Linear Algebra Problems

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Linear Algebra: Part I

1. Compute the matrix product

$$\begin{bmatrix} 10 & -2 & 3 \\ 5 & 2 & -1 \\ 1 & 4 & 10 \\ 0 & 9 & -5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2 \\ 8 & 0 \\ 1 & -3 \end{bmatrix}.$$

2. Suppose $\beta \in \mathbb{R}^p$. Given a collection of data points $x_i \in \mathbb{R}^p, y_i \in \mathbb{R}, i = 1, \ldots, n$, suppose the following equation holds for all i:

$$y_i = \beta^T x_i.$$

Let

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, X = \begin{bmatrix} | & | & \cdots & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & \vdots & | \end{bmatrix}.$$

Write the equation $y_i = \beta^T x_i$ above simultaneously for all i as a single matrix-vector equation involving X, Y, β, and matrix operations on these.

3. Let

$$A = \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \vdots & | \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Describe in words the effect of multiplying A on the right by the following matrix.

$$S = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

4. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4.$$

Write down the expressions

(a) $x^T y$ (this is called the *inner product* of x and y).

(b) xy^T (this is called the *outer product* of x and y).

5. Let
$$A = \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \vdots & | \end{bmatrix} \in \mathbb{R}^{n\times n}$$
and consequently
$$A^T = \begin{bmatrix} -- & a_1^T & -- \\ -- & a_2^T & -- \\ \vdots & \vdots & \vdots \\ -- & a_n^T & -- \end{bmatrix} \in \mathbb{R}^{n\times n}.$$
Write the matrix-matrix product AA^T as a sum of outer products.

6. Prove the following relations for vectors $x \in \mathbb{R}^n$(look up the definitions for these from the slides if you forgot).
   (a) $||x||_\infty \leq ||x||_2$
   (b) $||x||_2 \leq ||x||_1$

      (you can use the fact that for positive numbers a_i we have $\sqrt{\sum_{i=1}^n a_i} \leq \sum_{i=1}^n \sqrt{a_i}$).

   Finally, write down a vector for which $||x||_\infty = ||x||_2 = ||x||_1$.

Linear Algebra: Part II

1. Show that $\mathbb{P}_2$, the set of polynomials of degree ≤ 2, is a vector space with operations $+, \cdot$ defined as

$$(a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

   and

$$\alpha \cdot ((a_0 + a_1x + a_2x^2)) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 \tag{1}$$

2. Find a basis for the vector space $\mathbb{P}_2$ above. In other words, find a set of polynomials $\{p_1, p_2, p_3\}$ such that every polynomial of degree ≤ 2 can be written as $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

3. Using the basis

$$\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$$

   for $\mathbb{R}^2$, represent the vectors

   (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   (b) $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$

   as linear combinations of the basis vectors above.

4. Write

$$\operatorname{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

   as a set of vectors in $\mathbb{R}^3$ (you can write it out in words, too).

5. (a) Show that the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \right\}$$

   is linearly dependent.

   (b) Using your work, find a vector x such that

$$\begin{bmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   (c) Is the linear transformation represented by the matrix

$$\begin{bmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix}$$

   injective? If yes, explain why. If not, prove it.

   (d) Is the matrix

$$\begin{bmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix}$$

   invertible? Explain why or why not.

Linear Algebra: Part III

1. For each linear transformation below, find its matrix with respect to the standard basis. (Hint: construct the matrix

$$\begin{bmatrix} | & | & \cdots & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & \vdots & | \end{bmatrix}$$

to get the result).

   (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x + 2y \\ 2x - 5y \\ 7y \end{bmatrix}$$

   (b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T(\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} w + x + y + z \\ x - z \\ w + 3x + 6z \end{bmatrix}$$

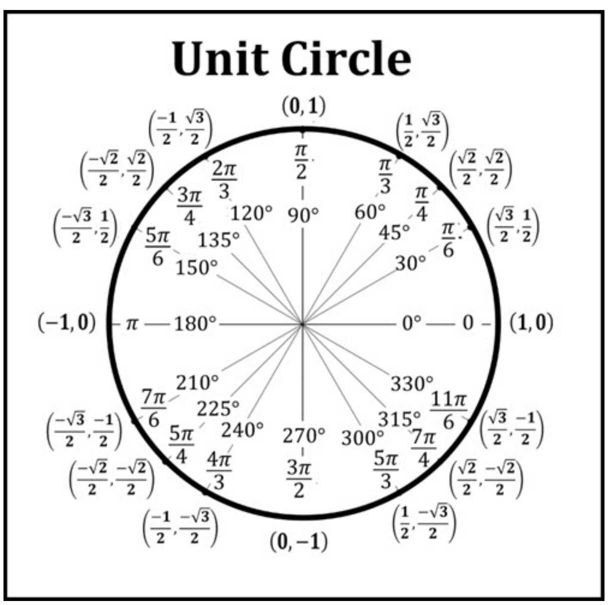
2. Find 3×3 matrices representing the following transformations on $\mathbb{R}^3$.

   (a) Project the vector onto the xy-plane (this has the effect of setting the third coordinate of the vector to zero).

   (b) Swaps the x and y coordinates of the vector.

3. Construct a non-zero matrix $A \in \mathbb{R}^{3\times 3}$ such that $A^2 = \mathbf{0}$. In words, describe what linear transformation this matrix performs. (A non-zero matrix contains at least one entry that is not equal to 0).

4. Use the information provided by the unit circle below to provide the matrix representing the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates the vector $60°$ counter-clockwise.



Linear Algebra: Part IV

1. Show the following statements hold for a matrix $A \in \mathbb{R}^{n\times n}$.
   (a) Use the fact that $\det(BC) = \det(B)\det(C)$ for any two matrices B, C to prove that if $\det(A) = 0$, then A is not an invertible matrix.
   (b) Use properties of the determinant to prove that if the columns of A are linearly dependent, then $\det(A) = 0$.

   These two facts together prove that if the columns of A are linearly dependent, then A is not invertible.

2. Compute the determinant of the 3×3 matrix $\begin{bmatrix} 4 & -1 & 0 \\ 2 & 0 & 3 \\ -1 & 4 & 2 \end{bmatrix}$ using the formula provided in the slides.

3. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

   by computing the characteristic polynomial $\det(A - \lambda I)$ and finding its roots.

   (Recall the formula for 2×2 determinants is given by $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$).

4. Given an SVD of the matrix $A \in \mathbb{R}^{n\times n}$, $A = U\Sigma V^T$, find an eigenvector decomposition of the matrix $A^T A$. That is, find matrices Q, Λ such that

$$A^T A = Q\Lambda Q^{-1}$$

   You can use the fact that for an orthogonal matrix, the transpose is the same as the inverse.