Special Differential Equations and Functions in Physics with Complex Analysis

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1 Introduction

There is the idea of special functions. Most people are familiar with some of these functions like sin, cos, exp, seen seemingly everywhere. We can go further though to some lesser known (but still "special") functions like the gamma function, the error function, the Riemann zeta function, dilogarithms, etc. They all share the quality of being deemed important and unsurprisingly then often have interesting properties and applications.

Here I want to focus on a set of special functions that notably show up in physics. In physics, we commonly come up with a differential equation that describes a system in question and of course wish to solve it to describe its evolution and behavior. It's not too unusual to just be given the solution in physics or engineering classes, where one might be told their special names along with a couple of interesting properties. The goal here though is to go through a few examples and derive some these special properties tied to the special functions. Not every kind of example is shown, but this should hopefully give a flavor as to methods used where in particular, complex analysis will be showcased as an important tool to get these results.

(I will also try to present this in a style that is more informal and conversational, compared to just definition, theorem, and corollary)

2 A General Start

A broad scope of equations in physics are described by linear second-order homogeneous ordinary differential equations of the general form

$$\frac{d^2}{dz^2}u(z) + P(z)\frac{d}{dz}u(z) + Q(z)u(z) = 0 \tag{1}$$

Some examples include:

- Simple Pendulum: $\frac{d^2}{dt^2}\theta - gL\sin(\theta) = 0$
- Damped Harmonic Oscillator: $\frac{d^2}{dx^2}x + 2\zeta\omega_0\frac{d}{dx}x + \omega_0^2 x = 0$
- Radial Equation for a Charge in a Hydrogenic Atom Coulomb Potential:

$$\frac{d^2}{dr^2}R + 2r\frac{d}{dr}R + [\frac{2mE}{\hbar^2} + \frac{2mZe^2}{4\pi\epsilon_0\hbar^2 r}\frac{\ell(\ell+1)}{r^2}]R = 0$$

Meanwhile second order *partial* differential equations often show up in physical systems as well.
Some examples include:

- Wave Equation: $\frac{\partial^2}{\partial t^2}u - c^2\nabla^2 u = 0$
- Stark Effect for the Hydrogen Atom Using Scaled Units and Parabolic Coordinates:

$$H\psi - E\psi = [-\frac{2}{\zeta+\eta}[\frac{\partial}{\partial\zeta}(\zeta\frac{\partial}{\partial\zeta}) + \frac{\partial}{\partial\eta}(\eta\frac{\partial}{\partial\eta})] - \frac{1}{2\zeta\eta}\frac{\partial^2}{\partial\phi^2} - \frac{2}{\zeta+\eta} + F\frac{\zeta-\eta}{2}\zeta - \eta]\psi - E\psi = 0$$

Before tackling each physics problem individually as we come across them, we can actually solve the general equation (1). A benefit of this is that when coming across a new problem, we can try to get it into an established general form (like equation (2) below) and from there we can just read off the solution in comparison with the general solution. Along the way, we might notice (and maybe expect prior with the help of symmetries in the problem and physical intuition) that the form of the equation and series solution match commonly recurring special cases and functions...

So then, with the requirement that $P(z)$ and $Q(z)$ are rational functions and that equation (1) has at most three singularities, a change of variables can transform the three singularities of equation (1) into $(0, 1, \infty)$ (see reference [2] for discussion on singularities. Basically a point z_0 is considered a singularity if u and u' can't be arbitrary values at z_0) in which we obtain **Gauss's hypergeometric equation**

$$z(1-z)\frac{d^2}{dz^2}u + [c - (a+b+1)z]\frac{d}{dz}u - abu = 0 \tag{2}$$

for constant $a, b, c \in \mathbb{C}$. Having at most 3 initial singularities restricts us to only quadratic order in coefficients in z. This can be solved by substituting in a general series solution $u(z) = \sum_{n=0}^{\infty} a_n z^{n+s}$, where after some algebraic manipulation and some reasoning concerning when terms should vanish, we get that

$$u(z) = a_0[1 + \sum_{n=1}^{\infty}\frac{a(a+1)(a+2)...(a+n-1)b(b+1)...(b+n-1)}{n!c(c+1)(c+2)...(c+n-1)}z^n] = a_0[1 + F(a,b;c;z)]$$

where $F(a,b;c;z)$ is called the **hypergeometric function**. The hypergeometric function is also sometimes written as ${}_2F_1(a,b;c;z)$. It can be shown that the function converges on the unit disc. At first thought, this might be a bit problematic for physics applications, but for the general hypergeometric function ${}_1F_1(a,b;c;z)$, if a or b are negative integers, the series will terminate (as eventually a term of zero in the product appears). For ${}_1F_1(a,b;c;z)$ solutions to physics equations, we will see then that a or b is negative, in which we don't need to worry about convergence. If we don't have negative parameters though, we will see in physical situations other elements come into play (like a decaying exponential multiplied to the series).

