Statistics Problems

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Statistics: Part I

1. Two fair dice are rolled and their results are recorded.

   (a) Write out the sample space Ω for this experiment (it should have 36 options).
   The sample space should be

$$\begin{bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{bmatrix}$$

   (b) Consider the random variables $X : \Omega \to \mathbb{R}$ that sums the two numbers on each die. What is $P(X \geq 7)$? We're assuming a uniform base measure. So we simply need to count the number of combinations that sum to get ≥ 7 and divide by 36. It should be $\frac{21}{36} = \frac{7}{12}$.

2. A group of 300 students at a high school were surveyed and asked if 1) they ride a bike to school and 2) they take any AP classes. Results are recorded below.

Bike/AP	Yes	No	Total
Yes	60	30	90
No	140	70	210
Total	200	100	300

   (a) What is $P(\text{Bike} \cap \text{AP})$?
   The probability should be

$$\frac{60}{300}.$$

   (b) What is $P(\text{AP}|\text{doesn't Bike})$?
   The probability should be

$$\frac{140}{210}.$$

   The denominator we can get straight from the table, and is the number of people who don't bike.

   (c) Are the events Bike and AP independent?
   We can calculate

$$P(Bike) \cdot P(AP) = P(Bike \cap AP)$$
$$\frac{90}{300} \cdot \frac{200}{300} = \frac{60}{300}$$
$$\frac{3}{10} \cdot \frac{2}{3} = \frac{1}{5}$$

This is true, so the events are independent.

3. Urn A has three red balls and two white balls, and urn B has two red balls and five white balls. A fair coin is tossed. If it lands heads up, a ball is drawn from urn A; otherwise, it's drawn from urn B.

   (a) Use the law of total probability to find $P(\text{red ball is drawn})$.
   The law of total probability tells us that

   $$P(red) = P(red \mid H)P(H) + P(red \mid T)P(T) = \frac{3}{5} \cdot \frac{1}{2} + \frac{2}{7} \cdot \frac{1}{2}$$

   (b) Use Bayes' rule to find $P(\text{heads was flipped} \mid \text{red ball was drawn})$.
   We have by Bayes' rule that

   $$P(heads \mid red) = \frac{P(red \mid heads)P(heads)}{P(red)} = \frac{\frac{3}{5} \cdot \frac{1}{2}}{\frac{3}{5} \cdot \frac{1}{2} + \frac{2}{7} \cdot \frac{1}{2}}.$$

   where the denominator was taken from the previous part above.

4. A player throws darts at a target. On each trial, independently of the other trials, he hits the bulls-eye with probability .05. How many times should he throw so that his probability of hitting the bulls-eye at least once is .5?

   Observe first that for any number of trials n, we have

   $$P(\text{hits at least once}) = 1 - P(\text{never hits})$$

   because these are complementary events. For n independent throws, the probability of never hitting the target is $.95^n$ because the throws are independent. We thus have

   $$P(\text{hits at least once}) = 1 - .95^n > .5$$

   and we can check that this is true for $n \geq 14$ (note that we round up).

Statistics: Part II

You will need to refer to your density cheat sheet to solve these problems.

1. Which is more likely: 9 heads in 10 tosses of a fair coin or 18 heads in 20 tosses? Use a binomial probability calculator to compare the quantities $\binom{10}{9}.5^9(1-.5)^1$ and $\binom{20}{18}.5^{18}(1-.5)^2$ to see that the first one is larger.

2. A multiple-choice test consists of 20 items, each with four choices. A student is able to eliminate one of the choices on each question as incorrect and chooses randomly from the remaining three choices. A passing grade is 12 items or more correct. What is the probability the student passes?

   Use a binomial calculator with $n = 20, p = .3333$ to compute

   $$P(X \geq 12) = .01297$$

3. Suppose that the random variable X has density

   $$p(x) = \begin{cases} cx^2, 0 \leq x \leq 5 \\ 0, \text{otherwise} \end{cases}.$$

   What value must c be?

   We must have

   $$\int_0^5 cx^2 = c\frac{x^3}{3}\Big|_0^5 = \frac{125c}{3} = 1$$

   and so $c = \frac{3}{125}$.

4. Suppose X is a exponential random variable with parameter $\lambda = 1$. Find the number c such that

   (a) $P(X \leq c) = 0.8$

   (b) $P(X \leq c) = 0.5$

   This second number is the *median* of the distribution P.

