho z - \rho\theta} x|x|^{p(1-z)-1}, \tag{2.10}$$

where we set $|x|^{p(1-z)-1} := \int_{[\frac{1}{n}, n]} \lambda^{p(1-z)-1} \, \mathrm{d}E_{|x|}(\lambda)$. We claim that $f \in \mathcal{F}(L^1, \mathcal{M})$, with

$$\|f\|_{\mathcal{F}} = \max\{e^{-\delta\theta^2 - \rho\theta} \|x\|_{L^p}^p, e^{\delta - \delta\theta^2 + \rho - \rho\theta}\}. \tag{2.11}$$

First, it follows from Lemma 2.36 that the function $z \mapsto |x|^{p(1-z)}$ is bounded and continuous on $\overline{\Omega}$ and analytic on Ω as a map into $L^1 \cap \mathcal{M}$. Since left-multiplication by a fixed operator is continuous and linear on $L^1 \cap \mathcal{M}$, it follows that the function $z \mapsto x|x|^{p(1-z)-1}$ has these properties too. Finally, since the scalar function $z \mapsto e^{\delta z^2 - \delta\theta^2 + \rho z - \rho\theta}$ is bounded and continuous on $\overline{\Omega}$ and analytic on Ω, the same properties easily follow for the function f (see Proposition A.6).

To show that $f \in \mathcal{F}(L^1, \mathcal{M})$, we now only need to show that $f(it) \to 0$ in $L^1(\mathcal{M}, \tau)$ and $f(1 + it) \to 0$ in $\mathcal{M}$ as $|t| \to \infty$. By Proposition 1.47(i), we note that $|f(z)| = e^{\delta \mathrm{Re}(z)^2 - \delta \mathrm{Im}(z)^2 - \delta\theta^2 + \rho \mathrm{Re}(z) - \rho\theta} |x|^{p(1-\mathrm{Re}(z))}$ for every $z \in \overline{\Omega}$. For any $t \in \mathbb{R}$, it follows that $\|f(it)\|_{L^1} = e^{-\delta t^2 - \delta\theta^2 - \rho\theta} \tau(|x|^p) = e^{-\delta t^2 - \delta\theta^2 - \rho\theta} \|x\|_{L^p}^p$, and $\|f(1 + it)\|_{L^\infty} = e^{\delta - \delta t^2 - \delta\theta^2 + \rho - \rho\theta}$. It is clear that these tend to 0 as $|t| \to \infty$, and that $\sup_{t \in \mathbb{R}} \|f(it)\|_{L^1} = e^{-\delta\theta^2 - \rho\theta} \|x\|_{L^p}^p$ and $\sup_{t \in \mathbb{R}} \|f(1 + it)\|_{L^\infty} = e^{\delta - \delta\theta^2 + \rho - \rho\theta}$, proving the desired formula for $\|f\|_{\mathcal{F}}$.

Moreover, since $p = \frac{1}{1-\theta}$, we have $f(\theta) = x|x|^{p(1-\theta)-1} = x$, so $x \in (L^1, \mathcal{M})_\theta$.

Letting $\delta \to 0$, (2.11) implies that $\|x\|_\theta \leq \max\{e^{-\rho\theta} \|x\|_{L^p}^p, e^{\rho - \rho\theta}\}$. As ρ varies, the right-hand side of this inequality is minimised when $e^{-\rho\theta} \|x\|_{L^p}^p = e^{\rho - \rho\theta}$. That is, $e^\rho = \|x\|_{L^p}^p$. (Here, we assume x is nonzero, otherwise we trivially have $\|x\|_\theta = \|x\|_{L^p}$.) With this choice of ρ, the inequality becomes

$$\|x\|_\theta \leq \|x\|_{L^p}^{p(1-\theta)} = \|x\|_{L^p}.$$

This shows that $L^p \subset (L^1, \mathcal{M})_\theta$, with $\|x\|_\theta \leq \|x\|_{L^p}$ for all $x \in L^p(\mathcal{M}, \tau)$.

Now, we suppose that $x \in (L^1, \mathcal{M})_\theta$, and we wish to show that $x \in L^p(\mathcal{M}, \tau)$ with $\|x\|_{L^p} \leq \|x\|_\theta$. By Proposition 2.38, it suffices to establish the inequality $|\tau(xy)| \leq \|x\|_\theta \|y\|_{L^{p'}}$ for all $y \in \mathcal{D}$. Therefore, fixing $y \in \mathcal{D}$, $\delta > 0$ and $\rho \in \mathbb{R}$, we define

$$g(z) = e^{\delta z^2 - \delta\theta^2 + \rho z - \rho\theta} y|y|^{p'z-1}. \tag{2.12}$$

Via the same argument as before, we have $g \in \mathcal{F}(\mathcal{M}, L^1)$ and $g(\theta) = y$. Moreover, $\|g(it)\|_{L^\infty} \leq e^{-\delta\theta^2-\rho\theta}$ and $\|g(1+it)\|_{L^1} \leq e^{\delta-\delta\theta^2+\rho-\rho\theta}\|y\|_{L^{p'}}^{p'}$ for all $t \in \mathbb{R}$. Finally, g is bounded and continuous on $\overline{\Omega}$ and analytic on Ω as a map into $L^1 \cap \mathcal{M}$. Therefore, for any $f \in \mathcal{F}(L^1, \mathcal{M})$ with $f(\theta) = x$, we can define $F(z) = \langle f(z), g(z)\rangle_{L^1+\mathcal{M}, L^1\cap\mathcal{M}} \in \mathbb{C}$ for every $z \in \overline{\Omega}$. Since the bilinear form $\langle\cdot,\cdot\rangle_{L^1+\mathcal{M}, L^1\cap\mathcal{M}}$ is continuous, it follows easily that F is bounded and continuous on $\overline{\Omega}$ and holomorphic on Ω (see Proposition A.3), so we can apply the three-line theorem. By Proposition 2.31, $|F(it)| \leq \|f(it)\|_{L^1}\|g(it)\|_{L^\infty} \leq e^{-\delta\theta^2-\rho\theta}\|f\|_{\mathcal{F}}$ and $|F(1+it)| \leq \|f(1+it)\|_{L^\infty}\|g(1+it)\|_{L^1} \leq e^{\delta-\delta\theta^2+\rho-\rho\theta}\|f\|_{\mathcal{F}}\|y\|_{L^{p'}}^{p'}$, for all $t \in \mathbb{R}$. By the three-line theorem (Theorem 2.24), it follows that

$$|\tau(xy)| = |F(\theta)| \leq \left(e^{-\delta\theta^2-\rho\theta}\|f\|_{\mathcal{F}}\right)^{1-\theta}\left(e^{\delta-\delta\theta^2+\rho-\rho\theta}\|f\|_{\mathcal{F}}\|y\|_{L^{p'}}^{p'}\right)^{\theta}.$$

Letting $\delta \to 0$, we get that

$$\begin{aligned}|\tau(xy)| &\leq \left(e^{-\rho\theta}\|f\|_{\mathcal{F}}\right)^{1-\theta}\left(e^{\rho-\rho\theta}\|f\|_{\mathcal{F}}\|y\|_{L^{p'}}^{p'}\right)^{\theta} \\ &= \|f\|_{\mathcal{F}}\|y\|_{L^{p'}}.\end{aligned}$$

Here, the last simplification follows from the equality $\theta = \frac{1}{p'}$.

Since we can choose f so that $\|f\|_{\mathcal{F}}$ is arbitrarily close to $\|x\|_\theta$, we conclude that $|\tau(xy)| \leq \|x\|_\theta\|y\|_{L^{p'}}$, proving that $x \in L^p(\mathcal{M}, \tau)$ with $\|x\|_{L^p} \leq \|x\|_\theta$. $\square$

We now greedily reap the benefits of what we have proven. The first important consequence follows immediately from Proposition 2.26.

Corollary 2.41. *For all $1 \leq p \leq \infty$, $L^p(\mathcal{M}, \tau)$ is a Banach space. In particular, $L^2(\mathcal{M}, \tau)$ is a Hilbert space.*

Next, Proposition 2.28 specialises to the following result, describing interpolation of L^p spaces for arbitrary $1 \leq p \leq \infty$.

Corollary 2.42. *For any $1 \leq p_0, p_1 \leq \infty$ and $0 < \theta < 1$, $(L^{p_0}, L^{p_1})_\theta = L^p$, where $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ (with equal norms).*

Thirdly, following Corollary 4.2 in [21], we establish the Banach duality of L^p spaces. We will not use this fact in this thesis, so we only sketch the proof.

Corollary 2.43. *For all $1 \leq p < \infty$, $L^p(\mathcal{M}, \tau)^* = L^{p'}(\mathcal{M}, \tau)$, where $\frac{1}{p} + \frac{1}{p'} = 1$.*

Proof. Since $L^2(\mathcal{M}, \tau)$ is a Hilbert space, we have $L^2(\mathcal{M}, \tau)^* = L^2(\mathcal{M}, \tau)$, so, in view of Proposition 2.34, we apply Proposition 2.27 to the compatible couple (L^1, L^2). Since $L^1(\mathcal{M}, \tau)^* = L^\infty(\mathcal{M}, \tau)$, Proposition 2.27 implies that for all $1 \leq p \leq 2$, $L^p(\mathcal{M}, \tau)^* = L^{p'}(\mathcal{M}, \tau)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Moreover, each such $L^p(\mathcal{M}, \tau)$ is reflexive, so $L^{p'}(\mathcal{M}, \tau)^* = L^p(\mathcal{M}, \tau)$. $\square$

Finally, we have the Riesz-Thorin property for noncommutative L^p spaces.

Corollary 2.44. *Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are von Neumann algebras with faithful, normal, semifinite traces τ_1 and τ_2. Let $1 \le p_1, p_2, q_1, q_2 \le \infty$, and suppose we have a linear map $T : L^{p_1}(\mathcal{M}_1, \tau_1) + L^{q_1}(\mathcal{M}_1, \tau_1)) \to L^{p_2}(\mathcal{M}_2, \tau_2) + L^{q_2}(\mathcal{M}_2, \tau_2)$, that restricts to bounded maps $T : L^{p_1} \to L^{p_2}$ and $T : L^{q_1} \to L^{q_2}$. Then fixing $0 < \theta < 1$ and letting*

$$\frac{1}{r_1} = \frac{1-\theta}{p_1} + \frac{\theta}{q_1}, \qquad \textit{and} \qquad \frac{1}{r_2} = \frac{1-\theta}{p_2} + \frac{\theta}{q_2},$$

T restricts to a bounded linear map $T : L^{r_1}(\mathcal{M}_1, \tau_1) \to L^{r_2}(\mathcal{M}_2, \tau_2)$, with

$$\|T\|_{L^{r_1}\to L^{r_2}} \le \|T\|^{1-\theta}_{L^{p_1}\to L^{p_2}} \|T\|^{\theta}_{L^{q_1}\to L^{q_2}}.$$

Proof. This follows from Corollary 2.42 and Proposition 2.26. □

Example 2.45. Let $\mathcal{H}$ be a separable Hilbert space. As in Subsection 1.1.4, consider the von Neumann algebra $B(\mathcal{H})$, with trace tr. We observed in Example 1.41 that $S(B(\mathcal{H}), \mathrm{tr}) = B(\mathcal{H})$, so all associated L^p spaces are contained in $B(\mathcal{H})$. We write $\mathcal{S}_p(\mathcal{H}) \coloneqq L^p(B(\mathcal{H}), \mathrm{tr})$, and we call $\mathcal{S}_p(\mathcal{H})$ a **Schatten ideal**.

Example 2.46. If $\mathcal{M} = L^\infty(X)$ for some σ-finite measure space (X, ν) and τ is the integral against ν, then Example 2.3.13(b) in [10] shows that $S(\mathcal{M}, \tau)$ is order-preservingly $*$-isomorphic to the space of (equivalence classes of) ν-measurable functions on X that are bounded except on some ν-finite set. Clearly, this class of functions contains $L^p(X, \nu)$ for every $1 \le p \le \infty$. Since any positive-valued measurable function on X can be approximated from below by essentially bounded functions, the extended trace τ on $S(\mathcal{M}, \tau)_+$ must coincide with the integral against ν, since both of these are normal traces. It follows that, under this isomorphism, $L^p(\mathcal{M}, \tau) \cong L^p(X, \nu)$.

2.4 Completely Bounded Maps

A critical notion in Chapter 4 will be the complete boundedness of a map between noncommutative L^p spaces. Defining this notion requires some menial labour, of which we spare the reader by occasionally providing proof sketches.

In this section, we consider operators on the Hilbert space $\mathcal{H}^{\oplus n}$, the direct sum of n copies of $\mathcal{H}$. For any $v \in \mathcal{H}^{\oplus n}$, we write $v = (v_1, \dots, v_n)$, and we see that $\mathcal{H}^{\oplus n}$ admits an inner product given by $\langle v, w\rangle_{\mathcal{H}^{\oplus n}} = \sum_{i=1}^n \langle v_i, w_i\rangle_{\mathcal{H}}$. Moreover,

$$\|v\|^2_{\mathcal{H}^{\oplus n}} = \sum_{i=1}^n \langle v_i, v_i\rangle_{\mathcal{H}} = \sum_{i=1}^n \|v_i\|^2_{\mathcal{H}}. \tag{2.13}$$

It follows from the completeness of $\mathcal{H}$ that $\mathcal{H}^{\oplus n}$ is a Hilbert space, and that a sequence is convergent in $\mathcal{H}^{\oplus n}$ if and only if each coordinate converges in $\mathcal{H}$. For every $i \in \{1, \ldots, n\}$, let $f_i \in B(\mathcal{H}, \mathcal{H}^{\oplus n})$ be the inclusion of $\mathcal{H}$ into the i-th coordinate, and let $q_i \in B(\mathcal{H}^{\oplus n}, \mathcal{H})$ be the projection $(v_1, \ldots, v_n) \mapsto v_i$. We have that $q_i f_i = \mathbf{1}_{\mathcal{H}}$, $\sum_{i=1}^n f_i q_i = \mathbf{1}_{\mathcal{H}^{\oplus n}}$, and $q_i^* = f_i$.

Now, given a vector space V, let $M_n(V)$ denote the vector space of $n \times n$ matrices with entries in V. If, additionally, V is a unital algebra, then $M_n(V)$ forms a unital algebra with matrix multiplication defined in the usual way. Finally, if V is a $*$-algebra, then for any $T = (T_{ij})_{1 \leq i,j \leq n} \in M_n(V)$, we define $T^* \in M_n(V)$ by $(T^*)_{ij} = (T_{ji})^*$, and this makes $M_n(V)$ a $*$-algebra.

To any operator $T \in B(\mathcal{H}^{\oplus n})$, we associate a matrix $(T_{ij})_{1 \leq i,j \leq n} \in M_n(B(\mathcal{H}))$, given by $T_{ij} = q_i T f_j$. For any $v \in \mathcal{H}^{\oplus n}$, we then compute that

$$Tv = \left(\sum_j T_{1j} v_j, \ldots, \sum_j T_{nj} v_j \right). \tag{2.14}$$

That is, T acts exactly as the matrix $(T_{ij})_{1 \leq i,j \leq n}$ should. From this, one quickly sees that the mapping $T \mapsto (T_{ij})_{1 \leq i,j \leq n}$ is a unital $*$-homomorphism from $B(\mathcal{H}^{\oplus n})$ to $M_n(B(\mathcal{H}))$. In fact, we have the following result.

Proposition 2.47. *The map $T \mapsto (T_{ij})_{1 \leq i,j \leq n}$ is an isomorphism of unital $*$-algebras $B(\mathcal{H}^{\oplus n}) \cong M_n(B(\mathcal{H}))$, and the norm on $B(\mathcal{H}^{\oplus n})$ is equivalent to the norm $T \mapsto \sum_{i,j=1}^n \|T_{ij}\|_{L^\infty}$.*

Proof. The homomorphism properties of this map are clear, and we claim that (2.14) defines an inverse map from $M_n(B(\mathcal{H}))$ to $B(\mathcal{H}^{\oplus n})$. We only need to check that for any matrix $(T_{ij})_{1 \leq i,j \leq n} \in M_n(B(\mathcal{H}))$, the associated operator T defined by (2.14) is bounded. For any $v \in \mathcal{H}^{\oplus n}$, we have

$$\begin{aligned}
\|Tv\|_{\mathcal{H}^{\oplus n}}^2 = \sum_{i=1}^n \left\| \sum_{j=1}^n T_{ij} v_j \right\|_{\mathcal{H}}^2 &\leq \sum_{i=1}^n \left(\sum_{j=1}^n \|T_{ij} v_j\|_{\mathcal{H}} \right)^2 \\
&\leq n \sum_{i,j=1}^n \|T_{ij} v_j\|_{\mathcal{H}}^2 \qquad \text{(Cauchy-Schwarz)} \\
&\leq n^2 \max\{\|T_{ij}\|_{L^\infty}^2\} \sum_{j=1}^n \|v_j\|_{\mathcal{H}}^2.
\end{aligned}$$

Thus, $\|T\|_{L^\infty} \leq n \max\{\|T_{ij}\|_{L^\infty}\}$. On the other hand, for any $\xi, \eta \in \mathcal{H}$,

$$|\langle T_{ij}\xi, \eta \rangle_{\mathcal{H}}| = |\langle T f_j \xi, f_i \eta \rangle_{\mathcal{H}^{\oplus n}}| \leq \|T\|_{L^\infty} \|\xi\|_{\mathcal{H}} \|\eta\|_{\mathcal{H}},$$

which proves that $\|T_{ij}\|_{L^\infty} \leq \|T\|_{L^\infty}$. The stated equivalence of norms now follows easily. $\square$

Hence, we identify $M_n(B(\mathcal{H}))$ with $B(\mathcal{H}^{\oplus n})$, so $M_n(B(\mathcal{H}))$ inherits the topologies on $B(\mathcal{H}^{\oplus n})$. Letting $M_n(\mathcal{M})$ be the unital $*$-algebra of $n \times n$ matrices with entries in $\mathcal{M}$, the following topological result will imply that $M_n(\mathcal{M})$ is a von Neumann algebra on $\mathcal{H}^{\oplus n}$.

Proposition 2.48. *A net $\{T_\alpha\}$ in $M_n(B(\mathcal{H}^n))$ converges weakly (resp. σ-weakly) to $T \in M_n(B(\mathcal{H}^{\oplus n}))$ if and only if $(T_\alpha)_{ij} \to T_{ij}$ weakly (resp. σ-weakly) for all $1 \leq i, j \leq n$.*

Proof. Suppose $T_\alpha \to T$ weakly. For any $\xi, \eta \in \mathcal{H}$, we have

$$\langle (T_\alpha)_{ij}\xi, \eta\rangle_{\mathcal{H}} = \langle T_\alpha f_j \xi, f_i \eta\rangle_{\mathcal{H}^{\oplus n}} \to \langle T f_j \xi, f_i \eta\rangle_{\mathcal{H}^{\oplus n}} = \langle T_{ij}\xi, \eta\rangle_{\mathcal{H}},$$

so $(T_\alpha)_{ij} \to T_{ij}$ weakly. Conversely, assume that $(T_\alpha)_{ij} \to T_{ij}$ weakly for all $1 \leq i, j \leq n$. For any $v, w \in \mathcal{H}^{\oplus n}$, we have

$$\langle T_\alpha v, w\rangle_{\mathcal{H}^{\oplus n}} = \sum_{i,j=1}^{n} \langle (T_\alpha)_{ij} v_j, w_i\rangle_{\mathcal{H}} \to \sum_{i,j=1}^{n} \langle T_{ij} v_j, w_i\rangle_{\mathcal{H}} = \langle Tv, w\rangle_{\mathcal{H}^{\oplus n}},$$

so $T_\alpha \to T$ weakly. Replacing the elements ξ, η with sequences in $\mathcal{H}$, the same argument proves the statement about σ-weak convergence. $\square$

Corollary 2.49. *$M_n(\mathcal{M})$ is a von Neumann algebra on $\mathcal{H}^{\oplus n}$.*

Proof. Since the von Neumann algebra $\mathcal{M}$ is weakly closed, Proposition 2.48 implies that $M_n(\mathcal{M})$ is weakly closed in $B(\mathcal{H}^{\oplus n})$. $\square$

To define a trace on $M_n(\mathcal{M})_+$, we will need the following lemma.

Lemma 2.50. *Suppose $T \in B(\mathcal{H}^{\oplus n})$ is positive. Then each diagonal entry of T is positive too. Moreover, if T is zero along its diagonal, then $T = 0$.*

Proof. Write $T = S^*S$, for some $S \in B(\mathcal{H}^{\oplus n})$. For every $i = 1, \ldots, n$, we have $T_{ii} = \sum_{j=1}^{n} (S^*)_{ij} S_{ji} = \sum_{j=1}^{n} (S_{ji})^* S_{ji}$, which is a sum of positive elements and therefore positive. Now, if the diagonal entries of T are all zero, this calculation implies that $S_{ji} = 0$ for all $1 \leq i, j \leq n$. Hence $S = 0$, and so $T = 0$. $\square$

As a result, the following definition makes sense.

Definition 2.51. For any $T \in M_n(\mathcal{M})_+$, define $\tau_n(T) = \sum_{i=1}^{n} \tau(T_{ii})$.

Proposition 2.52. *The function $\tau_n : M_n(\mathcal{M})_+ \to [0, \infty]$ is a faithful, normal, semifinite trace.*

Proof. The additive and scaling properties of τ_n follow immediately from the corresponding properties of τ. To see that $\tau_n(T^*T) = \tau_n(TT^*)$, we notice that $(T^*T)_{ii} = \sum_{j=1}^n (T_{ji})^* T_{ji}$, and so

$$\tau_n(T^*T) = \sum_{i,j=1}^n \tau((T_{ji})^*T_{ji}) = \sum_{i,j=1}^n \tau(T_{ji}(T_{ji})^*) = \tau_n(TT^*).$$

The faithfulness of τ_n follows from the faithfulness of τ and the second assertion in Lemma 2.50.

For the normality of τ_n, Lemma 2.50 implies that an increasing net in $M_n(\mathcal{M})_+$ has increasing, positive diagonal entries. Moreover, since the supremum of an increasing net is its strong (hence, weak) limit (Proposition 1.18), the normality of τ_n follows from Proposition 2.48.

Finally, we show that τ_n is semifinite. Suppose $T \in M_n(\mathcal{M})_+$ is nonzero. By Proposition 1.24, it suffices to find $S \in M_n(\mathcal{M})_+$ satisfying $0 < S \leq T$ and $\tau_n(S) < \infty$. Since $T^{1/2}$ is positive and nonzero, Lemma 2.50 implies that the diagonal matrix entries of $T^{1/2}$ are positive and at least one diagonal entry is nonzero. Without loss of generality, assume $(T^{1/2})_{11} > 0$. Writing $(T^{1/2})_{11} = u(T^{1/2})_{11}$ for its polar decomposition, notice that u is just projection onto $\overline{\mathrm{ran}((T^{1/2})_{11})}$. By Proposition 1.48, choose a nonzero projection $p \in P(\mathcal{M})$ with $p \leq u$ and $\tau(p) < \infty$, and let $P \in M_n(\mathcal{M})_+$ be the matrix whose only nonzero entry is $P_{11} = p$. Now, let $S = T^{1/2}PT^{1/2}$. Since $0 \leq P \leq \mathbf{1}_{\mathcal{H}^{\oplus n}}$, it follows from Proposition 1.16 that $0 \leq S \leq T$. For all $1 \leq i, j \leq n$, we compute that $S_{ij} = (T^{1/2})_{i1}p(T^{1/2})_{1j}$. Since $p \in L^1(\mathcal{M}, \tau)$, which is a normed $\mathcal{M}$-bimodule, it follows that $\tau_n(S) = \sum_{i=1}^n \tau(S_{ii}) < \infty$. Finally, by Proposition 1.29, we have $\mathrm{ran}(p) \subset \overline{\mathrm{ran}((T^{1/2})_{11})} = \ker((T^{1/2})_{11})^\perp$, so $S_{11} = (T^{1/2})_{11}p(T^{1/2})_{11} > 0$. □

Now, to identify the noncommutative L^p spaces associated to $(M_n(\mathcal{M}), \tau_n)$, we will take an approach closer to Kosaki's construction [19] than to the construction taken in the previous sections. First, recall that $\mathcal{M}_*$ is the space of σ-weakly continuous linear functionals on $\mathcal{M}$. We have the following matrix result.