We can naturally write a more general hypergeometric function by adding more constants to get

$$ {}_pF_q(a_1, a_2, ..., b_1, b_2, ...; z) = \sum_{n=0}^{\infty} \frac{a_1(a_1+1)...a_2(a_2+1)...}{b_1(b_1+1)...b_2(b_2+1)...} z^n $$

The only other important case of the general hypergeometric function we will consider though is the **confluent hypergeometric series**, ${}_1F_1(a; b; z)$. When singularities at $z = 1$ and ∞ from equation (2) merge (there is a "confluence") and we have the **confluent hypergeometric equation**

$$ z\frac{d^2}{dz^2}u + (c - z)\frac{d}{dz}u - au = 0 $$

whose general solution is

$$ u(z) = A\,{}_1F_1(a; c; z) + Bz^{1-c}\,{}_1F_1(1 + a - c; 2 - c; z) $$

with A and B constants.

With a general framework of 2nd order ordinary differential equations now, let's finally examine some physical systems.

3 Legendre Polynomials

For our first example, we will obtain the Legendre polynomials and the associated Legendre polynomials, with complex analysis coming into play in the later sections.

In quantum mechanics, the time independent Schrodinger equation tells us for a simplified central force problem with a particle of mass m,

$$ -\frac{\hbar^2}{2m}\nabla^2 u(r) + [V(r) - E]u(r) = 0 $$

Given the symmetry of the problem, it will be nice to switch to spherical coordinates where $r = (r, \theta, \phi)$. Meanwhile, recall the Laplacian in spherical coordinates

$$ \nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} $$

Now we may have jumped the gun, skipping ordinary differential equation physics examples (of which there are many) but we will see that we can get ordinary differential equations from separation of variables. As so, let $u(\mathbf{r}) = R(r)Y(\theta, \phi)$. We then have

$$ \frac{1}{R}[\frac{d}{dr}(r^2\frac{d}{dr}) + \frac{2mr^2}{\hbar^2}(E - V(r))]R(r) = -\frac{1}{Y}[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}]Y(\theta, \phi) $$

where it must be that the two sides equal a constant, say λ. We are able to consider just the radial or angular part then. Examining the angular part, we have

$$ [[\sin\theta\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial}{\partial\theta}) + \lambda\sin^2\theta] + \frac{\partial^2}{\partial\phi^2}]Y(\theta, \phi) = 0 $$

Letting $Y = g(\phi)f(\theta)$ and looking at just $f(\theta)$, we find that

$$ \sin^2\theta\frac{d^2 f}{d\theta^2} + \sin\theta\cos\theta\frac{df}{d\theta} + (\lambda\sin^2\theta - m^2)f = 0 $$

With some manipulation and various considerations (detailed in reference [2]) and changing from $\theta \to x$, we can obtain a slightly more generalized form of

$$(1-x^2)f''(x) - 2xf'(x) + \ell(\ell+1)f(x) = 0 \tag{3}$$

which is called **Legendre's equation**. We can compare this with Gauss's hypergeometric equation (2), and we see that the solution is

$$P_\ell(x) = {}_2F_1(-\ell, \ell+1; 1, \frac{1}{2}(1-x))$$

which are known as the **Legendre polynomials of order** ℓ. The fact that they are polynomials comes from the negative integer $-\ell$.

The first few Legendre polynomials follow:

ℓ	$P_\ell(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$

We define the **associated Legendre polynomials**, which are slightly more general than the Legendre polynomials, as

$$P_\ell^m(x) = (1-x^2)^{\frac{1}{2}|m|}\frac{d^{|m|}}{dx^{|m|}}P_\ell(x)$$

which are solutions to

$$\frac{d}{dx}[(1-x^2)\frac{d}{dx}P_\ell^m(x)] + [\ell(\ell+1) - \frac{m^2}{1-x^2}]P_\ell^m(x) = 0 \tag{4}$$

One can easily check that when $m = 0$, the associated Legendre polynomials are just the Legendre polynomials and equation (3) becomes equation (4).

With that, a little more work can then be done to cast the $Y(\theta, \phi)$ into the so-called spherical harmonics, and one can also follow similar calculations for obtaining $R(r)$.

4 Bessel Functions

Let's look at another example of a physical problem.

So-called *waveguides* are an instrumentation that can be used to transmit electromagnetic waves, and they have certain interesting properties like picking out propagation modes. We will consider cylindrical waveguides here and try to find out how electromagnetic radiation acts inside them.

From Maxwell's equations, we can obtain the wave equations for light in a vacuum

$$\nabla^2\mathbf{E} - \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \tag{5}$$

$$\nabla^2\mathbf{B} - \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2} = 0 \tag{6}$$

whose solutions are

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\rho, \theta)e^{i(kz-\omega t)} \tag{7}$$

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}(\rho, \theta)e^{i(kz-\omega t)} \tag{8}$$

where we've used cylindrical coordinates given the symmetry of the system. Recall that the Laplacian in cylindrical coordinates is

$$\nabla^2 = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}$$

It suffices to just consider E_z, so using plugging the above expression for ∇^2 and the solution (7) into equation (5), we get

$$[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + + \frac{1}{\rho^2}\frac{\partial^2}{\partial\theta^2} + (\mu_0\epsilon_0\omega^2 - k^2)^2]E_z(\rho,\theta) = 0$$

Multiplying by ρ^2, we can separate variables ρ and θ with $E_z(\rho,\theta) = u(\rho)v(\theta)$ to see that

$$\frac{1}{u}[\rho^2\frac{d^2u}{d\rho^2} + \rho\frac{2u}{d\rho} + \gamma^2\rho^2 u] = -\frac{1}{v}\frac{d^2v}{d\theta^2}$$