   For part a) we set up the equation

   $$\int_0^c e^{-x} = 0.8$$

   where we used the density $\lambda e^{-\lambda x}$ for the exponential (recalling it's zero when $x < 0$). Integrating, we get

   $$-e^{-x}\Big|_0^c = -e^{-c} - (-e^0) = 1 - e^{-c}$$

   Setting this equal to 0.8 and solving we have

   $$\begin{aligned} 1 - e^{-c} &= 0.8 \\ e^{-c} &= 0.2 \\ c &= -\log(0.2) \end{aligned}$$

   where log denotes the natural log. By the exact same reasoning, the median for part b) is $c = -\log(0.5)$.

5. The *cumulative distribution function* or *cdf* of a continuous random variable X is defined as
$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} p(u)du.$$
Write the quantity $P(a \leq X \leq b)$ as an expression that only involves terms of the cdf F_X. The quantity $F(b) - F(a)$ gives the desired result. We can think of this as subtracting the area to the left of a from the area to the left of b. What's left is the area (probability) between a and b.

6. Compute
   (a) $\lim_{x\to\infty} F_X(x)$
   (b) $\lim_{x\to-\infty} F_X(x)$

   Hint: no calculus needed.

   We must have a) 1 and b) 0. To build intuition for this, examine images of cdf plots.

Statistics: Part III

1. Let X be Poisson random variable with parameter λ. Use the fact that $\sum_{j=0}^{\infty} \frac{\lambda^j}{j!} = e^{\lambda}$ to compute $\mathbb{E}(X)$.

   A Poisson random variable with parameter λ is such that

   $$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

   and so the expectation should be

   $$\begin{aligned}
   &\sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} \\
   =&\sum_{k=1}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} \\
   =&\lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} \\
   =&\lambda \sum_{k=0}^{\infty} \frac{\lambda^k e^{-\lambda}}{k!} \\
   =&\lambda \cdot 1 \\
   =&\lambda
   \end{aligned}$$

2. Evaluate

   $$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}. \tag{1}$$

   (Hint: Look at the density for a normal random variable. You don't need to find an antiderivative).

   The density of a standard normal random variable (i.e. with $\mu = 0, \sigma = 1$ is

   $$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

   As this integrates to 1, we must have

   $$\begin{aligned}
   \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} &= 1 \\
   \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} &= 1 \\
   \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} &= \sqrt{2\pi}
   \end{aligned}$$

3. For a Gaussian random variable $X \sim \mathcal{N}(\mu, \sigma^2)$, it turns out that

   $$\begin{aligned}
   \mathbb{E}(X) &= \mu \\
   \mathrm{Var}(X) &= \sigma^2.
   \end{aligned}$$

   Use this information to compute $\mathbb{E}(X^2)$.

Using the definition of variance and the linearity of expectation, we get

$$\begin{aligned}
\mathrm{Var}(X) &= \sigma^2 \\
\mathbb{E}\Big[(X-\mu)^2\Big] &= \sigma^2 \\
\mathbb{E}\Big[X^2 - 2\mu X + \mu^2] &= \sigma^2 \\
\mathbb{E}(X^2) - 2\mu\mathbb{E}(X) + \mathbb{E}(\mu^2) &= \sigma^2 \\
\mathbb{E}(X^2) - 2\mu\mu + \mu^2 &= \sigma^2 \\
\mathbb{E}(X^2) &= \mu^2 + \sigma^2
\end{aligned}$$

4. Let X be a Bernoulli random variable with parameter p, i.e.

$$\begin{aligned}
P(X=1) &= p \\
P(X=0) &- 1 - p.
\end{aligned}$$

Calculate $\mathbb{E}(X)$ and $\mathrm{Var}(X)$.