Proposition 2.53. *Every matrix $(\Phi_{ij})_{1\leq i,j\leq n} \in M_n(\mathcal{M}_*)$ defines a σ-weakly continuous linear functional $\Phi \in M_n(\mathcal{M})_*$, given by $\Phi(T) = \sum_{i,j=1}^n \Phi_{ij}T_{ji}$. This defines an isomorphism of vector spaces $M_n(\mathcal{M}_*) \cong M_n(\mathcal{M})_*$, in which the norm on $M_n(\mathcal{M})_*$ is equivalent to the norm $\Phi \mapsto \sum_{i,j=1}^n \|\Phi_{ij}\|_{\mathcal{M}_*}$.*

Proof. It is clear that the formula $\Phi(T) = \sum_{i,j=1}^n \Phi_{ij} T_{ji}$ defines a linear functional on $M_n(\mathcal{M})$. To show that it is σ-weakly continuous, assume $\{T_\alpha\}$ is a net in $M_n(\mathcal{M})$ converging σ-weakly to $T \in M_n(\mathcal{M})$. By Proposition 2.48, $(T_\alpha)_{ij} \to T_{ij}$ σ-weakly for all $1 \leq i, j \leq n$. Since each Φ_{ij} is σ-weakly continuous, it follows that $\Phi(T_\alpha) \to \Phi(T)$.

It is also clear that the mapping $(\Phi_{ij})_{1 \leq i,j \leq n} \mapsto \Phi$ is linear, so we just need to construct an inverse map. Given $x \in B(\mathcal{H})$, let $x^{ij} \in M_n(B(\mathcal{H}))$ be the matrix with $(x^{ij})_{ij} = x$ and all other matrix entries zero. Given $\Phi \in M_n(\mathcal{M})_*$, define the matrix $(\Phi_{ij})_{1 \leq i,j \leq n}$ by $\Phi_{ij} x = \Phi x^{ji}$. Since Φ is σ-weakly continuous, as is the map $x \mapsto x^{ji}$ by Proposition 2.48, we have $\Phi_{ij} \in \mathcal{M}_*$. For any $T \in M_n(\mathcal{M})$, we have $T = \sum_{i,j=1}^n (T_{ji})^{ji}$, so $\Phi(T) = \sum_{i,j=1}^n \Phi((T_{ji})^{ji}) = \sum_{i,j=1}^n \Phi_{ij} T_{ji}$. Thus, we have constructed the desired inverse. The statement about norms now follows easily from the equivalence of norms in Proposition 2.47. □

In view of Theorem 2.22, we have now established a vector space isomorphism $M_n(L^1(\mathcal{M}, \tau)) \cong L^1(M_n(\mathcal{M}), \tau_n)$. Under this isomorphism, the matrix $(T_{ij})_{1 \leq i,j \leq n} \in M_n(L^1(\mathcal{M}, \tau))$ corresponds to the operator $T \in L^1(M_n(\mathcal{M}), \tau_n)$, whose action on $M_n(\mathcal{M})$ is given by $\langle T, S \rangle = \sum_{i,j=1}^n \tau(T_{ij} S_{ji})$, for all $S \in M_n(\mathcal{M})$. Moreover, Proposition 2.53 implies that the norm on $L^1(M_n(\mathcal{M}), \tau_n)$ is equivalent to the norm $T \mapsto \sum_{i,j=1}^n \|T_{ij}\|_{L^1}$.

With a moment's thought, we are assured that the embeddings of $M_n(\mathcal{M})$ and $M_n(L^1(\mathcal{M}, \tau))$ into the algebraic sum $L^1(M_n(\mathcal{M}), \tau_n) + M_n(\mathcal{M})$ agree on the set $M_n(L^1 \cap \mathcal{M})$, and the image of this set is $L^1(M_n(\mathcal{M}), \tau_n) \cap M_n(\mathcal{M})$. Therefore, we identify $L^1(M_n(\mathcal{M}), \tau_n) + M_n(\mathcal{M})$ with the vector space $M_n(L^1(\mathcal{M}, \tau) + \mathcal{M})$, and we see that the norm on $L^1(M_n(\mathcal{M}), \tau_n) + M_n(\mathcal{M})$ is equivalent to the norm $T \mapsto \sum_{i,j=1}^n \|T_{ij}\|_{L^1 + \mathcal{M}}$. The bilinear pairing between $L^1 \cap M_n(\mathcal{M})$ and $L^1 + M_n(\mathcal{M})$ is now given by

$$\langle A, B \rangle = \sum_{i,j=1}^n \tau(A_{ij} B_{ji}).$$

Theorem 2.54. *For any $1 \leq p \leq \infty$, this isomorphism identifies $M_n(L^p(\mathcal{M}, \tau))$ with $L^p(M_n(\mathcal{M}), \tau_n)$, and the norm on $L^p(M_n(\mathcal{M}), \tau_n)$ is equivalent to the norm $T \mapsto \sum_{i,j=1}^n \|T_{ij}\|_{L^p}$.*

Proof. We argue via complex interpolation. For any $1 \leq i, j \leq n$, define a linear map $\pi_{ij} : L^1(M_n(\mathcal{M}), \tau_n) + M_n(\mathcal{M}) \to L^1(\mathcal{M}, \tau) + \mathcal{M}$ sending $T \mapsto T_{ij}$. Since π_{ij} is continuous and linear, composition with π_{ij} sends $\mathcal{F}(L^1(M_n(\mathcal{M}), \tau_n), M_n(\mathcal{M}))$

to $\mathcal{F}(L^1(\mathcal{M},\tau),\mathcal{M})$. Hence, if $T \in L^p(M_n(\mathcal{M}),\tau_n)$, then $T_{ij} \in L^p(\mathcal{M},\tau)$. On the other hand, given a family of functions $\{f_{ij}\}_{1\le i,j\le n}$ in $\mathcal{F}(L^1(\mathcal{M},\tau),\mathcal{M})$, define a matrix-valued function $f(z) = (f_{ij}(z))_{1\le i,j\le n}$. By the equivalences of norms stated earlier in this section, it is clear that $f \in \mathcal{F}(L^1(M_n(\mathcal{M}),\tau_n), M_n(\mathcal{M}))$, so if each $T_{ij} \in L^p(\mathcal{M},\tau)$, then $T \in L^p(M_n(\mathcal{M}),\tau_n)$. The statement about norms now follows from the equivalences of norms established earlier in this section. □

Henceforth, we identify the vector space $M_n(L^p(\mathcal{M},\tau))$ with the Banach space $L^p(M_n(\mathcal{M}),\tau_n)$. At last, we can state what it means for a map between L^p spaces to be completely bounded.

Definition 2.55. For $i \in \{1,2\}$, let $\mathcal{M}_i$ be a von Neumann algebra on a separable Hilbert space $\mathcal{H}_i$ with a faithful, normal, semifinite trace τ_i, and fix $1 \le p_i \le \infty$. If $T : L^{p_1}(\mathcal{M}_1,\tau_1) \to L^{p_2}(\mathcal{M}_2,\tau_2)$ is a bounded linear map, then for any $n \in \mathbb{N}$, we define a bounded linear map $\mathrm{id}_n \otimes T : L^{p_1}(M_n(\mathcal{M}_1),(\tau_1)_n) \to L^{p_2}(M_n(\mathcal{M}_2),(\tau_2)_n)$ by $((\mathrm{id}_n \otimes T)A)_{ij} = TA_{ij}$. That is, $\mathrm{id}_n \otimes T$ applies T to each matrix entry. We say that T is **completely bounded** if $\|T\|_{CB} \coloneqq \sup_{n\in\mathbb{N}} \|\mathrm{id}_n \otimes T\| < \infty$.

In general, bounded maps may not be completely bounded (for a simple example, see Exercise 1.8 in [25]). The following example of a map that *is* completely bounded will be useful to us in Chapter 4.

Proposition 2.56. *For any unitary $u \in U(\mathcal{M})$ and any $1 \le p \le \infty$, the maps $L_u(x) = ux$ and $R_u(x) = xu$ are completely bounded on $L^p(\mathcal{M},\tau)$. In fact, for any $n \in \mathbb{N}$, $\mathrm{id}_n \otimes L_u$ and $\mathrm{id}_n \otimes R_u$ are isometries on $L^p(M_n(\mathcal{M}),\tau_n)$.*

Proof. For any $x \in L^p(\mathcal{M},\tau)$, $(ux)^*(ux) = x^*u^*ux = x^*x$, so $|ux| = |x|$ and $\|ux\|_{L^p} = \|x\|_{L^p}$. Similarly, $(xu)(xu)^* = xx^*$, so $|(xu)^*| = |x^*|$ and by Corollary 2.12, this implies $\|xu\|_{L^p} = \|x\|_{L^p}$. Therefore, L_u and R_u are continuous. Using the characterisation of adjoints at the beginning of the section, the same argument shows that $\|(\mathrm{id}_n \otimes L_u)A\|_{L^p} = \|A\|_{L^p} = \|(\mathrm{id}_n \otimes R_u)A\|_{L^p}$ for all $A \in L^p(M_n(\mathcal{M}),\tau_n) \cap M_n(\mathcal{M})$. From Proposition 2.34, this set is dense in $L^p(M_n(\mathcal{M}),\tau_n)$, so we are done. □

There is one more matrix construction that we will need.

Definition 2.57. Given $A \in M_n(\mathbb{C})$ and $x \in L^1(\mathcal{M},\tau) \cap \mathcal{M}$, $1 \le p \le \infty$, define $A \otimes x \in L^1(M_n(\mathcal{M}),\tau_n) \cap M_n(\mathcal{M})$ by $(A \otimes x)_{ij} = A_{ij}x$.

The result we will need is the following.

Theorem 2.58. *With A and x as above, one has $\|A \otimes x\|_{L^p} = \|A\|_{\mathcal{S}_p}\|x\|_{L^p}$ for all $1 \leq p < \infty$.*

Proof. First, we observe that for any $R \in M_n(\mathbb{C})$ and $y \in M_n(\mathcal{M})$,

$$((A \otimes x)(R \otimes y))_{ij} = \sum_{k=1}^{n} A_{ik} R_{kj} xy = (AR)_{ij} xy.$$

That is, $(A \otimes x)(R \otimes y) = (AR) \otimes (xy)$. Additionally, directly from the definition of matrix adjoints, we have $(R \otimes y)^* = R^* \otimes y^*$. Hence, if R and y are positive, then $R \otimes y = (R^{1/2} \otimes y^{1/2})^*(R^{1/2} \otimes y^{1/2})$ is positive too.

It follows from these calculations that $|A \otimes x| = |A| \otimes |x|$, and therefore $|A \otimes x|^q = |A|^q \otimes |x|^q$ for every $q \in \mathbb{N}$. Moreover, by the uniqueness of positive square roots, induction on m yields that $|A \otimes x|^{\frac{q}{2^m}} = |A|^{\frac{q}{2^m}} \otimes |x|^{\frac{q}{2^m}}$ for all $q, m \in \mathbb{N}$.

For arbitrary $p > 0$, choose a sequence of rationals $\{p_n\}_{n \in \mathbb{N}}$, each of the form $\frac{q}{2^m}$, so that $|A \otimes x|^{p_n} = |A|^{p_n} \otimes |x|^{p_n}$, and such that $p_n \to p$. As $n \to \infty$, Proposition 1.35(iv) implies $|A|^{p_n} \to |A|^p$ and $|x|^{p_n} \to |x|^p$ in their respective operator norm topologies. Since the norms of $|A|^{p_n}$ and $|x|^{p_n}$ are uniformly bounded, it follows from the equivalence of norms in Proposition 2.47 that $|A|^{p_n} \otimes |x|^{p_n} \to |A|^p \otimes |x|^p$ in operator norm as well. Since we also have $|A \otimes x|^{p_n} \to |A \otimes x|^p$ in operator norm, we conclude that $|A \otimes x|^p = |A|^p \otimes |x|^p$. Therefore,

$$\tau_n(|A \otimes x|^p) = \sum_{i=1}^{n} (|A|^p)_{ii} \tau(|x|^p) = \operatorname{tr}(|A|^p)\tau(|x|^p),$$

from which the equality $\|A \otimes x\|_{L^p} = \|A\|_{\mathcal{S}_p}\|x\|_{L^p}$ follows immediately. □

Chapter 3

Group von Neumann Algebras

In this chapter, we define and study the von Neumann algebras associated with locally compact groups. Under certain conditions, these von Neumann algebras admit faithful, normal, semifinite traces, and the associated L^p spaces will be central to the main results of Chapter 4. Therefore, guided by the exposition in [35], we will first briefly present the relevant theory of locally compact groups and the Haar measure. After this, we will construct the group von Neumann algebra and its associated trace.

3.1 Locally Compact Groups

Definition 3.1. A **locally compact group** is a group G, equipped with a Hausdorff topology in which

(i) the multiplication map from $G \times G$ to G is continuous;

(ii) the inversion map on G is continuous; and,

(iii) G is a **locally compact** topological space, i.e., for any $s \in G$, every neighbourhood of s contains a compact neighbourhood of s.

Usually, we denote the identity element of a group by the symbol e.

Examples 3.2. Any group equipped with the discrete topology is a locally compact group. This arises most naturally for finite or countable groups, for example $\mathbb{Z}$ or a finitely-generated free group. The additive group $\mathbb{R}^n$ and the unit circle $\mathbf{T}$ in $\mathbb{C}$ with its multiplicative structure are examples of nondiscrete locally compact groups.

It is clear that the inversion map is a homeomorphism of G onto itself. For any subset $W \subset G$, we will write W^{-1} for the image of W under this homeomorphism, and we say W is **symmetric** if $W = W^{-1}$. Moreover, for any $s \in G$, we write $sW = \{sw : w \in W\}$. Finally, given subsets $V, W \subset G$, we denote by VW the set $\{vw : v \in V, w \in W\}$. Also, for any $n \in \mathbb{N}$, we write $V^n = \{v_1 v_2 \dots v_n : v_i \in V\}$. Since VW is the image of $V \times W$ under the multiplication map, it follows that VW is compact whenever V and W are. Inductively, it follows that, if V is compact, so is V^n for every $n \in \mathbb{N}$.

We now present two important results about the space $C_c(G)$ of complex-valued compactly supported continuous functions on G. For $f \in C_c(G)$, we will write $\mathrm{supp}(f)$ for the support of the function f.

Theorem 3.3 (Lemma 1.62 in [35]). *Let G be a locally compact group. Any function $f \in C_c(G)$ is uniformly continuous in the following sense: for any $\varepsilon > 0$, there exists a neighbourhood V of e in G such that $|f(s) - f(r)| < \varepsilon$ whenever $s^{-1}r \in V$ or $sr^{-1} \in V$.*

Theorem 3.4 (Urysohn's Lemma, Lemma 2.12 in [27]). *Let G be a locally compact group. For any compact subset $K \subset G$ and any open neighbourhood V of K, there exists a function $f \in C_c(G)$ with $0 \leq f(s) \leq 1$ for all $s \in G$, $f(s) = 1$ for all $s \in K$, and $\mathrm{supp}(f) \subset V$.*

In this thesis, we will be especially interested in locally compact groups that are second-countable. Amongst discrete groups, these are precisely the countable groups. We also recall that $\mathbb{R}^n$ and $\mathbf{T}$ are second-countable. Such groups have the following useful properties.

Proposition 3.5. *If G is a second-countable, locally compact group, then G admits a countable basis consisting of precompact open sets.*

Proof. Let $\mathcal{B}$ be a countable basis for G and let $\mathcal{B}_0$ be the collection of precompact open sets in $\mathcal{B}$. We will show that $\mathcal{B}_0$ is a basis for G. Since G is locally compact, every open set in G is a union of precompact open sets, and therefore it suffices to show that every precompact open set in G is a union of sets in $\mathcal{B}_0$. But any subset of a precompact set is also precompact, so this is clear. □

Proposition 3.6. *If G is a second-countable, locally compact group, then G admits an exhaustion by compact subsets, i.e., a sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n$ and $K_n \subset \mathrm{int}(K_{n+1})$ for every $n \in \mathbb{N}$.*

Proof. In view of the previous proposition, choose a countable basis $\{U_i\}_{i=1}^{\infty}$ for G consisting of precompact open sets. Without loss of generality, assume $e \in U_1$. For each $n \in \mathbb{N}$, define $E_n = \bigcup_{i=1}^{n} \overline{U_i}$, so that $\{E_n\}$ is an increasing sequence of compact neighbourhoods of e. Inductively, we define $K_1 = E_1$ and $K_{n+1} = K_n E_{n+1}$ for every $n \in \mathbb{N}$. Since multiplication is continuous, one sees inductively that each K_n is a compact neighbourhood of e containing E_n. Therefore, $\bigcup_{n=1}^{\infty} K_n \supset \bigcup_{n=1}^{\infty} E_n = G$. Additionally, since E_{n+1} is a neighbourhood of e we have that $K_n \subset \operatorname{int}(K_{n+1})$. $\square$

3.1.1 The Haar Measure

Suppose X is a locally compact Hausdorff space. A **Radon measure** on X is a Borel measure satisfying the following properties:

- (finite on compact sets) for any compact set $K \subset X$, $\mu(K) < \infty$;
- (inner regularity on open sets) for any open set $U \subset X$,

$$\mu(U) = \sup\{\mu(K) : K \subset U,\ K \text{ compact}\};$$

- (outer regularity on measurable sets) for any Borel set $E \subset X$,

$$\mu(E) = \inf\{\mu(U) : E \subset U,\ U \text{ open}\}.$$

Definition 3.7. Given a locally compact group G, a **left Haar measure** on G is a nonzero Radon measure μ that is left-invariant, in the sense that for any Borel set $E \subset G$ and any $g \in G$, $\mu(gE) = \mu(E)$. A **right Haar measure** on G is a Radon measure $\tilde{\mu}$ that is right-invariant, in that for any Borel set $E \subset G$ and any $g \in G$, $\tilde{\mu}(Eg) = \tilde{\mu}(E)$.

The key result of this section is the following. For a proof of this fact, we direct the reader to Theorems 2.10 and 2.20 in [12].

Theorem 3.8. *Any locally compact group G admits a left (resp. right) Haar measure, and this left (resp. right) Haar measure is unique up to scaling by a positive constant.*

In many cases, we can construct Haar measures by hand. It is well-known that the Lebesgue measure on $\mathbb{R}^n$ is both a left and right Haar measure (see, for example, Chapter 1 of [32]). Moreover, if G is a discrete group, it is easily checked that the counting measure on G is both a left and right Haar measure.

Locally compact groups whose left and right Haar measures coincide are called **unimodular**. If G is unimodular, then for any function $f \in L^1(G,\mu)$ and any $r \in G$, we have

$$\int_G f(s)\,\mathrm{d}\mu(s) = \int_G f(rs)\,\mathrm{d}\mu(s) = \int_G f(sr)\,\mathrm{d}\mu(s). \tag{3.1}$$

One notes that if μ is a left Haar measure on G, the Borel measure $\tilde{\mu}$ defined by $\tilde{\mu}(E) = \mu(E^{-1})$ is a right Haar measure on G. If G is unimodular with Haar measure μ, then the uniqueness assertion in Theorem 3.8 implies that there exists a constant $C > 0$ such that $\mu(E) = C\mu(E^{-1})$ for every Borel set $E \subset G$. Letting F be a compact neighbourhood of e and setting $E = F \cap F^{-1}$, we have that E is a symmetric compact neighbourhood of e, and Proposition 3.9 below implies that $0 < \mu(E) = \mu(E^{-1}) < \infty$. Thus, $C = 1$. Consequently, for any $f \in L^1(G,\mu)$,

$$\int_G f(s)\,\mathrm{d}\mu(s) = \int_G f(s^{-1})\,\mathrm{d}\mu(s). \tag{3.2}$$

It is clear that any abelian group is unimodular, and we noted above that any discrete group is unimodular. More surprisingly, so is any compact group (see Lemma 1.65 in [35]). This thesis will give particular attention to unimodular groups, since their associated von Neumann algebras will admit faithful, normal, semifinite traces.

Suppose G is a second-countable, locally compact group. In view of Proposition 3.6, and the condition that a Haar measure is finite on compact sets, we infer that G has σ-finite Haar measure. Therefore, we enjoy familiar results such as Fubini's theorem and the separability of $L^2(G)$. We conclude this section with two more properties of the Haar measure, the first of which closes the proof of (3.2) for unimodular groups G.

Proposition 3.9. *Let G be a locally compact group with left (resp. right) Haar measure μ. If U is a nonempty open subset of G, then $\mu(U) > 0$.*

Proof. By way of contradiction, suppose $\mu(U) = 0$. Any compact set K admits a covering by finitely many left-translates (resp. right-translates) of U. Since each such translate has measure zero, it follows that $\mu(K) = 0$. Since this holds for all compact sets $K \subset G$, inner regularity implies that $\mu(G) = 0$, but, by definition, μ is nonzero. □

Proposition 3.10 (Theorem 3.14 in [27]). *Let G be a locally compact group with left Haar measure μ. For all $1 \leq p < \infty$, $C_c(G)$ is a dense subset of $L^p(G,\mu)$.*

3.2 The von Neumann Algebra $\mathcal{L}G$

In this section, we explore two equivalent perspectives on the group von Neumann algebra associated to a second-countable unimodular group. Throughout, let G be such a group, and let μ denote its Haar measure.

Our first characterisation of the group von Neumann algebra, informed by [16], is constructed through a group representation. For each $r \in G$, we define a left-translation operator $\lambda_r \in B(L^2(G))$ by $\lambda_r\xi(s) = \xi(r^{-1}s)$ for every $\xi \in L^2(G)$ and $s \in G$. Since μ is left-invariant,

$$\int_G |\lambda_r\xi(s)|^2 \,\mathrm{d}\mu(s) = \int_G |\xi(r^{-1}s)|^2 \,\mathrm{d}\mu(s) = \int_G |\xi(s)|^2 < \infty. \tag{3.3}$$

Thus, λ_r is an isometry on $L^2(G)$. In fact, the map $r \mapsto \lambda_r$ is a unitary representation of G. To see this, notice that for any $r, q, s \in G$ and $\xi \in L^2(G)$,

$$\lambda_r\lambda_q\xi(s) = \lambda_q\xi(r^{-1}s) = \xi(q^{-1}r^{-1}s) = \lambda_{rq}\xi(s). \tag{3.4}$$

That is, $\lambda_r\lambda_q = \lambda_{rq}$. Since λ_e is the identity map on $L^2(G)$, it follows that each left-translation operator λ_r is invertible, with $\lambda_r^{-1} = \lambda_{r^{-1}}$. Now, for any $r \in G$ and any $\xi, \eta \in L^2(G)$, the left-invariance of the Haar measure determines that

$$\langle \lambda_r\xi, \eta\rangle = \int_G \xi(r^{-1}s)\overline{\eta(s)} \,\mathrm{d}\mu(s) = \int_G \xi(s)\overline{\eta(rs)} \,\mathrm{d}\mu(s) = \langle \xi, \lambda_{r^{-1}}\eta\rangle. \tag{3.5}$$

That is, $\lambda_r^* = \lambda_r^{-1}$, so λ_r is a unitary operator on $L^2(G)$. Thus, the map $r \mapsto \lambda_r$ is a unitary representation of G into $B(L^2(G))$. We call it the **left regular representation** of G, and we denote the collection of left-translation operators by $\lambda(G)$. It is now easily seen that $\mathrm{span}(\lambda(G))$ is a unital $*$-subalgebra of $B(L^2(G))$, and we can make the following definition.

Definition 3.11. The **group von Neumann algebra** of G, denoted $\mathcal{L}G$, is the bicommutant $\lambda(G)''$. In view of Proposition 1.8, $\mathcal{L}G$ is just the closure of $\mathrm{span}(\lambda(G))$ in any of the topologies defined in Definition 1.1.