Examining the more interesting $u(\rho)$ solution, we have

$$\rho^2\frac{d^2u}{d\rho^2} + \rho\frac{du}{d\rho} + [(\mu_0\epsilon_0\omega^2 - k^2)^2\rho^2 - n^2]u = 0$$

with n an integer.
We can change variable to $x = (\mu_0\epsilon_0\omega^2 - k^2)\rho$ to get the form of **Bessel's equation**

$$x^2\frac{d^2}{dx^2} - x\frac{d}{dx}u + (x^2 - n^2)u = 0$$

It has two singularities at ∞ and 0, so changing it into a confluent hypergeometric equation with $u(x) = x^s e^{g(x)} f(x)$, we can obtain

$$zf''(z) + (2n + 1 - z)f'(z) + (n + \frac{1}{2})f(z) = 0$$

From which after some manipulations, we have

$$z\frac{d^2f}{dz^2} + (2n + 1 - \frac{2iz}{\alpha})\frac{df}{dz} - \frac{2i}{\alpha}(n + \frac{1}{2})f = 0$$

with the solution read off as

$$f(x) = A\,{}_1F_1(n + \frac{1}{2}; 2n + 1; 2ix) + B(-2ix)^{-2n}\,{}_1F_1(-n + \frac{1}{2}; -2n + 1; 2ix)$$

which implies

$$u(x) = e^{-ix}[A_n x^n\,{}_1F_1(n + \frac{1}{2}; 2n + 1; 2ix) + B_n x^{-n}\,{}_1F_1(-n + \frac{1}{2}; -2n + 1; 2ix)]$$

with A_n and B_n constants. We can left $a_\nu = 2^\nu\Gamma(\nu + 1)A_\nu$ and $b_\nu = 2^{-\nu}\Gamma(-\nu + 1)B_\nu$, where then $u(x)$ becomes

$$u(x) = a_\nu J_\nu(x) + b_\nu J_{-\nu}(x)$$

with $J_\nu(x)$ called a **Bessel function of the first kind of order** ν and

$$J_\nu(x) = \frac{e^{-ix}}{\Gamma(\nu + 1)}(\frac{x}{2})^\nu\,{}_1F_1(\nu + \frac{1}{2}; 2\nu + 1; 2ix) \tag{9}$$

where the convergence of the solution is controlled by the decaying exponential.

Similarly, one can also find the solution for $v(\theta)$ from which we just determine a_ν and b_ν from boundary conditions which the parameters of the waveguide determine (such as, at the (ideal) conducting inner radius of the waveguide, it must be that $E_z(\rho,\theta)) = 0$).

A function that is linearly independent with respect to $J_\nu(x)$ is also a solution. We define

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

as a **Bessel function of the second kind**, or **Neumann function**.

Finally, we can also define

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x)$$
$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x)$$

which are **Hankel functions of the first and second kind** and are seen in physics scattering problems. Meanwhile, more interesting results for the various Bessel functions which don't require complex analysis like the recursion relations, Bourget's hypothesis, etc. can also be shown.

5 New Ways to Look at Legendre Polynomials and Bessel Functions

With a base and some familiarity with the special functions and equations, we can finally introduce some complex analysis as one of the tools to get different forms of special functions and also derive some of their properties and relations.

5.1 Rodrigues' formula

Tied to the Legendre polynomials, we can now derive an important relation known as Rodrigues' formula.

Using the series representation of the Legendre polynomials, the identity $(k-m+1)_m = k!(k-m+1)_{n-k}/\Gamma(n-m+1)$, and the expansion $(1-z)^s = \sum_{k=0}^{\infty} \frac{(-s)(-s+1)...}{k!} z^k$, we have

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{\infty} \frac{(-n)_k(n+1)_k}{k!k!2^k} \sum_{m=0}^{\infty} \frac{(k-m+1)_m}{m!}(-1)^m x^m \\ &= \sum_{m=0}^{\infty} [\sum_{k=0}^{n} \frac{(-n)_k(n+1)_k}{k!2^k} \frac{(k-m+1)_{n-k}}{\Gamma(n-m+1)}] \frac{(-1)^m}{m!} x^m \end{aligned}$$

where we've introduced the (rising) Pochhammer notation, $(x)_m \coloneqq x(x+1)(x+2)...(x+n-1)$.
Some more manipulation with Pochhammer identities and using the divergence of $\Gamma(n-m+1)$ for $m > n$ (in which these terms will fall out of the series from the $\Gamma(n-m+1)$ in the denominator), we have

$$P_n(x) = \sum_{m=0}^{n} \frac{(-1)^m x^m n!}{m!\Gamma(n-m+1)} \sum_{k=0}^{n} \frac{(-n-1-k+1)_k}{k!2^k} \frac{(k-m+1)_{n-k}}{(n-k)!}$$

Considering the right hand side sum, we can write

$$w(t) = \sum_{p=0}^{\infty} c_p t^p \qquad \text{where } c_p = \sum_{r=0}^{p} \frac{(-p-1-r+1)_k}{r!2^r} \frac{(r-m+1)_{p-r}}{(p-r)!}$$

