Linear Algebra Problems

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Linear Algebra: Part I

1. Compute the matrix product

$$\begin{bmatrix} 10 & -2 & 3 \\ 5 & 2 & -1 \\ 1 & 4 & 10 \\ 0 & 9 & -5 \end{bmatrix} \cdot \begin{bmatrix} -2 & 2 \\ 8 & 0 \\ 1 & -3 \end{bmatrix}.$$

The result should be

$$\begin{bmatrix} -33 & 11 \\ 5 & 13 \\ 40 & -28 \\ 67 & 15 \end{bmatrix}$$

2. Suppose $\beta \in \mathbb{R}^p$. Given a collection of data points $x_i \in \mathbb{R}^p, y_i \in \mathbb{R}, i = 1, \ldots, n$, suppose the following equation holds for all i:

$$y_i = \beta^T x_i.$$

Let

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}, X = \begin{bmatrix} | & | & \cdots & | \\ x_1 & x_2 & \cdots & x_n \\ | & | & \vdots & | \end{bmatrix}.$$

Write the equation $y_i = \beta^T x_i$ above simultaneously for all i as a single matrix-vector equation involving X, Y, β, and matrix operations on these.

We can succinctly write $Y = X^T\beta$ to write all the equations simultaneously.

3. Let

$$A = \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \vdots & | \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Describe in words the effect of multiplying A on the right by the following matrix.

$$S = \begin{bmatrix} \sigma_1 & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & 0 & \cdots & 0 \\ 0 & 0 & \sigma_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \sigma_n \end{bmatrix}$$

This multiplication *scales the columns* of A, multiplying a_1 by σ_1, a_2 by σ_2, etc. .

4. Let

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4.$$

Write down the expressions

(a) $x^T y$ (this is called the *inner product* of x and y). $x^T y = x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4 \in \mathbb{R}$

(b) xy^T (this is called the *outer product* of x and y). We have

$$xy^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix} = \begin{bmatrix} x_1y_1 & x_1y_2 & x_1y_3 & x_1y_4 \\ x_2y_1 & x_2y_2 & x_2y_3 & x_2y_4 \\ x_3y_1 & x_3y_2 & x_3y_3 & x_3y_4 \\ x_4y_1 & x_4y_2 & x_4y_3 & x_4y_4 \end{bmatrix}$$

5. Let

$$A = \begin{bmatrix} | & | & \cdots & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & \vdots & | \end{bmatrix} \in \mathbb{R}^{n \times n}$$

and consequently

$$A^T = \begin{bmatrix} -- & a_1^T & -- \\ -- & a_2^T & -- \\ \vdots & \vdots & \vdots \\ -- & a_n^T & -- \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

Write the matrix-matrix product AA^T as a sum of outer products.

It turns to be that

$$AA^T = \sum_{i=1}^{n} a_i a_i^T.$$

This can be verified by writing out each entry of both sides fully.

6. Prove the following relations for vectors $x \in \mathbb{R}^n$(look up the definitions for these from the slides if you forgot).

(a) $||x||_\infty \leq ||x||_2$

(b) $||x||_2 \leq ||x||_1$

(you can use the fact that for positive numbers a_i we have $\sqrt{\sum_{i=1}^n a_i} \leq \sum_{i=1}^n \sqrt{a_i}$).

Finally, write down a vector for which $||x||_\infty = ||x||_2 = ||x||_1$.

For the first part, we have

$$\begin{aligned} ||x||_\infty &= \max_{i=1,\dots,n} |x_i| \\ &= \max_{i=1,\dots,n} \sqrt{|x_i|^2} \\ &= \max_{i=1,\dots,n} \sqrt{x_i^2} \\ &\leq \sqrt{\sum_{i=1}^{n} x_i^2} \\ &= ||x||_2 \end{aligned}$$

and for the second part we have

$$\begin{aligned}
||x||_2 &= \sqrt{\sum_{i=1}^{n} x_i^2} \\
&\leq \sum_{i=1}^{n} \sqrt{x_i^2} \quad \text{by the fact provided} \\
&= \sum_{i=1}^{n} |x_i| \\
&= ||x||_1
\end{aligned}$$

A vector where all are equal is $x = (1,0,\ldots,0)^\top$.

Linear Algebra: Part II

1. Show that $\mathbb{P}_2$, the set of polynomials of degree ≤ 2, is a vector space with operations $+, \cdot$ defined as

$$(a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$$

and

$$\alpha \cdot ((a_0 + a_1 x + a_2 x^2)) = \alpha a_0 + \alpha a_1 x + \alpha a_2 x^2 \tag{1}$$

We can verify that all the eight axioms are satisfied. We omit the notation for "$\cdot$" occasionally where it's obvious we perform multiplication. Let p, p_1, p_2, p_3 denote arbitrary elements of $\mathbb{P}_2$, and $\alpha, \beta \in \mathbb{R}$.