The group von Neumann algebra admits a second, equivalent definition in terms of convolutions of compactly supported continuous functions. For any $\xi, \eta \in C_c(G)$, the **convolution** of ξ and η is the function $\xi * \eta : G \to \mathbb{C}$ given by

$$(\xi * \eta)(s) = \int_G \xi(r)\eta(r^{-1}s) \,\mathrm{d}\mu(r).$$

Here, the integral converges because $\xi, \eta \in L^2(G)$. By the same formula, one can define convolutions for broader classes of functions, but in general the integral above may not be well-defined. When it is defined, certain properties are easily shown, like associativity and distributivity over addition of functions.

First, we will show that $C_c(G)$ is closed under convolutions. In fact, we will prove a slightly stronger result, which will be useful to us later.

Proposition 3.12. *For any $\xi \in L^2(G)$ and $\eta \in C_c(G)$, the convolution $\xi * \eta$ is a continuous function on G. If, moreover, $\xi \in C_c(G)$, then $\xi * \eta \in C_c(G)$.*

Proof. To see the first point, suppose $\{t_\alpha\}$ is a net in G converging to $t \in G$. The Cauchy-Schwarz inequality implies that for every α,

$$|\xi * \eta(t_\alpha) - \xi * \eta(t)| \leq \|\xi\|_{L^2} \left(\int_G |\eta(r^{-1}t_\alpha) - \eta(r^{-1}t)|^2 \,\mathrm{d}\mu(r) \right)^{1/2}. \tag{3.6}$$

Since η is compactly supported and t_α eventually lies in a compact neighbourhood W of t, the integrand on the right-hand side is eventually supported in the compact set $W \operatorname{supp}(\eta)^{-1}$. By Theorem 3.3, η is uniformly continuous, so the integrand on the right-hand side of (3.6) converges uniformly to 0 as $t_\alpha \to t$, and so the integral converges to 0 too, proving that $\xi * \eta$ is continuous.

Now, assume $\xi \in C_c(G)$. For any $s \in G$, if $\int_G \xi(r)\eta(r^{-1}s)\,\mathrm{d}\mu(r) \neq 0$, then there exists $r \in \operatorname{supp}(\xi)$ with $r^{-1}s \in \operatorname{supp}(\eta)$. Therefore, $s \in \operatorname{supp}(\xi)\operatorname{supp}(\eta)$, which is compact, so $\xi * \eta$ is compactly supported. □

Now, one can easily verify that convolution makes $C_c(G)$ into an algebra. In fact, since G is unimodular, we make $C_c(G)$ a $*$-algebra by setting $\xi^*(s) = \overline{\xi(s^{-1})}$ for all $\xi \in C_c(G)$ and $s \in G$. The unimodularity of G also gives rise to the following, equivalent characterisation of convolutions.

Proposition 3.13. *For any μ-measurable functions ξ, η on G and any $s \in G$ for which the following integrals are well-defined, we have*

$$\int_G \xi(r)\eta(r^{-1}s)\,\mathrm{d}\mu(r) = \int_G \xi(st^{-1})\eta(t)\,\mathrm{d}\mu(t).$$

Proof. This is obtained via the substitution $t = r^{-1}s$, which is allowed because G is unimodular. □

We stated earlier that convolutions are not *a priori* well-defined for broader classes of functions. We show here two circumstances where convolution is a meaningful operation.

Proposition 3.14. *If $\xi, \eta \in L^2(G)$, then $\xi * \eta$ is everywhere-defined and for every $s \in G$, $|\xi * \eta(s)| \leq \|\xi\|_{L^2}\|\eta\|_{L^2}$*

Proof. This follows from the Cauchy-Schwarz inequality and unimodularity. □

Proposition 3.15. *For any $\xi \in L^1(G)$ and $\eta \in L^2(G)$, we have $\xi * \eta \in L^2(G)$. Moreover, the convolution map $(\xi, \eta) \mapsto \xi * \eta$ is a bicontinuous, bilinear map from $L^1(G) \times L^2(G)$ to $L^2(G)$.*

Proof. We apply Minkowski's integral inequality (see Exercise 1.15 in [33]).

$$\begin{aligned}
\|\xi * \eta\|_{L^2} &= \left(\int_G \left| \int_G \xi(r)\eta(r^{-1}s)\,\mathrm{d}\mu(r) \right|^2 \mathrm{d}\mu(s) \right)^{1/2} \\
&\leq \int_G \left(\int_G \left|\xi(r)\eta(r^{-1}s)\right|^2 \mathrm{d}\mu(s) \right)^{1/2} \mathrm{d}\mu(r) \\
&\leq \int_G |\xi(r)| \left(\int_G \left|\eta(r^{-1}s)\right|^2 \mathrm{d}\mu(s) \right)^{1/2} \mathrm{d}\mu(r) \\
&= \|\xi\|_{L^1} \|\eta\|_{L^2}.
\end{aligned}$$

The bilinearity assertion follows from linearity of the integral, and bicontinuity follows from the bound $\|\xi * \eta\|_{L^2} \leq \|\xi\|_{L^1} \|\eta\|_{L^2}$. □

This shows that for any $\xi \in L^1(G)$, we obtain an operator $\lambda(\xi) \in B(L^2(G))$ given by $\lambda(\xi)\eta = \xi * \eta$, and $\|\lambda(\xi)\|_{L^\infty} \leq \|\xi\|_{L^1(G)}$. In further generality, if ξ is any Borel function on G, we denote by $\lambda(\xi)$ the (possibly unbounded) linear operator on $L^2(G)$ given by $\lambda(\xi)\eta = \xi * \eta$, whose domain consists of those $\eta \in L^2(G)$ for which $\xi * \eta$ is almost-everywhere defined and $\xi * \eta \in L^2(G)$. We make an observation about the operators $\lambda(\xi)$ when $\xi \in L^2(G)$.

Proposition 3.16. *If $\xi \in L^2(G)$, the operator $\lambda(\xi)$ is closed.*

Proof. Suppose η_k is a sequence in $\mathrm{dom}(\lambda(\xi))$ converging to η in $L^2(G)$ and $\lambda(\xi)\eta_k \to \psi \in L^2(G)$. We must show that $\xi * \eta = \psi$. Since $\xi * \eta_k \to \psi$ in $L^2(G)$, we only need to show that $\xi * \eta_k \to \xi * \eta$ pointwise. That is, we need $\xi * (\eta - \eta_k)$ to converge pointwise to 0, but this follows immediately from Proposition 3.14. □

Proposition 3.17. *The map $\lambda : C_c(G) \to B(L^2(G))$ is a $*$-homomorphism, and its range is a nondegenerate $*$-subalgebra of $B(L^2(G))$.*

Proof. We have already seen that this map is linear, and the associativity of convolution immediately implies that $\lambda(\xi * \eta) = \lambda(\xi)\lambda(\eta)$ for all $\xi, \eta \in C_c(G)$. To see that involutions are sent to adjoints, we apply the formula in Proposition 3.13 to see that for any $\eta, \chi \in L^2(G)$ and $\xi \in C_c(G)$,

$$\langle \lambda(\xi)\eta, \chi \rangle_{L^2} = \int_G \int_G \xi(st^{-1})\eta(t)\,\mathrm{d}\mu(t)\overline{\chi(s)}\,\mathrm{d}\mu(s)$$

Since G is second-countable, we apply Fubini's theorem, to see that

$$\begin{aligned}\langle\lambda(\xi)\eta,\chi\rangle_{L^2} &= \int_G \eta(t)\int_G \xi(st^{-1})\overline{\chi(s)}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t)\\ &= \int_G \eta(t)\overline{\int_G \xi^*(ts^{-1})\chi(s)\,\mathrm{d}\mu(s)}\,\mathrm{d}\mu(t)\\ &= \langle\eta,\lambda(\xi^*)\chi\rangle_{L^2}\end{aligned}$$

proving that $\lambda(\xi)^* = \lambda(\xi^*)$. We have shown that λ is a $*$-homomorphism and therefore $\lambda(C_c(G))$ is a $*$-subalgebra of $B(L^2(G))$. That this subalgebra is non-degenerate follows from the next lemma. □

Lemma 3.18. *$L^2(G)$ has an approximate identity consisting of functions in $C_c(G)$. That is, there exists a net $\{u_V\}$ of self-adjoint functions in $C_c(G)$ with $\|u_V\|_{L^1} = 1$, such that $\lambda(u_V) \to \mathbf{1}_{L^2(G)}$ strongly, i.e., for every $\xi \in L^2(G)$, $u_V * \xi \to \xi$ in $L^2(G)$.*

Proof. Let V be an open neighbourhood of e in G. By Urysohn's lemma, there exists a positive-valued function $u_V \in C_c(G)$ with $u_V(e) \neq 0$ and $\mathrm{supp}(u_V) \subset \overline{V}$. By replacing u_V with the function $s \mapsto u_V(s)u_V(s^{-1})$, we can force u_V to be self-adjoint in the $*$-algebra $C_c(G)$. Also, by scaling, we may assume that $\|u_V\|_{L^1} = 1$. We claim that the net $\{u_V\}$, indexed by the directed set of neighbourhoods of e in G, is an approximate identity for $L^2(G)$.

First, suppose $\xi \in C_c(G)$. Let $K = \mathrm{supp}\,\xi$, and fix a compact neighbourhood W of e in G. For any neighbourhood V of e, since $\int_G u_V(r)\,\mathrm{d}\mu(r) = 1$, we have

$$\begin{aligned}\|\xi - u_V * \xi\|_{L^2}^2 &= \int_G \left|\xi(s) - \int_G u_V(r)\xi(r^{-1}s)\,\mathrm{d}\mu(r)\right|^2 \mathrm{d}\mu(s)\\ &\le \int_G \Big(\int_G u_V(r)\big|\xi(s) - \xi(r^{-1}s)\big|\,\mathrm{d}\mu(r)\Big)^2 \mathrm{d}\mu(s)\end{aligned}$$

Eventually, we have $V \subset W$. In this case, the last integrand can only be nonzero when $r \in V \subset W$. Moreover, $\xi(s) - \xi(r^{-1}s)$ is nonzero only if $s \in K$ or $r^{-1}s \in K$. In either case, this implies $s \in WK$, which is compact. Since ξ is uniformly continuous on G, we see that as V shrinks, this last integral converges to 0. Now, to see that $u_V * \xi \to \xi$ for all $\xi \in L^2(G)$, we simply note that $C_c(G)$ is dense in $L^2(G)$, and the net $\{\lambda(u_V)\}$ is uniformly bounded in $B(L^2(G))$. □

The homomorphism properties in Proposition 3.17 continue to hold for larger classes of functions, and, in order to avoid restating arguments, we will often use these properties without proof. We note also that, since G is unimodular, the involution defined on $C_c(G)$ extends (via the same definition) to antilinear isometries on $L^p(G)$ for every $1 \leq p \leq \infty$.

Now, consider the von Neumann algebra $\lambda(C_c(G))''$. By Proposition 1.8, this is the closure of $\lambda(C_c(G))$ in any of the operator topologies introduced in Definition 1.1. Immediately, one may ask, for which Borel functions ξ on G is $\lambda(\xi) \in \lambda(C_c(G))''$? We explore this question in the next propositions.

Proposition 3.19. *For every $\xi \in L^1(G)$, one has $\lambda(\xi) \in \lambda(C_c(G))''$.*

Proof. Choose a net $\{\xi_\alpha\}$ in $C_c(G)$ converging to ξ in $L^1(G)$. By Proposition 3.15, $\lambda(\xi_\alpha) \to \lambda(\xi)$ with respect to the operator norm, and therefore strongly. $\square$

In general, for $\xi \in L^2(G)$, the operator $\lambda(\xi)$ may not be bounded. In what follows, we consider the functions in $L^2(G)$ that happen to define bounded convolution operators, and whose convolution operators lie in $\lambda(C_c(G))''$.

Definition 3.20. Let $\mathcal{J}$ denote the set of operators in $\lambda(C_c(G))''$ that take the form $\lambda(\xi)$, for some $\xi \in L^2(G)$.

Proposition 3.21. *$\mathcal{J}$ is a two-sided ideal of $\lambda(C_c(G))''$. In particular, for any $x \in \lambda(C_c(G))''$ and $\lambda(\xi) \in \mathcal{J}$, $x\lambda(\xi) = \lambda(x\xi)$ and $\lambda(\xi)x = \lambda((x^*\xi^*)^*)$.*

Proof. It is clear that $\mathcal{J}$ is a linear subspace of $\lambda(C_c(G))''$. Suppose $x \in \lambda(C_c(G))''$ and $\lambda(\xi) \in \mathcal{J}$. We wish to show that $x\lambda(\xi) = \lambda(x\xi)$. Since $x\xi \in L^2(G)$, it will follow that $x\lambda(\xi) \in \mathcal{J}$.

Since $\lambda(C_c(G))''$ is the strong closure of $\lambda(C_c(G))$, choose a net $\{\psi_\alpha\}$ in $C_c(G)$ such that $\lambda(\psi_\alpha) \to x$ strongly. In particular, for every $\eta \in L^2(G)$, we have that $\psi_\alpha * \xi * \eta \to x(\xi * \eta)$ in $L^2(G)$. Moreover, $\psi_\alpha * \xi \to x\xi$ in $L^2(G)$, so by Proposition 3.14, $\psi_\alpha * \xi * \eta \to (x\xi) * \eta$ uniformly. This shows that, for every $\eta \in L^2(G)$, $x(\xi * \eta) = (x\xi) * \eta$ almost everywhere. That is, $x\lambda(\xi) = \lambda(x\xi)$, so $\mathcal{J}$ is a left-ideal of $\mathcal{L}G$. By taking adjoints, the required formula for $\lambda(\xi)x$ follows immediately, proving that $\mathcal{J}$ is a two-sided ideal. $\square$

Remark 3.22. Since G is unimodular, everything discussed to this point can be dualised, by replacing the left-convolution map $\lambda(\xi)(\eta) = \xi * \eta$ with the right-convolution map $\rho(\xi)(\eta) = \eta * \xi$. From this, we construct the von Neumann algebra $\rho(C_c(G))''$. The von Neumann algebras $\lambda(C_c(G))''$ and $\rho(C_c(G))''$ are closely related: they are each other's commutants (see Theorem 10.A.8 in [1]).

Recall now the group von Neumann algebra $\mathcal{L}G$, generated by the left regular representation of G. We show here that this coincides with the von Neumann algebra $\lambda(C_c(G))''$.

Proposition 3.23. *The von Neumann algebras $\mathcal{L}G$ and $\lambda(C_c(G))''$ coincide.*

Proof. It suffices to establish the two inclusions $\lambda(G) \subset \lambda(C_c(G))''$ and $\lambda(C_c(G)) \subset \lambda(G)''$. For the first point, let $\{u_V\}$ be the approximate identity constructed in Lemma 3.18. Fixing $r \in G$, consider the net of functions $\{\lambda_r u_V\}$ in $C_c(G)$. For any $\xi \in L^2(G)$ and $s \in G$, the left-invariant of μ implies

$$\begin{aligned}\lambda_r u_V * \xi(s) &= \int_G u_V(r^{-1}t)\xi(t^{-1}s)d\mu(t)\\ &= \int_G u_V(t)\xi(t^{-1}r^{-1}s)d\mu(t) = \lambda_r(u_V * \xi)(s).\end{aligned}$$

That is, $\lambda(\lambda_r u_V) = \lambda_r\lambda(u_V)$. Since $\lambda(u_V) \to \mathbf{1}_{L^2(G)}$ strongly, it follows from Proposition 1.3 that $\lambda(\lambda_r u_V) \to \lambda_r$ strongly. Therefore, $\lambda_r \in \lambda(C_c(G))''$.

Now, to see the second point, we let $\xi \in C_c(G)$, and we wish to approximate $\lambda(\xi)$ strongly by linear combinations of translation operators. Our approach (which, to avoid tedium, we will only sketch) is akin to approximating an integral by Riemann sums. For any partition $\mathcal{P} = \{P_1, \ldots, P_k\}$ of $\operatorname{supp}(\xi)$ into Borel sets, we choose elements $s_i \in P_i$ and define $x_{\mathcal{P}} = \sum_{i=1}^k \mu(P_i)\xi(s_i)\lambda_{s_i}$. We claim that, as we refine the partition $\mathcal{P}$, $x_{\mathcal{P}} \to \lambda(\xi)$ strongly. That is, $x_{\mathcal{P}}\eta \to \xi * \eta$ for every $\eta \in L^2(G)$. In fact, we only need to check that this convergence holds when $\eta \in C_c(G)$. This is because $C_c(G)$ is dense in $L^2(G)$ and $x_{\mathcal{P}}$ is uniformly bounded, with $\|x_{\mathcal{P}}\|_{L^\infty} \leq \sum_{i=1}^k \mu(P_i)|\xi(s_i)| \leq \|\xi\|_{L^\infty(G)}\mu(\operatorname{supp}(\xi))$. For any $\eta \in C_c(G)$,

$$\|\xi * \eta - x_{\mathcal{P}}\eta\|_{L^2}^2 = \int_G \left| \int_G \xi(r)\eta(r^{-1}t)\,\mathrm{d}\mu(r) - \sum_{i=1}^k \mu(P_i)\xi(s_i)\eta(s_i^{-1}t) \right|^2 \mathrm{d}\mu(t).$$

This is now just a computation about Riemann sums of a compactly supported continuous function. As we refine the partition $\mathcal{P}$, the integrand converges to 0 uniformly in t, and since the integrand is only nonzero for t in the compact set $\operatorname{supp}(\xi)\operatorname{supp}(\eta)$, this implies the integral converges to 0, as desired. □

Example 3.24. Consider the case $G = \mathbb{Z}$. It is well-known that the Fourier transform is a unitary operator $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbf{T})$, and this induces an isomorphism $B(\ell^2(\mathbb{Z})) \to B(L^2(\mathbf{T}))$ given by $x \mapsto \mathcal{F}x\mathcal{F}^{-1}$. This isomorphism is isometric and is a homeomorphism in each of the topologies in Definition 1.1. It is also well-known that this isomorphism sends the translation operator λ_n to the multiplication operator by the function $z \mapsto z^n$. In fact, this isomorphism identifies the group von Neumann algebra $\mathcal{L}\mathbb{Z}$ with $L^\infty(\mathbf{T})$. More generally, in abstract harmonic analysis, any locally compact abelian group G admits a so-called Pontryagin dual group $\widehat{G}$ and a Fourier transform $L^2(G) \to L^2(\widehat{G})$ (see

Chapter 4 in [12]), through which we obtain an isomorphism $\mathcal{L}G \cong L^\infty(\widehat{G})$. To see this, Proposition 3.1 in [35] shows that the norm-closure of $\operatorname{span}(\lambda(G))$ is sent to the space $C_0(\widehat{G})$ of complex-valued continuous functions on $\widehat{G}$ vanishing at infinity. Taking σ-weak closures, Theorem 1.2 in Chapter III of [34] implies that $\mathcal{L}G \cong L^\infty(\widehat{G})$.

3.3 The Trace on $\mathcal{L}G$

In order to define noncommutative L^p spaces associated to the von Neumann algebra $\mathcal{L}G$, we now construct a faithful, normal, semifinite trace τ on $\mathcal{L}G_+$.

Definition 3.25. For any $x \in \mathcal{L}G_+$, if $x^{1/2} = \lambda(\xi) \in \mathcal{J}$, then set $\tau(x) = \|\xi\|_{L^2}^2$. Otherwise, set $\tau(x) = \infty$.

We need to show that this map is well-defined. Namely, we will show that there is at most one choice of function ξ for any operator $x \in \mathcal{L}G_+$.

Proposition 3.26. *For any $\xi, \eta \in L^2$, the function $\xi * \eta : G \to \mathbb{C}$ is continuous.*

Proof. In Proposition 3.12, this was proven for $\eta \in C_c(G)$. For arbitrary $\eta \in L^2(G)$, choose a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ in $C_c(G)$ that converges to η in $L^2(G)$. By Proposition 3.14, $\xi * \eta_k \to \xi * \eta$ uniformly as $k \to \infty$. As a uniform limit of continuous functions, $\xi * \eta$ is continuous. $\square$

Corollary 3.27. *Suppose $\xi_1, \xi_2 \in L^2(G)$ are such that $\lambda(\xi_1), \lambda(\xi_2) \in \mathcal{J}$. If $\lambda(\xi_1) = \lambda(\xi_2)$, then $\xi_1 = \xi_2$.*

Proof. By linearity, it suffices to prove that if $\xi \in L^2(G)$ and $\lambda(\xi) = 0$, then $\xi = 0$. By Proposition 3.26, $\xi * \xi^*$ is continuous, so $\xi * \xi^*(s) = 0$ for every $s \in G$. In particular,

$$0 = \xi * \xi^*(e) = \int \xi(r)\overline{\xi(r)}\, \mathrm{d}\mu(r) = \|\xi\|_{L^2}^2.$$

Thus, $\xi = 0$ in $L^2(G)$. $\square$

Therefore, the map τ is well-defined. Next, we show that τ defines a faithful, normal, semifinite trace. The following lemmas will be helpful.

Lemma 3.28. *For any $\lambda(\xi) \in \mathcal{J}$, we have $\xi \in \overline{\operatorname{ran}(\lambda(\xi))}$.*

Proof. In view of Remark 3.22, we dualise Lemma 3.18 and choose a net $\{\eta_\alpha\}$ in $C_c(G)$ such that $f * \eta_\alpha \to f$ in $L^2(G)$ for every $f \in L^2(G)$. In particular, $\xi = \lim_\alpha \xi * \eta_\alpha$, which proves the lemma, since $\xi * \eta_\alpha \in \operatorname{ran}(\lambda(\xi))$. $\square$

Lemma 3.29. *If $\{x_\alpha\}$ is an increasing net in $\mathcal{L}G_+$ with supremum $x \in \mathcal{L}G_+$, then $x_\alpha^{1/2} \to x^{1/2}$ strongly.*

Proof. By Proposition 1.18, we have $x_\alpha \to x$ strongly. Since the net $\{x_\alpha\}$ is uniformly bounded (namely, $\|x_\alpha\|_{L^\infty} \leq \|x\|_{L^\infty}$), Proposition 1.3 implies that for any polynomial p with complex coefficients, $p(x_\alpha) \to p(x)$ strongly. For every α, we have $\sigma(x_\alpha) \subset [0, \|x\|_{L^\infty}]$. Therefore, via the Stone-Weierstrauss theorem, choose a sequence of polynomials $\{p_n\}_{n=1}^\infty$ so that $p_n(\lambda) \to \lambda^{1/2}$ uniformly on the compact interval $[0, \|x\|_{L^\infty}]$. By Proposition 1.35(iv), it follows that $p_n(x) \to x^{1/2}$ and $p_n(x_\alpha) \to x_\alpha^{1/2}$ with respect to the operator norm as $n \to \infty$, and this convergence is uniform in α. Therefore, $x_\alpha^{1/2} \to x^{1/2}$ strongly. □

The following proof is adapted from Theorem 6.2.1 in Part I of [9].

Theorem 3.30. *The map τ in Definition 3.25 is a trace on $\mathcal{L}G$. Moreover, it is faithful, normal, and semifinite.*

Proof. First, we show that τ is additive. Let $x, y \in \mathcal{L}G_+$. Since $x, y \leq x + y$, Proposition 1.20 implies that there exist $a, b \in \mathcal{M}$ with $x^{1/2} = a(x+y)^{1/2}$ and $y^{1/2} = b(x+y)^{1/2}$, with $\ker(x+y) \subset \ker(a), \ker(b)$. If $(x+y)^{1/2} \in \mathcal{J}$, this implies that $x^{1/2}, y^{1/2} \in \mathcal{J}$ too. Contrapositively, if $\tau(x) = \infty$ or $\tau(y) = \infty$, then $\tau(x+y) = \infty$, as desired. Therefore, we may assume that $x^{1/2} = \lambda(\xi)$ and $y^{1/2} = \lambda(\eta)$ for some $\xi, \eta \in L^2(G)$. We wish to prove that $(x+y)^{1/2} = \lambda(\psi)$ with $\psi \in L^2(G)$ satisfying $\|\psi\|_{L^2}^2 = \|\xi\|_{L^2}^2 + \|\eta\|_{L^2}^2$.