One verify that this is a Cauchy product expansion where

$$w(t) = (1+\frac{t}{2})^{-n-1}(1+t)^{n-m}$$

As $w(t)$ is analytic near $t = 0$, we see that

$$c_p = \frac{1}{2\pi i} \oint_C \frac{w(t)}{t^{p+1}} dt$$

where C is a path that encloses the origin but not any singularities of $w(t)$.
Evaluating $p = n$, by the Cauchy's differential integral formula, we have

$$c_n = \frac{2^{n+1}}{2\pi i} \oint_C \frac{(1+t)^{n-m}}{(2t+t^2)^{n+1}} dt$$

Changing variables to $u = 2t + t^2$ and using the binomial expansion, we have

$$c_n = \frac{2^n}{2\pi i} \sum_{r=0}^{\infty} \frac{((\frac{n-m-1}{2}) - r + 1)_r}{r!} \oint_C u^{r-n-1} du$$

where closed curve C surrounds the point $u = 0$. In the region bounded by C the integrand is analytic for $r > n$ and has a pole of order $n + 1 - r$ for $r \leq n$. From the formula for residues we find that the residues of all integrals in the sum vanish except for the term $r = n$ in which case the residue is 1. Thus,

$$c_n = \frac{2^n(-\frac{1}{2}(n+m-1))_n}{n!}$$

Now since $\frac{1}{2}(n+m-1) < n$, we have $-\frac{1}{2}(n+m-1)_n = 0$ unless $n+m-1$ is an odd integer. Putting c_n back into the $P_n(x)$ series above with the re-indexing of $k = \frac{n-m}{2}$ to keep odd integers, we have

$$P_n(x) = \sum_{k=0}^{n/2} \frac{(-1)^{n-2k} x^{n-2k} 2^n}{(n-2k)!(2k)!} (k - n + 1/2)_n$$

Using the identity

$$\frac{(k-n+1/2)}{(2k)!} = 2^{-2n} \frac{(2n-2k)!}{(n-k)!} \frac{(-1)^{k-n}}{k!}$$

gets us

$$P_n(x) = \sum_{k=0}^{n/2} \frac{(-1)^k (2n-2k)!}{2^n k!(n-2k)!(n-k)!} x^{n-2k}$$

...which is another series representation of the Legendre polynomials. However, note that when we differentiate, we have

$$\frac{d^n}{dx^n} x^{2n-2k} = (n-2k+1)_n x^{n-2k}$$

and $(n-2k+1)_n = 0$ for $k \geq (n/2)+1$. Adding terms into the sum for $(n/2)+1 \leq k \leq n$, and letting $p = n - k$, we get

$$\begin{aligned} P_n(x) &= \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \sum_{p=0}^{n} \frac{(-1)^p n!}{p!(n-p)!} x^{2p} \\ &= \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n \end{aligned}$$

using a binomial expansion sum in the last line. Recalling the definition of the associated Legendre polynomials from earlier, we finally have **Rodrigues' formula**

$$P_n^m = (1-x^2)^{\frac{1}{2}|m|} \frac{d^{|m|}}{dx^{|m|}} P_n(x) = \frac{1}{2^n n!} (1-x^2)^{\frac{1}{2}|m|} \frac{d^{n+|m|}}{dx^{n+|m|}} (x^2-1)^n \tag{10}$$

which can be used to derive the orthogonality of the associated Legendre polynomials among other things.

5.2 Legendre Polynomial Integral Representations

Recall Cauchy's differentiation integral formula

$$\frac{d^n}{dz^n}f(z) = \frac{n!}{2\pi i}\oint_C \frac{f(t)}{(t-z)^{n+1}}dt$$

In trying to find an integral representation for the Legendre polynomials, we may notice that the above formula shares an nth derivative just like in the Rodrigues' formula. Playing with this idea, let's take $f(z) = \frac{1}{2^n n!}(z^2-1)^n$. Then we have

$$P_n(z) = \frac{1}{2^n n!}\frac{d^n}{dz^n}(z^2-1)^n = \frac{1}{2^n}\frac{1}{2\pi i}\oint_C \frac{(t^2-1)^n}{(t-z)^{n+1}}dt \tag{11}$$

with C a contour that encloses the point $t = z$. This is known as **Schlafli's integral** for the $P_n(z)$.

We can easily get recursion formulas for Legendre polynomials which follows from taking the derivative of equation (11) and carrying out some manipulations to get

$$zP_n'(z) + (n+1)P_n(z) - P_{n+1}'(z) = 0$$

While we are at it, we can derive another integral representation for Legendre polynomials.
Starting with Schlafli's integral for the $P_n(z)$, we can choose the contour C as a circle of radius $|\sqrt{z^2-1}|$ centered on z with the parameterization $t = z + |\sqrt{z^2-1}|e^{i\theta}$, as shown in figure 1. The phase from the absolute value can be absorbed into the $e^{i\theta}$ phase to get $t = z + \sqrt{z^2-1}e^{i\phi}$. Our integral then becomes

$$P_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi}(z + \sqrt{z^2-1}\cos(\phi))^n d\phi$$