   (a) $p_1 + p_2 = p_2 + p_1$
   (b) $(p_1 + p_2) + p_3 = p_1 + (p_2 + p_3)$
   (c) The zero polynomial $\vec{0} = 0 + 0x + 0x^2$ suffices.
   (d) We can define $-p$ to be the polynomial with all coefficients negated to get the result.
   (e) $1 \cdot p = p$
   (f) $(\alpha\beta)p = \alpha(\beta p)$
   (g) $\alpha(p_1 + p_2) = \alpha p_1 + \alpha p_2$
   (h) $(\alpha + \beta)p = \alpha p + \beta p$

2. Find a basis for the vector space $\mathbb{P}_2$ above. In other words, find a set of polynomials $\{p_1, p_2, p_3\}$ such that every polynomial of degree ≤ 2 can be written as $\alpha_1 p_1 + \alpha_2 p_2 + \alpha_3 p_3$ for some $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$.

   An easy choice of basis, in fact the standard basis for this vector space, is $p_1 = 1, p_2 = x, p_3 = x^2$, in other words the monomials.

3. Using the basis

$$\left\{ \begin{bmatrix} -5 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix} \right\}$$

   for $\mathbb{R}^2$, represent the vectors

   (a) $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
   We can check that

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 \\ 2 \end{bmatrix} + \frac{7}{6} \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

   (b) $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ We can check that

$$\begin{bmatrix} -2 \\ 2 \end{bmatrix} = 1 \begin{bmatrix} -5 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

   as linear combinations of the basis vectors above.

4. Write

$$\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

   as a set of vectors in $\mathbb{R}^3$ (you can write it out in words, too). In words, this set of vectors is the xy-plane, a subset of $\mathbb{R}^3$. In other words, the set of all vectors in $\mathbb{R}^3$ with third entry equal to 0. Formally,

$$\text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| z = 0 \right\}$$

5. (a) Show that the set of vectors

$$\left\{ \begin{bmatrix} 3 \\ 4 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \right\}$$

is linearly dependent. We can verify that $v_1 + 2v_2 - 3v_3 = 0$ where v_1, v_2, v_3 are shorthand for the three vectors. This suffices as a condition for linear dependence. To show precisely that one of the vectors is a linear combination of the others as in class, we can solve for any of the three, e.g.

$$v_1 = -2v_2 + 3v_3$$

(b) Using your work, find a vector x such that

$$\begin{bmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using the definitions of matrix-vector products, we have

$$\begin{bmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

by our work in part a).

(c) Is the linear transformation represented by the matrix

$$\begin{bmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix}$$

injective? If yes, explain why. If not, prove it. No. This mapping maps the vectors $\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ to the same point, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This means the transformation is not injective.

(d) Is the matrix

$$\begin{bmatrix} 3 & 0 & 1 \\ 4 & 1 & 2 \\ -4 & 2 & 0 \end{bmatrix}$$

invertible? Explain why or why not. No, the linear transformation represented by this matrix is not invertible (because it's not injective), and so the matrix is not invertible.

Linear Algebra: Part III

1. For each linear transformation below, find its matrix with respect to the standard basis. (Hint: construct the matrix

$$\begin{bmatrix} | & | & \cdots & | \\ T(e_1) & T(e_2) & \cdots & T(e_n) \\ | & | & \vdots & | \end{bmatrix}$$

to get the result).

(a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x + 2y \\ 2x - 5y \\ 7y \end{bmatrix}$$

Following the hint, the matrix should be $\begin{bmatrix} 1 & 2 \\ 2 & -5 \\ 0 & 7 \end{bmatrix}$ because $T(e_1) = [1,2,0]^T, T(e_2) = [2, -5, 7]^T$.

(b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by

$$T(\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} w + x + y + z \\ x - z \\ w + 3x + 6z \end{bmatrix}$$

Following the hint, the matrix should be $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 3 & 0 & 6 \end{bmatrix}$.

2. Find 3×3 matrices representing the following transformations on $\mathbb{R}^3$.

(a) Project the vector onto the xy-plane (this has the effect of setting the third coordinate of the vector to zero). We ought to have by the same reasoning as above

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) Swaps the x and y coordinates of the vector. We ought to have by the same reasoning as above

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3. Construct a non-zero matrix $A \in \mathbb{R}^{3\times 3}$ such that $A^2 = \mathbf{0}$. In words, describe what linear transformation this matrix performs. (A non-zero matrix contains at least one entry that is not equal to 0).

