Calculus Problems

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Calculus: Part I

1. Using the familiar set notation $\mathbb{R}, \mathbb{Q}, \cap, \cup, \setminus, ^C$, write at least three different expressions that represent the set of all irrational real numbers.

   Some options are $\mathbb{R} \setminus \mathbb{Q}, \mathbb{Q}^C, (\mathbb{R} \cup \mathbb{Q}) \setminus \mathbb{Q}, (\mathbb{R} \cap \mathbb{Q})^C$.

2. Let $A = [0,1], B = [0,1), C = [0,2], D = [0,2)$. What is

   (a) $A \cap B^C$? $\{1\}$

   (b) $(A \cup C) \cap (A \cup D)$? $[0,2)$

   (c) $A^C \cap D^C$? $(-\infty, 0) \cup [2, \infty)$

3. Let $A = \{1,2,3,4,5\}$ and $B = \{10, 20, 30\}$. Let $f : A \to B$.

   (a) Is it possible for f to be surjective? If yes, explicitly provide such a function f by providing values for $f(1), f(2), \ldots, f(5)$. If no, explain why.

   Yes, one such f is $f(1) = 10, f(2) = 20, f(3) = f(4) = f(5) = 30$.

   (b) Is it possible for f to be injective? If yes, explicitly provide such a function f by providing values for $f(1), f(2), \ldots, f(5)$. If no, explain why.

   No; we have 3 "boxes" and 5 "items". By pigeonhole principle, there must be two values in A mapping to same value in B.

4. Let f be a function given by the formula $f(x) = \frac{1}{1+x}$.

   (a) What is $(f \circ f)(x)$? (Hint: You may need to find a common denominator).

   We have
   $$(f \circ f)(x) = \frac{1}{1 + \frac{1}{1+x}} = \frac{1}{\frac{1+x}{1+x} + \frac{1}{1+x}} = \frac{1+x}{2+x}$$

   (b) What is the domain of the function $(f \circ f)(x)$?

   By inspection, -2 is not in the domain. By composition, -1 is not in the domain as $f(-1)$ is undefined. So the domain is $\mathbb{R} \setminus \{-1, -2\}$.

5. Let $f : \mathbb{R}^3 \to \mathbb{R}^3$ be given by
   $$f(x_1, x_2, x_3) = x_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 3 \\ -4 \end{bmatrix}. \tag{1}$$

   Is f injective? Surjective?

   The function f is not injective. To see this, observe that $f(0,0,0) = (0,0,0)$ and also that $f(1,-1,1) = (0,0,0)$. As two inputs map to the same element, the function is not injective.

   The function f is not surjective. To show this a counterexample suffices. It turns out that there are no inputs such that $f(x_1, x_2, x_3) = (0,0,4)$.

Calculus: Part II

1. Evaluate
$$\lim_{x\to 3}\frac{x^2-2x-3}{x-3}.$$
We have
$$\lim_{x\to 3}\frac{x^2-2x-3}{x-3}=\lim_{x\to 3}\frac{\cancel{(x-3)}(x+1)}{\cancel{x-3}}=\lim_{x\to 3}x+1=4.$$

2. Is the function
$$f(x)=\begin{cases}\frac{x^2-2x-3}{x-3}, x\neq 3\\ 3, x=3\end{cases}$$
continuous at $x=3$? Explain why or why not.
No, because using part a) we have $\lim_{x\to 3}f(x)\neq f(3)$.

3. Calculate the derivative of
$$f(x)=\sin(\cos(x)).$$
Using the chain rule with $f(x)=\sin(x), g(x)=\cos(x)$ yields
$$f'(g(x))g'(x)=\cos(\cos(x))(-\sin(x))$$

4. Calculate the derivative of
$$f(x)=\frac{(x-1)(x-4)}{(x-2)(x-3)}.$$
We could use product and quotient rules, but it seems easier to multiply out and use the quotient rule once.
$$f(x)=\frac{x^2-5x+4}{x^2-5x+6}$$
and so
$$\begin{aligned}f'(x)&=\frac{(x^2-5x+6)\cdot(2x-5)-(x^2-5x+4)\cdot(2x-5)}{(x^2-5x+6)^2}\\&=\frac{(2x-5)\big[(x^2-5x+6)-(x^2-5x+4)\big]}{(x^2-5x+6)^2}\\&=\frac{(4x-10)}{(x^2-5x+6)^2}\end{aligned}$$