or

$$P_n(z) = \frac{1}{\pi}\int_0^{\pi}(z + \sqrt{z^2-1}\cos(\phi))^n d\phi$$

which is known as **Laplace's integral representation** for $P_n(z)$

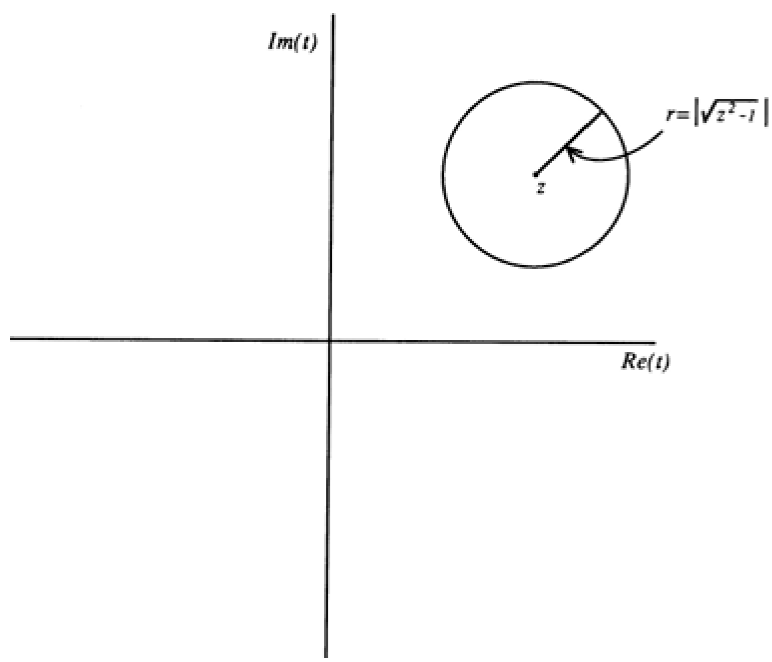

figure 1, from reference [2]

5.3 Bessel Function Integral Representations

Having gotten some neat ways to write down the Legendre polynomials with integrals, we can do the same for Bessel Functions. We start with a series representation of $J_\nu(x)$ which can be shown to be equivalent to equation (9)

$$J_\nu(x) = (\frac{x}{2})^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}(\frac{x}{2})^{2k} \tag{12}$$

where we will consider integer values of ν. Our strategy will be to find a set of functions such that

$$g(u,x) = \sum_{k=-\infty}^{\infty} J_k(x)u^k$$

where $g(u,x)$ is called a **generating function** for the Bessel functions.

Substituting equation (12) into $g(u,x)$, we have

$$g(u,x) = \sum_{m=0}^{\infty}\sum_{k=-\infty}^{\infty} \frac{(-1)^m}{m!\Gamma(k+m+1)}(\frac{x}{2})^{2m+k}u^k$$

Now since $\Gamma(k+m+1)$ diverges for $k < -m$, we can write

$$g(u,x) = \sum_{m=0}^{\infty}\sum_{k=-2m}^{\infty} \frac{(-1)^m}{m!\Gamma(k+m+1)}(\frac{x}{2})^{2m+k}u^k$$

Re-indexing $n = k+2m$, this becomes

$$g(u,x) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(n-m+1)}(\frac{x}{2})^{n}u^{n-2m}$$

Again, $\Gamma(n-m+1)$ diverges for $m > n$, so also rearranging some, we have

$$g(u,x) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}[(\frac{(-1)^m}{m!}(\frac{x}{2u})^m)(\frac{1}{(n-m)!}(\frac{xu}{2})^{n-m})]$$

This is actually the form of a Cauchy product of two series, where

$$g(u,x) = \sum_{k=0}^{\infty}\frac{1}{k!}(\frac{-x}{2u})^k \sum_{j=0}^{\infty}\frac{1}{j!}(\frac{xu}{2})^j = e^{\frac{1}{2}(u-u^{-1})}$$

We've found our function g where $e^{\frac{1}{2}(u-u^{-1})} = \sum_{k=-\infty}^{\infty} J_k(x)u^k$ which is a Laurent expansion of $\exp(\frac{1}{2}(u-u^{-1}))$ about $u=0$. It follows then that

$$J_n(x) = \frac{1}{2\pi i}\oint_C u^{-n-1}e^{\frac{1}{2}(u-u^{-1})}du$$

with C enclosing the singularity at zero. This also has the name of **Schlafli's integral** like with the Legendre polynomials and is valid for integer n.

Since n is an integer, we can take C as the unit circle on the origin. Then $u = e^{i\theta}$ and

$$\begin{aligned} J_n(x) &= \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}e^{ix\sin\theta}d\theta \\ &= \frac{1}{2\pi}\int_0^{\pi} e^{i(x\sin\theta-n\theta)}d\theta + \frac{1}{2\pi}\int_\pi^{2\pi} e^{i(x\sin\theta-n\theta)}d\theta \\ &= \frac{1}{\pi}\int_0^{\pi}\cos(x\sin\theta-n\theta)d\theta \end{aligned} \tag{13}$$

where we let $\theta \to 2\pi - \theta$ in the RHS integral. The final expression is called **Bessel's integral** and can be easily computed numerically to get values for the Bessel functions of integer order.