Since $x = (x+y)^{1/2}a^*a(x+y)^{1/2}$ and $y = (x+y)^{1/2}b^*b(x+y)^{1/2}$, we see that

$$x + y = (x+y)^{1/2}(a^*a + b^*b)(x+y)^{1/2}.$$

Since $(x+y)^{1/2}$ is self-adjoint, $\ker((x+y)^{1/2}) = \ker(x+y) \subset \ker(a^*a + b^*b)$. Taking orthogonal complements, this implies $\overline{\operatorname{ran}(a^*a + b^*b)} \subset \overline{\operatorname{ran}((x+y)^{1/2})}$. Moreover, since $\overline{\operatorname{ran}((x+y)^{1/2})} = \ker((x+y)^{1/2})^\perp$, we see that $(x+y)^{1/2}$ is injective on $\overline{\operatorname{ran}(x+y)^{1/2}}$, so $a^*a + b^*b$ must fix $\operatorname{ran}((x+y)^{1/2})$. That is,

$$(x+y)^{1/2} = (a^*a + b^*b)(x+y)^{1/2} = a^*x^{1/2} + b^*y^{1/2} = \lambda(a^*\xi + b^*\eta),$$

so we take $\psi = a^*\xi + b^*\eta$. Since $x^{1/2} = a(x+y)^{1/2}$, Corollary 3.27 implies $\xi = a\psi$ and similarly $\eta = b\psi$. Hence, $\psi = (a^*a + b^*b)\psi$, and

$$\|\psi\|_{L^2}^2 = \langle (a^*a + b^*b)\psi, \psi\rangle = \langle a\psi, a\psi\rangle + \langle b\psi, b\psi\rangle = \|\xi\|_{L^2}^2 + \|\eta\|_{L^2}^2.$$

This proves that τ is additive. Immediately, this implies that τ is monotonic.

To see homogeneity, let $x \in \mathcal{L}G_+$ and $\alpha \geq 0$. It is clear that $\tau(0) = 0$, so assume $\alpha > 0$. Then, $x^{1/2} \in \mathcal{J}$ if and only if $\alpha^{1/2}x^{1/2} \in \mathcal{J}$. In this case, writing $x^{1/2} = \lambda(\xi)$, we have $(\alpha x)^{1/2} = \lambda(\alpha^{1/2}\xi)$, and so $\tau(\alpha x) = \|\alpha^{1/2}\xi\|_{L^2}^2 = \alpha\|\xi\|_{L^2}^2$.

Next, for any $x \in \mathcal{L}G$, we must prove that $\tau(x^*x) = \tau(xx^*)$. If both sides are infinite, then we are done, so assume without loss of generality that $\tau(x^*x) < \infty$. That is, $|x| = \lambda(\xi)$ for some $\xi \in L^2(G)$. Let $x = u|x|$ be the polar decomposition of x. By Proposition 1.47(iii), $|x^*| = u|x|u^*$. Using the calculations in Proposition 3.21, $|x|u^* = (u\lambda(\xi))^* = \lambda((u\xi)^*)$, so $|x^*| = \lambda(u(u\xi)^*)$. Therefore, we wish to show that $\|u(u\xi)^*\|_{L^2} = \|\xi\|_{L^2}$. Since u is a partial isometry from $\overline{\operatorname{ran}(|x|)}$ to $\overline{\operatorname{ran}(x)}$, we just need $\xi \in \overline{\operatorname{ran}(|x|)}$ and $(u\xi)^* \in \overline{\operatorname{ran}(|x|)}$. The first point is resolved immediately by Lemma 3.28. For the second point, Lemma 3.28 determines that $(u\xi)^* \in \overline{\operatorname{ran}(\lambda((u\xi)^*))} = \overline{\operatorname{ran}(|x|u^*)} \subset \overline{\operatorname{ran}(|x|)}$. Thus, τ is a trace.

Faithfulness is clear, since $\|\xi\|_{L^2}^2 = 0$ if and only if $\xi = 0$.

To see normality, suppose $\{x_\alpha\}$ is an increasing net in $\mathcal{L}G_+$ with $x_\alpha \nearrow x$. We assume $\tau(x_\alpha)$ is uniformly bounded (otherwise the monotonicity of τ implies $\tau(x) = \infty$, as required). So, write $x_\alpha^{1/2} = \lambda(\xi_\alpha)$. Since $\|\xi_\alpha\|_{L^2}$ is uniformly bounded, the Banach-Alaoglu theorem applied to the Hilbert space $L^2(G)$ asserts that there exists a subnet $\{\xi_{\alpha_\beta}\}$ converging *weakly* to some $\xi \in L^2(G)$, in the sense that $\langle \xi_{\alpha_\beta}, \eta\rangle \to \langle \xi, \eta\rangle$ for every $\eta \in L^2(G)$, and that $\|\xi\|_{L^2} \leq \sup_\alpha \|\xi_\alpha\|_{L^2}$. Since we still have $x_{\alpha_\beta} \nearrow x$, we can simply pass to this subnet. For every $\eta, \psi \in C_c(G)$, we apply Proposition 3.17 to right-convolutions as in Remark 3.22 to see that

$$\langle \xi_\alpha * \psi, \eta\rangle = \langle \xi_\alpha, \eta * \psi^*\rangle \to \langle \xi, \eta * \psi^*\rangle = \langle \xi * \psi, \eta\rangle.$$

Since we have $|\langle \xi_\alpha * \psi, \eta\rangle| \leq \|x^{1/2}\|_{L^\infty}\|\psi\|_{L^2(G)}\|\eta\|_{L^2(G)}$ for every α, the same bound applies to $\langle \xi * \psi, \eta\rangle$. Since $C_c(G)$ is dense in $L^2(G)$, Proposition 3.16 and the closed graph theorem imply that $\lambda(\xi) \in B(L^2(G))$ and $\lambda(\xi)$ is the weak limit of the net $\{\lambda(\xi_\alpha)\}$. But by Lemma 3.29, $\lambda(\xi_\alpha) = x_\alpha^{1/2} \to x^{1/2}$ strongly. Therefore, $x^{1/2} = \lambda(\xi)$, so $\tau(x) = \|\xi\|_{L^2}^2 \leq \sup_\alpha \|\xi_\alpha\|_{L^2}^2 = \sup_\alpha \tau(x_\alpha)$.

Finally, we show that τ is semifinite. Fix $0 \neq x \in \mathcal{M}_+$. By Proposition 1.24, it suffices to construct a nonzero $y \in \mathcal{M}_+$ with $y \leq x$ and $\tau(y) < \infty$. To this end, choose some $\psi \in C_c(G)$ with $\psi \notin \ker(x^{1/2})$. (Since $C_c(G)$ is dense in $L^2(G)$ and $x^{1/2} \neq 0$, such a ψ must exist.) Scaling ψ if necessary, we assume $\|\lambda(\psi^*)\|_{L^\infty} \leq 1$. Letting $a = \lambda(\psi^*) \in \mathcal{J}$, it follows that $a^*a \leq \mathbf{1}_{L^2(G)}$ and hence $x^{1/2}a^*ax^{1/2} \leq x$. Therefore, let $y = x^{1/2}a^*ax^{1/2} = (ax^{1/2})^*(ax^{1/2})$. By Proposition 3.21, we have $ax^{1/2} = \lambda(x^{1/2}\psi)^*$. Since $x^{1/2}\psi \neq 0$, we now have $0 < y \leq x$. Finally, write $ax^{1/2} = u|ax^{1/2}|$ for the polar decomposition of $ax^{1/2}$. Since $\mathcal{J}$ is a two-sided ideal in $\mathcal{L}G$, we have $y^{1/2} = |ax^{1/2}| = u^*ax^{1/2} \in \mathcal{J}$, so $\tau(y) < \infty$. □

Having now established that $\mathcal{L}G$ is a von Neumann algebra with a faithful, normal, semifinite trace, we can now define the spaces $L^p(\mathcal{L}G, \tau)$ as in Definition 2.6. We will simply denote these spaces by $L^p(\mathcal{L}G)$. One notes that the positive operators in $\mathcal{J}$ are precisely the positive operators in $L^2(\mathcal{L}G) \cap \mathcal{L}G$. Moreover, for any $x \in \mathcal{L}G$, one has $x \in \mathcal{J}$ if and only if $|x| \in \mathcal{J}$, since $\mathcal{J}$ is a two-sided ideal of $\mathcal{L}G$. Also, by definition, $x \in L^2(\mathcal{L}G)$ if and only if $|x| \in L^2(\mathcal{L}G)$, so $\mathcal{J} = L^2(\mathcal{L}G) \cap \mathcal{L}G$. From Proposition 2.9, it now follows easily that $L^1(\mathcal{L}G) \cap \mathcal{L}G = \{yz : y, z \in \mathcal{J}\}$.

Recall from Proposition 2.11 that τ extends to a continuous linear functional on $L^1(\mathcal{L}G)$. Proposition 3.31 computes the trace on $L^1(\mathcal{L}G) \cap \mathcal{L}G$.

Proposition 3.31. *Suppose $\xi, \eta \in L^2(G)$ are such that $\lambda(\xi), \lambda(\eta) \in \mathcal{J}$. Then $\tau(\lambda(\xi)^*\lambda(\eta)) = \langle \eta, \xi \rangle$. Equivalently, $\tau(\lambda(\xi)\lambda(\eta)) = \int \xi(s)\eta(s^{-1})\,\mathrm{d}\mu(s)$.*

Proof. First, consider the case where $\xi = \eta$. Write $\lambda(\xi) = u|\lambda(\xi)|$ for the polar decomposition of $\lambda(\xi)$. Since $\mathcal{J}$ is a two-sided ideal of $\mathcal{L}G$ and $u^* \in \mathcal{L}G$, we have $|\lambda(\xi)| = u^*\lambda(\xi) = \lambda(u^*\xi) \in \mathcal{J}$. Therefore, $\lambda(u^*\xi) = (\lambda(\xi)^*\lambda(\xi))^{1/2}$, and so $\tau(\lambda(\xi)^*\lambda(\xi)) = \|u^*\xi\|_{L^2}^2$. To see that $\|u^*\xi\|_{L^2}^2 = \|\xi\|_{L^2}^2$, we need to show that $\xi \in \ker(u^*)^\perp = \overline{\operatorname{ran}(u)} = \overline{\operatorname{ran}(\lambda(\xi))}$, but this is exactly Lemma 3.28.

We prove the general case via polarisation. By an easy calculation using the homomorphism properties of λ, we have

$$\lambda(\xi)^*\lambda(\eta) = \frac{1}{4}\left(\sum_{k=0}^{3} i^k \lambda(\eta + i^k\xi)^*\lambda(\eta + i^k\xi)\right),$$

and therefore by the linearity of τ,

$$\tau(\lambda(\xi)^*\lambda(\eta)) = \frac{1}{4}\left(\sum_{k=0}^{3} i^k \|\eta + i^k\xi\|_{L^2}^2\right) = \langle \eta, \xi \rangle,$$

as required. □

Example 3.32. If G is a discrete group, then the trace τ defined above is a finite trace. To see this, for any $s \in G$, let χ_s denote the characteristic function of the singleton set $\{s\}$, and notice that $\lambda(\chi_e) = \mathbf{1}_{L^2(G)}$. Hence, $\tau(\mathbf{1}_{L^2(G)}) = \|\chi_e\|_{L^2}^2 = 1$, and for any $x \in \mathcal{L}G_+$, we have $x \leq \|x\|_{L^\infty}\mathbf{1}_{L^2(G)}$, so $\tau(x) \leq \|x\|_{L^\infty}\tau(\mathbf{1}_{L^2(G)}) < \infty$. Therefore, $\mathcal{L}G \subset L^1(\mathcal{L}G)$, so every $x \in \mathcal{L}G$ can be written as $\lambda(\xi)^*\lambda(\eta)$, with $\xi, \eta \in \ell^2(G)$. In this case, for any $s \in G$, the unimodularity of G implies

$$\langle x\chi_s, \chi_s \rangle = \langle \eta * \chi_s, \xi * \chi_s \rangle = \int \eta(ts^{-1})\overline{\xi(ts^{-1})}\,\mathrm{d}\mu(t) = \langle \eta, \xi \rangle = \tau(x).$$

In particular, if G is a finite group, then $\tau(x) = \frac{1}{|G|}\operatorname{tr}(x)$, where tr is the usual matrix trace, as in Examples 1.23.

Example 3.33. If G is an abelian group, then a routine argument shows that, under the isomorphism $\mathcal{L}G \cong L^\infty(\widehat{G})$, τ is integration against the Haar measure on $\widehat{G}$. Therefore, when G is abelian, $L^p(\mathcal{L}G) \cong L^p(\widehat{G})$. In this way, noncommutative L^p spaces associated with the group von Neumann algebra $\mathcal{L}G$ generalise the L^p spaces encountered in Fourier analysis.

Remark 3.34. We conclude this chapter with a brief remark on the matrix L^p spaces $L^p(M_n(\mathcal{L}G), \tau_n)$, which we will simply denote by $L^p(M_n(\mathcal{L}G))$. It is clear that we may identify the Hilbert space $L^2(G)^{\oplus n}$ with $L^2(G \times \mathbb{Z}/n\mathbb{Z})$. One might wonder whether $M_n(\mathcal{L}G) \cong \mathcal{L}(G \times \mathbb{Z}/n\mathbb{Z})$. Alas, this is never true for $n \geq 2$, even when G is the trivial group, since $\mathcal{L}(\mathbb{Z}/n\mathbb{Z})$ consists only of matrices that are constant down their diagonals (see Example 3.3.4 in [16]).

Chapter 4

Schur and Fourier Multipliers

For a second-countable, unimodular locally compact group G, we can now define Herz-Schur multipliers and Fourier multipliers as certain operators on $B(L^2(G))$ and $\mathcal{L}G$ respectively, each induced by a continuous function on G. These were the operators studied by Bożejko and Fendler [3]. Drawing on results from Chapter 2 about duality, density and interpolation, we will observe that these multipliers extend/restrict to mappings on their respective noncommutative L^p spaces, and that these L^p-multipliers are closely related to one another. Guided by the work in [5], we then prove two inequalities, which together establish that the L^p-extensions of Herz-Schur and Fourier multipliers have the same completely bounded norms when G is amenable. Each inequality depends on a certain reduction of the multipliers to a suitable subset of G. The first inequality uses restrictions to finite subsets, while the second uses truncations onto compact subsets. Throughout this chapter, we assume that G is a second-countable, unimodular group.

4.1 Herz-Schur Multipliers

First, we characterise Herz-Schur multipliers for arbitrary second-countable groups, and we show that this generalises the classical notion of Schur multipliers.

For a separable Hilbert space $\mathcal{H}$, recall from Example 2.45 that the Schatten ideal $\mathcal{S}_p(\mathcal{H})$, $1 \leq p < \infty$, is the space of operators $x \in B(\mathcal{H})$ with $\mathrm{tr}(|x|^p) < \infty$. Moreover, we write $\mathcal{S}_\infty(\mathcal{H}) = B(\mathcal{H})$. Since $\mathcal{S}_p(\mathcal{H}) \subset B(\mathcal{H})$ for all $1 \leq p \leq \infty$, Corollary 2.42 implies that $\mathcal{S}_p(\mathcal{H}) \subset \mathcal{S}_q(\mathcal{H})$ whenever $p \leq q$, and when $q < \infty$, Proposition 2.34 implies $\mathcal{S}_p(\mathcal{H})$ is dense in $\mathcal{S}_q(\mathcal{H})$.

In the case when $\mathcal{H} = L^2(X)$ for a σ-finite measure space (X, ν), we recall the following concrete description of $\mathcal{S}_2(L^2(X, \nu))$.

Proposition 4.1 (Proposition 9.3.1 in Chapter III of [13])**.** *A bounded linear operator $x \in B(L^2(X))$ lies in the Schatten ideal $\mathcal{S}_2(L^2(X))$ if and only if x is an integral operator with integral kernel in $L^2(X \times X)$. That is, there exists a function $(s,t) \mapsto x_{s,t}$ in $L^2(X \times X)$ such that for every $\xi \in L^2(X)$ and almost every $s \in X$, one has $x\xi(s) = \int x_{s,t}\xi(t)\,\mathrm{d}\nu(t)$. This determines an isomorphism $L^2(X \times X) \cong \mathcal{S}_2(L^2(X))$. Moreover, if x and y are Hilbert-Schmidt operators with kernels $x_{s,t}$ and $y_{s,t}$, then*

$$\mathrm{tr}(xy) = \int_X \int_X x_{s,t} y_{t,s}\,\mathrm{d}\nu(s)\,\mathrm{d}\nu(t). \tag{4.1}$$

Moreover, x^* has integral kernel $(s,t) \mapsto \overline{x_{t,s}}$. Thus, $\|x\|_{\mathcal{S}_2} = \|x_{s,t}\|_{L^2(X\times X)}$.

We call $\mathcal{S}_2(L^2(X))$ the class of **Hilbert-Schmidt** operators. Moving forward, if $x \in B(L^2(X))$ is an integral operator, we write $(s,t) \mapsto x_{s,t}$ for its integral kernel. For example, for any $\lambda(\xi) \in \mathcal{L}G$, $\lambda(\xi)_{s,t} = \xi(st^{-1})$ by Proposition 3.13.

Any function $\phi \in L^\infty(G)$ induces a bounded mapping on $L^2(G \times G)$, sending $\xi \in L^2(G \times G)$ to the function $(s,t) \mapsto \phi(st^{-1})\xi(s,t)$. Therefore, viewing ξ as the integral kernel of a Hilbert-Schmidt operator, ϕ induces a bounded map on $\mathcal{S}_2(L^2(G))$, which we denote by M_ϕ.

Let us consider how to extend M_ϕ to a map on $\mathcal{S}_p(L^2(G))$, for arbitrary $1 \le p \le \infty$. Fixing such a p, suppose M_ϕ restricts to a map $\mathcal{S}_2 \cap \mathcal{S}_p \to \mathcal{S}_2 \cap \mathcal{S}_p$ and that this map is bounded, with norm $C_{\phi,p}$, with respect to the $\mathcal{S}_p$-norm. If $1 \le p \le 2$, then $\mathcal{S}_p \subset \mathcal{S}_2$, so this condition merely says that M_ϕ restricts to a bounded map on $\mathcal{S}_p$ with norm $C_{\phi,p}$. If $2 < p < \infty$, then since $\mathcal{S}_2$ is a dense subset of $\mathcal{S}_p$, M_ϕ extends uniquely to a bounded map on $\mathcal{S}_p$ with norm $C_{\phi,p}$. If $p = \infty$, the situation requires more care, because $\mathcal{S}_2$ is not dense in $B(L^2(G))$. Nevertheless, we will construct a canonical extension of M_ϕ to $B(L^2(G))$. To do this, for any $\phi \in L^\infty(G)$, define $\tilde{\phi} \in L^\infty(G)$ by $\tilde{\phi}(s) = \phi(s^{-1})$, for all $s \in G$.

Proposition 4.2. *Given $\phi \in L^\infty(G)$, suppose the map M_ϕ is bounded with respect to the operator norm. Then M_ϕ extends uniquely to a σ-weakly continuous, norm-bounded linear map on $B(L^2(G))$, with norm $C_{\phi,\infty}$.*

Proof. For any $x, y \in \mathcal{S}_1(L^2(G)) \subset \mathcal{S}_2(L^2(G))$, the trace formula (4.1) implies

$$\begin{aligned}\mathrm{tr}((M_\phi x)y) &= \int_G \int_G \phi(st^{-1}) x_{s,t} y_{t,s}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) \\ &= \int_G \int_G x_{s,t} \tilde{\phi}(ts^{-1}) y_{t,s}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) \\ &= \mathrm{tr}\left(x(M_{\tilde{\phi}} y)\right). \end{aligned} \tag{4.2}$$

Thus, by Proposition 2.37, $|\operatorname{tr}(x(M_{\tilde{\phi}}y))| = |\operatorname{tr}((M_\phi x)y)| \leq C_{\phi,\infty}\|x\|_{L^\infty}\|y\|_{\mathcal{S}_1}$. From Proposition 2.38, it follows that that $M_{\tilde{\phi}}$ restricts to a bounded map T on $\mathcal{S}_1(L^2(G))$ with norm at most $C_{\phi,\infty}$. By Corollary 2.23, T dualises to a σ-weakly continuous, norm-bounded map T^* on $B(L^2(G))$, with $\|T^*\| = \|T\| \leq C_{\phi,\infty}$.

In fact, by (4.2), T^* must extend M_ϕ, so we must have $\|T^*\| = C_{\phi,\infty}$. Recall from Proposition 2.35 that $\mathcal{S}_2(L^2(G))$ is σ-weakly dense in $B(L^2(G))$. Therefore, T^* is the unique σ-weakly continuous extension of M_ϕ to $B(L^2(G))$. □

This leads us to the definition of Herz-Schur multipliers. For this, we consider only the space $C_b(G)$ of bounded continuous complex-valued functions on G.

Definition 4.3. Let $\phi \in C_b(G)$. For $1 \leq p \leq \infty$, if M_ϕ is bounded on $\mathcal{S}_2 \cap \mathcal{S}_p$ with respect to the $\mathcal{S}_p$-norm, then we let $M_{\phi,p}$ denote the restriction/extension of M_ϕ to $\mathcal{S}_p$ constructed above, and we call $M_{\phi,p}$ a **Schur multiplier** on $\mathcal{S}_p$. In the special case $p = \infty$, we call $M_{\phi,\infty}$ a **Herz-Schur** multiplier and write $\phi \in B_2(G)$.

The following important (and surprising) fact about $B_2(G)$ was proven in [3] (p.299) using an unpublished characterisation of Schur multipliers by Gilbert. This characterisation is explained in the proof of Theorem 1.7 in [20].

Theorem 4.4. *If $\phi \in B_2(G)$, then the Herz-Schur multiplier $M_{\phi,\infty}$ is automatically completely bounded with $\|M_{\phi,\infty}\|_{CB} = \|M_{\phi,\infty}\|$.*

Example 4.5. Classically (for example, in [26]), Schur multipliers are defined on ℓ^2-spaces. Given a set X, the space $\ell^2(X)$ has an orthonormal basis $\{\chi_s\}_{s\in X}$, where χ_s denotes the characteristic function of the set $\{s\}$. Any $A \in B(\ell^2(X))$ can be written as a matrix $(A_{s,t})_{s,t\in X}$, by setting $A_{s,t} = \langle A\chi_t, \chi_s\rangle = A\chi_t(s)$. (At this moment, $A_{s,t}$ does not *a priori* denote the integral kernel of A.) For any $\xi \in \ell^2(X)$, one has $\xi = \sum_{t\in X} \xi(t)\chi_t$, so

$$A\xi(s) = \sum_{t\in X} A\chi_t(s)\xi(t) = \sum_{t\in X} A_{s,t}\xi(t). \tag{4.3}$$

This shows that, in fact, A is an integral operator whose integral kernel coincides with the matrix entries $A_{s,t}$. A bounded linear map on $B(\ell^2(X))$ of the form $(A_{s,t})_{s,t\in X} \mapsto (\psi(s,t)A_{s,t})_{s,t\in X}$ for $\psi \in \ell^\infty(X \times X)$ is called a Schur multiplier.

Now, consider a countable, discrete group G and a function $\phi \in B_2(G)$. In view of the above discussion, when $x \in \mathcal{S}_2(\ell^2(G))$, the map M_ϕ acts by multiplying each (s,t)-entry in the matrix for x by $\phi(st^{-1})$. Extending σ-weakly, we see that $M_{\phi,\infty}$ also acts by entrywise multiplication of matrices, so $M_{\phi,\infty}$ is a Schur multiplier, in the sense of [26].

Using interpolation theory, we will show that every $\phi \in B_2(G)$ defines a completely bounded Schur multiplier on $\mathcal{S}_p(L^2(G))$, for all $1 \leq p \leq \infty$. To do this, we must briefly consider matrix transposes on $L^2(G)$. Observe that the mapping $\xi \mapsto \overline{\xi}$, given by $\overline{\xi}(s) = \overline{\xi(s)}$, is antilinear and isometric on $L^2(G)$. Now, for any $x \in B(L^2(G))$, we define $x^{\mathrm{T}} \in B(L^2(G))$ by $x^{\mathrm{T}}\xi = \overline{x^*\overline{\xi}}$ for every $\xi \in L^2(G)$. It is clear that the mapping $x \mapsto x^{\mathrm{T}}$ is isometric on $B(L^2(G))$. Moreover, it follows from Proposition 4.1 that for any $x \in \mathcal{S}_2$, x^{T} has integral kernel $(x^{\mathrm{T}})_{s,t} = x_{t,s}$. These ideas lead to the following lemma.