5. Find the value x that maximizes the function
$$\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-5)^2}{2}}$$
by taking the derivative and setting it equal to zero, and solving for x.
By the chain rule, the derivative is
$$\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-5)^2}{2}}\cdot\big[-(x-5)\big]=0\implies x=5$$

Calculus: Part III

1. Find an antiderivative of
$$f(x) = x^2 - 5x^3 + \cos(x)$$
One antiderivative is $F(x) = \frac{x^3}{3} - \frac{5x^4}{4} + \sin(x)$

2. Find an antiderivative of the function
$$f(x) = \frac{1}{x+5}.$$
One antiderivative is $F(x) = \log(x+5)$ where we use log to mean the natural logarithm. .

3. Find an antiderivative of
$$f(x) = x\sqrt{16 - x^2}.$$
Let $u = 16 - x^2$ so that $du = -2xdx$. Then by u-substitution we have
$$\begin{aligned} &\int -\frac{1}{2}du\sqrt{u} \\ &= -\frac{1}{2}\int u^{1/2}du \\ &= -\frac{1}{2}\cdot\frac{2}{3}u^{3/2} \\ &= -\frac{1}{3}(16 - x^2)^{3/2}. \end{aligned}$$

4. Evaluate the definite integral
$$\int_0^{\pi/2} cos(x)dx.$$
We have
$$\sin(x)\mid_0^{\pi/2} = 1 - 0 = 1.$$

5. Evaluate the definite integral
$$\int_{-\infty}^{\infty} xe^{-x^2}dx.$$
By u-substitution with $u = -x^2, du = -2xdx$ we have
$$\begin{aligned} &-\frac{1}{2}\int_{-\infty}^{\infty} e^u du \\ &= -\frac{1}{2}e^{-x^2}\mid_{-\infty}^{\infty} = 0 \end{aligned}$$

Calculus: Part IV

1. Calculate the partial derivatives $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ of
$$f(x,y) = \cos(x^2 + 2y) - e^{4x-y} + y^3.$$
We have
$$\frac{\partial}{\partial x} f(x,y) = -\sin(x^2 + 2y) \cdot (2x) - e^{4x-y} \cdot 4$$
$$\frac{\partial}{\partial y} f(x,y) = -\sin(x^2 + 2y) \cdot (2) - e^{4x-y} \cdot (-1) + 3y^2$$

2. Calculate the gradient of
$$-\frac{1}{2}\log 2\pi - \frac{1}{2}\log \sigma^2 - \frac{1}{2\sigma^2}(x-\mu)^2$$
with respect to the variables μ, σ^2. Evaluate the gradient at the point $(\mu, \sigma^2) = (0, 1)$. The partial derivative w/r/t μ is
$$\frac{1}{\sigma^2}(x-\mu)$$
and the partial derivative w/r/t σ^2 is
$$-\frac{1}{2\sigma^2} - \frac{(x-\mu)^2}{2} \cdot -(\sigma^2)^{-2}$$
so the gradient is
$$\nabla f = \begin{bmatrix} \frac{\partial}{\partial \mu} \\ \frac{\partial}{\partial \sigma^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sigma^2}(x-\mu) \\ -\frac{1}{2\sigma^2} + \frac{(x-\mu)^2}{2\sigma^4} \end{bmatrix}$$
Evaluating at (0,1) yields
$$\nabla f(0,1) = \begin{bmatrix} x \\ \frac{x^2-1}{2} \end{bmatrix} \tag{2}$$

3. Calculate
$$\int_0^1 \int_{-1}^2 x e^{xy} dx dy.$$
We first swap the order of integration to make it easier
$$\int_0^1 \int_{-1}^2 x e^{xy} dx dy = \int_{-1}^2 \int_0^1 x e^{xy} dy dx. \tag{3}$$
The inner integral is
$$\int_0^1 x e^{xy} dy = e^{xy} \Big|_{y=0}^{y=1} = e^x - 1$$
Now,
$$\begin{aligned} \int_{-1}^2 e^x - 1 &= e^x - x \Big|_{-1}^2 \\ &= (e^2 - 2) - (e^{-1} + 1) \\ &= e^2 - \frac{1}{e} - 3 \end{aligned}$$

4. **(Challenge)** Compute the gradient of the *matrix function* $f : \mathbb{R}^{n\times n} \to \mathbb{R}$ given by

$$f(\mathbf{X}) = a^\top \mathbf{X} b$$

for fixed vectors $a, b \in \mathbb{R}^n$. A gradient of a matrix function is computed the same way as a gradient of a vector function: taking the partial derivative with respect to each element of the matrix.