To examine another integral representation of $J_\nu(x)$, we start again with the series form of

$$J_\nu(x) = (\frac{x}{2})^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}(\frac{x}{2})^{2k}$$

We try to find a function $f(x,t)$ such that

$$J_\nu(x) = \oint_C f(x,t)dt = 2\pi i[sum\ of\ residues]$$

Note that

$$\frac{1}{2\pi i}\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\nu+1)}(\frac{x}{2})^{2k+\nu} = [sum\ of\ residues] = \sum_{k=0}^{\infty} a_{-1}(t_k)$$

and that $\Gamma(-t)$ has simple poles at $t = 0, 1, 2, 3, ...$ with

$$\Gamma(-t) = \frac{(-1)^n}{n!}\frac{1}{-t+n} + h(t) \qquad \text{for } n-1 < \mathrm{Re}(t) < n+1$$

where h is analytic in the given region. (The Γ expresion follows from an integral representation of Γ obtained by considering a keyhole contour integral for the usual $\int_C s^{z-1}e^s ds$, followed by some manipulations and a series expansion of exp)
We see then that

$$\lim_{t\to n}[(t-n)f(x,t)] = a_{-1}(t_n) = \lim_{t\to n}[\frac{-1}{2\pi i}\frac{(t-n)\Gamma(-t)}{\Gamma(\nu+t+1)}(\frac{x}{2})^{2t+\nu}]$$

So it follows that

$$f(x,t) = \frac{(t-n)\Gamma(-t)}{\Gamma(\nu+t+1)}(\frac{x}{2})^{2t+\nu}$$

and

$$J_\nu(x) = \frac{-1}{2\pi i}\oint_C \frac{\Gamma(-t)}{\Gamma(\nu+t+1)}(\frac{x}{2})^{2t+\nu}dt$$

with C shown in figure 2.

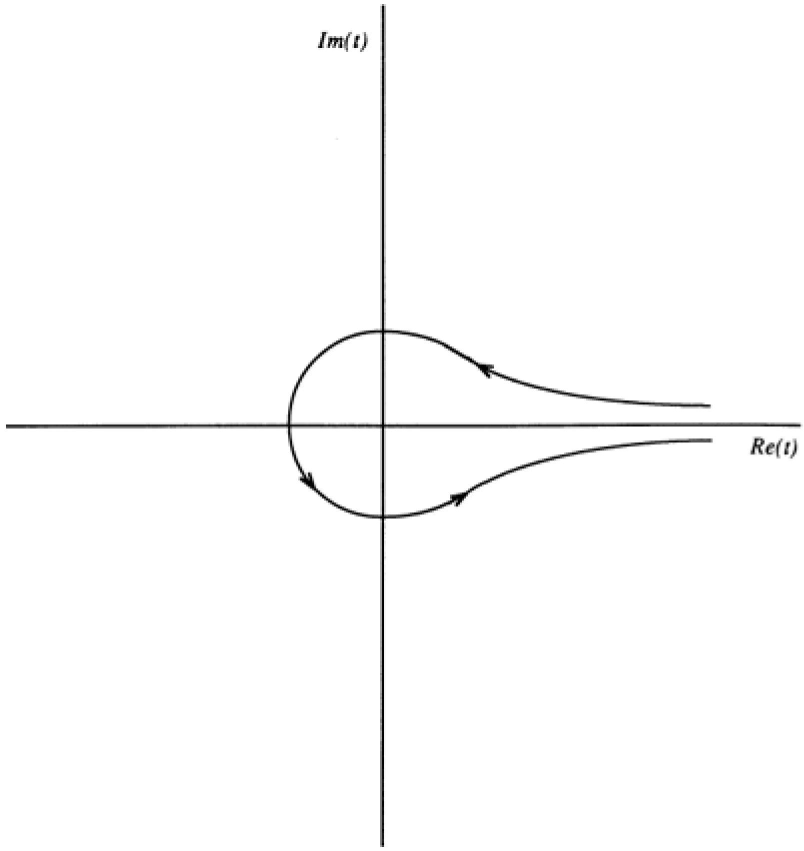

figure 2, from reference [2]

We can deform the contour to get figure 3 below and then extend and shrink the two half circles to get the so-called **Barnes's representation**

$$J_{\nu}(x) = \frac{1}{2\pi i}\int_{-i\infty}^{+\infty} \frac{\Gamma(-t)}{\Gamma(\nu+t+1)}(\frac{x}{2})^{2t+\nu}dt$$

given that $\mathrm{Re}(\nu) > 0$.