Lemma 4.6. *If $\phi \in B_2(G)$, then so is $\tilde{\phi}$, and in this case $\|M_{\phi,\infty}\| = \|M_{\tilde{\phi},\infty}\|$.*

Proof. For any $x \in \mathcal{S}_2(L^2(G))$, we have

$$(M_{\tilde{\phi}}(x^{\mathrm{T}}))_{s,t} = \phi(ts^{-1})x_{t,s} = ((M_\phi x)^{\mathrm{T}})_{s,t},$$

so $M_{\tilde{\phi}}(x^{\mathrm{T}}) = (M_\phi x)^{\mathrm{T}}$. Since the mapping $x \mapsto x^{\mathrm{T}}$ is isometric with respect to the operator norm, we conclude that $\tilde{\phi} \in B_2(G)$ with $\|M_{\tilde{\phi},\infty}\| = \|M_{\phi,\infty}\|$. □

Proposition 4.7. *If $\phi \in B_2(G)$, then ϕ defines a completely bounded Schur multipler on $\mathcal{S}_p(L^2(G))$ for every $1 \leq p \leq \infty$, with $\|M_{\phi,p}\|_{CB} \leq \|M_{\phi,\infty}\|$. Moreover, the Schur multiplier $M_{\phi,p}$ is just the restriction of $M_{\phi,\infty}$ to $\mathcal{S}_p(L^2(G))$.*

Proof. The $p = \infty$ case holds by Theorem 4.4. For $p = 1$, the preceding lemma established that $\tilde{\phi} \in B_2(G)$ with $\|M_{\tilde{\phi},\infty}\| = \|M_{\phi,\infty}\|$. As in Proposition 4.2, it follows that ϕ defines a Schur multiplier on $\mathcal{S}_1$ and $M_{\tilde{\phi},\infty}$ is the dual map to $M_{\phi,1}$. Hence, for any $n \in \mathbb{N}$, we have

$$\|\mathrm{id}_n \otimes M_{\phi,1}\| = \|\mathrm{id}_n \otimes M_{\tilde{\phi},\infty}\| = \|M_{\tilde{\phi},\infty}\| = \|M_{\phi,\infty}\|,$$

where the second equality holds due to Theorem 4.4. Therefore, $M_{\phi,1}$ is completely bounded, with $\|M_{\phi,1}\|_{CB} = \|M_{\phi,\infty}\|$. Since $M_{\phi,1}$ is the restriction of M_ϕ, which is the restriction of $M_{\phi,\infty}$, the Riesz-Thorin property in Corollary 2.44 determines that $M_{\phi,\infty}$ restricts to a Schur multiplier $M_{\phi,p}$ on $\mathcal{S}_p(L^2(G))$, for every $1 < p < \infty$. For any $n \in \mathbb{N}$, Corollary 2.44 calculates that

$$\|\mathrm{id}_n \otimes M_{\phi,p}\| \leq \|\mathrm{id}_n \otimes M_{\phi,1}\|^{1/p}\|\mathrm{id}_n \otimes M_{\phi,\infty}\|^{1-1/p} \leq \|M_{\phi,\infty}\|.$$

Therefore, $\|M_{\phi,p}\|_{CB} \leq \|M_{\phi,\infty}\|$. □

Remark 4.8. It is now clear that, for $\phi \in C_b(G)$, we have $\phi \in B_2(G)$ if and only if ϕ defines a Schur multiplier on $\mathcal{S}_1$, and in this case $\|\mathrm{id}_n \otimes M_{\phi,1}\| = \|\mathrm{id}_n \otimes M_{\phi,\infty}\|$ for all $n \in \mathbb{N}$.

From now on, when $\phi \in B_2(G)$, we will often write M_ϕ for the Herz-Schur multiplier $M_{\phi,\infty}$, since each $M_{\phi,p}$ is simply the restriction of $M_{\phi,\infty}$ to $\mathcal{S}_p(L^2(G))$.

4.1.1 Restrictions to Finite Subsets

The central result that we will prove in the remainder of this section relates the completely bounded norm of a Schur multiplier on $\mathcal{S}_p$ to the multipliers obtained by restrictions to finite subsets of G. The key to this result is understanding how Schur multipliers behave under a change of measure.

For an arbitrary Radon measure ν on G, we can define Schur multipliers on $\mathcal{S}_p(L^2(G,\nu))$ in exactly the same manner as before. If $\phi \in C_b(G)$ defines a Schur multiplier on $\mathcal{S}_p(L^2(G,\nu))$, we denote the associated Schur multiplier by $M_{\phi,\nu,p}$.

Suppose $F \subset G$ contains only finitely many points, and let ν be the counting measure on F. Then the space $L^2(G,\nu)$ with respect to this counting measure is unitarily equivalent to $\ell^2(F)$. Since this space is finite-dimensional, we have that $\mathcal{S}_2(\ell^2(F)) = B(\ell^2(F))$, and any $\phi \in C_b(G)$ defines a Schur multiplier on $B(\ell^2(F))$, which we denote by $M_{\phi,F}$. The desired relation between M_ϕ and $M_{\phi,F}$ is contained in the following result, established by Lafforgue and de la Salle in Theorem 1.19 of [20], which we will prove at the end of the section.

Theorem 4.9. *Let $\phi \in C_b(G)$ and $1 \leq p < \infty$. Assume that there exists a constant C, such that for any subset $F \subset G$ containing only finitely many points, the map $M_{\phi,F}$ is completely bounded on $\mathcal{S}_p(\ell^2 F)$ with $\|M_{\phi,F}\|_{CB(\mathcal{S}_p)} \leq C$. Then ϕ defines a completely bounded Schur multiplier on $\mathcal{S}_p(L^2(G))$ with $\|M_{\phi,p}\|_{CB} \leq C$.*

To prove the theorem, we require some notions from measure theory. We say that a net $\{\nu_\alpha\}$ of Radon measures on G converges **vaguely** to a Radon measure ν on G if $\int_G f\,\mathrm{d}\nu_\alpha \to \int_G f\,\mathrm{d}\nu$ for every $f \in C_c(G)$. Also, we let $M_{\mathrm{fin}}(G)$ denote the set of finitely-supported Radon measures on G. That is, $M_{\mathrm{fin}}(G)$ is the set of positive linear combinations of measures δ_s, where δ_s is the Dirac measure at the point $s \in G$. As a first step towards proving Theorem 4.9, we show that the hypothesis to the theorem implies the seemingly stronger statement that $\|M_{\phi,\nu,p}\|_{CB} \leq C$ for every $\nu \in M_{\mathrm{fin}}(G)$.

The following results about finite-rank operators will be useful to us. One notes that if $x \in B(L^2(G))$ has finite rank and polar decomposition $x = u|x|$, then $|x|$ also has finite rank, since u is a partial isometry with initial space $\overline{\mathrm{ran}(|x|)}$ and final space $\overline{\mathrm{ran}(x)}$. Therefore, $\mathrm{tr}(|x|) < \infty$, so $x \in \mathcal{S}_1(L^2(G))$.

Lemma 4.10. *If $x \in B(L^2(G))$ has finite rank, then there exists an orthonormal family $\{e_i\}_{i=1}^N$ in $L^2(G)$ and constants $\{a_{ij}\}_{1\leq i,j\leq N}$ in $\mathbb{C}$ so that, for all $\xi \in L^2(G)$, $x\xi = \sum_{i,j=1}^N a_{ij}\langle \xi, e_j\rangle e_i$, and x has integral kernel $x_{s,t} = \sum_{i,j=1}^N a_{ij} e_i(s)\overline{e_j(t)}$.*

Proof. As a first step, let $\{e_i\}_{i=1}^k$ be an orthonormal basis for $\operatorname{ran}(x)$. We extend this to an orthonormal basis $\{e_i\}_{i=1}^N$ for $\operatorname{ran}(x) + \operatorname{ran}(x^*x)$. Now, if $\xi \in L^2(G)$, then since $x\xi \in \operatorname{ran}(x)$, we have

$$x\xi = \sum_{i=1}^k \langle x\xi, e_i\rangle e_i = \sum_{i=1}^k \langle \xi, x^*e_i\rangle e_i.$$

Since $x^*e_i \in \operatorname{ran}(x^*x)$, we have $x^*e_i = \sum_{j=1}^N \langle x^*e_i, e_j\rangle e_j$. Altogether, we have

$$x\xi = \sum_{i=1}^k \left\langle \xi, \sum_{j=1}^N \langle x^*e_i, e_j\rangle e_j \right\rangle e_i = \sum_{i=1}^k \sum_{j=1}^N \overline{\langle x^*e_i, e_j\rangle} \langle \xi, e_j\rangle e_i.$$

Therefore, setting $a_{ij} = \overline{\langle x^*e_i, e_j\rangle} = \langle xe_j, e_i\rangle$ when $1 \le i \le k$ and $a_{ij} = 0$ for $i > k$, we get the desired decomposition.

For the second statement, we notice that for every $\xi \in L^2(G)$ and $s \in G$,

$$x\xi(s) = \sum_{i,j=1}^N a_{ij}\langle \xi, e_j\rangle e_i(s) = \int_G \sum_{i,j=1}^N a_{ij}\xi(t)\overline{e_j(t)}e_i(s)\,\mathrm{d}\mu(t).$$

That is, $x_{s,t} = \sum_{i,j=1}^N a_{ij}e_i(s)\overline{e_j(t)}$, as required. $\square$

In this situation, we will concisely write $x = \sum_{i,j=1}^N a_{ij}\langle \cdot, e_j\rangle e_i$, and we call this the matrix decomposition of x with respect to the orthonormal family $\{e_i\}_{i=1}^N$. This decomposition respects $\mathcal{S}_p$-norms, as we demonstrate now.

Lemma 4.11. *Let $\{e_i\}_{i=1}^N$ be an orthonormal family in a Hilbert space $\mathcal{H}$, and let $x = \sum_{i,j=1}^N a_{ij}\langle \cdot, e_j\rangle e_i$, with $a_{ij} \in \mathbb{C}$. For any $1 \le p < \infty$, $\|x\|_{\mathcal{S}_p(\mathcal{H})} = \|(a_{ij})\|_{\mathcal{S}_p(\mathbb{C}^N)}$.*

Proof. Let $\{\xi_i\}_{i=1}^N$ be an orthonormal basis for $\mathbb{C}^N$, and let $u : \mathbb{C}^N \to \mathcal{H}$ be the linear map given by $u\xi_i = e_i$. Since $\langle u\xi_i, e_j\rangle_{\mathcal{H}} = \langle \xi_i, \xi_j\rangle$, we have $u^*e_j = \xi_j$, so $u^*u = \mathrm{id}_{\mathbb{C}^N}$, and uu^* is orthogonal projection onto $\operatorname{span}\{e_i\}_{i=1}^N$, which contains $\ker(x)^\perp = \ker(|x|)^\perp = \operatorname{ran}(|x|)$. Moreover, we observe that, for any $1 \le j \le N$, $u^*xu\xi_j = u^*xe_j = u^*(\sum_{i=1}^N a_{ij}e_i) = \sum_{i=1}^N a_{ij}\xi_i$. That is, with respect to the basis $\{\xi_i\}_{i=1}^N$, the linear operator u^*xu is given by the matrix $(a_{ij})_{1\le i,j\le n}$.

Now, we claim that $u^*|x|u = |u^*xu|$. Since $|x| = uu^*|x|$, we have

$$(u^*|x|u)^2 = u^*|x|^2u = u^*x^*xu = (u^*xu)^*(u^*xu),$$

so $u^*|x|u = |u^*xu|$. Now, define the spectral measure $E : \mathcal{B} \to P(M_N(\mathbb{C}))$ by $E(M) = u^*E_{|x|}(M)u$. It is routine to check that $\int_{\mathbb{R}} \lambda\,\mathrm{d}E(\lambda) = u^*|x|u = |u^*xu|$, so E is the spectral measure associated to the operator $|u^*xu|$ in Theorem 1.34. Consequently, for every $1 \leq p < \infty$, $|u^*xu|^p = \int_{[0,\infty)} \lambda^p \, \mathrm{d}E(\lambda) = u^*|x|^p u$.

Finally, since $\langle |x|^p e_i, e_i \rangle = \langle u^*|x|^p u\xi_i, \xi_i \rangle$ for every $i = 1, \ldots, n$, we have

$$\operatorname{tr}(|x|^p)^{1/p} = \operatorname{tr}(u^*|x|^p u)^{1/p} = \operatorname{tr}(|u^*xu|^p)^{1/p} = \|(a_{ij})\|_{\mathcal{S}_p(\mathbb{C}^N)},$$

as required. □

Proposition 4.12. *Let $\phi \in C_b(G)$, $1 \leq p < \infty$, and $n \in \mathbb{N}$. For any $\nu \in M_{\mathrm{fin}}(G)$, the norm $\|\mathrm{id}_n \otimes M_{\phi,\nu,p}\|$ depends only on the support of ν.*

Proof. Let $\nu \in M_{\mathrm{fin}}(G)$ and write F for the support of ν.

First, let us prove the result for $n = 1$. Fix $x \in B(L^2(G,\nu))$. For any $t \in G$, recall that χ_t denotes the characteristic function of the singleton set $\{t\}$. For any $\xi \in L^2(G,\nu)$, one has $\xi = \sum_{t \in F} \xi(t)\chi_t$, so for every $s \in F$, we have $x\xi(s) = \sum_{t\in F} \xi(t) x\chi_t(s) = \int_G \frac{1}{\nu(\{t\})}\xi(t)x\chi_t(s)\,\mathrm{d}\nu(t)$. Thus, x has integral kernel $x_{s,t} = \frac{1}{\nu(\{t\})} x\chi_t(s)$. Therefore, $(M_{\phi,\nu}x)\xi(s) = \sum_{t\in F}\phi(st^{-1})\xi(t)x\chi_t(s)$. In particular, $(M_{\phi,\nu}x)\chi_t(s) = \phi(st^{-1})x\chi_t(s)$.

Now, for each $s \in F$, let $e_s = \nu(\{s\})^{-1/2}\chi_s$, so that $\{e_s\}_{s\in F}$ forms an orthonormal basis for $L^2(G,\nu)$. Notice that $\langle xe_t, e_s\rangle = \frac{\nu(\{s\})^{1/2}}{\nu(\{t\})^{1/2}} x\chi_t(s)$ for all $s, t \in F$. Thus, as in Lemma 4.10, the matrix decomposition of x with respect to this orthonormal family is given by

$$x = \sum_{s,t\in F} \langle xe_t, e_s\rangle \langle \cdot, e_t\rangle e_s = \sum_{s,t\in F} \frac{\nu(\{s\})^{1/2}}{\nu(\{t\})^{1/2}} x\chi_t(s)\langle \cdot, e_t\rangle e_s.$$

Letting $x_t^s = \frac{\nu(\{s\})^{1/2}}{\nu(\{t\})^{1/2}} x\chi_t(s)$, Lemma 4.11 implies $\|x\|_{\mathcal{S}_p(L^2(G,\nu))} = \|(x_t^s)\|_{\mathcal{S}_p(\mathbb{C}^N)}$, where $N = |F|$. Similarly,

$$M_{\phi,\nu}x = \sum_{s,t\in F} \frac{\nu(\{s\})^{1/2}}{\nu(\{t\})^{1/2}} (M_{\phi,\nu}x)\chi_t(s)\langle\cdot, e_t\rangle e_s = \sum_{s,t\in F} \phi(st^{-1}) x_t^s \langle \cdot, e_t\rangle e_s,$$

and so $\|M_{\phi,\nu}x\|_{\mathcal{S}_p(L^2(G,\nu))} = \|(\phi(st^{-1})x_t^s)\|_{\mathcal{S}_p(\mathbb{C}^N)}$. We now see that

$$\|M_{\phi,\nu,p}\| = \sup_{0\neq (x_t^s)\in\mathcal{S}_p(\mathbb{C}^N)} \frac{\|(\phi(st^{-1})x_t^s)\|_{\mathcal{S}_p(\mathbb{C}^N)}}{\|(x_t^s)\|_{\mathcal{S}_p(\mathbb{C}^N)}},$$

which depends only on the support of ν, concluding the proof for $n = 1$.

The argument for $n \geq 2$ is similar. Given $A \in \mathcal{S}_p(L^2(G,\nu)^{\oplus n})$, we write each matrix entry A_{ij} in the same way as above to see that $\|A\|_{\mathcal{S}_p} = \|((A_{ij})_t^s)\|_{\mathcal{S}_p(\mathbb{C}^{nN})}$ and $\|(\mathrm{id}_n \otimes M_{\phi,\nu,p})A\|_{\mathcal{S}_p} = \|((\phi(st^{-1})(A_{ij})_t^s)\|_{\mathcal{S}_p(\mathbb{C}^{nN})}$. □

As a result, the hypothesis in Theorem 4.9 implies that for every $\nu \in M_{\text{fin}}(G)$, $M_{\phi,\nu}$ is completely bounded with $\|M_{\phi,\nu}\|_{CB} \leq C$. To transfer the complete boundedness of such multipliers to the complete boundedness of Schur multipliers with respect to the Haar measure, we will use the following result.

Proposition 4.13. *There exists a net $\{\mu_\alpha\}$ in $M_{\text{fin}}(G)$ converging vaguely to the Haar measure μ. Moreover, $\mu_\alpha \times \mu_\alpha \to \mu \times \mu$ vaguely on $G \times G$.*

Proof. By Proposition 3.6, choose an exhaustion $\{K_n\}_{n\in\mathbb{N}}$ of G by compact sets. Consider the directed set $\mathcal{V}$ of neighbourhoods of e in G. We will index our net by the directed set $\mathbb{N} \times \mathcal{V}$. For every $n \in \mathbb{N}$ and $V \in \mathcal{V}$, the compact set K_n can be partitioned into finitely many Borel subsets $\{E_1, \ldots, E_N\}$, each contained inside a left-translate of V. For each $i = 1, \ldots, N$, choose a point $s_i \in E_i$ and set $\mu_{n,V} = \sum_{i=1}^N \mu(E_i)\delta_{s_i}$.

Fix $f \in C_c(G)$, and choose n sufficiently large so that $\operatorname{supp}(f) \subset K_n$. Since $\{E_1, \ldots, E_n\}$ partitions K_n, the triangle inequality implies that

$$\left| \int_G f(s)\,\mathrm{d}\mu(s) - \sum_{i=1}^N \mu(E_i) f(s_i) \right| \leq \sum_{i=1}^N \int_{E_i} |f(s) - f(s_i)|\,\mathrm{d}\mu(s).$$

Each integral on the right-hand side is nonzero only when $E_i \cap \operatorname{supp}(f) \neq \emptyset$, in which case $E_i \subset \operatorname{supp}(f)V^{-1}V$. For any $\varepsilon > 0$, Theorem 3.3 implies that for sufficiently small V, each integral on the right-hand side is at most $\mu(E_i)\varepsilon$, and hence the right-hand side is at most $\mu(\operatorname{supp}(f)V^{-1}V)\varepsilon$. Eventually, V is contained inside a compact set W, and so $\mu(\operatorname{supp}(f)V^{-1}V) \leq \mu(\operatorname{supp}(f)W^{-1}W) < \infty$. This concludes the proof.

The second statement follows in exactly the same way, since any neighbourhood of (e, e) in $G \times G$ contains a set of the form $V \times V$, with $V \in \mathcal{V}$. □

We now need only two more results before proving Theorem 4.9. The following lemma will help in establishing the first.

Lemma 4.14. *If $x = \langle \cdot, \xi \rangle \eta$ for $\xi, \eta \in L^2(G)$, then $\|x\|_{\mathcal{S}_1} = \|\xi\|_{L^2}\|\eta\|_{L^2}$.*

Proof. By scaling, we may assume that $\|\xi\|_{L^2} = \|\eta\|_{L^2} = 1$. For any $f, g \in L^2(G)$, we have $\langle xf, g \rangle = \langle f, \xi \rangle \langle \eta, g \rangle = \langle f, \langle g, \eta \rangle \xi \rangle$, so $x^* = \langle \cdot, \eta \rangle \xi$ and

$$x^*xf = \langle \langle f, \xi \rangle \eta, \eta \rangle \xi = \|\eta\|_{L^2}^2 \langle f, \xi \rangle \xi.$$

Since $\|\eta\|_{L^2} = \|\xi\|_{L^2} = 1$, it follows that $x^*x =: p_\xi$ is just orthogonal projection onto $\operatorname{span}\{\xi\}$. Therefore, $|x| = p_\xi$ and so $\operatorname{tr}(|x|) = \operatorname{tr}(p_\xi) = 1$, as required. □

Given functions $\psi, \chi \in C_c(G)$, define $\psi \otimes \chi \in C_c(G \times G)$ by $(s,t) \mapsto \psi(s)\chi(t)$. We identify the linear span of such functions with $C_c(G) \otimes C_c(G)$.

Proposition 4.15. *For any $1 \leq p < \infty$, the finite-rank operators in $B(L^2(G))$ with integral kernel in $C_c(G) \otimes C_c(G)$ are dense in $\mathcal{S}_p(L^2(G))$.*

Proof. By Proposition 2.34, $\mathcal{S}_1(L^2(G))$ is dense in $\mathcal{S}_p(L^2(G))$, so we only need to prove the proposition for $p = 1$. First, we claim that the finite-rank operators on $L^2(G)$ are dense in $\mathcal{S}_1(L^2(G))$. Fix an orthonormal basis $\{\xi_k\}_{k \in \mathbb{N}}$ for $L^2(G)$, and let p_n denote the orthogonal projection onto $\operatorname{span}\{\xi_1, \ldots, \xi_n\}$. Now, for any $x \in \mathcal{S}_1(L^2(G))$ with polar decomposition $x = u|x|$, since $p_n \nearrow \mathbf{1}_{L^2(G)}$ it follows that $|x|^{1/2} p_n |x|^{1/2} \nearrow |x|$. By the normality of tr and the bimodule properties of $\mathcal{S}_1(L^2(G))$, it follows that $u|x|^{1/2} p_n |x|^{1/2} \to x$ in $\mathcal{S}_1$. Since p_n has finite rank, so does $u|x|^{1/2} p_n |x|^{1/2}$, proving the density claim. Therefore, we are left to show that the finite rank operators with integral kernel in $C_c(G) \otimes C_c(G)$ are dense amongst all finite rank operators, with respect to the $\mathcal{S}_1$-norm.

Suppose x has finite rank. Let $\{e_i\}_{i=1}^{\infty}$ be an orthonormal family in $L^2(G)$ such that $x = \sum_{i,j=1}^{N} a_{ij} \langle \cdot, e_j \rangle e_i$, and choose sequences $\{e_{i,n}\}_{n=1}^{\infty}$ in $C_c(G)$ such that $e_{i,n} \to e_i$ in $L^2(G)$. For each $n \in \mathbb{N}$, define $x_n = \sum_{i,j=1}^{N} a_{ij} \langle \cdot, e_{j,n} \rangle e_{i,n}$, so that x_n is a finite-rank operator with integral kernel in $C_c(G) \otimes C_c(G)$, by Lemma 4.10. We need to show that $x_n \to x$ in $\mathcal{S}_1(L^2(G))$. By the triangle inequality,

$$\begin{aligned} \|x - x_n\|_{\mathcal{S}_1} &= \left\| \sum_{i,j=1}^{N} a_{ij} \big(\langle \cdot, e_j \rangle e_i - \langle \cdot, e_{j,n} \rangle e_{i,n} \big) \right\|_{\mathcal{S}_1} \\ &\leq \sum_{i,j=1}^{N} |a_{ij}| \left(\|\langle \cdot, e_j - e_{j,n} \rangle e_i\|_{\mathcal{S}_1} + \|\langle \cdot, e_{j,n} \rangle (e_i - e_{i,n})\|_{\mathcal{S}_1} \right) \\ &= \sum_{i,j=1}^{N} |a_{ij}| \left(\|e_j - e_{j,n}\|_{L^2} \|e_i\|_{L^2} + \|e_{j,n}\|_{L^2} \|e_{i,n} - e_i\|_{L^2} \right). \end{aligned}$$

The last equality holds due to Lemma 4.14. Since $e_{i,n} \to e_i$ in $L^2(G)$ for every $i = 1, \ldots, N$, it follows that $\|x - x_n\|_{\mathcal{S}_1} \to 0$ as $n \to \infty$. □

Our final result before the proof of Theorem 4.9 is a simple observation.