You can verify that the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

works. We can describe what this matrix does by looking at the three columns: 1) it maps $(1,0,0)^\top$ to zeo, so it sets the first coordinate of any input to zero, 2) ditto with second coordinate, 3) it maps $(0,0,1)^\top$ to $(1,0,0)^\top$; this swaps the third coordinate and the first coordinate. We would write out

$$T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} z \\ 0 \\ 0 \end{bmatrix}$$

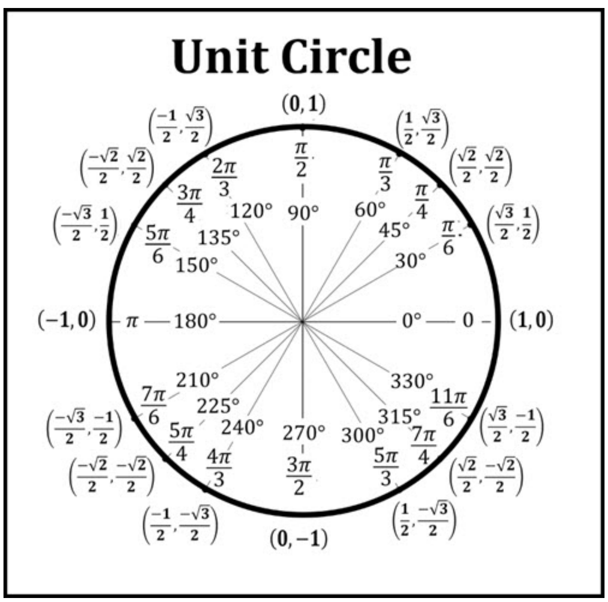
4. Use the information provided by the unit circle below to provide the matrix representing the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that rotates the vector 60° counter-clockwise. We see from the chart that such a transformation maps

$$T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} \frac{-\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

and so the matrix should be

$$\begin{bmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$



Linear Algebra: Part IV

1. Show the following statements hold for a matrix $A \in \mathbb{R}^{n\times n}$.

   (a) Use the fact that $\det(BC) = \det(B)\det(C)$ for any two matrices B, C to prove that if $\det(A) = 0$, then A is not an invertible matrix.

   If $\det(A) = 0$, then for any other matrix B we have $\det(AB) = \det(A)\det(B) = 0$. This means there cannot exist a matrix A^{-1} for that would imply $\det(AA^{-1}) = \det(I) = 1$, a contradiction.

   (b) Use properties of the determinant to prove that if the columns of A are linearly dependent, then $\det(A) = 0$.

   Without loss of generality, suppose $a_1 = \alpha_2 a_2 + \cdots + \alpha_n a_n$. Then using multilinearity of the determinant we have

$$\det\left(\begin{bmatrix} | & \cdots & | \\ a_1 & \cdots & a_n \\ | & \cdots & | \end{bmatrix}\right) = \begin{bmatrix} | & \cdots & | \\ \alpha_2 a_2 + \cdots + \alpha_n a_n & \cdots & a_n \\ | & \cdots & | \end{bmatrix}\Bigg)$$

$$= \sum_{j=2}^{n} \alpha_j \det\left(\begin{bmatrix} | & \cdots & | \\ a_j & \cdots & a_n \\ | & \cdots & | \end{bmatrix}\right)$$

   where each matrix in the sum above is the original matrix A with the first column replaced by the jth column. Now, using the third property of determinants, that if any two columns are the same the determinant is zero, we see that all of these determinants are zero, and hence the answer is zero.

   These two facts together prove that if the columns of A are linearly dependent, then A is not invertible.

2. Compute the determinant of the 3×3 matrix $\begin{bmatrix} 4 & -1 & 0 \\ 2 & 0 & 3 \\ -1 & 4 & 2 \end{bmatrix}$ using the formula provided in the slides.

   The determinant turns out to be -41.

3. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

   by computing the characteristic polynomial $\det(A - \lambda I)$ and finding its roots.

   (Recall the formula for 2×2 determinants is given by $\det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc$).

   We have

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix}$$

   and so using the formula, the determinant is

$$-\lambda(-3-\lambda) + 2 = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

   so the zeroes are given by $\lambda = -1, -2$, and thus these are the eigenvalues.

4. Given an SVD of the matrix $A \in \mathbb{R}^{n\times n}$, $A = U\Sigma V^T$, find an eigenvector decomposition of the matrix $A^T A$. That is, find matrices Q, Λ such that

$$A^T A = Q\Lambda Q^{-1}$$

You can use the fact that for an orthogonal matrix, the transpose is the same as the inverse.

We'll need to use the fact that for any three matrices, we have $(ABC)^\top = C^\top B^\top A^\top$. Writing out the product we have

$$\begin{aligned} A^\top A &= (U\Sigma V^\top)^\top (U\Sigma V^\top) \\ &= V^{\top\top}\Sigma^\top U^\top U\Sigma V^\top \end{aligned}$$

Now, the transpose of a transpose yields back the original matrix. Also, recall that in the SVD both U and V are orthogonal so $U^\top U = UU^\top = I$ and likewise with V. Simplifying,

$$= V\Sigma^\top \Sigma V^\top$$

Finally, again by orthogonality we have $V^\top = V^{-1}$ and so we have

$$= V\Sigma^\top \Sigma V^{-1}$$

which is an eigendecomposition.