We can compute

$$\begin{aligned}
\mathbb{E}(X) &= 0 \cdot P(X=0) + 1 \cdot P(X=1) \\
&= 0 + p = p
\end{aligned}$$

and using this,

$$\begin{aligned}
\mathrm{Var}(X) &= \mathbb{E}(X-\mu)^2 \\
&= \mathbb{E}(X-p)^2 \\
&= (0-p)^2 P(X=0) + (1-p)^2 P(X=1) \\
&= p^2(1-p) + p(1-p)^2 \\
&= p(1-p)\big[p + 1 - p\big] \\
&= p(1-p).
\end{aligned}$$

5. Suppose that $\mathbb{E}(X) = \mu$ and $\mathrm{Var}(X) = \tau^2$. Let $Z = \frac{X-\mu}{\tau}$. Compute
   (a) $\mathbb{E}(Z)$
   (b) $\mathrm{Var}(Z)$

We have

$$\begin{aligned}
\mathbb{E}(Z) &= \mathbb{E}(\frac{X-\mu}{\tau}) \\
&= \frac{1}{\tau} \cdot \big[\mathbb{E}(X) - \mu\big] \\
&= \frac{1}{\tau} \cdot \big[0\big] \\
&= 0
\end{aligned}$$

by linearity and

$$\begin{aligned}\mathrm{Var}(Z) &= \mathbb{E}\Big(Z-0\Big)^2\\ &= \mathbb{E}\frac{(X-\mu)^2}{\tau^2}\\ &= \frac{1}{\tau^2}\Big(\mathbb{E}(X^2)-2\mu\mathbb{E}(X)+\mu^2\Big)\\ &= \frac{1}{\tau^2}\Big(\mathbb{E}(X^2)-\mu^2\Big)\\ &= \frac{1}{\tau^2}\Big(\mu^2+\tau^2-\mu^2\Big)\\ &= 1\end{aligned}$$

In the second to last line, we used the result of question three which holds even for random variables which aren't Gaussian.

6. Use the definition of variance and covariance to prove that

$$\mathrm{Var}(\alpha X+\beta Y)=\alpha^2\mathrm{Var}(X)+\beta^2\mathrm{Var}(Y)+2\alpha\beta\mathrm{Cov}(X,Y)$$

for random variables X,Y and scalars $\alpha,\beta\in\mathbb{R}$.

We can do all of this strictly from the definitions. First let $\mu=\mathbb{E}(X),\nu=\mathbb{E}(Y)$ so that the mean of the random variable is $\mathbb{E}(\alpha X+\beta Y)=\alpha\mu+\beta\nu$ by linearity. For ease, call this number c and we'll come back to it later.

$$\begin{aligned}\mathrm{Var}(\alpha X+\beta Y) &= \mathbb{E}\Big[\alpha X+\beta Y-c\Big]^2\\ &= \mathbb{E}((\alpha X+\beta Y)^2)-2c\mathbb{E}(\alpha X+\beta Y)+\mathbb{E}(c^2)\\ &= \mathbb{E}(\alpha^2X^2+2\alpha\beta XY+\beta^2Y^2)-2cc+c^2\\ &= \alpha^2\mathbb{E}(X^2)+2\alpha\beta\mathbb{E}(XY)+\beta^2\mathbb{E}(Y^2)-c^2\end{aligned}$$

where all we've used is FOIL (several times) and linearity of expectation, as well as the definition of the number c. Now we'll plug back in for c

$$\begin{aligned}&= \alpha^2\mathbb{E}(X^2)+2\alpha\beta\mathbb{E}(XY)+\beta^2\mathbb{E}(Y^2)-(\alpha\mu+\beta\nu)^2\\ &= \alpha^2\mathbb{E}(X^2)+2\alpha\beta\mathbb{E}(XY)+\beta^2\mathbb{E}(Y^2)-\alpha^2\mu^2-2\alpha\beta\mu\nu-\beta^2\nu^2\\ &= \alpha^2\Big(\mathbb{E}(X^2)-\mu^2\Big)+\beta^2\Big(\mathbb{E}(Y^2)-\nu^2\Big)+2\alpha\beta\Big(\mathbb{E}(XY)-\mu\nu\Big)\end{aligned}$$

Finally, we rely on a few facts, that you can show on your own: for any random variables X,Y we have

$$\begin{aligned}\mathrm{Var}(X) &= E(X^2)-\big[\mathbb{E}(X)\big[^2\\ \mathrm{Var}(Y) &= E(Y^2)-\big[\mathbb{E}(Y)\big[^2\\ \mathrm{Cov}(X,Y) &= E(XY)-\mathbb{E}(X)\mathbb{E}(Y).\end{aligned}$$

You should try to prove these on their own as mini-problems. But plugging these in to the above yields

$$\alpha^2\mathrm{Var}(X)+\beta^2\mathrm{Var}(Y)+2\alpha\beta\mathrm{Cov}(X,Y)$$

as desired.