Calculus Problems

Noah Kochanski

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Calculus: Part I

1. Using the familiar set notation $\mathbb{R}, \mathbb{Q}, \cap, \cup, \setminus, ^C$, write at least three different expressions that represent the set of all irrational real numbers.
2. Let $A = [0,1], B = [0,1), C = [0,2], D = [0,2)$. What is
   (a) $A \cap B^C$?
   (b) $(A \cup C) \cap (A \cup D)$?
   (c) $A^C \cap D^C$?
3. Let $A = \{1,2,3,4,5\}$ and $B = \{10, 20, 30\}$. Let $f : A \to B$.
   (a) Is it possible for f to be surjective? If yes, explicitly provide such a function f by providing values for $f(1)$,$f(2), \ldots, f(5)$. If no, explain why.
   (b) Is it possible for f to be injective? If yes, explicitly provide such a function f by providing values for $f(1)$,$f(2), \ldots, f(5)$. If no, explain why.
4. Let f be a function given by the formula $f(x) = \frac{1}{1+x}$.
   (a) What is $(f \circ f)(x)$? (Hint: You may need to find a common denominator).
   (b) What is the domain of the function $(f \circ f)(x)$?
5. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$f(x_1, x_2, x_3) = x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}. \tag{1}$$

   Is f injective? Surjective?

Calculus: Part II

1. Evaluate
$$\lim_{x \to 3} \frac{x^2 - 2x - 3}{x - 3}.$$

2. Is the function
$$f(x) = \begin{cases} \frac{x^2-2x-3}{x-3}, x \neq 3 \\ 3, x = 3 \end{cases}$$
continuous at $x = 3$? Explain why or why not.

3. Calculate the derivative of
$$f(x) = \sin(\cos(x)).$$

4. Calculate the derivative of
$$f(x) = \frac{(x-1)(x-4)}{(x-2)(x-3)}.$$

5. Find the value x that maximizes the function
$$\frac{1}{\sqrt{2\pi}} e^{-\frac{(x-5)^2}{2}}$$
by taking the derivative and setting it equal to zero, and solving for x.

Calculus: Part III

1. Find an antiderivative of
$$f(x) = x^2 - 5x^3 + cos(x)$$

2. Find an antiderivative of the function
$$f(x) = \frac{1}{x+5}.$$

3. Find an antiderivative of
$$f(x) = x\sqrt{16 - x^2}.$$

4. Evaluate the definite integral
$$\int_0^{\pi/2} cos(x)dx.$$

5. Evaluate the definite integral
$$\int_{-\infty}^{\infty} xe^{-x^2}dx.$$

Calculus: Part IV

1. Calculate the partial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ of

$$f(x,y) = \cos(x^2 + 2y) - e^{4x-y} + y^3.$$

2. Calculate the gradient of

$$-\frac{1}{2}\log 2\pi - \frac{1}{2}\log \sigma^2 - \frac{1}{2\sigma^2}(x-\mu)^2$$

with respect to the variables μ, σ^2. Evaluate the gradient at the point $(\mu, \sigma^2) = (0, 1)$.

3. Calculate

$$\int_0^1 \int_{-1}^2 xe^{xy} dxdy.$$

4. **(Challenge)** Compute the gradient of the *matrix function* $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ given by

$$f(\mathbf{X}) = a^\top \mathbf{X} b$$

for fixed vectors $a, b \in \mathbb{R}^n$. A gradient of a matrix function is computed the same way as a gradient of a vector function: taking the partial derivative with respect to each element of the matrix.

5. **(Challenge)** Let $X \sim \mathcal{N}(\mu, \Sigma)$ be a random vector with the given n-dimensional multivariate normal distribution. Use the change of variable formula to find the density of the random variable $Y = \mathbf{A}X$ for some invertible matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$. What is the distribution of Y?

(For your reference, if $Y = T(X)$ and X has density p_X, the the density of Y is given by

$$p_Y(y) = p_X(T^{-1}(y))\left|\det JT^{-1}(y)\right|$$

where JT^{-1} is the Jacobian of the inverse of the T.)