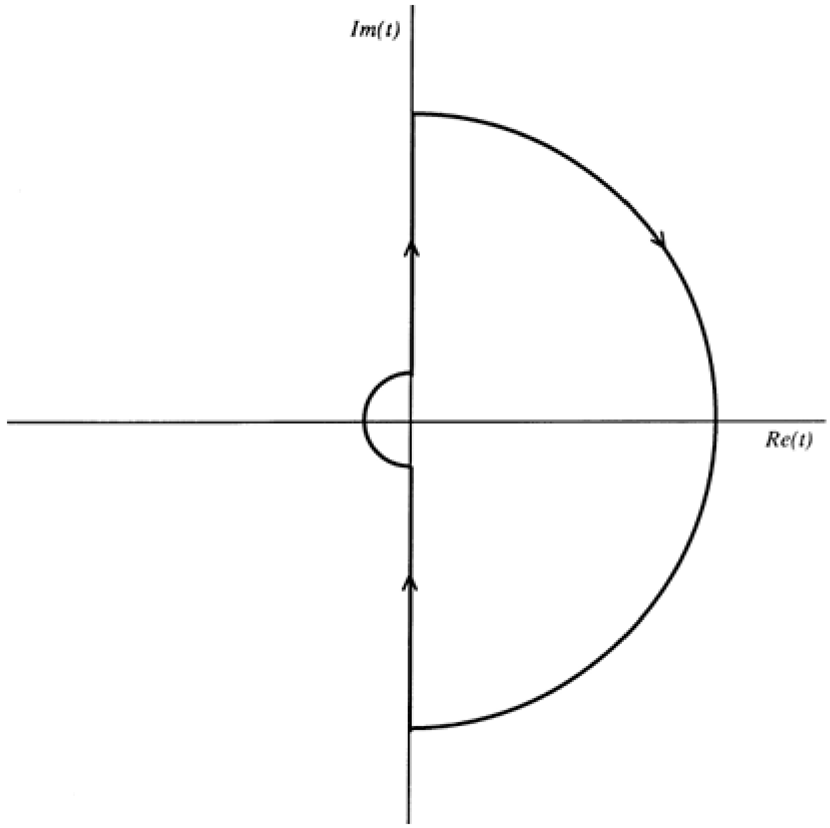

figure 3, from reference [2]

We can also obtain a more general Bessel integral representation that allows to use non-integer orders, which is done in reference [2] using again the same series representation for $J_\nu(x)$ but plugging in the integral expression for Γ and doing some manipulation and calculation with the result. The final expression is

$$J_\nu(x) = \frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)}(\frac{x}{2})^\nu \int_{-1}^{1} e^{-ixt}(1-t^2)^{\nu-\frac{1}{2}}dt \tag{14}$$

6 More Special Functions and their Integral Representations

We've touched on a couple of examples, but there are many other physics systems with their differential equations and solutions which can be considered to give us other special functions. From there, complex analysis can again be used to find integral representations and other relations. Here is just a short list of two more functions with their names and corresponding hypergeometric function.

Hermite Polynomials:

- $H_n(\rho) = \frac{n!(-1)^{-\frac{1}{2}n}}{(\frac{n}{2})!}\ {}_1F_1(-\frac{n}{2};\frac{1}{2};\rho^2)$ for even n
- $H_n(\rho) = \frac{2n!(-1)^{\frac{1}{2}(1-n)}}{(\frac{n-1}{2})!}\ {}_1F_1(-\frac{n-1}{2};\frac{3}{2};\rho^2)$ for odd n
- Physics Example: Appears when considering the harmonic oscillator in quantum mechanics

Laguerre Polynomials:

- $L_q(x) = q!\, {}_1F_1(-q, 1; x) = e^x \frac{d^q}{dx^q}(x^q + e^{-x})$
- Physics Example: They can describe the radial part of a hydrogenic atom

Other related functions follow like Associated Laguerre polynomials, Hankel functions, Neumann functions, Spherical Harmonics, etc.

One question someone could ask is *"Can we get integral representation of say the hypergeometric function?"*. The answer is yes. See reference [2] for more info.

7 Asymptotics

Given we find a solution to a problem in terms of special functions, we might want to know how the solution behaves at large values of the argument. For example, in quantum mechanics, we will often care about describing the particle as it's far away from a source potential. As an aside, interesting phenomenon can arise in considering these types of asymptotics when complex numbers are thrown into the mix. We will consider positive real arguments here though. See reference [1] for more information concerning asymptotics in the complex plane, with the so-called "Stokes phenomenon".

Working through just one asymptotic expression with the help of complex analysis, we consider scattering problems in quantum mechanics whose solutions often incorporate Bessel functions. Our detectors will usually be well approximated as "far away" from the potential source (eg. Coulomb potential) we are scattering off of, so we might like to know the large (real) argument asymptotic behavior of Bessel functions. It suffices to consider the asymptotic behavior of $J_\nu(s)$ (from which we can get the spherical Bessel function, Spherical Hankel functions, etc).

Theorem:

$$J_\nu(s) = \sqrt{\frac{2}{\pi s}} \cos(s - \frac{\pi\nu}{2} - \frac{\pi}{4}) + O(s^{-3/2}) \text{ as } s \to \infty$$

Proof: We will evaluate a contour integral of the Bessel function using expression 14 obtained earlier of

$$J_\nu(s) = \frac{1}{\Gamma(\nu + 1/2)\sqrt{\pi}} (\frac{s}{2})^\nu \int_{-1}^{1} e^{ist}((1-t^2)^{\nu-1/2} dt$$

We can ignore the constants in front and just consider $I(s) = \int_{-1}^{1} e^{ist}((1-t^2)^{\nu-1/2} dt$ for the time being.

First, we consider the contour Γ in the limit $R \to \infty$ and $\epsilon \to 0$ and pick the branch of $(1-z^2)^{\nu-1/2}$ that is positive when $z = x \in (-1, 1)$:

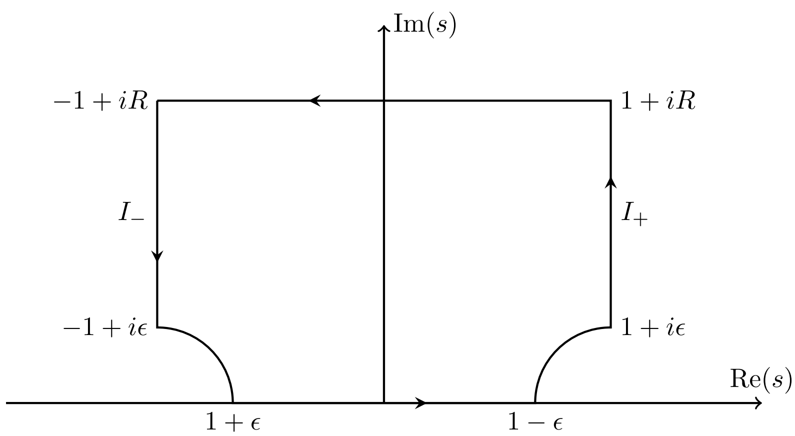

figure 4

The points ± 1 are outside the contour, so Cauchy's Integral Theorem says that $\int_\Gamma I(z)dz = 0$, which we will evaluate piecewise.