Lemma 4.16. *For any $n \in \mathbb{N}$ and $1 \leq p < \infty$, the map from $M_n(\mathbb{C})$ to $[0, \infty)$ given by $T \mapsto \|T\|_{\mathcal{S}_p}$ is continuous with respect to the operator norm on $M_n(\mathbb{C})$.*

Proof. For any $S, T \in M_n(\mathbb{C})$, we have

$$\big| \|T\|_{\mathcal{S}_p} - \|S\|_{\mathcal{S}_p} \big| \leq \|T - S\|_{\mathcal{S}_p} = \operatorname{tr}(|T - S|^p)^{1/p} \leq n^{1/p} \|T - S\|_{L^\infty},$$

where the last inequality holds because $|\operatorname{tr}(A)| \leq n \|A\|_{L^\infty}$ for all $A \in M_n(\mathbb{C})$. □

With these tools, we now prove the promised result. Our proof closely follows the argument for Theorem 1.19 in [20].

Proof of Theorem 4.9. We begin with some reductions. By Proposition 4.15, it suffices to establish the inequality $\|(\mathrm{id}_n \otimes M_\phi)A\|_{\mathcal{S}_p} \leq C\|A\|_{\mathcal{S}_p}$, for every $A \in \mathcal{S}_1(L^2(G)^{\oplus n})$ whose matrix entries have finite rank and integral kernel in $C_c(G) \otimes C_c(G)$. To do this, by Proposition 2.38, it suffices to prove the inequality $|\mathrm{tr}(((\mathrm{id}_n \otimes M_\phi)A)B)| \leq C\|A\|_{\mathcal{S}_p}\|B\|_{\mathcal{S}_{p'}}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, for every $B \in \mathcal{S}_1(L^2(G)^{\oplus n})$ whose matrix entries have finite rank and integral kernel in $C_c(G) \otimes C_c(G)$. By Lemma 4.10, choose a finite orthonormal family $\{e_k\}_{k=1}^N$ in $C_c(G)$ so that we can write the matrix entries as $A_{ij} = \sum_{k,l=1}^N a_{ij,kl}\langle \cdot, e_l\rangle e_k$ and $B_{ij} = \sum_{k,l=1}^N b_{ij,kl}\langle \cdot, e_l\rangle e_k$. Also, we write $K_{ij} = \sum_{k,l=1}^N a_{ij,kl} e_k \otimes \overline{e_l}$ and $\tilde{K}_{ij} = \sum_{k,l=1}^N b_{ij,kl} e_k \otimes \overline{e_l}$ for the integral kernels of A_{ij} and B_{ij} respectively.

Recall that $f_i : L^2(G) \to L^2(G)^{\oplus n}$ denotes inclusion into the i-th coordinate. Writing $e_{k,i} = f_i(e_k)$, the set $\{e_{k,i}\}$ is an orthonormal family in $L^2(G,\mu)^{\oplus n}$, and $A = \sum_{i,j=1}^n \sum_{k,l=1}^N a_{ij,kl}\langle \cdot, e_{l,j}\rangle_{\mathcal{H}^{\oplus n}} e_{k,i}$, and $B = \sum_{i,j=1}^n \sum_{k,l=1}^N b_{ij,kl}\langle \cdot, e_{l,j}\rangle_{\mathcal{H}^{\oplus n}} e_{k,i}$.

Let $\{\mu_\alpha\}$ be the net in $M_{\mathrm{fin}}(G)$ constructed in Proposition 4.13, converging vaguely to the Haar measure μ. For each α, we define $A^\alpha \in \mathcal{S}_p(L^2(G,\mu_\alpha)^{\oplus n})$ by $A^\alpha_{ij} = \sum_{k,l=1}^N a_{ij,kl}\langle \cdot, e_l\rangle e_k$, with inner products taken with respect to μ_α. That is, A^α_{ij} is the integral operator with integral kernel K_{ij} in $L^2(G,\mu_\alpha)$. Similarly, we define $B^\alpha \in \mathcal{S}_{p'}(L^2(G,\mu_\alpha)^{\oplus n})$ by $B^\alpha_{ij} = \sum_{k,l=1}^N b_{ij,kl}\langle \cdot, e_l\rangle e_k$.

In view of Proposition 4.12, for every α, one has

$$|\mathrm{tr}_n(((\mathrm{id}_n \otimes M_{\phi,\mu_\alpha})A^\alpha)B^\alpha)| \leq C\|A^\alpha\|_{\mathcal{S}_p}\|B^\alpha\|_{\mathcal{S}_{p'}}.$$

We will now show that $\mathrm{tr}_n(((\mathrm{id}_n \otimes M_{\phi,\mu_\alpha})A^\alpha)B^\alpha) \to \mathrm{tr}_n(((\mathrm{id}_n \otimes M_\phi)A)B)$, $\|A^\alpha\|_{\mathcal{S}_p} \to \|A\|_{\mathcal{S}_p}$, and $\|B^\alpha\|_{\mathcal{S}_{p'}} \to \|B\|_{\mathcal{S}_{p'}}$. This will conclude the proof.

To see the first point, observe that

$$\begin{aligned}\mathrm{tr}_n(((\mathrm{id}_n \otimes M_{\phi,\mu_\alpha})A^\alpha)B^\alpha) &= \sum_{i,j=1}^n \mathrm{tr}((M_{\phi,\mu_\alpha}A^\alpha_{ij})B^\alpha_{ji}) \\ &= \sum_{i,j=1}^n \int_G \int_G \phi(st^{-1})K_{ij}(s,t)\tilde{K}_{ji}(t,s)\,\mathrm{d}\mu_\alpha(s)\,\mathrm{d}\mu_\alpha(t),\end{aligned}$$

and similarly,

$$\mathrm{tr}_n(((\mathrm{id}_n \otimes M_\phi)A)B) = \sum_{i,j=1}^n \int_G \int_G \phi(st^{-1})K_{ij}(s,t)\tilde{K}_{ji}(t,s)\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t).$$

Since $K_{ij}, \tilde{K}_{ji} \in C_c(G \times G)$, the function $(s,t) \mapsto \phi(st^{-1})K_{ij}(s,t)\tilde{K}_{ji}(t,s)$ is compactly supported and continuous as well. By Proposition 4.13, $\mu_\alpha \times \mu_\alpha$ converges vaguely to $\mu \times \mu$, so we have $\mathrm{tr}_n(((\mathrm{id}_n \otimes M_{\phi,\mu_\alpha})A^\alpha)B^\alpha) \to \mathrm{tr}_n(((\mathrm{id}_n \otimes M_\phi)A)B)$.

Now, we show that $\|A^\alpha\|_{\mathcal{S}_p} \to \|A\|_{\mathcal{S}_p}$. Via the Gram-Schmidt process, we modify the linearly independent family $\{e_i\}_{i=1}^N$ into a family $\{e_i^\alpha\}_{i=1}^N$ that is orthonormal with respect to μ_α. Since $K_{ij} \in \mathrm{span}\{e_k \otimes \overline{e_l}\} = \mathrm{span}\{e_k^\alpha \otimes \overline{e_l^\alpha}\}$, we can write $A_{ij}^\alpha = \sum_{k,l=1}^N a_{ij,kl}^\alpha \langle \cdot, e_l^\alpha \rangle e_k^\alpha$, and we claim that $\lim_\alpha a_{ij,kl}^\alpha = a_{ij,kl}$ for all i, j, k, and l. The proof of this fact is long and is therefore deferred to Corollary 4.18. By Lemmas 4.11 and 4.16, it follows that $\|A^\alpha\|_{\mathcal{S}_p} \to \|A\|_{\mathcal{S}_p}$. By the same argument, $\|B^\alpha\|_{\mathcal{S}_{p'}} \to \|B\|_{\mathcal{S}_{p'}}$, concluding the proof. $\square$

Now, we only need to justify the claim that $\lim_\alpha a_{ij,kl}^\alpha = a_{ij,kl}$. Let $\langle \cdot, \cdot \rangle_\alpha$ and $\| \cdot \|_\alpha$ denote the inner product and norm on $L^2(G, \mu_\alpha)$. We begin with a lemma.

Lemma 4.17. *For every $i = 1, \dots, N$, we have $\lim_\alpha \|e_i - e_i^\alpha\|_\alpha = 0$, and for every $1 \le j < i$, we have $\lim_\alpha \langle e_i, e_j^\alpha \rangle_\alpha = 0$.*

Proof. We prove this by induction on i. Explicitly, in the Gram-Schmidt process, we set $e_1^\alpha = \frac{e_1}{\|e_1\|_\alpha}$ and then

$$e_i^\alpha = \frac{e_i - \sum_{k=1}^{i-1} \langle e_i, e_k^\alpha \rangle_\alpha e_k^\alpha}{\|e_i - \sum_{k=1}^{i-1} \langle e_i, e_k^\alpha \rangle_\alpha e_k^\alpha\|_\alpha}. \tag{4.4}$$

Since the set $\{e_i\}_{i=1}^N$ is linearly independent, this is well-defined.

For the base case, since $e_1^\alpha = \frac{e_1}{\|e_1\|_\alpha}$, we have

$$\|e_1 - e_1^\alpha\|_\alpha = \left| 1 - \frac{1}{\|e_1\|_\alpha} \right| \|e_1\|_\alpha.$$

Since μ_α converges vaguely to μ, we have $\|e_1\|_\alpha \to 1$, and so the right-hand side converges to 0. The lemma's second statement holds vacuously when $i = 1$.

Now, suppose the result holds for all $i < k$, with $k \ge 2$. We first show that $\langle e_k, e_j^\alpha \rangle_\alpha \to 0$ for every $j < k$. By the triangle and Cauchy-Schwarz inequalities,

$$|\langle e_k, e_j^\alpha \rangle_\alpha| \le |\langle e_k, e_j \rangle_\alpha| + \|e_k\|_\alpha \|e_j - e_j^\alpha\|_\alpha.$$

Since μ_α converges vaguely to μ, we have $\langle e_k, e_j \rangle_\alpha \to 0$ and $\|e_k\|_\alpha \to 1$. By assumption $\|e_j - e_j^\alpha\|_\alpha \to 0$, so we conclude that $\langle e_k, e_j^\alpha \rangle_\alpha \to 0$.

Now, we show that $\|e_k - e_k^\alpha\|_\alpha \to 0$. Let $C_{\alpha,k} = \|e_k - \sum_{j=1}^{k-1} \langle e_k, e_j^\alpha \rangle e_j^\alpha\|_\alpha$. Since $\lim_\alpha \langle e_k, e_j^\alpha \rangle_\alpha = 0$, it follows that $\lim_\alpha C_{\alpha,k} = \lim_\alpha \|e_k\|_\alpha = 1$.

Since $e_k^\alpha = \frac{e_k - \sum_{j=1}^{k-1}\langle e_k, e_j^\alpha\rangle_\alpha e_j^\alpha}{C_{\alpha,k}}$, we have

$$e_k = C_{\alpha,k}e_k^\alpha + \sum_{j=1}^{k-1}\langle e_k, e_j^\alpha\rangle_\alpha e_j^\alpha. \tag{4.5}$$

Since $\|e_j^\alpha\|_\alpha = 1$ for every $1 \le j \le N$, the triangle inequality now implies that $\|e_k - e_k^\alpha\|_\alpha \le |C_{\alpha,k} - 1| + \sum_{j=1}^{k-1} |\langle e_k, e_j^\alpha\rangle|$. Since $\lim_\alpha C_{\alpha,k} = 1$ and $\lim_\alpha\langle e_k, e_j^\alpha\rangle = 0$ for every $j < k$, we conclude that $\lim_\alpha \|e_k - e_k^\alpha\|_\alpha = 0$. □

Corollary 4.18. *Writing* $A_{ij}^\alpha = \sum_{k,l=1}^N a_{ij,kl}^\alpha\langle\cdot, e_l^\alpha\rangle_\alpha e_k^\alpha$, *we have* $\lim_\alpha a_{ij,kl}^\alpha = a_{ij,kl}$.

Proof. Recall that $A_{ij}^\alpha = \sum_{k,l=1}^N a_{ij,kl}\langle\cdot, e_l\rangle_\alpha e_k$. Rewriting each e_l and e_k via (4.5), we extract $a_{ij,kl}^\alpha$ as the coefficient of $\langle\cdot, e_l^\alpha\rangle_\alpha e_k^\alpha$. Explicitly,

$$\begin{aligned} a_{ij,kl}^\alpha = C_{\alpha,l}C_{\alpha,k}a_{ij,kl} &+ C_{\alpha,l}\sum_{q=k+1}^N \langle e_q, e_k^\alpha\rangle_\alpha a_{ij,ql} \\ &+ C_{\alpha,k}\sum_{r=l+1}^N \overline{\langle e_r, e_l^\alpha\rangle_\alpha} a_{ij,kr} + \sum_{q=k+1}^N\sum_{r=l+1}^N \langle e_q, e_k^\alpha\rangle_\alpha \overline{\langle e_r, e_l^\alpha\rangle_\alpha} a_{ij,qr}. \end{aligned}$$

Since $\lim_\alpha C_{\alpha,l} = \lim_\alpha C_{\alpha,k} = 1$ and $\lim_\alpha\langle e_q, e_k^\alpha\rangle_\alpha = \lim_\alpha\langle e_r, e_l^\alpha\rangle_\alpha = 0$ whenever $q > k$ and $r > l$, we conclude that $\lim_\alpha a_{ij,kl}^\alpha = a_{ij,kl}$. □

4.2 Fourier Multipliers

In classical Fourier analysis, an L^2-Fourier multiplier on $\mathbf{T}$ with symbol $\phi \in \ell^\infty(\mathbb{Z})$ is a bounded linear map A_ϕ on $L^2(\mathbf{T})$ such that $\widehat{A_\phi f} = \phi\hat{f}$ for every $f \in L^2(\mathbf{T})$. (Here, $f \mapsto \hat{f}$ denotes the Fourier transform.) That is, A_ϕ acts by pointwise multiplication of Fourier coefficients. If A_ϕ extends/restricts continuously to a map on $L^p(\mathbf{T})$ for $1 \le p \le \infty$, we call this extension an L^p-Fourier multiplier on $\mathbf{T}$. Importantly, A_ϕ restricts to an L^∞-Fourier multiplier if and only if it extends to an L^1-Fourier multiplier (one can see this using the same argument as the Proposition in Subsection IV.3.1 of [31]).

For abelian locally compact groups, Fourier multipliers can be defined similarly, using the Fourier transform from $L^2(G)$ to $L^2(\widehat{G})$ (see Theorem 4.25 in [12]). To generalise this notion to arbitrary locally compact groups, we must consider the so-called Fourier algebra $A(G)$, a Banach algebra introduced by Eymard [11], consisting of certain continuous functions on G. When G is abelian, $A(G)$ coincides isometrically with the image of $L^1(\widehat{G})$ under the Fourier transform (see Remark 2.4.5 in [18]). Therefore, we say that $\phi \in C_b(G)$ defines a Fourier multiplier if pointwise multiplication by ϕ is a bounded map on $A(G)$.

Eymard showed (Theorem 3.10 in [11]) that for any locally compact group G, the Banach dual of $A(G)$ is isometrically isomorphic to the von Neumann algebra $\mathcal{L}G$. This is the key insight through which we will study Fourier multipliers. Indeed, Proposition 1.2 in [8] shows that the space of functions defining Fourier multipliers is exactly the class $MA(G)$ described in the following definition.

Definition 4.19. For $\phi \in C_b(G)$, we put $\phi \in MA(G)$ if there exists a σ-weakly continuous linear map T_ϕ on $\mathcal{L}G$ such that $T_\phi \lambda_s = \phi(s)\lambda_s$ for all $s \in G$. Since the linear span of the operators λ_s is σ-weakly dense in $\mathcal{L}G$ by definition, the map T_ϕ is necessarily unique, and we call it the **Fourier multiplier** associated to ϕ. By Proposition 1.4, the map T_ϕ is automatically bounded. If T_ϕ is completely bounded, then we write $\phi \in M_{CB}A(G)$.

Example 4.20. Suppose the map A_ϕ defined above restricts to an L^∞-multiplier on $\mathbf{T}$. For any oscillation $e_n \in L^\infty(\mathbf{T})$ given by $e_n(z) = z^n$, we have that $A_\phi e_n = \phi(n)e_n$. Therefore, under the isomorphism $\mathcal{L}\mathbb{Z} \cong L^\infty(\mathbf{T})$ established in Example 3.24, T_ϕ corresponds to the L^∞-multiplier on $\mathbf{T}$ with symbol ϕ.

Bożejko and Fendler [3] proved that for any locally compact group G, the class of functions $M_{CB}A(G)$ defining completely bounded Fourier multipliers coincides with the space $B_2(G)$ of functions defining Herz-Schur multipliers, and for any $\phi \in M_{CB}A(G)$ we have $\|T_\phi\|_{CB} = \|M_\phi\|_{CB}$. We will not detail their proof here. However, recalling that $\mathcal{L}G \subset B(L^2(G))$, we will show that when $\phi \in B_2(G)$, the Fourier multiplier T_ϕ is just the restriction of the Schur multiplier M_ϕ to $\mathcal{L}G$.

To do this, we define some notation that will be useful in the remainder of the thesis. Suppose $F \subset G$ is compact. Let χ_F be the characteristic function of F, and let $P_F \in B(L^2(G))$ be the multiplication operator by χ_F. We note that P_F is an orthogonal projection, and we think of it as the projection onto $L^2(F)$.

Lemma 4.21. *If $\{K_n\}_{n\in\mathbb{N}}$ is an exhaustion of G by compact sets, then we have $P_{K_n} \to \mathbf{1}_{L^2(G)}$ in any of the locally convex topologies in Subsection 1.1.1.*

Proof. Fix $\xi \in L^2(G)$. We have

$$\|\xi - P_{K_n}\xi\|_{L^2} = \left(\int \left|\big(1 - \chi_{K_n}(s)\big)\xi(s)\right|^2 \,\mathrm{d}\mu(s)\right)^{1/2},$$

which converges to 0 by the dominated convergence theorem. This shows that $P_{K_n} \to \mathbf{1}_{L^2(G)}$ strongly. Since $\|P_{K_n}\|_{L^\infty} \leq 1$, the last assertion in Proposition 1.2 implies that $P_{K_n} \to \mathbf{1}_{L^2(G)}$ σ-strongly, and hence in all weaker topologies. $\square$

Let us make some elementary computations about the projection P_F. First, suppose $x \in B(\mathcal{H})$ is an integral operator with kernel $(s,t) \mapsto x_{s,t}$. For example, we might take $x = \lambda(\xi)$, so that $x_{s,t} = \xi(st^{-1})$. For any $\eta \in L^2(G)$ and $s \in G$,

$$xP_F\eta(s) = \int x_{s,t}\chi_F(t)\eta(t)\,\mathrm{d}\mu(t), \tag{4.6}$$

so that xP_F has integral kernel $(xP_F)_{s,t} = x_{s,t}\chi_F(t)$. Similarly,

$$P_Fx\eta(s) = \int \chi_F(s)x_{s,t}\eta(t)\,\mathrm{d}\mu(t), \tag{4.7}$$

so that P_Fx has integral kernel $(P_Fx)_{s,t} = \chi_F(s)x_{s,t}$. This fact will be important in the proof of the following proposition.

Proposition 4.22. *If $\phi \in B_2(G)$, then the Schur multiplier M_ϕ restricts to a σ-weakly continuous linear map on $\mathcal{L}G$ with $M_\phi\lambda_r = \phi(r)\lambda_r$ for every $r \in G$. Hence, $\phi \in M_{CB}A(G)$ with $T_\phi = M_\phi|_{\mathcal{L}G}$ and $\|T_\phi\|_{CB} \leq \|M_\phi\|_{CB}$.*

Proof. Let K_n be an exhaustion of G by compact sets. Also, let $\{u_V\}$ be an approximate identity as in Lemma 3.18. Fixing $r \in G$, let $\xi_V = \lambda_r u_V$. As in the proof of Proposition 3.23, we have that $\lambda(\xi_V) \to \lambda_r$ strongly. In fact, since the net $\{\lambda(\xi_V)\}$ is uniformly bounded, this implies σ-strong (hence, σ-weak) convergence.

Since $\lambda(\xi_V)$ is an integral operator with kernel $(s,t) \mapsto \xi_V(st^{-1})$, it follows that $P_{K_n}\lambda(\xi_V)$ has integral kernel $(s,t) \mapsto \chi_{K_n}(s)\xi_V(st^{-1})$. Since K_n is compact and $\xi_V \in C_c(G)$, this kernel lies in $L^2(G \times G)$, so $P_{K_n}\lambda(\xi_V) \in \mathcal{S}_2$. By definition, $(M_\phi P_{K_n}\lambda(\xi_V))_{s,t} = \chi_{K_n}(s)\phi(st^{-1})\xi_V(st^{-1})$, which is precisely the integral kernel of $P_{K_n}\lambda(\phi\xi_V)$. This proves the equality between $M_\phi(P_{K_n}\lambda(\xi_V))$ and $P_{K_n}\lambda(\phi\xi_V)$.

Recalling that M_ϕ is σ-weakly continuous, we take σ-weak limits as $n \to \infty$ and see that $M_\phi\lambda(\xi_V) = \lambda(\phi\xi_V)$. Since $\lambda(\xi_V) \to \lambda_r$ σ-weakly, it follows that $\lambda(\phi\xi_V) \to M_\phi\lambda_r$ σ-weakly.

Since each $\|\xi_V\|_{L^1(G)} = 1$ and $\mathrm{supp}(\xi_V) \subset rV$, the continuity of ϕ implies that $\|\phi\xi_V - \phi(r)\xi_V\|_{L^1(G)} \to 0$ as V shrinks towards $\{e\}$. It follows from Proposition 3.15 that $\|\lambda(\phi\xi_V) - \phi(r)\lambda(\xi_V)\|_{L^\infty} \to 0$. Therefore, in the σ-weak topology, $\lim_V \lambda(\phi\xi_V) = \lim_V \phi(r)\lambda(\xi_V)$. That is, $M_\phi\lambda_r = \phi(r)\lambda_r$, as required.

Now, recall from Proposition 3.23 that $\mathcal{L}G$ is the σ-weak closure of the linear span of $\lambda(G)$. It follows that $\mathcal{L}G$ is invariant under M_ϕ. Thus, $\phi \in MA(G)$ and $T_\phi = M_\phi|_{\mathcal{L}G}$. Since M_ϕ is completely bounded, it follows that T_ϕ is completely bounded too, with $\|T_\phi\|_{CB} \leq \|M_\phi\|_{CB} = \|M_\phi\|$. □

Recall that $\mathcal{J}$ is the two-sided ideal in $\mathcal{L}G$ comprised of convolution operators $\lambda(\xi)$ with $\xi \in L^2(G)$. The argument above that proved $M_\phi\lambda(\xi_V) = \lambda(\phi\xi_V)$ also proves that Fourier multipliers act in the following way on $\mathcal{J}$.

Proposition 4.23. *If $\phi \in M_{CB}A(G)$ and $\lambda(\xi) \in \mathcal{J}$, then $T_\phi\lambda(\xi) = \lambda(\phi\xi)$.*

We now work to extend the Fourier multiplier T_ϕ to $L^p(\mathcal{L}G)$ for all $1 \leq p < \infty$.