A simple 2×2 case should be enough to illustrate the solution. We have

$$\begin{aligned} f(\mathbf{X}) &= a^\top \mathbf{X} b \\ &= [a_1, a_2] \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\ &= a_1 b_1 x_{11} + a_1 b_2 x_{12} + a_2 b_1 x_{21} + a_2 b_2 x_{22} \end{aligned}$$

and taking the partial derivative with respect to each of the four components yields

$$\begin{aligned} \nabla f(\mathbf{X}) &= \begin{bmatrix} a_1 b_1 & a_1 b_2 \\ a_2 b_1 & a_2 b_2 \end{bmatrix} \\ &= ab^\top \end{aligned}$$

and it turns out this formula holds for any dimension n, not just the $n = 2$ case shown here.

5. **(Challenge)** Let $X \sim \mathcal{N}(\mu, \Sigma)$ be a random vector with the given n-dimensional multivariate normal distribution. Use the change of variable formula to find the density of the random variable $Y = \mathbf{A}X$ for some invertible matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$. What is the distribution of Y?

(For your reference, if $Y = T(X)$ and X has density p_X, the the density of Y is given by

$$p_Y(y) = p_X(T^{-1}(y)) \left| \det JT^{-1}(y) \right|$$

where JT^{-1} is the Jacobian of the inverse of the T.)

Up to constants (that don't change in our example), the density $p_X(x)$ is given by

$$\det(\Sigma)^{-1/2} \exp\left[(x-\mu)^\top \Sigma^{-1} (x-\mu) \right]$$

Observe also that $T^{-1}(y) = \mathbf{A}^{-1} y$ and $\det JT^{-1}(y) = \det A^{-1}$. Plugging in to the change of variable formula yields

$$p_Y(y) = \det(\Sigma)^{-1/2} \exp\left[(\mathbf{A}^{-1}y - \mu)^\top \Sigma^{-1} (\mathbf{A}^{-1}y - \mu) \right] \cdot |\det \mathbf{A}^{-1}|.$$

Observe that $\det \mathbf{A}^{-1} = \det \mathbf{A}^{-T}$ and so $\det \mathbf{A}^{-1} = (\det \mathbf{A})^{-1/2} \cdot (\det \mathbf{A}^T)^{-1/2}$ by properties of determinants. We can thus rewrite $\det(\Sigma)^{-1/2} |\det \mathbf{A}^{-1}| = ((\det \mathbf{A}) \det(\Sigma) (\det \mathbf{A}^T))^{-1/2}$ which is equivalent to $(\det(A \Sigma A^\top)^{-1/2}$.

Observe also that for the exponent we have

$$\begin{aligned} (\mathbf{A}^{-1}y - \mu)^\top \Sigma^{-1} (\mathbf{A}^{-1}y - \mu) &= (\mu - \mathbf{A}^{-1}y)^\top \Sigma^{-1} (\mu - \mathbf{A}^{-1}y) \\ &= (\mu - \mathbf{A}^{-1}y)^\top \mathbf{A}^\top \mathbf{A}^{-\top} \Sigma^{-1} \mathbf{A}^{-1} \mathbf{A} (\mu - \mathbf{A}^{-1}y) \\ &= (\mathbf{A}\mu - y)^\top \mathbf{A}^{-\top} \Sigma^{-1} \mathbf{A}^{-1} (\mathbf{A}\mu - y) \\ &= (y - \mathbf{A}\mu)^\top (\mathbf{A} \Sigma \mathbf{A}^\top)^{-1} (y - \mathbf{A}\mu) \end{aligned}$$

All of this work implies that the density of Y corresponds to that of a $\mathcal{N}(A\mu, A\Sigma A^\top)$ distribution.