The contribution from the line from $1 + iR$ to $-1 + iR$ will vanish due to the exponential and the semi-arcs will likewise vanish.

Next, considering the right hand side vertical line and substituting in $z = 1 + iy$, we have

$$I_+(s) = ie^{is} \int_0^\infty e^{-sy}(1 - (1 + iy)^2)^{\nu - 1/2} dy$$

The left hand hand side verticle line component, $I_-(s)$, follows a similar expression. One might see that the forms of $I_\pm(s)$ are almost identical to the form of I, but the switch from e^{ist} (which oscillates as $s \to \infty$) to e^{-sy} (a decaying exponential as $s \to \infty$) will allow in fact allow for easier computations. Next we will need a lemma.

Lemma: Fix a and m, with $a > 0$ and $m < -1$. Then as $s \to \infty$, we have

$$\int_0^a e^{-sx} x^m dx = s^{-m-1}\Gamma(m+1) + O(e^{-cs})$$

for some positive c.

Proof of Lemma: Take

$$\int_0^a e^{-sx} x^m dx = \int_0^\infty e^{-sx} x^m dx - \int_a^\infty e^{-sx} x^m dx$$

With a change of variables of x to x/s, the first integral on the takes the form of the Gamma function

$$\int_0^\infty e^{-sx} x^m dx \to s^{-m-1} \int_0^\infty e^{-x} x^m dx = s^{-m-1}\Gamma(m+1)$$

For $c < a$, the other integral evaluates as

$$\int_a^\infty e^{-sx} x^m dx = e^{-cs} \int_a^\infty e^{-s(x-c)} x^m dx = O(e^{-cs})$$

where we have convergence of the last integral to a positive number by the comparison test with $\int_a^\infty e^{-t} dt$. (Note that $f(x) = O(g(x))$ for $x \in [a, b]$ means that $|f(x)| \leq Mg(x)$ for $x \in [a, b]$ and a positive real number M. In the limit of $x \to \infty$, we have $f(x) = O(g(x))$ if there exists some x_0 such that $|f(x)| \leq Mg(x)$ for all $x \geq x_0$.)

$\square$

Returning to the theorem, we see that

$$(1 - (1 + iy)^2))^{\nu - 1/2} = (-2iy)^{\nu - 1/2} + O(y^{\nu + 1/2}) \qquad \text{for } 0 \leq y \leq 1$$

For the equation in question, $m = \nu \mp 1/2$, so applying the lemma above, we see that

$$I_+(s) = i(-2i)^{\nu - 1/2} e^{is} s^{-\nu - 1/2}\Gamma(\nu + 1/2) + O(s^{-\nu - 3/2})$$

and

$$I_-(s) = i(2i)^{\nu - 1/2} e^{is} s^{-\nu - 1/2}\Gamma(\nu + 1/2) + O(s^{-\nu - 3/2})$$

Finally, restoring back the constants in $J_\nu(s)$ and using the relations $i = e^{i\pi/2}$, $-i = e^{-i\pi/2}$, we have

$$\begin{aligned}
J_\nu(s) &= \frac{(s/2)^\nu}{\Gamma(\nu + 1/2)\sqrt{\pi}}[-I_-(s) - I_+(s)] \\
&= \frac{(s/2)^\nu}{\sqrt{\pi}}[(2^{\nu+1/2})(e^{+i(s - \pi\nu/2 - \pi/2)})(s^{-\nu-1/2}) - (2^{\nu+1/2})(e^{-i(s - \pi\nu/2 - \pi/2)})(s^{-\nu-1/2}) + O(s^{-\nu-3/2})] \\
&= \sqrt{\frac{2}{\pi s}} \cos(s - \frac{\pi\nu}{2} - \frac{\pi}{4}) + O(s^{-3/2})
\end{aligned}$$

which proves the theorem.

$\square$

Interestingly enough as a side note, we can also get asymptotic expressions for the hypergeometric function ${}_2F_1(a, b; c; z)$ and also the confluent hypergeometric function ${}_1F_1(a; b; z)$.

8 Closing Comments

Having gone through the processes of obtaining a couple of special functions along with a few of their properties and various forms, as a short final word, more can also be said in terms of theory and results which were not touched on here. We saw the generating function for $J_\nu(x)$ and there exists generating functions for other special functions too. Orthogonality of the various special functions and more recursion relations can also be shown. Bessel functions can be shown to follow a multiplication theorem. etc.

References

[1] Meyer, R. E. (1989), "A simple explanation of the Stokes phenomenon", SIAM Rev., 31 (3): 435–445, doi:10.1137/1031090
[2] Seaborn, James "Hypergeometric Functions and Their Applications"
[3] Stein and Shakarchi, "Complex Analysis"