Proposition 4.24. *Suppose $\phi \in M_{CB}A(G)$. Then T_ϕ extends to a completely bounded linear map $T_{\phi,1}$ on $L^1(\mathcal{L}G)$ with $\|T_{\phi,1}\|_{CB} = \|T_\phi\|_{CB}$.*

Proof. First, suppose $x, y \in L^1(\mathcal{L}G)\cap\mathcal{L}G$, which is contained in $\mathcal{J}$ by the remarks preceding Proposition 3.31. Therefore, write $x = \lambda(\xi)$ and $y = \lambda(\eta)$, for some $\xi, \eta \in L^2(G)$. Recalling that $\tilde{\phi} \in C_b(G)$ is defined by $\tilde{\phi}(s) = \phi(s^{-1})$, it follows from Lemma 4.6, paired with Bożejko and Fendler's result [3], that $\tilde{\phi} \in M_{CB}A(G)$ and $\|T_{\tilde{\phi}}\|_{CB} = \|T_\phi\|_{CB}$. By Proposition 4.23 and Proposition 3.31,

$$\begin{aligned}\tau((T_\phi x)y) = \tau(\lambda(\phi\xi)\lambda(\eta)) &= \int_G \phi(s)\xi(s)\eta(s^{-1})\,\mathrm{d}\mu(s)\\ &= \int_G \xi(s)\tilde{\phi}(s^{-1})\eta(s^{-1})\,\mathrm{d}\mu(s)\\ &= \tau(x(T_{\tilde{\phi}}y)).\end{aligned}$$

Therefore, by Proposition 2.16, $|\tau((T_\phi x)y)| = |\tau(x(T_{\tilde{\phi}}y))| \leq \|T_{\tilde{\phi}}\|\|x\|_{L^1}\|y\|_{L^\infty}$. By Proposition 2.38, it follows that $T_\phi x \in L^1(\mathcal{L}G)$ with $\|T_\phi x\|_{L^1} \leq \|T_{\tilde{\phi}}\|\|x\|_{L^1}$. Since $L^1(\mathcal{L}G)\cap\mathcal{L}G$ is dense in $L^1(\mathcal{L}G)$, we deduce that T_ϕ extends to a bounded map $T_{\phi,1}$ on $L^1(\mathcal{L}G)$, with dual map $T_{\tilde{\phi}}$. Thus, $\|T_{\phi,1}\|_{CB} = \|T_{\tilde{\phi}}\|_{CB} = \|T_\phi\|_{CB}$. □

Corollary 4.25. *If $\phi \in M_{CB}A(G)$, then T_ϕ extends to completely bounded maps $T_{\phi,p}$ on $L^p(\mathcal{L}G)$ for every $1 \leq p \leq \infty$ with $\|T_{\phi,p}\|_{CB} \leq \|\phi\|_{M_{CB}A(G)}$.*

Proof. By the Riesz-Thorin property in Corollary 2.44, the maps T_ϕ and $T_{\phi,1}$ induce bounded maps $T_{\phi,p}$ on $L^p(\mathcal{L}G)$ for all $1 < p < \infty$, and for any $n \in \mathbb{N}$,

$$\|\mathrm{id}_n \otimes T_{\phi,p}\| \leq \|\mathrm{id}_n \otimes T_{\phi,1}\|^{1/p}\|\mathrm{id}_n \otimes T_\phi\|^{1-1/p} \leq \|T_\phi\|_{CB}.$$

This establishes the inequality $\|T_{\phi,p}\|_{CB} \leq \|T_\phi\|_{CB}$. □

Importantly, for each $\phi \in M_{CB}A(G)$, this construction ensures that the maps $T_{\phi,p}$ all agree on the intersections of their domains.

Example 4.26. Consider again the example $G = \mathbb{Z}$. Recall from Example 3.33 that we have an isomorphism $L^p(\mathcal{L}\mathbb{Z}) \cong L^p(\mathbf{T})$. By Example 4.20 and a density argument, it is clear that under this isomorphism, $T_{\phi,p}$ corresponds to the L^p-Fourier multiplier with symbol ϕ.

Our principal goal herein is to extend Bożejko and Fendler's result in [3] to L^p spaces. Namely, for a certain class of locally compact groups, we will show that $\|M_{\phi,p}\|_{CB} = \|T_{\phi,p}\|_{CB}$ for all $\phi \in M_{CB}A(G)$ and $1 \leq p < \infty$.

4.2.1 Truncations

In the remainder of this section, we define maps $j_{p,F} : L^p(\mathcal{L}G) \to \mathcal{S}_p(L^2(G))$ that *intertwine* Fourier and Schur multipliers, in the sense that $j_{p,F} \circ T_{\phi,p} = M_{\phi,p} \circ j_{p,F}$. These will be a crucial tool in establishing the second key inequality of the thesis.

Recall that for any compact set $F \subset G$, we defined the projection P_F onto $L^2(F) \subset L^2(G)$. For any integral operator $x \in B(L^2(G))$ with kernel $(s,t) \mapsto x_{s,t}$, we calculated in (4.6) and (4.7) that $(xP_F)_{s,t} = x_{s,t}\chi_F(t)$ and $(P_F x)_{s,t} = \chi_F(s)x_{s,t}$. In particular, $P_F x P_F$ is an integral operator with $(P_F x P_F)_{s,t} = \chi_F(s)\chi_F(t)x_{s,t}$. The importance of the mapping $x \mapsto P_F x P_F$ is explained in the following proposition: it intertwines Fourier and Schur multipliers.

Proposition 4.27. *Fix $\phi \in M_{CB}A(G)$ and $F \subset G$ compact. For any $x \in \mathcal{L}G$, $P_F(T_\phi x)P_F = M_\phi(P_F x P_F)$.*

Proof. First, suppose $x = \lambda(\xi)$ for some $\xi \in C_c(G)$. Then $P_F x P_F$ has integral kernel $(s,t) \mapsto \chi_F(s)\chi_F(t)\xi(st^{-1})$, which lies in $L^2(G \times G)$. Since $T_\phi x = \lambda(\phi\xi)$, we see that $P_F(T_\phi x)P_F$ has integral kernel $(s,t) \mapsto \chi_F(s)\chi_F(t)\phi(st^{-1})\xi(st^{-1})$, which is indeed the integral kernel of $M_\phi(P_F x P_F)$.

Since $\lambda(C_c(G))$ is σ-weakly dense in $\mathcal{L}G$, and the multipliers T_ϕ and M_ϕ are σ-weakly continuous, the desired equality must hold for all $x \in \mathcal{L}G$. $\square$

Eventually, we will construct similar maps from $L^p(\mathcal{L}G)$ to $\mathcal{S}_p(L^2(G))$ for all $1 \le p \le \infty$, that intertwine Fourier and Schur multipliers in the same way. Importantly, these maps will be completely bounded. First, let $j_{\infty,F} : \mathcal{L}G \to B(L^2(G))$ denote the map $x \mapsto P_F x P_F$. Now, recall the pure tensors in Definition 2.57. For any $A \in M_n(\mathcal{M})$, we have $\mathrm{id}_n \otimes j_{\infty,F}(A) = (\mathbf{1}_{M_n(\mathbb{C})} \otimes P_F)A(\mathbf{1}_{M_n(\mathbb{C})} \otimes P_F)$. Moreover, identifying $L^2(G)^{\oplus n}$ with $L^2(G \times \mathbb{Z}/n\mathbb{Z})$, we notice that $\mathbf{1}_{M_n(\mathbb{C})} \otimes P_F = P_{\tilde{F}}$, where $\tilde{F} = F \times \mathbb{Z}/n\mathbb{Z}$. With this in mind, we have the following proposition, the proof of which was adapted from a suggestion by my supervisor.

Proposition 4.28. *Fix $n \in \mathbb{N}$. If $A \in L^2(M_n(\mathcal{L}G)) \cap M_n(\mathcal{L}G)$, then $P_{\tilde{F}}A$ and $AP_{\tilde{F}}$ are Hilbert-Schmidt operators with $\|P_{\tilde{F}}A\|_{\mathcal{S}_2} = \|AP_{\tilde{F}}\|_{\mathcal{S}_2} = \mu(F)^{1/2}\|A\|_{L^2}$.*

Proof. By the remarks preceding Proposition 3.31, $L^2(\mathcal{L}G) \cap \mathcal{L}G = \mathcal{J}$, so we write $A_{ij} = \lambda(\xi_{ij})$ with $\xi_{ij} \in L^2(G)$. Using Proposition 3.31, we compute

$$\|A\|_{L^2}^2 = \tau_n(A^*A) = \sum_{i,j=1}^{n} \tau((A_{ij})^*A_{ij}) = \sum_{i,j=1}^{n} \|\xi_{ij}\|_{L^2(G)}^2.$$

Now, we claim that $P_{\tilde{F}}A \in \mathcal{S}_2(L^2(G \times \mathbb{Z}/n\mathbb{Z}))$. For any $\eta \in L^2(G \times \mathbb{Z}/n\mathbb{Z})$ and $j \in \mathbb{Z}/n\mathbb{Z}$, we define $\eta_j \in L^2(G)$ by $\eta_j(s) = \eta(s, j)$, so that $\eta = (\eta_0, \dots, \eta_{n-1})$ in the correspondence $L^2(G \times \mathbb{Z}/n\mathbb{Z}) \cong L^2(G)^{\oplus n}$. Then for any $s \in G$, $k \in \mathbb{Z}/n\mathbb{Z}$,

$$\begin{aligned} A\eta(s,k) &= \sum_{j=1}^{n} A_{kj}\eta_j(s) \\ &= \sum_{j=1}^{n} \int_G \xi_{kj}(st^{-1})\eta_j(t)\,\mathrm{d}\mu(t), \end{aligned}$$

so A has integral kernel $A_{(s,k),(t,j)} = \xi_{kj}(st^{-1})$. It follows that $P_{\tilde{F}}A$ has integral kernel $((s,k),(t,j)) \mapsto \chi_F(s)\xi_{kj}(st^{-1})$. Since F is compact and each $\xi_{kj} \in L^2(G)$, this kernel is in $L^2((G\times\mathbb{Z}/n\mathbb{Z})\times(G\times\mathbb{Z}/n\mathbb{Z}))$. In particular, using the calculation of the $\mathcal{S}_2$-norm in Proposition 4.1,

$$\begin{aligned} \|P_{\tilde{F}}A\|_{\mathcal{S}_2}^2 &= \sum_{k,j=1}^{n} \int_G \int_G |\chi_F(s)\xi_{kj}(st^{-1})|^2\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) \\ &= \sum_{k,j=1}^{n} \mu(F)\|\xi_{kj}\|_{L^2(G)}^2 \\ &= \mu(F)\|A\|_{L^2}^2. \end{aligned}$$

Similarly, $AP_{\tilde{F}}$ has integral kernel $((s,k),(t,j)) \mapsto \chi_F(t)\xi_{kj}(st^{-1})$, and a similar calculation as above yields that $\|AP_{\tilde{F}}\|_{\mathcal{S}_2} = \mu(F)^{1/2}\|A\|_{L^2}$. □

Corollary 4.29. *Fix $n \in \mathbb{N}$ and $F \subset G$ compact. If $A \in L^1(M_n(\mathcal{L}G)) \cap M_n(\mathcal{L}G)$, then $P_{\tilde{F}}AP_{\tilde{F}} \in \mathcal{S}_1(L^2(G)^{\oplus n})$ with $\|P_{\tilde{F}}AP_{\tilde{F}}\|_{\mathcal{S}_1} \le \mu(F)\|A\|_{L^1}$.*

Proof. Write $|A| = U|A|$ for the polar decomposition of A. By Proposition 2.18 and Proposition 4.28, we have

$$\|P_{\tilde{F}}AP_{\tilde{F}}\|_{\mathcal{S}_1} \le \left\|P_{\tilde{F}}U|A|^{1/2}\right\|_{\mathcal{S}_2} \left\||A|^{1/2}P_{\tilde{F}}\right\|_{\mathcal{S}_2} \le \mu(F)\left\|U|A|^{1/2}\right\|_{L^2}\left\||A|^{1/2}\right\|_{L^2}.$$

Applying Proposition 2.14, we conclude that $\|P_{\tilde{F}}AP_{\tilde{F}}\|_{\mathcal{S}_1} \le \mu(F)\|A\|_{L^1}$. □

By Proposition 2.34, $L^1(\mathcal{L}G) \cap \mathcal{L}G$ is dense in $L^1(\mathcal{L}G)$. The preceding result therefore implies that the map $x \mapsto P_FxP_F$ extends uniquely to a completely bounded map $j_{1,F} : L^1(\mathcal{L}G) \to \mathcal{S}_1(L^2(G))$ with completely bounded norm at most $\mu(F)$. Moreover, by Proposition 4.27, this extension intertwines Fourier and Schur multipliers, in the sense that $j_{1,F} \circ T_{\phi,1} = M_{\phi,1} \circ j_{1,F}$.

Trivially, for any $A \in M_n(\mathcal{L}G)$, $\|P_{\tilde{F}}AP_{\tilde{F}}\|_{L^\infty} \le \|A\|_{L^\infty}$. That is, $j_{\infty,F}$ is also completely bounded, with norm at most 1. By interpolating between $j_{1,F}$ and $j_{\infty,F}$, we will obtain the desired maps from $L^p(\mathcal{L}G)$ to $\mathcal{S}_p(L^2(G))$.

Corollary 4.30. *For any $1 \leq p < \infty$, the map on $L^p(\mathcal{L}G) \cap \mathcal{L}G$ given by $x \mapsto P_F x P_F$ extends to a completely bounded map $j_{p,F} : L^p(\mathcal{L}G) \to \mathcal{S}_p(L^2(G))$ that intertwines Fourier and Schur multipliers and satisfies $\|j_{p,F}\|_{CB} \leq \mu(F)^{1/p}$.*

Proof. Since $j_{1,F}$ and $j_{\infty,F}$ agree on $L^1(\mathcal{L}G) \cap \mathcal{L}G$, the Riesz-Thorin property in Corollary 2.44 induces bounded maps $j_{p,F} : L^p(\mathcal{L}G) \to \mathcal{S}_p(L^2(G))$. Since $\mathrm{id}_n \otimes j_{p,F}$ is obtained by interpolating $\mathrm{id}_n \otimes j_{1,F}$ and $\mathrm{id}_n \otimes j_{\infty,F}$, we get that

$$\|\mathrm{id}_n \otimes j_{p,F}\| \leq \|\mathrm{id}_n \otimes j_{1,F}\|^{1/p} \|\mathrm{id}_n \otimes j_{\infty,F}\|^{1-1/p} \leq \mu(F)^{1/p},$$

as required. The intertwining property follows immediately from Proposition 4.27 and the density of $L^1(\mathcal{L}G) \cap \mathcal{L}G$ in $L^p(\mathcal{L}G)$ established in Proposition 2.34. □

4.3 The First Inequality

In this section, we establish the first main result of this thesis, that for any $\phi \in M_{CB}A(G)$, the completely bounded norm of the Schur multiplier $M_{\phi,p}$ is bounded above by the completely bounded norm of the Fourier multiplier $T_{\phi,p}$. For $p = \infty$ (the case of Bożejko and Fendler), this is the nontrivial inequality. Our proof, which follows Caspers and de la Salle's argument [5], is driven forwards by Theorem 4.9 and thereby bears close resemblance to Bożejko and Fendler's method. We will need the following technical result. Let $\mathbb{Q}_{>1} \coloneqq \mathbb{Q} \cap (1, \infty)$.

Lemma 4.31. *Let $p, p' \in \mathbb{Q}_{>1}$ be such that $\frac{1}{p} + \frac{1}{p'} = 1$. There exist nets $\{x_\alpha\}, \{y_\alpha\}$ in $L^1(\mathcal{L}G) \cap \mathcal{L}G$ with $\|x_\alpha\|_{L^p} = 1$ and $\|y_\alpha\|_{L^{p'}} = 1$, such that $\tau((T_\phi x_\alpha) y_\alpha) \to \phi(e)$ for every $\phi \in M_{CB}A(G)$.*

Proof. First, write $p = \frac{r}{q}$, with $r, q \in \mathbb{N}$, so that $p' = \frac{r}{r-q}$, and note that $r > q$.

Consider the directed set $\{V_\alpha\}$ of symmetric open neighbourhoods of e in G. For every V_α, since the preimage of V_α under the multiplication map is an open neighbourhood of (e, e) in $G \times G$, choose neighbourhoods U_α and $\tilde{U}_\alpha$ of e so that $U_\alpha \tilde{U}_\alpha \subset V_\alpha$. Now, let $W_\alpha = U_\alpha^{-1} \cap \tilde{U}_\alpha$, so that W_α is a neighbourhood of e satisfying $W_\alpha^{-1} W_\alpha \subset V_\alpha$. By Theorem 3.4, choose a positive-valued function $\psi_\alpha \in C_c(G)$ with $\psi_\alpha(e) \neq 0$ and $\mathrm{supp}(\psi_\alpha) \subset W_\alpha$. Defining $\eta_\alpha = \psi_\alpha^* * \psi_\alpha$, Proposition 3.12 implies $\eta_\alpha \in C_c(G)$ and $\mathrm{supp}(\eta_\alpha) \subset W_\alpha^{-1} W_\alpha \subset V_\alpha$. By Proposition 3.31, $\tau(\lambda(\eta_\alpha)) = \|\psi_\alpha\|_{L^2}^2$, so we obtain a net $\{\lambda(\eta_\alpha)\}$ of positive operators in $L^1(\mathcal{L}G) \cap \mathcal{L}G$. Normalising η_α if necessary, we may assume $\|\lambda(\eta_\alpha)\|_{L^r} = 1$.

Now, define $x_\alpha = \lambda(\eta_\alpha)^q$ and $y_\alpha = \lambda(\eta_\alpha)^{r-q}$. Then

$$\|x_\alpha\|_{L^p}^p = \tau(\lambda(\eta_\alpha)^{qp}) = \tau(\lambda(\eta_\alpha)^r) = 1,$$

and

$$\|y_\alpha\|_{L^{p'}}^{p'} = \tau(\lambda(\eta_\alpha)^{(r-q)p'}) = \tau(\lambda(\eta_\alpha)^r) = 1.$$

Since $\lambda : C_c(G) \to \mathcal{L}G$ is a $*$-homomorphism, we have $x_\alpha = \lambda(f_\alpha)$, where f_α is the q-fold convolution of η_α with itself. One sees inductively that $f_\alpha \in C_c(G)$, with $\operatorname{supp}(f_\alpha) \subset (V_\alpha)^q$. Therefore, by Proposition 4.23, $T_\phi x_\alpha = \lambda(\phi f_\alpha)$.

Similarly, $y_\alpha = \lambda(g_\alpha)$, where $g_\alpha \in C_c(G)$ is the $(r-q)$-fold convolution of η_α with itself. Therefore, by Proposition 3.31,

$$\tau((T_\phi x_\alpha)y_\alpha) = \tau(\lambda(\phi f_\alpha)\lambda(g_\alpha)) = \int_G \phi(s) f_\alpha(s) g_\alpha(s^{-1})\,\mathrm{d}\mu(s).$$

Consider the special case where $\phi \equiv 1$ on G. (In this case, T_ϕ is the identity map on $\mathcal{L}G$, which is completely bounded.) Let $\tilde{g}_\alpha(s) = g_\alpha(s^{-1})$, $s \in G$. Since f_α and g_α take positive values, Proposition 3.31 implies that

$$\|f_\alpha \tilde{g}_\alpha\|_{L^1(G)} = \int_G f_\alpha(s) g_\alpha(s^{-1})\,\mathrm{d}\mu(s) = \tau(x_\alpha y_\alpha) = \tau(\lambda(\eta_\alpha)^r) = 1.$$

Returning to the general case, we have

$$\begin{aligned}
|\tau((T_\phi x_\alpha)y_\alpha) - \phi(e)| &= \left| \int_G \phi(s) f_\alpha(s) g_\alpha(s^{-1})\,\mathrm{d}\mu(s) - \int_G \phi(e) f_\alpha(s) g_\alpha(s^{-1})\,\mathrm{d}\mu(s) \right| \\
&= \left| \int_G (\phi(s) - \phi(e)) f_\alpha(s) g_\alpha(s^{-1})\,\mathrm{d}\mu(s) \right| \\
&\le \|f_\alpha \tilde{g}_\alpha\|_{L^1(G)} \sup_{s \in V_\alpha^{\min\{r,q\}}} |\phi(s) - \phi(e)| \\
&= \sup_{s \in (V_\alpha)^{\min\{r,q\}}} |\phi(s) - \phi(e)|,
\end{aligned}$$

where the inequality holds because $\operatorname{supp}(f_\alpha \tilde{g}_\alpha) \subset (V_\alpha)^{\min\{r,q\}}$. Since ϕ is continuous, this supremum must tend to 0 as V_α shrinks. Thus, $\lim_\alpha \tau((T_\phi x_\alpha)y_\alpha) = \phi(e)$, as desired. $\square$

Theorem 4.32. *Let G be a locally compact, second-countable, unimodular group. Suppose $\phi \in M_{CB}A(G)$, and fix $1 \le p < \infty$. Then $\|M_{\phi,p}\|_{CB} \le \|T_{\phi,p}\|_{CB}$.*

Proof. For $p = 1$, Bożejko and Fendler's result [3], together with Remark 4.8 and Proposition 4.24, implies that $\|M_{\phi,1}\|_{CB} = \|T_{\phi,1}\|_{CB}$. For $1 < p < \infty$, recall from Proposition 4.7 and Corollary 4.25 that the maps $M_{\phi,p}$ and $T_{\phi,p}$ can be obtained via interpolation. Therefore, the completely bounded norms $\|M_{\phi,p}\|_{CB}$ and $\|T_{\phi,p}\|_{CB}$ are logarithmically convex in p, hence continuous on the open interval $(1, \infty)$. Therefore, it suffices to prove the desired inequality for all p in a dense subset of $(1, \infty)$. In particular, we will assume that $p \in \mathbb{Q}_{>1}$.

We wish to prove that $\|M_{\phi,p}\|_{CB} \leq \|T_{\phi,p}\|_{CB}$. In view of Theorem 4.9 and Proposition 2.38, it suffices to prove the following: for any $n \in \mathbb{N}$, any subset $F \subset G$ containing only finitely many points, and any $A, B \in B(\ell^2(F)^{\oplus n})$,

$$|\operatorname{tr}_n(((\mathrm{id}_n \otimes M_{\phi,F})A)B)| \leq \|T_{\phi,p}\|_{CB}\|A\|_{\mathcal{S}_p}\|B\|_{\mathcal{S}_{p'}}, \tag{4.8}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Using Proposition 4.1, we have

$$\begin{aligned}\operatorname{tr}_n(((\mathrm{id}_n \otimes M_{\phi,F})A)B) &= \sum_{i,j=1}^{n} \operatorname{tr}((M_{\phi,F}A_{ij})B_{ji}) \\ &= \sum_{i,j=1}^{n} \sum_{s,t\in F} \phi(st^{-1})(A_{ij})_{s,t}(B_{ji})_{t,s}. \end{aligned}\tag{4.9}$$

Now, for all $s, t \in F$, define $\phi_{s,t} \in C_b(G)$ by $\phi_{s,t}(g) = \phi(sgt^{-1})$. We claim that $\phi_{s,t} \in M_{CB}A(G)$. Indeed, for each $g \in G$, we have

$$\begin{aligned}\phi_{s,t}(g)\lambda_g &= \phi(sgt^{-1})\lambda_g \\ &= \lambda_{s^{-1}}\phi(sgt^{-1})\lambda_{sgt^{-1}}\lambda_t \\ &= \lambda_{s^{-1}}T_\phi(\lambda_s\lambda_g\lambda_{t^{-1}})\lambda_t.\end{aligned}$$

Since T_ϕ is σ-weakly continuous, so is the mapping $x \mapsto \lambda_{s^{-1}}T_\phi(\lambda_s x\lambda_{t^{-1}})\lambda_t$ on $\mathcal{L}G$. Therefore, the Fourier multiplier $T_{\phi_{s,t}}$ is given by $T_{\phi_{s,t}}x = \lambda_{s^{-1}}T_\phi(\lambda_s x\lambda_{t^{-1}})\lambda_t$, for $x \in \mathcal{L}G$. By assumption, T_ϕ is completely bounded, and by Proposition 2.56, so is multiplication by a unitary. Therefore, the Fourier multiplier $T_{\phi_{s,t}}$ is completely bounded and $\phi_{s,t} \in M_{CB}A(G)$. Choose nets $\{x_\alpha\}, \{y_\alpha\}$ in $L^1(\mathcal{L}G) \cap \mathcal{L}G$ as in Lemma 4.31, so that $\tau((T_{\phi_{s,t}}x_\alpha)y_\alpha) \to \phi_{s,t}(e) = \phi(st^{-1})$. Picking up from (4.9), we have

$$\begin{aligned}\operatorname{tr}_n(((\mathrm{id}_n \otimes M_{\phi,F})A)B) &= \sum_{i,j=1}^{n} \sum_{s,t\in F} \phi(st^{-1})(A_{ij})_{s,t}(B_{ji})_{t,s} \\ &= \lim_\alpha \sum_{i,j=1}^{n} \sum_{s,t\in F} \tau\left(\lambda_{s^{-1}}T_\phi(\lambda_s x_\alpha\lambda_{t^{-1}})\lambda_t y_\alpha\right)(A_{ij})_{s,t}(B_{ji})_{t,s} \\ &= \lim_\alpha \sum_{i,j=1}^{n} \sum_{s,t\in F} \tau\big(T_\phi((A_{ij})_{s,t}\lambda_s x_\alpha\lambda_{t^{-1}})(B_{ji})_{t,s}\lambda_t y_\alpha\lambda_{s^{-1}}\big).\end{aligned}$$

The last line holds due to the cyclicity property in Proposition 2.11. Now, letting $m = |F|$, we will define matrices $A^\alpha, B^\alpha \in M_{mn}(\mathcal{L}G))$ for each α. We will regard A^α and B^α as $n \times n$ matrices whose entries are themselves $m \times m$ matrices with entries in $\mathcal{L}G$ and rows and columns indexed by F. Explicitly, we set

$(A_{ij}^\alpha)_{s,t} = (A_{ij})_{s,t}\lambda_s x_\alpha \lambda_{t^{-1}}$ and $(B_{ij}^\alpha)_{s,t} = (B_{ij})_{s,t}\lambda_s y_\alpha \lambda_{t^{-1}}$. Then the above chain of equalities says that

$$\begin{aligned}
|\operatorname{tr}_n(((\mathrm{id}_n \otimes M_{\phi,F})A)B)| &= \lim_\alpha \left| \sum_{i,j=1}^n \sum_{s,t\in F} \tau\left(T_\phi(A_{ij}^\alpha)_{s,t}(B_{ji}^\alpha)_{t,s}\right)\right| \\
&= \lim_\alpha \left|\tau_{mn}\left(((\mathrm{id}_{mn} \otimes T_\phi)A^\alpha)B^\alpha\right)\right| \\
&\leq \lim_\alpha \|T_{\phi,p}\|_{CB}\|A^\alpha\|_{L^p}\|B^\alpha\|_{L^{p'}},
\end{aligned}$$

where the last equality follows from Proposition 2.37. The proof will be concluded once we show that $\|A^\alpha\|_{L^p} = \|A\|_{\mathcal{S}_p}$ and $\|B^\alpha\|_{L^p} = \|B\|_{\mathcal{S}_p}$. To this end, define $U \in M_{mn}(\mathcal{L}G)$, with rows and columns indexed as A^α and B^α were, by

$$(U_{ij})_{s,t} = \begin{cases} \lambda_s, & \text{if } s = t \text{ and } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Since U is unitary and $A^\alpha = U(A \otimes x_\alpha)U^*$, Proposition 2.56 and Theorem 2.58 imply that

$$\|A^\alpha\|_{L^p} = \|A \otimes x_\alpha\|_{L^p} = \|A\|_{\mathcal{S}_p}\|x_\alpha\|_{L^p} = \|A\|_{\mathcal{S}_p},$$

since $\|x_\alpha\|_{L^p} = 1$. Similarly, $\|B_\alpha\|_{\mathcal{S}_{p'}} = \|B\|_{\mathcal{S}_{p'}}$, concluding the proof of (4.8). □

4.4 The Second Inequality for Amenable Groups

Our approach to proving the reverse inequality, that $\|T_{\phi,p}\|_{CB} \leq \|M_{\phi,p}\|_{CB}$, will demand one additional assumption on the group: amenability. Neuwirth and Ricard [23], and Caspers and de la Salle [5] required this same assumption. In fact, Caspers and de la Salle proved (Theorem 2.1 in [5]) that an essential construction in their argument, which closely mirrors Neuwirth and Ricard's approach, is only possible when the group is amenable. The approach taken in this thesis will simplify their construction, on account of the assumed second-countability of the group. Nevertheless, our approach will resemble that in [5], so this author believes that a similar result for non-amenable groups would require substantially different methods.

There are several equivalent definitions of an amenable group. We will present just one of them, which is shown to be equivalent to several others in Theorem 4.10 in [24].

Definition 4.33. A locally compact group G is **amenable** if it satisfies the following Fölner condition: for any compact set $K \subset G$ and any $\varepsilon > 0$, there exists a compact set $W \subset G$ such that, for every $g \in K$,

$$\frac{\mu(gW \triangle W)}{\mu(W)} < \varepsilon.$$

All compact groups are amenable (just take $W = G$ in the above definition), as are all abelian locally compact groups (see Proposition 0.15 in [24]). In our case, where the group G is second-countable, the Fölner condition implies the existence of a so-called Fölner sequence.

Proposition 4.34. *If G is amenable, then G admits a sequence $\{F_n\}_{n\in\mathbb{N}}$ of compact subsets such that*

$$\frac{\mu(gF_n \cap F_n)}{\mu(F_n)} \to 1,$$

for every $g \in G$ as $n \to \infty$. Such a sequence is called a ***Fölner sequence****.*

Proof. By Proposition 3.6, choose an exhaustion of G by compact sets $\{K_n\}_{n\in\mathbb{N}}$. By the amenability condition, to each set K_n we can associate a compact set F_n such that

$$\frac{\mu(gF_n \triangle F_n)}{\mu(F_n)} < \frac{1}{n}$$

for every $g \in K_n$. Since every $g \in G$ eventually lies in K_n, it follows that

$$\frac{\mu(gF_n \triangle F_n)}{\mu(F_n)} \to 0$$

for every $g \in G$ as $n \to \infty$, which implies that

$$\frac{\mu(gF_n \cap F_n)}{\mu(F_n)} \to 1$$

for every $g \in G$ as $n \to \infty$. □

As a parenthetical remark, we mention that if G is amenable, then any Fourier multiplier is automatically completely bounded, i.e., $MA(G) = M_{CB}A(G)$. This is proven in Corollary 1.8 of [8].

To establish the required inequality when G is amenable, our key construction is the following. By Proposition 4.34, choose a Fölner sequence $\{F_n\}_{n=1}^{\infty}$. Fixing $1 < p, p' < \infty$ such that $\frac{1}{p} + \frac{1}{p'} = 1$, define $j_{p,F_n} : L^p(\mathcal{L}G) \to \mathcal{S}_p(L^2(G))$ as in Corollary 4.30. We define maps $i_n : L^p(\mathcal{L}G) \to \mathcal{S}_p(L^2(G))$ by

$$i_n(x) = \frac{j_{p,F_n}(x)}{\mu(F_n)^{1/p}}$$

for all $x \in L^p(\mathcal{L}G)$, and $i'_n : L^{p'}(\mathcal{L}G) \to \mathcal{S}_{p'}(L^2(G))$ by

$$i'_n(y) = \frac{j_{p',F_n}(y)}{\mu(F_n)^{1/p'}},$$

for all $y \in L^{p'}(\mathcal{L}G)$. By Corollary 4.29, these maps are complete contractions, in that $\|i_n\|_{CB}, \|i'_n\|_{CB} \le 1$, and they intertwine Schur and Fourier multipliers. Also, for $x \in L^1(\mathcal{L}G) \cap \mathcal{L}G$, note that $i_n(x) = \frac{P_{F_n} x P_{F_n}}{\mu(F_n)^{1/p}} \in \mathcal{S}_1(L^2(G))$, and a similar calculation holds for $i'_n(y)$ when $y \in L^1(\mathcal{L}G) \cap \mathcal{L}G$. The following lemma propels the argument for the reverse inequality forward.

Lemma 4.35. *Fix $k \in \mathbb{N}$. For any $A, B \in L^1(M_k(\mathcal{L}G)) \cap M_k(\mathcal{L}G)$,*

$$\lim_{n\to\infty} \operatorname{tr}_k \left((\mathrm{id}_k \otimes i_n(A))(\mathrm{id}_k \otimes i'_n(B))\right) = \tau_k(AB). \tag{4.10}$$

Proof. Consider first the $k = 1$ case. For any $x, y \in L^1(\mathcal{L}G) \cap \mathcal{L}G \subset \mathcal{J}$, we can write $x = \lambda(\xi)$ and $y = \lambda(\eta)$, with $\xi, \eta \in L^2(G)$. For each $n \in \mathbb{N}$, we have $P_{F_n} x P_{F_n}, P_{F_n} y P_{F_n} \in \mathcal{S}_1(L^2(G)) \subset \mathcal{S}_2(L^2(G))$, so by Proposition 4.1,

$$\operatorname{tr}\left(i_n(x) i'_n(y)\right) = \frac{1}{\mu(F_n)} \int_G \int_G (P_{F_n} x P_{F_n})_{s,t} (P_{F_n} y P_{F_n})_{t,s} \, \mathrm{d}\mu(t) \, \mathrm{d}\mu(s).$$

Applying the calculations of integral kernels in Subsection 4.2.1, and then the change-of-variables $r = st^{-1}$, we get

$$\begin{aligned}
\operatorname{tr}\left(i_n(x) i'_n(y)\right) &= \frac{1}{\mu(F_n)} \int_G \int_G \chi_{F_n}(s) \chi_{F_n}(t) \xi(st^{-1}) \eta(ts^{-1}) \, \mathrm{d}\mu(t) \, \mathrm{d}\mu(s) \\
&= \frac{1}{\mu(F_n)} \int_G \xi(r) \eta(r^{-1}) \left(\int_{F_n} \chi_{F_n}(r^{-1}s) \, \mathrm{d}\mu(s) \right) \mathrm{d}\mu(r) \\
&= \int_G \xi(r) \eta(r^{-1}) \frac{\mu(F_n \cap rF_n)}{\mu(F_n)} \, \mathrm{d}\mu(r).
\end{aligned}$$

By Proposition 4.34, the integrand converges pointwise to $\xi(r)\eta(r^{-1})$ as $n \to \infty$. Since the absolute value of the integrand is bounded above by $|\xi(r)\eta(r^{-1})|$ and $\xi, \eta \in L^2(G)$, the dominated convergence theorem applies. Therefore, by Proposition 3.31,

$$\lim_{n\to\infty} \operatorname{tr}(i_n(x) i'_n(y)) = \int_G \xi(r) \eta(r^{-1}) d\mu(r) = \tau(xy),$$

as desired. Now, fix arbitrary $k \in \mathbb{N}$, and $A, B \in L^1(M_k(\mathcal{L}G)) \cap M_k(\mathcal{L}G)$. The above calculation for the $k = 1$ case implies

$$\begin{aligned}
\lim_{n\to\infty} \operatorname{tr}_k \left((\mathrm{id}_k \otimes i_n(A))(\mathrm{id}_k \otimes i'_n(B))\right) &= \lim_{n\to\infty} \sum_{i,j=1}^{k} \operatorname{tr}\left(i_n(A_{ij}) i'_n(B_{ji})\right) \\
&= \sum_{i,j=1}^{k} \tau(A_{ij} B_{ji}) = \tau_k(AB),
\end{aligned}$$

concluding the proof of (4.10). □

Finally, we prove the second inequality.

Theorem 4.36. *Let G be an amenable, unimodular, second-countable locally compact group. Fixing $1 \leq p < \infty$ and $\phi \in M_{CB}A(G)$, we have*

$$\|T_{\phi,p}\|_{CB} \leq \|M_{\phi,p}\|_{CB}.$$

Proof. For $p = 1$, the result follows from Bożejko and Fendler's result in [3], paired with Remark 4.8 and Proposition 4.24. Therefore, we will assume $1 < p < \infty$.

Letting $\frac{1}{p} + \frac{1}{p'} = 1$, define maps i_n and i'_n as above. Fixing $k \in \mathbb{N}$, define $W_p = \{B \in L^1(M_k(\mathcal{L}G)) \cap M_k(\mathcal{L}G) : \|B\|_{L^p} \leq 1\}$, and define $W_{p'}$ similarly. For any $A \in L^1(M_k(\mathcal{L}G)) \cap M_k(\mathcal{L}G)$, Proposition 2.38 and then Lemma 4.35 imply that

$$\begin{aligned}
\|A\|_{L^p} &= \sup_{B \in W_{p'}} |\tau_k(AB)| \\
&= \sup_{B \in W_{p'}} \lim_{n\to\infty} |\mathrm{tr}_k\left((\mathrm{id}_k \otimes i_n(A))(\mathrm{id}_k \otimes i'_n(B))\right)| \\
&\leq \lim_{n\to\infty} \sup_{B \in W_{p'}} |\mathrm{tr}_k\left((\mathrm{id}_k \otimes i_n(A))(\mathrm{id}_k \otimes i'_n(B))\right)| \\
&\leq \lim_{n\to\infty} \|\mathrm{id}_k \otimes i_n(A)\|_{\mathcal{S}_p}.
\end{aligned}$$

The last inequality holds by Proposition 2.37, since $\mathrm{id}_k \otimes i'_n$ is a contraction. In fact, since $\|\mathrm{id}_k \otimes i_n(A)\|_{\mathcal{S}_p} \leq \|A\|_{L^p}$ for every $n \in \mathbb{N}$, this proves that for every $A \in L^1(M_k(\mathcal{L}G)) \cap M_k(\mathcal{L}G)$, $\|A\|_{L^p} = \lim_{n\to\infty} \|\mathrm{id}_k \otimes i_n(A)\|_{\mathcal{S}_p}$. Since $L^1(M_k(\mathcal{L}G)) \cap M_k(\mathcal{L}G)$ is dense in $L^p(M_k(\mathcal{L}G))$ and $i_n \circ T_{\phi,p} = M_{\phi,p} \circ i_n$, it follows that

$$\begin{aligned}
\|\mathrm{id}_k \otimes T_{\phi,p}\| &= \sup_{A \in W_p} \|(\mathrm{id}_k \otimes T_{\phi,p})A\|_{L^p} \\
&= \sup_{A \in W_p} \lim_{n\to\infty} \|(\mathrm{id}_k \otimes M_{\phi,p})(\mathrm{id}_k \otimes i_n)A\|_{\mathcal{S}_p} \\
&\leq \|\mathrm{id}_k \otimes M_{\phi,p}\|,
\end{aligned}$$

where the last inequality holds because $\mathrm{id}_k \otimes i_n$ is a contraction. Taking suprema over all $k \in \mathbb{N}$, we conclude that $\|T_{\phi,p}\|_{CB} \leq \|M_{\phi,p}\|_{CB}$. □

Combining Theorem 4.32 and Theorem 4.36, we arrive at the desired result.

Theorem 4.37. *Let G be an amenable, unimodular, second-countable locally compact group. For any $1 \leq p < \infty$ and $\phi \in M_{CB}A(G)$, we have the equality*

$$\|M_{\phi,p}\|_{CB} = \|T_{\phi,p}\|_{CB}.$$

Appendix A

Banach-Valued Complex Analysis

In this appendix, we prove some basic results about Banach-valued complex analysis, with a particular eye towards the three-line theorem (Theorem 2.24). We can connect the theory of such functions to the theory of holomorphic functions via the following result.

Lemma A.1. *Let X be a Banach space and $U \subset \mathbb{C}$ an open set. For any complex-analytic function $f : U \to X$ and any continuous linear functional $\phi \in X^*$, the function $\phi \circ f$ is holomorphic on U.*

Proof. Fix any $z \in U$. Applying first the linearity of ϕ and then the continuity of ϕ, we have

$$\lim_{w \to z} \frac{\phi(f(w)) - \phi(f(z))}{w - z} = \lim_{w \to z} \phi\left(\frac{f(w) - f(z)}{w - z}\right) = \phi(f'(z)).$$

□

The same proof yields the following, more general, observation.

Proposition A.2. *Let $T : X \to Y$ be a continuous linear map between Banach spaces. If $U \subset \mathbb{C}$ is any open set and $f : U \to X$ is complex-analytic, then so is $T \circ f : U \to Y$.*

The following is an immediate consequence of Lemma A.1, since a product $X \times Y$ of Banach spaces is a Banach space with norm $\|(x, y)\|_{X \times Y} = \|x\|_X \|y\|_Y$.

Proposition A.3. *Suppose X and Y are Banach spaces with a continuous bilinear form $\langle \cdot, \cdot \rangle$ on $X \times Y$. If $U \subset \mathbb{C}$ is open and $f : U \to X$ and $g : U \to Y$ are complex-analytic, then the function $U \to \mathbb{C}$ given by $z \mapsto \langle f(z), g(z) \rangle$ is holomorphic on U.*

Using this, we can deduce the Banach-valued analogue to the maximum modulus principle. Taking $X = \mathbb{C}$, we get the usual maximum modulus principle for holomorphic functions (see, for example, Section 12.1 in [27]).

Proposition A.4. *Let X be a Banach space and $U \subset \mathbb{C}$ a bounded open set. Let $\overline{U}$ denote the closure of U and ∂U denote the boundary of U. If $f : \overline{U} \to X$ is continuous on $\overline{U}$ and complex-analytic on U, then $\|f(z_0)\|_X \leq \max_{z\in\partial U} \|f(z)\|_X$ for any $z_0 \in \overline{U}$.*

Proof. By the Hahn-Banach theorem, we have that for every $x \in X$,

$$\|x\|_X = \sup_{\substack{\phi\in X^* \\ \|\phi\|\leq 1}} |\phi(x)|.$$

But for every $\phi \in X^*$, $\phi \circ f$ is continuous on $\overline{U}$ and, by Lemma A.1, it is holomorphic on U. From the classical maximum modulus principle, we therefore have the inequality $|\phi(f(z_0))| \leq \max_{z\in\partial U} |\phi(f(z))|$. Therefore,

$$\|f(z_0)\|_X = \sup_{\substack{\phi\in X^* \\ \|\phi\|\leq 1}} |\phi(f(z_0))| \leq \max_{z\in\partial U} \sup_{\substack{\phi\in X^* \\ \|\phi\|\leq 1}} |\phi(f(z))| = \max_{z\in\partial U} \|f(z)\|.$$

$\square$

From this, we can deduce the three-line theorem.

Corollary A.5 (Three-Line Theorem)**.** *Let X be a Banach space, and let Ω denote the open strip $\{z \in \mathbb{C} : 0 < \mathrm{Re}(z) < 1\}$, with closure $\overline{\Omega}$. Assume a function $f : \overline{\Omega} \to X$ is bounded and continuous on $\overline{\Omega}$ and complex-analytic on Ω. Let $M_0 = \sup_{t\in\mathbb{R}} \|f(it)\|$ and $M_1 = \sup_{t\in\mathbb{R}} \|f(1 + it)\|$. Then for any $z \in \Omega$, one has $|f(z)| \leq M_0^{1-\theta} M_1^{\theta}$, where $\theta = \mathrm{Re}(z)$.*

Proof. Let $B = \sup_{z\in\overline{\Omega}} \|f(z)\|$, and fix $\rho \in \mathbb{R}$. For $\varepsilon > 0$, define $f_\varepsilon : \overline{\Omega} \to X$ by $f_\varepsilon(z) = e^{\varepsilon z^2+\rho z} f(z)$. We will first show that $\|f_\varepsilon(z)\| \leq \max\{M_0, M_1 e^{\varepsilon+\rho}\}$ for every $z \in \overline{\Omega}$.

For every $t \in \mathbb{R}$, we have the estimates $\|f_\varepsilon(it)\| = e^{-\varepsilon t^2}\|f(it)\| \leq M_0$ and $\|f_\varepsilon(1 + it)\| = e^{\varepsilon-\varepsilon t^2+\rho}\|f(1 + it)\| \leq e^{\varepsilon+\rho} M_1$. Moreover, for any $z \in \Omega$, writing $z = \theta + iy$, we have

$$\|f_\varepsilon(z)\| = e^{\varepsilon\theta^2-\varepsilon y^2+\rho\theta}\|f(z)\| \leq e^{\varepsilon+\rho-\varepsilon y^2} B. \tag{A.1}$$

Since $M_1 \leq B$, choose $y_0 \geq 0$ so that $e^{-\varepsilon y_0^2} B = M_1$, and let R be the rectangle $\{z \in \bar{\Omega} : |\mathrm{Im}(z)| \leq y_0\}$. From (A.1), one sees that for all $z \in \partial R$,

$$\|f_\varepsilon(z)\| \leq \max\{M_0, M_1 e^{\varepsilon+\rho}\}.$$

Therefore, by the maximum modulus principle, the same inequality holds for all $z \in R$. Moreover, for all $z \in \overline{\Omega} \setminus R$, we have $|\mathrm{Im}(z)| > y_0$, so by (A.1), $|f_\varepsilon(z)| \leq M_1 e^{\varepsilon+\rho}$ too.

This establishes the inequality $\|f_\varepsilon(z)\| \leq \max\{M_0, M_1 e^{\varepsilon+\rho}\}$ for all $z \in \overline{\Omega}$. Recall that $f(z) = e^{-\varepsilon z^2 - \rho z} f_\varepsilon(z)$. Writing $z = \theta + iy$, with $\theta \in [0,1]$, $y \in \mathbb{R}$, we deduce that

$$\|f(\theta + iy)\| \leq e^{-\varepsilon\theta^2 + \varepsilon y^2 - \rho\theta} \max\{M_0, M_1 e^{\varepsilon+\rho}\}$$

Letting $\varepsilon \to 0$, we get

$$\|f(\theta + iy)\| \leq \max\{M_0 e^{-\rho\theta}, M_1 e^{\rho(1-\theta)}\}. \tag{A.2}$$

First, assume $M_0, M_1 > 0$. We note that $M_0 e^{-\rho\theta} = M_1 e^{\rho(1-\theta)}$ precisely when $\frac{M_0}{M_1} = e^\rho$. In this case, $M_0 e^{-\rho\theta} = M_0 \left(\frac{M_1}{M_0}\right)^\theta = M_0^{1-\theta} M_1^\theta$, and (A.2) becomes $\|f(\theta + iy)\| \leq M_0^{1-\theta} M_1^\theta$, as desired.

Now, if $M_0 = 0$ or $M_1 = 0$, then for any sufficiently small $\delta > 0$, we have $\sup_{t\in\mathbb{R}} \|f(it)+\delta\| \neq 0$ and $\sup_{t\in\mathbb{R}} \|f(1+it)+\delta\| \neq 0$. The function $f_\delta(z) = f(z)+\delta$ satisfies all the hypotheses in the theorem, so the above argument determines that for all $\theta \in [0,1]$ and $y \in \mathbb{R}$,

$$\begin{aligned} \|f_\delta(\theta + iy)\| &\leq (\sup_{t\in\mathbb{R}} \|f(it) + \delta\|)^{1-\theta} (\sup_{t\in\mathbb{R}} \|f(1 + it) + \delta\|)^\theta \\ &\leq (M_0 + \delta)^{1-\theta} (M_1 + \delta)^\theta \end{aligned}$$

Fixing θ and y, and letting δ tend to zero, we conclude that

$$\|f(\theta + iy)\| \leq M_0^{1-\theta} M_1^\theta = 0.$$

That is, $f(z) = 0$ for all $z \in \overline{\Omega}$. □

Finally, we prove some elementary calculations.

Proposition A.6. *Let X be a Banach space and $U \subset \mathbb{C}$. Suppose we have functions $f : U \to X$ and $\lambda : U \to \mathbb{C}$. Define $\lambda f : U \to X$ by $\lambda f(z) = \lambda(z) f(z)$.*

(i) If f and λ are bounded functions, then λf is bounded too.

(ii) If f and λ are continuous functions, then λf is continuous too.

(iii) If U is an open set and f and λ are analytic, then λf is analytic too, with $(\lambda f)' = \lambda' f + \lambda(f')$.

Proof. (i) is clear, and (ii) and (iii) follow from the inequality,

$$\|\lambda(z)f(z) - \lambda(w)f(w)\|_X \leq |\lambda(z) - \lambda(w)| \|f(z)\|_X + |\lambda(w)| \|f(z) - f(w)\|_X.$$

□

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